

The light asymptotic limit of conformal blocks in $\mathcal{N} = 1$ super Liouville field theory

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ABSTRACT: Analytic expressions for the two dimensional $\mathcal{N} = 1$ SLFT blocks in the light semi-classical limit are found for both Neveu-Schwarz and Ramond sectors. The calculations are done by using the duality between $SU(2)$ $\mathcal{N} = 2$ super-symmetric gauge theories living on R^4/Z_2 space and two dimensional $\mathcal{N} = 1$ super Liouville field theory. It is shown that in the light asymptotic limit only a restricted set of Young diagrams contributes to the partition function. This enables us to sum up the instanton series explicitly and find closed expressions for the corresponding $\mathcal{N} = 1$ SLFT four point blocks in the light asymptotic limit.

KEYWORDS: Conformal Field Theory, Supersymmetric Gauge Theory, Supersymmetry and Duality

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1 Introduction

Two dimensional conformal field theory (CFT) [1] is relevant in statistical physics while studying second order phase transitions, and it is also an important building block in String theory [2]. An important example of CFT is Liouville field theory (LFT) [3] which is a bosonic field theory with exponential interaction. This theory is endowed with the spin two conserved currents that are the holomorphic and anti-holomorphic components of the stress energy tensor. The Fourier components of these currents obey the Virasoro algebra. There

are more general CFTs which in addition to the spin two currents include also conserved currents with higher spins [4]. The corresponding symmetry algebra is called \mathcal{W} algebra. Important examples of theories that enjoy \mathcal{W} symmetry are Toda field theories. These theories generalize LFT to the case of several interacting scalar fields.

As a first step on the way of constructing a full fledged quantum theory it is instructive to investigate its quasi classical limit. In both Liouville and Toda theories one can distinguish three types of quasi classical limits. These are mini-superspace, heavy and light limits. All three are the large central charge limits. They differ from each other by the behavior of primary fields under consideration. The primary fields are given by the vertex operators $V_\alpha = e^{i\alpha\phi}$. In the light limit we choose $\alpha = \eta b$ and send b to zero. Thus we take the large central charge limit keeping the conformal dimension finite.

The AGT correspondence [5] connects $2d$ conformal blocks in LFT to the Nekrasov Partition Function [6–8] of the four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories. The AGT correspondence is a powerful tool not only for deriving correlation functions in $2d$ CFTs but also for studying gauge theories by applying CFT methods. The Nekrasov partition function can be represented as a sum over Young diagrams [6, 8, 9] which according to the AGT correspondence can be used to compute conformal blocks in $2d$ LFT. In [10] the $U(N)$ Nekrasov partition function in the light asymptotic limit was considered. It was proved that in this limit for a specific choice of fields in the Nekrasov partition function contribute only Young diagrams whose number of rows does not exceed $(N - 1)$. This simplification makes it possible to write an explicit formula for the partition function in this limit. After applying AGT duality a large class of \mathcal{W}_N light conformal blocks for arbitrary N 's has been obtained.

$\mathcal{N} = 1$ super Liouville field theory (SLFT) [11] is an important example of $\mathcal{N} = 1$ super conformal field theory (SCFT) [12–15]. In [16, 17] an AGT like correspondence between the $\mathcal{N} = 1$ SLFT and the $U(2)$ super-symmetric gauge theories living on the space R^4/Z_2 is given.

Besides the spin two conserved currents (energy-momentum tensor) SLFT includes also spin $3/2$ currents (the super-currents). These currents generate super conformal symmetry which in $2d$ is described by the Neveu-Schwarz-Ramond algebra [12, 14, 15]. If upon encircling a field by the super-current an extra multiplier -1 is produced, one refers to this field as a Ramond field. Those fields which are local with respect to the super current are called Neveu-Schwarz fields.

In this paper different $\mathcal{N} = 1$ SLFT blocks in the light limit are derived by using the above mentioned duality between super Yang-Mills theory and $2d$ SCFT. We obtained that in the case of SLFT the analysis of the light limit is more subtle and complicated compare to the bosonic Liouville theory. In particular we found that in the light limit to the conformal blocks contribute not only one row diagrams. For instance the instanton partition functions that correspond to the conformal blocks with four Ramond fields also get contribution from the diagrams, like those in figures (3(b)) and (3(c)) below.

The paper is organized as follows. In section 2 the expression for the instanton partition functions of $\mathcal{N} = 2$ SYM on R^4/Z_2 [19, 20] is reviewed. In section 3 we bring known facts for $\mathcal{N} = 1$ SLFT and its light asymptotic limit that will be useful for us. In subsection 4.1

the map between $\mathcal{N} = 1$ super Liouville conformal blocks and $\mathcal{N} = 2$ SYM on R^4/Z_2 is given. In subsection 4.2 the rules for the light asymptotic limit are written. In section 5 we present new results on various partition function in the light limit. In section 6 by using these partition functions we give the corresponding conformal blocks in the light limit. In appendix B some technical points on the instanton partition function of SU(2) gauge theories on R^4/Z_2 are reviewed. In appendix C we proved that in the light limit to the instanton partition function contribute only the Young diagrams depicted at figure 3 and, in appendix D computations of these partition functions in the light limit are given.

2 The partition functions of $\mathcal{N} = 2$ SYM on R^4/Z_2

Let us consider $\mathcal{N} = 2$ SYM theory with a U(2) gauge group on the space R^4/Z_2 . The instanton part of the partition function for this theory can be represented as (see [19, 20])

$$Z_{(u_1, u_2), (v_1, v_2)}^{(q_1, q_2)}(\vec{a}^{(0)}, \vec{a}^{(1)}, \vec{a}^{(2)} | q) = \sum_{\{\vec{Y}^{\vec{q}}\}} F_{\vec{Y}^{\vec{q}}(u_1, u_2), (v_1, v_2)}^{(q_1, q_2)}(\vec{a}^{(0)}, \vec{a}^{(1)}, \vec{a}^{(2)}) q^{\frac{|\vec{Y}|}{2}}. \quad (2.1)$$

The sum goes over the pairs of Young diagrams $\vec{Y}^{\vec{q}} = (Y_1^{q_1}, Y_2^{q_2})$ colored in chess like order. To each diagram one ascribes a \mathbb{Z}_2 charge $q_i, i = 1, 2$ which indicates the color of the corner and takes values 0 or 1 (white or black correspondingly). $|\vec{Y}|$ is the total number of boxes in Y_1 and Y_2 , and q is the instanton counting parameter. Let us clarify our conventions on gauge theory parameters $a_i^{(0,1,2)}, i = 1, 2$. The parameters $a_i^{(1)}$ are expectation values of the scalar field in vector multiplet. Without loss of generality we will assume that the “center of mass” of these expectation values is zero

$$\vec{a}^{(1)} = \frac{1}{2} (a_1^{(1)} + a_2^{(1)}) = 0, \quad (2.2)$$

since a nonzero center of mass can be absorbed by shifting hypermultiplet masses. Furthermore $a_i^{(0)} (a_i^{(2)})$ are the masses of fundamental (anti-fundamental) hypers.

The expansion coefficient of the instanton partition function (2.1) is given by

$$F_{\vec{Y}^{\vec{q}}(u_1, u_2), (v_1, v_2)}^{(q_1, q_2)}(\vec{a}^{(0)}, \vec{a}^{(1)}, \vec{a}^{(2)}) = \prod_{i=1}^2 \prod_{j=1}^2 \frac{Z_{bf}(u_i, a_i^{(0)}, \emptyset | q_j, a_j^{(1)}, Y_j) Z_{bf}(q_i, a_i^{(1)}, Y_i | v_j, a_j^{(2)}, \emptyset)}{Z_{bf}(q_i, a_i^{(1)}, Y_i | q_j, a_j^{(1)}, Y_j)}, \quad (2.3)$$

where

$$Z_{bf}(x, a, \lambda | y, b, \mu) = \prod_{s \in \lambda^*} (a - b - \epsilon_1 L_\mu(s) + \epsilon_2 (1 + A_\lambda(s))) \prod_{s \in \mu^*} (a - b + \epsilon_1 (1 + L_\lambda(s)) - \epsilon_2 A_\mu(s)). \quad (2.4)$$

Here ϵ_1 and ϵ_2 are the Ω -background parameters. We will use the notation $\epsilon = \epsilon_1 + \epsilon_2$. $A_\lambda(s) (L_\lambda(s))$ is the arm-length (leg-length) of the square s towards the Young diagram λ , defined as oriented vertical (horizontal) distance of the square s to outer boundary of the

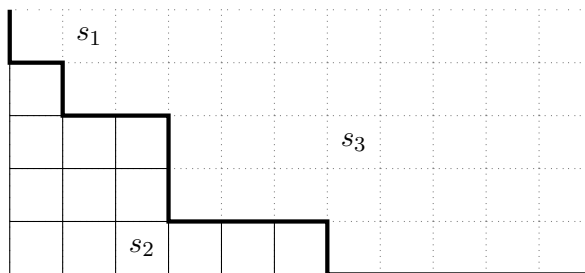


Figure 1. Arm and leg length with respect to the Young diagram whose borders are outlined by dark black: $A(s_1) = -2$, $L(s_1) = -2$, $A(s_2) = 2$, $L(s_2) = 3$, $A(s_3) = -3$, $L(s_3) = -4$.

Young tableau λ (see figure 1). λ^* , μ^* are subsets of boxes λ and μ respectively such that, a box of λ (μ) belongs to λ^* (μ^*) if and only if the replacement

$$\epsilon_1, \epsilon_2 \rightarrow 1; a \rightarrow x; b \rightarrow y \quad (i = 1, 2) \quad (2.5)$$

in the first (second) multiplier of (2.4) results in 0 (mod 2) (remind that u_i and v_i ($i = 1, 2$) take values 0 or 1). For more details see appendix B.

According to the duality between $\mathcal{N} = 2$ SYM on R^4/Z_2 and $\mathcal{N} = 1$ SLFT these partition functions are directly related to four point conformal blocks in $\mathcal{N} = 1$ SLFT. Before describing this relation let us briefly recall few facts about $\mathcal{N} = 1$ SLFT itself.

3 Known facts on $\mathcal{N} = 1$ SLFT and its light asymptotic limit

Super-Liouville field theory is a supersymmetric generalization of the bosonic Liouville theory, which is known to be the theory of matter induced gravity in two dimensions. Similarly SLFT describes 2d supergravity, induced by supersymmetric matter. Super-Liouville field theory on a two-dimensional surface with metric g_{ab} is given by the Lagrangian density

$$\mathcal{L} = \frac{1}{2\pi} g^{ab} \partial_a \varphi \partial_b \varphi + \frac{1}{2\pi} (\psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi}) + 2i\mu b^2 \bar{\psi} \psi e^{b\varphi} + 2\pi \mu^2 b^2 e^{2b\varphi}. \quad (3.1)$$

There are two kinds of fields in $2d$ $\mathcal{N} = 1$ SLFT called Neveu-Schwarz and Ramond fields to be specified below. The symmetries of the theory are generated by the energy-momentum tensor and the superconformal currents

$$T = -\frac{1}{2} (\partial \varphi \partial \varphi - Q \partial^2 \varphi + \psi \partial \psi), \quad (3.2)$$

$$G = i(\psi \partial \varphi - Q \partial \psi). \quad (3.3)$$

Commutation relation of the Neveu-Schwarz-Ramond algebra are

$$[L_m, L_n] = (m - n) L_{m+n} + \frac{c}{12} m(m^2 - 1) \delta_{m+n}, \quad (3.4)$$

$$[L_m, G_k] = \frac{m - 2k}{2} G_{m+k}, \quad (3.5)$$

$$\{G_k, G_l\} = 2L_{l+k} + \frac{c}{3} \left(k^2 - \frac{1}{4} \right) \delta_{k+l}, \quad (3.6)$$

with the central charge

$$c_L = \frac{3}{2} + 3Q^2, \text{ where } Q = b + b^{-1}. \quad (3.7)$$

Here L_m and G_k are the Laurent series coefficients of the currents T and G respectively. For the Ramond algebra k and l take integer and for the Neveu-Schwarz algebra half-integer values.

It is known that in the Neveu-Schwarz sector at the light asymptotic limit the symmetry algebra reduces to the finite subalgebra generated by $L_0, L_{\pm 1}, G_{\pm 1/2}$ only. Notice that for this subalgebra the central extension terms in (3.4) and (3.6) disappear. For the mentioned values of m and l (3.4)–(3.6) is obviously closed. For the Ramond sector its light asymptotic limit is more subtle and needs to be clarified yet.

NS primary fields $\Phi_\alpha(z, \bar{z})$ in this theory, $\Phi_\alpha(z, \bar{z}) = e^{\alpha\varphi(z, \bar{z})}$, have conformal dimensions

$$\Delta_\alpha^{NS} = \frac{1}{2}\alpha(Q - \alpha). \quad (3.8)$$

Introduce also the field that is the highest component of the NS superfield build from Φ_α

$$\Phi_{\tilde{\alpha}}(z, \bar{z}) = G_{-1/2}\bar{G}_{-1/2}\Phi_\alpha(z, \bar{z}), \quad (3.9)$$

with dimension

$$\tilde{\Delta}_\alpha^{NS} = \Delta_\alpha^{NS} + 1/2, \quad (3.10)$$

and as well as the Ramond primary fields defined as

$$R_\alpha^\pm(z, \bar{z}) = \sigma^\pm(z, \bar{z})e^{\alpha\varphi(z, \bar{z})} \quad (3.11)$$

where σ^\pm is the spin field with dimension $1/16$. Thus the dimension of a Ramond operator is

$$\Delta_\alpha^R = \frac{1}{16} + \frac{1}{2}\alpha(Q - \alpha). \quad (3.12)$$

4 $\mathcal{N} = 1$ super Liouville conformal blocks and their relation to the $\mathcal{N} = 2$ SYM on R^4/Z_2

Let us schematically denote by $\langle \Psi_1(\infty)\Psi_2(1)\Psi_3(q)\Psi_4(0) \rangle_{\Delta^\Psi}$ conformal block of Ψ_i , $i = 1 \dots 4$, fields with intermediary field Ψ of conformal weight Δ^Ψ .

Four point blocks where all four fields are bosonic primaries Φ_i with conformal weights Δ_{α_i} are connected with the Z_{inst} partition function in the following way (see [17])

$$\diamond Z_{(0,0),(0,0)}^{(0,0)} = q^{\Delta_1^{NS} + \Delta_2^{NS} - \Delta^{NS}} (1 - q)^U \langle \Phi_4(\infty)\Phi_3(1)\Phi_1(q)\Phi_2(0) \rangle_{\Delta^{NS}} \quad (4.1)$$

and for $\tilde{\Delta} = \Delta + \frac{1}{2}$

$$\blacklozenge Z_{(0,0),(0,0)}^{(1,1)} = \frac{q^{\Delta_1^{NS} + \Delta_2^{NS} - \tilde{\Delta}^{NS}}}{2} (1 - q)^U \langle \Phi_4(\infty)\Phi_3(1)\Phi_1(q)\Phi_2(0) \rangle_{\tilde{\Delta}^{NS}}. \quad (4.2)$$

The index \diamond shows that the number of black and white boxes (the number of boxes in both diagrams together) are equal and the index \blacklozenge show the number differ by one. In the expressions (4.1) and (4.2) U is given by

$$U = \alpha_2 (Q - \alpha_3) . \tag{4.3}$$

We will see that in the light asymptotic limit U is just one. So in this limit the corresponding partition function gives the four point conformal block for bosonic fields.

Let us look at the $\langle R\Phi\Phi R \rangle$ type conformal block. According to [16] this conformal blocks are connected to the instanton partition function in the following way

$$\diamond Z_{(0,0),(0,0)}^{(0,1)} = q^{\Delta_3^R + \Delta_4^{NS} - \Delta^R} (1 - q)^{(U - \frac{3}{8} + \Delta_1 - \Delta_2 - \Delta_3 + \Delta_4)} \langle R_2^+(\infty) \Phi_1(1) \Phi_4(q) R_3^+(0) \rangle_{\Delta^R} \tag{4.4}$$

Now let us look at the $\langle RRRR \rangle$ conformal blocks [16]. For the partition functions with equal numbers of black and white cells

$$\diamond Z_{(1,0),(1,0)}^{(0,0)}(q) = (1 - q)^U \left(G_{sl(2)}(q) H_-(q) + \tilde{G}_{sl(2)}(q) \tilde{H}_-(q) \right) , \tag{4.5}$$

$$\diamond Z_{(0,1),(0,1)}^{(0,0)}(q) = (1 - q)^U \left(G_{sl(2)}(q) H_+(q) + \tilde{G}_{sl(2)}(q) \tilde{H}_+(q) \right) , \tag{4.6}$$

$$\diamond Z_{(1,0),(0,1)}^{(0,0)}(q) = (1 - q)^U \left(G_{sl(2)}(q) F_-(q) + \tilde{G}_{sl(2)}(q) \tilde{F}_-(-q) \right) , \tag{4.7}$$

$$\diamond Z_{(0,1),(1,0)}^{(0,0)}(q) = (1 - q)^U \left(G_{sl(2)}(q) F_+(q) + \tilde{G}_{sl(2)}(q) \tilde{F}_+(-q) \right) . \tag{4.8}$$

For the partition functions whose numbers of black and white boxes differ by one

$$\blacklozenge Z_{(1,0),(1,0)}^{(1,1)}(q) = (1 - q)^U \left(\tilde{G}_{sl(2)}(q) H_+(q) + G_{sl(2)}(q) \tilde{H}_+(q) \right) , \tag{4.9}$$

$$\blacklozenge Z_{(0,1),(0,1)}^{(1,1)}(q) = (1 - q)^U \left(\tilde{G}_{sl(2)}(q) H_-(q) + G_{sl(2)}(q) \tilde{H}_-(q) \right) , \tag{4.10}$$

$$\blacklozenge Z_{(1,0),(0,1)}^{(1,1)}(q) = (1 - q)^U \left(\tilde{G}_{sl(2)}(q) F_+(q) + G_{sl(2)}(q) \tilde{F}_+(-q) \right) , \tag{4.11}$$

$$\blacklozenge Z_{(0,1),(1,0)}^{(1,1)}(q) = (1 - q)^U \left(\tilde{G}_{sl(2)}(q) F_-(q) + G_{sl(2)}(q) \tilde{F}_-(-q) \right) . \tag{4.12}$$

Here H_{\pm} , F_{\pm} , \tilde{H}_{\pm} and \tilde{F}_{\pm} are related to the conformal blocks containing four Ramond fields, for their definition see appendix A. $G(q)$ and $\tilde{G}(q)$ are certain conformal blocks of the $su(\hat{2})_2$ WZW model, which are given by

$$G(q) = (1 - q)^{-\frac{3}{8}} \sqrt{\frac{1}{2} (1 + \sqrt{1 - q})} , \tag{4.13}$$

$$\tilde{G}(q) = (1 - q)^{-\frac{3}{8}} \sqrt{\frac{1}{2} (1 - \sqrt{1 - q})} . \tag{4.14}$$

Below is given the map that connects the gauge parameters of the instanton partition functions for $\mathcal{N} = 2$ SYM on R^4/Z_2 to the primary fields in the $\mathcal{N} = 1$ SLFT conformal blocks.

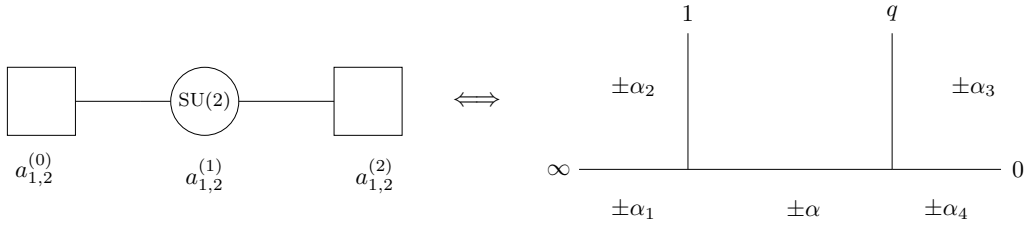


Figure 2. On the left: the quiver diagram for the conformal $SU(2)$ gauge theory. On the right: the diagram of the conformal block for the dual $\mathcal{N} = 1$ SLFT .

4.1 The map relating partition functions to conformal blocks

First of all, the instanton counting parameter q gets identified with the cross ratio of insertion points, as already anticipated in formulas (4.5)–(4.12), for CFT block. The Liouville parameter b is related to the Ω -background parameters via

$$b = \sqrt{\frac{\epsilon_1}{\epsilon_2}}. \quad (4.15)$$

The map between the gauge parameters (2.1) and conformal block parameters can be established from the following rules (see figure 2). First define the rescaled gauge parameters

$$A_i^{(0)} = \frac{a_i^{(0)}}{\sqrt{\epsilon_1 \epsilon_2}}; \quad A_i^{(1)} = \frac{a_i^{(1)}}{\sqrt{\epsilon_1 \epsilon_2}}; \quad A_i^{(2)} = \frac{a_i^{(2)}}{\sqrt{\epsilon_1 \epsilon_2}}, \quad (4.16)$$

where $i = 1, 2$.

Then

- The differences between the “centers of masses” of the successive rescaled gauge parameters (4.16) give the charges of the “vertical” entries of the conformal block:

$$\bar{A}^{(1)} - \bar{A}^{(0)} = \alpha_2; \quad \bar{A}^{(2)} - \bar{A}^{(1)} = \alpha_3. \quad (4.17)$$

- The rescaled gauge parameters with the subtracted centers of masses give the momenta of the “horizontal” entries of the conformal block:

$$\begin{aligned} A_i^{(0)} - \bar{A}^{(0)} &= (-)^{i+1} \left(\alpha_1 - \frac{Q}{2} \right); \\ A_i^{(1)} - \bar{A}^{(1)} &= (-)^{i+1} \left(\alpha - \frac{Q}{2} \right); \\ A_i^{(2)} - \bar{A}^{(2)} &= (-)^{i+1} \left(\alpha_4 - \frac{Q}{2} \right). \end{aligned} \quad (4.18)$$

Using (2.2) and (4.16)–(4.18) we obtain the relation between the gauge and conformal

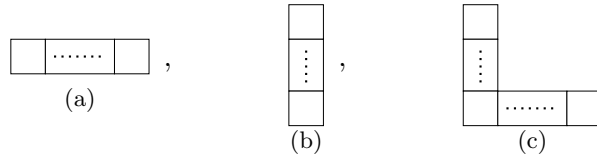


Figure 3. The possible nonempty Young diagrams.

parameters:

$$\begin{aligned}
 \frac{a_i^{(0)}}{\sqrt{\epsilon_1 \epsilon_2}} &= (-)^{i+1} \left(\alpha_1 - \frac{Q}{2} \right) - \alpha_2; \\
 \frac{a_i^{(1)}}{\sqrt{\epsilon_1 \epsilon_2}} &= (-)^{i+1} \left(\alpha - \frac{Q}{2} \right); \\
 \frac{a_i^{(2)}}{\sqrt{\epsilon_1 \epsilon_2}} &= (-)^{i+1} \left(\alpha_4 - \frac{Q}{2} \right) + \alpha_3.
 \end{aligned}
 \tag{4.19}$$

4.2 Light asymptotic limit of the gauge parameters

In this paper we are interested in so called “light” asymptotic limit i.e. the central charge is sent to infinity (i.e. $b \rightarrow 0$) while keeping the dimensions finite. It follows from (3.8) and (3.12) that to reach this limit one can simply put

$$\alpha = b\eta; \quad \alpha_l = b\eta_l; \quad \text{where } l = 1; 2; 4,
 \tag{4.20}$$

by keeping all the parameters η finite. If we exchange α with $Q - \alpha$ the conformal dimension remains the same (see (3.8) and (3.12)), so for α_3 we can take as its light asymptotic limit

$$Q - \alpha_3 = b\eta_3
 \tag{4.21}$$

By taking the limit in this way we get rid of the U(1) factor defined in (4.3). Using (4.20), (4.21) we can rewrite the AGT map (4.19) as

$$a_i^{(0)} = (-)^{i+1} \left(\epsilon_1 \eta_1 - \frac{\epsilon}{2} \right) - \epsilon_1 \eta_2
 \tag{4.22}$$

$$a_i^{(1)} = (-)^{i+1} \left(\epsilon_1 \eta - \frac{\epsilon}{2} \right);
 \tag{4.23}$$

$$a_i^{(2)} = (-)^{i+1} \left(\epsilon_1 \eta_4 - \frac{\epsilon}{2} \right) + \epsilon - \epsilon_1 \eta_3.
 \tag{4.24}$$

5 Partition function in the light asymptotic limit

We have shown in appendix C that for the light asymptotic limit only a restricted set of Young diagrams contributes to the instanton partition function. This set varies depending on the charges and the differences of black and white cells of the related Young diagrams. Below are given all pairs of Y_1 and Y_2 for which the coefficient of the instanton expansion (2.1) is non zero in the light limit. In order to compute these coefficients for a given pair of diagrams Y_1 and Y_2 one makes use of (2.3), (2.4), (4.22)–(4.24) and then goes to the light limit $\epsilon_1 \rightarrow 0$. The results are given below (detailed calculation for some of the coefficients can be found in appendix D).

5.1 Partition functions corresponding to conformal blocks with four Neveu-Schwarz fields

The expansion coefficient $\diamond F_{(0,0),(0,0)}^{(0,0)}$ does not vanish in the light asymptotic limit if Y_2 is a empty Young diagram and Y_1 (see figure 3(a)) has only one row with $2k$ boxes, where k can be zero or any positive integer. It is equal to

$$\diamond F_{(0,0),(0,0)}^{(0,0)} = \frac{\left(\frac{1}{2}(\eta - \eta_4 + \eta_3)\right)_k \left(\frac{1}{2}(\eta - \eta_1 + \eta_2)\right)_k}{k! (\eta)_k}. \quad (5.1)$$

For more details see appendix D. Inserting (5.1) in (2.1), we derive

$$\diamond Z_{(0,0),(0,0)}^{(0,0)}(q) = {}_2F_1(A, B; \eta; q). \quad (5.2)$$

Here A and B are

$$A = \frac{1}{2}(\eta - \eta_1 + \eta_2) \text{ and } B = \frac{1}{2}(\eta - \eta_4 + \eta_3), \quad (5.3)$$

and ${}_2F_1(a, b; c; x)$ is the hypergeometric function. It has the series expansion

$${}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k! (c)_k} x^k, \text{ where } (u)_k = u(u+1)\dots(u+k-1). \quad (5.4)$$

In the case of $\diamond F_{(0,0),(0,0)}^{(1,1)}$ for some set of pairs Y_1, Y_2 one gets large coefficients of order $\frac{1}{\epsilon_1}$. Thus one should take into account these pairs and neglect those pairs whose contributions are of order $O(1)$ or bigger. An analyses quite similar to the one presented in the appendix C, shows that Y_2 should be an empty and Y_1 must have a single row with $2k+1$ boxes (see figure 3(a)). Their contribution is

$$\diamond F_{(0,0),(0,0)}^{(1,1)} = \frac{1}{\epsilon_1 \epsilon_2} \frac{\left(\frac{1}{2}(\eta - \eta_4 + \eta_3 + 1)\right)_k \left(\frac{1}{2}(\eta - \eta_1 + \eta_2 + 1)\right)_k}{2k! (\eta)_{k+1}}. \quad (5.5)$$

After inserting it in (2.1), we will get

$$\diamond Z_{(0,0),(0,0)}^{(1,1)}(q) = \frac{1}{\epsilon_1 \epsilon_2} \frac{\sqrt{q}}{2\eta} {}_2F_1\left(A + \frac{1}{2}, B + \frac{1}{2}; \eta + 1; q\right). \quad (5.6)$$

5.2 Partition function corresponding to the conformal block with two Neveu-Schwarz and two Ramond fields

The coefficients of $\diamond Z_{(0,0),(0,0)}^{(1,0)}$ do not vanish in the light limit if Y_2 is empty and Y_1 (see figure 3(a)) is a diagram with only one row with $2k$ boxes. Their contributions are

$$\diamond F_{(0,0),(0,0)}^{(1,0)} = \frac{\left(\frac{1}{2}(\eta - \eta_4 + \eta_3 + 1)\right)_k \left(\frac{1}{2}(\eta - \eta_1 + \eta_2 + 1)\right)_k}{k! \left(\eta + \frac{1}{2}\right)_k}. \quad (5.7)$$

The corresponding partition function is

$$\diamond Z_{(0,0),(0,0)}^{(1,0)}(q) = {}_2F_1\left(A + \frac{1}{2}, B + \frac{1}{2}; \eta + \frac{1}{2}; q\right). \quad (5.8)$$

The case of $\diamond Z_{(0,0),(0,0)}^{(0,1)}$ is more subtle. Its coefficient do not vanish if Y_1 (see figure 3(a)) is a one row diagram with $2k$ boxes and Y_2 (see figure 3(b)) is a one column diagram with $2m$ boxes. Here one should consider the cases $m = 0$ and $m \neq 0$ separately:

- when $m = 0$

$$\diamond F_{(0,0),(0,0)}^{(0,1)} = \frac{\left(\frac{1}{2}(\eta - \eta^{(4)} + \eta^{(3)})\right)_k \left(\frac{1}{2}(\eta - \eta^{(1)} + \eta^{(2)})\right)_k}{k! \left(\eta + \frac{1}{2}\right)_k}; \quad (5.9)$$

- when $m \neq 0$

$$\diamond F_{(0,0),(0,0)}^{(0,1)} = \frac{1}{2m+1} \frac{\left(\frac{1}{2}(\eta - \eta^{(4)} + \eta^{(3)})\right)_k \left(\frac{1}{2}(\eta - \eta^{(1)} + \eta^{(2)})\right)_k}{k! \left(\eta - \frac{1}{2}\right)_k}. \quad (5.10)$$

The corresponding instanton partition function is

$$\diamond Z_{(0,0),(0,0)}^{(0,1)}(q) = {}_2F_1\left(A, B; \eta + \frac{1}{2}; q\right) + \frac{\tanh^{-1}(\sqrt{q})}{\sqrt{q}} {}_2F_1\left(A, B; \eta - \frac{1}{2}; q\right). \quad (5.11)$$

5.3 Partition functions corresponding to conformal blokes with four Ramond fields

$\diamond F_{(0,1),(0,1)}^{(0,0)}$ differs from zero in the light asymptotic limit if Y_2 (see figure 3(b)) is a single column diagram with $2m$ boxes, and Y_1 (see figure 3(a)) a single row diagram with $2k$ boxes, where m and k can be zero or any positive integer. Their contribution is

$$\diamond F_{(0,1),(0,1)}^{(0,0)} = \left(\frac{(1/2)_m}{m!}\right)^2 \frac{\left(\frac{1}{2}(\eta - \eta_4 + \eta_3)\right)_k \left(\frac{1}{2}(\eta - \eta_1 + \eta_2)\right)_k}{k! (\eta)_k}. \quad (5.12)$$

Its instanton partition function is

$$\diamond Z_{(0,1),(0,1)}^{(0,0)}(q) = \frac{2}{\pi} K(q) {}_2F_1(A, B; \eta; q). \quad (5.13)$$

$K(x)$ and $E(x)$ are complete elliptic integrals of the first and second kind correspondingly. They can be expressed in terms of the Gauss hypergeometric function, as

$$K(x) = \frac{\pi}{2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) \quad \text{and} \quad E(x) = \frac{\pi}{2} {}_2F_1\left(\frac{1}{2}, -\frac{1}{2}; 1; x\right) \quad (5.14)$$

In the case of $\diamond F_{(1,0),(1,0)}^{(0,0)}$ for pairs of Young diagrams Y_2, Y_1 , with Y_2 empty and Y_1 (see figure 3(c)) possessing one column with $2m$ boxes and other $2k$ columns with only one box, one gets large coefficients of order $\frac{1}{\epsilon_1}$ in the light limit. In total Y_1 consists of $2m + 2k$ boxes. These pairs give the main contribution. These terms are

$$\diamond F_{(1,0),(1,0)}^{(0,0)} = \frac{\epsilon_2}{\epsilon_1} \frac{\left(\frac{1}{2}\right)_m \left(-\frac{1}{2}\right)_m \left(\frac{1}{2}(\eta - \eta_4 + \eta_3 + 1)\right)_k \left(\frac{1}{2}(\eta - \eta_1 + \eta_2 + 1)\right)_k}{(m-1)! m! k! \eta (\eta + 1)_k}. \quad (5.15)$$

Its partition function is given by

$$\diamond Z_{(1,0),(1,0)}^{(0,0)}(q) = \frac{\epsilon_2}{\epsilon_1} \frac{(E(q) - K(q))}{\pi \eta} {}_2F_1\left(A + \frac{1}{2}, B + \frac{1}{2}; \eta + 1; q\right). \quad (5.16)$$

$\diamond F_{(0,1),(1,0)}^{(0,0)}$ differs from zero if Y_2 is empty and Y_1 is a one row diagram (see figure 3(a)) with $2k$ boxes. Their contribution is

$$\diamond F_{(0,1),(1,0)}^{(0,0)} = \frac{\left(\frac{1}{2}(\eta - \eta_4 + \eta_3 + 1)\right)_k \left(\frac{1}{2}(\eta - \eta_1 + \eta_2)\right)_k}{k! (\eta)_k}. \quad (5.17)$$

Its instanton partition function is given by

$$\diamond Z_{(0,1),(1,0)}^{(0,0)}(q) = {}_2F_1\left(A, B + \frac{1}{2}; \eta; q\right). \quad (5.18)$$

$\diamond F_{(1,0),(0,1)}^{(0,0)}$ is not zero if Y_2 is empty and Y_1 (see figure 3(a)) is a one row diagram with $2k$ boxes. Their contribution is

$$\diamond F_{(1,0),(0,1)}^{(0,0)} = \frac{\left(\frac{1}{2}(\eta - \eta_1 + \eta_2 + 1)\right)_k \left(\frac{1}{2}(\eta - \eta_4 + \eta_3)\right)_k}{k! (\eta)_k}. \quad (5.19)$$

Its partition function is given by

$$\diamond Z_{(1,0),(0,1)}^{(0,0)}(q) = {}_2F_1\left(A + \frac{1}{2}, B; \eta; q\right). \quad (5.20)$$

In the case of $\diamond F_{(0,1),(0,1)}^{(1,1)}$ for some set of pairs Y_1, Y_2 one gets large coefficients of order $\frac{1}{\epsilon_1}$ in the light limit. These coefficients will give the main contribution in the partition function. These terms are obtained when Y_2 is empty and Y_1 (see figure 3(c)) has one column with $2m + 1$ boxes and $2k$ columns with only one box, the total number of boxes is equal to $2m + 2k + 1$. They are given by

$$\diamond F_{(0,1),(0,1)}^{(1,1)} = \frac{\epsilon_2}{\epsilon_1} \left(\frac{\left(\frac{1}{2}\right)_m}{m!}\right)^2 \frac{\left(\frac{1}{2}(\eta - \eta_4 + \eta_3 + 1)\right)_k \left(\frac{1}{2}(\eta - \eta_1 + \eta_2 + 1)\right)_k}{-2\eta k! (\eta + 1)_k}. \quad (5.21)$$

For its partition function, we receive

$$\diamond Z_{(0,1),(0,1)}^{(1,1)}(q) = -\frac{\epsilon_2 \sqrt{q}}{\epsilon_1 \pi \eta} K(q) {}_2F_1\left(A + \frac{1}{2}, B + \frac{1}{2}; \eta + 1; q\right). \quad (5.22)$$

$\diamond F_{(1,0),(1,0)}^{(1,1)}$ differs from zero if Y_2 is a one column diagram (see figure 3(b)) with $2m + 1$ boxes and Y_1 is a one row diagram (see figure 3(a)) with $2k$ boxes. Their contribution is

$$\diamond F_{(1,0),(1,0)}^{(1,1)} = \frac{1}{(2 + 2m)(1 + 2m)} \left(\frac{\left(\frac{3}{2}\right)_m}{m!}\right)^2 \frac{\left(\frac{1}{2}(\eta - \eta_4 + \eta_3)\right)_k \left(\frac{1}{2}(\eta - \eta_1 + \eta_2)\right)_k}{k! (\eta)_k}. \quad (5.23)$$

For the corresponding instanton partition function, we will get

$$\diamond Z_{(1,0),(1,0)}^{(1,1)}(q) = -\frac{2(E(q) - K(q))}{\pi \sqrt{q}} {}_2F_1(A, B; \eta; q). \quad (5.24)$$

Both $\diamond L F_{(1,0),(0,1)}^{(1,1)}$ and $\diamond L F_{(0,1),(1,0)}^{(1,1)}$ do not vanish if Y_2 is empty and Y_1 (see figure 3(a)) is a one row diagram with $2k + 1$ boxes. Their contributions are

$$\diamond L F_{(0,1),(1,0)}^{(1,1)} = \frac{\left(\frac{1}{2}(\eta - \eta_1 + \eta_2 + 1)\right)_k \left(\frac{1}{2}(\eta - \eta_4 + \eta_3)\right)_{k+1}}{k! (\eta)_k}, \quad (5.25)$$

$$\diamond L F_{(1,0),(0,1)}^{(1,1)} = \frac{\left(\frac{1}{2}(\eta - \eta_1 + \eta_2)\right)_{k+1} \left(\frac{1}{2}(\eta - \eta_4 + \eta_3 + 1)\right)_k}{k! (\eta)_k}. \quad (5.26)$$

Their partition functions are

$$\diamond L Z_{(0,1),(1,0)}^{(1,1)}(q) = \frac{B}{\eta} \sqrt{q} {}_2F_1\left(A + \frac{1}{2}, B + 1; \eta + 1; q\right). \quad (5.27)$$

$$\diamond L Z_{(1,0),(0,1)}^{(1,1)}(q) = \frac{A}{\eta} \sqrt{q} {}_2F_1\left(A + 1, B + \frac{1}{2}; \eta + 1; q\right). \quad (5.28)$$

6 Conformal blocks for $\mathcal{N} = 1$ SLFT in the light asymptotic limit

Applying (5.2) and (5.6) to (4.1) and (4.2) we will get the conformal blocks with all four fields being NS in the light limit:

$$\langle \Phi_4(\infty) \Phi_3(1) \Phi_1(q) \Phi_2(0) \rangle_{\Delta_{NS}}^L = q^{\frac{1}{2}(\eta - \eta^{(2)} - \eta^{(1)})} {}_2F_1(A, B; \eta; q), \quad (6.1)$$

$$\langle \Phi_4(\infty) \Phi_3(1) \Phi_1(q) \Phi_2(0) \rangle_{\Delta_{NS}}^L = \frac{q^{\frac{1}{2}(1 + \eta - \eta^{(2)} - \eta^{(1)})}}{\eta} {}_2F_1\left(A + \frac{1}{2}, B + \frac{1}{2}; \eta + 1; q\right). \quad (6.2)$$

These results are in agreement with [21].

By applying (5.11) for (4.4) we get the conformal blocks with two R fields and two NS fields

$$\begin{aligned} \langle R_2^+(\infty) \Phi_1(1) \Phi_4(q) R_3^+(0) \rangle_{\Delta_R}^L &= q^{\frac{1}{2}(\eta - \eta^{(3)} - \eta^{(4)})} (1 - q)^{-\frac{1}{2}(\eta^{(1)} - \eta^{(2)} - \eta^{(3)} + \eta^{(4)} - 1)} \\ &\times \left({}_2F_1\left(A, B; \eta + \frac{1}{2}; q\right) + \frac{\tanh^{-1}(\sqrt{q})}{\sqrt{q}} {}_2F_1\left(A, B; \eta - \frac{1}{2}; q\right) \right), \end{aligned} \quad (6.3)$$

where the intermediate field is a Ramond field.

As it was already mentioned the conformal blocks with four R fields are expressed in terms of H_{\pm} , \tilde{H}_{\pm} , F_{\pm} , \tilde{F}_{\pm} . Their connection to the instanton partition is given in (4.5)–(4.12). Applying (5.13)–(5.28), we can derive them (see appendix (E.1)–(E.8)). Their

expressions get slightly simplified when one takes $q = \sin^2(t)$ with $t \in (0, \frac{\pi}{2})$.

$$H_-^L(\sin^2(t)) = \frac{\epsilon_2 \cos(\frac{t}{2})(E(\sin^2(t)) - \cos(t)K(\sin^2(t))) {}_2F_1(A + \frac{1}{2}, B + \frac{1}{2}; \eta + 1; \sin^2(t))}{\epsilon_1 \pi \eta \sqrt[4]{\cos(t)}}, \quad (6.4)$$

$$\tilde{H}_-^L(\sin^2(t)) = -\frac{\epsilon_2 \sin(t)(\cos(t)K(\sin^2(t)) + E(\sin^2(t))) {}_2F_1(A + \frac{1}{2}, B + \frac{1}{2}; \eta + 1; \sin^2(t))}{\epsilon_1 \sqrt{2} \pi \eta \sqrt[4]{\cos(t)} \sqrt{\cos(t) + 1}}, \quad (6.5)$$

$$H_+^L(\sin^2(t)) = \frac{\sec(\frac{t}{2})(\cos(t)K(\sin^2(t)) + E(\sin^2(t))) {}_2F_1(A, B; \eta; \sin^2(t))}{\pi \sqrt[4]{\cos(t)}}, \quad (6.6)$$

$$\tilde{H}_+^L(\sin^2(t)) = \frac{\csc(\frac{t}{2})(\cos(t)K(\sin^2(t)) - E(\sin^2(t))) {}_2F_1(A, B; \eta; \sin^2(t))}{\pi \sqrt[4]{\cos(t)}}, \quad (6.7)$$

$$F_+^L(\sin^2(t)) = \frac{\sec(\frac{t}{2})(\eta(\cos(t) + 1) {}_2F_1(A, B + \frac{1}{2}; \eta; \sin^2(t)) - A \sin^2(t) {}_2F_1(A + 1, B + \frac{1}{2}; \eta + 1; \sin^2(t)))}{2\eta \sqrt[4]{\cos(t)}}, \quad (6.8)$$

$$F_-^L(\sin^2(t)) = \frac{\sec(\frac{t}{2})(\eta(\cos(t) + 1) {}_2F_1(A + \frac{1}{2}, B; \eta; \sin^2(t)) - B \sin^2(t) {}_2F_1(A + \frac{1}{2}, B + 1; \eta + 1; \sin^2(t)))}{2\eta \sqrt[4]{\cos(t)}}, \quad (6.9)$$

$$\tilde{F}_+^L(-\sin^2(t)) = \frac{\sin(t)(A(\cos(t) + 1) {}_2F_1(A + 1, B + \frac{1}{2}; \eta + 1; \sin^2(t)) - \eta {}_2F_1(A, B + \frac{1}{2}; \eta; \sin^2(t)))}{\sqrt{2} \eta \sqrt[4]{\cos(t)} \sqrt{\cos(t) + 1}}, \quad (6.10)$$

$$\tilde{F}_-^L(-\sin^2(t)) = \frac{\sin(t)(B(\cos(t) + 1) {}_2F_1(A + \frac{1}{2}, B + 1; \eta + 1; \sin^2(t)) - \eta {}_2F_1(A + \frac{1}{2}, B; \eta; \sin^2(t)))}{\sqrt{2} \eta \sqrt[4]{\cos(t)} \sqrt{\cos(t) + 1}}. \quad (6.11)$$

Summary. With the help of the AGT like correspondence between $SU(2)$ $\mathcal{N} = 2$ supersymmetric gauge theories living on R^4/Z_2 space and two dimensional $\mathcal{N} = 1$ SLFT proposed in [16, 17], analytic expressions are found for the various four point super-conformal blocks in the light asymptotic limit. Namely we have found light blocks when:

- all four insertions are NS fields see (6.1), (6.2);
- two of the insertions are NS and the other two are Ramond fields see (6.3);
- all four are Ramond fields see (6.4)–(6.11).

The first result of the list above is not new, it has been found in [21] via a direct, CFT approach. The remaining cases, to my knowledge, are analyzed for the first time and could be helpful for better understanding of the subtleties of the light limit in Ramond sector.

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A Super Liouville conformal blocks of four R -fields

Here, following [16] we define the functions H_{\pm} , F_{\pm} , \tilde{H}_{\pm} and \tilde{F}_{\pm} , which are used in the main text. The OPEs for two Ramond fields can be written as

$$R_1^{\pm}(z)R_2^{\pm}(0) = z^{\Delta - \Delta_1 - \Delta_2} \sum_{N=0}^{\infty} z^N |N; \pm\pm\rangle, \quad (A.1)$$

$$R_1^{\pm}(z)R_2^{\mp}(0) = z^{\Delta - \Delta_1 - \Delta_2} \sum_{N=0}^{\infty} z^N |N; \pm\mp\rangle. \quad (A.2)$$

In the NS sector at level zero there is only one state, namely the NS primary state of dimension Δ . Thus $|N; \pm\rangle$ states are proportional to this NS state

$$|0; \pm\rangle = \gamma_{\pm}|0\rangle. \tag{A.3}$$

By definition

$$|N; \pm\rangle = \begin{cases} |N; ++\rangle \pm |N; --\rangle & \text{if } N \in \mathbb{Z} \\ |N; +-\rangle \mp i|N; -+\rangle & \text{if } N \in \mathbb{Z} + 1/2 \end{cases}. \tag{A.4}$$

In this notations

$$|0; \pm\rangle = \Gamma_{\pm}|0\rangle \quad \text{where} \quad \Gamma_{\pm} = (\gamma_+ \pm \gamma_-). \tag{A.5}$$

$H_{\pm}, F_{\pm}, \tilde{H}_{\pm}$ and \tilde{F}_{\pm} are related to the conformal blocks with four Ramond fields in the following way (below q is the cross ratio of insertion points)

$$F_{\pm} = \frac{1}{\Gamma_{\pm}\Gamma_{\pm}} \sum_{N=0,1,\dots} q^N \langle N; \pm|N; \pm\rangle; \quad H_{\pm} = \frac{1}{\Gamma_{\pm}\Gamma_{\mp}} \sum_{N=0,1,\dots} q^N \langle N; \pm|N; \mp\rangle, \tag{A.6}$$

$$\tilde{F}_{\pm} = \frac{(-i)}{\Gamma_{\pm}\Gamma_{\pm}} \sum_{N=\frac{1}{2},\frac{3}{2},\dots} q^N \langle N; \pm|N; \pm\rangle; \quad \tilde{H}_{\pm} = \frac{1}{\Gamma_{\pm}\Gamma_{\mp}} \sum_{N=\frac{1}{2},\frac{3}{2},\dots} q^N \langle N; \pm|N; \mp\rangle, \tag{A.7}$$

where conformal blocks are divided by Γ_{\pm} so that if one takes the normalization $\langle 0|0\rangle = 1$, then the expansion of F_{\pm} starts as $1 + F_{\pm}q + \dots$. For more details and explanation the reader should consult [16].

B Restriction rules

Let us look at (2.4). To see whether a box of $\lambda(\mu)$ is in $\lambda^*(\mu^*)$ or not we replace

$$\epsilon_1, \epsilon_2 \rightarrow 1; \quad a_i^{(0)} \rightarrow u_i; \quad a_i^{(1)} \rightarrow q_i; \quad a_i^{(2)} \rightarrow v_i \quad (i = 1, 2) \tag{B.1}$$

and evaluate a factor corresponding to a box of $\lambda(\mu)$. If the result is equal to 0 (mod 2) then the chosen box belongs to $\lambda^*(\mu^*)$ otherwise not. Let us apply this constraint for each of the bifundamentals appearing in (2.3):

- For $Z_{bf}(u_i, a_i^{(0)}, \emptyset | q_j, a_j^{(1)}, Y_j)$, a box $s \in Y_j$ is also in Y_j^* iff

$$u_i + q_j + 1 + L_{\emptyset}(s) + A_{Y_j}(s) = 0 \pmod{2}. \tag{B.2}$$

- For $Z_{bf}(q_i, a_i^{(1)}, Y_i | v_j, a_j^{(2)}, \emptyset)$ a box $s \in Y_i$ is also in Y_i^* iff

$$q_i + v_j + 1 + L_{\emptyset}(s) + A_{Y_i}(s) = 0 \pmod{2}. \tag{B.3}$$

- For $Z_{bf}(q_i, a_i^{(1)}, Y_i | q_j, a_j^{(1)}, Y_j)$:
a box $s \in Y_i$ is also in Y_i^* iff

$$q_i + q_j + 1 + L_{Y_j}(s) + A_{Y_i}(s) = 0 \pmod{2}, \tag{B.4}$$

a box $s \in Y_j$ is also in Y_j^* iff

$$q_j + q_i + 1 + L_{Y_i}(s) + A_{Y_j}(s) = 0 \pmod{2}, \tag{B.5}$$

where $i, j = 1, 2$.

C Proof of the restrictions on the Young diagrams for $\diamond_L Z_{(0,0),(0,0)}^{(0,0)}$ and $\diamond_L Z_{(0,1),(0,1)}^{(0,0)}$

Here we prove, as we mentioned in section 5, that in the light asymptotic limit contribute only diagrams depicted in figure 3. We will give all details for the cases of $\diamond_L Z_{(0,0),(0,0)}^{(0,0)}$ and $\diamond_L Z_{(0,1),(0,1)}^{(0,0)}$. The proofs for the other cases are quite similar. Let us compute the factors in (2.3).

Inserting (4.22) and (4.23) in (2.4), we obtain for the first factor of the numerator in (2.3):

$$\begin{aligned}
 Z_{bf}(u_i, a_i^{(0)}, \emptyset \mid q_j, a_j^{(1)}, Y_j) = & \tag{C.1} \\
 & \prod_{s \in Y_j^*} \left(\epsilon_1 \left((-)^{i+1} \left(\eta_1 - \frac{1}{2} \right) - \eta_2 + (-)^j \left(\eta - \frac{1}{2} \right) + L_{\emptyset}(s) + 1 \right) \right. \\
 & \left. + \epsilon_2 \left(-A_{Y_j}(s) + \frac{(-)^{j+1} - (-)^{i+1}}{2} \right) \right).
 \end{aligned}$$

In the same way the second factor of the numerator in (2.3) is given by

$$\begin{aligned}
 Z_{bf}(q_i, a_i^{(1)}, Y_i \mid v_j, a_j^{(2)}, \emptyset) = & \tag{C.2} \\
 & \prod_{s \in Y_i^*} \left(\epsilon_1 \left((-)^{i+1} \left(\eta - \frac{1}{2} \right) + (-)^j \left(\eta_4 - \frac{1}{2} \right) + \eta_3 - L_{\emptyset}(s) - 1 \right) \right. \\
 & \left. + \epsilon_2 \left(A_{Y_i}(s) + \frac{(-)^{j+1} - (-)^{i+1}}{2} \right) \right).
 \end{aligned}$$

and for the denominator of (2.3) we will get

$$\begin{aligned}
 Z_{bf}(q_i, a_i^{(1)}, Y_i \mid q_j, a_j^{(1)}, Y_j) = & \tag{C.3} \\
 & \prod_{s \in Y_i^*} \left(\epsilon_1 \left(((-)^{i+1} - (-)^{j+1}) \left(\eta - \frac{1}{2} \right) - L_{Y_j}(s) \right) \right. \\
 & \left. + \epsilon_2 \left(A_{Y_i}(s) + \frac{(-)^{j+1} - (-)^{i+1}}{2} + 1 \right) \right) \\
 & \prod_{s \in Y_j^*} \left(\epsilon_1 \left(((-)^{i+1} - (-)^{j+1}) \left(\eta - \frac{1}{2} \right) + 1 + L_{Y_i}(s) \right) \right. \\
 & \left. + \epsilon_2 \left(-A_{Y_j}(s) + \frac{(-)^{j+1} - (-)^{i+1}}{2} \right) \right).
 \end{aligned}$$

The instanton expansion coefficients (2.3) are proportional to ϵ_1^N . We will show that $N > 0$ for all pairs of Young diagram, except those depicted in figure 3. This means that all other diagrams do not contribute in (2.1) in the light limit ($\epsilon_1 \rightarrow 0$).

Note that in (C.1) for some boxes from Y_j^* the coefficient in front of ϵ_2 vanishes. Denote the number of such boxes by n_1 . Similarly the numbers of boxes of this kind

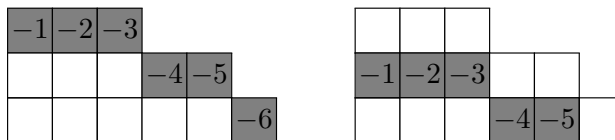


Figure 4. The left diagram shows that there are $Y_{i,1}$ boxes such that $A_{Y_i} = 0$ (painted gray). The numbers are the leg-length of this boxes towards the empty diagram. The right diagram shows that there are $Y_{i,2}$ boxes with $A_{Y_i} = 1$ (painted grey) and again the numbers are the leg-length of these boxes towards the empty diagram.

	$Y_{1,1} = 2m$	$Y_{1,1} = 2m + 1$		$Y_{1,2} = 2k$	$Y_{1,2} = 2k + 1$		$Y_{2,1} = 2l$	$Y_{2,1} = 2l + 1$
$u_1 + q_1 = \text{even}$	$n_{1,1} = m$	$n_{1,1} = m + 1$	$u_2 + q_1 = \text{even}$	$n_{1,2} = k$	$n_{1,2} = k$	$u_2 + q_2 = \text{even}$	$n_{1,3} = l$	$n_{1,3} = l + 1$
$u_1 + q_1 = \text{odd}$	$n_{1,1} = m$	$n_{1,1} = m$	$u_2 + q_1 = \text{odd}$	$n_{1,2} = k$	$n_{1,2} = k + 1$	$u_2 + q_2 = \text{odd}$	$n_{1,3} = l$	$n_{1,3} = l$
$v_1 + q_1 = \text{even}$	$n_{2,1} = m$	$n_{2,1} = m + 1$	$v_2 + q_1 = \text{even}$	$n_{2,2} = k$	$n_{2,2} = k$	$v_2 + q_2 = \text{even}$	$n_{2,3} = l$	$n_{2,3} = l + 1$
$v_1 + q_1 = \text{odd}$	$n_{2,1} = m$	$n_{2,1} = m$	$v_2 + q_1 = \text{odd}$	$n_{2,2} = k$	$n_{2,2} = k + 1$	$v_2 + q_2 = \text{odd}$	$n_{2,3} = l$	$n_{2,3} = l$

Table 1. Depending on q_i, u_i and v_i , n_1 and n_2 take different values. One can get them from this table by $n_1 = n_{1,1} + n_{1,2} + n_{1,3}$ and $n_2 = n_{2,1} + n_{2,2} + n_{2,3}$.

in (C.2) and (C.3) are denoted by n_2 and n_3 respectively. It is obvious that

$$N = n_1 + n_2 - n_3. \tag{C.4}$$

First we explain how to compute the number n_1 . As we mentioned already, (C.1) is proportional to ϵ_1 whenever the term proportional to ϵ_2 vanishes. This occurs when

$$A_{Y_j}(s) = \frac{1}{2} \left((-1)^{j+1} - (-1)^{i+1} \right), \quad s \in Y_j. \tag{C.5}$$

Note that the chosen box s belongs to the same diagram towards which its arm-length is evaluated, hence the arm-length must always be positive or zero. From (C.5) we can see that the only possible values for i and j that give positive or zero arm-lengths in (2.3) are:

$$j = 1; \quad i = 1; \quad A_{Y_1}(s) = 0; \quad (s \in Y_1), \tag{C.6}$$

$$j = 1; \quad i = 2; \quad A_{Y_1}(s) = 1; \quad (s \in Y_1), \tag{C.7}$$

$$j = 2; \quad i = 2; \quad A_{Y_2}(s) = 0; \quad (s \in Y_2). \tag{C.8}$$

(C.6) implies that only the boxes that have zero arm-length contribute to n_1 . It is obvious from the left diagram of figure 4 that there are exactly $Y_{1,1}$ boxes in Y_1 for which the arm-length vanishes (here and below we denote by $Y_{i,k}$ the number of boxes in the k 'th row of diagram Y_i). But not all these boxes obey the restriction (B.2), which can be written as

$$u_1 + q_1 + 1 + L_\emptyset(s) = 0 \pmod{2}, \quad A_{Y_1}(s) = 0 \quad (s \in Y_1). \tag{C.9}$$

From the first picture of figure 4 one can see that $L_\emptyset(s) = -1, -2, \dots, -Y_{1,1}$. Using this we obtain the number of boxes in $Y_{1,1}$ which are in Y_1^* , denoted by $n_{1,1}$. The results are presented in table 1. Correspondingly, the number of boxes satisfying (C.7) with unit arm-lengths in Y_1 is equal to $Y_{1,2}$, and finally, the number of the boxes obeying (C.8) with zero arm-lengths in Y_2 is equal to $Y_{2,1}$. But not all of $Y_{1,2}$ and $Y_{2,1}$ boxes are in Y_1^* and Y_2^*

respectively. We should impose also the constraint (B.2). With the same steps one can get the number of boxes in Y_1^* and Y_2^* denoted by $n_{1,2}$ and $n_{1,3}$ correspondingly. The results again are summarized in table 1. Obviously

$$n_1 = n_{1,1} + n_{1,2} + n_{1,3}. \quad (\text{C.10})$$

Now let us compute n_2 . From (C.2) we see that the term proportional to ϵ_2 vanishes if

$$A_{Y_i}(s) = \frac{1}{2} \left((-)^{i+1} - (-)^{j+1} \right); \quad s \in Y_i, \quad (\text{C.11})$$

where again the arm-length is towards its own diagram. This means that it is always positive or zero. Therefore

$$i = 1; \quad j = 1; \quad A_{Y_1}(s) = 0; \quad (s \in Y_1); \quad (\text{C.12})$$

$$i = 1; \quad j = 2; \quad A_{Y_1}(s) = 1; \quad (s \in Y_1); \quad (\text{C.13})$$

$$i = 2; \quad j = 2; \quad A_{Y_2}(s) = 0; \quad (s \in Y_2). \quad (\text{C.14})$$

Again in the Y_1 diagram there are $Y_{1,1}$ and $Y_{1,2}$ boxes with zero and unit arm-length and $Y_{2,1}$ boxes in Y_2 with zero arm-length (see figure 4). All the boxes that contribute to n_2 must obey (B.3). The results is displayed in table 1.

Let us calculate n_3 . In (C.3) the term proportional to ϵ_2 vanishes if

$$A_{Y_i}(s) = \frac{1}{2} \left((-)^{i+1} - (-)^{j+1} \right) - 1; \quad (s \in Y_i), \quad (\text{C.15})$$

$$A_{Y_j}(s) = \frac{1}{2} \left((-)^{j+1} - (-)^{i+1} \right); \quad (s \in Y_j), \quad (\text{C.16})$$

Again both arm-lengths should be positive. This implies

$$i = 1; \quad j = 2; \quad A_{Y_1}(s) = 0; \quad (s \in Y_1), \quad (\text{C.17})$$

$$j = 1; \quad i = 1; \quad A_{Y_1}(s) = 0; \quad (s \in Y_1), \quad (\text{C.18})$$

$$j = 1; \quad i = 2; \quad A_{Y_1}(s) = 1; \quad (s \in Y_1), \quad (\text{C.19})$$

$$j = 2; \quad i = 2; \quad A_{Y_2}(s) = 0; \quad (s \in Y_2), \quad (\text{C.20})$$

Let us apply the constraint (B.4) and (B.5) for the boxes defined above. The result is

$$s \in Y_1 \text{ with } A_{Y_1}(s) = 0 \text{ is also in } Y_1^* \text{ if } q_2 + q_1 + 1 + L_{Y_2}(s) = 0 \pmod{2}; \quad (\text{C.21})$$

$$s \in Y_1 \text{ with } A_{Y_1}(s) = 0 \text{ is also in } Y_1^* \text{ if } 1 + L_{Y_1}(s) = 0 \pmod{2}; \quad (\text{C.22})$$

$$s \in Y_1 \text{ with } A_{Y_1}(s) = 1 \text{ is also in } Y_1^* \text{ if } q_2 + q_1 + L_{Y_2}(s) = 0 \pmod{2}; \quad (\text{C.23})$$

$$s \in Y_2 \text{ with } A_{Y_2}(s) = 0 \text{ is also in } Y_2^* \text{ if } 1 + L_{Y_2}(s) = 0 \pmod{2}, \quad (\text{C.24})$$

Let us denote by $n_{3,j}$ $j = 1, 2, 3, 4$ the number of boxes that obey (C.21)–(C.24) correspondingly. Obviously

$$n_3 = n_{3,1} + n_{3,2} + n_{3,3} + n_{3,4}. \quad (\text{C.25})$$

It is not difficult to see from (C.21)–(C.24) that $n_{3,j}$ obey the constraints

$$\text{For both } Y_{1,1} = 2m \text{ or } Y_{1,1} = 2m + 1, \quad n_{3,2} \leq m; \quad (\text{C.26})$$

$$\text{For both } Y_{2,1} = 2l \text{ or } Y_{2,1} = 2l + 1, \quad n_{3,4} \leq l; \quad (\text{C.27})$$

$$n_{3,1} + n_{3,2} \leq Y_{1,1}. \quad (\text{C.28})$$

The first two constraints are a consequence of (C.22) and (C.24) respectively. The third constraint can be seen from (C.21) and (C.22).

The case $\diamond F_{(0,0),(0,0)}^{(0,0)}$. From the above analysis it is obvious that N depends on the parity (odd or even) of the numbers $Y_{1,1}$, $Y_{1,2}$ and $Y_{2,1}$. We will consider each case separately.

1. If $Y_{1,1} = 2m$, $Y_{1,2} = 2k$, $Y_{2,1} = 2l$. Using table 1 for n_1 and n_2 and (C.27), (C.28) for n_3 we will get

$$n_1 + n_2 = 2m + 2k + 2l, \quad \text{and} \quad n_3 \leq 2m + 2k + l. \quad (\text{C.29})$$

Substituting this into (C.4) we obtain $N \geq l$. In the light asymptotic limit $\epsilon_1 \rightarrow 0$ the contribution of a pair of diagrams for which $N > 0$ is negligible compared to the case with $N = 0$. Thus we are interested in pairs of diagrams for which $l = 0$. This means that $Y_{2,1} = 0$. Recalling that $Y_{2,1}$ is the number of boxes in the first row of Y_2 , we obtain that Y_2 is an empty Young diagram.

Using (C.23) we can express $n_{3,3}$ in terms of $Y_{1,2}$ and get $n_3 \leq 2m + k$ thus, $N \geq k$ and $k = 0$, $Y_{1,2} = 0$, hence Y_2 is a one row diagram with $2m$ boxes.

2. $Y_{1,1} = 2m$, $Y_{1,2} = 2k$, $Y_{2,1} = 2l + 1$

$$n_1 + n_2 = 2m + 2k + 2l + 2 \quad \text{and} \quad n_3 \leq 2m + 2k + l \quad (\text{C.30})$$

so that $N \geq l + 2$ and thus $N > 0$. The contribution of these pairs in the instanton partition function (2.1) is negligible compared to the first case where we had pairs of diagrams with $N = 0$.

3. If $Y_{1,1} = 2m$, $Y_{1,2} = 2k + 1$, $Y_{2,1} = 2l + 1$ then

$$n_1 + n_2 = 2m + 2k + 2l + 2 \quad \text{and} \quad n_3 \leq 2m + 2k + 1 + l \quad (\text{C.31})$$

so, $N > 0$ and in this case there is no contribution.

4. $Y_{1,1} = 2m$, $Y_{1,2} = 2k + 1$, $Y_{2,1} = 2l$ then

$$n_1 + n_2 = 2m + 2l + 2k \quad \text{and} \quad n_3 \leq 2m + 2k + 1 + l \quad (\text{C.32})$$

so we have two possibilities $l = 0, 1$ that may give a non positive N .

- (a) When $l = 0$ Y_2 is empty, then by using (C.23)

$$n_1 + n_2 = 2m + 2k \quad \text{and} \quad n_3 \leq 2m + k. \quad (\text{C.33})$$

It seems that for $k = 0$, which is $Y_{1,2} = 1$, one may have a contribution in the partition function. For this case we are able to calculate n_3 precisely using (C.21)–(C.24). The result is $n_3 = 2m - 1$. This means that in fact $N = 1$, thus we get no contribution.

(b) When $l = 1$, a careful examination shows that $N > 0$, therefore no contribution too.

5. $Y_{1,1} = 2m + 1, Y_{1,2} = 2k, Y_{2,1} = 2l$ then

$$n_1 + n_2 = 2m + 2 + 2k + 2l \quad \text{and} \quad n_3 \leq 2m + 1 + 2k + l \quad (\text{C.34})$$

so $N > 0$, no contribution.

6. If $Y_{1,1} = 2m + 1, Y_{1,2} = 2k, Y_{2,1} = 2l + 1$ then

$$n_1 + n_2 = 2m + 2 + 2k + 2l + 2 \quad \text{and} \quad n_3 \leq 2m + 1 + 2k + l \quad (\text{C.35})$$

so $N > 0$, no contribution.

7. If $Y_{1,1} = 2m + 1, Y_{1,2} = 2k + 1, Y_{2,1} = 2l$ then

$$n_1 + n_2 = 2m + 2 + 2k + 2l \quad \text{and} \quad n_3 \leq 2m + 1 + 2k + 1 + l \quad (\text{C.36})$$

Thus the only possibility is $l = 0$. This means that $Y_{2,1} = 0$ so Y_2 is an empty Young diagram.

Using (C.23) we can see that $n_3 \leq 2m + 1 + k$ which means that $N > 0$, thus no contribution.

8. If $Y_{1,1} = 2m + 1, Y_{1,2} = 2k + 1, Y_{2,1} = 2l + 1$ then

$$n_1 + n_2 = 2m + 2 + 2k + 2l + 2 \quad \text{and} \quad n_3 \leq 2m + 1 + 2k + 1 + l \quad (\text{C.37})$$

so $N > 0$, no contribution.

We conclude that Y_2 is empty and Y_1 is a one row diagram with even number of boxes.

The case $\diamond F_{(0,1),(0,1)}^{(0,0)}$.

1. If $Y_{1,1} = 2m, Y_{1,2} = 2k, Y_{2,1} = 2l$. Using table 1 for n_1 and n_2 and (C.27), (C.28) for n_3 we will get

$$n_1 + n_2 = 2m + 2l + 2k, \quad \text{and} \quad n_3 \leq 2m + 2k + l, \quad (\text{C.38})$$

$l = 0$ and Y_2 is an empty diagram. By using (C.23) we can express $n_{3,3}$ in terms of $Y_{1,2}$, thus we get $n_3 \leq 2m + k$, hence $k = 0, Y_{1,2} = 0$ so, Y_1 is a one row diagram with $2m$ boxes.

2. $Y_{1,1} = 2m, Y_{1,2} = 2k, Y_{2,1} = 2l + 1$

$$n_1 + n_2 = 2m + 2k + 2l \quad \text{and} \quad n_3 \leq 2m + 2k + l, \quad (\text{C.39})$$

thus, $l = 0$, $Y_{2,1}$ may be possible. Using (C.23) we get that $n_3 \leq 2m + k$, hence $k = 0$. One can check that when Y_1 has one row with even number of boxes and Y_2 one column with even number of boxes then $N = 0$. So this kind of pairs do contribute.

3. If $Y_{1,1} = 2m, Y_{1,2} = 2k + 1, Y_{2,1} = 2l + 1$ then

$$n_1 + n_2 = 2m + 2k + 2 + 2l \quad \text{and} \quad n_3 \leq 2m + 2k + 1 + l, \quad (\text{C.40})$$

$N > 0$, no contribution.

4. $Y_{1,1} = 2m, Y_{1,2} = 2k + 1, Y_{2,1} = 2l$ then

$$n_1 + n_2 = 2m + 2k + 2 + 2l \quad \text{and} \quad n_3 \leq 2m + 2k + 1 + l, \quad (\text{C.41})$$

thus, no contribution.

5. $Y_{1,1} = 2m + 1, Y_{1,2} = 2k, Y_{2,1} = 2l$, then

$$n_1 + n_2 = 2m + 2 + 2k + 2l \quad \text{and} \quad n_3 \leq 2m + 1 + 2k + l \quad (\text{C.42})$$

so, no contribution.

6. If $Y_{1,1} = 2m + 1, Y_{1,2} = 2k, Y_{2,1} = 2l + 1$ then

$$n_1 + n_2 = 2m + 2 + 2k + 2l \quad \text{and} \quad n_3 \leq 2m + 1 + 2k + l \quad (\text{C.43})$$

so $N > 0$, no contribution.

7. If $Y_{1,1} = 2m + 1, Y_{1,2} = 2k + 1, Y_{2,1} = 2l$ then

$$n_1 + n_2 = 2m + 2 + 2k + 2 + 2l \quad \text{and} \quad n_3 \leq 2m + 1 + 2k + 1 + l \quad (\text{C.44})$$

$N > 0$, thus no contribution.

8. If $Y_{1,1} = 2m + 1, Y_{1,2} = 2k + 1, Y_{2,1} = 2l + 1$ then

$$n_1 + n_2 = 2m + 2 + 2k + 2 + 2l \quad \text{and} \quad n_3 \leq 2m + 1 + 2k + 1 + l \quad (\text{C.45})$$

so $N > 0$, no contribution.

We have shown that in the light asymptotic limit to the instanton partition function contribute only Young diagrams considered in cases 1 and 2. Combining these two cases we see that Y_1 is a one row diagram with even number of boxes and Y_2 is a one column diagram with even number of boxes.

Some instanton partition functions (for example $\diamond Z_{(0,0),(0,0)}^{(1,1)}$) for some set of pairs Y_1, Y_2 have large expansion coefficients of order $\frac{1}{\epsilon_1}$. These cases are similar to the ones we discussed above but here we should take into account the pairs with $N = -1$ and neglect the ones with $N > -1$.

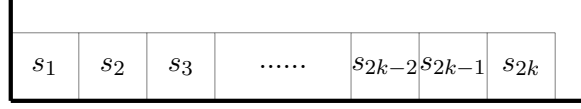


Figure 5. The bold line corresponds to Y_2 - an empty diagram; the thin lines indicate Y_1 - a one row diagram.

D The calculation of $\diamond_L F_{(0,0),(0,0)}^{(0,0)}$ and $\diamond_L F_{(0,1),(0,1)}^{(0,0)}$

Let us calculate $\diamond_L F_{(0,0),(0,0)}^{(0,0)}$. As we know from appendix C, Y_2 is empty and Y_1 (see figure 3(a)) is a one row diagram with even number of boxes. Let us look at $Z_{bf}(a_2^{(0)}, \emptyset | a_1^{(1)}, Y_1)$. By using (C.1) we will get

$$Z_{bf}(a_2^{(0)}, \emptyset | a_1^{(1)}, Y_1) = \prod_{s \in Y_2^*} (\epsilon_1 (-\eta_1 - \eta_2 - \eta + 2 + L_\emptyset(s)) + \epsilon_2), \quad (\text{D.1})$$

where we used the fact that the arm-length $A_{Y_1}(s) = 0$ when $s \in Y_1^*$. One can see from figure 5 that $L_\emptyset(s_1) = -1$, $L_\emptyset(s_2) = -2 \dots L_\emptyset(s_{2k}) = -2k$. If a box of Y_1 is also in Y_1^* we must use (B.2) which, in this case can be written as $1 + L_\emptyset(s_j) = 0 \pmod{2}$. We see that the leg-lengths must be odd numbers so, $Y_1^* = \{s_1, s_3, \dots, s_{2j-1}, \dots, s_{2k-1}\}$. Thus $L_\emptyset(s_{2j-1}) = 1 - 2j$ where $j = 1, \dots, k$. Inserting this into (D.1) we will get

$$Z_{bf}(a_2^{(0)}, \emptyset | a_1^{(1)}, Y_1) = \prod_{j=1}^k (\epsilon_1 (-\eta_1 - \eta_2 - \eta + 3 - 2j) + \epsilon_2). \quad (\text{D.2})$$

The next step is to take $\epsilon_1 \rightarrow 0$. The result is

$$Z_{bf}(a_2^{(0)}, \emptyset | a_1^{(1)}, Y_1) \xrightarrow{\epsilon_1 \rightarrow 0} \epsilon_2^k, \quad (\text{D.3})$$

all the other bifundamentals are derived with the same steps. Here are the results:

$$Z_{bf}(a_1^{(0)}, \emptyset | a_1^{(1)}, Y_1) \xrightarrow{\epsilon_1 \rightarrow 0} \epsilon_1^k \prod_{j=1}^k (\eta_1 - \eta_2 - \eta + 2 - 2j); \quad (\text{D.4})$$

$$Z_{bf}(a_1^{(1)}, Y_1 | a_2^{(2)}, \emptyset) \xrightarrow{\epsilon_1 \rightarrow 0} (-\epsilon_2)^k; \quad (\text{D.5})$$

$$Z_{bf}(a_1^{(1)}, Y_1 | a_1^{(2)}, \emptyset) \xrightarrow{\epsilon_1 \rightarrow 0} \epsilon_1^k \prod_{j=1}^k (-\eta_4 + \eta_3 + \eta - 2 + 2j). \quad (\text{D.6})$$

To get the light asymptotic limit for the denominator of (2.1) one must use (C.3) and the constraint rules (B.4) and (B.5). The result will be

$$Z_{bf}(a_2^{(1)}, \emptyset | a_1^{(1)}, Y_1) \xrightarrow{\epsilon_1 \rightarrow 0} \epsilon_2^k; \quad (\text{D.7})$$

$$Z_{bf}(a_1^{(1)}, Y_1 | a_2^{(1)}, \emptyset) \xrightarrow{\epsilon_1 \rightarrow 0} \epsilon_1^k \prod_{j=1}^k (2\eta - 2 + 2j); \quad (\text{D.8})$$

$$Z_{bf}(a_1^{(1)}, Y_1 | a_1^{(1)}, Y_1) \xrightarrow{\epsilon_1 \rightarrow 0} (\epsilon_2 \epsilon_1)^k \prod_{j=0}^{k-1} (2 + 2j). \quad (\text{D.9})$$

Now taking the product of (D.3)–(D.6) and dividing it to the product of (D.7)–(D.9) one gets (5.1).

Now I will derive $\diamond F_{(0,1),(0,1)}^{(0,0)}$. As we know from appendix C, Y_2 is a Young diagram with only one column (see 3(b)) containing $2m$ boxes and Y_1 a one row Young diagram (see 3(a)) with $2k$ boxes. The bifundamentals are derived in the same way as in the first case. The results for the numerator of (2.3) are:

$$Z_{bf}(a_2^{(0)}, \emptyset | a_2^{(1)}, Y_2) \xrightarrow{\epsilon_1 \rightarrow 0} (-\epsilon_2)^m \prod_{i=1}^m (2i-1); \quad (\text{D.10})$$

$$Z_{bf}(a_1^{(0)}, \emptyset | a_2^{(1)}, Y_2) \xrightarrow{\epsilon_1 \rightarrow 0} (-\epsilon_2)^m \prod_{i=1}^m (2i-1); \quad (\text{D.11})$$

$$Z_{bf}(a_2^{(0)}, \emptyset | a_1^{(1)}, Y_1) \xrightarrow{\epsilon_1 \rightarrow 0} \epsilon_2^k; \quad (\text{D.12})$$

$$Z_{bf}(a_1^{(0)}, \emptyset | a_1^{(1)}, Y_1) \xrightarrow{\epsilon_1 \rightarrow 0} \epsilon_1^k \prod_{j=1}^k (\eta_1 - \eta_2 - \eta + 2 - 2j); \quad (\text{D.13})$$

$$Z_{bf}(a_2^{(1)}, Y_2 | a_2^{(2)}, \emptyset) \xrightarrow{\epsilon_1 \rightarrow 0} \epsilon_2^m \prod_{i=1}^m (2i-1); \quad (\text{D.14})$$

$$Z_{bf}(a_2^{(1)}, Y_2 | a_1^{(2)}, \emptyset) \xrightarrow{\epsilon_1 \rightarrow 0} \epsilon_2^m \prod_{i=1}^m (2i-1); \quad (\text{D.15})$$

$$Z_{bf}(a_1^{(1)}, Y_1 | a_2^{(2)}, \emptyset) \xrightarrow{\epsilon_1 \rightarrow 0} (-\epsilon_2)^k; \quad (\text{D.16})$$

$$Z_{bf}(a_1^{(1)}, Y_1 | a_1^{(2)}, \emptyset) \xrightarrow{\epsilon_1 \rightarrow 0} \epsilon_1^k \prod_{j=1}^k (\eta - \eta_4 + \eta_3 - 2 + 2j) \quad (\text{D.17})$$

and for the denominator:

$$Z_{bf}(a_2^{(1)}, Y_2 | a_1^{(1)}, Y_1) \xrightarrow{\epsilon_1 \rightarrow 0} (-\epsilon_2)^k \epsilon_2^m \prod_{i=1}^m 2i; \quad (\text{D.18})$$

$$Z_{bf}(a_2^{(1)}, Y_2 | a_2^{(1)}, Y_2) \xrightarrow{\epsilon_1 \rightarrow 0} (-\epsilon_2)^m \epsilon_2^m \prod_{i=1}^m 2i \prod_{i=1}^m (2i-1); \quad (\text{D.19})$$

$$Z_{bf}(a_1^{(1)}, Y_1 | a_1^{(1)}, Y_1) \xrightarrow{\epsilon_1 \rightarrow 0} (-\epsilon_2)^k \epsilon_1^k \prod_{j=1}^k 2j; \quad (\text{D.20})$$

$$Z_{bf}(a_1^{(1)}, Y_1 | a_2^{(1)}, Y_2) \xrightarrow{\epsilon_1 \rightarrow 0} (-\epsilon_2)^m \epsilon_1^k \prod_{j=1}^k (2\eta - 2 + 2j) \prod_{i=1}^m (2i-1), \quad (\text{D.21})$$

by dividing the numerator to the denominator one gets (5.12).

E $H_{\pm}, \tilde{H}_{\pm}, F_{\pm}, \tilde{F}_{\pm}$ without variable exchange

$$H_{-}^L(q) = \frac{\epsilon_2 \sqrt{\sqrt{1-q} + 1} (E(q) - \sqrt{1-q}K(q)) {}_2F_1(A + \frac{1}{2}, B + \frac{1}{2}; \eta + 1; q)}{\epsilon_1 \sqrt{2\pi\eta} \sqrt[8]{1-q}}, \quad (\text{E.1})$$

$$\tilde{H}_-^L(q) = -\frac{\epsilon_2 \sqrt{\sqrt{1-q}+1}(1-q)^{3/8} \sqrt{q} (\sqrt{1-q}K(q)+E(q)) {}_2F_1\left(A+\frac{1}{2}, B+\frac{1}{2}; \eta+1; q\right)}{\epsilon_1 \sqrt{2\pi\eta} (-q + \sqrt{1-q} + 1)}, \quad (\text{E.2})$$

$$H_+^L(q) = \frac{\sqrt{2}\sqrt{\sqrt{1-q}+1}(1-q)^{3/8} (\sqrt{1-q}K(q)+E(q)) {}_2F_1(A, B; \eta; q)}{\pi (-q + \sqrt{1-q} + 1)}, \quad (\text{E.3})$$

$$\tilde{H}_+^L(q) = \frac{\sqrt{2}\sqrt{\sqrt{1-q}+1} (\sqrt{1-q}K(q)-E(q)) {}_2F_1(A, B; \eta; q)}{\pi \sqrt[8]{1-q}\sqrt{q}}, \quad (\text{E.4})$$

$$F_+^L(q) = \frac{\sqrt{\sqrt{1-q}+1}(1-q)^{3/8}}{\sqrt{2\eta} (-q + \sqrt{1-q} + 1)} \times \left(\eta (\sqrt{1-q} + 1) {}_2F_1\left(A, B + \frac{1}{2}; \eta; q\right) - Aq {}_2F_1\left(A + 1, B + \frac{1}{2}; \eta + 1; q\right) \right), \quad (\text{E.5})$$

$$F_-^L(q) = \frac{\sqrt{\sqrt{1-q}+1}(1-q)^{3/8}}{\sqrt{2\eta} (-q + \sqrt{1-q} + 1)} \times \left(\eta (\sqrt{1-q} + 1) {}_2F_1\left(A + \frac{1}{2}, B; \eta; q\right) - Bq {}_2F_1\left(A + \frac{1}{2}, B + 1; \eta + 1; q\right) \right), \quad (\text{E.6})$$

$$\tilde{F}_+^L(-q) = \frac{\sqrt{\sqrt{1-q}+1}(1-q)^{3/8} \sqrt{q}}{\sqrt{2\eta} (-q + \sqrt{1-q} + 1)} \times \left(A (\sqrt{1-q} + 1) {}_2F_1\left(A + 1, B + \frac{1}{2}; \eta + 1; q\right) - \eta {}_2F_1\left(A, B + \frac{1}{2}; \eta; q\right) \right), \quad (\text{E.7})$$

$$\tilde{F}_-^L(-q) = \frac{\sqrt{\sqrt{1-q}+1}(1-q)^{3/8} \sqrt{q}}{\sqrt{2\eta} (-q + \sqrt{1-q} + 1)} \times \left(B (\sqrt{1-q} + 1) {}_2F_1\left(A + \frac{1}{2}, B + 1; \eta + 1; q\right) - \eta {}_2F_1\left(A + \frac{1}{2}, B; \eta; q\right) \right). \quad (\text{E.8})$$

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References

- [1] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, *Nucl. Phys.* **B 241** (1984) 333 [[INSPIRE](#)].
- [2] M. Green, J. Schwarz and E. Witten, *Superstring theory: volume 1. Introduction*, Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge U.K. (1988).
- [3] A.M. Polyakov, *Quantum geometry of bosonic strings*, *Phys. Lett.* **B 103** (1981) 207.
- [4] A.B. Zamolodchikov, *Infinite additional symmetries in two-dimensional conformal quantum field theory*, *Theor. Math. Phys.* **65** (1985) 1205 [[INSPIRE](#)].
- [5] L.F. Alday, D. Gaiotto and Y. Tachikawa, *Liouville correlation functions from four-dimensional gauge theories*, *Lett. Math. Phys.* **91** (2010) 167 [[arXiv:0906.3219](#)] [[INSPIRE](#)].

- [6] N.A. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, *Adv. Theor. Math. Phys.* **7** (2003) 831 [[hep-th/0206161](#)] [[INSPIRE](#)].
- [7] A. Losev, N. Nekrasov and S.L. Shatashvili, *Testing Seiberg-Witten solution*, in the proceedings of the *Strings, branes and dualities*, May 26–June 14, Cargese, France (1997).
- [8] N. Nekrasov and A. Okounkov, *Seiberg-Witten theory and random partitions*, *Prog. Math.* **244** (2006) 525 [[hep-th/0306238](#)] [[INSPIRE](#)].
- [9] R. Flume and R. Poghossian, *An algorithm for the microscopic evaluation of the coefficients of the Seiberg-Witten prepotential*, *Int. J. Mod. Phys. A* **18** (2003) 2541 [[hep-th/0208176](#)] [[INSPIRE](#)].
- [10] H. Poghosyan, R. Poghossian and G. Sarkissian, *The light asymptotic limit of conformal blocks in Toda field theory*, *JHEP* **05** (2016) 087 [[arXiv:1602.04829](#)] [[INSPIRE](#)].
- [11] A.M. Polyakov, *Quantum geometry of fermionic strings*, *Phys. Lett.* **B 103** (1981) 211.
- [12] A.B. Zamolodchikov and R.G. Poghossian, *Operator algebra in two-dimensional superconformal field theory* (in Russian), *Sov. J. Nucl. Phys.* **47** (1988) 929 [[INSPIRE](#)].
- [13] D. Friedan, Z.-a. Qiu and S.H. Shenker, *Superconformal invariance in two-dimensions and the tricritical Ising model*, *Phys. Lett.* **151B** (1985) 37 [[INSPIRE](#)].
- [14] M.A. Bershadsky, V.G. Knizhnik and M.G. Teitelman, *Superconformal symmetry in two-dimensions*, *Phys. Lett.* **151B** (1985) 31 [[INSPIRE](#)].
- [15] H. Eichenherr, *Minimal operator algebras in superconformal quantum field theory*, *Phys. Lett.* **151B** (1985) 26 [[INSPIRE](#)].
- [16] A. Belavin and B. Mukhametzhanov, *$N = 1$ superconformal blocks with Ramond fields from AGT correspondence*, *JHEP* **01** (2013) 178 [[arXiv:1210.7454](#)] [[INSPIRE](#)].
- [17] A. Belavin, V. Belavin and M. Bershtein, *Instantons and 2d superconformal field theory*, *JHEP* **09** (2011) 117 [[arXiv:1106.4001](#)] [[INSPIRE](#)].
- [18] V. Belavin and B. Feigin, *Super Liouville conformal blocks from $N = 2$ $SU(2)$ quiver gauge theories*, *JHEP* **07** (2011) 079 [[arXiv:1105.5800](#)] [[INSPIRE](#)].
- [19] F. Fucito, J.F. Morales and R. Poghossian, *Multi instanton calculus on ALE spaces*, *Nucl. Phys.* **B 703** (2004) 518 [[hep-th/0406243](#)] [[INSPIRE](#)].
- [20] F. Fucito, J.F. Morales and R. Poghossian, *Instanton on toric singularities and black hole countings*, *JHEP* **12** (2006) 073 [[hep-th/0610154](#)] [[INSPIRE](#)].
- [21] V.A. Belavin, *$N = 1$ supersymmetric conformal block recursion relations*, *Theor. Math. Phys.* **152** (2007) 1275 [[hep-th/0611295](#)] [[INSPIRE](#)].