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Defects in the tri-critical Ising model

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Abstract: We consider two different conformal field theories with central charge $c = 7/10$. One is the diagonal invariant minimal model in which all fields have integer spins; the other is the local fermionic theory with superconformal symmetry in which fields can have half-integer spin. We construct new conformal (but not topological or factorised) defects in the minimal model. We do this by first constructing defects in the fermionic model as boundary conditions in a fermionic theory of central charge $c = 7/5$, using the folding trick as first proposed by Gang and Yamaguchi [1]. We then act on these with interface defects to find the new conformal defects. As part of the construction, we find the topological defects in the fermionic theory and the interfaces between the fermionic theory and the minimal model. We also consider the simpler case of defects in the theory of a single free fermion and interface defects between the Ising model and a single fermion as a prelude to calculations in the tri-critical Ising model.

Keywords: Conformal Field Theory, Boundary Quantum Field Theory, Conformal and W Symmetry

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1 Introduction

A conformal defect is a local line of discontinuity in a conformal field theory or between two different conformal field theories. The simplest situation is that of a cylinder with the defect wrapping the cylinder once, or equivalently of a plane with the defect placed on the unit circle. The defect can be represented as an operator D mapping from the Hilbert space of the theory inside the unit circle to that of the theory outside the circle and it is easy to state the condition that the defect is conformal:

$$(L_m - \bar{L}_{-m})\, D = D(L_m - \bar{L}_{-m})\ . \tag{1.1}$$

There are very few cases, however, in which the general solution to this equation can be found as this is equivalent (via the folding trick) to the equations of a conformal boundary condition in the folded model. If the central charges of the holomorphic and anti-holomorphic Virasoro algebras of two theories inside and outside the defect are $(c_1, \bar{c}_1)$ and $(c_2, \bar{c}_2)$, then the folded model has central charges ($c_{\text{tot}} = c_1 + \bar{c}_2$, $\bar{c}_{\text{tot}} = \bar{c}_1 + c_2$). A conformal defect between the two theories can only exist if $c_1 - \bar{c}_2 = \bar{c}_1 - c_2$, or equivalently a conformal boundary condition on the folded model can only exist if $c_{\text{tot}} = \bar{c}_{\text{tot}}$. Conformal boundary conditions have been completely classified for minimal models of the Virasoro Algebra [2, 3] (which have $c < 1$) and for free boson theories [4–9] (with $c = 1$) but not for higher values of c. Since the tri-critical Ising model (TCIM) has $c = \bar{c} = 7/10$, the folded model has $c_{\text{tot}} = 7/5$, the general conformal boundary condition for the folded model is not known and so the general solution to (1.1) is not known for the TCIM. From now on, we will only consider theories with $c = \bar{c}$ and will not mention $\bar{c}$ again.

Some particular solutions to (1.1) are known - these are the Topological and Factorising defects. A defect is topological if it satisfies

$$L_m\, D = D L_m\ , \quad \bar{L}_m\, D = D\bar{L}_m\ . \tag{1.2}$$

Such defects are classified for unitary minimal models such as the TCIM for which there are 6 fundamental topological defects.

A defect is factorising if it satisfies

$$(L_m - \bar{L}_{-m})\, D = 0\ ,\quad D(L_m - \bar{L}_{-m}) = 0\ , \tag{1.3}$$

which is equivalent to a cut in the worldsheet separating the inner and outer theories with a conformal boundary condition for each theory. Conformal boundary conditions have again been classified for unitary minimal models and so all factorising defects are also known for the TCIM: there are 6 fundamental conformal boundary conditions for the TCIM leading to 36 fundamental factorising defects.

Sums of topological and factorising defects will of course also satisfy (1.1) but there is strong evidence that these do not exhaust the list of conformal defects for the TCIM. Notably, one can consider relevant perturbations of topological defects in the TCIM. Such perturbations define a renormalisation group flow in the space of defects, with the defect at the IR fixed point also being a conformal defect. Both perturbative and numerical TCSA (truncated conformal space approach) calculations suggest [10] that there are non-topological, non-factorising, conformal defects which can be found this way.

In [1], Gang and Yamaguchi considered defects in TCIM constructed as GSO projections of boundary states in a folded supersymmetric model. This provided candidate expressions for new non-topological non-factorising conformal defects, but their construction also produced factorising defects which did not agree with the known expressions. For this reason, we think that their paper deserves re-examination: if the construction produces factorising defects which fall outside the known classification, then it is quite possible that the new candidate defects they proposed are also incorrect.

In this paper we take a first step towards re-examining the results of [1] through a related, but different, route to constructing defects in TCIM. We adapt ideas from Gaiotto [11] and from Gang and Yamaguchi [1] to construct defects in TCIM from conformal boundaries in the folded version of the Neveu-Schwarz sector of a fermionic theory through the use of interfaces between the fermionic theory and the TCIM. The idea is that there is a fermionic $c = 7/10$ theory (SVIR_3) with superconformal symmetry which is related to the TCIM. Superconformal defects in SVIR_3 are then equivalent to superconformal boundary conditions in $\mathrm{SVIR}_3{}^{\otimes 2}$=$\mathrm{SVIR}_3\otimes\mathrm{SVIR}_3$. As $\mathrm{SVIR}_3{}^{\otimes 2}$ has central charge $c = 7/5$ less than 3/2, its *superconformal* boundary conditions can be classified. We propose a set of boundary conditions B for $\mathrm{SVIR}_3{}^{\otimes 2}$ (related to but not the same as those in [1]), leading to superconformal defects D' in SVIR_3. We also propose a set of topological interfaces I between SVIR_3 and TCIM. One can then sandwich the superconformal defects D' between the topological interface defects I interpolating between the TCIM and the fermionic model, leading to conformal defects $D = ID'I^\dagger$ in the TCIM. One can then easily show (by considering the defect entropy) that there are defects D that are not expressible in terms of elementary topological and factorised defects in TCIM, and hence are new defects. This construction is summarised in figure 1.

As a warm-up exercise, in section 2 we consider first the simpler case of the $c = 1/2$ Ising model and the related free-fermion. We propose defects in the free-fermion model which preserve the fermion algebra and interfaces between the Ising model and the free-fermion model.

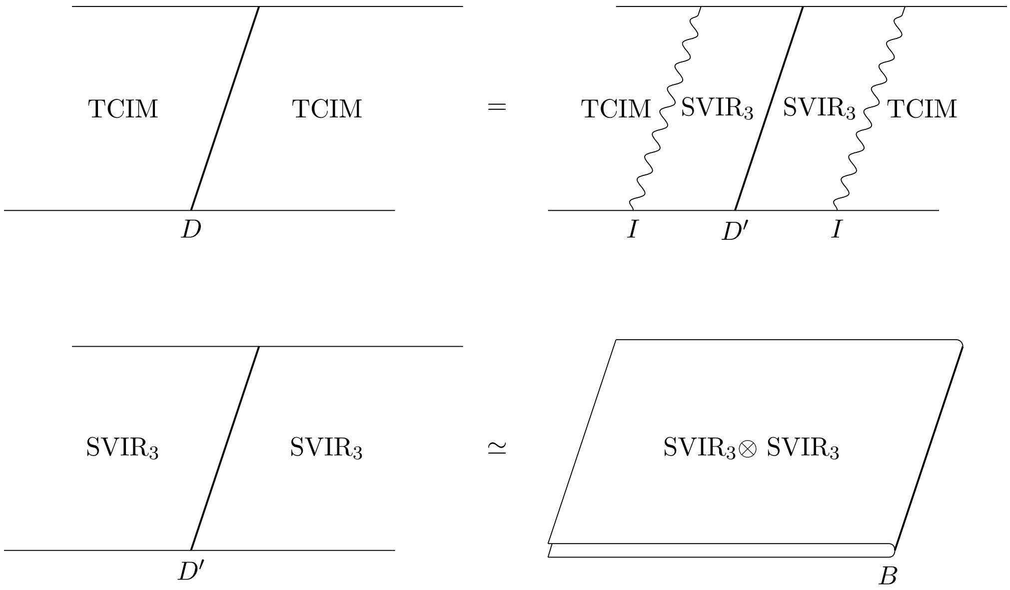

Figure 1. The equivalence between a boundary condition B in the folded fermionic model, a defect D' in the fermionic model and a defect D in the TCIM. I is an interface defect.

We then turn in section 3 to TCIM, the tri-critical Ising model, and SVIR$_3$, the fermionic model at $c = 7/10$, and propose superconformal topological defects in SVIR$_3$ and topological interfaces between SVIR$_3$ and TCIM.

Next, in section 4, we consider SVIR$_3^{\otimes 2}$ and propose a set of superconformal boundary conditions in this model. This is related to, but not the same as, the construction in [1]. By considering partition functions of SVIR$_3^{\otimes 2}$ on the cylinder and comparing them with partition functions of SVIR$_3$ on the torus, we show how to interpret some of these boundary conditions in terms of known defects in SVIR$_3$, and how some are new defects in SVIR$_3$.

Finally, we consider the construction of conformal defects in TCIM from the superconformal defects in SVIR$_3$ in section 6 and show that some of these cannot be constructed as superpositions of known topological and factorised defects in TCIM. We also explain that we are unable to compare our results with those of [1] which do not seem compatible with our general approach.

We end with some comments on the new defects and on possible further work.

2 The Ising model and the free fermion

In this section we recall the basics of the Ising model, set up our definition of the Neveu-Schwarz free fermion theory, and propose a set of defects in the free fermion theory and a set of interfaces between the Ising model and the free-fermion theory. We present some results exhibiting the consistency of these proposals. We give our conventions for the characters of the free fermion and Ising model in appendix B.

2.1 The Ising model

There is a single modular invariant unitary conformal field theory with $c = 1/2$, the Ising model. This is the first non-trivial value of c in the minimal unitary series $c(m) = 1 - 6/(m(m+1))$ for the Virasoro algebra, corresponding to $m = 3$.

The Virasoro algebra with $c = 1/2$ has three unitary irreducible highest weight representations with $h \in \{0,\ 1/2,\ 1/16\}$ with characters[1] $\chi_h^{(3)}$. As the Ising model is a diagonal modular invariant theory, the theory correspondingly has three primary fields ($I \equiv \phi_0$, $\epsilon \equiv \phi_{1/2}$, $\sigma \equiv \phi_{1/16}$), three topological defects (D_0, $D_{1/2}$, $D_{1/16}$) and three elementary conformal boundary conditions (B_0, $B_{1/2}$, $B_{1/16}$), each labelled by the set of highest weight representations [3]. The bulk Hilbert space is

$$\mathcal{H}_{\text{Ising}} = (\mathcal{H}_0 \otimes \bar{\mathcal{H}}_0) \oplus (\mathcal{H}_{1/2} \otimes \bar{\mathcal{H}}_{1/2}) \oplus (\mathcal{H}_{1/16} \otimes \bar{\mathcal{H}}_{1/16}) \ , \tag{2.1}$$

and the partition function on the torus is

$$\text{Tr}_{\mathcal{H}_{\text{Ising}}}(q^{L_0 - c/24}\bar{q}^{\bar{L}_0 - c/24}) = |\chi_0^{(3)}(q)|^2 + |\chi_{1/2}^{(3)}(q)|^2 + |\chi_{1/16}^{(3)}(q)|^2 \ , \tag{2.2}$$

which is modular invariant.

Modular invariance is, however, not a necessary condition to have a well defined field theory on the plane or on the cylinder, and amongst the possible field theories one can consider is the "Neveu-Schwarz free fermion" (FF) defined below.

2.2 The Neveu-Schwarz free fermion

We will consider the case of a symmetric theory with $c = \bar{c} = 1/2$ of a holomorphic fermion $\psi(z)$ and an anti-holomorphic fermion $\bar{\psi}(\bar{z})$. We shall also only consider the Neveu-Schwarz sector, in which the fermions on the plane have mode decompositions

$$\psi(z) = \sum_{m \in \mathbb{Z}} \psi_m z^{-m-1/2} \ , \quad \bar{\psi}(\bar{z}) = \sum_{m \in \mathbb{Z}} \bar{\psi}_m \bar{z}^{-m-1/2} \ , \tag{2.3}$$

and anti-commutators

$$\{\psi_m, \psi_n\} = \{\bar{\psi}_m, \bar{\psi}_n\} = \delta_{m+n,0} \ , \quad \{\psi_m, \bar{\psi}_n\} = 0 \ . \tag{2.4}$$

The Hilbert space, $\mathcal{H}_{\text{FF}}$, of the Neveu-Schwarz fermion is the Fock space generated by the action of negative fermion modes acting on the unique vacuum state $|0\rangle$,

$$\mathcal{H}_{\text{FF}} = \mathcal{H}_{NS} \otimes \bar{\mathcal{H}}_{NS} = (\mathcal{H}_0 \oplus \mathcal{H}_{1/2}) \otimes (\bar{\mathcal{H}}_0 \oplus \bar{\mathcal{H}}_{1/2}) \ . \tag{2.5}$$

The partition function on the cylinder is

$$Z_{\text{FF}} = \text{Tr}_{\mathcal{H}_{\text{FF}}}(q^{L_0 - c/24}\bar{q}^{\bar{L}_0 - c/24}) = |\chi_{NS}(q)|^2 \ , \tag{2.6}$$

where

$$\chi_{NS}(q) = \chi_0^{(3)}(q) + \chi_{1/2}^{(3)}(q) \ . \tag{2.7}$$

Z_{FF} is invariant under the modular transformation $\tau \to -1/\tau$, that is $q = \exp(2\pi i\tau) \to \tilde{q} = \exp(-2\pi i/\tau)$. The function is not invariant under $\tau \to 1 + \tau$ but we shall in general consider the theory defined on a right torus with q real, and the theory is well defined on such a space.

[1] We denote the character of the representation of weight h in the m-th unitary Virasoro minimal model by $\chi_h^{(m)}$.

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2.2.1 Defects in the free fermion theory

We say a defect $D_{\epsilon,\epsilon'}$ in the free-fermion model conserves the fermion algebra (up to automorphism) if

$$\psi_m D_{\epsilon,\epsilon'} = \epsilon D_{\epsilon,\epsilon'} \psi_m \ , \quad \bar\psi_n D_{\epsilon,\epsilon'} = \epsilon' D_{\epsilon,\epsilon'} \bar\psi_m \ , \tag{2.8}$$

where $\epsilon = \pm 1$ and $\epsilon' = \pm 1$. These conditions entirely determine the defect operators up to normalisation constants $\alpha_{\epsilon,\epsilon'}$ as

$$D_{++} = \alpha_{++}\mathbf{1} \ , \quad D_{+-} = \alpha_{+-}(-1)^{\bar F} \ , \quad D_{-+} = \alpha_{-+}(-1)^{F} \ , \quad D_{--} = \alpha_{--}(-1)^{F+\bar F} \ , \tag{2.9}$$

where $\mathbf{1}$ is the identity operator on $\mathcal{H}_{\rm FF}$.

We would like to impose two conditions: firstly, the Cardy condition that the trace over the cylinder with the insertion of a defect D_A is an integer combination of characters of the free fermion in the dual channel, ie the constants $N^A_{\alpha\beta}$ defined by

$$\mathrm{Tr}_{\mathcal{H}_{\rm FF}}(D_A\, q^{L_0 - c/24}\, \bar q^{\bar L_0 - c/24}) = \sum_{\alpha,\beta} N^A_{\alpha\beta}\, \chi_\alpha(\tilde q)\, \chi_\beta(\bar{\tilde q}) \ , \tag{2.10}$$

should be non-negative integers. This reflects the requirement that the torus partition function can be interpreted as a trace over a space which carries a representation of the free-fermion algebra. Note that we will allow α and β in the sum in (2.10) to run over NS and R which is necessary as the defect can change the periodicities of the field ψ and $\bar\psi$ - see the appendix for details.

The Hilbert space of the Neveu-Schwarz fermions is $\mathcal{H}_{\rm FF}$ and so we have

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{H}}(D_{++}\, q^{L_0-c/24}\bar q^{\bar L_0 - c/24}) &= \alpha_{++}|\chi_{NS}(q)|^2 &&= \alpha_{++}|\chi_{NS}(\tilde q)|^2 \\
\mathrm{Tr}_{\mathcal{H}}(D_{+-}\, q^{L_0-c/24}\bar q^{\bar L_0 - c/24}) &= \alpha_{+-}\chi_{NS}(q)\chi_{\widetilde{NS}}(\bar q) &&= \sqrt{2}\alpha_{+-}\chi_{NS}(\tilde q)\chi_R(\bar{\tilde q}) \\
\mathrm{Tr}_{\mathcal{H}}(D_{-+}\, q^{L_0-c/24}\bar q^{\bar L_0 - c/24}) &= \alpha_{-+}\chi_{\widetilde{NS}}(q)\chi_{NS}(\bar q) &&= \sqrt{2}\alpha_{-+}\chi_R(\tilde q)\chi_{NS}(\bar{\tilde q}) \\
\mathrm{Tr}_{\mathcal{H}}(D_{--}\, q^{L_0-c/24}\bar q^{\bar L_0 - c/24}) &= \alpha_{--}\chi_{\widetilde{NS}}(q)\chi_{\widetilde{NS}}(\bar q) &&= 2\alpha_{--}|\chi_R(\tilde q)|^2
\end{aligned} \tag{2.11}$$

Secondly, we would like the structure constants M^C_{AB} in the algebra of defect operators,

$$D_A\, D_B \ = \sum_C M^C_{AB} D_C \ , \tag{2.12}$$

to be non-negative integers. We clearly have

$$D_{\epsilon,\epsilon'}\, D_{\eta,\eta'} = \frac{\alpha_{\epsilon,\epsilon'}\ \alpha_{\eta,\eta'}}{\alpha_{\epsilon\eta,\epsilon'\eta'}} D_{\epsilon\eta,\epsilon'\eta'} \ . \tag{2.13}$$

The simplest solution that makes the right-hand side of equations (2.11) and the coefficients in equation (2.13) integers is

$$\alpha_{++} = \alpha_{--} = 1 \ , \quad \alpha_{+-} = \alpha_{-+} = \sqrt{2} \ . \tag{2.14}$$

It is, at first sight, surprising that this means that the operator $(-1)^F$ is not represented by a defect, the defect instead being $D_{-+} = \sqrt{2}(-1)^F$. However, this seems necessary for there to be an integer number of operators that create the D_{-+} defect. The

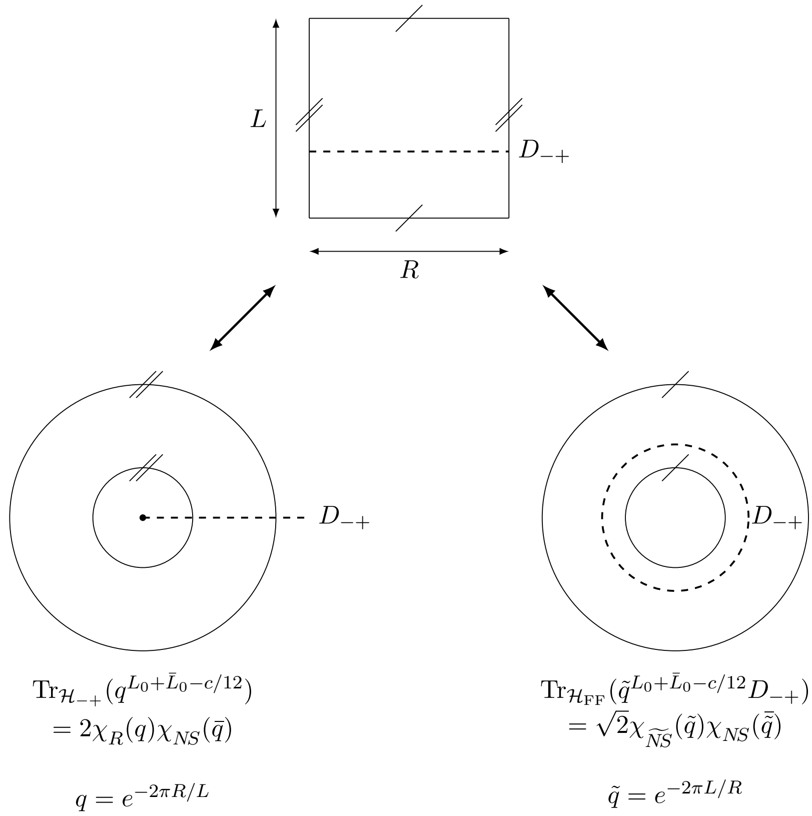

Figure 2. The trace over the space of fields on which the D_{-+} defect can end is related by a modular transformation to the trace with the defect inserted.

primary operators that create the defect, i.e. the operators on which the defect can end, are counted by the partition function on the torus with a single defect inserted, as shown in figure 2, and the proposal here ensures that this space is two-dimensional, which is the smallest dimension possible given that this space has to carry a representation of the Ramond algebra.

The space of fields on which the D_{-+} defect can end is $\mathcal{H}_{-+}$. This is related by a modular S transformation to the trace over the free fermion space with the insertion of D_{-+} and so (taking q to be real) $\mathrm{Tr}_{\mathcal{H}_{-+}}(q^{L_0+\bar{L}_0-c/12}) = \mathrm{Tr}_{\mathcal{H}_{\mathrm{FF}}}(\tilde{q}^{L_0+\bar{L}_0-c/12}D_{-+}) = \sqrt{2}\chi_{\widetilde{NS}}(\tilde{q})\chi_{NS}(\tilde{q}) = 2\chi_R(q)\chi_{NS}(q)$, i.e. a two dimensional space of primary operators on which the defect can end.

2.2.2 Boundary states for the free fermion theory

We would like to define boundary conditions which preserve the free fermion algebra, up to automorphism. This implies that a boundary state $\|B\rangle\rangle$ must satisfy the condition

$$(\psi_m - i\epsilon\bar{\psi}_{-m})\|B\rangle\rangle \ , \tag{2.15}$$

for $\epsilon = \pm 1$. The space of Ishibashi states for a fixed choice of ϵ is one-dimensional, that is there are only two Ishibashi states of interest,

$$|NS, \epsilon\rangle\rangle = \prod_{m=0}^{\infty} e^{i\epsilon\psi_{-m-1/2}\bar{\psi}_{-m-1/2}}|0\rangle \ . \tag{2.16}$$

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The overlaps of these Ishibashi states are

$$\begin{aligned}\langle\langle NS,\pm|q^{\frac{1}{2}(L_0+\bar{L}_0-c/12)}|NS,\pm\rangle\rangle &= \chi_{NS}(q) = \chi_{NS}(\tilde{q}) \ ,\\ \langle\langle NS,\pm|q^{\frac{1}{2}(L_0+\bar{L}_0-c/12)}|NS,\mp\rangle\rangle &= \chi_{\widetilde{NS}}(q) = \sqrt{2}\chi_R(\tilde{q}) \ .\end{aligned} \tag{2.17}$$

Note that we use 'double kets with single vertical bar' $|i\rangle\rangle$ to denote Ishibashi states and 'double kets with double vertical bars' $\|B\rangle\rangle$ to denote (elementary) boundary states. Since we would like the cylinder partition function of two physical boundary states to be expressible in the crossed channel as the trace of the Hamiltonian along a strip, it should be a non-negative integer combination of the characters $\chi_\alpha(\tilde{q})$. It would be nice to be able to choose the boundary states to be $|NS,+\rangle\rangle$ and $|NS,-\rangle\rangle$, but the mutual overlap is not an integer multiple of $\chi_R(\tilde{q})$ and so we have to make a choice for the boundary states. We can make either of the two choices

$$1.: \quad \{\|A\rangle\rangle = |NS,+\rangle\rangle \ , \ \ \|B\rangle\rangle = \sqrt{2}|NS,-\rangle\rangle\} \ , \tag{2.18}$$

$$2.: \quad \{\|A\rangle\rangle = |NS,-\rangle\rangle \ , \ \ \|B\rangle\rangle = \sqrt{2}|NS,+\rangle\rangle\} \ . \tag{2.19}$$

These then have cylinder partition functions

$$\begin{aligned}Z_{AA} &= \langle\langle A\|q^{\frac{1}{2}(L_0+\bar{L}_0-c/12)}\|A\rangle\rangle = \chi_{NS}(\tilde{q}) \ ,\\ Z_{AB} &= \langle\langle A\|q^{\frac{1}{2}(L_0+\bar{L}_0-c/12)}\|B\rangle\rangle = 2\chi_R(\tilde{q}) \ ,\\ Z_{BB} &= \langle\langle B\|q^{\frac{1}{2}(L_0+\bar{L}_0-c/12)}\|B\rangle\rangle = 2\chi_{NS}(\tilde{q}) \ .\end{aligned} \tag{2.20}$$

Since $\chi_{NS} = \chi_0 + \chi_{1/2}$, the identity Virasoro representation appears twice in Z_{BB} and so this is not actually an elementary defect with respect to the Virasoro algebra. The usual conclusion [12–14] would be that we need to introduce the Ramond sector with Ishibashi states $|R,\pm\rangle\rangle$. This will allow us to write the boundary state $\|B\rangle\rangle$ as the superposition of two states $\|B_\pm\rangle\rangle$, which take the form

$$1.: \quad \|B_\pm\rangle\rangle = \frac{1}{\sqrt{2}}|NS,-\rangle\rangle \pm \frac{1}{2^{1/4}}|R,-\rangle\rangle \ , \tag{2.21}$$

$$2.: \quad \|B_\pm\rangle\rangle = \frac{1}{\sqrt{2}}|NS,+\rangle\rangle \pm \frac{1}{2^{1/4}}|R,+\rangle\rangle \ . \tag{2.22}$$

These, however, do not have cylinder partition functions that can be interpreted as the trace over a representation of the fermion algebra. They can be considered as sums over different spin structures for the fermion, or as partition functions for the Ising model obtained from the free fermion by projecting onto even fermion number states. The Ising model has three fundamental boundary conditions, usually denoted $(-)$, (f) and $(+)$ for Ising spin fixed down, free, or fixed up and their boundary states can be identified as

$$\|f\rangle\rangle = \|A\rangle\rangle \ , \ \ \|\pm\rangle\rangle = \|B_\pm\rangle\rangle \ , \tag{2.23}$$

and the cylinder partition functions are traces over representations of the Virasoro algebra,

$$\begin{aligned} \langle\!\langle A \| q^{\frac{1}{2}(L_0+\bar{L}_0-c/12)} \| B_\pm \rangle\!\rangle &= \chi_R(\tilde{q}) = \chi^{(3)}_{1/16}(\tilde{q}) \,, \\ \langle\!\langle B_\pm \| q^{\frac{1}{2}(L_0+\bar{L}_0-c/12)} \| B_\pm \rangle\!\rangle &= \frac{1}{2}\big(\chi_{NS}(\tilde{q}) + \chi_{\widetilde{NS}}(\tilde{q})\big) = \chi^{(3)}_0(\tilde{q}) \\ \langle\!\langle B_\pm \| q^{\frac{1}{2}(L_0+\bar{L}_0-c/12)} \| B_\mp \rangle\!\rangle &= \frac{1}{2}\big(\chi_{NS}(\tilde{q}) - \chi_{\widetilde{NS}}(\tilde{q})\big) = \chi^{(3)}_{1/2}(\tilde{q}) \end{aligned} \tag{2.24}$$

There is, however, no requirement for us to extend the space of boundary conditions in this way: it is perfectly consistent to ask that the fermion boundary condition be defined for a single choice of spin structure and we will work with the boundary conditions $\|A\rangle\!\rangle$ and $\|B\rangle\!\rangle$.

This agrees with the analysis in [15, 16] in which the free fermion is stated to have two boundary conditions, "free" and "fixed",[2] distinguished by having two and one ground states respectively, with the "fixed" boundary condition having a two-dimensional space of weight zero fields, one bosonic and one fermionic. This fermionic weight zero field also arises in the Ising model, where the "ϵ" defect can end at the junction of a $(+)$ and a $(-)$ boundary condition in a field of weight zero. Such a space with one bosonic and one fermionic degree of freedom, $\mathbb{C}^{1|1}$, also arises when constructing spin fields in the Ising model in [18], and we think it is probable that a treatment of boundary conditions in the manner of [18] will give our result in a rigorous manner.

2.3 Interfaces between the Ising model and the free fermion

We would now like to consider the case of topological interfaces between the Ising model and the Neveu-Schwarz free fermion. Consider an operator I from the Hilbert space of the free fermion to the Hilbert space of the Ising model; the operator $I^\dagger$ will map from the Ising model to the free fermion model. The topological conditions,

$$L_m\, I = I\, L_m \;,\;\; \bar{L}_m\, I = I\, \bar{L}_m \;, \tag{2.25}$$

mean that I must be a sum of projectors on representations of the Virasoro algebra. Since the Hilbert spaces of the two theories are

$$\begin{aligned} \mathcal{H}_{\text{Ising}} &= (\mathcal{H}_0 \otimes \bar{\mathcal{H}}_0) \,\oplus\, (\mathcal{H}_{1/2} \otimes \bar{\mathcal{H}}_{1/2}) \,\oplus\, (\mathcal{H}_{1/16} \otimes \bar{\mathcal{H}}_{1/16}) \,, \\ \mathcal{H}_{\text{FF}} &= (\mathcal{H}_0 \oplus \mathcal{H}_{1/2}) \otimes (\bar{\mathcal{H}}_0 \oplus \bar{\mathcal{H}}_{1/2}) \,, \end{aligned} \tag{2.26}$$

we see that the operator I is determined up to two constants, a and b, as

$$I_{a,b} = a P_0 \bar{P}_0 + b P_{1/2} \bar{P}_{1/2} \,, \tag{2.27}$$

where $P_0\bar{P}_0$ is the projector onto $\mathcal{H}_0 \otimes \bar{\mathcal{H}}_0$ and $P_{1/2}\bar{P}_{1/2}$ is the projector onto $\mathcal{H}_{1/2} \otimes \bar{\mathcal{H}}_{1/2}$.

As before, we would like the coefficients $M^\gamma_{A\beta}$ and $M^\gamma_{\beta A}$ in the algebra of defect operators

$$D_A\, I_\beta = \sum_\gamma\, M^\gamma_{A\beta}\, I_\gamma \;,\;\; I_\beta\, D_A = \sum_\gamma\, M^\gamma_{\beta A}\, I_\gamma \;, \tag{2.28}$$

[2] Note that this is not the same as the usual naming conventions for the Ising model, for which "free" and "fixed" would have one and two ground states respectively, but since these are interchanged by duality [17], this should not be of concern.

to be non-negative integer coefficients. The known Ising defects [19, 20] and free fermion defects can be expressed in terms of the projectors $P_h\bar{P}_{\bar{h}}$ as

$$\begin{aligned}
\text{Ising}: D_0 \quad &= P_0\bar{P}_0 + P_{1/2}\bar{P}_{1/2} + P_{1/16}\bar{P}_{1/16} \ , \\
D_{1/2} \quad &= P_0\bar{P}_0 + P_{1/2}\bar{P}_{1/2} - P_{1/16}\bar{P}_{1/16} \ , \\
D_{1/16} &= \sqrt{2}(P_0\bar{P}_0 - P_{1/2}\bar{P}_{1/2}) \ , \\
\text{Free fermion}: D_{++} &= (P_0 + P_{1/2})(\bar{P}_0 + \bar{P}_{1/2}) \ , \\
D_{+-} &= \sqrt{2}(P_0 + P_{1/2})(\bar{P}_0 - \bar{P}_{1/2}) \ , \\
D_{-+} &= \sqrt{2}(P_0 - P_{1/2})(\bar{P}_0 + \bar{P}_{1/2}) \ , \\
D_{++} &= (P_0 - P_{1/2})(\bar{P}_0 - \bar{P}_{1/2}) \ .
\end{aligned} \tag{2.29}$$

This gives the algebra

$$D_0 \, I_{a,b} = I_{a,b} \ , \quad D_{1/2} \, I_{a,b} = I_{a,b} \ , \quad D_{1/16} \, I_{a,b} = \sqrt{2} I_{a,-b} \ , \tag{2.30}$$

$$I_{a,b} \, D_{++} = I_{a,b} \ , \quad I_{a,b} \, D_{+-} = \sqrt{2} I_{a,-b} \ , \quad I_{a,b} \, D_{-+} = \sqrt{2} I_{a,-b} \ , \quad I_{a,b} \, D_{--} = I_{a,b} \ . \tag{2.31}$$

We would also like the action of the interfaces on boundary states to give integer combinations of boundary states in the other model, that is the coefficients $M^B_{\alpha A}$ and $\tilde{M}^B_{\alpha A}$ in the algebra

$$I_\alpha \, \|B^{\text{FF}}_A\rangle\!\rangle = \sum_B M^B_{\alpha A} \, \|B^{\text{Ising}}_B\rangle\!\rangle \ , \quad I^\dagger_\alpha \, \|B^{\text{Ising}}_A\rangle\!\rangle = \sum_B \tilde{M}^B_{\alpha A} \, \|B^{\text{FF}}_B\rangle\!\rangle \ , \tag{2.32}$$

should be non-negative integers. If $I_{a,b}$ is an interface that acts from the free-fermion space to the Ising model and $I^\dagger_{a,b}$ acts in the opposite direction, we have

$$\begin{aligned}
I_{a,b} \, \|A\rangle\!\rangle &= \frac{a+b}{2\sqrt{2}} \big(\|B_0\rangle\!\rangle + \|B_{1/2}\rangle\!\rangle\big) + \frac{a-b}{2} \|B_{1/16}\rangle\!\rangle \ , \\
I_{a,b} \, \|B\rangle\!\rangle &= \frac{a-b}{2} \big(\|B_0\rangle\!\rangle + \|B_{1/2}\rangle\!\rangle\big) + \frac{a+b}{\sqrt{2}} \|B_{1/16}\rangle\!\rangle \ , \\
I^\dagger_{a,b} \, \|B_0\rangle\!\rangle &= \frac{a+b}{2\sqrt{2}} \|A\rangle\!\rangle + \frac{a-b}{4} \|B\rangle\!\rangle \ , \\
I^\dagger_{a,b} \, \|B_{1/2}\rangle\!\rangle &= \frac{a+b}{2\sqrt{2}} \|A\rangle\!\rangle + \frac{a-b}{4} \|B\rangle\!\rangle \ , \\
I^\dagger_{a,b} \, \|B_{1/16}\rangle\!\rangle &= \frac{a-b}{2} \|A\rangle\!\rangle + \frac{a+b}{2\sqrt{2}} \|B\rangle\!\rangle \ .
\end{aligned} \tag{2.33}$$

The general solution to the integrality conditions is $a = \sqrt{2}m + 2n$, $b = \sqrt{2}m - 2n$, for integers m and n. This suggests that there are two elementary interfaces, $I = I_{\sqrt{2},\sqrt{2}}$ and $I' = I_{2,-2}$ satisfying

$$\begin{aligned}
I \, \|A\rangle\!\rangle &= \|B_0\rangle\!\rangle + \|B_{1/2}\rangle\!\rangle \ , & I^\dagger \, \|B_0\rangle\!\rangle &= I^\dagger \, \|B_{1/2}\rangle\!\rangle = \|A\rangle\!\rangle \ , \\
I \, \|B\rangle\!\rangle &= 2\|B_{1/16}\rangle\!\rangle \ , & I^\dagger \, \|B_{1/16}\rangle\!\rangle &= \|B\rangle\!\rangle \ , \\
I' \, \|A\rangle\!\rangle &= 2\|B_{1/16}\rangle\!\rangle \ , & I'^\dagger \, \|B_0\rangle\!\rangle &= I'^\dagger \, \|B_{1/2}\rangle\!\rangle = \|B\rangle\!\rangle \ , \\
I' \, \|B\rangle\!\rangle &= 2\big(\|B_0\rangle\!\rangle + \|B_{1/2}\rangle\!\rangle\big) \ , & I'^\dagger \, \|B_{1/16}\rangle\!\rangle &= 2\|A\rangle\!\rangle \ ,
\end{aligned} \tag{2.34}$$

and that every other interface is formed by a linear combination of these two.

Finally, we would like the product of interfaces to be expressible as a non-negative integer combination of topological defects, that is the constants $M^A_{\alpha\beta}$ and $\tilde{M}^A_{\alpha\beta}$ in

$$I_\alpha\, I^\dagger_\beta = \sum_A M^A_{\alpha\beta}\, D^{\rm Ising}_A\ , \quad I^\dagger_\alpha\, I_\beta = \sum_A \tilde{M}^A_{\alpha\beta}\, D^{\rm FF}_A\ , \tag{2.35}$$

should also be non-negative integers. We have

$$\begin{aligned} I_{a,b}\, I^\dagger_{c,d} &= \frac{ac+bd}{4}\big(D_0 + D_{1/2}\big) + \frac{ac-bd}{2\sqrt{2}} D_{1/16}\ , \\ I^\dagger_{c,d}\, I_{a,b} &= \frac{ac+bd}{4}\big(D_{++} + D_{--}\big) + \frac{ac-bd}{4\sqrt{2}}\big(D_{+-} + D_{-+}\big)\ , \end{aligned} \tag{2.36}$$

We see that two elementary interfaces I and I' do lead to an algebra with integer coefficients:

$$\begin{aligned} I\, I^\dagger &= D_0 + D_{1/2}\ , & I\, I'^\dagger &= 2D_{1/16}\ , \\ I^\dagger\, I &= D_{++} + D_{--}\ , & I'^\dagger\, I &= D_{+-} + D_{-+}\ , \\ I\, D_{++} &= I\, D_{--} = I\ , & I\, D_{+-} &= I\, D_{-+} = I'\ , \\ D_0\, I &= D_{1/2}\, I = I\ , & D_{1/16}\, I &= I'\ . \end{aligned} \tag{2.37}$$

2.4 Consistency tests

The main object of this section was to construct topological interface operators between the Ising model and the free-fermion model (as a warm-up for the tri-critical Ising case). Since these interfaces are topological, one should be able to move the interfaces past field insertions without changing their conformal properties. As an example, we can ask whether we can pull the interface past the free fermion field, so that the free fermion field's insertion point is now on the Ising model side of the interface. Since the free fermion $\psi(z)$ is not a local field in the Ising model, it must arise as a defect creation operator, that is as the termination point of a defect, which is the defect $D_{1/2}$. This is shown in figure 3. This is only possible if there is a one-dimensional space of zero-weight interface-interface-defect junctions. These are counted by the partition function with insertions of I, $D_{1/2}$ and $I^\dagger$. We can calculate this using $I\, I^\dagger\, D_{1/2} = (D_0 + D_{1/2})\, D_{1/2} = (D_0 + D_{1/2})$,

$$\begin{aligned} {\rm Tr}_{\mathcal{H}_{IID_{1/2}}}(q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24}) &= {\rm Tr}_{\mathcal{H}_{\rm FF}}(I^\dagger D_{1/2} I\, q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24}) \\ &= {\rm Tr}_{\mathcal{H}_{\rm Ising}}(I I^\dagger D_{1/2}\, q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24}) \\ &= {\rm Tr}_{\mathcal{H}_{\rm Ising}}(D_0\, q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24}) \\ &\quad + {\rm Tr}_{\mathcal{H}_{\rm Ising}}(D_{1/2}\, q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24}) \\ &= \Big(|\chi^{(3)}_0(q)|^2 + |\chi^{(3)}_{1/2}(q)|^2 + |\chi^{(3)}_{1/16}(q)|^2\Big) \\ &\quad + \Big(|\chi^{(3)}_0(q)|^2 + |\chi^{(3)}_{1/2}(q)|^2 - |\chi^{(3)}_{1/16}(q)|^2\Big) \\ &= 2|\chi^{(3)}_0(q)|^2 + 2|\chi^{(3)}_{1/2}(q)|^2 \\ &= |\chi^{(3)}_0(\tilde{q}) + \chi^{(3)}_{1/2}(\tilde{q})|^2 + 2|\chi^{(3)}_{1/16}(\tilde{q})|^2\ . \end{aligned} \tag{2.38}$$

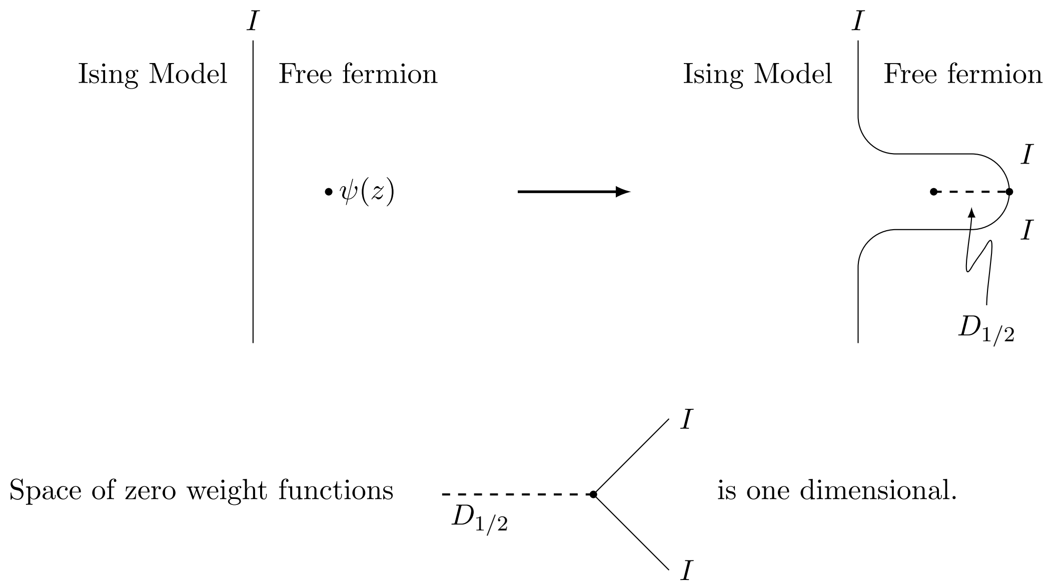

Figure 3. The ψ field in the free-fermion model is realised as a defect-creation operator in the Ising model. The associated space of three-defect junctions is one-dimensional.

The coefficient of $|\chi_0^{(3)}(\tilde{q})|^2$ counts the dimension of weight zero junction fields so this is indeed one-dimensional.

As a second example, consider a defect D_{-+} terminating in the FF model on fields of conformal weights $(1/16, 0)$. Since the partition function with D_{-+} is given from (2.11) and (2.14) as

$$\mathrm{Tr}_{\mathcal{H}_{\mathrm{FF}}}(D_{-+}q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24}) = 2\chi_R(\tilde{q})\chi_{NS}(\bar{\tilde{q}}) \;, \tag{2.39}$$

the space of such fields is two dimensional. What happens when the interface is pulled past this defect-terminating field? The only way a field of weights $(1/16, 0)$ can arise in the Ising model is as a field on the end of a $D_{1/16}$ defect, and the space of such fields is only one dimensional. The resolution is that the space of zero-weight fields at the point where the D_{-+} defect crosses the interface I to become the $D_{1/16}$ defect is two-dimensional, as calculated by

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{H}_{ID_{-+}ID_{1/16}}}(q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24}) &= \mathrm{Tr}_{\mathcal{H}_{\mathrm{Ising}}}(ID_{-+}I^\dagger D_{1/16}\, q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24}) \\
&= \mathrm{Tr}_{\mathcal{H}_{\mathrm{Ising}}}\big((2D_0+2D_{1/2})q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24}\big) \\
&= 2|\chi_0(\tilde{q})|^2 + \dots \;.
\end{aligned} \tag{2.40}$$

In every case we have considered, the result of pulling local fields through the interface result, in a similar fashion, in consistent interpretations in terms of the space of interface-defect junctions.

We now turn to the main objects of interest, the tri-critical Ising model and the related supersymmetric theory.

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name	I	ϵ	σ	$\hat{I}$	$\hat{\epsilon}$	$\hat{\sigma}$
Kac labels (r, s)	$(1, 1)$	$(3, 1)$	$(2, 1)$	$(1, 3)$	$(1, 2)$	$(2, 2)$
weight h	0	$3/2$	$7/16$	$3/5$	$1/10$	$3/80$
Entropy of defect D_h	1	1	$\sqrt{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{\sqrt{2}}$
Entropy of boundary B_h	$\left(\frac{5-\sqrt{5}}{40}\right)^{\frac{1}{4}}$	$\left(\frac{5-\sqrt{5}}{40}\right)^{\frac{1}{4}}$	$\left(\frac{5-\sqrt{5}}{10}\right)^{\frac{1}{4}}$	$\left(\frac{5+2\sqrt{5}}{20}\right)^{\frac{1}{4}}$	$\left(\frac{5+2\sqrt{5}}{20}\right)^{\frac{1}{4}}$	$\left(\frac{5+2\sqrt{5}}{5}\right)^{\frac{1}{4}}$

Table 1. TCIM data.

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3 The tri-critical Ising model and the fermionic model at $c = 7/10$

The tri-critical Ising model (TCIM for short) is the unitary minimal model of the Virasoro algebra with $c = 7/10$ with diagonal modular invariant and all the local fields have integer spin. The Virasoro algebra with $c = 7/10$ has six unitary highest weight representations, and correspondingly the TCIM has six primary fields, six elementary topological defects and six elementary conformal boundary conditions, each labelled by these representations.

The TCIM is the unique local field theory with $c = 7/10$ with a modular invariant partition function but we can define a related model with local fields with integer and half-integer spins which is well-defined on the cylinder, in the same way the FF model is related to the Ising model. This fermionic model has superconformal symmetry and we will denote it by SVIR_3. There are four irreducible unitary highest-weight representations of the super Virasoro algebra with $c = 7/10$ of which two are Neveu-Schwarz and two are Ramond. The local field theory SVIR_3 is the diagonal theory formed from the Neveu-Schwarz representations.

Our aim is to find interfaces between the TCIM and SVIR_3 theories and use these to construct defects in TCIM from defects in SVIR_3. Before we do that we will first have to construct defects in SVIR_3 and we will do this by identifying them with boundary conditions on the doubled model, $\text{SVIR}_3{}^{\otimes 2}$. This is the main technical challenge of the paper. Before we come to this point, we first introduce some notation and recall some facts about the TCIM and SVIR_3 models. (We list our conventions for the representations, their labels, characters, fusion rules, etc, in appendix C.)

3.1 TCIM

The tri-critical Ising model is the unitary, diagonal modular invariant theory at $c = 7/10$ and is the $m = 4$ member of the unitary minimal series. The Virasoro algebra has six unitary highest weight representations at $c = 7/10$ with characters $\chi_h^{(4)}$; associated to each of these there is a topological defect $D_{\hat{\epsilon}} \equiv D_{1/10} \equiv D_{(1,2)}$, etc and a conformal boundary condition $|\hat{\epsilon}\rangle\rangle$ etc. We list these together with their defect and boundary entropies in table 1.

The shorthand names are chosen to make clear the fusion rules, which are (Lee-Yang)×(Ising). The fields I, ϵ and σ have Ising fusion rules, and I and $\hat{I}$ have Lee-Yang

name	I_{NS}	φ_R	φ_{NS}	I_R
Kac labels $(r.s)$	$(1,1)\equiv(2,4)$	$(1,2)\equiv(2,3)$	$(1,3)\equiv(2,2)$	$(1,4)\equiv(2,1)$
weight h	0	3/80	1/10	7/16

Table 2. SVIR_3 data.

rules, that is

$$\hat{I}\star\hat{I} = I + \hat{I}\ , \tag{3.1}$$

and in general

$$X\star\hat{Y} = (\widehat{X\star Y})\ ,\ \ \hat{X}\star\hat{Y} = (X\star Y) + (\widehat{X\star Y})\ . \tag{3.2}$$

3.2 SVIR_3

The SVIR_3 model has local fields $G(z)$ and $\bar{G}(\bar{z})$ of conformal weights $(3/2,0)$ and $(0,3/2)$ respectively, which are fermionic and generate two copies of the $c = 7/10$ superconformal algebra. This is the $m = 3$ member of the superconformal unitary minimal series for which $c(m) = 3/2(1-8/(m(m+2)))$. There are two Neveu-Schwarz representations and two Ramond representations, which we label as in table 2.

The fusion rules are (Lee-Yang)×(Free-fermion). The local field theory SVIR_3 consists of the Neveu-Schwarz fields and, in terms of representations of the Virasoro algebra, has Hilbert space

$$\mathcal{H}_{\mathrm{SVIR}_3} = (\mathcal{H}_0\oplus\mathcal{H}_{3/2})\otimes(\bar{\mathcal{H}}_0\oplus\bar{\mathcal{H}}_{3/2})\ \ \oplus\ \ (\mathcal{H}_{1/10}\oplus\mathcal{H}_{3/5})\otimes(\bar{\mathcal{H}}_{1/10}\oplus\bar{\mathcal{H}}_{3/5})\ . \tag{3.3}$$

The partition function of the Neveu-Schwarz sector of SVIR_3 model can be expressed in terms of the characters ch^3_h of the superconformal algebra and $\chi^{(4)}_h$ of the Virasoro algebra,

$$\begin{aligned}\mathrm{Tr}_{\mathcal{H}_{\mathrm{SVIR}_3}}(q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24}) &= |\mathrm{ch}^3_0(q)|^2 + |\mathrm{ch}^3_{1/10}(q)|^2\\ &= |\chi^{(4)}_0(q)+\chi^{(4)}_{3/2}(q)|^2 + |\chi^{(4)}_{1/10}(q)+\chi^{(4)}_{3/5}(q)|^2\ .\end{aligned} \tag{3.4}$$

It is not fully modular invariant, but it is invariant under $\tau\to-1/\tau$ and $\tau\to\tau+2$.

The operators $(-1)^F$ and $(-1)^{\bar{F}}$ each have eigenvalue $+1$ on the state $|0\rangle$ and value -1 on the state $|1/10\rangle$. The field $\phi_{1/10}$ is still a bosonic field, though, as $(-1)^{F+\bar{F}}$ has eigenvalue $+1$.

The Ramond representations of the $N=1$ superconformal algebra do not correspond to local fields in the SVIR_3 theory - as fields, they can only arise as defect-creation fields (that is fields on which defect lines end) or fields at which two or more defects join.

3.2.1 Superconformal defects in SVIR_3

We will consider superconformal topological defects in SVIR_3, that is defects which preserve the $N=1$ algebra up to automorphism and so satisfy

$$L_m\,D = D\,L_m\ ,\ \ G_m\,D = \epsilon D\,G_m\ ,\ \ \bar{L}_m\,D = D\,\bar{L}_m\ ,\ \ \bar{G}_m\,D = \epsilon' D\,\bar{G}_m\ , \tag{3.5}$$

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where ϵ and ϵ' are ± 1. A defect satisfying (3.5) is determined up to two constants as

$$D_{\epsilon\epsilon'} = a(P_0 + \epsilon P_{3/2})(\bar{P}_0 + \epsilon' \bar{P}_{3/2}) + b(P_{1/10} + \epsilon P_{3/5})(\bar{P}_{1/10} + \epsilon' \bar{P}_{3/5}) , \tag{3.6}$$

where $P_i \bar{P}_j$ is the projector onto the Virasoro representations $\mathcal{H}_i \otimes \bar{\mathcal{H}}_j$.

We have found a complete set of eight elementary defects satisfying these conditions, namely

$$\{D_I, D_\varphi, \sqrt{2}(-1)^F D_I, \sqrt{2}(-1)^F D_\varphi, \sqrt{2}(-1)^{\bar{F}} D_I, \sqrt{2}(-1)^{\bar{F}} D_\varphi, (-1)^{F+\bar{F}} D_I, (-1)^{F+\bar{F}} D_\varphi\} , \tag{3.7}$$

where the two fundamental defects with $\epsilon = \epsilon' = 1$ are $D_I = \mathbf{1}$ (the identity operator on $\mathcal{H}_{\mathrm{SVIR}_3}$) and D_φ, given by

$$\begin{aligned} D_I &= (P_0 + P_{3/2})(\bar{P}_0 + \bar{P}_{3/2}) + (P_{1/10} + P_{3/5})(\bar{P}_{1/10} + \bar{P}_{3/5}) , \\ D_\varphi &= \frac{1+\sqrt{5}}{2}(P_0 + P_{3/2})(\bar{P}_0 + \bar{P}_{3/2}) + \frac{1-\sqrt{5}}{2}(P_{1/10} + P_{3/5})(\bar{P}_{1/10} + \bar{P}_{3/5}) . \end{aligned} \tag{3.8}$$

These defects have a product algebra with non-negative integer structure constants, the defect product algebra being determined by

$$D_\varphi \, D_\varphi \; = \; D_I + D_\varphi . \tag{3.9}$$

Further, the trace over the Hilbert space with any number of defects inserted can be expressed as an integer combination of characters of the superconformal algebra in the dual channel. All such traces are determined by the elementary traces

$$\mathrm{Tr}(q^H \bar{q}^{\bar{H}}) \; = \; |\mathrm{ch}^3_{1,1}(\tilde{q})|^2 + |\mathrm{ch}^3_{1,3}(\tilde{q})|^2 , \tag{3.10}$$

$$\mathrm{Tr}(\sqrt{2}(-1)^F q^H \bar{q}^{\bar{H}}) \; = \; 2\left[\mathrm{ch}^3_{1,2}(\tilde{q})\mathrm{ch}^3_{1,3}(\bar{\tilde{q}}) + \mathrm{ch}^3_{1,4}(\tilde{q})\mathrm{ch}^3_{1,1}(\bar{\tilde{q}})\right] , \tag{3.11}$$

$$\mathrm{Tr}((-1)^{F+\bar{F}} q^H \bar{q}^{\bar{H}}) \; = \; 2\left[|\mathrm{ch}^3_{1,2}(\tilde{q})|^2 + |\mathrm{ch}^3_{1,4}(\tilde{q})|^2\right] , \tag{3.12}$$

$$\mathrm{Tr}(D_\varphi(-1)^F q^H \bar{q}^{\bar{H}}) \; = \; |\mathrm{ch}^3_{1,3}(\tilde{q})|^2 + \mathrm{ch}^3_{1,3}(\tilde{q})\mathrm{ch}^3_{1,1}(\bar{\tilde{q}}) + \mathrm{ch}^3_{1,1}(\tilde{q})\mathrm{ch}^3_{1,3}(\bar{\tilde{q}}) , \tag{3.13}$$

$$\mathrm{Tr}(\sqrt{2}(-1)^F D_\varphi \, q^H \bar{q}^{\bar{H}}) \; = \; 2\left[\mathrm{ch}^3_{1,2}(\tilde{q})(\mathrm{ch}^3_{1,1}(\bar{\tilde{q}}) + \mathrm{ch}^3_{1,3}(\bar{\tilde{q}})) + \mathrm{ch}^3_{1,4}(\tilde{q})\mathrm{ch}^3_{1,3}(\bar{\tilde{q}})\right] , \tag{3.14}$$

$$\mathrm{Tr}((-1)^{F+\bar{F}} D_\varphi \, q^H \bar{q}^{\bar{H}}) \; = \; 2\left[|\mathrm{ch}^3_{1,2}(\tilde{q})|^2 + \mathrm{ch}^3_{1,2}(\tilde{q})\mathrm{ch}^3_{1,4}(\bar{\tilde{q}}) + \mathrm{ch}^3_{1,4}(\tilde{q})\mathrm{ch}^3_{1,2}(\bar{\tilde{q}})\right] , \tag{3.15}$$

where $H = L_0 - \frac{7}{240}$, $\bar{H} = \bar{L}_0 - \frac{7}{240}$.

3.2.2 Boundary states for SVIR_3

As with the free fermion, we shall consider boundary states with components in the Neveu-Schwarz sector only. Some of these will not be elementary with respect to the Virasoro algebra, with the introduction of the Ramond sector allowing this degeneracy to be lifted, but we do not need this sector to construct non-conformal defects in TCIM.

There are two gluing conditions for the superconformal algebra, $\pm$, which we take to be

$$(G_m + i\epsilon \bar{G}_{-m})|h, \epsilon\rangle\!\rangle = 0 . \tag{3.16}$$

Given that there are two Neveu-Schwarz representations, there are then four Ishibashi states,

$$\begin{aligned} |0,\pm\rangle &= |0\rangle \mp \frac{i}{2c/3} G_{-3/2}\bar{G}_{-3/2}\,|0\rangle + \frac{1}{c/2} L_{-2}\bar{L}_{-2}\,|0\rangle + \dots \\ |\frac{1}{10},\pm\rangle &= |\frac{1}{10}\rangle \mp \frac{i}{1/5} G_{-1/2}\bar{G}_{-1/2}\,|\frac{1}{10}\rangle + \frac{1}{1/5} L_{-1}\bar{L}_{-1}\,|\frac{1}{10}\rangle + \dots \end{aligned} \tag{3.17}$$

Their normalisation is given by

$$\langle\!\langle h,\pm| q^{\frac{1}{2}(L_0+\bar{L}_0-c/12)} |h',\pm\rangle\!\rangle = \delta_{h,h'}\,\mathrm{ch}^3_h(q)\;, \tag{3.18}$$

$$\langle\!\langle h,\pm|(-1)^F q^{\frac{1}{2}(L_0+\bar{L}_0-c/12)} |h',\pm\rangle\!\rangle = \delta_{h,h'}\,\widetilde{\mathrm{ch}}^3_h(q)\;. \tag{3.19}$$

In addition, the fermion parity operators act on the Ishibashi states as

$$(-1)^F|h,\pm\rangle\!\rangle = \varepsilon(h)|h,\mp\rangle\!\rangle\;,\quad (-1)^{\bar{F}}|h,\pm\rangle\!\rangle = \varepsilon(h)|h,\mp\rangle\!\rangle\;,\quad (-1)^{F+\bar{F}}|h,\pm\rangle\!\rangle = |h,\pm\rangle\!\rangle\;, \tag{3.20}$$

where $\varepsilon(h) = \pm 1$ is the fermion parity of the highest weight state $|h\rangle$. For SVIR$_3$ we take $\varepsilon(0) = 1$ and $\varepsilon(1/10) = -1$.

We find that there are four consistent fundamental boundary conditions, which we call $\|I_{NS}\rangle\!\rangle$, $\|\varphi_{NS}\rangle\!\rangle$, $\|I_R\rangle\!\rangle$, and $\|\varphi_R\rangle\!\rangle$. There are, again, two choices for the way to construct these boundary states, corresponding to the two gluing conditions. This means that one consistent choice is

$$\begin{aligned} \|I_{NS}\rangle\!\rangle &= \left(\frac{5-\sqrt{5}}{10}\right)^{\frac14} |0,+\rangle\!\rangle + \left(\frac{5+\sqrt{5}}{10}\right)^{\frac14} |\frac{1}{10},+\rangle\!\rangle\;, \\ \|\varphi_{NS}\rangle\!\rangle = D_\varphi\|I_{NS}\rangle\!\rangle &= \left(\frac{\sqrt{5}+2}{\sqrt{5}}\right)^{\frac14} |0,+\rangle\!\rangle - \left(\frac{\sqrt{5}-2}{\sqrt{5}}\right)^{\frac14} |\frac{1}{10},+\rangle\!\rangle\;, \\ \|I_R\rangle\!\rangle = \sqrt{2}(-1)^F\|I_{NS}\rangle\!\rangle &= \left(\frac{2(5-\sqrt{5})}{5}\right)^{\frac14} |0,-\rangle\!\rangle - \left(\frac{2(5+\sqrt{5})}{5}\right)^{\frac14} |\frac{1}{10},-\rangle\!\rangle\;, \\ \|\varphi_R\rangle\!\rangle = \sqrt{2}(-1)^F\|\varphi_{NS}\rangle\!\rangle &= \left(\frac{4(\sqrt{5}+2)}{\sqrt{5}}\right)^{\frac14} |0,-\rangle\!\rangle + \left(\frac{4(\sqrt{5}-2)}{\sqrt{5}}\right)^{\frac14} |\frac{1}{10},-\rangle\!\rangle\;. \end{aligned} \tag{3.21}$$

and the other is given by using the Ishibashi states of the opposite gluing condition.

These have overlaps/cylinder partition functions as follows:

	$\|I_{NS}\rangle\!\rangle$	$\|\varphi_{NS}\rangle\!\rangle$	$\|I_R\rangle\!\rangle$	$\|\varphi_R\rangle\!\rangle$
$\langle\!\langle I_{NS}\|$	$\mathrm{ch}^3_{1,1}(\tilde{q})$	$\mathrm{ch}^3_{1,3}(\tilde{q})$	$2\,\mathrm{ch}^3_{1,4}(\tilde{q})$	$2\,\mathrm{ch}^3_{1,2}(\tilde{q})$
$\langle\!\langle \varphi_{NS}\|$		$\mathrm{ch}^3_{1,1}(\tilde{q})+\mathrm{ch}^3_{1,3}(\tilde{q})$	$2\,\mathrm{ch}^3_{1,2}(\tilde{q})$	$2\,\mathrm{ch}^3_{1,2}(\tilde{q})+2\,\mathrm{ch}^3_{1,4}(\tilde{q})$
$\langle\!\langle I_R\|$			$2\,\mathrm{ch}^3_{1,1}(\tilde{q})$	$2\,\mathrm{ch}^3_{1,3}(\tilde{q})$
$\langle \varphi_R\|$				$2\,\mathrm{ch}^3_{1,1}(\tilde{q})+2\,\mathrm{ch}^3_{1,3}(\tilde{q})$

(3.22)

The boundary states $\|I_R\rangle\!\rangle$ and $\|\varphi_R\rangle\!\rangle$ are not fundamental as conformal boundary conditions, since the identity representation appears twice in their cylinder partition functions.

If we allow the introduction of the Ramond sector then they can each be written as a superposition of two boundary states, but the cylinder partition functions will not then be expressible as a sum of characters of the super Virasoro algebra.

3.3 Interface operators

The common sectors of $\mathcal{H}_{\rm TCIM}$ and $\mathcal{H}_{\rm SVIR_3}$ are, in terms of representations of the Virasoro algebra,

$$(\mathcal{H}_0 \otimes \bar{\mathcal{H}}_0) \;\oplus\; (\mathcal{H}_{3/2} \otimes \bar{\mathcal{H}}_{3/2}) \;\oplus\; (\mathcal{H}_{1/10} \otimes \bar{\mathcal{H}}_{1/10}) \;\oplus\; (\mathcal{H}_{3/5} \otimes \bar{\mathcal{H}}_{3/5})\,. \tag{3.23}$$

This means that a topological interface operator I satisfying (2.25) and acting from the space of TCIM to SVIR$_3$ is a constant on each of these four sectors and so is determined by four constants

$$I(a,b,c,d) = a\, P_0\bar{P}_0 + b\, P_{3/2}\bar{P}_{3/2} + c\, P_{1/10}\bar{P}_{1/10} + d\, P_{3/5}\bar{P}_{3/5}\;, \tag{3.24}$$

as well as a map identifying the Virasoro highest weights states of weights $(3/2,3/2)$ and $(3/5,3/5)$. We take this to be

$$\begin{aligned} |(1,3)\rangle_{\rm TCIM} &= i\,\xi_{1,3} G_{-3/2}\bar{G}_{-3/2}\,|0\rangle_{\rm SVIR_3}\;,\\ |(3,1)\rangle_{\rm TCIM} &= i\,\xi_{3,1} G_{-1/2}\bar{G}_{-1/2}\,|\frac{1}{10}\rangle_{\rm SVIR_3}\;, \end{aligned} \tag{3.25}$$

where $\xi_{1,3}$ and $\xi_{3,1}$ are signs.

Requiring integer coefficients in the expansions

$$I_\alpha\,\|B_A^{\rm SVIR_3}\rangle\!\rangle = \sum_B M^B_{\alpha A}\|B_B^{\rm TCIM}\rangle\!\rangle\;,\;\; I_\alpha^\dagger\,\|B_A^{\rm TCIM}\rangle\!\rangle = \sum_B \tilde{M}^B_{\alpha A}\|B_B^{\rm SVIR_3}\rangle\!\rangle\;, \tag{3.26}$$

allows us to solve for (a,b,c,d). In particular, from the definitions in (3.21), the expansions (3.17) and the identifications (3.25), we get

$$I(a,b,c,d)^\dagger\,\|B_0\rangle\!\rangle = m_1\|I_{NS}\rangle\!\rangle + m_2\|\varphi_{NS}\rangle\!\rangle + m_3\|I_R\rangle\!\rangle + m_4\|\varphi_R\rangle\!\rangle\;, \tag{3.27}$$

with

$$\begin{aligned} a &= \sqrt{2}m_1 + \frac{1+\sqrt{5}}{\sqrt{2}}m_2 + 2m_3 + (1+\sqrt{5})m_4\;,\\ -\xi_{1,3}\,b &= \sqrt{2}m_1 + \frac{1+\sqrt{5}}{\sqrt{2}}m_2 - 2m_3 - (1+\sqrt{5})m_4\;,\\ c &= \sqrt{2}m_1 + \frac{1-\sqrt{5}}{\sqrt{2}}m_2 - 2m_3 - (1-\sqrt{5})m_4\;,\\ -\xi_{3,1}d &= \sqrt{2}m_1 + \frac{1-\sqrt{5}}{\sqrt{2}}m_2 + 2m_3 + (1-\sqrt{5})m_4\;. \end{aligned} \tag{3.28}$$

We will choose $\xi_{1,3} = \xi_{3,1} = -1$. This means that any interface can be expressed as a combination

$$I(a,b,c,d) = m_1\, I + m_2\, I_2 + m_3\, I_3 + m_4\, I_4\;, \tag{3.29}$$

where the interfaces are given by

$$\begin{aligned} I &= I(\sqrt{2},\sqrt{2},\sqrt{2},\sqrt{2}) \\ I_2 &= I\Big(\frac{1+\sqrt{5}}{\sqrt{2}},\frac{1+\sqrt{5}}{\sqrt{2}},\frac{1-\sqrt{5}}{\sqrt{2}},\frac{1-\sqrt{5}}{\sqrt{2}}\Big) \\ I_3 &= I(2,-2,-2,2) \\ I_4 &= I(1{+}\sqrt{5},-1{-}\sqrt{5},-1{+}\sqrt{5},1{-}\sqrt{5}) \end{aligned} \tag{3.30}$$

These act on the boundary states as follows

$$\begin{aligned} I\,\|I_{NS}\rangle\!\rangle &= \|B_{(1,1)}\rangle\!\rangle + \|B_{(3,1)}\rangle\!\rangle\;, & I^\dagger\,\|B_{(1,1)}\rangle\!\rangle &= I^\dagger\,\|B_{(3,1)}\rangle\!\rangle = \|I_{NS}\rangle\!\rangle\;, \\ I\,\|\varphi_{NS}\rangle\!\rangle &= \|B_{(1,2)}\rangle\!\rangle + \|B_{(1,3)}\rangle\!\rangle\;, & I^\dagger\,\|B_{(1,2)}\rangle\!\rangle &= I^\dagger\,\|B_{(1,3)}\rangle\!\rangle = \|\varphi_{NS}\rangle\!\rangle\;, \\ I\,\|I_R\rangle\!\rangle &= 2\,\|B_{(2,1)}\rangle\!\rangle\;, & I^\dagger\,\|B_{(2,1)}\rangle\!\rangle &= \|I_R\rangle\!\rangle\;, \\ I\,\|\varphi_R\rangle\!\rangle &= 2\,\|B_{(2,2)}\rangle\!\rangle\;, & I^\dagger\,\|B_{(2,2)}\rangle\!\rangle &= \|\varphi_R\rangle\!\rangle\;, \end{aligned} \tag{3.31}$$

and satisfy the relations

$$I_2 = D_{1/10}\,I = I\,D_\varphi\;,\quad I_3 = D_{7/16}\,I = I\cdot\sqrt{2}(-1)^F\;,\quad I_4 = D_{1/10}\,I = I\cdot\sqrt{2}(-1)^F D_\varphi\;. \tag{3.32}$$

Requiring $I\,I^\dagger$ be expressible as a sum of topological defects in TCIM and $I^\dagger\,I$ be expressible as a sum of topological defects in SVIR_3 provides a strong constraint, just as it did in the Ising/FF case. Taking a,b,c,d to be real, we have, for example,

$$\begin{aligned} I(a,b,c,d)\,I(a,b,c,d)^\dagger \;=\;& \frac{(5-\sqrt{5})(a^2+b^2)+(5+\sqrt{5})(c^2+d^2)}{40}(D_0+D_{3/2}) \\ &+\frac{a^2+b^2-c^2-d^2}{4\sqrt{5}}(D_{1/10}+D_{3/5}) \\ &+\frac{(5-\sqrt{5})(a^2-b^2)-(5+\sqrt{5})(c^2-d^2)}{20\sqrt{2}}D_{7/16} \\ &+\frac{a^2-b^2+c^2-d^2}{2\sqrt{10}}D_{3/80}\;. \end{aligned} \tag{3.33}$$

We find the interfaces given by (3.29) indeed give integer coefficients, and the fundamental interface I given by (3.30) satisfies

$$I\,I^\dagger = D_0 + D_{3/2}\;,\quad I^\dagger\,I = D_I + (-1)^{F+\bar{F}}D_I\;, \tag{3.34}$$

$$D_{3/2}\,I = I\cdot(-1)^{F+\bar{F}} = I\;. \tag{3.35}$$

To summarise, we have found a set of four elementary boundary conditions for SVIR_3, a set of eight elementary topological superconformal defects in SVIR_3 and a set of four fundamental interfaces between TCIM and SVIR_3. All we need to do now is to find superconformal defects in SVIR_3 and use the interface operators to generate defects in TCIM.

4 The doubled model, $\mathrm{SVIR}_3{}^{\otimes 2}$

The basic idea behind finding superconformal non-topological defects in SVIR_3 is that every superconformal defect in SVIR_3 is equivalent to a superconformal boundary condition on $\mathrm{SVIR}_3{}^{\otimes 2}$. Since the central charge of $\mathrm{SVIR}_3{}^{\otimes 2}$ is $7/5$, which is less than $3/2$, there are only a finite set of fundamental superconformal boundary conditions, and that it is quite possible that not all of these are topological.

We first have to identify the local fields in $\mathrm{SVIR}_3{}^{\otimes 2}$. Since we are only considering Neveu-Schwarz sectors, the partition function of $\mathrm{SVIR}_3{}^{\otimes 2}$ is exactly the square of the partition function of SVIR_3. This has been identified in [1] as the Neveu-Schwarz sector of the D_6–E_6 modular invariant of the $N = 1$ super Virasoro algebra, and so we can consider $\mathrm{SVIR}_3{}^{\otimes 2}$ as a model with sVir_{10} symmetry. There are numerous character identities relating characters of the superconformal algebras sVir_3 at $c = 7/10$ and sVir_{10} at $c = 7/5$; these allow one to write the partition function in terms of characters of sVir_{10} as

$$\begin{aligned} Z_{\mathrm{SVIR}_3{}^{\otimes 2}} &= (Z_{\mathrm{SVIR}_3})^2 = \left(|\mathrm{ch}^3_{1,1}|^2 + |\mathrm{ch}^3_{1,3}|^2\right)^2 \\ &= |\mathrm{ch}^{10}_{1,1} + \mathrm{ch}^{10}_{1,5} + \mathrm{ch}^{10}_{1,7} + \mathrm{ch}^{10}_{1,11}|^2 + |\mathrm{ch}^{10}_{3,1} + \mathrm{ch}^{10}_{3,5} + \mathrm{ch}^{10}_{3,7} + \mathrm{ch}^{10}_{3,11}|^2 \\ &\quad + 2|\mathrm{ch}^{10}_{(5,1)} + \mathrm{ch}^{10}_{(5,5)}|^2 \; . \end{aligned} \tag{4.1}$$

We will be constructing boundary states which respect the sVir_{10} symmetry, and the building blocks are Ishibashi states for the sVir_{10} algebra. There are two Ishibashi states (one for each gluing condition) for each diagonal term (those with $h = \bar{h}$) in the partition function (4.1). Hence there are 24 Ishibashi states in total.

The diagonal terms in the partition function (4.1) can be identified by Kac labels (r, s) with r and s taking values in the exponents of D_6 and a subset of the exponents of E_6 respectively. These exponents are $\{1, 3, 5, 5', 7, 9\}$ and $\{1, 4, 5, 7, 8, 11\}$ (the label 5 appears twice, as do fields with labels $(5, s)$ in the Neveu-Schwarz sector (4.1)). This over-counts the set of fields by a factor of two, as it does not take into account the symmetry of the Kac labels $(r, s) \simeq (10-r, 12-s)$. Furthermore, it also includes the Ramond sector (those with $s = 4$ or $s = 8$). the result is that we can label the Neveu-Schwarz diagonal fields in (4.1) by (r, s) with $r \in \{1, 3, 5, 5', 7, 9\}$ and $s \in \{1, 5, 7, 11\}$ modulo $(r, s) \simeq (10 - r, 12 - s)$.

The full chiral algebra of $\mathrm{SVIR}_3{}^{\otimes 2}$ is, of course, larger than sVir_{10} and so can be described as an extension of the sVir_{10} by super-primary fields, i.e. as a super W-algebra. From the character decomposition of the full chiral algebra $\mathrm{sVir}_3 \otimes \mathrm{sVir}_3$,

$$(\mathrm{ch}^3_{1,1})^2 = \mathrm{ch}^{10}_{1,1} + \mathrm{ch}^{10}_{1,5} + \mathrm{ch}^{10}_{1,7} + \mathrm{ch}^{10}_{1,11} \; , \tag{4.2}$$

we see that the chiral algebra contains three super-primary fields $\mathcal{W}^{(3/2)}$, $\mathcal{W}^{(7/2)}$, and $\mathcal{W}^{(10)}$ of weights $h^{(10)}_{1,5} = 3/2$, $h^{(10)}_{1,7} = 7/2$, and $h^{(10)}_{1,11} = 10$ respectively. Expressions for these fields are given in appendix A. Extending sVir_{10} by the field of weight $3/2$ just recovers the full $\mathrm{sVir}_3 \times \mathrm{sVir}_3$ algebra. Extending sVir_{10} by the field of weight $7/2$ gives an algebra SW(7/2) which has been considered before in [21–23]. Extending sVir_{10} by the field of weight 10 gives a new algebra SW(10). We will return to these algebras when we consider the boundary states in section 5.

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(r,s)	$(1,1)$	$(3,5)$	$(5,5)$	$(5',5)$
$\rho(\lvert r,s\rangle)$	$\lvert 0\rangle\langle 0\rvert$	$\lvert\frac{1}{10}\rangle\langle\frac{1}{10}\rvert$	$\lvert 0\rangle\langle\frac{1}{10}\rvert$	$\lvert\frac{1}{10}\rangle\langle 0\rvert$
$\rho'(\lvert r,s\rangle)$	$\lvert 0\rangle\langle 0\rvert$	$-\lvert\frac{1}{10}\rangle\langle\frac{1}{10}\rvert$	$-\lvert 0\rangle\langle\frac{1}{10}\rvert$	$\lvert\frac{1}{10}\rangle\langle 0\rvert$

Table 3. Images of the highest weight states.

Finally, we note here that it is also possible to express a single copy of the SVIR_3 partition function in terms of characters of the $c = 7/5$ algebra:

$$\begin{aligned} Z_{\mathrm{SVIR}_3} &= (\mathrm{ch}^3_{1,1})^2 + (\mathrm{ch}^3_{1,3})^2 \\ &= \mathrm{ch}^{10}_{1,1} + \mathrm{ch}^{10}_{1,5} + \mathrm{ch}^{10}_{1,7} + \mathrm{ch}^{10}_{1,11} + \mathrm{ch}^{10}_{3,1} + \mathrm{ch}^{10}_{3,5} + \mathrm{ch}^{10}_{3,7} + \mathrm{ch}^{10}_{3,11} \ , \end{aligned} \tag{4.3}$$

where we note again that q is real. The reason is that one can embed the $c = 7/5$ superconformal algebra into the two (holomorphic and anti-holomorphic) copies of the $c = 7/10$ algebra in SVIR_3.

4.1 Relating boundary conditions on $\mathrm{SVIR}_3^{\otimes 2}$ to defects in SVIR_3

Central to the idea of the folding procedure is that a boundary condition in $\mathrm{SVIR}_3^{\otimes 2}$ is equivalent to a defect in SVIR_3. The folding condition is simple if the boundary is along the real axis with the folded model living entirely in the upper half plane: for each field $\phi(z,\bar z)$ in SVIR_3, there are two copies $\phi^{(a)}$ in the folded theory, with the $a = 1$ copy being the original field, $\phi^{(1)}(z,\bar z) = \phi(z,\bar z)$ and the $a = 2$ copy being the folded field, $\phi^{(2)}(z,\bar z) = \phi(\bar z, z)$. When we consider boundary states and defect operators, it is more usual for the boundary/defect to be on the unit circle, and we can define a map ρ from states in $\mathrm{SVIR}_3^{\otimes 2}$ to defects in SVIR_3 as follows.

If the boundary lies on the unit circle with $\mathrm{SVIR}_3^{\otimes 2}$ defined on the exterior of the unit circle, copy $a = 1$ corresponds to the superconformal algebra on the outside of the unit circle and copy $a = 2$, the inside. If $\rho(|B\rangle\rangle) = \hat D$, then we can define

$$\begin{aligned} \rho(G^1_m \| B\rangle\rangle) &= G_m \hat D \ , \\ \rho(\bar G^1_m \| B\rangle\rangle) &= \bar G_m \hat D \ , \\ \rho(G^2_m \| B\rangle\rangle) &= -i(-1)^{F+\bar F} \hat D (-1)^{F+\bar F} \bar G_{-m} \ , \\ \rho(\bar G^2_m \| B\rangle\rangle) &= i(-1)^{F+\bar F} \hat D (-1)^{F+\bar F} G_{-m} \ . \end{aligned} \tag{4.4}$$

The signs in (4.4) are determined by finding a family of Möbius maps which interpolate the identity map preserving the real axis and a map which sends the real axis to the unit circle, as explained in appendix D.

To complete the definition, we need the image of the highest weight states in $\mathrm{SVIR}_3^{\otimes 2}$ which are tensor products, $|h_1\rangle \otimes |h_2\rangle$. We can take the simplest choice, i.e. $\rho(|h_1\rangle \otimes |h_2\rangle) = |h_1\rangle\langle h_2|$, but as we will see later, it will be helpful to define in addition the map $\rho'(|h_1\rangle \otimes |h_2\rangle) = |h_1\rangle\langle h_2|(-1)^F$. These maps are summarised in table 3.

We can now identify the 24 known conformal defects in SVIR_3 (8 topological and 16 factorised) with boundary conditions on $\mathrm{SVIR}_3^{\otimes 2}$, and in particular the gluing conditions satisfied by the boundary conditions.

For a topological defect, we have two free signs given in equation (3.5). These imply that the corresponding boundary state satisfies

$$\left.\begin{aligned} G_m D &= \eta D G_m \\ \bar{G}_m D &= \eta' D \bar{G}_m \end{aligned}\right\} \Rightarrow \begin{cases} (G^1_m + i\eta \bar{G}^2_{-m})\|B\rangle\rangle = 0 \\ (G^2_m + i\eta' \bar{G}^1_{-m})\|B\rangle\rangle = 0 \end{cases} \tag{4.5}$$

For a factorising defect, we have again two free signs coming from the two gluing conditions of the two boundary states, which imply for the corresponding defect

$$\left.\begin{aligned} (G_m + i\eta\, \bar{G}_{-m})\|A,\eta\rangle\rangle\langle\langle B,\eta'\| = 0 \\ \|A,\eta\rangle\rangle\langle\langle B,\eta'\|(G_m - i\eta'\, \bar{G}_{-m}) = 0 \end{aligned}\right\} \Rightarrow \begin{cases} (G^1_m + i\eta \bar{G}^1_{-m})\|B\rangle\rangle = 0 \\ (G^2_m - i\eta' \bar{G}^2_{-m})\|B\rangle\rangle = 0 \end{cases} \tag{4.6}$$

It is clear from equations (4.5) and (4.6) that we will not be able to find a way to express all the gluing conditions that arise as gluing conditions on a single set of combinations $G^1_m \pm G^2_m$ and $\bar{G}^1_m \pm \bar{G}^2_m$, and so, in the next section, we consider exactly how we can organise the boundary states of $\mathrm{SVIR}_3{}^{\otimes 2}$ corresponding to the known defects into boundary states of the algebra sVir_{10}.

4.2 Relating the boundary conditions on $\mathrm{SVIR}_3{}^{\otimes 2}$ to boundary conditions for sVir_{10}

To view $\mathrm{SVIR}_3{}^{\otimes 2}$ as a model of the $c = 7/5$ superconformal algebra sVir_{10} we need to define an embedding of sVir_{10} into the two copies of the $c = 7/10$ algebra sVir_3 in SVIR_3. An embedding of sVir_{10} into $\mathrm{sVir}_3 \times \mathrm{sVir}_3$ is defined by four signs $\{\alpha, \beta, \gamma, \delta\}$:

$$\iota_{\alpha\beta\gamma\delta}(G_m) = \alpha G^1_m + \beta G^2_m\ , \quad \iota_{\alpha\beta\gamma\delta}(\bar{G}_m) = \gamma \bar{G}^1_m + \delta \bar{G}^2_m\ . \tag{4.7}$$

We will denote the combined map $(\rho \circ \iota_{\alpha\beta\gamma\delta})$ by $\rho_{\alpha\beta\gamma\delta}$.

We also need to define a map from the Ishibashi states with respect to sVir_{10} to the states in $\mathrm{SVIR}_3{}^{\otimes 2}$. The Ishibashi states are determined by a highest weight states and a gluing condition, and the highest weight condition depends on the choice of embedding, so we need to take some care over this.

For the moment we restrict attention to the gluing condition. Suppose we have an Ishibashi state $|h,\epsilon\rangle\rangle$ of sVir_{10} satisfying

$$(G_m + i\epsilon \bar{G}_{-m})|h,\epsilon\rangle\rangle = 0\ . \tag{4.8}$$

With the embedding $\iota_{\alpha\beta\gamma\delta}$, this state satisfies

$$(\alpha G^1_m + \beta G^2_m + i\epsilon\gamma \bar{G}^1_{-m} + i\epsilon\delta \bar{G}^2_{-m})|h,\epsilon\rangle\rangle = 0\ . \tag{4.9}$$

Hence a purely transmitting defect with gluing conditions $\{\eta, \eta'\}$ corresponds to $\eta = \alpha\delta\epsilon$ and $\eta' = \beta\gamma\epsilon$, that is $\alpha\beta\gamma\delta = \eta\eta'$ and a pure reflecting defects with gluing conditions $\{\eta, \eta'\}$ corresponds to $\eta = \alpha\gamma\epsilon$ and $\eta' = -\beta\delta\epsilon$, that is $\alpha\beta\gamma\delta = -\eta\eta'$. Hence we find that there are two equivalence classes of embeddings, given by $\alpha\beta\gamma\delta = \pm 1$. An embedding with $\alpha\beta\gamma\delta = 1$ will correspond to transmitting defects with $\eta\eta' = 1$ and reflecting defects with

$\eta\eta' = -1$, whereas an embedding with $\alpha\beta\gamma\delta = -1$ will correspond to transmitting defects with $\eta\eta' = -1$ and reflecting defects with $\eta\eta' = 1$.

The result is that we can expect two sets of boundary states with respect to two different choices of embeddings sVir_{10}, with one set giving half the defects of SVIR_3 and the other set giving the other half. The precise expressions for these boundary states in terms of Ishibashi states will of course depend on the definitions of the highest weight states. There are eight highest weight states with respect to sVir_{10} whose definitions depend on the embedding, namely those with Kac labels $(1,5)$, $(1,7)$, $(1,11)$, $(3,1)$, $(3,7)$, $(3,11)$, $(5,1)$, and $(5',1)$. For example, the state $|1,5\rangle$ of weight $(3/2,3/2)$ is given by

$$\iota_{\alpha\beta\gamma\delta}(|1,5\rangle) = \frac{i\eta_{1,5}}{4c/3}(\alpha G^1_{-3/2} - \beta G^2_{-3/2})(\gamma\bar{G}^1_{-3/2} - \delta\bar{G}^2_{-3/2})|0\rangle \ , \tag{4.10}$$

where $c = 7/10$ and $\eta_{1,5}$ is a free sign, and the state $|3,1\rangle$ of weight $(7/10, 7/10)$ is given by

$$\iota_{\alpha\beta\gamma\delta}(|3,1\rangle) = \frac{i\eta_{3,1}}{2h}(\alpha G^1_{-1/2} - \beta G^2_{-1/2})(\gamma\bar{G}^1_{-1/2} - \delta\bar{G}^2_{-1/2})\,|3,5\rangle \ , \tag{4.11}$$

where $h = 1/10$ and $\eta_{3,1}$ is a free sign. We have a free sign $\eta_{r,s}$ for each of these eight such states. Expanded expressions for these states and others can be found in appendix E.

Using these facts, it is possible to construct the boundary states corresponding to all the known topological and factorising defects in SVIR_3. We find that they can all be written in terms of the states $\|(a,b)_{NS}\rangle\rangle$ and $\|(a,b)_{\widetilde{NS}}\rangle\rangle$ defined in [1] in (at least) two ways. We illustrate this in the next two sections with the case of the identity defect in SVIR_3, and the factorising defect $\|I_{NS}\rangle\rangle\langle\langle I_{NS}\|$, before summarising the results in section 4.4.

4.2.1 The identity defect in SVIR_3

The identity defect in SVIR_3 takes the very simple form of a sum over an orthonormal basis of $\mathcal{H}_{\mathrm{SVIR}_3}$:

$$\mathbf{1} = \sum_\psi |\psi\rangle\langle\psi| \ . \tag{4.12}$$

Expanding this out,

$$\begin{aligned}\mathbf{1} \ =& \ |0\rangle\langle 0| + \frac{1}{2c/3}\left(G_{-3/2}|0\rangle\langle 0|G_{3/2} + \bar{G}_{-3/2}|0\rangle\langle 0|\bar{G}_{3/2}\right) + \dots \\ &+ |\tfrac{1}{10}\rangle\langle\tfrac{1}{10}| + \frac{1}{1/5}\left(G_{-1/2}|\tfrac{1}{10}\rangle\langle\tfrac{1}{10}|G_{1/2} + \bar{G}_{-1/2}|\tfrac{1}{10}\rangle\langle\tfrac{1}{10}|\bar{G}_{1/2}\right) + \dots\end{aligned} \tag{4.13}$$

This must arise from a combination of the Ishibashi states $|(1,1),\epsilon\rangle\rangle$, $|(1,5),\epsilon\rangle\rangle$, $|(1,7),\epsilon\rangle\rangle$, $|(1,11),\epsilon\rangle\rangle$, $|(3,1),\epsilon\rangle\rangle$, $|(3,5),\epsilon\rangle\rangle$, $|(3,7),\epsilon\rangle\rangle$, and $|(3,11)\epsilon\rangle\rangle$. Since the Identity defect satisfies (3.5) with $\eta = \eta' = 1$, the gluing condition ϵ and embedding $\iota_{\alpha\beta\gamma\delta}$ satisfy $\epsilon = \alpha\delta = \beta\gamma$ and $\alpha\beta\gamma\delta = 1$.

The simplest choice is $\alpha = \beta = \gamma = \delta = \epsilon = 1$. This still leaves the signs $\eta_{r,s}$ free. Given the freedom to choose these signs, the boundary state $\|D_I\rangle\rangle$ can be expressed as sum over all the Ishibashi states in the NS sector with the $+$ gluing condition:

$$\begin{aligned}\mathbf{1} &= \rho_{++++}(\|D_I\rangle\rangle) \ , \\ \|D_I\rangle\rangle &= |(1,1),+\rangle\rangle + |(1,5),+\rangle\rangle + |(1,7),+\rangle\rangle + |(1,11),+\rangle\rangle \\ &\quad + |(3,1),+\rangle\rangle + |(3,5),+\rangle\rangle + |(3,7),+\rangle\rangle + |(3,11),+\rangle\rangle \ ,\end{aligned} \tag{4.14}$$

where ρ acts as in table 3. Looking at the explicit expressions in appendix E, this fixes in particular $\eta_{1,5} = 1$, $\eta_{3,1} = 1$ and $\eta_{3,7} = -1$.

We could, by choosing the signs of η in a different fashion, instead have the equally symmetric expression

$$\begin{aligned} \mathbf{1} &= \rho'_{++++}(\|D_I\rangle\rangle) \;, \\ \|D_I\rangle\rangle &= |(1,1),+\rangle\rangle - |(1,5),+\rangle\rangle + |(1,7),+\rangle\rangle - |(1,11),+\rangle\rangle \\ &\quad + |(3,1),+\rangle\rangle - |(3,5),+\rangle\rangle + |(3,7),+\rangle\rangle - |(3,11),+\rangle\rangle \;, \end{aligned} \tag{4.15}$$

where ρ' acts as in table 3. Looking again at the explicit expressions in appendix E, this implies the opposite choices,

$$\eta_{1,5} = -1 \;, \quad \eta_{3,1} = -1 \;, \quad \eta_{3,7} = 1 \;. \tag{4.16}$$

Note that in equations (4.14) and (4.15) we have suppressed all the information regarding the choices of signs for the descendent states $|1,5\rangle$ etc, and have only kept the information regarding the choice of the signs for the maps of the highest weight states from table 3.

4.2.2 The factorising defect $\|I_{NS}\rangle\rangle\langle\langle I_{NS}\|$ in SVIR$_3$

We will take the boundary state $\|I_{NS}\rangle\rangle$ to be given in terms of the Ishibashi states in SVIR$_3$ as

$$\|I_{NS}\rangle\rangle = \left(\frac{5-\sqrt{5}}{10}\right)^{\frac{1}{4}} |0,+\rangle\rangle + \left(\frac{5+\sqrt{5}}{10}\right)^{\frac{1}{4}} |\tfrac{1}{10},+\rangle\rangle \;, \tag{4.17}$$

which is one of the two possibilities shown in (3.21). This means the factorising defect $\|I_{NS}\rangle\rangle\langle\langle I_{NS}\|$ is given as

$$\begin{aligned} \|I_{NS}\rangle\rangle\langle\langle I_{NS}\| &= \left(\frac{5-\sqrt{5}}{10}\right)^{\frac{1}{2}} |0,+\rangle\rangle\langle\langle 0,+| + \left(\frac{1}{5}\right)^{\frac{1}{4}} |0,+\rangle\rangle\langle\langle \tfrac{1}{10},+| \\ &\quad + \left(\frac{1}{5}\right)^{\frac{1}{4}} |\tfrac{1}{10},+\rangle\rangle\langle\langle 0,+| \left(\frac{5+\sqrt{5}}{10}\right)^{\frac{1}{2}} |\tfrac{1}{10},+\rangle\rangle\langle\langle \tfrac{1}{10},+| \;. \end{aligned} \tag{4.18}$$

Since this factorising defect satisfies (4.6) with $\eta = \eta' = 1$, the gluing condition ϵ and embedding $\iota_{\alpha\beta\gamma\delta}$ satisfy $\epsilon = \alpha\gamma = -\beta\delta$ and $\alpha\beta\gamma\delta = -1$.

The simplest choice, which we take from now on, is $\alpha = \beta = \gamma = \eta = 1$, $\delta = -1$. Given the freedom to choose the signs $\eta_{r,s}$, we can express this factorising defect in many ways, but the one that we will use later is this:

$$\begin{aligned} \|I_{NS}\rangle\rangle\langle\langle I_{NS}\| &= \rho_{+++-}(\|I_{NS}I_{NS}\rangle\rangle) \;, \\ \|I_{NS}I_{NS}\rangle\rangle &= \left(\frac{5-\sqrt{5}}{10}\right)^{\frac{1}{2}} \big(|(1,1),+\rangle\rangle + |(1,5),+\rangle\rangle + |(1,7),+\rangle\rangle + |(1,11),+\rangle\rangle\big) \\ &\quad + \left(\frac{1}{5}\right)^{\frac{1}{4}} \big(|(5,1),+\rangle\rangle + |(5,5),+\rangle\rangle + |(5',1),+\rangle\rangle + |(5',5),+\rangle\rangle\big) \\ &\quad + \left(\frac{5+\sqrt{5}}{10}\right)^{\frac{1}{2}} \big(|(3,1),+\rangle\rangle + |(3,5),+\rangle\rangle + |(3,7),+\rangle\rangle + |(3,11),+\rangle\rangle\big) \;, \end{aligned} \tag{4.19}$$

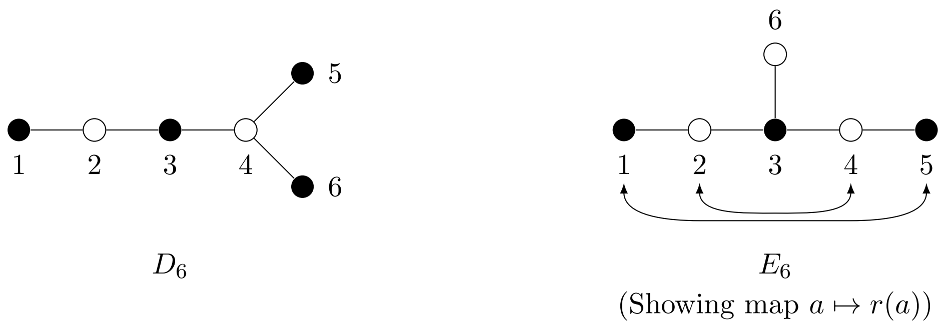

Figure 4. The Dynkin diagrams of D_6 and E_6 showing the bi-colouration and the map r.

where ρ acts as in table 3 and the signs $\eta_{1,5}$,$\eta_{3,1}$ and $\eta_{3,7}$ are again as in equation (4.16). We cannot say whether the remaining signs are also fixed as for the identity defect as we have not calculated them. Note that in equation (4.19) we have again suppressed all the information regarding the choices of signs for the descendent states $|1,5\rangle$ etc, and have only kept the information regarding the choice of the signs for the maps of the highest weight states from table 3.

4.3 Boundary conditions in $\mathrm{SVIR}_3{}^{\otimes 2}$

The boundary states are linear combinations of the Ishibashi states corresponding to diagonal ($h = \bar{h}$) Neveu-Schwarz fields. As noted above, there are 24 of these, $|(r,s),\pm\rangle\rangle$, labelled by (r,s) odd exponents of D_6 and E_6 modulo the Kac symmetry, and a gluing condition.

A set of boundary states for the GSO projection of $\mathrm{SVIR}_3{}^{\otimes 2}$ was proposed in [1] although there are some difficulties with these, as explained in 7. The states in [1] have both Neveu-Schwarz and Ramond contributions; here we only need components in the Neveu-Schwarz sector.

The boundary conditions themselves are labelled by pairs of nodes on the Dynkin diagrams of D_6 and E_6, together with a choice of gluing condition. However, this again over-counts the number of Neveu-Schwarz boundary states in two ways. Firstly, the nodes on the E_6 diagram that are related by the diagram symmetry, $r : 1 \mapsto 5$ and $r : 2 \mapsto 4$ lead to the same Neveu-Schwarz contribution. (We can think of this as replacing the E_6 Dynkin diagram by the F_4 diagram, with the nodes related by the Z_2 symmetry corresponding to the short simple roots of F_4). This gives 24 pairs of nodes. Secondly, we can bi-colour the Dynkin diagrams and split these 24 pairs into those with nodes of the same colour and those with nodes of opposite colour, giving two sets of 12 pairs of nodes, combined with the gluing condition.

With the bi-colouration as in figure 4, and making a choice for the representatives of the nodes related by the Z_2 symmetry of the E_6 diagram $a \mapsto r(a)$, we take the nodes with the same colouration to be

$$\mathcal{N}_e = \{(1,1),(3,1),(5,1),(6,1),(2,2),(4,2),(1,3),(3,3),(5,3),(6,3),(2,6),(4,6)\}\ , \quad (4.20)$$

and the nodes with opposite colouration to be

$$\mathcal{N}_o = \{(2,1),(4,1),(1,2),(3,2),(5,2),(6,2)(2,3),(4,3),(1,6),(3,6),(5,6),(6,6)\}\ . \quad (4.21)$$

The key ingredient in the boundary states proposed in [1] are matrices $\Psi^{(a,b)}_{(r,s)}$,

$$\Psi^{(a,b)}_{(r,s)} = \frac{\psi^r_a(D_6)\,\psi^s_b(E_6)}{\sqrt{S^{(8)}_{1r}S^{(10)}_{1s}}} \ , \tag{4.22}$$

formed from eigenvectors $\psi^r_a(G)$ of the adjacency matrices of the Dynkin diagram of G and modular S-matrices $S^{(k)}_{rs}$ for affine $su(2)$ characters at level k. We give the vectors $\psi^r_a(G)$ and a table of numerical values of $\Psi^{(a,b)}_{(r,s)}$ in appendix F for convenience. These matrices have the property that under the Kac-symmetry

$$\Psi^{(a,b)}_{(r,s)} = \begin{cases} \Psi^{(a,b)}_{(10-r,12-s)} & (a,b)\in\mathcal{N}_e\ ,\ \text{same colouration,} \\ -\Psi^{(a,b)}_{(10-r,12-s)} & (a,b)\in\mathcal{N}_o\ ,\ \text{opposite colouration.} \end{cases} \tag{4.23}$$

Following [1], we can define boundary states $\|(a,b)_{NS}\rangle\rangle$ using these matrices, but we take a slightly different choice to [1],

$$\|(a,b)_{NS}\rangle\rangle = \sum_{\substack{r\in\{1,3,5,5',7,9\} \\ s\in\{1,7\}}} \Psi^{(a,b)}_{(r,s)}\,|(r,s),+\rangle\rangle \ , \tag{4.24}$$

where in [1] the sum over s is $s\in\{1,5\}$. The sums are over exactly the same representations, but the choice of different representatives results in expressions which differ by a sign for $s=7$ when the nodes are of opposite colour (eg the labels $(1,7)$ and $(9,5)$ denote the same representation, but $\Psi^{(1,2)}_{(1,7)} = -\Psi^{(1,2)}_{(9,5})$.

Our choice of representatives was motivated by the fact the E_6 $\hat{su}(2)_{10}$ WZW model has an extended symmetry algebra consisting of the representations $(1)\oplus(7)$ and so our choice seems natural when considering fusion of the sVir_{10} model. We think it results in more natural expression for the final boundary states.

One consequence is that (unlike the situation in [1]) our choice of representatives results in sets of states which only differ by factors of $\sqrt{2}$,

$$\|(a,6)_{NS}\rangle\rangle = \sqrt{2}\,\|(a,1)_{NS}\rangle\rangle\ ,\ \ \|(a,3)_{NS}\rangle\rangle = \sqrt{2}\,\|(a,2)_{NS}\rangle\rangle\ . \tag{4.25}$$

These two different ways of expressing the same boundary state may seem redundant, but it helps a great deal when it comes to providing consistent descriptions of all the possible boundary states for $\text{SVIR}_3{}^{\otimes 2}$.

We also define states $\|(a,b)_{\widetilde{NS}}\rangle\rangle$ in a slightly different way to [1], as

$$\|(a,b)_{\widetilde{NS}}\rangle\rangle = (-1)^F\|(a,b)_{NS}\rangle\rangle\ . \tag{4.26}$$

These differ from the states defined in [1] by an extra sign for each of the Ishibashi states which corresponds to a fermionic highest weight state, that is for the states $|r,s\rangle$ with $(r,s)\in\{(1,5),(1,7),(3,1),(3,11),(5,5),(5',5)\}$ which again simplifies the identification of the known boundary states.

$\alpha\beta\gamma\delta = -1,\ \alpha{=}\beta{=}\gamma{=}1,\ \delta{=}{-1},\ \text{map} = \rho_{+++-}$	
Defect	Boundary states
$\|I_{NS}\rangle\!\rangle\langle\!\langle I_{NS}\|$	$\sqrt{2}\,\|(1,1)_{NS}\rangle\!\rangle\,,\quad \|(1,6)_{NS}\rangle\!\rangle$
$\|I_{NS}\rangle\!\rangle\langle\!\langle \varphi_{NS}\|$	$\sqrt{2}\,\|(5,1)_{NS}\rangle\!\rangle\,,\quad \|(6,6)_{NS}\rangle\!\rangle$
$\|\varphi_{NS}\rangle\!\rangle\langle\!\langle I_{NS}\|$	$\sqrt{2}\,\|(6,1)_{NS}\rangle\!\rangle\,,\quad \|(6,6)_{NS}\rangle\!\rangle$
$\|\varphi_{NS}\rangle\!\rangle\langle\!\langle \varphi_{NS}\|$	$\sqrt{2}\,\|(3,1)_{NS}\rangle\!\rangle\,,\quad \|(3,6)_{NS}\rangle\!\rangle$
$\sqrt{2}(-1)^F$	$\sqrt{2}\,\|(2,6)_{NS}\rangle\!\rangle\,, 2\,\|(2,1)_{NS}\rangle\!\rangle$
$\sqrt{2}(-1)^{\bar F}$	$\sqrt{2}\,\|(2,6)_{\widetilde{NS}}\rangle\!\rangle\,, 2\,\|(2,1)_{\widetilde{NS}}\rangle\!\rangle$
$\sqrt{2}(-1)^F D_\varphi$	$\sqrt{2}\,\|(4,6)_{NS}\rangle\!\rangle\,, 2\,\|(4,1)_{NS}\rangle\!\rangle$
$\sqrt{2}(-1)^{\bar F} D_\varphi$	$\sqrt{2}\,\|(4,6)_{\widetilde{NS}}\rangle\!\rangle\,, 2\,\|(4,1)_{\widetilde{NS}}\rangle\!\rangle$
$(-1)^F\|I_{NS}\rangle\!\rangle\langle\!\langle I_{NS}\|(-1)^F$	$\sqrt{2}\,\|(1,1)_{\widetilde{NS}}\rangle\!\rangle\,,\quad \|(1,6)_{\widetilde{NS}}\rangle\!\rangle$
$(-1)^F\|I_{NS}\rangle\!\rangle\langle\!\langle \varphi_{NS}\|(-1)^F$	$\sqrt{2}\,\|(5,1)_{\widetilde{NS}}\rangle\!\rangle\,,\quad \|(6,6)_{\widetilde{NS}}\rangle\!\rangle$
$(-1)^F\|\varphi_{NS}\rangle\!\rangle\langle\!\langle I_{NS}\|(-1)^F$	$\sqrt{2}\,\|(6,1)_{\widetilde{NS}}\rangle\!\rangle\,,\quad \|(6,6)_{\widetilde{NS}}\rangle\!\rangle$
$(-1)^F\|\varphi_{NS}\rangle\!\rangle\langle\!\langle \varphi_{NS}\|(-1)^F$	$\sqrt{2}\,\|(3,1)_{\widetilde{NS}}\rangle\!\rangle\,,\quad \|(3,6)_{\widetilde{NS}}\rangle\!\rangle$

$\alpha\beta\gamma\delta = 1,\ \alpha{=}\beta{=}\gamma{=}\delta{=}1,\ \text{map} = \rho'_{++++}$	
Defect	Boundary states
$\|I_{NS}\rangle\!\rangle\langle\!\langle I_R\|$	$\sqrt{2}\,\|(1,6)_{NS}\rangle\!\rangle\,,\ 2\,\|(1,1)_{NS}\rangle\!\rangle$
$\|I_{NS}\rangle\!\rangle\langle\!\langle \varphi_R\|$	$\sqrt{2}\,\|(5,6)_{NS}\rangle\!\rangle\,,\ 2\,\|(6,1)_{NS}\rangle\!\rangle$
$\|\varphi_{NS}\rangle\!\rangle\langle\!\langle I_R\|$	$\sqrt{2}\,\|(6,6)_{NS}\rangle\!\rangle\,,\ 2\,\|(6,1)_{NS}\rangle\!\rangle$
$\|\varphi_{NS}\rangle\!\rangle\langle\!\langle \varphi_R\|$	$\sqrt{2}\,\|(3,6)_{NS}\rangle\!\rangle\,,\ 2\,\|(3,1)_{NS}\rangle\!\rangle$
$\|I_R\rangle\!\rangle\langle\!\langle I_{NS}\|$	$\sqrt{2}\,\|(1,6)_{\widetilde{NS}}\rangle\!\rangle\,,\ 2\,\|(1,1)_{\widetilde{NS}}\rangle\!\rangle$
$\|I_R\rangle\!\rangle\langle\!\langle \varphi_{NS}\|$	$\sqrt{2}\,\|(5,6)_{\widetilde{NS}}\rangle\!\rangle\,,\ 2\,\|(6,1)_{\widetilde{NS}}\rangle\!\rangle$
$\|\varphi_R\rangle\!\rangle\langle\!\langle I_{NS}\|$	$\sqrt{2}\,\|(6,6)_{\widetilde{NS}}\rangle\!\rangle\,,\ 2\,\|(6,1)_{\widetilde{NS}}\rangle\!\rangle$
$\|\varphi_R\rangle\!\rangle\langle\!\langle \varphi_{NS}\|$	$\sqrt{2}\,\|(3,6)_{\widetilde{NS}}\rangle\!\rangle\,,\ 2\,\|(3,1)_{\widetilde{NS}}\rangle\!\rangle$
$\mathbf{1}$	$\sqrt{2}\,\|(2,1)_{NS}\rangle\!\rangle\,,\quad \|(2,6)_{NS}\rangle\!\rangle$
$(-1)^{F+\bar F}$	$\sqrt{2}\,\|(2,1)_{\widetilde{NS}}\rangle\!\rangle\,,\quad \|(2,6)_{\widetilde{NS}}\rangle\!\rangle$
D_φ	$\sqrt{2}\,\|(4,1)_{NS}\rangle\!\rangle\,,\quad \|(4,6)_{NS}\rangle\!\rangle$
$(-1)^{F+\bar F}D_\varphi$	$\sqrt{2}\,\|(4,1)_{\widetilde{NS}}\rangle\!\rangle\,,\quad \|(4,6)_{\widetilde{NS}}\rangle\!\rangle$

Table 4. Identifications of the boundary states corresponding to the known defects.

4.4 Identifying known defects

With the definitions (4.25) and (4.26) we can identify all the known defects in SVIR$_3$. These split into two sets, those with $\alpha\beta\gamma\delta = 1$ for which we need the highest-weight state map ρ', supplemented by suitable choices of signs for the descendent states, and those with $\alpha\beta\gamma\delta = -1$ for which we need the map ρ, shown in table 4.

Note that these two sets are not defined at the same time as they use different embeddings; we cannot describe the defects $\mathbf{1}$ and $\|I_{NS}\rangle\!\rangle\langle\!\langle I_{NS}\|$ as supersymmetric boundary conditions for SVIR$_3^{\otimes 2}$ at the same time.

As an example, we show here the overlap of the boundary states $\sqrt{2}\,\|(2,1)_{NS}\rangle\!\rangle = \|(2,6)_{NS}\rangle\!\rangle$ (representing the identity defect) with $\sqrt{2}\,\|(2,1)_{\widetilde{NS}}\rangle\!\rangle = \|(2,6)_{\widetilde{NS}}\rangle\!\rangle$ (representing the defect $(-1)^{F+\bar F}$), $2\,\|(1,1)_{NS}\rangle\!\rangle = \sqrt{2}\,\|(1,6)_{NS}\rangle\!\rangle$ (representing $\|I_{NS}\rangle\!\rangle\langle\!\langle I_R\|$), and $2\,\|(1,1)_{\widetilde{NS}}\rangle\!\rangle = \sqrt{2}\,\|(1,6)_{\widetilde{NS}}\rangle\!\rangle$ (representing $\|I_R\rangle\!\rangle\langle\!\langle I_{NS}\|$), all of which have $\alpha\beta\gamma\delta = 1$. We have exactly the expected results:

$$\begin{aligned}
\langle\!\langle (2,6)_{NS}\|q^H\|(2,6)_{NS}\rangle\!\rangle &= \big(\mathrm{ch}^{10}_{1,1} + \mathrm{ch}^{10}_{1,5} + \mathrm{ch}^{10}_{1,7} + \mathrm{ch}^{10}_{1,11} \\
&\qquad + \mathrm{ch}^{10}_{3,1} + \mathrm{ch}^{10}_{3,5} + \mathrm{ch}^{10}_{3,7} + \mathrm{ch}^{10}_{3,11}\big)(\tilde q) \\
&= \big(\mathrm{ch}^{3}_{1,1}(\tilde q)\big)^2 + \big(\mathrm{ch}^{3}_{1,3}(\tilde q)\big)^2\,, \qquad (4.27)\\
\langle\!\langle (2,6)_{NS}\|q^H\|(2,6)_{\widetilde{NS}}\rangle\!\rangle &= 2\big(\mathrm{ch}^{10}_{1,4} + \mathrm{ch}^{10}_{1,8} + \mathrm{ch}^{10}_{3,4} + \mathrm{ch}^{10}_{3,8}\big)(\tilde q) \\
&= 2\big(\mathrm{ch}^{3}_{1,2}(\tilde q)\big)^2 + 2\big(\mathrm{ch}^{3}_{1,4}(\tilde q)\big)^2\,, \qquad (4.28)\\
2\,\langle\!\langle (2,6)_{NS}\|q^H\|(1,1)_{NS}\rangle\!\rangle &= 2\big(\mathrm{ch}^{10}_{2,4} + \mathrm{ch}^{10}_{2,8}\big)(\tilde q) \\
&= 2\,\mathrm{ch}^{3}_{1,4}(\sqrt{\tilde q})\,, \qquad (4.29)\\
2\,\langle\!\langle (2,6)_{NS}\|q^H\|(1,1)_{\widetilde{NS}}\rangle\!\rangle &= 2\big(\mathrm{ch}^{10}_{2,1} + \mathrm{ch}^{10}_{2,5} + \mathrm{ch}^{10}_{2,7} + \mathrm{ch}^{10}_{2,11}\big)(\tilde q) \\
&= 2\,\mathrm{ch}^{3}_{1,4}(\sqrt{\tilde q})\,, \qquad (4.30)
\end{aligned}$$

where $H = (L_0 + \bar{L}_0 - 7/60)$, $q = \exp(-4\pi L)$ and $\tilde{q} = \exp(-\pi/L)$. Note that the overlaps of $\|(2,6)_{NS}\rangle\!\rangle$ with $2\,\|(1,1)_{NS}\rangle\!\rangle$ and $2\,\|(1,1)_{\widetilde{NS}}\rangle\!\rangle$ are the same, $2\,\mathrm{ch}^3_{1,4}(\sqrt{\tilde{q}})$, thanks to two different identities relating the characters of sVir_{10} and sVir_3. This is a function of $\sqrt{\tilde{q}}$ since geometrically it corresponds to a strip of width $2L$, as shown in figure 5.

However, if we consider defects with different values of $\alpha\beta\gamma\delta$ we do not get sensible results. The overlap of the boundary state in $\mathrm{SVIR}_3{}^{\otimes 2}$ corresponding to the identity defect with the boundary state corresponding to the factorising defect $\|I_{NS}\rangle\!\rangle\langle\!\langle I_{NS}\|$ will give the partition function on the strip of width $2L$ and boundary conditions I_{NS} on both sides, that is

$$\mathrm{Tr}_{\mathrm{SVIR}_3}\left(q^H\, D_I\, \|I_{NS}\rangle\!\rangle\langle\!\langle I_{NS}\|\right) = \mathrm{ch}^3_{1,1}(\sqrt{\tilde{q}})\ . \tag{4.31}$$

But $\mathrm{ch}^3_{1,1}(\sqrt{q}) = q^{-7/480}(1 + q^{3/4} + q + q^{5/4} + q^{3/2} + \ldots)$ cannot be expressed as a sum of characters of the $c = 7/5$ algebra, and so it is not possible for the two defects D_I and $\|I_{NS}\rangle\!\rangle\langle\!\langle I_{NS}\|$ to be represented as boundary states for sVir_{10} at the same time. If we look at table 4, we see that D_I corresponds to $\|(2,6)_{NS}\rangle\!\rangle$ defined with embedding ι_{++++} but $\|I_{NS}\rangle\!\rangle\langle\!\langle I_{NS}\|$ corresponds to $\|(1,6)_{NS}\rangle\!\rangle$ with embedding ι_{+++-}, and so their overlap being calculated as

$$\langle\!\langle (2,6)_{NS}\|q^H\|(1,6)_{NS}\rangle\!\rangle = \sqrt{2}\,\mathrm{ch}^3_{1,4}(\sqrt{\tilde{q}}) \tag{4.32}$$

has nothing to do with the required quantity.

4.5 Identifying new defects

Now that we have identified all the known defects, we can see that they all correspond to the nodes 1 and 6 on the E_6 diagram. If we instead use the nodes 2 and 3 on the E_6 diagram, we find new defects which are neither topological nor factorising.

The transmission coefficient was defined in [24] in such a way that a topological defect has $T = 1$ and a factorising boundary condition has $T = 0$, and thus it lets us quickly identify these amongst the boundary states. We can calculate the T (transmission) coefficient and the defect entropy for these new defects using the expressions in appendix E. If the defect obtained from a boundary state $\|\Psi\rangle\!\rangle$ of the following form,

$$\|\Psi\rangle\!\rangle = A|(1,1)\epsilon\rangle\!\rangle + B|(1,5)\epsilon\rangle\!\rangle + \ldots\ , \tag{4.33}$$

and the sign is $\eta_{1,5} = -1$, then the transmission coefficient is

$$T = \frac{1}{2}\Big[1 - \epsilon B/A\Big]\ , \tag{4.34}$$

whether the map is ρ_{+++-} or ρ'_{++++}.

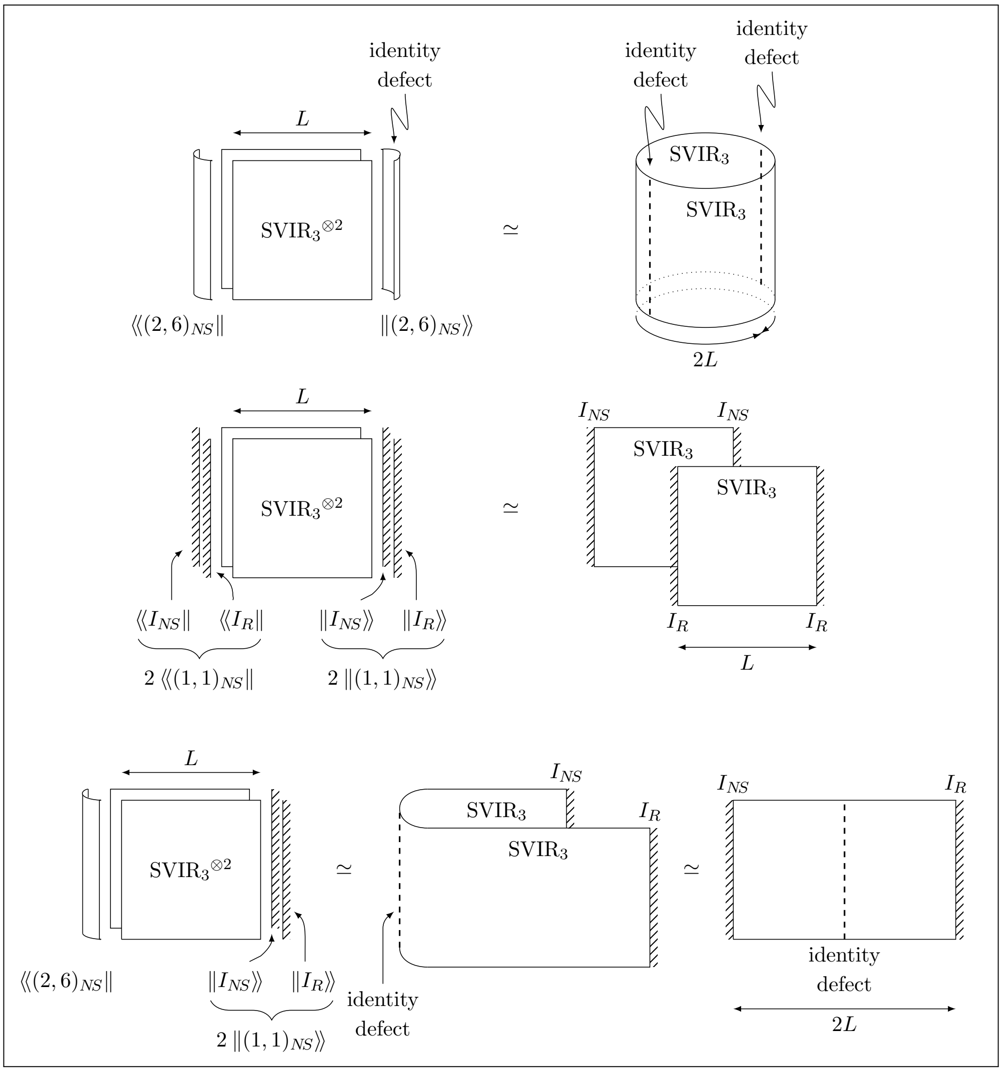

Figure 5. Different boundary conditions on $\mathrm{SVIR_3}^{\otimes 2}$ result in different geometrical set-ups for $\mathrm{SVIR_3}$.

The g values are independent of the embedding and choice of signs η given by

$$g\big(\|(a,1)_{NS/\widetilde{NS}}\rangle\!\rangle\big) = \begin{cases} \dfrac{S^{(8)}_{a1}}{\sqrt{S^{(8)}_{1,1}}} = \sqrt{1+\frac{1}{\sqrt{5}}}\,\sin\left(\frac{a\pi}{10}\right) & \text{for } a=1,2,3,4 \\ \dfrac{S^{(8)}_{5,1}}{2\sqrt{S^{(8)}_{1,1}}} = \frac{1}{2}\sqrt{1+\frac{1}{\sqrt{5}}} & \text{for } a=5,6 \end{cases} \tag{4.35}$$

$$g\big(\|(a,2)_{NS/\widetilde{NS}}\rangle\!\rangle\big) = \sqrt{2+\sqrt{3}}\quad g\big(\|(a,1)_{NS/\widetilde{NS}}\rangle\!\rangle\big)\,, \tag{4.36}$$

$$g\big(\|(a,3)_{NS/\widetilde{NS}}\rangle\!\rangle\big) = (1+\sqrt{3})\quad g\big(\|(a,1)_{NS/\widetilde{NS}}\rangle\!\rangle\big)\,, \tag{4.37}$$

$$g\big(\|(a,6)_{NS/\widetilde{NS}}\rangle\!\rangle\big) = \sqrt{2}\quad g\big(\|(a,1)_{NS/\widetilde{NS}}\rangle\!\rangle\big)\,, \tag{4.38}$$

where the last two relations follow from (4.25).

T	g		boundary states defined with ι_{+++-} and map ρ
1	$\sqrt{2}$	$1.414\ldots$	$2\,\|(2,1)_{NS}\rangle\rangle,\ 2\,\|(2,1)_{\widetilde{NS}}\rangle\rangle$
	$\frac{1+\sqrt{5}}{\sqrt{2}}$	$2.288\ldots$	$2\,\|(4,1)_{NS}\rangle\rangle,\ 2\,\|(4,1)_{\widetilde{NS}}\rangle\rangle$
0	$\left(\frac{5-\sqrt{5}}{10}\right)^{1/2}$	$0.5257\ldots$	$\|(1,6)_{NS}\rangle\rangle,\ \|(1,6)_{\widetilde{NS}}\rangle\rangle$
	$\left(\frac{5+\sqrt{5}}{10}\right)^{1/2}$	$0.8506\ldots$	$\|(5,6)_{NS}\rangle\rangle,\ \|(6,6)_{NS}\rangle\rangle,\ \|(5,6)_{\widetilde{NS}}\rangle\rangle,\ \|(6,6)_{\widetilde{NS}}\rangle\rangle$
	$\left(\frac{5+2\sqrt{5}}{5}\right)^{1/2}$	$1.3763\ldots$	$\|(3,6)_{NS}\rangle\rangle,\ \|(3,6)_{\widetilde{NS}}\rangle\rangle$
$\frac{\sqrt{3}-1}{2}$	$\sqrt{2+\sqrt{3}}$	$1.9318\ldots$	$\|(2,3)_{NS}\rangle\rangle,\ \|(2,3)_{\widetilde{NS}}\rangle\rangle$
		$3.1258\ldots$	$\|(4,3)_{NS}\rangle\rangle,\ \|(4,3)_{\widetilde{NS}}\rangle\rangle$
$\frac{3-\sqrt{3}}{2}$		$1.4363\ldots$	$2\,\|(1,2)_{NS}\rangle\rangle,\ 2\,\|(1,2)_{\widetilde{NS}}\rangle\rangle$
		$3.7603\ldots$	$2\,\|(3,2)_{NS}\rangle\rangle,\ 2\,\|(3,2)_{\widetilde{NS}}\rangle\rangle$
		$2.3240\ldots$	$2\,\|(5,2)_{NS}\rangle\rangle,\ 2\,\|(6,2)_{NS}\rangle\rangle,\ 2\,\|(5,2)_{\widetilde{NS}}\rangle\rangle,\ 2\,\|(6,2)_{\widetilde{NS}}\rangle\rangle$
T	g		boundary states defined with ι_{++++} and map ρ'
1	1	1	$\|(2,6)_{NS}\rangle\rangle,\ \|(2,6)_{\widetilde{NS}}\rangle\rangle$
	$\frac{1+\sqrt{5}}{2}$	$1.618\ldots$	$\|(4,6)_{NS}\rangle\rangle,\ \|(4,6)_{\widetilde{NS}}\rangle\rangle$
0	$\left(\frac{5-\sqrt{5}}{5}\right)^{1/2}$	$0.7434\ldots$	$2\,\|(1,1)_{NS}\rangle\rangle,\ 2\,\|(1,1)_{\widetilde{NS}}\rangle\rangle$
	$\left(\frac{5+\sqrt{5}}{5}\right)^{1/2}$	$1.2030\ldots$	$2\,\|(5,1)_{NS}\rangle\rangle,\ 2\,\|(6,1)_{NS}\rangle\rangle,\ 2\,\|(5,1)_{\widetilde{NS}}\rangle\rangle,\ 2\,\|(6,1)_{\widetilde{NS}}\rangle\rangle$
	$\left(\frac{10-2\sqrt{5}}{5}\right)^{1/2}$	$1.9465\ldots$	$2\,\|(3,1)_{NS}\rangle\rangle,\ 2\,\|(3,1)_{\widetilde{NS}}\rangle\rangle$
$\frac{\sqrt{3}-1}{2}$	$1+\sqrt{3}$	$2.732\ldots$	$2\,\|(2,2)_{NS}\rangle\rangle,\ 2\,\|(2,2)_{\widetilde{NS}}\rangle\rangle$
		$4.4205\ldots$	$2\,\|(4,2)_{NS}\rangle\rangle,\ 2\,\|(4,2)_{\widetilde{NS}}\rangle\rangle$
$\frac{3-\sqrt{3}}{2}$		$1.0156\ldots$	$\|(1,3)_{NS}\rangle\rangle,\ \|(1,3)_{\widetilde{NS}}\rangle\rangle$
		$2.6589\ldots$	$\|(3,3)_{NS}\rangle\rangle,\ \|(3,3)_{\widetilde{NS}}\rangle\rangle$
		$1.6433\ldots$	$\|(5,3)_{NS}\rangle\rangle,\ \|(6,3)_{\widetilde{NS}}\rangle\rangle$

Table 5. T and g values for the $\mathrm{SVIR}_3^{\otimes 2}$ boundary states.

Note that $\|\Psi\rangle\rangle$ and $(-1)^F\|\Psi\rangle\rangle$ have the same value of g and T. We find that the boundary states $\|(a,b)_{NS/\widetilde{NS}}\rangle\rangle$ only take four different values for T as in table 5, but a large range of g values. We also list the g values for the known topological and factorising defects in SVIR_3 in the same table.

If the g value of a boundary state cannot be expressed as a sum of the g values of known topological and factorising defects, then this boundary state must correspond to a "new" defect.

Again, these defects fall into two sets - those defined from the boundary state using embedding ι_{++++} and map ρ', and those defined with embedding ι_{+++-} and map ρ. With each set, the boundary states satisfy Cardy's condition, that is, the overlaps of any two boundary states corresponding to the same embedding ρ, or ρ', are non-negative integer combinations of characters of sVir_{10}. The overlaps of states corresponding to different maps do not satisfy Cardy's condition.

Further, the overlaps involving the known (topological and factorising) defects can be expressed in terms of the characters of sVir_3, but those involving the new defects can not.

As an example, we consider the overlaps of the boundary states $\sqrt{2}\,\|(2,1)_{NS}\rangle\!\rangle = \|(2,6)_{NS}\rangle\!\rangle$ (representing the identity defect) with $\sqrt{2}\,\|(1,2)_{NS}\rangle\!\rangle = \|(1,3)_{NS}\rangle\!\rangle$ and $\sqrt{2}\,\|(1,1)_{NS}\rangle\!\rangle = \|(1,6)_{NS}\rangle\!\rangle$ (representing $\|I_{NS}\rangle\!\rangle\langle\!\langle I_{NS}\|$) with $\sqrt{2}\,\|(2,2)_{NS}\rangle\!\rangle = \|(2,3)_{NS}\rangle\!\rangle$.

We have

$$\begin{aligned}
\langle\!\langle (2,6)_{NS}\|q^H\|(2,6)_{NS}\rangle\!\rangle &= \big(\mathrm{ch}^{10}_{1,1} + \mathrm{ch}^{10}_{1,5} + \mathrm{ch}^{10}_{1,7} + \mathrm{ch}^{10}_{1,11} \\
&\qquad + \mathrm{ch}^{10}_{3,1} + \mathrm{ch}^{10}_{3,5} + \mathrm{ch}^{10}_{3,7} + \mathrm{ch}^{10}_{3,11}\big)(\tilde q)\ , \\
&= \big(\mathrm{ch}^3_{1,1}(\tilde q)\big)^2 + \big(\mathrm{ch}^3_{1,3}(\tilde q)\big)^2 && (4.39)\\
\langle\!\langle (2,6)_{NS}\|q^H\|(1,3)_{NS}\rangle\!\rangle &= \big(\mathrm{ch}^{10}_{2,2} + \mathrm{ch}^{10}_{2,4} + 2\,\mathrm{ch}^{10}_{2,6} + \mathrm{ch}^{10}_{2,8} + \mathrm{ch}^{10}_{2,10}\big)(\tilde q)\ , && (4.40)\\
\langle\!\langle (1,6)_{NS}\|q^H\|(1,6)_{NS}\rangle\!\rangle &= \big(\mathrm{ch}^{10}_{1,1} + \mathrm{ch}^{10}_{1,5} + \mathrm{ch}^{10}_{1,7} + \mathrm{ch}^{10}_{1,11}\big)(\tilde q) \\
&= \big(\mathrm{ch}^3_{1,1}(\tilde q)\big)^2\ , && (4.41)\\
\langle\!\langle (1,6)_{NS}\|q^H\|(2,3)_{NS}\rangle\!\rangle &= \big(\mathrm{ch}^{10}_{2,2} + \mathrm{ch}^{10}_{2,4} + 2\,\mathrm{ch}^{10}_{2,6} + \mathrm{ch}^{10}_{2,8} + \mathrm{ch}^{10}_{2,10}\big)(\tilde q)\ , && (4.42)
\end{aligned}$$

where $H = (L_0 + \bar L_0 - 7/60)$, $q = \exp(-4\pi L)$ and $\tilde q = \exp(-\pi/L)$.

Since $h^{(10)}_{2,2} = \frac{1}{80} \neq h^{(3)}_{r,s} + h^{(3)}_{r',s'}$ for any (r,s), (r',s') in sVir_3, the overlap $\langle\!\langle (2,6)_{NS}\|q^H\|(1,3)_{NS}\rangle\!\rangle$ cannot be expressed as a sum of products of characters $\mathrm{ch}^3_{r,s}(\tilde q)\,\mathrm{ch}^3_{r',s'}(\tilde q)$. In addition, since $h^{(10)}_{2,2} - \frac{7}{120} = -\frac{11}{240} \neq \frac{1}{2}(h^{(3)}_{r,s} - \frac{7}{240})$ for any (r,s) in sVir_3, it cannot be expressed as a sum of characters $\mathrm{ch}^3_{r,s}(\sqrt{\tilde q})$.

Note that $\langle\!\langle (2,6)_{NS}\|q^H\|(1,3)_{NS}\rangle\!\rangle = \langle\!\langle (1,6)_{NS}\|q^H\|(2,3)_{NS}\rangle\!\rangle$, which suggest that these overlaps are related by the insertion of a topological defect in the doubled model labelled by the Dynkin nodes $(2,1)$.

Just for reference, we give the overlaps of the new boundary states with themselves to show that they satisfy Cardy's condition, but also cannot be expressed in terms of characters of sVir_3:

$$\begin{aligned}
\langle\!\langle (1,3)_{NS}\|q^H\|(1,3)_{NS}\rangle\!\rangle &= \big(\mathrm{ch}^{10}_{1,1} + 2\mathrm{ch}^{10}_{1,3} + 3\mathrm{ch}^{10}_{1,5} + 3\mathrm{ch}^{10}_{1,7} + 2\mathrm{ch}^{10}_{1,9} + \mathrm{ch}^{10}_{1,11}\big)(\tilde q)\ , \\
\langle\!\langle (2,3)_{NS}\|q^H\|(2,3)_{NS}\rangle\!\rangle &= \big(\mathrm{ch}^{10}_{1,1} + 2\mathrm{ch}^{10}_{1,3} + 3\mathrm{ch}^{10}_{1,5} + 3\mathrm{ch}^{10}_{1,7} + 2\mathrm{ch}^{10}_{1,9} + \mathrm{ch}^{10}_{1,11} \\
&\quad + \big(\mathrm{ch}^{10}_{3,1} + 2\mathrm{ch}^{10}_{3,3} + 3\mathrm{ch}^{10}_{3,5} + 3\mathrm{ch}^{10}_{3,7} + 2\mathrm{ch}^{10}_{3,9} + \mathrm{ch}^{10}_{3,11}\big)(\tilde q)\ ,
\end{aligned} \tag{4.43}$$

4.5.1 New factorising defects in SVIR_3

While the boundary state $\|(1,6)_{\widetilde{NS}}\rangle\!\rangle$ can be identified as the defect $(-1)^F\|I_{NS}\rangle\!\rangle\langle\!\langle I_{NS}\|(-1)^F$, this is not actually the product of two boundary states in SVIR_3. The state $(-1)^F\|I_{NS}\rangle\!\rangle$ does not satisfy Cardy's constraint - for example, its overlap with $\|I_{NS}\rangle\!\rangle$ is not an integer combination of characters in the crossed channel:

$$\langle\!\langle I_{NS}\|q^H(-1)^F\|I_{NS}\rangle\!\rangle = \sqrt{2}\,\mathrm{ch}^3_{1,4}(\tilde q)\ . \tag{4.44}$$

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The defect $(-1)^F \|I_{NS}\rangle\!\rangle\langle\!\langle I_{NS}\|(-1)^F$ does however satisfy the constraint - for example

$$\langle\!\langle(1,6)_{NS}\|q^H\|(1,6)_{\widetilde{NS}}\rangle\!\rangle = \langle\!\langle I_{NS}\|q^H(-1)^F\|I_{NS}\rangle\!\rangle\langle\!\langle I_{NS}\|(-1)^F q^H\|I_{NS}\rangle\!\rangle = 2\,\mathrm{ch}^3_{1,4}(\tilde{q})^2\ . \tag{4.45}$$

Conversely, the factorising defect $\|I_R\rangle\!\rangle\langle\!\langle I_R\|$ does not arise in the tables 4, The resolution seems to be that these factorising defects are not fundamental and instead we have

$$\begin{aligned} \|(1,6)_{\widetilde{NS}}\rangle\!\rangle &\simeq (-1)^F\|I_{NS}\rangle\!\rangle\langle\!\langle I_{NS}\|(-1)^F\ , \\ \|I_R\rangle\!\rangle\langle\!\langle I_R\| &= 2(-1)^F\|I_{NS}\rangle\!\rangle\langle\!\langle I_{NS}\|(-1)^F \simeq 2\|(1,6)_{\widetilde{NS}}\rangle\!\rangle\ . \end{aligned} \tag{4.46}$$

This illustrates the possibility that each known factorising and topological defect in SVIR$_3$ gives rise to a superconformal boundary state in SVIR$_3{}^{\otimes 2}$, but the converse need not to be true.

5 Boundary states in SVIR$_3{}^{\otimes 2}$ and extended algebras

The boundary states we have discussed can be understood from the point of view of extended superconformal algebras. There are two relevant algebras, $SW(3/2)$ and $SW(10)$ which is a subalgebra of $SW(3/2)$.

5.1 The algebra $SW(3/2)$

The first case to consider is boundary states which preserve the whole algebra $SW(3/2)$. Since this is the same as sVir$_3 \otimes$ sVir$_3$, we expect to recover the known topological and factorised defects. This algebra contains not only the superconformal generator, $G(z)$, but a fermionic primary field of weight 3/2, $\mathcal{W}^{(3/2)}(z)$ and its superpartner of weight 2, $\mathcal{T}(z)$. Details are given in appendix A.

As with the field $G(z)$, we have a choice for the gluing conditions of the field $\mathcal{W}^{(3/2)}$, so that we can define gluing conditions (ϵ,ϵ') where the Ishibashi state $|(h,\tilde{h})\epsilon,\epsilon'\rangle\!\rangle$ satisfies

$$(G_m + i\epsilon\bar{G}_{-m})|(h,\tilde{h})\epsilon,\epsilon'\rangle\!\rangle = 0\ ,\ \ (\mathcal{W}^{(3/2)}_m + i\epsilon'\bar{\mathcal{W}}^{(3/2)}_{-m})|(h,\tilde{h})\epsilon,\epsilon'\rangle\!\rangle = 0\ . \tag{5.1}$$

The SVIR$_3{}^{\otimes 2}$ model can be thought of as a model of $SW(3/2)$ in two ways. With the embedding $\alpha\beta\gamma\delta = 1$, the partition function is diagonal in the four characters of this algebra,

$$Z = |\chi_1|^2 + |\chi_3|^2 + |\chi_5|^2 + |\chi_{5'}|^2\ . \tag{5.2}$$

With the embedding $\alpha\beta\gamma\delta = -1$, the partition function is not diagonal, but is instead

$$Z = |\chi_1|^2 + |\chi_3|^2 + \chi_5\bar{\chi}_{5'} + \chi_{5'}\bar{\chi}_5\ . \tag{5.3}$$

Either way, we would expect to have four Ishibashi states for each set of gluing conditions, and hence four boundary states per gluing condition, but this is not quite the case. The $SW(3/2)$ algebra relations include

$$\{G_m, \mathcal{W}^{(3/2)}_n\} = 2\mathcal{T}_{m+n} \tag{5.4}$$

and so an Ishibashi state $|(h,\tilde{h})\epsilon,\epsilon'\rangle\rangle$ satisfies

$$(\mathcal{T}_0 - \epsilon\epsilon'\bar{\mathcal{T}}_0)|(h,\tilde{h})\epsilon,\epsilon'\rangle\rangle = 0\ . \tag{5.5}$$

This means that for either embedding, there are only two choices of gluing conditions for the representations with $h = \frac{1}{10}$, so that rather than having 16 Ishibashi states, in fact we only have 12 different Ishibashi states.

This means we will have 12 independent combinations of these Ishibashi states into boundary states, which is exactly what we find. There are 12 mutually consistent boundary states which preserve this algebra with $\alpha\beta\gamma\delta = 1$ and 12 (different) consistent states with $\alpha\beta\gamma\delta = -1$ which correspond to the known topological and factorising boundary defects of SVIR_3.

Since we can choose $\mathcal{W}^{(3/2)}$ and $\bar{\mathcal{W}}^{(3/2)}$ so that

$$|1,5\rangle = i\eta_{1,5}\mathcal{W}^{(3/2)}_{-3/2}\,\bar{\mathcal{W}}^{(3/2)}_{-3/2}\,|0\rangle\ \ , \tag{5.6}$$

then we have

$$\mathcal{W}^{(3/2)}_{3/2}\,|1,5\rangle = i\eta_{1,5}\bar{\mathcal{W}}^{(3/2)}_{-3/2}\,|0\rangle\ \ , \tag{5.7}$$

and so the coefficients of $|1,1\rangle\rangle$ and $|1,5\rangle\rangle$ are related by this gluing condition. As we can read off table 8, $\Psi^{(a,b)}_{(1,5)} = \Psi^{(a,b)}_{(1,1)}$ for (a,b) equal to $(2,6)$ and $(4,6)$, and $\Psi^{(a,b)}_{(1,5)} = -\Psi^{(a,b)}_{(1,1)}$ for (a,b) equal to $(1,1)$, $(3,1)$, $(5,1)$ and $(6,1)$.

5.2 The algebra $SW(10)$

When we turn to the algebra $SW(10)$ which has a superprimary field $\mathcal{W}$ of weight 10. Since this algebra is invariant under $\mathcal{W}^{(10)} \to -\mathcal{W}^{(10)}$, we can again choose the gluing condition

$$\mathcal{W}^{(10)}_m|h\rangle\rangle = \pm\bar{\mathcal{W}}^{(10)}_{-m}|h\rangle\rangle\ . \tag{5.8}$$

Since we can choose $\mathcal{W}^{(10)}$ and $\bar{\mathcal{W}}^{(10)}$ so that

$$|(1,11)\rangle = \eta_{1,11}\mathcal{W}^{(10)}_{-10}\bar{\mathcal{W}}^{(10)}_{-10}\,|0\rangle\ \ , \tag{5.9}$$

then we have

$$\mathcal{W}^{(10)}_{10}\,|(1,11)\rangle = \eta_{1,11}\bar{\mathcal{W}}^{(10)}_{-10}\,|0\rangle\ \ , \tag{5.10}$$

and so the coefficients of $|(1,1)\rangle\rangle$ and $|(1,11)\rangle\rangle$ are related by this gluing condition. As we can read off table 8, $\Psi^{(a,b)}_{(1,11)} = \Psi^{(a,b)}_{(1,1)}$ for boundaries (a,b) in $\mathcal{I}_e$ with $b = 2$ and $b = 6$, but $\Psi^{(a,b)}_{(1,11)} = -\Psi^{(a,b)}_{(1,1)}$ for $b = 1$ and $b = 3$. Likewise, the coefficients $\Psi^{(a,b)}_{(1,5)} = \pm\Psi^{(a,b)}_{(1,7)}$, $\Psi^{(a,b)}_{(3,1)} = \pm\Psi^{(a,b)}_{(3,11)}$ and $\Psi^{(a,b)}_{(3,5)} = \pm\Psi^{(a,b)}_{(3,7)}$.

There are now 8 different representations of $SW(10)$ appearing in $\text{SVIR}_3{}^{\otimes 2}$, as each representation of $SW(3/2)$ splits into exactly two representations of $SW(10)$. By the same reasoning as for the algebra $SW(3/2)$, 4 of these representations will allow all four choices of gluing condition but 4 will only allow 2 choices, so that there are 24 different Ishibashi states, leading to two sets of 24 mutually consistent boundary states. These are exactly the full set of boundary states we have found. We can say that the boundary states we have discussed in this paper are precisely those which preserve the algebra $SW(10)$.

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6 Defects in TCIM from defects in $\text{SVIR}_3{}^{\otimes 2}$

We have found a set of non-topological, non-factorising defects in SVIR_3 from the boundary states $|(a,2)_{NS/\widetilde{NS}}\rangle\rangle$ and $|(a,3)_{NS/\widetilde{NS}}\rangle\rangle$ in $\text{SVIR}_3{}^{\otimes 2}$. We can now use these to construct defects in TCIM by using the interface operators constructed in section 3.3: if $\hat{D}_{\text{SVIR}_3}$ is a defect in SVIR_3, then D_{TCIM} defined by

$$\hat{D}_{TCIM} = I \cdot \hat{D}_{\text{SVIR}_3} \cdot I^\dagger \,, \tag{6.1}$$

is a defect in TCIM. The T and g values of $\hat{D}_{TCIM}$ are easy to find, they are just

$$T(\hat{D}_{TCIM}) = T(\hat{D}_{\text{SVIR}_3}) \,,\;\; g(\hat{D}_{TCIM}) = 2g(\hat{D}_{\text{SVIR}_3}) \,. \tag{6.2}$$

It is very unlikely that this defect in TCIM is fundamental - it is instead very likely that it is the superposition of two (possibly identical) defects. This is exactly what happened in the free fermion case, where the map from free fermion defects to Ising model defects always resulted in the superposition of two [or more] defects of the same g value, as below, and it is also true for the identifiable topological and factorising defects in SVIR_3, eg

$$\begin{aligned} I \cdot D_1 \cdot I^\dagger &= D_0 + D_{3/2} \,, \\ I \cdot D_\varphi \cdot I^\dagger &= D_{1/10} + D_{3/5} \,, \\ I \cdot \sqrt{2}(-1)^F \cdot I^\dagger &= 2D_{7/16} \,, \\ I \cdot \sqrt{2}(-1)^F D_\varphi \cdot I^\dagger &= 2D_{3/80} \,, \\ I \cdot |I_{NS}\rangle\rangle\langle\langle I_{NS}| \cdot I^\dagger &= \left(|B_0\rangle\rangle + |B_{3/2}\rangle\rangle\right)\left(\langle\langle B_0| + \langle\langle B_{3/2}|\right) \,, \end{aligned} \tag{6.3}$$

When we come to the new defects in SVIR_3, we cannot say for certain whether they are fundamental or not, but given the results above it is very likely that they are not. Looking at table 5, the simplest conformal non-topological defects we can construct come from the boundary states $\|(1,3)_{NS/\widetilde{NS}}\rangle\rangle$ with $g = 2.03\ldots$. Let us denote these by $\mathcal{D}^\pm$,

$$\mathcal{D}^+ = I \cdot \rho(\|(1,3)_{NS}\rangle\rangle) \cdot I^\dagger \,,\;\; \mathcal{D}^- = I \cdot \rho(\|(1,3)_{\widetilde{NS}}\rangle\rangle) \cdot I^\dagger \,. \tag{6.4}$$

From the fact that $\|(1,3)_{\widetilde{NS}}\rangle\rangle = (-1)^F\|(1,3)_{NS}\rangle\rangle$, it follows that $\rho(\|(1,3)_{\widetilde{NS}}\rangle\rangle) = (-1)^F\rho(\|(1,3)_{NS}\rangle\rangle)(-1)^{\bar{F}}$ and so

$$\mathcal{D}^- = (-1)^F\mathcal{D}^+(-1)^F = \frac{1}{2}D_{2,1}\cdot\mathcal{D}^+\cdot D_{2,1} \,. \tag{6.5}$$

Using this relation, we have

$$\langle 0|\,\mathcal{D}^+\,|0\rangle = \langle 0|\,\mathcal{D}^-\,|0\rangle \;\;,\;\; \langle 0|\,\mathcal{D}^+\,|\tfrac{1}{10}\rangle = -\,\langle 0|\,\mathcal{D}^-\,|\tfrac{1}{10}\rangle \;\;, \tag{6.6}$$

and from the explicit expressions for the boundary state coefficients in table 8, we thus see that $\mathcal{D}^\pm$ are distinct, different operators.

$$\begin{aligned} \langle 0|\,\mathcal{D}^+\,|0\rangle = \langle 0|\,\mathcal{D}^-\,|0\rangle = 2\Psi^{(1,3)}_{(1,1)} = 2(15)^{-1/4}\left(\frac{(3+\sqrt{3})(\sqrt{5}-1)}{2(\sqrt{3}-1)}\right)^{1/2} = 2.031\ldots \,, \\ \langle 0|\,\mathcal{D}^+\,|\tfrac{1}{10}\rangle = -\,\langle 0|\,\mathcal{D}^-\,|\tfrac{1}{10}\rangle = -2(15)^{-1/4}\sqrt{2\sqrt{3}-3} = -0.692\ldots \,. \end{aligned} \tag{6.7}$$

6.1 Comparisons with the results of Gang and Yamaguchi

We can now attempt to compare our results for non-topological, non-factorising defects with those of Gang and Yamaguchi. The simplest such defects we have found are $\mathcal{D}^{\pm}$ defined in (6.4) with $g = 2.031\ldots$ and $T = (3-\sqrt{3})/2$. Looking at the list of proposed defects in section 3.2 of [1] the only candidates to which we can hope to relate $\mathcal{D}^{\pm}$ are those from the boundary state $|(1,3)\rangle_{A_\pm}$ which have the same value of T and half the g-value.

From the definitions in equation (3.9) of [1], the states $|(1,3)\rangle_{A_\pm}$ have equal and opposite components in the Ramond sector. Since our defects have no components in the Ramond sector, we must consider the sum $|(1,3)\rangle_{A_+} + |(1,3)\rangle_{A_-}$ which has the same T and g values as each of $\mathcal{D}^{\pm}$.

There are no precise definitions given in [1] on how to obtain a defect from a boundary state, but we can see that $|(1,3)\rangle_{A_+} + |(1,3)\rangle_{A_-}$ has zero overlap with the states $|(5,3,5)_{10}\rangle$ and $|(5',3,5)_{10}\rangle$ (in the notation of [1]) which are equivalent to (in our notation) $|(5,5)\rangle$ and $|(5',5)\rangle$. This means that whatever map $\tilde{\rho}$ is required to obtain a defect from a boundary state in the formalism of [1], the corresponding defect has zero matrix elements between $\langle 0|$ and $|1/10\rangle$

$$\langle 0|\,\tilde{\rho}\Big(\,|(1,3)\rangle_{A_+} + |(1,3)\rangle_{A_-}\,\Big)\,|\tfrac{1}{10}\rangle = 0\,, \tag{6.8}$$

and so cannot be equal to either $\mathcal{D}^+$ or $\mathcal{D}^-$.

Gang and Yamaguchi do not give details on the precise map $\tilde{\rho}$ required to obtain a defect from a boundary state in their formalism. We can be sure that the method we use cannot work, as this will result in defects which are not GSO projected, that is defects which are not maps from the TCIM to the TCIM. To illustrate this, we consider the states used in [1] in the representation

$$\Big[\mathcal{H}^3_{1,3}\otimes\mathcal{H}^3_{1,3}\Big]^{\otimes 2} = \Big[\mathcal{H}^{10}_{3,1}\oplus\mathcal{H}^{10}_{3,5}\oplus\mathcal{H}^{10}_{3,7}\oplus\mathcal{H}^{10}_{3,11}\Big]^{\otimes 2} \tag{6.9}$$

The paper [1] uses coset representations, and each highest weight representation $\mathcal{H}^{10}_{r,s} \equiv \mathcal{H}^{10}_{10-r,12-s}$ of sVir_{10} splits into two coset representations,

$$\begin{aligned}
\mathcal{H}^{10}_{3,1} &= \mathcal{H}_{(3,1,1)_{10}}\oplus\mathcal{H}_{(3,3,1)_{10}}\;, \quad \mathcal{H}^{10}_{3,5} = \mathcal{H}_{(3,1,5)_{10}}\oplus\mathcal{H}_{(3,3,5)_{10}}\;,\\
\mathcal{H}^{10}_{3,7} &= \mathcal{H}_{(7,1,5)_{10}}\oplus\mathcal{H}_{(7,3,5)_{10}}\;, \quad \mathcal{H}^{10}_{3,11} = \mathcal{H}_{(7,1,1)_{10}}\oplus\mathcal{H}_{(7,3,1)_{10}}\;.
\end{aligned} \tag{6.10}$$

Only four of these coset representations appear in the boundary states of [1], with conformal weights as follows

Representation	Weight
$(3,3,5)_{10}$	$1/5$
$(3,1,1)_{10}$	$6/5$
$(7,3,5)_{10}$	$6/5$
$(7,1,1)_{10}$	$31/5$

(6.11)

In our terms, these can be identified with $\mathrm{SVIR}_3^{\otimes 2}$ descendants of the sVir_{10} highest weight states,

$$\begin{aligned} |(3,3,5)_{10}\rangle &= |(3,5)\rangle \ , & |(7,3,5)_{10}\rangle &= |(3,7)\rangle \ , \\ |(3,1,1)_{10}\rangle &= \frac{i\eta}{7/5} G_{-1/2}\bar{G}_{-1/2}\,|(3,1)\rangle \ , & |(7,1,1)_{10}\rangle &= \frac{i\eta'}{57/5} G_{-1/2}\bar{G}_{-1/2}\,|(3,11)\rangle \ , \end{aligned} \tag{6.12}$$

where η and η' are undetermined signs. Further, given an embedding $\iota_{\alpha\beta\gamma\delta}$, the states $|(3,7)\rangle$ and $|(3,1)\rangle$ can be identified from appendix E as

$$\begin{aligned} |(3,1)\rangle &= \frac{i\eta_{3,1}}{2/5}(\alpha G^1_{-1/2} - \beta G^2_{-1/2})(\gamma \bar{G}^1_{-1/2} - \delta \bar{G}^2_{-1/2})\,|(3,5)\rangle \ , \\ |(3,7)\rangle &= \frac{\eta_{3,7}}{7/5}(L^1_{-1} - L^2_{-1} + \frac{\alpha\beta}{1/5} G^1_{-1/2}G^2_{-1/2})\left(\bar{L}^1_{-1} - \bar{L}^2_{-1} + \frac{\gamma\delta}{1/5}\bar{G}^1_{-1/2}\bar{G}^2_{-1/2}\right)|(3,5)\rangle \ . \end{aligned} \tag{6.13}$$

This means that the state $|(3,1,1)_{10}\rangle$ is

$$\begin{aligned} |(3,1,1)_{10}\rangle = \frac{-\eta\,\eta_{3,1}}{(2/5)(7/5)}&(L^1_{-1} - L^2_{-1} - 2\alpha\beta G^1_{-1/2}G^2_{-1/2}) \\ &\times (\bar{L}^1_{-1} - \bar{L}^2_{-1} - 2\gamma\delta \bar{G}^1_{-1/2}\bar{G}^2_{-1/2})\,|(3,5)\rangle \,. \end{aligned} \tag{6.14}$$

Putting these together with the results in appendix E, and the fact that the boundary state

$$|(3,3,5)_{10}\rangle\rangle = |(3,5)\rangle + \frac{1}{2/5} L_{-1}\bar{L}_{-1}\,|(3,5)\rangle + \dots \ , \tag{6.15}$$

we can find the expansion up to level one of a defect given by a combination of boundary states constructed from the four states (6.12):

$$|\Psi\rangle\rangle = A|(3,3,5)_{10}\rangle\rangle + B|(3,1,1)_{10}\rangle\rangle + C|(7,3,5)_{10}\rangle\rangle + D|(7,1,1)_{10}\rangle\rangle \ , \tag{6.16}$$

$$\begin{aligned} \rho_{\alpha\beta\gamma\delta}\big(|\Psi\rangle\rangle\big) = {} & A|\tfrac{1}{10}\rangle\langle\tfrac{1}{10}| \\ & + \left(\frac{A}{2/5} - \frac{B\eta\eta_{3,1}}{(2/5)(7/5)} + \frac{C\eta_{3,7}}{7/5}\right)\left[L_{-1}\bar{L}_{-1}|\tfrac{1}{10}\rangle\langle\tfrac{1}{10}| + |\tfrac{1}{10}\rangle\langle\tfrac{1}{10}|\bar{L}_1 L_1\right] \\ & + \left(\frac{A}{2/5} + \frac{B\eta\eta_{3,1}}{(2/5)(7/5)} - \frac{C\eta_{3,7}}{7/5}\right)\left[L_{-1}|\tfrac{1}{10}\rangle\langle\tfrac{1}{10}|L_1 + \bar{L}_{-1}|\tfrac{1}{10}\rangle\langle\tfrac{1}{10}|\bar{L}_1\right] \\ & + i\alpha\beta\frac{(B\eta\eta_{3,1} + C\eta_{3,7})}{7/25}\left[G_{-1/2}|\tfrac{1}{10}\rangle\langle\tfrac{1}{10}|\bar{G}_{1/2}L_1 - \bar{L}_{-1}G_{-1/2}|\tfrac{1}{10}\rangle\langle\tfrac{1}{10}|\bar{G}_{1/2}\right] \\ & + i\gamma\delta\frac{(B\eta\eta_{3,1} + C\eta_{3,7})}{7/25}\left[L_{-1}\bar{G}_{-1/2}|\tfrac{1}{10}\rangle\langle\tfrac{1}{10}|G_{1/2} - \bar{G}_{-1/2}|\tfrac{1}{10}\rangle\langle\tfrac{1}{10}|\bar{G}_{1/2}\bar{L}_1\right] \\ & + \alpha\beta\gamma\delta\left(\frac{2B\eta\eta_{3,1}}{7/25} - \frac{C\eta_{3,7}}{7/125}\right)\left[G_{-1/2}\bar{G}_{-1/2}|\tfrac{1}{10}\rangle\langle\tfrac{1}{10}|\bar{G}_{1/2}G_{1/2}\right] + \dots \end{aligned} \tag{6.17}$$

The expression (6.17) is only GSO projected if $B\eta\eta_{3,1} + C\eta_{3,7} = 0$, otherwise it is not. We can fix $\eta\eta_{3,1}$ and $\eta_{3.7}$ by comparing (6.17) with equation (3.20) of [1]. Equation (3.20) says that the expression (6.17) should be purely transmitting for $A = B = 1, C = -1$ and purely reflecting for $B = -1, A = C = 1$, from which we deduce that $\eta\eta_{3,1} = \eta_{3.7} = 1$. We

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can now decide if the defects arising from the boundary states of [1] are GSO projected or not by looking at the ratio of the coefficients B and C of the states $|(3,1,1)_{10}\rangle\rangle$ and $|(7,3,5)_{10}\rangle\rangle$. If this ratio is -1, the resulting defect can be GSO projected, if it is not -1 then it is not GSO projected:

$B = -C$, GSO projected	$B \neq -C$, not GSO projected
$\lvert(2,6)\rangle_{A\pm}\,, \lvert(4,6)\rangle_{A\pm}$	$\lvert(1,3)\rangle_{A\pm}\,, \lvert(3,3)\rangle_{A\pm}\,, \lvert(5,3)\rangle_{A\pm}\,, \lvert(6,3)\rangle_{A\pm}$
$\lvert(1,1)\rangle_B\,, \lvert(3,1)\rangle_B\,, \lvert(5,1)\rangle_B\,, \lvert(6,1)\rangle_B$	$\lvert(2,2)\rangle_B\,, \lvert(4,2)\rangle_B$

(6.18)

Those which are GSO projected correspond to topological or factorising defects; none of the "new" defects proposed in [1] lead to GSO projected defects in our formalism, and so it is difficult for us to make a stronger comparison with the proposals of [1].

7 Conclusions

We have constructed GSO-projected defects in the tri-critical Ising model from defects in the Neveu-Schwarz sector of the supersymmetric tri-critical Ising model using interface operators. Our construction uses many elements from the paper of Gang and Yamaguchi [1] but in the end the defects we propose are not the same as theirs. There is some doubt over the complete validity of their approach as it leads to factorised defects outside the normal classification, and using our methods would result in the new defects proposed in [1] not being properly GSO projected, but we must stress that we have not shown that their non-topological defects are incorrect, simply that they are not the same as ours.

As part of our construction, we found evidence for two non-commensurate sets of boundary states in $\mathrm{SVIR}_3^{\otimes 2}$ corresponding to two inequivalent embeddings of $c = 7/5$ algebra into two copies of $c = 7/10$. We have identified half of these boundary states as known objects, the remaining half are new and lead to non-topological and non-factorising defects in the tri-critical Ising model We hope that these new defects will include the conjectured 'C' defect in [10].

We think it should be possible to derive the boundary states we have proposed for $\mathrm{SVIR}_3^{\otimes 2}$ using topological field theory methods, in the spirit of as well as compare our method with the construction of fermionic models of Novak and Runkel [18] using topological field theory methods which incorporate spin structure.

The next steps would be to extend our approach of explicit construction of boundary states and consideration of extended algebras to include the Ramond sector of the supersymmetric tri-critical Ising model and obtain defects in the tri-critical Ising model using GSO projection, rather than interface operators and compare these directly with the results of [1].

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A The chiral algebra of $\mathrm{SVIR}_3^{\otimes 2}$

The chiral algebra of $\mathrm{SVIR}_3^{\otimes 2}$ is, of course, generated by two copies of the superconformal algebra, with superconformal fields G^1 and G^2. It can also be viewed as generated by a superconformal generator $G(z)$ and three super-primary fields $\mathcal{W}^{(3/2)}$, $\mathcal{W}^{(7/2)}$ and $\mathcal{W}^{(10)}$, of weights 3/2, 7/2 and 10 respectively.

The choice of G is fixed by the two signs α and β,

$$G(z) = \alpha G^1(z) + \beta G^2(z) \ . \tag{A.1}$$

With this choice, the super-primary fields $\mathcal{W}^{(3/2)}$ and $\mathcal{W}^{(7/2)}$ can be defined by the states

$$|\mathcal{W}^{3/2}\rangle = (\alpha G^1_{-3/2} - \beta G^2_{-3/2})|0\rangle \tag{A.2}$$

$$|\mathcal{W}^{7/2}\rangle = \Big[\alpha\Big(L^1_{-2}G^1_{-3/2} - \frac{3}{4}G^1_{-7/2}\Big) + \beta\Big(L^2_{-2}G^2_{-3/2} - \frac{3}{4}G^2_{-7/2}\Big) - \frac{17}{2}(\beta L^1_{-2}G^2_{-3/2} + \alpha L^2_{-2}G^1_{-3/2})\Big]|0\rangle \tag{A.3}$$

– the expression for $|\mathcal{W}^{10}\rangle$ is too lengthy to give here, and is not unique due to null states in the vacuum representation at $c = 7/10$. The states $|G\rangle$ and $|\mathcal{W}^{(7/2)}\rangle$ are even under interchanging $G^1 \leftrightarrow G^2$, and $|\mathcal{W}^{(3/2)}\rangle$ and $|\mathcal{W}^{(10)}\rangle$ are odd.

Since G and $\mathcal{W}^{3/2}$ generate the whole chiral algebra on their own, the fields $\mathcal{W}^{7/2}$ and $\mathcal{W}^{(10)}$ can be expressed in terms of G and $\mathcal{W}^{3/2}$, but they can also be considered as fields in their own right.

The super-partner to $\mathcal{W}^{(3/2)}$ is $\mathcal{T}(z)$ defined by

$$|\mathcal{T}\rangle = \frac{1}{2}G_{-1/2}\,|\mathcal{W}^{(3/2)}\rangle = (L^1_{-2} - L^2_{-2})\,|0\rangle \ , \ \ \mathcal{T}(z) = T^1(z) - T^2(z) \ . \tag{A.4}$$

The representations of $\mathcal{W}^{(3/2)}$ are thus labelled by the eigenvalues h of $L_0 = L^1_0 + L^2_0$ and $\tilde{h}$ of $\mathcal{T}_0 = L^1_0 - L^2_0$; in terms of the eigenvalues h^i of L^i_0 we clearly have

$$h = h^1 + h^2 \ , \ \ \tilde{h} = h^1 - h^2 \ . \tag{A.5}$$

Since there are two NS representations of sVir_3, there are four NS representations of $SW(3/2)$ at $c = 7/5$ which we label $\{1, 3, 5, 5'\}$ with highest weight eigenvalues and characters as in table 6.

There are two interesting subalgebras of this chiral algebra - the super W-algebra SW(7/2), where the superconformal algebra is extended by the single field $\mathcal{W}^{7/2}$ of weight 7/2, and the super W-algebra SW(10), where the superconformal algebra is extended by the single field $\mathcal{W}^{7/2}$ of weight 7/2.

These can be proven to be closed algebras without calculating the commutation relations explicitly.

In the first case, $SW(7/2)$ consists of all fields in $\mathrm{SVIR}_3^{\otimes 2}$ which are invariant under interchanging the fields G^1 and G^2. SW(7/2) was considered as an abstract super W-algebra in [21–23] where it was shown to be consistent for $c = 7/5$.

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label	$(h,\tilde h)$	character
1	$(0,0)$	$\chi_1 = \mathrm{ch}^{10}_{1,1} + \mathrm{ch}^{10}_{1,5} + \mathrm{ch}^{10}_{1,7} + \mathrm{ch}^{10}_{1,11} = (\mathrm{ch}^{3}_{1,1})^2$
3	$(\frac{1}{5},0)$	$\chi_3 = \mathrm{ch}^{10}_{3,1} + \mathrm{ch}^{10}_{3,5} + \mathrm{ch}^{10}_{3,7} + \mathrm{ch}^{10}_{3,11} = (\mathrm{ch}^{3}_{1,3})^2$
5	$(\frac{1}{10},\frac{1}{10})$	$\chi_5 = \mathrm{ch}^{10}_{5,1} + \mathrm{ch}^{10}_{5,5} = \mathrm{ch}^{3}_{1,1}\mathrm{ch}^{3}_{1,3}$
$5'$	$(\frac{1}{10},-\frac{1}{10})$	$\chi_{5'} = \mathrm{ch}^{10}_{5,1} + \mathrm{ch}^{10}_{5,5} = \mathrm{ch}^{3}_{1,1}\mathrm{ch}^{3}_{1,3}$

Table 6. The NS representations of $SW(3/2)$ at $c = 7/5$.

label	h	character
1	0	$\chi_1 = \mathrm{ch}^{10}_{1,1} + \mathrm{ch}^{10}_{1,1}$
$\tilde{1}$	$\frac{3}{2}$	$\chi_{\tilde{1}} = \mathrm{ch}^{10}_{1,5} + \mathrm{ch}^{10}_{1,7}$
3	$\frac{7}{10}$	$\chi_3 = \mathrm{ch}^{10}_{3,1} + \mathrm{ch}^{10}_{3,11}$
$\tilde{3}$	$\frac{1}{5}$	$\chi_{\tilde{3}} = \mathrm{ch}^{10}_{3,5} + \mathrm{ch}^{10}_{3,7}$
5	$\frac{13}{5}$	$\chi_5 = \mathrm{ch}^{10}_{5,1}$
$\tilde{5}$	$\frac{1}{10}$	$\chi_{\tilde{5}} = \mathrm{ch}^{10}_{5,5}$
$5'$	$\frac{13}{5}$	$\chi_{5'} = \mathrm{ch}^{10}_{5,1}$
$\tilde{5}'$	$\frac{1}{10}$	$\chi_{\tilde{5}'} = \mathrm{ch}^{10}_{5,5}$

Table 7. The NS representations of $SW(10)$ at $c = 7/5$.

In the second case, $SW(10)$ is closed as the fusion rules of the superconformal algebra at $c = 7/2$ are $[1,11] * [1,11] = [1,1]$, that is the field $\mathcal{W}^{(10)}$ is a simple current for the superconformal algebra. There are eight representations of $SW(10)$ as each representation of $SW(3/2)$ is reducible into two representations of $SW(10)$ with labels and characters as in table 7. We give the value of h, the eigenvalue of L_0, only, the eigenvalue of $\mathcal{W}^{(10)}$ being too hard to calculate.

B Conventions for the free-fermion and the Ising model

If a free fermion is single-valued on a path around the origin, then it has an expansion over modes ψ_m with $m \in \mathbb{Z} + 1/2$,

$$\psi(z) = \sum_{m\in\mathbb{Z}+1/2} \psi_m z^{-m-1/2} \,. \tag{B.1}$$

These modes form the Neveu-Schwarz free-fermion algebra,

$$\{\psi_m, \psi_n\} = \delta_{m+n,0} \tag{B.2}$$

which has a single unitary irreducible representation, $\mathcal{H}_{NS}$ with character

$$\chi_{NS}(q) = \mathrm{Tr}_{\mathcal{H}_{NS}}(q^{L_0 - c/24}) = q^{-1/48} \prod_{m=0}^{\infty} (1 + q^{m+1/2}) \,. \tag{B.3}$$

We also define $\chi_{\widetilde{NS}}$ as the trace with the insertion of $(-1)^F$,

$$\chi_{\widetilde{NS}}(q) = \mathrm{Tr}_{\mathcal{H}_{NS}}(q^{L_0-c/24}(-1)^F) = q^{-1/48}\prod_{m=0}^{\infty}(1-q^{m+1/2}) \,. \tag{B.4}$$

If a free fermion instead changes sign on a path around the origin, it has an expansion in modes ψ_m, $m\in\mathbb{Z}$,

$$\psi(z) = \sum_{m\in\mathbb{Z}} \psi_m z^{-m-1/2} \,. \tag{B.5}$$

These satisfy the Ramond free-fermion algebra $\{\psi_m,\psi_n\} = \delta_{m+n,0}$. This algebra has two unitary irreducible highest weight representations, $\mathcal{H}_{R_\pm}$ with highest weights $|1/16\rangle_\pm$ satisfying

$$\psi_0\,|1/16\rangle_\pm = \pm\frac{1}{\sqrt{2}}\,|1/16\rangle_\pm \,. \tag{B.6}$$

These have the same character

$$\chi_R(q) = \mathrm{Tr}_{\mathcal{H}_{R_\pm}}(q^{L_0-c/24}) = q^{1/24}\prod_{m=1}^{\infty}(1+q^m) \,. \tag{B.7}$$

Note that there is an alternative and widely used convention which includes a factor of $\sqrt{2}$ in the definition, ${\chi_R}^{\text{alternative}} = \sqrt{2}\chi_R$. This alternative definition has the advantage of making the modular S matrix in equation (B.10) symmetric, but the disadvantage that it is not the trace of $q^{L_0-c/24}$ over a representation.

The three unitary irreducible highest weight representations of the Virasoro algebra with $c=1/2$ have weights $h\in\{0,\,1/2,\,1/16\}$ and their characters are

$$\chi_0^{(3)} = \frac{1}{2}(\chi_{NS}+\chi_{\widetilde{NS}}) \,,\quad \chi_{1/2}^{(3)} = \frac{1}{2}(\chi_{NS}-\chi_{\widetilde{NS}}) \,,\quad \chi_{1/16}^{(3)} = \chi_R \,. \tag{B.8}$$

These characters are related under modular transformation $\tau\to -1/\tau$, that is $q = \exp(2\pi i\tau)\to\tilde{q} = \exp(-2\pi i/\tau)$, by

$$\begin{pmatrix}\chi_0^{(3)}\\ \chi_{1/2}^{(3)}\\ \chi_{1/16}^{(3)}\end{pmatrix}(q) = \begin{pmatrix}1/2 & 1/2 & 1/\sqrt{2}\\ 1/2 & 1/2 & -1/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2} & 0\end{pmatrix}\begin{pmatrix}\chi_0^{(3)}\\ \chi_{1/2}^{(3)}\\ \chi_{1/16}^{(3)}\end{pmatrix}(\tilde{q}) \tag{B.9}$$

$$\begin{pmatrix}\chi_{NS}\\ \chi_{\widetilde{NS}}\\ \chi_R\end{pmatrix}(q) = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & \sqrt{2}\\ 0 & 1/\sqrt{2} & 0\end{pmatrix}\begin{pmatrix}\chi_{NS}\\ \chi_{\widetilde{NS}}\\ \chi_R\end{pmatrix}(\tilde{q}) \tag{B.10}$$

C Conventions for the super Virasoro minimal models

C.1 Characters

For the super Virasoro algebra with the central charge $0\le c<3/2$, irreducible modules are unitary at discrete points, and corresponding highest weight modules are labelled by (c,h), both of which are parametrised by the integers m, r, s as

$$c = \frac{3}{2}\left(1-\frac{8}{m(m+2)}\right) \tag{C.1}$$

with $m = 2, 3, 4, \ldots$, and

$$h^{(m)}_{r,s} = \frac{((m+2)r - ms)^2 - 4}{8m(m+2)} + \frac{1}{32}\left(1 - (-1)^{r-s}\right) \tag{C.2}$$

where $1 \leq r \leq m-1$ and $1 \leq s \leq m+1$. As usual, due to the Kac table symmetry $h^{(m)}_{r,s} = h^{(m)}_{m-r,m+2-s}$, we need identification of the Kac labels $(r,s) \sim (m-r, m+2-s)$. When we denote the super Virasoro algebra by sVir_m, it is understood to take the irreducible modules with (c,h) specified by (C.1) and (C.2). When $r - s \in 2\mathbb{Z}$, a representation is in the Neveu-Schwarz sector, which corresponds to the modes G_n with $n \in \mathbb{Z} + 1/2$, and when $r - s \in 2\mathbb{Z} + 1$, a representation is in the Ramond sector, which corresponds to G_n with $n \in \mathbb{Z}$.

Since sVir is Z_2 graded by fermion parity of the generators (L_n are bosonic and G_m are fermionic), it is natural to consider Z_2 graded modules. We may introduce an operator $(-1)^F$ on a module and take a basis, in which the highest weight state $|h^{(m)}_{r,s}\rangle$ is an eigenvector of $(-1)^F$ with the eigenvalue $\varepsilon(r,s) = \pm 1$ and $\{(-1)^F, G_m\} = 0$.

For a highest weight module $\mathcal{H}_{NS}$ in the Neveu-Schwarz sector, which is generated from $|h^{(m)}_{r,s}\rangle$, we define its character by

$$\mathrm{ch}^m_{r,s}(q) \equiv \mathrm{Tr}_{\mathcal{H}_{NS}}\, q^{L_0 - \frac{c}{24}} = q^{-\frac{c}{24}} \sum_{n=-\infty}^{\infty} \left(q^{h(2mn+r,s)} - q^{h(2mn-r,s)}\right) \prod_{l=1}^{\infty} \frac{1 + q^{l-\frac{1}{2}}}{1 - q^l},$$

where $h(r,s) = h^{(m)}_{r,s}$ and the explicit formula on the right hand side is given in [25]. We also define the following quantity associated to this module

$$\begin{aligned}\widetilde{\mathrm{ch}}^m_{r,s}(q) &\equiv \mathrm{Tr}_{\mathcal{H}_{NS}}\, (-1)^F q^{L_0 - \frac{c}{24}} \\ &= \varepsilon(r,s)\, q^{-\frac{c}{24}} \sum_{n=-\infty}^{\infty} (-1)^{mn} \left(q^{h(2mn+r,s)} - (-1)^{rs} q^{h(2mn-r,s)}\right) \prod_{l=1}^{\infty} \frac{1 - q^{l-\frac{1}{2}}}{1 - q^l},\end{aligned}$$

where $\varepsilon(r,s) = \pm$ is the eigenvalue of $(-1)^F$ on the highest weight vector. Note that when $\varepsilon(r,s) = 1$, the series expansion of $\widetilde{\mathrm{ch}}^{(m)}_{r,s}$ always starts from $q^{h-\frac{c}{24}}(1 - \ldots)$.

Due to the zero modes, some care is needed when defining a character for a highest module $\mathcal{H}_R$ in the Ramond sector, which is generated from $|h^{(m)}_{r,s}\rangle$. When $h^{(m)}_{r,s} \neq c/24$, L_0 eigensubspaces of $\mathcal{H}_R$ are two-dimensional in which we can take two basis vectors to carry opposite fermion parity. On the other hand, L_0 eigensubspaces are one-dimensional when $h^{(m)}_{r,s} = c/24$, which happens for $m \in 2\mathbb{Z}$ and $(r,s) = (m/2, m/2+1)$ — we call this representation the fixed point of a Kac table. We simply define the following function

$$\mathrm{ch}^m_{r,s}(q) = q^{-\frac{c}{24}} \sum_{n=-\infty}^{\infty} \left(q^{h(2mn+r,s)} - q^{h(2mn-r,s)}\right) \prod_{l=1}^{\infty} \frac{1 + q^l}{1 - q^l},$$

for $r - s \in 2\mathbb{Z} + 1$. The expansion of this function is of the form $q^{h-\frac{c}{24}}(1 + \ldots)$. We view this as a "character" of $\mathcal{H}_R$ in the sense that

$$\begin{aligned}\mathrm{Tr}_{\mathcal{H}_R}\, q^{L_0 - \frac{c}{24}} &= 2\, \mathrm{ch}^m_{r,s}(q) \qquad \text{when } h \neq c/24 \\ \mathrm{Tr}_{\mathcal{H}_R}\, q^{L_0 - \frac{c}{24}} &= \mathrm{ch}^m_{r,s}(q) \qquad \text{when } h = c/24.\end{aligned}$$

We choose this normalisation so that it is easy to see if Cardy's condition is satisfied or not.

C.2 Modular transformations

When we denote a representation of sVir by its Kac label, we take the "bottom half" of the Kac table. That is, we take the following sets of Kac labels

$$(r,s)\in\mathcal{I}_{NS} \text{ when } r+s\in 2\mathbb{Z} \text{ and } \begin{cases} 1\le s\le m+1 & \text{for } r<\frac{m}{2} \\ 1\le s\le \frac{m}{2} & \text{for } r=\frac{m}{2} \end{cases}$$

$$(r,s)\in\mathcal{I}_{R} \text{ when } r+s\in 2\mathbb{Z}+1 \text{ and } \begin{cases} 1\le s\le m+1 & \text{for } r<\frac{m}{2} \\ 1\le s\le \frac{m}{2}+1 & \text{for } r=\frac{m}{2} \end{cases}$$

Note that we take this convention only to make it clear that two distinct Kac labels correspond to different representations, and there is no "physical" reason to do so. For example, in the D_6–E_6 theory, it may be more natural to take $r\in\{1,3,5,5',7,9\}$ and $s\in\{1,7\}$.

We define the modular S-matrix elements as follows

$$\begin{aligned} \mathrm{ch}^{NS}_{r,s}(\tilde q) &= \sum_{(r',s')\in\mathcal{I}_{NS}} S^{[NS,NS]}_{(r,s)(r',s')}\,\mathrm{ch}^{NS}_{r',s'}(q)\ , \\ \widetilde{\mathrm{ch}}^{\widetilde{NS}}_{r,s}(\tilde q) &= \sum_{(r',s')\in\mathcal{I}_{R}} S^{[\widetilde{NS},R]}_{(r,s)(r',s')}\,\mathrm{ch}^{R}_{r',s'}(q)\ , \\ \mathrm{ch}^{R}_{r,s}(\tilde q) &= \sum_{(r',s')\in\mathcal{I}_{NS}} S^{[R,\widetilde{NS}]}_{(r,s)(r',s')}\,\widetilde{\mathrm{ch}}^{\widetilde{NS}}_{r',s'}(q)\ , \end{aligned}$$

which can be written explicitly as

$$\begin{aligned} S^{[NS,NS]}_{(r_1,s_1)(r_2,s_2)} &= \frac{4}{\sqrt{m(m+2)}}\sin\left(\frac{\pi r_1 r_2}{m}\right)\sin\left(\frac{\pi s_1 s_2}{m+2}\right)\ , \\ S^{[\widetilde{NS},R]}_{(r_1,s_1)(r_2,s_2)} &= \varepsilon(r_1,s_1)\,(-1)^{\frac{r_1-s_1}{2}}\,\frac{4\sqrt{2}\,G(r_2,s_2)}{\sqrt{m(m+2)}}\sin\left(\frac{\pi r_1 r_2}{m}\right)\sin\left(\frac{\pi s_1 s_2}{m+2}\right)\ , \\ S^{[R,\widetilde{NS}]}_{(r_1,s_1)(r_2,s_2)} &= \varepsilon(r_2,s_2)\,(-1)^{\frac{r_2-s_2}{2}}\,\frac{2\sqrt{2}}{\sqrt{m(m+2)}}\sin\left(\frac{\pi r_1 r_2}{m}\right)\sin\left(\frac{\pi s_1 s_2}{m+2}\right)\ , \end{aligned}$$

where

$$G(r,s)=\begin{cases} \frac{1}{2} & \text{if } r=\frac{m}{2} \text{ and } s=\frac{m}{2}+1 \\ 1 & \text{otherwise} \end{cases}\ .$$

Note that $\mathrm{ch}^R_i(q)$ is not quite a modular function but $\sqrt{2}\mathrm{ch}^R_i(q)$ is. Therefore, the above modular S matrix is non-symmetric but squares to $\mathbf{1}$. It is possible to make the S matrix symmetric by introducing the modified Ramond character $\sqrt{2}\mathrm{ch}^R_i(q)$ when i is not the fixed point, but we do not do that in this paper.

In terms of $\hat{su}(2)_k$ modular S matrix elements

$$S^{(k)}_{ij}=\sqrt{\frac{2}{k+2}}\sin\left(\frac{ij\pi}{k+2}\right)$$

where $i, j = 1, 2, \ldots, k+1$, sVir S matrix elements can be written as

$$
\begin{aligned}
S^{[NS,NS]}_{(r_1,s_1)(r_2,s_2)} &= 2S^{(m-2)}_{r_1 r_2} S^{(m)}_{s_1 s_2} \,, \\
S^{[\widetilde{NS},R]}_{(r_1,s_1)(r_2,s_2)} &= \varepsilon(r_1,s_1)\,(-1)^{\frac{r_1-s_1}{2}}\,\sqrt{2}G(r_2,s_2)\,2S^{(m-2)}_{r_1 r_2} S^{(m)}_{s_1 s_2} \,, \\
S^{[R,\widetilde{NS}]}_{(r_1,s_1)(r_2,s_2)} &= \varepsilon(r_2,s_2)\,(-1)^{\frac{r_2-s_2}{2}}\,\sqrt{2}S^{(m-2)}_{r_1 r_2} S^{(m)}_{s_1 s_2} \,.
\end{aligned}
$$

C.3 Fermion parity assignment of NS highest weight vectors

In most cases, a choice of $\varepsilon(r,s)$ for NS highest weight vectors is irrelevant. Usually, NS highest weight vectors $|r,s\rangle$ are taken to be bosonic (i.e. $G_{-1/2}|r,s\rangle$ and $G_{-3/2}|0\rangle$ are fermionic). However, we take the following convention:

- For m odd,

$$
\begin{aligned}
r+s \in 4\mathbb{Z}+2 &\to |r,s\rangle \text{ bosonic i.e. } \varepsilon(r,s)=1 \\
r+s \in 4\mathbb{Z} &\to |r,s\rangle \text{ fermionic i.e. } \varepsilon(r,s)=-1
\end{aligned}
$$

  (In particular, $|1,3\rangle = |2,2\rangle$ with $h = \frac{1}{10}$ is fermionic in $m=3$.)

- For $m = 10$ with the D_6-E_6 bulk partition function,

$$
\begin{aligned}
(r,s) = (1,5),(1,7),(3,1),(3,11),(5,5),(5,7),(7,1),(7,11),(9,5),(9,7) &\to \text{fermionic} \\
\text{others} &\to \text{bosonic}
\end{aligned}
$$

The first choice for m odd cases makes all the fusion coefficients $\left(N_{\widetilde{NS}\,\widetilde{NS}}{}^{\widetilde{NS}}\right)_{ij}{}^{k}$ non-negative. However, there is no obvious procedure to make all these coefficients non-negative for m even cases. The second choice for $m = 10$ comes from two observations: modular transformations of the bulk partition function and character identities between $m = 3$ and $m = 10$.

- Consider the D_6-E_6 bulk partition function,

$$
\begin{aligned}
Z &= \frac{1}{2}\left(Z_{NS} + Z_{\widetilde{NS}}\right) + Z_R \\
Z_{NS} &= \left|\mathrm{ch}^{10}_{1,1} + \mathrm{ch}^{10}_{1,5} + \mathrm{ch}^{10}_{1,7} + \mathrm{ch}^{10}_{1,11}\right|^2 + \left|\mathrm{ch}^{10}_{3,1} + \mathrm{ch}^{10}_{3,5} + \mathrm{ch}^{10}_{3,7} + \mathrm{ch}^{10}_{3,11}\right|^2 \\
&\quad + 2\left|\mathrm{ch}^{10}_{5,1} + \mathrm{ch}^{10}_{5,5}\right|^2 \\
Z_R &= 2\left|\mathrm{ch}^{10}_{1,4} + \mathrm{ch}^{10}_{1,8}\right|^2 + 2\left|\mathrm{ch}^{10}_{3,4} + \mathrm{ch}^{10}_{3,8}\right|^2 + 4\left|\mathrm{ch}^{10}_{5,4}\right|^2
\end{aligned}
$$

  If we demand $Z_{\widetilde{NS}}$ to have the same form as Z_{NS}, we need

$$
\begin{aligned}
\varepsilon(1,1) = \varepsilon(1,11) = -\varepsilon(1,5) &= -\varepsilon(1,7) \\
\varepsilon(3,1) = \varepsilon(3,11) = -\varepsilon(3,5) &= -\varepsilon(3,7) \\
\varepsilon(5,1) &= -\varepsilon(5,5)
\end{aligned}
$$

  to ensure modular S transformation $\frac{1}{2}Z_{\widetilde{NS}} \leftrightarrow Z_R$.

- From the NS character identities between $m = 3$ and $m = 10$, if we want something similar for $\widetilde{NS}$ characters, that is (again with q real)

$$\left(\widetilde{\mathrm{ch}}^{3}_{1,1}\right)^2 = \widetilde{\mathrm{ch}}^{10}_{1,1} + \widetilde{\mathrm{ch}}^{10}_{1,5} + \widetilde{\mathrm{ch}}^{10}_{1,7} + \widetilde{\mathrm{ch}}^{10}_{1,11}$$
$$\left(\widetilde{\mathrm{ch}}^{3}_{1,3}\right)^2 = \widetilde{\mathrm{ch}}^{10}_{3,1} + \widetilde{\mathrm{ch}}^{10}_{3,5} + \widetilde{\mathrm{ch}}^{10}_{3,7} + \widetilde{\mathrm{ch}}^{10}_{3,11}$$

then they fix $\varepsilon(1,1) = 1$, $\varepsilon(3,1) = -1$, etc. Furthermore, if we take $\varepsilon(1,3) = -1$ for $m = 3$,

$$\widetilde{\mathrm{ch}}^{3}_{1,1} \cdot \widetilde{\mathrm{ch}}^{3}_{1,3} = \widetilde{\mathrm{ch}}^{10}_{5,1} + \widetilde{\mathrm{ch}}^{10}_{5,5}$$

fixes $\varepsilon(5,1) = 1$ and $\varepsilon(5,5) = -1$.

The above arguments fix $\varepsilon(r,s)$ of the NS representations with (r,s) appearing in the D_6-E_6 bulk partition function. For the other NS representations, we simply pick $\varepsilon(r,s) = 1$.

D The folding map relating boundaries and defects

We want to relate two copies of the superconformal algebra defined on the exterior of the unit circle with one copy outside and one inside. We shall do this by considering a family of Möbius maps $w \mapsto z(w)$, such that the image of the real axis changes smoothly from the real axis to the unit circle. We can take such a map to be defined by

$$w = 2iR\left(\frac{z - i/R}{z - i/R + 2iR}\right) . \tag{D.1}$$

For $R = \infty$, this is the identity map; for $R = 1$ this maps the real axis to the unit circle. This map further has the property that the derivative at the origin is 1,

$$\left.\frac{\partial z}{\partial w}\right|_{w=0} = 1 . \tag{D.2}$$

The map relating generators of the folded model, G^2, and the unfolded model, $\bar{G}$, is

$$G^2(w)|_{w=a} = \bar{G}(\bar{w})|_{w=\bar{a}} . \tag{D.3}$$

We would like to relate the modes G^2_m and $\bar{G}_m$ in the expansions of the fields

$$z^{3/2}G^2(z) = \sum_m G^2_m z^{-m} , \quad \bar{z}^{3/2}\bar{G}(\bar{z}) = \sum_m \bar{G}_m \bar{z}^{-m} , \tag{D.4}$$

when $R = 1$. Under the map (D.1), the relation becomes

$$\left. z^{3/2}G^2(z)\right|_{w=a} = +\left(\frac{(2iR - a)(2aR^2 + 2iR - a)}{(2iR + a)(2aR^2 - 2iR - a)}\right)^{3/2} \left.\bar{z}^{3/2}\bar{G}(\bar{z})\right|_{w=\bar{a}} , \tag{D.5}$$

where the '+' sign is chosen so that the map is correct at $R = \infty$. At $R = 1$, we have

$$\bar{z}|_{w=a} = \frac{1}{a} , \tag{D.6}$$

and so,

$$a^{3/2}G^2(a) = -ia^{-3/2}\bar{G}(1/a) , \quad G^2_m = -i\bar{G}_{-m} , \tag{D.7}$$

where the factor of $-i$ comes from requiring the relation (D.5) continue smoothly to $R = 1$. Likewise, we find

$$\bar{G}^2_m = +iG_{-m} . \tag{D.8}$$

E Explicit expansions of boundary states

We give here the explicit expansions of some boundary states for $\mathrm{SVIR}_3^{\otimes 2}$ and their images under the maps ι and ρ. These expressions are needed to fix the constants $\eta_{1,5}$, $\eta_{3,1}$ etc as well as to calculate the transmission coefficient T. Throughout this section, we shall use $c' = 2c$.

$$|(1,1)\rangle = |0\rangle \tag{E.1}$$

$$\begin{aligned}
|(1,1)\epsilon\rangle\!\rangle = &\ |0\rangle - \frac{i\epsilon}{2c'/3} G_{-3/2}\bar{G}_{-3/2}\,|0\rangle + \frac{1}{c'/2} L_{-2}\bar{L}_{-2}\,|0\rangle - \frac{i\epsilon}{2c'} G_{-5/2}\bar{G}_{-5/2}\,|0\rangle \\
&+ \frac{1}{2c'} L_{-3}\bar{L}_{-3}\,|0\rangle - \frac{3i\epsilon}{c'(c'+12)} L_{-2}G_{-3/2}\bar{L}_{-2}\bar{G}_{-3/2}\,|0\rangle \\
&- \frac{81 i\epsilon}{c'(c'+12)(21+4c')} \left[L_{-2}G_{-3/2} - \frac{c'+12}{9} G_{-7/2} \right] \\
&\times \left[\bar{L}_{-2}\bar{G}_{-3/2} - \frac{c'+12}{9} \bar{G}_{-7/2} \right] |0\rangle + \dots
\end{aligned} \tag{E.2}$$

$$\begin{aligned}
\iota(|(1,1)\epsilon\rangle\!\rangle) = &\ |0\rangle - \frac{i\epsilon}{4c/3} (\alpha G^1_{-3/2} + \beta G^2_{-3/2})(\gamma \bar{G}^1_{-3/2} + \delta \bar{G}^2_{-3/2})\,|0\rangle \\
&+ \frac{1}{c} (L^1_{-2} + L^2_{-2})(\bar{L}^1_{-2} + \bar{L}^2_{-2})\,|0\rangle \\
&- \frac{i\epsilon}{4c} (\alpha G^1_{-5/2} + \beta G^2_{-5/2})(\gamma \bar{G}^1_{-5/2} + \delta \bar{G}^2_{-5/2})\,|0\rangle \\
&+ \frac{1}{4c} (L^1_{-3} + L^2_{-3})(\bar{L}^1_{-3} + \bar{L}^2_{-3})\,|0\rangle \\
&- \frac{3i\epsilon}{4c(c+6)} (L^1_{-2} + L^2_{-2})(\alpha G^1_{-3/2} + \beta G^2_{-3/2}) \\
&\times (\bar{L}^1_{-2} + \bar{L}^2_{-2})(\gamma \bar{G}^1_{-3/2} + \delta \bar{G}^2_{-3/2})\,|0\rangle \\
&- \frac{81 i\epsilon}{4c(c+6)(21+8c)} \\
&\times \left[(L^1_{-2} + L^2_{-2})(\alpha G^1_{-3/2} + \beta G^2_{-3/2}) - \frac{2(c+6)}{9} (\alpha G^1_{-7/2} + \beta G^2_{-7/2}) \right] \\
&\times \left[(\bar{L}^1_{-2} + \bar{L}^2_{-2})(\gamma \bar{G}^1_{-3/2} + \delta \bar{G}^2_{-3/2}) - \frac{2(c+6)}{9} (\gamma \bar{G}^1_{-7/2} + \delta \bar{G}^2_{-7/2}) \right] |0\rangle \\
&+ \dots
\end{aligned} \tag{E.3}$$

$$\begin{aligned}
\rho(|(1,1)\epsilon\rangle\!\rangle) = &\ |0\rangle\langle 0| \\
&- \frac{i\epsilon}{4c/3} \Big[\alpha\gamma G_{-3/2}\bar{G}_{-3/2}|0\rangle\langle 0| + i\alpha\delta G_{-3/2}|0\rangle\langle 0|G_{3/2} \\
&\qquad + i\beta\gamma \bar{G}_{-3/2}|0\rangle\langle 0|\bar{G}_{3/2} + \beta\delta |0\rangle\langle 0|\bar{G}_{3/2}G_{3/2} \Big] \\
&+ \frac{1}{c} \Big[L_{-2}\bar{L}_{-2}|0\rangle\langle 0| + L_{-2}|0\rangle\langle 0|L_2 + \bar{L}_{-2}|0\rangle\langle 0|\bar{L}_2 + |0\rangle\langle 0|\bar{L}_2 L_2 \Big] \\
&- \frac{i\epsilon}{4c} \Big[\alpha\gamma G_{-5/2}\bar{G}_{-5/2}|0\rangle\langle 0| + i\alpha\delta G_{-5/2}|0\rangle\langle 0|G_{3/2} \\
&\qquad + i\beta\gamma \bar{G}_{-5/2}|0\rangle\langle 0|\bar{G}_{3/2} + \beta\delta |0\rangle\langle 0|\bar{G}_{3/2}G_{3/2} \Big]
\end{aligned}$$

$$+\frac{1}{4c}\Big[L_{-3}\bar{L}_{-3}|0\rangle\!\langle 0| + L_{-3}|0\rangle\!\langle 0|L_3 + \bar{L}_{-3}|0\rangle\!\langle 0|\bar{L}_3 + |0\rangle\!\langle 0|\bar{L}_3 L_3\Big] + \dots \tag{E.4}$$

$$|(1,5)\rangle = |\tfrac{3}{2}\rangle \tag{E.5}$$

$$|(1,5)\epsilon\rangle\!\rangle = |\tfrac{3}{2}\rangle - \frac{i\epsilon}{3}G_{-1/2}\bar{G}_{-1/2}\,|\tfrac{3}{2}\rangle + \frac{1}{3}L_{-1}\bar{L}_{-1}\,|\tfrac{3}{2}\rangle + \dots \tag{E.6}$$

$$\iota(|(1,5)\rangle) = \frac{i\,\eta_{1,5}}{4c/3}(\alpha G^1_{-3/2} - \beta G^2_{-3/2})(\gamma\bar{G}^1_{-3/2} - \delta\bar{G}^2_{-3/2})\,|0\rangle \tag{E.7}$$

$$\iota(|(1,5)\epsilon\rangle\!\rangle) = \frac{i\,\eta_{1,5}}{4c/3}(\alpha G^1_{-3/2} - \beta G^2_{-3/2})(\gamma\bar{G}^1_{-3/2} - \delta\bar{G}^2_{-3/2})\,|0\rangle - \frac{\epsilon\eta_{1,5}}{c}(L^1_{-1} - L^2_{-1})(\bar{L}^1_{-1}\bar{L}^2_{-1})\,|0\rangle + \dots \tag{E.8}$$

$$\rho(|(1,5)\epsilon\rangle\!\rangle) = \frac{i\,\eta_{1,5}}{4c/3}\Big[\alpha\gamma G_{-1/2}\bar{G}_{-3/2}|0\rangle\!\langle 0| + \beta\delta|0\rangle\!\langle 0|\bar{G}_{3/2}G_{3/2} - i\alpha\delta G_{-3/2}|0\rangle\!\langle 0|G_{3/2} - i\beta\gamma\bar{G}_{-3/2}|0\rangle\!\langle 0|\bar{G}_{3/2}\Big] \tag{E.9}$$
$$- \frac{\epsilon\eta_{1,5}}{c}\Big[L_{-2}\bar{L}_{-2}|0\rangle\!\langle 0| - L_{-2}|0\rangle\!\langle 0|L_2 - \bar{L}_{-2}|0\rangle\!\langle 0|\bar{L}_2 + |0\rangle\!\langle 0|\bar{L}_2 L_2\Big] + \dots$$

We now consider the sector corresponding to $\mathcal{H}_{1/10}\otimes\mathcal{H}_{1/10}$. We give the results in terms of a state of weight $2h$, but of course in this particular case $h = 1/10$, the states are identified as

$$|2h\rangle = |\tfrac{1}{5}\rangle = |(3,5)\rangle\ ,\quad |2h{+}\tfrac{1}{2}\rangle = |\tfrac{7}{10}\rangle = |(3,1)\rangle\ ,\quad |2h{+}1\rangle = |\tfrac{6}{5}\rangle = |(3,7)\rangle\ , \tag{E.10}$$

and the constants are $\eta \equiv \eta_{3,1}$ and $\eta' \equiv \eta_{3,7}$.

$$|(2h)\epsilon\rangle\!\rangle = |2h\rangle - \frac{i\epsilon}{4h}G_{-1/2}\bar{G}_{-1/2}\,|2h\rangle + \frac{1}{4h}L_{-1}\bar{L}_{-1}\,|2h\rangle + \dots \tag{E.11}$$

$$\iota(|(2h)\epsilon\rangle\!\rangle) = |2h\rangle - \frac{i\epsilon}{4h}(\alpha G^1_{-1/2} + \beta G^2_{-1/2})(\gamma\bar{G}^1_{-1/2} + \delta\bar{G}^2_{-1/2})\,|2h\rangle + \frac{1}{4h}(L^1_{-1} + L^2_{-1})(\bar{L}^1_{-1} + \bar{L}^2_{-1})\,|2h\rangle + \dots \tag{E.12}$$

$$\rho(|(2h)\epsilon\rangle\!\rangle) = |h\rangle\!\langle h| - \frac{i\epsilon}{4h}\Big[\alpha\gamma G_{-1/2}\bar{G}_{-1/2}|h\rangle\!\langle h| + i\alpha\delta G_{-1/2}|h\rangle\!\langle h|G_{1/2} + i\beta\gamma\bar{G}_{-1/2}|h\rangle\!\langle h|\bar{G}_{1/2} + \beta\delta|h\rangle\!\langle h|\bar{G}_{1/2}G_{1/2}\Big]$$
$$+ \frac{1}{4h}\Big[L_{-1}\bar{L}_{-1}|h\rangle\!\langle h| + L_{-1}|h\rangle\!\langle h|L_1 + \bar{L}_{-1}|h\rangle\!\langle h|\bar{L}_1 + |h\rangle\!\langle h|\bar{L}_1 L_1\Big] + \dots$$

$$|\Big(2h{+}\frac{1}{2}\Big)\epsilon\rangle\!\rangle = |2h{+}\tfrac{1}{2}\rangle - \frac{i\epsilon}{4h{+}1}G_{-1/2}\bar{G}_{-1/2}\,|2h{+}\tfrac{1}{2}\rangle + \dots \tag{E.13}$$

$$\iota\Big(\,|2h{+}\tfrac{1}{2}\rangle\Big) = \frac{i\eta}{4h}(\alpha G^1_{-1/2} - \beta G^2_{-1/2})(\gamma\bar{G}^1_{-1/2} - \delta\bar{G}^2_{-1/2})\,|2h\rangle \tag{E.14}$$

$$\iota\Big(|\Big(2h{+}\frac{1}{2}\Big)\epsilon\rangle\!\rangle\Big) = \frac{i\eta}{4h}(\alpha G^1_{-1/2} - \beta G^2_{-1/2})(\gamma\bar{G}^1_{-1/2} - \delta\bar{G}^2_{-1/2})\,|2h\rangle - \frac{\epsilon\eta}{4h(4h+1)}(L^1_{-1} - L^2_{-1} - 2\alpha\beta G^1_{-1/2}G^2_{-1/2}) \times (\bar{L}^1_{-1} - \bar{L}^2_{-1} - 2\gamma\delta\bar{G}^1_{-1/2}\bar{G}^2_{-1/2})\,|2h\rangle + \dots \tag{E.15}$$

$$\rho(|\left(2h+\frac{1}{2}\right)\epsilon\rangle\!\rangle) = \frac{\eta}{4h}\Big[i\alpha\gamma G_{-1/2}\bar{G}_{-1/2}|h\rangle\langle h| + i\beta\delta|h\rangle\langle h|\bar{G}_{1/2}G_{1/2}$$
$$+ \alpha\delta G_{-1/2}|h\rangle\langle h|G_{1/2} + \beta\gamma\bar{G}_{-1/2}|h\rangle\langle h|\bar{G}_{1/2}\Big]$$
$$- \frac{\epsilon\eta}{4h(4h{+}1)}\Big[L_{-1}\bar{L}_{-1}|h\rangle\langle h| {+} L_{-1}|h\rangle\langle h|L_1 {+} \bar{L}_{-1}|h\rangle\langle h|\bar{L}_1 {+} |h\rangle\langle h|\bar{L}_1L_1\Big]$$
$$+ \frac{2i\epsilon\eta\alpha\beta}{4h(4h{+}1)}\Big[G_{-1/2}|h\rangle\langle h|\bar{G}_{1/2}L_1 - \bar{L}_{-1}G_{-1/2}|h\rangle\langle h|\bar{G}_{1/2}\Big]$$
$$+ \frac{2i\epsilon\eta\gamma\delta}{4h(4h{+}1)}\Big[L_{-1}\bar{G}_{-1/2}|h\rangle\langle h|G_{1/2} - \bar{G}_{-1/2}|h\rangle\langle h|\bar{G}_{1/2}\bar{L}_1\Big]$$
$$+ \frac{4\epsilon\eta\alpha\beta\gamma\delta}{4h(4h{+}1)}\Big[G_{-1/2}\bar{G}_{-1/2}|h\rangle\langle h|\bar{G}_{1/2}G_{1/2}\Big] + \dots$$
$$|(2h{+}1)\epsilon\rangle\!\rangle = |2h{+}1\rangle + \dots \tag{E.16}$$
$$\iota(|(2h{+}1)\epsilon\rangle\!\rangle) = \frac{\eta'}{4h{+}1}\left(L^1_{-1} - L^2_{-1} + \frac{\alpha\beta}{2h}G^1_{-1/2}G^2_{-1/2}\right)$$
$$\times\left(\bar{L}^1_{-1} - \bar{L}^2_{-1} + \frac{\gamma\delta}{2h}\bar{G}^1_{-1/2}\bar{G}^2_{-1/2}\right)|2h\rangle + \dots \tag{E.17}$$
$$\rho(|(2h{+}1)\epsilon\rangle\!\rangle) = \frac{\eta'}{4h{+}1}\Big[L_{-1}\bar{L}_{-1}|h\rangle\langle h| {-} L_{-1}|h\rangle\langle h|L_1 {-} \bar{L}_{-1}|h\rangle\langle h|\bar{L}_1 {+} |h\rangle\langle h|\bar{L}_1L_1\Big] + \dots$$
$$+ \frac{i\eta'\alpha\beta}{2h(4h{+}1)}\Big[G_{-1/2}|h\rangle\langle h|\bar{G}_{1/2}L_1 - \bar{L}_{-1}G_{-1/2}|h\rangle\langle h|\bar{G}_{1/2}\Big]$$
$$+ \frac{i\eta'\gamma\delta}{2h(4h{+}1)}\Big[L_{-1}\bar{G}_{-1/2}|h\rangle\langle h|G_{1/2} - \bar{G}_{-1/2}|h\rangle\langle h|\bar{G}_{1/2}\bar{L}_1\Big]$$
$$- \frac{\eta'\alpha\beta\gamma\delta}{4h^2(4h{+}1)}\Big[G_{-1/2}\bar{G}_{-1/2}|h\rangle\langle h|\bar{G}_{1/2}G_{1/2}\Big] + \dots \tag{E.18}$$

F The matrices $\Psi^{(a,b)}_{(r,s)}$

The matrices $\Psi^{(a,b)}_{(r,s)}$ are given in terms of the eigenvectors of adjacency matrices of the Dynkin diagrams of D_6 and E_6 in equation (4.22):

$$\Psi^{(a,b)}_{(r,s)} = \frac{\psi^r_a(D_6)\,\psi^s_b(E_6)}{\sqrt{S^{(8)}_{1r}S^{(10)}_{1s}}}\,, \tag{F.1}$$

We repeat here for convenience the vectors $\psi^r_a(G)$ given in [1].

The eigenvectors of the D_6 adjacency matrix $\psi^r_a(D_6)$ are given by

$$\psi^r_a(D_6) = \sqrt{2}S^{(8)}_{ar} \quad \text{for } a,r\neq 5 \qquad\qquad \psi^{5^\pm}_a(D_6) = S^{(8)}_{a5} \quad \text{for } a\neq 5$$

$$\psi^r_{5^\pm}(D_6) = \frac{1}{\sqrt{2}}S^{(8)}_{5r} \quad \text{for } r\neq 5 \qquad\qquad \psi^{5^{\epsilon'}}_{5^\epsilon}(D_6) = \frac{1}{2}\left(S^{(8)}_{55} - \epsilon\epsilon'\right)$$

where $a = 1,2,3,4,5^+,5^-$ ($a = 5^\pm$ correspond to 5 and 6 nodes on the D_6 Dynkin diagram), $r \in \mathcal{E}(D_6) = \{1,3,5,5',7,9\}$ ($r = 5^\pm$ above correspond to 5 and $5'$), and $S^{(8)}_{ij}$ is the $\widehat{\mathrm{su}}(2)_8$

modular S matrix elements,

$$S_{ij}^{(k)} = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi i j}{k+2}\right)$$

Explicitly, the entries in $\psi_a^r(D_6)$ are

$a \setminus r$	1	3	5^+ $(=5)$	5^- $(=5')$	7	9
1	$\frac{-1+\sqrt{5}}{2\sqrt{10}}$	$\frac{1}{2}\sqrt{\frac{3}{5}+\frac{1}{\sqrt{5}}}$	$\frac{1}{\sqrt{5}}$	$\frac{1}{\sqrt{5}}$	$\frac{1}{2}\sqrt{\frac{3}{5}+\frac{1}{\sqrt{5}}}$	$\frac{-1+\sqrt{5}}{2\sqrt{10}}$
2	$\frac{1}{2}\sqrt{1-\frac{1}{\sqrt{5}}}$	$\frac{1}{2}\sqrt{1+\frac{1}{\sqrt{5}}}$	0	0	$-\frac{1}{2}\sqrt{1+\frac{1}{\sqrt{5}}}$	$-\frac{1}{2}\sqrt{1-\frac{1}{\sqrt{5}}}$
3	$\frac{1}{2}\sqrt{\frac{3}{5}+\frac{1}{\sqrt{5}}}$	$\frac{-1+\sqrt{5}}{2\sqrt{10}}$	$-\frac{1}{\sqrt{5}}$	$-\frac{1}{\sqrt{5}}$	$\frac{-1+\sqrt{5}}{2\sqrt{10}}$	$\frac{1}{2}\sqrt{\frac{3}{5}+\frac{1}{\sqrt{5}}}$
4	$\frac{1}{2}\sqrt{1+\frac{1}{\sqrt{5}}}$	$-\frac{1}{2}\sqrt{1-\frac{1}{\sqrt{5}}}$	0	0	$\frac{1}{2}\sqrt{1-\frac{1}{\sqrt{5}}}$	$-\frac{1}{2}\sqrt{1+\frac{1}{\sqrt{5}}}$
5^+ $(=5)$	$\frac{1}{\sqrt{10}}$	$-\frac{1}{\sqrt{10}}$	$\frac{1}{10}\left(-5+\sqrt{5}\right)$	$\frac{1}{10}\left(5+\sqrt{5}\right)$	$-\frac{1}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$
5^- $(=6)$	$\frac{1}{\sqrt{10}}$	$-\frac{1}{\sqrt{10}}$	$\frac{1}{10}\left(5+\sqrt{5}\right)$	$\frac{1}{10}\left(-5+\sqrt{5}\right)$	$-\frac{1}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$

The eigenvectors of the E_6 adjacency matrix $\psi_b^s(E_6)$ are given by

$b \setminus s$	1	4	5	7	8	11
1	a	$\frac{1}{2}$	b	b	$\frac{1}{2}$	a
2	b	$\frac{1}{2}$	a	$-a$	$-\frac{1}{2}$	$-b$
3	c	0	$-d$	$-d$	0	c
4	b	$-\frac{1}{2}$	a	$-a$	$\frac{1}{2}$	$-b$
5	a	$-\frac{1}{2}$	b	b	$-\frac{1}{2}$	a
6	d	0	$-c$	c	0	$-d$

where

$$a = \frac{1}{2}\sqrt{\frac{3-\sqrt{3}}{6}} \qquad b = \frac{1}{2}\sqrt{\frac{3+\sqrt{3}}{6}}$$
$$c = \frac{1}{2}\sqrt{\frac{3+\sqrt{3}}{3}} \qquad d = \frac{1}{2}\sqrt{\frac{3-\sqrt{3}}{3}}$$

Putting these together, we can calculate the entries of Ψ. Since it is helpful to have an overview of the properties of Ψ when discussing the boundary states from the extended algebra point of view, we include a table of the approximate numerical values in table 8.

G Character identities

Relations expressing products of NS characters of sVir_3 as sums of NS characters of sVir_{10}:

$$\mathrm{ch}^{10}_{1,1} + \mathrm{ch}^{10}_{1,5} + \mathrm{ch}^{10}_{1,7} + \mathrm{ch}^{10}_{1,11} = (\mathrm{ch}^{3}_{1,1})^2 \tag{G.1}$$

$$\mathrm{ch}^{10}_{3,1} + \mathrm{ch}^{10}_{3,5} + \mathrm{ch}^{10}_{3,7} + \mathrm{ch}^{10}_{3,11} = (\mathrm{ch}^{3}_{1,3})^2 \tag{G.2}$$

$$\mathrm{ch}^{10}_{5,1} + \mathrm{ch}^{10}_{5,5} = \mathrm{ch}^{3}_{1,1} \cdot \mathrm{ch}^{3}_{1,3} \tag{G.3}$$

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(a,b)	(r,s) $(1,1)$	$(1,5)$	$(1,7)$	$(1,11)$	$(3,1)$	$(3,5)$	$(3,7)$	$(3,11)$	$(5,1)$	$(5,5)$	$(5',1)$	$(5',5)$
$(1,1)$	0.3717	0.3717	0.3717	0.3717	0.6015	0.6015	0.6015	0.6015	0.4729	0.4729	0.4729	0.4729
$(1,2)$	0.7182	−0.1924	−0.1924	0.7182	1.162	−0.3114	−0.3114	1.162	0.9135	−0.2448	0.9135	−0.2448
$(1,3)$	1.016	−0.2721	−0.2721	1.016	1.643	−0.4403	−0.4403	1.643	1.292	−0.3462	1.292	−0.3462
$(1,6)$	0.5257	0.5257	0.5257	0.5257	0.8507	0.8507	0.8507	0.8507	0.6687	0.6687	0.6687	0.6687
$(2,1)$	0.7071	−0.7071	0.7071	−0.7071	0.7071	−0.7071	0.7071	−0.7071	0	0	0	0
$(2,2)$	1.366	0.3660	−0.3660	−1.366	1.366	0.3660	−0.3660	−1.366	0	0	0	0
$(2,3)$	1.932	0.5176	−0.5176	−1.932	1.932	0.5176	−0.5176	−1.932	0	0	0	0
$(2,6)$	1.000	−1.000	1.000	−1.000	1.000	−1.000	1.000	−1.000	0	0	0	0
$(3,1)$	0.9732	0.9732	0.9732	0.9732	0.2298	0.2298	0.2298	0.2298	−0.4729	−0.4729	−0.4729	−0.4729
$(3,2)$	1.880	−0.5038	−0.5038	1.880	0.4438	−0.1189	−0.1189	0.4438	−0.9135	0.2448	−0.9135	0.2448
$(3,3)$	2.659	−0.7125	−0.7125	2.659	0.6277	−0.1682	−0.1682	0.6277	−1.292	0.3462	−1.292	0.3462
$(3,6)$	1.376	1.376	1.376	1.376	0.3249	0.3249	0.3249	0.3249	−0.6687	−0.6687	−0.6687	−0.6687
$(4,1)$	1.144	−1.144	1.144	−1.144	−0.4370	0.4370	−0.4370	0.4370	0	0	0	0
$(4,2)$	2.210	0.5922	−0.5922	−2.210	−0.8443	−0.2262	0.2262	0.8443	0	0	0	0
$(4,3)$	3.126	0.8376	−0.8376	−3.126	−1.194	−0.3199	0.3199	1.194	0	0	0	0
$(4,6)$	1.618	−1.618	1.618	−1.618	−0.6180	0.6180	−0.6180	0.6180	0	0	0	0
$(5,1)$	0.6015	0.6015	0.6015	0.6015	−0.3717	−0.3717	−0.3717	−0.3717	−0.2923	−0.2923	0.7651	0.7651
$(5,2)$	1.162	−0.3114	−0.3114	1.162	−0.7182	0.1924	0.1924	−0.7182	−0.5646	0.1513	1.478	−0.3961
$(5,3)$	1.643	−0.4403	−0.4403	1.643	−1.016	0.2721	0.2721	−1.016	−0.7984	0.2139	2.090	−0.5601
$(5,6)$	0.8507	0.8507	0.8507	0.8507	−0.5257	−0.5257	−0.5257	−0.5257	−0.4133	−0.4133	1.082	1.082
$(6,1)$	0.6015	0.6015	0.6015	0.6015	−0.3717	−0.3717	−0.3717	−0.3717	0.7651	0.7651	−0.2923	−0.2923
$(6,2)$	1.162	−0.3114	−0.3114	1.162	−0.7182	0.1924	0.1924	−0.7182	1.478	−0.3961	−0.5646	0.1513
$(6,3)$	1.643	−0.4403	−0.4403	1.643	−1.016	0.2721	0.2721	−1.016	2.090	−0.5601	−0.7984	0.2139
$(6,6)$	0.8507	0.8507	0.8507	0.8507	−0.5257	−0.5257	−0.5257	−0.5257	1.082	1.082	−0.4133	−0.4133

Table 8. Numerical values of the boundary state coefficients $\Psi^{(a,b)}_{(r,s)}$.

Relations expressing products of Ramond characters for sVir_3 as sums of Ramond characters for sVir_{10}:

$$\mathrm{ch}^{10}_{3,4} + \mathrm{ch}^{10}_{3,8} = (\mathrm{ch}^{3}_{1,2})^2 \tag{G.4}$$

$$\mathrm{ch}^{10}_{1,4} + \mathrm{ch}^{10}_{1,8} = (\mathrm{ch}^{3}_{1,4})^2 \tag{G.5}$$

$$\mathrm{ch}^{10}_{5,4} = \mathrm{ch}^{3}_{1,2} \cdot \mathrm{ch}^{3}_{1,4} \tag{G.6}$$

Relations expressing Ramond characters for sVir_3 at $\sqrt{q}$ as sums of characters for sVir_{10}. Note that the same characters for sVir_3 can be expressed as sums of characters in both the Ramond and NS sectors of sVir_{10}.

$$\mathrm{ch}^{10}_{2,4} + \mathrm{ch}^{10}_{2,8} = \mathrm{ch}^{3}_{1,4}(\sqrt{q}) \tag{G.7}$$

$$\mathrm{ch}^{10}_{2,1} + \mathrm{ch}^{10}_{2,5} + \mathrm{ch}^{10}_{2,7} + \mathrm{ch}^{10}_{2,11} = \mathrm{ch}^{3}_{1,4}(\sqrt{q}) \tag{G.8}$$

$$\mathrm{ch}^{10}_{4,4} + \mathrm{ch}^{10}_{4,8} = \mathrm{ch}^{3}_{1,2}(\sqrt{q}) \tag{G.9}$$

$$\mathrm{ch}^{10}_{4,1} + \mathrm{ch}^{10}_{4,5} + \mathrm{ch}^{10}_{4,7} + \mathrm{ch}^{10}_{4,11} = \mathrm{ch}^{3}_{1,2}(\sqrt{q}) \tag{G.10}$$

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