

Off-diagonal Bethe Ansatz solution of the τ_2 -model

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ABSTRACT: The generic quantum τ_2 -model (also known as Baxter-Bazhanov-Stroganov (BBS) model) with periodic boundary condition is studied via the off-diagonal Bethe Ansatz method. The eigenvalues of the corresponding transfer matrix (solutions of the recursive functional relations in τ_j -hierarchy) with generic site-dependent inhomogeneity parameters are given in terms of an inhomogeneous $T - Q$ relation with polynomial Q -functions. The associated Bethe Ansatz equations are obtained. Numerical solutions of the Bethe Ansatz equations for small number of sites indicate that the inhomogeneous $T - Q$ relation does indeed give the complete spectrum.

KEYWORDS: Lattice Integrable Models, Bethe Ansatz

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1 Introduction

Among quantum integrable models, the τ_2 (BBS)-model [1] plays a special role for its unique properties, e.g., it is one of the simplest quantum integrable models associated with cyclic representation of the Weyl algebra; it allows to include multiple inhomogeneity parameters on each single site without breaking the integrability of the model; and more interestingly, the τ_2 -model under certain parameter constraint is highly related to some other integrable models such as the chiral Potts model [2–7] and the relativistic quantum Toda chain model [8]. Many papers have appeared in literature for such connections and many efforts have been made to obtain the solutions of chiral Potts model by solving the τ_2 -model with a recursive functional relation [9–12]. However, it was found that only in the super-integrable sub-sector [2] the algebraic Bethe Ansatz method can be applied on this model to obtain Baxter’s $T - Q$ [13] solutions and Bethe Ansatz equations, while for the generic τ_2 -model, though its integrability [1] was proven, there is no simple Q -operator solution in terms of Baxter’s $T - Q$ relation. The Q -operator is in fact a very complicated function defined in high genus space and its concrete form is still hard to be derived.

In this paper, we adopt the off-diagonal Bethe Ansatz method [14–17] (for comprehensive introduction, see [18]) to study the quantum τ_2 -model. It seems that the situation of the generic τ_2 -model is quite similar to the quantum XYZ model with an odd number

of sites [14–18], in which there is also no simple polynomial solutions of the Q -function in terms of Baxter’s $T - Q$ relation. However, by including an extra off-diagonal term in the $T - Q$ relation (i.e., the inhomogeneous $T - Q$ relation), we show that the eigenvalues of the generic τ_2 transfer matrix can be expressed explicitly in terms of a trigonometric polynomial Q function and thus a proper set of Bethe Ansatz equations can be derived.

The structure of the paper is the following. In the subsequent section, we give a brief introduction of the τ_2 transfer matrix. In section 3, we study the fundamental properties of the transfer matrix and its fusion hierarchy. In section 4, we give the eigenvalues of the transfer matrix and the associated Bethe Ansatz equations. Concluding remarks are given in section 5 and the detailed proofs about the inhomogeneous $T - Q$ relation and its degenerate case are given in appendices A & B.

2 Transfer matrix

Let $R(u) \in \text{End}(\mathbf{C}^2 \otimes \mathbf{C}^2)$ be the six-vertex R -matrix

$$R(u) = \begin{pmatrix} \sinh(u + \eta) & 0 & 0 & 0 \\ 0 & \sinh u \sinh \eta & 0 & 0 \\ 0 & \sinh \eta \sinh u & 0 & 0 \\ 0 & 0 & 0 & \sinh(u + \eta) \end{pmatrix}, \quad (2.1)$$

with the crossing parameter η taking the special values:¹

$$\eta = 2i\pi/p, \quad p = 2l + 1, \quad l = 1, 2, \dots \quad (2.2)$$

The R -matrix satisfies the quantum Yang-Baxter equation (QYBE) [13, 19] and has played an important role in the quantum integrable systems and the quantum group theories [20]. Moreover, the R -matrix becomes some projectors when the spectral parameter u takes some special values:

$$\text{Antisymmetric-fusion conditions : } R(-\eta) = -2 \sinh \eta P^{(-)}, \quad (2.3)$$

$$\text{Symmetric-fusion conditions : } R(\eta) = 2 \sinh \eta \text{Diag}(\cosh \eta, 1, 1, \cosh \eta) P^{(+)}, \quad (2.4)$$

where $P^{(+)}$ ($P^{(-)}$) is the symmetric (anti-symmetric) projector of the tensor space $\mathbf{C}^2 \otimes \mathbf{C}^2$.

Let \mathbf{V} denote a p -dimensional linear space (i.e. the local Hilbert space) with an orthonormal basis $\{|m\rangle \mid m \in \mathbb{Z}_p\}$. X and Z are two $p \times p$ matrices acting on the basis as follows:

$$X|m\rangle = q^m|m\rangle, \quad Z|m\rangle = |m + 1\rangle, \quad q = e^{-\eta}, \quad m \in \mathbb{Z}_p. \quad (2.5)$$

Here and below we adopt the standard notations: for any matrix $A \in \text{End}(\mathbf{V})$, A_n is an embedding operator in the tensor space $\mathbf{V} \otimes \mathbf{V} \otimes \dots$, which acts as A on the n -th space and as identity on the other factor spaces. Then the embedding operators $\{X_n, Z_n \mid n = 1, \dots, N\}$ satisfy the ultra-local Weyl algebra:

$$X_n Z_m = q^{\delta_{nm}} Z_m X_n, \quad X_n^p = Z_n^p = 1, \quad \forall n, m \in \{1, \dots, N\}. \quad (2.6)$$

¹It corresponds to the case that $q = e^{-\eta}$ is a p -root of unity: $q^p = 1$. The generalization to the case of $\eta = 2i\pi p'/p$ with two coprime positive integers p' and p is straightforward.

The τ_2 -model can be described by an quantum spin chain [1]. With each site n of the quantum chain, the associated L -operator $L_n(u) \in \text{End}(\mathbf{C}^2 \otimes \mathbf{V})$ defined in the most general cyclic representation of $U_q(sl_2)$, is given by [1]

$$L_n(u) = \begin{pmatrix} e^u d_n^{(+)} X_n + e^{-u} d_n^{(-)} X_n^{-1} & (g_n^{(+)} X_n^{-1} + g_n^{(-)} X_n) Z_n \\ (h_n^{(+)} X_n^{-1} + h_n^{(-)} X_n) Z_n^{-1} & e^u f_n^{(+)} X_n^{-1} + e^{-u} f_n^{(-)} X_n \end{pmatrix} \\ = \begin{pmatrix} A_n(u) & B_n(u) \\ C_n(u) & D_n(u) \end{pmatrix}, \quad n = 1, \dots, N, \quad (2.7)$$

where $d_n^{(+)}$, $d_n^{(-)}$, $g_n^{(+)}$, $g_n^{(-)}$, $h_n^{(+)}$, $h_n^{(-)}$, $f_n^{(+)}$ and $f_n^{(-)}$ are some parameters associated with the n -th site. These parameters are subjected to two constraints:

$$g_n^{(-)} h_n^{(-)} = f_n^{(-)} d_n^{(+)}, \quad g_n^{(+)} h_n^{(+)} = f_n^{(+)} d_n^{(-)}, \quad n = 1, \dots, N. \quad (2.8)$$

It was shown [1] that the L -operators satisfy the relations:

$$R(u-v)(L_n(u) \otimes 1)(1 \otimes L_n(v)) = (1 \otimes L_n(v))(L_n(u) \otimes 1)R(u-v), \quad n = 1, \dots, N, \quad (2.9)$$

where the R -matrix $R(u)$ is given by (2.1). The corresponding one-row monodromy matrix $T(u)$ is thus defined as:

$$T(u) = \begin{pmatrix} \mathbf{A}(u) & \mathbf{B}(u) \\ \mathbf{C}(u) & \mathbf{D}(u) \end{pmatrix} = L_N(u) L_{N-1}(u) \cdots L_1(u), \quad (2.10)$$

which satisfies the quadratic relation known as the Yang-Baxter algebra

$$R(u-v)(T(u) \otimes 1)(1 \otimes T(v)) = (1 \otimes T(v))(T(u) \otimes 1)R(u-v). \quad (2.11)$$

The transfer matrix $t(u)$ of the τ_2 -model with periodic boundary condition is then given by the partial trace of the monodromy matrix $T(u)$ in the auxiliary space, namely,

$$t(u) = \text{tr}(T(u)) = \mathbf{A}(u) + \mathbf{D}(u). \quad (2.12)$$

The quadratic relation (2.11) leads to the fact that the transfer matrices with different spectral parameters are mutually commutative [19], i.e., $[t(u), t(v)] = 0$, which guarantees the integrability of the model by treating $t(u)$ as the generating functional of the conserved quantities.

The aim of this paper is to construct the eigenvalues $\Lambda(u)$ of the transfer matrix $t(u)$ for generic inhomogeneity parameters $\{d_n^{(\pm)}, f_n^{(\pm)}, g_n^{(\pm)}, h_n^{(\pm)} | n = 1, \dots, N\}$ obeying the constraints (2.8).

3 Properties of the transfer matrix

3.1 Asymptotic behaviors and average values

Following [21, 22], let us introduce the operator \mathcal{Q} which commutes with the transfer matrix

$$\mathcal{Q} = \prod_{n=1}^N X_n, \quad [\mathcal{Q}, t(u)] = 0, \quad \mathcal{Q}^p = \text{id}. \quad (3.1)$$

The explicit expression (2.7) of the L -operator and the definition (2.10) of the monodromy matrix $T(u)$ imply that the transfer matrix $t(u)$ given by (2.12) enjoys the asymptotic behavior:

$$\lim_{u \rightarrow \pm\infty} t(u) = e^{\pm Nu} \left\{ D^{(\pm)} \mathcal{Q}^{\pm 1} + F^{(\pm)} \mathcal{Q}^{\mp 1} \right\} + \dots, \quad (3.2)$$

where $D^{(\pm)}$ and $F^{(\pm)}$ are four constants related to the inhomogeneous parameters as follows:

$$D^{(\pm)} = \prod_{n=1}^N d_n^{(\pm)}, \quad F^{(\pm)} = \prod_{n=1}^N f_n^{(\pm)}. \quad (3.3)$$

Moreover, (2.7) allows us to derive the quasi-periodicity

$$L_n(u + i\pi) = -\sigma^z L_n(u) \sigma^z, \quad (3.4)$$

which leads to the quasi-periodicity of the transfer matrix $t(u)$

$$t(u + i\pi) = (-1)^N t(u). \quad (3.5)$$

The above relation implies that the transfer matrix $t(u)$ can be expressed in terms of e^u as a Laurent polynomial of the form

$$t(u) = e^{Nu} t_N + e^{(N-2)u} t_{N-1} + \dots + e^{-Nu} t_0, \quad (3.6)$$

where $\{t_n | n = 0, 1, \dots, N\}$ form the $N + 1$ conserved charges. In particular, t_N and t_0 are given by

$$\begin{aligned} t_N &= D^{(+)} \mathcal{Q} + F^{(+)} \mathcal{Q}^{-1}, \\ t_0 &= D^{(-)} \mathcal{Q}^{-1} + F^{(-)} \mathcal{Q}, \end{aligned}$$

where the constants $D^{(\pm)}$ and $F^{(\pm)}$ are given by (3.3).

The property (2.3) of the R -matrix and the relation (2.11) enables one to introduce the quantum determinant [23–25] of the associated Yang-Baxter algebra

$$\text{Det}_q(T(u)) = \mathbf{A}(u)\mathbf{D}(u - \eta) - \mathbf{B}(u)\mathbf{C}(u - \eta). \quad (3.7)$$

Direct calculation shows that it is proportional to the identity operator and has the factorized form:

$$\text{Det}_q(T(u)) = \prod_{n=1}^N \text{Det}_q(L_n(u)) = a(u)d(u - \eta) \times \text{id} \stackrel{\text{def}}{=} \delta(u) \times \text{id}, \quad (3.8)$$

$$a(u) = e^{-\frac{N}{2}\eta} \left\{ D^{(+)} F^{(+)} \right\}^{\frac{1}{2}} \prod_{n=1}^N \left(e^{u+\eta} - e^{-u-\eta} e^{2\eta} \frac{g_n^{(-)} h_n^{(+)}}{d_n^{(+)} f_n^{(+)}} \right), \quad (3.9)$$

$$d(u) = e^{-\frac{N}{2}\eta} \left\{ D^{(+)} F^{(+)} \right\}^{\frac{1}{2}} \prod_{n=1}^N \left(e^u - e^{-u} \frac{g_n^{(+)} h_n^{(-)}}{d_n^{(+)} f_n^{(+)}} \right), \quad (3.10)$$

where $D^{(\pm)}$ and $F^{(\pm)}$ are given by (3.3).

Let us define the average value $\mathcal{O}(u)$ of the matrix elements of the monodromy matrix $T(u)$ (or the L -operators $L_n(u)$) using the averaging procedure [26]:

$$\mathcal{O}(u) = \prod_{m=1}^p \mathcal{O}(u - m\eta), \quad (3.11)$$

where the operator $O(u)$ can be $\{\mathbf{A}(u), \mathbf{B}(u), \mathbf{C}(u), \mathbf{D}(u)\}$ or $\{A_n(u), B_n(u), C_n(u), D_n(u) \mid n = 1, \dots, N\}$. It was shown [26] that

$$\mathcal{T}(u) = \begin{pmatrix} \mathcal{A}(u) & \mathcal{B}(u) \\ \mathcal{C}(u) & \mathcal{D}(u) \end{pmatrix} = \mathcal{L}_N(u) \mathcal{L}_{N-1}(u) \cdots \mathcal{L}_1(u), \quad (3.12)$$

where the average value of each L -operator is given by

$$\begin{aligned} \mathcal{L}_n(u) &= \begin{pmatrix} \mathcal{A}_n(u) & \mathcal{B}_n(u) \\ \mathcal{C}_n(u) & \mathcal{D}_n(u) \end{pmatrix} \\ &= \begin{pmatrix} e^{pu} \{d_n^{(+)}\}^p + e^{-pu} \{d_n^{(-)}\}^p & \{g_n^{(+)}\}^p + \{g_n^{(-)}\}^p \\ \{h_n^{(+)}\}^p + \{h_n^{(-)}\}^p & e^{pu} \{f_n^{(+)}\}^p + e^{-pu} \{f_n^{(-)}\}^p \end{pmatrix}, \end{aligned} \quad (3.13)$$

and $n = 1, \dots, N$. It is remarked that the average values of the matrix elements are Laurent polynomials of e^{pu} , which implies

$$\mathcal{T}(u + \eta) = \mathcal{T}(u), \quad \mathcal{L}_n(u + \eta) = \mathcal{L}_n(u), \quad n = 1, \dots, N, \quad (3.14)$$

$$\lim_{u \rightarrow \pm\infty} \mathcal{A}(u) = e^{\pm pNu} \{D^{(\pm)}\}^p, \quad (3.15)$$

$$\lim_{u \rightarrow \pm\infty} \mathcal{D}(u) = e^{\pm pNu} \{F^{(\pm)}\}^p, \quad (3.16)$$

where the constants $D^{(\pm)}$ and $F^{(\pm)}$ are given by (3.3).

3.2 Fusion hierarchy and truncation identity

The transfer matrix $t(u)$ given by (2.12) is constructed by tracing over a spin- $\frac{1}{2}$ (i.e., two-dimensional) auxiliary space. Using the fusion procedure [23, 27, 28], the arbitrary higher spin- j ($j = 1, \frac{3}{2}, 2, \dots$) transfer matrices $t^{(j)}(u)$ which correspond to spin- j auxiliary spaces and the same quantum space, i.e., the N -tensor space $\mathbf{V} \otimes \mathbf{V} \otimes \dots$ can be constructed. These transfer matrices $\{t^{(j)}(u) \mid j = \frac{1}{2}, 1, \frac{3}{2}, 2, \dots\}$ (including the transfer matrix $t(u)$ given by (2.12) as the first one: $t(u) = t^{(\frac{1}{2})}(u)$) commute with each other

$$[t^{(j)}(u), t^{(j')}(v)] = 0, \quad j, j' \in \frac{1}{2}, 1, \frac{3}{2}, \dots, \quad (3.17)$$

and obey the fusion hierarchy relations [1, 21, 27, 28]

$$\begin{aligned} t^{(\frac{1}{2})}(u) t^{(j-\frac{1}{2})}(u - j\eta) &= t^{(j)}\left(u - \left(j - \frac{1}{2}\right)\eta\right) + \delta(u) t^{(j-1)}\left(u - \left(j + \frac{1}{2}\right)\eta\right), \\ j &= \frac{1}{2}, 1, \frac{3}{2}, \dots, \end{aligned} \quad (3.18)$$

The truncation identity (3.20) of the spin- $\frac{p}{2}$ transfer matrix leads to the fact that the corresponding eigenvalue $\Lambda_k^{(\frac{p}{2})}(u)$ satisfies the relation

$$\Lambda_k^{(\frac{p}{2})}(u) = \mathcal{A}(u) + \mathcal{D}(u) + \delta \left(u - \left(\frac{p-1}{2} \right) \eta \right) \Lambda_k^{(\frac{p-2}{2})}(u), \quad (4.6)$$

where the functions $\mathcal{A}(u)$ and $\mathcal{D}(u)$ are given by (3.12)–(3.13).

It is believed [1, 21, 22] that the quasi-periodicity (4.2), the asymptotic behavior (4.3), the analytic property (4.4) and the truncation identity (4.6) completely determine the eigenvalues $\{\Lambda_k(u) | k = 1, 2, \dots, p\}$ of the fundamental transfer matrix $t(u)$ given by (2.12).

4.2 T-Q relation

4.2.1 Generic case

Following the method developed in [14–17] (or for details we refer the reader to [18]), let us introduce the following inhomogeneous $T - Q$ relation

$$\Lambda_k(u) = e^{\phi_k} a(u) \frac{Q(u-\eta)}{Q(u)} + e^{-\phi_k} d(u) \frac{Q(u+\eta)}{Q(u)} + 2^{(1-p)N} c_k \frac{F_k(u)}{Q(u)}, \quad (4.7)$$

where ϕ_k is a generic complex number,² the functions $a(u)$ and $d(u)$ are given by (3.9) and (3.10), the function $F_k(u)$ is given by

$$F_k(u) = \mathcal{A}(u) + \mathcal{D}(u) - e^{p\phi_k} \bar{\mathcal{A}}(u) - e^{-p\phi_k} \bar{\mathcal{D}}(u), \quad (4.8)$$

$$\bar{\mathcal{A}}(u) = \prod_{m=1}^p a(u - m\eta), \quad \bar{\mathcal{D}}(u) = \prod_{m=1}^p d(u - m\eta), \quad (4.9)$$

and the function $Q(u)$ is a trigonometric polynomial of degree $(p-1)N$

$$Q(u) = \prod_{j=1}^{(p-1)N} \sinh(u - \lambda_j). \quad (4.10)$$

Here the $(p-1)N + 1$ parameters c_k and $\{\lambda_j | j = 1, \dots, (p-1)N\}$ satisfy the associated Bethe Ansatz equations (BAEs)

$$e^{\phi_k} a(\lambda_j) Q(\lambda_j - \eta) + e^{-\phi_k} d(\lambda_j) Q(\lambda_j + \eta) + 2^{(1-p)N} c_k F_k(\lambda_j) = 0, \quad j = 1, \dots, (p-1)N, \quad (4.11)$$

$$\begin{aligned} & q^k D^{(+)} + q^{-k} F^{(+)} - 2 \left\{ D^{(+)} F^{(+)} \right\}^{\frac{1}{2}} \cosh \left(\phi_k + \frac{3}{2} N \eta \right) \\ & = c_k e^{\sum_{j=1}^{(p-1)N} \lambda_j} \left\{ \left\{ D^{(+)} \right\}^p + \left\{ F^{(+)} \right\}^p - 2(-1)^N \left\{ D^{(+)} F^{(+)} \right\}^{\frac{p}{2}} \cosh p\phi_k \right\}, \end{aligned} \quad (4.12)$$

² ϕ_k is chosen such that the degree of the trigonometric polynomial $F_k(u)$ given by (4.8) is pN .

n	k	λ_1	λ_2	λ_3	λ_4	c_k
1	0	$-0.41481 - 0.48777i$	$-0.41481 + 0.48777i$	$0.16543 - 0.35104i$	$0.16543 + 0.35104i$	$0.07290 + 0.00000i$
2	0	$-0.66826 - 1.49724i$	$-0.04032 - 0.48519i$	$0.06867 + 1.53930i$	$0.14115 + 0.44313i$	$0.07290 + 0.00000i$
3	0	$-0.66826 + 1.49724i$	$-0.04032 + 0.48519i$	$0.06867 - 1.53930i$	$0.14115 - 0.44313i$	$0.07290 + 0.00000i$
4	1	$-0.38066 - 1.28929i$	$-0.18674 + 0.42053i$	$0.33201 + 0.59239i$	$0.48476 + 1.21602i$	$-0.08872 + 0.02966i$
5	1	$-0.21413 - 0.44969i$	$0.07446 + 0.72464i$	$0.16553 - 0.57279i$	$0.22352 + 1.23748i$	$-0.08872 + 0.02966i$
6	1	$-0.22477 - 0.43023i$	$0.06454 + 0.56084i$	$0.18032 - 0.56761i$	$0.22930 + 1.37666i$	$-0.08872 + 0.02966i$
7	2	$-0.38066 + 1.28929i$	$-0.18674 - 0.42053i$	$0.33201 - 0.59239i$	$0.48476 - 1.21602i$	$-0.08872 - 0.02966i$
8	2	$-0.21413 + 0.44969i$	$0.07446 - 0.72464i$	$0.16553 + 0.57279i$	$0.22352 - 1.23748i$	$-0.08872 - 0.02966i$
9	2	$-0.22477 + 0.43023i$	$0.06454 - 0.56084i$	$0.18032 + 0.56761i$	$0.22930 - 1.37666i$	$-0.08872 - 0.02966i$

Table 1. The Bethe roots solved from the Bethe Ansatz equations (4.11)–(4.13) for $p = 3$, $N = 2$ and $\phi_k = 0$ with the inhomogeneity parameters $d_1^{(\pm)} = 2$, $f_1^{(\pm)} = 1/2$, $g_1^{(\pm)} = 3$, $h_1^{(\pm)} = 1/3$, $d_2^{(\pm)} = \sqrt{3}$, $f_2^{(\pm)} = 1/\sqrt{3}$, $g_2^{(\pm)} = \sqrt{2}$ and $h_2^{(\pm)} = 1/\sqrt{2}$.

$$\begin{aligned}
 & q^{-k} D^{(-)} + q^k F^{(-)} - (-1)^N e^{\phi_k - \frac{N}{2}\eta} \left\{ \frac{G^{(-)} H^{(+)}}{\{D^{(+)} F^{(+)}\}^{\frac{1}{2}}} + e^{-2\phi_k + N\eta} \frac{G^{(+)} H^{(-)}}{\{D^{(+)} F^{(+)}\}^{\frac{1}{2}}} \right\} \\
 &= c_k e^{-\sum_{j=1}^{(p-1)N} \lambda_j} \\
 & \times \left\{ \{D^{(-)}\}^p + \{F^{(-)}\}^p - e^{p\phi_k} \frac{\{G^{(-)} H^{(+)}\}^p}{\{D^{(+)} F^{(+)}\}^{\frac{p}{2}}} - e^{-p\phi_k} \frac{\{G^{(+)} H^{(-)}\}^p}{\{D^{(+)} F^{(+)}\}^{\frac{p}{2}}} \right\}. \quad (4.13)
 \end{aligned}$$

Here the constants $D^{(\pm)}$ and $F^{(\pm)}$ are given by (3.3) and $G^{(\pm)}$ and $H^{(\pm)}$ read

$$G^{(\pm)} = \prod_{n=1}^N g_n^{(\pm)}, \quad H^{(\pm)} = \prod_{n=1}^N h_n^{(\pm)}. \quad (4.14)$$

Notice that for a given ϕ_k , either (4.12) or (4.13) only serves as a selection rule (see the remarks in the end of appendix A).

It can be shown that the inhomogeneous $T - Q$ relation (4.7) does indeed satisfy (4.2)–(4.4) and (4.6) provided that the $(p-1)N + 1$ parameters c_k and $\{\lambda_j | j = 1, \dots, (p-1)N\}$ obey the BAEs (4.11)–(4.13). The proof is given in appendix A. Hence $\{\Lambda_k(u) | k = 1, \dots, p\}$ given by the $T - Q$ relation (4.7) become the eigenvalues of the transfer matrix $t(u)$ of the τ_2 -model with periodic boundary condition.

Numerical solutions of the Bethe Ansatz equations and exact diagonalization of the transfer matrix are performed for $p = 3$, $N = 2$ and $N = 3$ and arbitrarily chosen inhomogeneity parameters. The Bethe roots for given ϕ_k are shown in table 1 & 2 ($N = 2$) and table 3 & 4 ($N = 3$) respectively. The $\Lambda(u)$ curves calculated from exact diagonalization and from the $T - Q$ relation coincide exactly (figure 1 & 2), which imply that the inhomogeneous $T - Q$ relation does indeed give the complete and correct spectrum of the generic τ_2 transfer matrix.

With the help of the determinant representation (4.5), we can obtain the eigenvalues $\{\Lambda^{(j)}(u) | j = 1, \frac{3}{2}, \dots, \frac{p}{2}\}$ of the higher spin- j transfer matrices $\{t^{(j)}(u) | j = 1, \frac{3}{2}, \dots, \frac{p}{2}\}$.

n	k	λ_1	λ_2	λ_3	λ_4	c_k
1	0	$-1.11708 + 0.45660i$	$-0.21978 - 0.33183i$	$0.12531 + 0.21565i$	$0.14087 - 0.52391i$	$0.08911 + 0.01654i$
2	0	$-1.24428 + 1.38621i$	$-0.01821 - 0.44011i$	$0.04372 - 1.51711i$	$0.14809 + 0.38754i$	$0.08911 + 0.01654i$
3	0	$-1.32148 + 1.26038i$	$-0.00660 + 0.42108i$	$0.12388 - 0.38684i$	$0.13352 - 1.47809i$	$0.08911 + 0.01654i$
4	1	$-0.79677 - 1.04629i$	$-0.63157 + 0.66772i$	$0.09800 + 0.34342i$	$1.41424 + 1.03879i$	$-0.21364 + 0.11605i$
5	1	$-0.54458 - 0.56692i$	$-0.39125 + 1.02018i$	$0.18039 - 0.47610i$	$0.83934 + 1.02649i$	$-0.21364 + 0.11605i$
6	1	$-0.50863 - 0.52803i$	$-0.23822 + 1.02316i$	$0.21632 - 0.51790i$	$0.61444 + 1.02642i$	$-0.21364 + 0.11605i$
7	2	$-0.16368 - 0.27977i$	$-0.15197 - 1.49634i$	$0.56109 + 1.39587i$	$0.74134 - 0.43992i$	$-0.09374 - 0.03046i$
8	2	$-0.17493 + 0.27850i$	$0.03704 - 0.71000i$	$0.53931 + 0.65255i$	$0.58536 - 1.04121i$	$-0.09374 - 0.03046i$
9	2	$-0.18197 + 0.26803i$	$0.06268 - 0.56937i$	$0.53093 + 0.61846i$	$0.57514 - 1.13728i$	$-0.09374 - 0.03046i$

Table 2. The Bethe roots solved from the Bethe Ansatz equations (4.11)–(4.13) for $p = 3$, $N = 2$ and $\phi_k = 1$ with the inhomogeneity parameters $d_1^{(\pm)} = 2$, $f_1^{(\pm)} = 1/2$, $g_1^{(\pm)} = 3$, $h_1^{(\pm)} = 1/3$, $d_2^{(\pm)} = \sqrt{3}$, $f_2^{(\pm)} = 1/\sqrt{3}$, $g_2^{(\pm)} = \sqrt{2}$ and $h_2^{(\pm)} = 1/\sqrt{2}$.

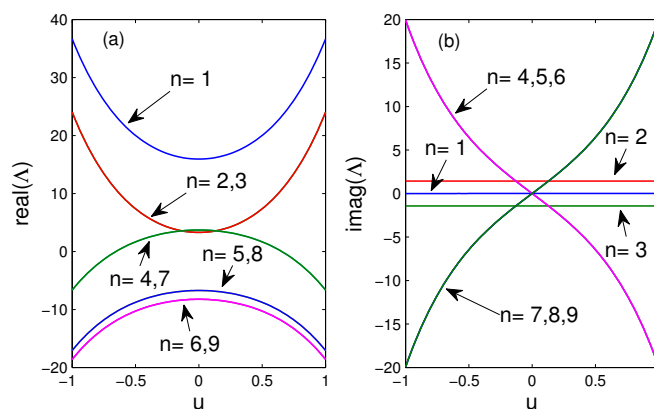


Figure 1. Real (a) and imaginary (b) parts of the eigenvalues $\Lambda(u)$ for $p = 3$, $N = 2$, $d_1^{(\pm)} = 2$, $f_1^{(\pm)} = 1/2$, $g_1^{(\pm)} = 3$, $h_1^{(\pm)} = 1/3$, $d_2^{(\pm)} = \sqrt{3}$, $f_2^{(\pm)} = 1/\sqrt{3}$, $g_2^{(\pm)} = \sqrt{2}$ and $h_2^{(\pm)} = 1/\sqrt{2}$. The curves calculated from exact diagonalization coincide with those derived from the inhomogeneous $T - Q$ relation.

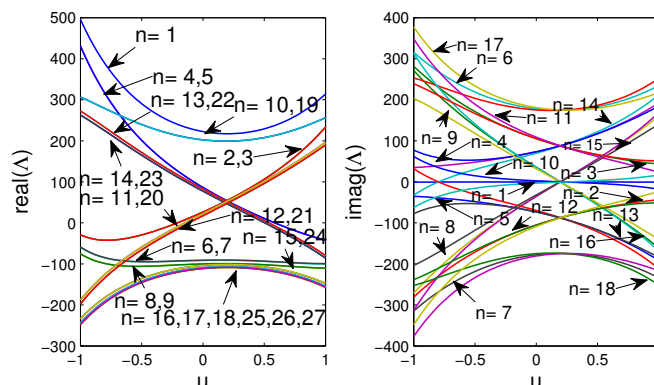


Figure 2. Real (a) and imaginary (b) parts of the eigenvalues $\Lambda(u)$ for $p = 3$, $N = 3$, $d_{1,2,3}^{(+)} = \{2, 0.2, 3\}$, $f_{1,2,3}^{(-)} = \{0.6, 4, 0.5\}$, $g_{1,2,3}^{(-)} = \{1, 0.4, 5\}$, $h_{1,2,3}^{(-)} = \{1.2, 2, 0.3\}$, $d_{1,2,3}^{(-)} = \{3, 1, 1.5\}$, $f_{1,2,3}^{(+)} = \{0.4, 0.8, 1\}$, $g_{1,2,3}^{(+)} = \{4, 0.1, 2\}$ and $h_{1,2,3}^{(+)} = \{0.3, 8, 0.75\}$. The curves calculated from exact diagonalization coincide with those derived from the inhomogeneous $T - Q$ relation.

n	k	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	c_k
1	0	-1.40457 - 0.54217i	-1.14643 + 0.48574i	0.21213 - 0.51244i	0.26215 + 0.52628i	2.23260 + 0.58946i	2.25229 - 0.54688i	0.11101 + 0.00000i
2	0	-1.37944 + 0.44821i	-1.08099 - 1.53477i	0.15169 - 0.57167i	0.25002 - 1.51348i	2.22412 + 0.58246i	2.24278 - 0.55234i	-0.11101 - 0.00000i
3	0	-1.34404 + 1.57012i	-1.20296 - 0.43961i	0.17315 + 0.54669i	0.31772 + 1.41399i	2.22188 + 0.59253i	2.24243 - 0.54213i	-0.11101 - 0.00000i
4	0	-1.40652 - 0.53734i	-1.16460 + 0.49706i	0.16933 - 0.47789i	0.21026 + 1.53395i	2.25354 + 0.54696i	2.34616 - 1.56275i	0.11101 - 0.00000i
5	0	-1.40978 - 0.54088i	-1.16556 + 0.47933i	0.19408 + 0.51898i	0.27686 - 1.43479i	2.21747 - 0.59576i	2.29509 - 1.56846i	-0.11101 + 0.00000i
6	0	-1.34697 + 1.55928i	-1.19166 - 0.45593i	0.16445 + 0.47419i	0.18455 - 0.54318i	2.25285 + 0.53773i	2.34494 + 1.56950i	-0.11101 - 0.00000i
7	0	-1.37026 + 0.45267i	-1.08588 - 1.50001i	0.16322 + 0.56858i	0.18792 - 0.51432i	2.21829 - 0.58781i	2.29488 - 1.56071i	-0.11101 - 0.00000i
8	0	-1.37070 + 0.44175i	-1.04295 - 1.51148i	-0.00426 - 1.54904i	0.21005 + 0.50411i	2.26182 + 0.54164i	2.35421 - 1.56857i	-0.11101 - 0.00000i
9	0	-1.33477 + 1.56165i	-1.18061 - 0.43623i	0.13757 + 1.54896i	0.25827 - 0.51762i	2.22542 - 0.59242i	2.30229 - 1.56434i	0.11101 - 0.00000i
10	1	-2.36717 - 0.64080i	-1.31215 + 0.72735i	0.21062 - 0.51316i	0.25655 + 0.52768i	1.95573 - 1.23307i	1.97213 + 0.77216i	0.15671 - 0.11901i
11	1	-2.25583 + 1.55193i	-1.41980 - 0.27051i	0.17387 + 0.54910i	0.31057 + 1.39847i	1.94548 - 1.22217i	1.96142 + 0.77494i	-0.15671 + 0.11901i
12	1	-2.26897 + 0.33840i	-1.32210 - 1.28008i	0.15296 - 0.57835i	0.24772 - 1.50723i	1.94115 - 1.23792i	1.96495 + 0.76376i	-0.15671 + 0.11901i
13	1	-2.38318 - 0.63525i	-1.32403 + 0.71167i	0.19433 + 0.51841i	0.27035 - 1.42332i	1.94420 + 0.71893i	2.01404 - 0.25029i	0.15671 - 0.11901i
14	1	-2.37051 - 0.62354i	-1.33281 + 0.72828i	0.17173 - 0.47490i	0.20767 + 1.53107i	1.97436 - 1.27414i	2.06528 - 0.24660i	0.15671 - 0.11901i
15	1	-2.24810 + 0.32841i	-1.31046 - 1.25036i	0.01507 - 1.55164i	0.20876 + 0.50544i	1.98173 - 1.28189i	2.06872 - 0.251140i	-0.15671 + 0.11901i
16	1	-2.24610 + 1.53155i	-1.40132 - 0.26168i	0.14240 + 1.55026i	0.25296 - 0.51903i	1.94771 + 0.72751i	2.02007 - 0.24686i	-0.15671 + 0.11901i
17	1	-2.26756 + 1.53313i	-1.40348 - 0.28154i	0.16668 + 0.47080i	0.18558 - 0.54463i	1.97161 - 1.28526i	2.06290 - 0.25235i	0.15671 - 0.11901i
18	1	-2.24957 + 0.35238i	-1.34102 - 1.25755i	0.16508 + 0.57440i	0.18837 - 0.51321i	1.93836 + 0.72705i	2.01449 - 0.24291i	0.15671 - 0.11901i
19	2	-1.96565 - 0.85298i	-1.71352 + 0.65345i	0.21076 - 0.51328i	0.25633 + 0.52691i	1.95534 - 0.73070i	1.97245 + 1.27644i	0.15671 + 0.11901i
20	2	-1.90791 + 0.17785i	-1.68438 - 1.40896i	0.15356 - 0.57754i	0.24708 - 1.50880i	1.94392 - 0.73195i	1.96344 + 1.26765i	-0.15671 - 0.11901i
21	2	-1.93248 + 1.31745i	-1.74683 - 0.32459i	0.17394 + 0.54847i	0.31135 + 1.40051i	1.94926 - 0.72122i	1.96047 + 1.28081i	-0.15671 - 0.11901i
22	2	-1.97499 - 0.84383i	-1.72780 + 0.65878i	0.17166 - 0.47433i	0.20771 + 1.53004i	1.97521 - 0.77013i	2.06392 + 0.25931i	0.15671 + 0.11901i
23	2	-1.97821 - 0.85630i	-1.72547 + 0.64443i	0.19443 + 0.51789i	0.26961 - 1.42148i	1.94102 + 1.22018i	2.01433 + 0.25512i	0.15671 + 0.11901i
24	2	-1.91801 + 1.30696i	-1.72979 - 0.32555i	0.14265 + 1.54942i	0.25294 - 0.51873i	1.94809 + 1.23029i	2.01983 + 0.25904i	-0.15671 - 0.11901i
25	2	-1.89125 + 0.17854i	-1.66717 - 1.39606i	0.01519 - 1.54823i	0.20867 + 0.50484i	1.98138 - 0.77737i	2.06890 + 0.25654i	-0.15671 - 0.11901i
26	2	-1.93408 + 1.29994i	-1.73940 - 0.33954i	0.16686 + 0.47060i	0.18568 - 0.54466i	1.97229 - 0.77985i	2.06435 + 0.25334i	0.15671 + 0.11901i
27	2	-1.90020 + 0.19277i	-1.68723 - 1.38868i	0.16492 + 0.57395i	0.18835 - 0.51320i	1.93588 + 1.23266i	2.01399 + 0.26233i	0.15671 + 0.11901i

Table 3. The Bethe roots solved from the Bethe Ansatz equations (4.11)–(4.13) for $p = 3$, $N = 3$ and $\phi_k = 0$ with the inhomogeneity parameters $d_{1,2,3}^{(+)} = \{2, 0, 2, 3\}$, $f_{1,2,3}^{(-)} = \{0, 6, 4, 0, 5\}$, $g_{1,2,3}^{(-)} = \{1, 0, 4, 5\}$, $h_{1,2,3}^{(-)} = \{1, 2, 2, 0, 3\}$, $d_{1,2,3}^{(+)} = \{3, 1, 1, 5\}$, $f_{1,2,3}^{(+)} = \{0, 4, 0, 8, 1\}$, $g_{1,2,3}^{(+)} = \{4, 0, 1, 2\}$ and $h_{1,2,3}^{(+)} = \{0, 3, 8, 0, 75\}$.

n	k	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	c_k
1	0	-1.18077 - 0.63093i	-0.77548 + 0.42106i	0.20506 - 0.48775i	0.41948 + 0.54567i	1.84319 - 0.63232i	2.12005 + 0.78427i	0.03770 - 0.00000i
2	0	-1.10529 + 0.34161i	-0.62163 - 1.46040i	0.05537 - 0.62831i	0.37145 - 1.52447i	1.82712 - 0.62376i	2.10450 + 0.75374i	-0.03770 - 0.00000i
3	0	-1.09702 - 1.53557i	-0.90813 - 0.34431i	0.08547 + 0.52810i	0.62553 + 1.18619i	1.84220 - 0.61528i	2.08348 + 0.78086i	0.03770 + 0.00000i
4	0	-1.18259 - 0.61106i	-0.80832 + 0.44954i	0.12775 - 0.40794i	0.16347 - 1.54513i	1.92326 + 0.29737i	2.40796 - 1.32436i	-0.03770 + 0.00000i
5	0	-1.19765 - 0.62056i	-0.80874 + 0.41520i	0.27139 + 0.55866i	0.41527 - 1.22888i	1.66113 + 1.39627i	2.29013 - 0.52074i	0.03770 + 0.00000i
6	0	-1.10378 - 1.56716i	-0.87367 - 0.36252i	0.10949 + 0.38463i	0.16998 - 0.54536i	1.91951 + 0.28868i	2.41000 - 1.33987i	-0.03770 - 0.00000i
7	0	-1.07093 + 0.34409i	-0.63932 - 1.35059i	0.16445 + 0.63300i	0.22628 - 0.53500i	1.67027 + 1.41723i	2.28078 - 0.50872i	0.03770 + 0.00000i
8	0	-1.08780 + 0.31120i	-0.46010 - 1.19085i	-0.36385 + 1.47511i	0.19124 + 0.44571i	1.92905 + 0.28943i	2.42300 - 1.33060i	0.03770 - 0.00000i
9	0	-1.06678 - 1.55890i	-0.87270 - 0.31728i	0.22250 + 1.47739i	0.37070 - 0.49410i	1.68121 + 1.39916i	2.29661 - 0.50628i	0.03770 + 0.00000i
10	1	-1.45639 - 0.68240i	-0.79897 + 0.53967i	0.20441 - 0.49025i	0.40545 + 0.55500i	1.67944 - 0.96068i	1.83770 + 0.82124i	0.03029 - 0.01164i
11	1	-1.33685 - 1.52407i	-0.96823 - 0.26543i	0.07943 + 0.53134i	0.60852 + 1.14853i	1.67359 - 0.92257i	1.81519 + 0.81477i	0.03029 - 0.01164i
12	1	-1.32102 + 0.27890i	-0.69844 - 1.31874i	0.04696 - 0.65235i	0.37397 - 1.51493i	1.63642 - 0.95097i	1.83377 + 0.79908i	-0.03029 + 0.01164i
13	1	-1.48351 - 0.67132i	-0.83065 + 0.52522i	0.27356 + 0.56560i	0.39269 - 1.21484i	1.54772 + 0.85431i	1.97184 - 0.27640i	0.03029 - 0.01164i
14	1	-1.46147 - 0.65184i	-0.83885 + 0.56232i	0.13313 - 0.40377i	0.16379 - 1.54201i	1.87621 - 0.08166i	1.99885 - 1.24207i	-0.03029 + 0.01164i
15	1	-1.28987 + 0.24571i	-0.62143 - 1.14415i	-0.29341 + 1.56827i	0.19063 + 0.44950i	1.88066 - 0.09019i	2.00508 - 1.24657i	0.03029 - 0.01164i
16	1	-1.30182 - 1.56123i	-0.93594 - 0.24033i	0.22228 + 1.48496i	0.35819 - 0.50515i	1.55627 + 0.87479i	1.97267 - 0.27048i	0.03029 - 0.01164i
17	1	-1.34976 - 1.56888i	-0.93296 - 0.28003i	0.11438 + 0.37805i	0.17059 - 0.54750i	1.87114 - 0.09041i	1.99826 - 1.25026i	0.03029 - 0.01164i
18	1	-1.27444 + 0.29316i	-0.74475 - 1.23106i	0.16793 + 0.65364i	0.22647 - 0.53682i	1.52925 + 0.87589i	1.96718 - 0.27224i	0.03029 - 0.01164i
19	2	-1.35404 - 0.79738i	-0.89775 + 0.51195i	0.20503 - 0.48909i	0.40262 + 0.55358i	1.63222 - 0.72436i	1.88358 + 1.16274i	0.03029 + 0.01164i
20	2	-1.23264 + 0.20519i	-0.79546 - 1.39083i	0.05445 - 0.64797i	0.36607 - 1.52397i	1.59919 - 0.70425i	1.88004 + 1.13766i	-0.03029 - 0.01164i
21	2	-1.28301 + 1.50079i	-1.03057 - 0.29809i	0.08365 + 0.52808i	0.60821 + 1.16322i	1.63587 - 0.68972i	1.85750 + 1.15474i	-0.03029 - 0.01164i
22	2	-1.36454 - 0.77340i	-0.93180 + 0.53133i	0.13333 - 0.40114i	0.16253 - 1.54377i	1.83391 + 0.10660i	2.03823 - 0.84378i	-0.03029 - 0.01164i
23	2	-1.37931 - 0.79345i	-0.92608 + 0.49850i	0.27348 + 0.56123i	0.38926 - 1.19929i	1.42181 + 1.05767i	2.09248 + 0.09277i	0.03029 + 0.01164i
24	2	-1.24340 + 1.47355i	-0.99698 - 0.27755i	0.22686 + 1.48133i	0.35590 - 0.50156i	1.44032 + 1.08390i	2.08896 + 0.09936i	-0.03029 - 0.01164i
25	2	-1.20219 + 0.18087i	-0.67017 - 1.27621i	-0.33153 - 1.52955i	0.18985 + 0.44636i	1.84149 + 0.09855i	2.04421 - 0.84418i	-0.03029 - 0.01164i
26	2	-1.28768 + 1.45716i	-0.99898 - 0.31718i	0.11502 + 0.37783i	0.17053 - 0.54662i	1.83151 + 0.09548i	2.04125 - 0.84925i	0.03029 + 0.01164i
27	2	-1.19567 + 0.22143i	-0.81573 - 1.31025i	0.16976 + 0.64979i	0.22613 - 0.53479i	1.40246 + 1.09819i	2.08470 + 0.09306i	0.03029 + 0.01164i

Table 4. The Bethe roots solved from the Bethe Ansatz equations (4.11)–(4.13) for $p = 3$, $N = 3$ and $\phi_k = 1$ with the inhomogeneity parameters $d_{1,2,3}^{(+)} = \{2, 0, 2, 3\}$, $f_{1,2,3}^{(-)} = \{0, 6, 4, 0, 5\}$, $g_{1,2,3}^{(-)} = \{1, 0, 4, 5\}$, $h_{1,2,3}^{(-)} = \{1, 2, 2, 0, 3\}$, $d_{1,2,3}^{(+)} = \{3, 1, 1, 5\}$, $f_{1,2,3}^{(+)} = \{0, 4, 0, 8, 1\}$, $g_{1,2,3}^{(+)} = \{4, 0, 1, 2\}$ and $h_{1,2,3}^{(+)} = \{0, 3, 8, 0, 75\}$.

4.2.2 Degenerate case

For generic inhomogeneous parameters $\{d_n^{(\pm)}, f_n^{(\pm)}, g_n^{(\pm)}, h_n^{(\pm)} | n = 1, \dots, N\}$ obeying the constraint (2.8), the inhomogeneous term in the $T - Q$ relation (4.7) does not vanish. In this subsection we consider some special case such that the inhomogeneous term vanishes. In this case, the inhomogeneous parameters $\{d_n^{(\pm)}, f_n^{(\pm)}, g_n^{(\pm)}, h_n^{(\pm)} | n = 1, \dots, N\}$ have to obey some further constraints besides (2.8) as follows:

$$e^{2M\eta} G^{(-)} H^{(+)} = (-1)^N F^{(+)} F^{(-)}, \quad (4.15)$$

or

$$e^{2M\eta} G^{(-)} H^{(+)} = (-1)^N D^{(+)} D^{(-)}, \quad (4.16)$$

and

$$F_k^{(p(N-2l))}(\{d_n^{(\pm)}, f_n^{(\pm)}, g_n^{(\pm)}, h_n^{(\pm)}\}) = 0, \quad l = 1, \dots, N-1. \quad (4.17)$$

Here $D^{(\pm)}$, $F^{(\pm)}$, $G^{(-)}$ and $H^{(+)}$ are given by (3.3) and (4.14), and each $F_k^{(p(N-2l))}$ (given in (A.2) below) is a polynomial of the inhomogeneous parameters $\{d_n^{(\pm)}, f_n^{(\pm)}, g_n^{(\pm)}, h_n^{(\pm)} | n = 1, \dots, N\}$. It is noted that in (4.15) (or (4.16)), M is some non-negative integer. The corresponding inhomogeneous $T - Q$ relation (4.7) then reduces to the conventional one [13]:

$$\Lambda_k(u) = e^{-(\frac{N}{2}-M+k)\eta} a(u) \left\{ \frac{D^{(+)}}{F^{(+)}} \right\}^{\frac{1}{2}} \frac{\bar{Q}(u-\eta)}{\bar{Q}(u)} + e^{(\frac{N}{2}-M+k)\eta} d(u) \left\{ \frac{F^{(+)}}{D^{(+)}} \right\}^{\frac{1}{2}} \frac{\bar{Q}(u+\eta)}{\bar{Q}(u)}, \quad (4.18)$$

or

$$\Lambda_k(u) = e^{-(\frac{N}{2}-M+k)\eta} a(u) \left\{ \frac{F^{(+)}}{D^{(+)}} \right\}^{\frac{1}{2}} \frac{\bar{Q}(u-\eta)}{\bar{Q}(u)} + e^{(\frac{N}{2}-M+k)\eta} d(u) \left\{ \frac{D^{(+)}}{F^{(+)}} \right\}^{\frac{1}{2}} \frac{\bar{Q}(u+\eta)}{\bar{Q}(u)}, \quad (4.19)$$

where the function $\bar{Q}(u)$ becomes [14–18, 32–39]

$$\bar{Q}(u) = \prod_{j=1}^M \sinh(u - \lambda_j). \quad (4.20)$$

The M parameters $\{\lambda_j | j = 1, \dots, M\}$ satisfy the associated BAEs

$$e^{-(N-2M+2k)\eta} \frac{D^{(+)} a(\lambda_j)}{F^{(+)} d(\lambda_j)} = -\frac{\bar{Q}(\lambda_j + \eta)}{\bar{Q}(\lambda_j - \eta)}, \quad j = 1, \dots, M, \quad (4.21)$$

or

$$e^{-(N-2M+2k)\eta} \frac{F^{(+)} a(\lambda_j)}{D^{(+)} d(\lambda_j)} = -\frac{\bar{Q}(\lambda_j + \eta)}{\bar{Q}(\lambda_j - \eta)}, \quad j = 1, \dots, M. \quad (4.22)$$

The proof is given in appendix B.

5 Conclusions

The most general cyclic representations of the quantum τ_2 -model (also known as Baxter-Bazhanov-Stroganov (BBS) model) with periodic boundary condition has been studied via the off-diagonal Bethe Ansatz method [18]. Based on the the truncation identity (3.20) of the fused transfer matrices, we construct the inhomogeneous $T - Q$ relation (4.7) of the eigenvalue of the fundamental transfer matrix $t(u)$ and the associated BAEs (4.11)–(4.13).

It should be noted that for generic inhomogeneity parameters $\{d_n^{(\pm)}, f_n^{(\pm)}, g_n^{(\pm)}, h_n^{(\pm)} | n = 1, \dots, N\}$ obeying the constraint (2.8), the inhomogeneous term (i.e., the third term) in the $T - Q$ relation (4.7) *does not* vanish, as long as one takes a polynomial Q function. However, if these inhomogeneity parameters satisfy the further constraints (4.15) and (4.17) (or (4.16) and (4.17)), the corresponding $T - Q$ relation reduces to the conventional one (4.18) (or (4.19)).

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A Proof of the $T - Q$ relation

In this appendix we prove that the inhomogeneous $T - Q$ relation (4.7) does satisfy (4.2)–(4.4) and (4.6) if the $(p - 1)N + 1$ parameters c_k and $\{\lambda_j | j = 1, \dots, (p - 1)N\}$ obey the BAEs (4.11)–(4.13).

From the construction (4.7) of the $T - Q$ relation and the definitions (4.8)–(4.10), one can easily check that the $T - Q$ relation satisfies the quasi-periodicity property (4.2). The BAEs (4.12)–(4.13) ensure that the asymptotic behavior (4.3) is also fulfilled. Moreover the BAEs (4.11) imply that the functions given by the $T - Q$ relation (4.7) actually satisfy (4.4). So far, the $T - Q$ relation already makes (4.2)–(4.4) satisfied.

Let us consider the function $F_k(u)$ given by (4.8). For generic inhomogeneity parameters $\{d_n^{(\pm)}, f_n^{(\pm)}, g_n^{(\pm)}, h_n^{(\pm)} | n = 1, \dots, N\}$ satisfying the constraint (2.8), we know that, as a function of e^u , $F_k(u)$ is a Laurent polynomial of degree pN with the form

$$\begin{aligned}
 F_k(u) &= \mathcal{A}(u) + \mathcal{D}(u) - e^{p\phi_k} \bar{\mathcal{A}}(u) - e^{-p\phi_k} \bar{\mathcal{D}}(u) \\
 &= F_k^{(pN)} e^{pNu} + F_k^{(pN-2)} e^{(pN-2)u} + \dots + F_k^{(-pN)} e^{-pNu} \\
 &= \mathcal{F}_k^{(0)} \prod_{j=1}^{pN} \left\{ \frac{e^u}{e^{z_j}} - \frac{e^{z_j}}{e^u} \right\}, \tag{A.1}
 \end{aligned}$$

where $\{z_j \bmod (2i\pi) | j = 1, \dots, pN\}$ are the zeros of $F_k(u)$ which are all different from each other and the constant $\mathcal{F}_k^{(0)}$ is related to the asymptotic behaviors of the function. The relations (3.14) and the definition (4.9) imply that the function $F_k(u)$ actually is a Laurent polynomial of e^{pu} with a degree N (i.e., there are only $N + 1$ non-vanishing coefficients in the expansion (A.1)), namely,

$$F_k(u) = \sum_{l=0}^N F_k^{(p(N-2l))}(\{d_n^{(\pm)}, f_n^{(\pm)}, g_n^{(\pm)}, h_n^{(\pm)}\}) e^{p(N-2l)u} \\ \stackrel{\text{def}}{=} \sum_{l=0}^N F_k^{(p(N-2l))} e^{p(N-2l)u}, \tag{A.2}$$

where the $N + 1$ non-vanishing coefficients $\{F_k^{(p(N-2l))} | l = 0, 1, \dots, N\}$ are polynomials of the inhomogeneity parameters $\{d_n^{(\pm)}, f_n^{(\pm)}, g_n^{(\pm)}, h_n^{(\pm)} | n = 1, \dots, N\}$. Moreover, it follows that

$$F_k(z_j) = F_k(z_j + m\eta) = 0, \quad m \in \mathbb{Z}. \tag{A.3}$$

Let us introduce the function $g(u)$ which is given by

$$g(u) = \Lambda_k^{(\frac{p}{2})}(u) - \mathcal{A}(u) - \mathcal{D}(u) - \delta \left(u - \left(\frac{p-1}{2} \right) \eta \right) \Lambda_k^{(\frac{p-2}{2})}(u), \tag{A.4}$$

where $\Lambda_k^{(\frac{p}{2})}(u)$ and $\Lambda_k^{(\frac{p-2}{2})}(u)$ are given by the determinant representation (4.5) with $\Lambda_k(u)$ given by (4.7). From the above definition, one knows that the function $g(u)$ as a function of e^u is a Laurent polynomial of degree pN of similar form as (A.1). Hence $g(u)$ is uniquely determined by its $pN + 1$ points values such as $+\infty$ (or $-\infty$) and $\{z_j | j = 1, \dots, pN\}$. Thanks to the property (A.3), we have

$$\Lambda_k(z_j + m\eta) = e^{\phi_k} a(z_j + m\eta) \frac{Q(z_j + m\eta - \eta)}{Q(z_j + m\eta)} + e^{-\phi_k} d(z_j + m\eta) \frac{Q(z_j + m\eta + \eta)}{Q(z_j + m\eta)}, \\ m \in \mathbb{Z}, \quad j = 1, \dots, pN. \tag{A.5}$$

Substituting the above relations into (4.5) and noting the fact $p\eta = 2i\pi$, after some tedious calculation, we have

$$\Lambda_k^{(\frac{p}{2})}(z_j) = e^{p\phi_k} \bar{\mathcal{A}}(z_j) + e^{-p\phi_k} \bar{\mathcal{D}}(z_j) + \delta \left(z_j - \left(\frac{p-1}{2} \right) \eta \right) \Lambda_k^{(\frac{p-2}{2})}(z_j) \\ = \mathcal{A}(z_j) + \mathcal{D}(z_j) + \delta \left(z_j - \left(\frac{p-1}{2} \right) \eta \right) \Lambda_k^{(\frac{p-2}{2})}(z_j), \quad j = 1, \dots, pN. \tag{A.6}$$

In deriving the second equality, we have used the fact: $F_k(z_j) = 0$. Then (A.6) implies that the function $g(u)$ vanishes at the points z_j , namely,

$$g(z_j) = 0, \quad j = 1, \dots, pN. \tag{A.7}$$

The BAEs (4.12)–(4.13) imply that the functions given by the $T - Q$ relation (4.7) also satisfy (4.3), which give rise to

$$\lim_{u \rightarrow \pm\infty} g(u) = 0. \tag{A.8}$$

(A.7)–(A.8) imply that $g(u) = 0$. Namely, the inhomogeneous $T - Q$ relation (4.7) does satisfy (4.6). Therefore we can conclude that $\{\Lambda_k(u)|k = 1, \dots, p\}$ given by the $T - Q$ relation (4.7) are the eigenvalues of the transfer matrix $t(u)$ of the τ_2 -model with periodic boundary condition provided that the $(p - 1)N + 1$ parameters c_k and $\{\lambda_j|j = 1, \dots, (p - 1)N\}$ satisfy the BAEs (4.11)–(4.13).

Some remarks are in order. Due to the fact that $g(u)$ given by (A.4) as a function of e^u is a Laurent polynomial of degree pN , the relations

$$g(u) = 0, \quad \text{when } u = z_1, \dots, z_{pN}, +\infty,$$

are already sufficient to ensure $g(u) = 0$. This implies that the BAEs (4.11)–(4.12) are sufficient to guarantee that the $T - Q$ relation (4.7) satisfy (4.2), (4.4), (4.6) and (4.3) with the $u \rightarrow +\infty$ limit. Then the BAE (4.13) only plays a role of the selection rule such that the $u \rightarrow -\infty$ behavior also matches.

B Proof of the degenerate case

In this appendix we show that the inhomogeneous $T - Q$ relation (4.7) does reduce to the conventional one (4.18) (or (4.19)) when the inhomogeneity parameters satisfy the constraints (4.15) and (4.17) or (4.16) and (4.17).

Suppose that the inhomogeneous $T - Q$ relation (4.7) can be reduced to the conventional one, namely,

$$\Lambda_k(u) = e^{\phi_k} a(u) \frac{\bar{Q}(u - \eta)}{\bar{Q}(u)} + e^{-\phi_k} d(u) \frac{\bar{Q}(u + \eta)}{\bar{Q}(u)}, \quad (\text{B.1})$$

where the Q -function is

$$\bar{Q}(u) = \prod_{j=1}^M \sinh(u - \lambda_j),$$

and M is a non-negative integer to be specified by (4.15) (or (4.16)). The asymptotic behavior (4.3) of $\Lambda_k(u)$ now becomes

$$q^k D^{(+)} + q^{-k} F^{(+)} - 2 \left\{ D^{(+)} F^{(+)} \right\}^{\frac{1}{2}} \cosh \left(\phi_k + \left(\frac{N}{2} - M \right) \eta \right) = 0, \quad (\text{B.2})$$

$$q^{-k} D^{(-)} + q^k F^{(-)} - (-1)^N \times \left\{ e^{\phi_k + (\frac{N}{2} + M)\eta} \frac{G^{(-)} H^{(+)}}{\{D^{(+)} F^{(+)}\}^{\frac{1}{2}}} + e^{-\phi_k - (\frac{N}{2} + M)\eta} \frac{G^{(+)} H^{(-)}}{\{D^{(+)} F^{(+)}\}^{\frac{1}{2}}} \right\} = 0. \quad (\text{B.3})$$

Only when the inhomogeneity parameters obey the constraint (4.15) or (4.16), there does exist a solution to the above two equations:

$$\begin{cases} e^{\phi_k} = e^{-(\frac{N}{2} - M + k)\eta} \left\{ \frac{D^{(+)}}{F^{(+)}} \right\}^{\frac{1}{2}}, & \text{under constraint (4.15),} \\ e^{\phi_k} = e^{-(\frac{N}{2} - M + k)\eta} \left\{ \frac{F^{(+)}}{D^{(+)}} \right\}^{\frac{1}{2}}, & \text{under constraint (4.16).} \end{cases} \quad (\text{B.4})$$

It is easy to check that both solutions give rise to $F_k^{(pN)} = F_k^{(-pN)} = 0$. Together with (4.17), we have that in each constrained case ((4.15) and (4.17) or (4.16) and (4.17)) the function $F_k(u)$ indeed vanishes, namely, $F_k(u) = 0$. Substituting the solution (B.4) into (B.1), we obtain the conventional $T - Q$ relation (4.18) or (4.19) respectively.

Substituting (4.18) into (4.5) and noting the fact $p\eta = 2i\pi$, after some tedious calculation, we have

$$\begin{aligned} \Lambda_k^{(\frac{p}{2})}(u) &= e^{p\phi_k} \bar{\mathcal{A}}(u) + e^{-p\phi_k} \bar{\mathcal{D}}(u) + \delta\left(u - \left(\frac{p-1}{2}\right)\eta\right) \Lambda_k^{(\frac{p-2}{2})}(u) \\ &= \mathcal{A}(u) + \mathcal{D}(u) + \delta\left(u - \left(\frac{p-1}{2}\right)\eta\right) \Lambda_k^{(\frac{p-2}{2})}(u). \end{aligned} \tag{B.5}$$

In deriving the second equality, we have used the fact: $F_k(u) = 0$ when the inhomogeneity parameters $\{d_n^{(\pm)}, f_n^{(\pm)}, g_n^{(\pm)}, h_n^{(\pm)} | n = 1, \dots, N\}$ satisfy the constraints (2.8), (4.15) and (4.17). Similarly we can prove that the reduced $T - Q$ relation (4.19) satisfies (4.3) and (4.6) provided that the inhomogeneity parameters obey the constraints (2.8), (4.16) and (4.17).

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