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On the singlet projector and the monodromy relation for $\mathrm{psu}(2,2|4)$ spin chains and reduction to subsectors

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Abstract: As a step toward uncovering the relation between the weak and the strong coupling regimes of the $\mathcal{N} = 4$ super Yang-Mills theory beyond the spectral level, we have developed in a previous paper [arXiv:1410.8533] a novel group theoretic interpretation of the Wick contraction of the fields, which allowed us to compute a much more general class of three-point functions in the SU(2) sector, as in the case of strong coupling [arXiv:1312.3727], directly in terms of the determinant representation of the partial domain wall partition function. Furthermore, we derived a non-trivial identity for the three point functions with monodromy operators inserted, being the discrete counterpart of the global monodromy condition which played such a crucial role in the computation at strong coupling. In this companion paper, we shall extend our study to the entire $\mathrm{psu}(2,2|4)$ sector and obtain several important generalizations. They include in particular (i) the manifestly conformally covariant construction, from the basic principle, of the singlet-projection operator for performing the Wick contraction and (ii) the derivation of the monodromy relation for the case of the so-called "harmonic R-matrix", as well as for the usual fundamental R-matrtix. The former case, which is new and has features rather different from the latter, is expected to have important applications. We also describe how the form of the monodromy relation is modified as $\mathrm{psu}(2,2|4)$ is reduced to its subsectors.

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Contents

1 Introduction

Although the idea and the use of AdS/CFT [1–3] have been expanded into enormous varieties of directions, still the understanding of the essence of the dynamical mechanism of this remarkable duality remains as one of the most important unsolved problems. For such a fundamental task, it is best to study deeply the prototypical example which has been studied most vigorously, namely the duality between the $\mathcal{N} = 4$ super Yang-Mills theory in 4 dimensions and the type IIB string theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$.

In this context, undoubtedly the pivotal work which pointed the way to analyze this strong/weak duality was the discovery of Minahan and Zarembo [4] that the one-loop dilatation operator for the gauge invariant composite operators made out of six scalars of super Yang-Mills theory takes exactly the form of the Hamiltonian of the integrable SO(6) spin chain system. This made (a part of) the integrability structure of the super Yang-Mills theory quite explicit, at least as far as the spectrum of the composite operators are concerned. Two years later, the power and the relevance of integrability was revealed also on the strong coupling side by the works [5, 6]. These works opened up the way to use integrability as an extremely powerful tool, not relying on the structure of supersymmetry and suitable for studying the dynamical aspects. Various results obtained along this way up to around 2010 are summarized in the review [7].

Subsequent developments can be classified into several categories. One is the more sophisticated way of computing the spectrum of composite operators, even at finite coupling. The reader should be referred to the most advanced approach [8, 9] and the references therein. Advancements in another category are the computation and understanding of other observables, such as the scattering amplitudes (for reviews, see [10–13]) and the Wilson loops (see for example [14, 15] and references therein).

In the realm of the gauge-invariant composite operators, properties beyond the spectrum level have been vigorously pursued. In particular, study of the three-point functions, the main interest behind the present work, have been advanced using the power of "integrability". Below let us briefly summarize the highlights of the recent developments in this category both at weak and strong couplings.

At weak coupling, a systematic procedure called "tailoring" has been developed [16–20], which essentially reduces the computations to those of certain scalar products of spin chain states. To actually bring them into a tractable form a technical improvement was needed [21], and then a special class of three-point functions for non-BPS operators have been expressed explicitly in terms of Slavnov determinants [22]. Furthermore, the semi-classical limit of such three-point functions with large charges were successfully evaluated in a remarkably compact form [18, 23–25].

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On the other hand, at strong coupling, the lack of the method for quantization of strings in curved space-time only permits one to deal with the semi-classical limit of large charges. Even in such a situation, the lack of the knowledge of the appropriate vertex operators and of the saddle point configuration made the complete computation quite difficult [27, 28]. Finally, applying the idea of the state-operator correspondence and with the use of the finite-gap integration method, the computation was rendered possible for the GKP string [29] and for the more interesting case of three-point functions of rather general class in the SU(2) sector, which includes the very special ones computed by the tailoring technique at weak coupling [30].

Now in order to understand the connection between the weak and the strong coupling regimes, it is important to find the common structures in the two regimes. Since the super Yang-Mills theory at weak coupling is completely well-defined, it would be better to start by re-analyzing the three-point functions on this side of the duality.

In a previous communication [31], we have developed two novel viewpoints and applied them explicitly to a class of three point functions in the SU(2) sector which are much more general than had been treated by the tailoring procedure. Let us summarize these two ideas as (I) and (II) below:

(I) One is the group theoretic reinterpretation of the Wick contraction of basic fields as a singlet projection of a tensor product of two fields.

  (Ia) In the case of the SU(2) sector, one can apply this idea with respect to the more refined $\mathrm{SU}(2)_L\times$ $\mathrm{SU}(2)_R$ structure present in that sector. This feature can be succinctly referred to as "double spin-chain" and it leads to the factorization of the left and the right sector and simplifies various formulas. This formalism allowed us to study a class of three-point functions of operators built upon more general spin-chain vacua than the special configuration discussed so far in the literature. This formulation makes the correspondence with the strong coupling computation done in [30] quite apparent.

  (Ib) Another conspicuous advantage of the new interpretation of the Wick contraction is that one can avoid the scalar products of off-shell states which appear in the tailoring prescription. Emergence of such an object required the trick [21] to turn one of the off-shell states into an on-shell state in order to write it in terms of the Slavnov determinant. In contrast, in our formulation one can directly obtain the expression in terms of the partial domain wall partition functions, and for a certain class of correlators it can be readily expressed as a determinant even for two off-shell states.

(II) The second new idea formulated explicitly in [31] is so-called the monodromy relation, which can be obtained by inserting the monodromy matrices Ω's inside the two-point or the three-point functions and using the unitarity and the crossing relations. This produces a relation between correlation functions of different operators and hence acts like the Schwinger-Dyson equation. In particular, in the special limit where the spectral parameter u goes to ∞, it reduces to the Ward identity for $\mathrm{SU}(2)_{\mathrm{L}}$ and

$\mathrm{SU}(2)_{\mathrm{R}}$. Moreover, for the three-point function in the semiclassical limit, where the operators carry large quantum numbers, it takes the classical relation $\Omega_1\Omega_2\Omega_3 = 1$, which is precisely of the form of the monodromy relation that follows from the integrability of the string theory, which played such a crucial role in the computation at strong coupling [27, 30]. Thus, its super Yang-Mills counterpart should also be considered as a major part of the concept of "integrability" beyond the spectral level.

Now the main purpose of the present paper is to extend these two main ideas explicitly to the full psu(2,2|4) sector and discuss how various general formulas are modified and simplified when we reduce them to the various subsectors. As for the monodromy relation (II), for the sake of clarity of presentation, we shall mainly concentrate on the case of the two-point functions. However, the extension to the three-point functions of our interest is straightforward, as was demonstrated in the case of SU(2) sector in [31], and the form of the result will also be briefly presented.

As far as the basic ideas (I) and (II) sketched above are concerned, similar ideas on the Wick contraction and the monodromy relations have also been discussed independently by [32].[1] Their work was based largely on the work by [33], which observed and utilized certain similarity of the Wick-contracting operator to the string field theory (SFT) vertex in the spirit of the string bit formulation. In this fashion, the work of [32] discussed already the full $psu(2,2|4)$ sector making use of the similar vertex, as well as the same oscillator representations and some associated basic formulas, as [33].

However, as far as the result (I) for the $psu(2,2|4)$ is concerned, the exponential form of the vertex written down by [33] was guessed by an analogy with the delta-function overlap in SFT and unfortunately was not $psu(2,2|4)$ singlet. The work of [32], which was based on [33], modified certain parts of the exponent and checked that it is a singlet projector a posteriori. However, there are two points that one wishes to improve on.

One is the understanding of why the singlet projector is of a simple exponential form, which was assumed in the work of [33] and hence in [32]. Such a form may be natural as the oscillator description of the δ-function overlap familiar in SFT context is indeed exponential. However, the analogy should be taken with care. For one thing the discrete indices of the oscillators for the string case are the Fourier mode numbers, whereas the similar indices in the super Yang-Mills case designate the location along the spin chain. Furthermore, in the case of the string the parts to be identified are rather homogeneous and hence it is natural to employ the (oscillator representation of) a delta-function to connect them. On the other hand, in the case of the $psu(2,2|4)$ spin chain the adjacent "string bits" can be quite different and the analogy to the delta function overlap is not intuitively obvious. In fact it is a simple exercise to construct the singlet state in the case of the spin j representation of SU(2) and confirm that it does *not* take an exponential form.

Therefore the surest way to obtain the desired vertex which effects the Wick contraction is to construct the most general singlet projector systematically in the space of tensor product of two spin chains. We shall show that the singlet projector exists for $su(2,2|4)$

[1]They did not discuss, however, the explicit advantage described in (Ia) and (Ib) gained by the new interpretation of the Wick contraction.

(as well as its restriction psu$(2,2|4)$) but not for u$(2,2|4)$ and, strictly speaking, for each sector of the representation of the su$(2,2|4)$ with a definite central charge C, the singlet vertex is not of a simple exponential form. However, provided that one is interested only in a sector with one definite value of the central charge,[2] one is allowed to use the simple exponential form, which is much more tractable. (We shall further elaborate on this later.) In this connection, we shall also explain in an appendix how the simple non-exponential singlet projector constructed for the SU(2) subsector in our previous work can be obtained from the general exponential projector for psu$(2,2|4)$.

The second point is that one wishes to improve the situation that the singlet projector of [32] is not manifestly conformally invariant, which is not useful for the treatment of the computation of the correlation functions of the local composite operators. In the present work we shall construct the version of the singlet projector which is manifestly conformally invariant and hence much simpler to use.

To explain what we mean by this, it is instructive to recall the following basic facts. In constructing the representations of the superalgebra u$(2,2|4)$, there are basically two different schemes, depending on which maximal bosonic subgroups of the supergroup U$(2,2|4)$ to make use of:

$$(E): \quad \mathrm{U}(2,2|4) \supset \mathrm{U}(1)_E \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{SU}(4)\,, \tag{1.1}$$
$$(D): \quad \mathrm{U}(2,2|4) \supset \mathrm{U}(1)_D \times \mathrm{SL}(2,C) \times \overline{\mathrm{SL}(2,C)} \times \mathrm{SU}(4) \tag{1.2}$$

Their difference resides in the choice of the subgroups in the SO$(4,2)$ part. The one, which we shall call E-scheme, makes use of the compact subgroups with the AdS energy E being diagonal. On the other hand, in the scheme to be called D-scheme, the dilatation generator D is diagonal and the rest of the subgroup chosen in SO$(4,2)$ is the non-compact Lorentz group SL$(2,C) \times \overline{\mathrm{SL}(2,C)}$. Therefore, the D-scheme is manifestly conformal covariant. It is well-known and fully discussed in [34] that these two schemes are connected by a non-unitary similarity transformation generated by the operator $U = e^{(\pi/4)(P_0 - K_0)}$ such that $U^{-1}DU = iE$.

In the treatment of [33], and hence [32], the oscillators appropriate for the E-scheme are used as basic building blocks for the generators of u$(2,2|4)$ and the relevant vertex operators. Since the D-scheme is more natural for the main purpose of computing the correlation functions of the basic super Yang-Mills fields, they transformed various quantities to that scheme by the similarity transformation using the operator U. However, since U does not map an individual component group, such as SU$(2)_\mathrm{L}$, in the E-scheme to a definite component group, such as SL(2,C), in the D-scheme,[3] the mapping does not make the description manifestly conformally covariant.

In our construction, to be described fully in section 2, we will stick to the D-scheme throughout, by using the oscillators which transform covariantly under the maximal subgroups shown in (1.2). This will make the entire description quite transparent without the need of the operator U.

[2]This is the case for the $\mathcal{N}=4$ super Yang-Mills, since the basic fields all carry $C=0$.

[3]Obviously, any similarity transformation, unitary or non-unitary, does not change the structure of the group.

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Let us next turn to the psu$(2,2|4)$ version of the monodromy relation (II). There are two natural types of monodromy matrices depending on the choice of the auxiliary space. One is the simpler and the fundamental one, for which the auxiliary space is taken to be $\mathbb{C}^{4|4}$. For this case, the derivation of the monodromy relation is a straightforward extension of the one for the SU(2) sector given in our previous work [31] and agrees with the description given in [32]. Another type is the monodromy relation associated with the so-called harmonic R-matrix, for which the structure of the auxiliary space is the same as that of the physical quantum space [35–38]. This case may be useful for obtaining local conserved quantities as well as for the study of scattering amplitudes [39–46]. In the present work, we shall derive the monodromy relation for this more complicated case as well, which was not discussed in [32]. As was demonstrated in [31], the monodromy relation for the three-point functions can be straightforwardly derived once that for the two-point functions is established, in this article we shall concentrate on the case of two-point functions.

Now the monodromy relations for the entire psu$(2,2|4)$ sector is practically too complicated to analyze at present. In this sense, it is of interest to look first at such relations for simpler subsectors. This has already been done for the SU(2) sector in our previous work. In the present work, we shall first sketch how this result can be rederived by the reduction of the relations for the psu$(2,2|4)$ sector and then apply similar techniques to the non-compact SL(2) subsector with much more detailed expositions. Under such reductions we shall see that certain non-trivial shifts in the spectral parameters are produced. It should be stated that all the discussions in this paper are at the tree level. It will be an important future task to extend some of the basic concepts to the loop level.

Having explained the essence of the new findings of the present work, let us briefly summarize the organization of the rest of this article.

In section 2, we start with a review (section 2.1), where we present the representation of the generators of u$(2,2|4)$ in terms of the oscillators, which transform covariantly under the maximal subgroups $\mathrm{U}(1)_{\mathrm{D}} \times \mathrm{SL}(2,\mathrm{C}) \times \overline{\mathrm{SL}(2,\mathrm{C})}$ of SO(4, 2) (the D-scheme choice discussed above) and $\mathrm{U}(1)_{\mathrm{J}} \times \mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}}$ of SO(6) respectively. With this set-up, in section 2.2 we solve the conditions for the most general singlet state in the tensor product of two Hilbert spaces. This gives a state the form of which is not quite an exponential in the tensor product of oscillators. We shall then explain that nevertheless for the application to the super Yang-Mills fields with $C = 0$, one can promote it to a simple exponential form. As a check, we compute the relevant 2-point functions of basic super Yang-Mills fields using this singlet state. (The demonstration that it reduces to a simple non-exponential form for the SU(2) subsector obtained in our previous work will be given in appendix B.)

In section 3, we derive explicitly the formulas for the monodromy relations for the correlation functions in the psu$(2,2|4)$ spin chain systems, first in the case of the fundamental R-matrix and then in the case of the harmonic R-matrix, which is more involved (some of the details are relegated to appendix C.).

In section 4, we explain how the monodromy relations for the psu$(2,2|4)$ can be reduced to the ones for the subsectors. In particular, we study the case of the compact SU(2) subsector and the non-compact SL(2) subsector and see that the reduction produces certain shifts in the spectral parameters.

In the final discussion section (section 5), we shall summarize the essential ideas and methods employed to obtain the new results in this work and discuss how they should be utilized to try to capture the principles through which to relate the super Yang-Mills theory and the string theory in AdS spaces.

As already indicated, three appendices (including appendix A where we list all the generators of $u(2,2|4)$ in the D-scheme notation for convenience) are provided to supplement the discussions given in the main text.

2 Conformally covariant oscillator description of psu(2, 2|4) and the singlet projector for the contraction of basic fields

We begin by constructing the singlet projector for the full $psu(2,2|4)$ sector from the first principle, with which one can efficiently perform the Wick contraction between the basic fields of the $\mathcal{N}=4$ super Yang-Mills fields. As already emphasized in our previous work [31], the use of this object is quite natural and versatile in computing fairly general class of correlation functions of gauge-invariant composite operators made out of SYM fields, at least at the tree level and possibly at the higher loop levels.

2.1 Oscillator representation of the generators of u(2,2|4) in the D-scheme

In the case of the SU(2) subsector discussed in our previous paper, the construction of the singlet projector was nothing but the elementary problem of forming a singlet state out of two spin 1/2 particles, once we regard the SU(2) spin chain as a double-chain associated with the two distinct SU(2) groups,[4] $\mathrm{SU}(2)_{\mathrm{L}}$ and $\mathrm{SU}(2)_{\mathrm{R}}$, acting on the chain. In the case of the full $psu(2,2|4)$ spin chain, however, the structure of the algebra and its representation are sufficiently involved to render the general construction non-trivial. Luckily, as our aim is to be able to perform the Wick contraction of only the basic SYM fields, we may restrict ourselves to the singleton representation, which can be realized by a minimal set of oscillators [47, 48]. However, before introducing the oscillators, we must recall that there are basically two different bases for the representations of the $u(2,2|4)$ algebra, depending on which maximal subgroups of the conformal group SO(2,4) are used and the properties of the oscillators depend on such bases. Let us describe and compare them in some detail below following [33].

2.1.1 E-scheme and the D-scheme

As already mentioned in the introduction, we shall call these two schemes E-scheme and D-scheme, where "E" and "D" stand for the energy and the dilatation respectively, for which the subgroups taken are shown below:

$$(\mathrm{E}): \quad \mathrm{SO}(2,4) \supset \mathrm{SO}(2)_E \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \,, \tag{2.1}$$

$$(\mathrm{D}): \quad \mathrm{SO}(2,4) \supset \mathrm{SO}(1,1)_D \times \mathrm{SL}(2,\mathbb{C}) \times \overline{\mathrm{SL}(2,\mathbb{C})} \,. \tag{2.2}$$

[4] It should be clear that these two groups, belonging to SU(4), are quite different from the $\mathrm{SU}(2)_{\mathrm{L}}\times\mathrm{SU}(2)_{\mathrm{R}}$ groups which will appear as a part of the maximal subgroups of SO(2,4) in the E-scheme described below in (2.1).

For the E-scheme, the maximal subgroups are all compact, including the $SO(2)_E$ factor, the eigenvalue of which is identified with the AdS energy.[5] Thus, in the context of AdS/CFT, this scheme is useful in describing the states and their spectra on the gravity/string side. On the other hand, for the D-scheme, the maximal subgroups are all non-compact, consisting of the dilatation and the Lorentz groups. As the interpretation of SO(2,4) as the conformal group in four dimensions is manifest in this scheme, D-scheme is more natural in discussing the correlation functions in the SYM theory. Accordingly, the set of oscillators used in these two schemes are different, each set transforming covariantly under the respective maximal subgroups.

Before introducing them and discussing their difference, it is useful to first recall how the SO(2,4) algebra and its representations are described according to these two schemes. From the point of view of the conformal algebra in four dimensions, the commutation relations of the generators of SO(2,4) are given by

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\mu\sigma}M_{\rho\nu} - \eta_{\nu\sigma}M_{\rho\mu})\,, \tag{2.3}$$
$$[M_{\mu\nu}, P_\rho] = -i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu)\,, \tag{2.4}$$
$$[M_{\mu\nu}, K_\rho] = -i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu)\,, \tag{2.5}$$
$$[D, M_{\mu\nu}] = [P_\mu, P_\nu] = [K_\mu, K_\nu] = 0\,, \tag{2.6}$$
$$[-iD, P_\mu] = P_\mu\,,\ [-iD, K_\mu] = -K_\mu\,, \tag{2.7}$$
$$[P_\mu, K_\nu] = 2i(\eta_{\mu\nu}D + M_{\mu\nu})\,, \tag{2.8}$$

where $\mu\,,\nu = 0, 1, 2, 3$ and the metric signature is taken to be $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. $M_{\mu\nu}$, P_μ, K_μ and D are, respectively, the Lorentz, the momentum, the special conformal and the dilatation generators. This set of commutation relations can be expressed more compactly as

$$[J_{KL}, J_{MN}] = -i(\eta_{KM}J_{LN} - \eta_{LM}J_{KN} + \eta_{KN}J_{ML} - \eta_{LN}J_{MK})\,, \tag{2.9}$$
$$J_{\mu\nu} := M_{\mu\nu}\,,\ J_{\mu-1} := \frac{1}{2}(P_\mu + K_\mu)\,,\ J_{\mu4} := \frac{1}{2}(P_\mu - K_\mu)\,,\ J_{-14} := D\,, \tag{2.10}$$

for which the structure of SO(2,4) is manifest. In this representation, the range of six-dimensional indices and the metric are taken to be $M, N = -1, 0, 1\cdots, 3, 4$ and $\eta_{MN} := \mathrm{diag}(-1, -1, 1, 1, 1, 1)$.

Now consider this algebra from the point of view of the E-scheme. It is easy to find that the generators of the compact maximal subgroups $U(1)_E$, $SU(2)_L$ and $SU(2)_R$ are given respectively by

$$E := J_{0-1} = \frac{1}{2}(P_0 + K_0)\,, \tag{2.11}$$
$$L_m := \frac{1}{2}\left(\frac{1}{2}\epsilon_{mnl}M_{nl} + M_{m4}\right)\,, \tag{2.12}$$
$$R_m := \frac{1}{2}\left(\frac{1}{2}\epsilon_{mnl}M_{nl} - M_{m4}\right)\,. \tag{2.13}$$

[5] Strictly speaking, one considers its universal cover.

where $m, n, l = 1, 2, 3$. Obviously, L_m and R_m commute with E and hence carry zero energy. The rest of the generators of SO(2,4) carry either positive or negative energy and thus the generators of the entire algebra are decomposed in the following fashion:

$$\begin{aligned} &\mathrm{so}(2,4) = \mathcal{E}^+ \oplus \mathcal{E}^0 \oplus \mathcal{E}^- \,, \\ &[E, \mathcal{E}^\pm] = \pm\mathcal{E}^\pm \,, \quad [E, \mathcal{E}^0] = 0 \,, \quad [\mathcal{E}^0, \mathcal{E}^\pm] \subset \pm\mathcal{E}^\pm \,, \quad [\mathcal{E}^+, \mathcal{E}^-] \subset \mathcal{E}^0 \,. \end{aligned} \tag{2.14}$$

Let $|e, j_L, j_R\rangle$ be the simultaneous eigenstate of the energy E and the third components L_3 and R_3 of $\mathrm{SU}(2)_L$ and $\mathrm{SU}(2)_R$ respectively with the eigenvalues denoted by e, j_L and j_R. Namely

$$E|e, j_L, j_R\rangle = e|e, j_L, j_R\rangle \,, \quad L_3|e, j_L, j_R\rangle = j_L|e, j_L, j_R\rangle \,, \quad R_3|e, j_L, j_R\rangle = j_R|e, j_L, j_R\rangle \,. \tag{2.15}$$

The physically relevant unitary positive energy representations are built upon the lowest weight state among the set $\{\ |e, j_L, j_R\rangle\}$, which is annihilated by all the energy-lowering generators belonging to $\mathcal{E}^-$. We denote them by L^{ij}, where the i and j are actually the spinor indices of $\mathrm{SU}(2)_L$ and $\mathrm{SU}(2)_R$ respectively and run from 1 to 2. Therefore we have four annihilation operators in total and the lowest weight state is characterized by

$$L^{ij}|e, j_L, j_R\rangle = 0 \,. \tag{2.16}$$

By acting onto this vacuum the four raising operators belonging to $\mathcal{E}^+$, which we denote by L_{ij}, one obtains unitary representations in the E-scheme.

Next consider the algebra $\mathrm{so}(2,4)$ from the D-scheme point of view. In this scheme, the generators of the maximal subgroups are given by

$$D = J_{-14} \,, \tag{2.17}$$

$$\mathcal{M}_m := \frac{1}{2}\left(\frac{1}{2}\epsilon_{mnl}M_{nl} + iM_{0m}\right) , \tag{2.18}$$

$$\mathcal{N}_m := \frac{1}{2}\left(\frac{1}{2}\epsilon_{mnl}M_{nl} - iM_{0m}\right) . \tag{2.19}$$

Here, $\mathcal{M}_m$ and $\mathcal{N}_m$ denote the generators of the Lorentz group $\mathrm{SL}(2,\mathbb{C}) \times \overline{\mathrm{SL}(2,\mathbb{C})}$. In this scheme, as is apparent from the commutation relations (2.7), P_μ and K_μ are, respectively the raising or lowering operators. Hence, the decomposition of the conformal algebra $\mathrm{so}(2,4)$ is of the structure

$$\begin{aligned} &\mathrm{so}(2,4) = \mathcal{D}^+ \oplus \mathcal{D}^0 \oplus \mathcal{D}^- \,, \\ &[-iD, \mathcal{D}^\pm] = \pm\mathcal{D}^\pm \,,\ [-iD, \mathcal{D}^0] = 0 \,,\ [\mathcal{D}^0, \mathcal{D}^\pm] \subset \pm\mathcal{D}^\pm \,,\ [\mathcal{D}^+, \mathcal{D}^-] \subset \mathcal{D}^0 \,, \end{aligned} \tag{2.20}$$

where $P_\mu \in \mathcal{D}^+$, $K_\mu \in \mathcal{D}^-$ and $D, \mathcal{M}_m, \mathcal{N}_m \in \mathcal{D}^0$.

From the point of view of CFT in four dimensions, which is directly expressed in the D-scheme, the multiplets of operators are built upon the conformal primaries placed at the origin $x^\mu = 0$. They carry definite dilatation charges, belong to the definite Lorentz representations, and are annihilated by the lowering operators K_μ. Using the state-operator correspondence, such a primary state, denoted by $|\Delta, j_{\mathcal{M}}, \bar{j}_{\mathcal{N}}\rangle$ with Δ and $(j_{\mathcal{M}}, \bar{j}_{\mathcal{N}})$ being the dilatation charge and the Lorentz spins, is characterized by

$$\begin{aligned} -iD|\Delta, j_{\mathcal{M}}, j_{\mathcal{N}}\rangle &= \Delta|\Delta, j_{\mathcal{M}}, j_{\mathcal{N}}\rangle\,, \quad K_\mu|\Delta, j_{\mathcal{M}}, j_{\mathcal{N}}\rangle = 0\,, \\ \mathcal{M}_3|\Delta, j_{\mathcal{M}}, j_{\mathcal{N}}\rangle &= j_{\mathcal{M}}|\Delta, j_{\mathcal{M}}, j_{\mathcal{N}}\rangle\,, \quad \mathcal{N}_3|\Delta, j_{\mathcal{M}}, j_{\mathcal{N}}\rangle = j_{\mathcal{N}}|\Delta, j_{\mathcal{M}}, j_{\mathcal{N}}\rangle\,. \end{aligned} \tag{2.21}$$

Then the module is built up by the descendants generated by the multiplicative actions of the raising operators P_μ. It should be emphasized that such a representation relevant for discussing the correlation functions is non-unitary, since the anti-hermitian operator $-iD$ has real eigenvalues.

Now let us give a brief description of the relation between the E-scheme used in [32, 33] and the D-scheme to be employed exclusively in this work. It is well-known by the work of [34] that there exists a non-unitary similarity transformation between the generators of these two schemes. The correspondence between $E = J_{-1,0}$ and $-iD = -iJ_{-1,4}$ indicates that such a transformation should rotate the non-compact 0-th direction into the compact 4-th direction and indeed it is effected by the operator

$$U = \exp\left(\frac{\pi}{2}M_{04}\right) = \exp\left(\frac{\pi}{4}(P_0 - K_0)\right)\,. \tag{2.22}$$

Explicit transformations are given by

$$U^{-1}(-iD)U = E\,, \quad U^{-1}L_mU = \mathcal{M}_m\,, \quad U^{-1}R_mU = \mathcal{N}_m\,, \tag{2.23}$$

$$U^{-1}P_\mu U \in \mathcal{E}^+\,, \quad U^{-1}K_\mu U \in \mathcal{E}^-\,, \tag{2.24}$$

$$|\Delta, j_{\mathcal{M}}, j_{\mathcal{N}}\rangle = U|e, j_L, j_R\rangle\,, \quad \text{with } \Delta = e\,,\ j_{\mathcal{M}} = j_L\,,\ j_{\mathcal{N}} = j_R\,. \tag{2.25}$$

As already mentioned, for the purpose of discussing the CFT correlation functions, the D-scheme is much more natural and if one starts from the E-scheme description as in [32, 33], one must necessarily manipulate with the operator U in the intermediate step. Also, in the oscillator representations of the generators, to be elaborated below, the D-scheme oscillators always keep the conformal covariance manifest as opposed to those in the E-scheme. We shall make the comparison more explicit later.

2.1.2 Oscillator representation in the D-scheme

Having argued the advantage of the D-scheme for our purpose, let us now introduce appropriate oscillators for this scheme and express the generators as their quadratic combinations.

For this purpose, it is useful to rewrite first the generators of the SO(2,4) algebra using the dotted and the undotted spinor indices of the Lorentz group. We will adopt the following conventions for the conversions of vectors and the tensors:

$$P_{\alpha\dot\beta} := -\frac{1}{2}(\sigma^\mu)_{\alpha\dot\beta}P_\mu\,, \qquad K^{\dot\alpha\beta} := +\frac{1}{2}(\bar\sigma^\mu)^{\dot\alpha\beta}K_\mu\,, \tag{2.26}$$

$$M_\alpha{}^\beta := \frac{i}{2}(\sigma^{\mu\nu})_\alpha{}^\beta M_{\mu\nu}\,, \qquad \bar{M}^{\dot\alpha}{}_{\dot\beta} := \frac{i}{2}(\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}M_{\mu\nu}\,, \tag{2.27}$$

where the Lorentz sigma matrices are defined in terms of the Paul matrices in the following way

$$(\sigma^\mu)_{\alpha\dot\beta} = (-1, \sigma^i)_{\alpha\dot\beta}\,, \qquad (\bar\sigma^\mu)^{\dot\alpha\beta} = \epsilon^{\dot\alpha\dot\gamma}\epsilon^{\beta\delta}(\sigma^\mu)_{\delta\dot\gamma} = (-1, -\sigma^i)^{\dot\alpha\beta}\,, \tag{2.28}$$

$$(\sigma^{\mu\nu})_\alpha{}^\beta = (\sigma^{[\mu}\bar\sigma^{\nu]})_\alpha{}^\beta\,, \qquad (\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} = (\bar\sigma^{[\mu}\sigma^{\nu]})^{\dot\alpha}{}_{\dot\beta}\,. \tag{2.29}$$

Our convention for the epsilon tensor, with which the indices are raised and lowered, will be $\epsilon_{12} = \epsilon_{\dot 1\dot 2} = -\epsilon^{12} = -\epsilon^{\dot 1\dot 2} = 1$.

In this notation, the SO(2,4) commutation relations take the form

$$[M_\alpha{}^\beta, J_\gamma] = \delta^\beta_\gamma J_\alpha - \frac{1}{2}\delta_\alpha{}^\beta J_\gamma\,, \qquad [M_\alpha{}^\beta, J^\gamma] = -\delta^\gamma_\alpha J^\beta + \frac{1}{2}\delta_\alpha{}^\beta J^\gamma\,, \tag{2.30}$$

$$[\bar M^{\dot\alpha}{}_{\dot\beta}, J_{\dot\gamma}] = -\delta^{\dot\alpha}_{\dot\gamma} J_{\dot\beta} + \frac{1}{2}\delta^{\dot\alpha}{}_{\dot\beta} J^{\dot\gamma}\,, \qquad [\bar M^{\dot\alpha}{}_{\dot\beta}, J_{\dot\gamma}] = \delta^{\dot\gamma}_{\dot\beta} J^{\dot\alpha} - \frac{1}{2}\delta^{\dot\alpha}{}_{\dot\beta} J_{\dot\gamma}\,, \tag{2.31}$$

$$[D, P_{\alpha\dot\beta}] = iP_{\alpha\dot\beta}\,,\ [D, K^{\dot\alpha\beta}] = -iK^{\dot\alpha\beta}\,, \qquad [D, M_\alpha{}^\beta] = [D, \bar M^{\dot\alpha}{}_{\dot\beta}] = 0 \tag{2.32}$$

$$[P_{\alpha\dot\beta}, K^{\dot\gamma\delta}] = \delta^\delta_\alpha \bar M^{\dot\gamma}{}_{\dot\beta} - \delta^{\dot\gamma}_{\dot\beta} M_\alpha{}^\delta + i\delta^\delta_\alpha\delta^{\dot\gamma}_{\dot\beta} D\,, \tag{2.33}$$

where J^γ and $J_{\dot\gamma}$ generically stand for quantities with undotted and dotted spinor indices.

With this preparation, it is now quite natural to introduce two sets of bosonic oscillators, undotted and dotted, which transform under $\mathrm{SL}(2,\mathbb{C})$ and $\overline{\mathrm{SL}(2,\mathbb{C})}$ respectively

$$[\mu^\alpha, \lambda_\beta] = \delta^\alpha_\beta,\ (\alpha\,, \beta = 1,2)\,,\quad [\tilde\mu^{\dot\alpha}, \tilde\lambda_{\dot\beta}] = \delta^{\dot\alpha}_{\dot\beta}\ (\dot\alpha\,, \dot\beta = \dot 1, \dot 2)\,. \tag{2.34}$$

In terms of these oscillators, the conformal generators can be expressed rather simply as[6]

$$M_\alpha{}^\beta = \lambda_\alpha\mu^\beta - \frac{1}{2}\delta^\beta_\alpha\lambda_\gamma\mu^\gamma\,, \qquad \bar M^{\dot\alpha}{}_{\dot\beta} = -\tilde\lambda_{\dot\beta}\tilde\mu^{\dot\alpha} + \frac{1}{2}\delta^{\dot\alpha}_{\dot\beta}\tilde\lambda_{\dot\gamma}\tilde\mu^{\dot\gamma}\,, \tag{2.35}$$

$$P_{\alpha\dot\beta} = \lambda_\alpha\tilde\lambda_{\dot\beta}\,,\ K^{\dot\alpha\beta} = \tilde\mu^{\dot\alpha}\mu^\beta\,, \qquad D = \frac{i}{2}(\lambda_\alpha\mu^\alpha + \tilde\lambda_{\dot\alpha}\tilde\mu^{\dot\alpha} + 2)\,. \tag{2.36}$$

Essentially the same oscillator representation was given in [49, 50]. We shall follow [50] with slight changes of signs and conventions.

Now to construct the SU(4) R-symmetry generators, we introduce four sets of fermionic oscillators satisfying the anti-commutation relations

$$\{\xi^a, \bar\xi_b\} = \delta^a_b\,,\ (a, b = 1,2,3,4)\,. \tag{2.37}$$

Then the SU(4) generators can be constructed as

$$R_a{}^b = \bar\xi_a\xi^b - \frac{1}{4}\delta^b_a\bar\xi_c\xi^c\,, \tag{2.38}$$

which indeed satisfy the correct commutation relations $[R_a{}^b, R_c{}^d] = \delta_b{}^c R_a{}^d - \delta_a{}^d R_c{}^b$. It is easy to check that under $R_a{}^b$ the oscillators $\bar\xi_b$ and ξ^a transform under the fundamental and anti-fundamental representations. As the SU(4) indices of any generator will be carried by these fundamental oscillators, this guarantees that a generator J_c (J^c) having a lower (upper) index transforms as a fundamental (anti-fundamental), i.e.

$$[R_a{}^b, J_c] = \delta^b_c R_a{}^b - \frac{1}{4}\delta_a{}^b J_c\,,\quad [R_a{}^b, J^c] = -\delta^c_a J^b + \frac{1}{4}\delta_a{}^b J^c\,. \tag{2.39}$$

[6]Recall that $-iD$ has positive real eigenvalues in our convention.

The remaining generators, namely the fermionic supersymmetry and superconformal generators, are expressed in a very simple way where the transformation properties are directly expressed by those of the constituent oscillators:

$$Q^a_\alpha := \lambda_\alpha \xi^a \,, \qquad \bar{Q}_{\dot{\alpha} a} = \tilde{\lambda}_{\dot{\alpha}} \bar{\xi}_a \,, \tag{2.40}$$

$$S^\alpha_a = \mu^\alpha \bar{\xi}_a \,, \qquad \bar{S}^{\dot{\alpha} a} = \tilde{\mu}^{\dot{\alpha}} \xi^a \,. \tag{2.41}$$

2.1.3 Central charge and hyper charge

The 30 bosonic and 32 fermionic generators constructed in terms of the oscillators above constitute the generators of the psu$(2,2|4)$. Actually, they do not close under (anti-) commutation. The closure requires the operator called the central charge given by

$$C = \frac{1}{2}(\lambda_\alpha \mu^\alpha - \tilde{\lambda}_{\dot{\alpha}} \tilde{\mu}^{\dot{\alpha}} + \bar{\xi}_a \xi^a) - 1 \,. \tag{2.42}$$

As the name indicates, C commutes with all the generators of psu$(2,2|4)$ and hence it takes a constant value for an irreducible representation. In particular, as we shall describe shortly, for the basic fields of the $\mathcal{N}=4$ SYM (i.e. for the field strength multiplet) of our interest, C vanishes. Thus in this sector, we can neglect this operator. Another additional operator of interest is the so-called the hypercharge operator[7] given by

$$B = \frac{1}{2}\bar{\xi}_a \xi^a \,. \tag{2.43}$$

This is essentially the fermion number operator. One notices that B does not appear in all the (anti-) commutation relations of psu(2,2|4) and thus it can be regarded as an outer automorphism of psu(2,2|4). By adding B and C to psu(2,2|4), we obtain the closed algebra called u(2,2|4).

The generators of u(2,2|4) can be expressed succinctly in terms of the oscillators as

$$J^A{}_B = \bar{\zeta}^A \zeta_B \,, \quad \bar{\zeta}^A = \begin{pmatrix} \lambda_\alpha \\ i\tilde{\mu}^{\dot{\alpha}} \\ \bar{\xi}_a \end{pmatrix}^A \,, \quad \zeta_A = \begin{pmatrix} \mu^\alpha \\ i\tilde{\lambda}_{\dot{\alpha}} \\ \xi^a \end{pmatrix}_A \,. \tag{2.44}$$

One can check that ζ's satisfy the graded commutator of the form

$$[\zeta_A, \bar{\zeta}^B] = \zeta_A \bar{\zeta}^B - (-1)^{|A||B|} \bar{\zeta}_B \zeta^A = \delta_A{}^B \,, \tag{2.45}$$

where $|A|$ is 1 for fermions and 0 for bosons. Hereafter, for simplicity, all the commutators should be interpreted as graded commutator as above. Then, the graded commutators between the generators of u(2,2|4) are neatly summarized in the following form:

$$[J^A{}_B, J^C{}_D] = \delta^C{}_B J^A{}_D - (-1)^{(|A|+|B|)(|C|+|D|)} \delta^A{}_D J^C{}_D \,. \tag{2.46}$$

[7] The definition of the hypercharge is ambiguous in the sense that we can add the central charge to it. For example, in the literature [49], the hypercharge is defined by $Z := \frac{1}{2}(\lambda_\alpha \mu^\alpha - \tilde{\lambda}_{\dot{\alpha}} \tilde{\mu}^{\dot{\alpha}})$ and it plays a role of the chirality operator. The relation to our definition is $Z = C - B + 1$.

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It is useful to write down the elements $J^A_{\ B}$ of u(2,2|4) in a matrix form in the following way:

$$J^A_{\ B} := \left(\begin{array}{cc|c} J_\alpha^{\ \beta} & J_{\alpha\dot\beta} & J_\alpha^{\ b} \\ J^{\dot\alpha\beta} & J^{\dot\alpha}_{\ \dot\beta} & J^{\dot\alpha b} \\ \hline J_a^{\ \beta} & J_{a\dot\beta} & J_a^{\ b} \end{array}\right)_{AB} = \left(\begin{array}{cc|c} Y_\alpha^{\ \beta} & iP_{\alpha\dot\beta} & Q^b_\alpha \\ iK^{\dot\alpha\beta} & Y^{\dot\alpha}_{\ \dot\beta} & i\bar S^{\dot\alpha b} \\ \hline S^\beta_a & i\bar Q_{\dot\beta a} & W_a^{\ b} \end{array}\right)_{AB} , \tag{2.47}$$

$$Y_\alpha^{\ \beta} = \lambda_\alpha\mu^\beta = M_\alpha^{\ \beta} + \frac{1}{2}\delta_\alpha^{\ \beta}(-iD + C - B)\,, \tag{2.48}$$

$$Y^{\dot\alpha}_{\ \dot\beta} = -\tilde\mu^{\dot\alpha}\tilde\lambda_{\dot\beta} = \bar M^{\dot\alpha}_{\ \dot\beta} + \frac{1}{2}\delta^{\dot\alpha}_{\ \dot\beta}(iD + C - B)\,, \tag{2.49}$$

$$W_a^{\ b} = \bar\xi_a\xi^b = R_a^{\ b} + \frac{1}{2}\delta_a^{\ b}B\,. \tag{2.50}$$

From this one can see that the central charge and the hypercharge are related to the trace and supertrace in the following way

$$\mathrm{tr}J := \sum_A J^A_{\ A} = 2C\,, \qquad \mathrm{str}J := \sum_A (-1)^A J^A_{\ A} = 2C - 4B\,. \tag{2.51}$$

As we shall see in section 2.2, the singlet projector we shall construct will be valid for the su(2,2|4) algebra as well as for psu(2,2|4), where the generators $\hat J^A{}_B$ of the former is obtained from u(2,2|4) by imposing the supertraceless condition as

$$\hat J^A_{\ B} := J^A_{\ B} - \frac{\mathrm{str}J}{8}(-1)^{|A|}\delta^A_{\ B}\,. \tag{2.52}$$

In particular this gives $\sum_A (-1)^{|A|}\hat J^A_{\ A} = 0$, which tells us that the hypercharge B is completely removed from su(2,2|4).

2.1.4 Oscillator vacuum and the representations of the fundamental SYM fields

We now move on to the oscillator representation for the fundamental fields which appear in $\mathcal{N} = 4$ SYM. For this purpose, we define the Fock vacuum $|0\rangle$ to be the state annihilated by all the annihilation operators:

$$\mu^\alpha|0\rangle = \tilde\mu^{\dot\alpha}|0\rangle = \xi^a|0\rangle = 0\,. \tag{2.53}$$

To be more precise, $|0\rangle$ is a tensor product of two vacua, one for the bosonic oscillators and the other for the fermionic ones. Namely,

$$|0\rangle = |0\rangle_B \otimes |0\rangle_F\,, \tag{2.54}$$

Then the Fock space is built upon this vacuum by acting by the creation operators $\lambda_\alpha, \tilde\lambda_{\dot\alpha}, \bar\xi_a$. However, not all the states produced this way correspond to the fields of $\mathcal{N} = 4$ SYM. The relevant ones are only those carrying zero central charge. This can be explicitly checked by the expressions of the basic $\mathcal{N} = 4$ SYM fields in terms of the oscillators given by [49]

$$F_{\alpha\beta}(0) \leftrightarrow \lambda_\alpha\lambda_\beta|0\rangle\,, \tag{2.55}$$

$$\psi_{\alpha a}(0) \leftrightarrow \lambda_\alpha \bar{\xi}_a|0\rangle\,, \tag{2.56}$$

$$\phi_{ab}(0) \leftrightarrow \bar{\xi}_a\bar{\xi}_b|0\rangle\,, \tag{2.57}$$

$$\bar{\psi}^a_{\dot{\alpha}}(0) \leftrightarrow \frac{1}{3!}\epsilon^{abcd}\tilde{\lambda}_{\dot{\alpha}}\bar{\xi}_b\bar{\xi}_c\bar{\xi}_d|0\rangle\,, \tag{2.58}$$

$$\bar{F}_{\dot{\alpha}\dot{\beta}}(0) \leftrightarrow \frac{1}{4!}\epsilon^{abcd}\tilde{\lambda}_{\dot{\alpha}}\tilde{\lambda}_{\dot{\beta}}\bar{\xi}_a\bar{\xi}_b\bar{\xi}_c\bar{\xi}_d|0\rangle\,. \tag{2.59}$$

From the form of C given in (2.42) it is clear that they all carry $C = 0$. Also, it is easy to check that these oscillator expressions of the fields carry the correct Lorentz and R-symmetry quantum numbers.

In addition to these fundamental fields, we need to express their derivatives. The field at the general position x is obtained by the action of the translation operator $e^{iP\cdot x}$ as

$$|\mathcal{O}(0)\rangle \to |\mathcal{O}(x)\rangle := e^{iP\cdot x}|\mathcal{O}(0)\rangle\,. \tag{2.60}$$

From the oscillator representation $P_{\alpha\dot{\beta}} = \lambda_\alpha\tilde{\lambda}_{\dot{\beta}}$, we see that the derivatives of a field can be expressed as

$$\partial_{(\alpha(\dot{\beta}}\cdots\partial_{\gamma)\dot{\delta})}\mathcal{O}(x) \cong (i\lambda_\alpha\tilde{\lambda}_{\dot{\beta}})\ldots(i\lambda_\gamma\tilde{\lambda}_{\dot{\delta}})|\mathcal{O}(x)\rangle\,, \tag{2.61}$$

where $\partial_{\alpha\dot{\beta}} = \partial/\partial x^{\dot{\beta}\alpha}$ and $x^{\dot{\alpha}\beta} := x^\mu(\bar{\sigma}_\mu)^{\dot{\alpha}\beta}$ and we have used $P\cdot x = P_{\alpha\dot{\beta}}x^{\dot{\beta}\alpha}$ and $\partial_{\alpha\dot{\beta}}e^{iP\cdot x} = iP_{\alpha\dot{\beta}}e^{iP\cdot x}$. Notice that the spinor indices $(\alpha,\gamma,\ldots)$ and $(\dot{\beta},\dot{\delta},\ldots)$ are symmetrized as the bosonic oscillators λ mutually commute. Also note that we can replace some combinations of the (covariant) derivatives by appropriate fields without derivatives using the equations of motion and the Bianchi identities. For example, we can set $\partial_{\alpha\dot{\beta}}\partial^{\dot{\beta}\alpha}\phi \propto \Box\phi$ and $\epsilon^{\alpha\beta}\partial_{\alpha\dot{\alpha}}\psi^a_\beta$ to zero due to the free equations of motion.[8] As a result, we can express fields with derivatives by expressions where all the spinor indices are totally symmetrized. Therefore, the independent fields with derivatives are simply generated by acting $P_{\alpha\dot{\beta}} = \lambda_\alpha\tilde{\lambda}_{\dot{\beta}}$ on the oscillator representations for the fundamental fields (2.55)–(2.59). Since $P_{\alpha\dot{\beta}}$ commutes with the central charge, these states with derivatives are still within the subspace with vanishing central charge.

2.1.5 Various "vacua" and their relations

It is an elementary exercise in quantum mechanics to construct the singlet state from two spin 1/2 states by forming a suitable combination of the highest and the lowest states. It is a slightly more involved exercise to extend this to the case of the general spin j, but the structure is similar: one combines the states built upon the lowest weight states and those built upon the highest weight states with simple weights. Indeed, up to an overall constant, the singlet state is given by

$$|\mathbf{1}_j\rangle = \sum_{l=0}^{2j}(-1)^l|-(j-l)\rangle\otimes|j-l\rangle \tag{2.62}$$

[8] In the interacting case, it is possible to replace the combinations of covariant derivatives such as $\mathcal{D}_{\alpha\dot{\beta}}\mathcal{D}^{\dot{\beta}\alpha}\phi$ and $\epsilon^{\alpha\beta}\mathcal{D}_{\alpha\dot{\alpha}}\psi^a_\beta$ by the fields without derivatives using the equations of motion as well.

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This indicates that for the construction of the singlet state for much more complicated case of psu(2,2|4), the basic idea should be the same and one would combine the Fock states built upon the lowest weight oscillator vacuum $|0\rangle$, already introduced, with the states built upon the highest weight oscillator vacuum $|\bar{0}\rangle$, which should be defined to be annihilated by the creation operators as

$$\lambda_\alpha|\bar{0}\rangle = \tilde{\lambda}_{\dot{\alpha}}|\bar{0}\rangle = \bar{\xi}_a|\bar{0}\rangle = 0\,. \tag{2.63}$$

Just as for $|0\rangle$ given in (2.54), the more precise definition of $|\bar{0}\rangle$ is

$$|\bar{0}\rangle \equiv |\bar{0}\rangle_B \otimes |\bar{0}\rangle_F\,. \tag{2.64}$$

From (2.63) it immediately follows that $|\bar{0}\rangle$ is annihilated by $P_{\alpha\dot{\beta}} = \lambda_\alpha\tilde{\lambda}_{\dot{\beta}}$ and thus the Fock space built on $|\bar{0}\rangle$ is a highest weight module as opposed to the lowest weight module built on $|0\rangle$.

There is an essential difference between the bosonic sector and the fermionic sector. For the bosonic sector, $|0\rangle_B$ and $|\bar{0}\rangle_B$ cannot be related by the action of a finite number of oscillators,[9] but for the fermionic sector one can readily identify $|\bar{0}\rangle_F = \bar{\xi}_1\bar{\xi}_2\bar{\xi}_3\bar{\xi}_4|0\rangle_F$.

It will turn out, however, that as for the fermionic oscillator Fock space describing the R-symmetry quantum numbers, "vacua" slightly different from $|0\rangle$ and $|\bar{0}\rangle$ will be more useful and more physical. To introduce them, we rename the fermionic oscillators in the following way so that half of the creation (annihilation) operators are switched to annihilation (creation) operators:[10]

$$c^i = \xi^i \ (i=1,2)\,, \qquad\qquad d^i = \bar{\xi}_{i+2} \ (i=1,2)\,, \tag{2.65}$$

$$\bar{c}_i = \bar{\xi}_i \ (i=1,2)\,, \qquad\qquad \bar{d}_i = \xi^{i+2} \ (i=1,2)\,. \tag{2.66}$$

We define the state $|Z\rangle$ as annihilated by the new annihilation operators c^i and d^i, while $|\bar{Z}\rangle$ is defined to be annihilated by the new creation operators $\bar{c}_i$ and $\bar{d}_i$.

$$c^i|Z\rangle = d^i|Z\rangle = 0\,, \tag{2.67}$$

$$\bar{c}_i|\bar{Z}\rangle = \bar{d}_i|\bar{Z}\rangle = 0\,. \tag{2.68}$$

As states built on the original vacuum $|0\rangle$, these new "vacua" can be written as

$$|Z\rangle = d^1d^2|0\rangle = \bar{\xi}_3\bar{\xi}_4|0\rangle\,, \tag{2.69}$$

$$|\bar{Z}\rangle = \bar{c}_1\bar{c}_2|0\rangle = \bar{\xi}_1\bar{\xi}_2|0\rangle \tag{2.70}$$

and they are related as $|\bar{Z}\rangle = -\bar{c}_1\bar{c}_2\bar{d}_1\bar{d}_2|Z\rangle$ or $|Z\rangle = -c^1c^2d^1d^2|\bar{Z}\rangle$. Now if we recall that the SO(6) scalars are represented by $\bar{\xi}_a\bar{\xi}_b|0\rangle$ as shown in (2.57), $|Z\rangle$ and $|\bar{Z}\rangle$ correspond to some physical scalars. To be definite let us identify them as states carrying $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$ quantum numbers of the SU(2) sector, where the generators are given by[11]

$$J^L_+ = c^1d^1\,, \quad J^L_- = \bar{d}_1\bar{c}_1\,, \quad J^L_3 = \frac{1}{2}(d^1\bar{d}_1 - \bar{c}_1c^1)\,, \tag{2.71}$$

[9] Actually, by using the operator $U^2 = \exp[\frac{\pi}{2}(P_0 - K_0)]$ one can map $|0\rangle$ to $|\bar{0}\rangle$ and exchange the role of the annihilation and the creation operators.

[10] Such a transformation is sometimes called a particle-hole transformation.

[11] Of course the choice of $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$ in SU(4) is not unique. We are simply taking a convenient one. Incidentally, our normalization for $J_\pm$ is $J_\pm = J_1 \pm iJ_2$.

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$$J_+^R = c^2 d^2\,, \quad J_-^R = \bar{d}_2 \bar{c}_2\,, \quad J_3^R = \frac{1}{2}(d^2\bar{d}_2 - \bar{c}_2 c^2)\,. \tag{2.72}$$

Then, it is easy to see that $|Z\rangle$ and $|\bar{Z}\rangle$ carry the quantum numbers $(\frac{1}{2}, \frac{1}{2})$ and $(-\frac{1}{2}, -\frac{1}{2})$ respectively and hence can be identified with, say, $\phi_1 + i\phi_2$ and its complex conjugate. The (de-)excitations of these vacua in the SU(2) sector with the quantum numbers $(\frac{1}{2}, -\frac{1}{2})$ and $(-\frac{1}{2}, \frac{1}{2})$ respectively, which are often denoted by $|X\rangle$ and $|-\bar{X}\rangle$, are given by

$$|X\rangle = J_-^R|Z\rangle = J_+^L|\bar{Z}\rangle = \bar{c}_2 d^1|0\rangle\,, \tag{2.73}$$

$$|-\bar{X}\rangle = J_-^L|Z\rangle = J_+^R|\bar{Z}\rangle = d^2\bar{c}_1|0\rangle\,. \tag{2.74}$$

Now in order to construct the singlet projector in section 2.2, it will turn out to be convenient to define the scalar states similar to the above, except that their bosonic part of the vacuum is switched from $|0\rangle_B$ to $|\bar{0}\rangle_B$. We will place a line over the kets (or the corresponding bra) to denote such scalar states. For example,

$$\overline{|Z\rangle} \equiv |\bar{0}\rangle_B \otimes d^1 d^2|0\rangle_F\,, \tag{2.75}$$

$$\overline{|\bar{Z}\rangle} \equiv |\bar{0}\rangle_B \otimes \bar{c}^1\bar{c}^2|0\rangle_F\,. \tag{2.76}$$

In this more precise notation, the previously defined $|Z\rangle$ and $|\bar{Z}\rangle$ are written as

$$|Z\rangle = |0\rangle_B \otimes d^1 d^2|0\rangle_F\,, \tag{2.77}$$

$$|\bar{Z}\rangle = |0\rangle_B \otimes \bar{c}^1\bar{c}^2|0\rangle_F\,. \tag{2.78}$$

Since overlined scalar states differ only in the bosonic sector, the properties of such states under the action of the fermionic oscillators are exactly the same as the un-overlined ones. For example, $c^i\overline{|Z\rangle} = 0$, etc., just as in (2.67) and (2.68).

$$\bar{c}_i\overline{|Z\rangle} = \bar{d}_i\overline{|Z\rangle} = 0\,, \quad \overline{|Z\rangle} := \xi^1\xi^2|\bar{0}\rangle\,, \tag{2.79}$$

$$c^i\overline{|\bar{Z}\rangle} = d^i\overline{|\bar{Z}\rangle} = 0\,, \quad \overline{|\bar{Z}\rangle} := \xi^3\xi^4|\bar{0}\rangle\,. \tag{2.80}$$

As we will need them later, it should be convenient to list the properties of the bra (or dual) vacua, which evidently follow from those of the ket vacua. $\langle 0|$ and $\langle\bar{0}|$ have the properties

$$\langle 0|\lambda_\alpha = \langle 0|\tilde{\lambda}_{\dot{\alpha}} = \langle 0|\bar{\xi}_a = 0\,, \quad \langle 0|0\rangle = 1\,, \tag{2.81}$$

$$\langle\bar{0}|\mu^\alpha = \langle\bar{0}|\tilde{\mu}^{\dot{\alpha}} = \langle\bar{0}|\xi^a = 0\,, \quad \langle\bar{0}|\bar{0}\rangle = 1\,. \tag{2.82}$$

This means that the dual Fock space is generated either by the action of $(\mu^\alpha, \tilde{\mu}^{\dot{\alpha}}, \xi_a)$ on $\langle 0|$ or by the action of $(\lambda_\alpha, \tilde{\lambda}_{\dot{\alpha}}, \bar{\xi}^a)$ on $\langle\bar{0}|$. As for the properties of the scalar bra vacua under the action of the fermionic oscillators, they satisfy

$$\langle Z|\bar{c}_i = \langle Z|\bar{d}_i = 0\,, \quad \langle\bar{Z}|c^i = \langle\bar{Z}|d^i = 0\,, \tag{2.83}$$

$$\langle Z|Z\rangle = \langle\bar{Z}|\bar{Z}\rangle = 1\,, \tag{2.84}$$

and exactly the same equations hold for the overlined bra states $\overline{\langle Z|}$ and $\overline{\langle\bar{Z}|}$.

2.1.6 Comparison with the E-scheme formulation

Before we start the systematic construction of the singlet projector using these oscillator representations, let us end this subsection with some comments on the difference between the oscillator representation we use and the one employed in [32, 33]. They basically work in the E-scheme, where the oscillators are covariant under the compact subgroup shown in (2.1). This in turn means that the representations corresponding to local composite operators are obtained indirectly by the use of the complicated operator $U = \exp[\frac{\pi}{4}(P_0 - K_0)]$. To be a little more specific, let us display the E-scheme oscillators and the Fock vacuum used in [32, 33]. The difference from ours is in the bosonic oscillators, which are given by

$$[a_i, \bar{a}^j] = \delta_i^j \,, (i,j=1,2) \; [b_s, \bar{b}^t] = \delta_s^t \,, \; (s,t=1,2) \tag{2.85}$$

$$a_i|0\rangle_E = b_s|0\rangle_E = 0 \,. \tag{2.86}$$

Then the SO(2,4) generators are expressed as bi-linears of these oscillators as

$$L_j^i = \bar{a}^i a_j - \frac{1}{2}\delta_j^i(\bar{a}^k a_k) \,, \; R_t^s = \bar{b}^s b_t - \frac{1}{2}\delta_t^s(\bar{b}^u b_u) \,, \tag{2.87}$$

$$E = \frac{1}{2}(\bar{a}^i a_i + \bar{b}^s b_s) + 1 \,, \; L_{is} = a_i b_s \,, \; L^{is} = \bar{a}^i \bar{b}^s \,, \tag{2.88}$$

where L_j^i, R_t^s are $SU(2)_L \times SU(2)_R$ generators, E is the AdS energy and L_{is}, L^{is} are the elements of $\mathcal{E}^-$, $\mathcal{E}^+$ respectively. Hence, the bosonic oscillators $a_i, \bar{a}^i$ transform covariantly under $SU(2)_L$ as a doublet and $b_s, \bar{b}^s$ are doublets of $SU(2)_R$. To convert them to the D-scheme oscillators, one needs to employ the similarity transformation using the operator U. However, as we mentioned in the introduction, any similarity transformation preserves the structure of the algebra and hence, for example, $SU(2)_L$ does not become a Lorentz group $SL(2,\mathbb{C})$. This is reflected in the transformation of the oscillators themselves. By using the explicit oscillator representation of U, we easily find, for example, $U^{-1}\bar{a}^i U = \frac{1}{\sqrt{2}}(\bar{a}^i + b_i)$ etc., which is not informative as far as the useful re-interpretation to the D-scheme is concerned. Thus although the E- and the D- schemes are connected by a similarity transformation U, the conformal covariance cannot be made manifest by just such a transformation. Hence for the purpose of dealing with the local composite operators, the use of D-scheme is much more transparent and indeed in what follows we shall never need the operator U.

2.2 Construction of the singlet projector for psu(2, 2|4)

2.2.1 Singlet condition and its solution

We shall now give a detailed construction of the singlet projector ${}_{\rm psu}\langle \mathbf{1}_{12}|$ for the states in the product of a pair of Hilbert spaces $\mathcal{H}_1 \otimes \mathcal{H}_2$, which satisfies the defining equation for the singlet projector

$${}_{\rm psu}\langle \mathbf{1}_{12}|(J^A{}_B \otimes \mathbf{1} + \mathbf{1} \otimes J^A{}_B) = 0 \,, \qquad J^A{}_B \in {\rm psu}(2,2|4) \,. \tag{2.89}$$

In order to find the most general singlet projector, we must proceed systematically. As it will become clear below, actually the desired singlet projector satisfying the relation above can be constructed for su(2, 2|4) as well as for psu(2, 2|4), but not for u(2, 2|4). Recall that the generators $J^A{}_B$ of su$(2,2|4)$ are obtained from the generator of u$(2,2|4)$, to be tentatively denoted by $\hat{J}^A{}_B$, by making them supertraceless, i.e.

$$J^A{}_B = \hat{J}^A{}_B - \frac{1}{8}\delta^A_B(-1)^{|A|}(\mathrm{Str}\hat{J})\,. \tag{2.90}$$

Because of this condition, when we interchange the order of the two conjugate oscillators making up any diagonal generator $J^A{}_A$, constant terms produced from the (anti-)commutation relations precisely cancel. This property will be of crucial importance for the construction of the true singlet projector.

First, let us begin by identifying the building block for the sector involving the oscillator pair λ_α and μ^α. Since the generators $J^A{}_B$ are quadratic in the oscillators, the building block which would realize the relation (2.89) in the above sector should be of the form

$${}_{\lambda\mu}\langle \mathbf{1}_{12}| \propto (\langle Z| \otimes \overline{\langle \bar{Z}|})(\mu^\alpha)^{n_\mu} \otimes (\lambda_\beta)^{n_\lambda}) \tag{2.91}$$

Now consider a useful combination of generators $\mathcal{J}^{(1)} \equiv \lambda_1\mu^1 - \lambda_2\mu^2 = \mu^1\lambda_1 - \mu^2\lambda_2$, which belongs to su$(2,2|4)$ and hence the interchange of the order of λ_α and μ^α does not produce any constant. When we apply $\mathcal{J}^{(1)} \otimes \mathbf{1}$, we should use the form $\mathcal{J}^{(1)} \equiv \lambda_1\mu^1 - \lambda_2\mu^2$ since λ_α annihilates $\langle Z|$. Then we easily obtain

$$(\langle Z| \otimes \overline{\langle \bar{Z}|})(\mu^\alpha)^{n_\mu} \otimes (\lambda_\beta)^{n_\lambda})(\mathcal{J}^{(1)} \otimes \mathbf{1}) = n_\mu(\delta^\alpha_1 - \delta^\alpha_2)(\langle Z| \otimes \overline{\langle \bar{Z}|})(\mu^\alpha)^{n_\mu} \otimes (\lambda_\beta)^{n_\lambda})\,. \tag{2.92}$$

On the other hand, when we apply $\mathbf{1} \otimes \mathcal{J}^{(1)}$, since $\overline{\langle \bar{Z}|}$ is annihilated by μ^α, we should use the form $\mathcal{J}^{(1)} = \mu^1\lambda_1 - \mu^2\lambda_2$. Then, we get

$$(\langle Z| \otimes \overline{\langle \bar{Z}|})(\mu^\alpha)^{n_\mu} \otimes (\lambda_\beta)^{n_\lambda})(\mathbf{1} \otimes \mathcal{J}^{(1)}) = -n_\lambda(\delta^1_\beta - \delta^2_\beta)(\langle Z| \otimes \overline{\langle \bar{Z}|})(\mu^\alpha)^{n_\mu} \otimes (\lambda_\beta)^{n_\lambda})\,. \tag{2.93}$$

In order for the sum of (2.92) and (2.93) to vanish, we must have $n_\mu = n_\lambda$ and $\alpha = \beta$. Hence, the form of the oscillator factor should actually be the combination $(\mu^\alpha \otimes \lambda_\alpha)^{n_{\mu_\alpha}}$

We can apply the same logic to the sectors consisting of other conjugate pairs, namely $(\tilde{\mu}^{\dot{\alpha}}, \tilde{\lambda}_{\dot{\alpha}}), (\bar{c}_i, c^i)$ and $(\bar{d}_j, d^j)$ and find similar conditions. In this way, we find that the necessary form for the singlet projector for su$(2,2|4)$ can be written as

$${}_{\mathrm{su}}\langle \mathbf{1}_{12}| = \sum_{\mathbf{n}} f(\mathbf{n})\langle \mathbf{n}| \tag{2.94}$$

$$\langle \mathbf{n}| \equiv \langle Z| \otimes \overline{\langle \bar{Z}|} \prod_{\alpha,\dot{\beta},i,j} \frac{(\mu^\alpha \otimes \lambda_\alpha)^{n_{\lambda_\alpha}}}{n_{\mu_\alpha}!} \frac{(\tilde{\mu}^{\dot{\alpha}} \otimes \tilde{\lambda}_{\dot{\alpha}})^{n_{\tilde{\lambda}_{\dot{\alpha}}}}}{n_{\tilde{\mu}_{\dot{\alpha}}}!} \frac{(c^i \otimes \bar{c}_i)^{n_{c_i}}}{n_{c_i}!} \frac{(d^j \otimes \bar{d}_j)^{n_{d_j}}}{n_{d_j}!}\,, \tag{2.95}$$

$$f(\mathbf{n}) = f(n_{\lambda_1}, n_{\lambda_2}, n_{\tilde{\lambda}_{\dot{1}}}, n_{\tilde{\lambda}_{\dot{2}}}, \ldots)\,, \tag{2.96}$$

where $f(\mathbf{n})$ at this stage is an arbitrary function and is to be determined by the requirement of the singlet condition. As for the sum over the powers n_{μ_α} etc, we shall allow them to be arbitrary non-negative integers.

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To see what conditions should be satisfied by the function $f(\mathbf{n})$, let us focus first on a simple generator in the (μ, λ) sector of the form $J_\alpha{}^\beta = \lambda_\alpha \mu^\beta$, where $\alpha \neq \beta$. Since λ_α is the annihilation operator for the bra state $\langle Z|$, just as before, we easily get

$$\begin{aligned}\sum_{\mathbf{n}} f(\mathbf{n})\langle \mathbf{n}|(\lambda_\alpha \mu^\beta \otimes 1) &= \sum_{\mathbf{n}} f(\mathbf{n})\langle \mathbf{n}|(\lambda_\alpha \otimes 1)(\mu^\beta \otimes 1) \\ &= \sum_{\mathbf{n}} f(\mathbf{n})\langle \mathbf{n}|\frac{n_{\lambda_\alpha}}{n_{\lambda_\alpha}!}(\mu^\alpha \otimes \lambda_\alpha)^{n_{\lambda_\alpha}-1}(1 \otimes \lambda_\alpha)(\mu^\beta \otimes 1)\cdots \\ &= \sum_{\mathbf{n}} f(n_{\lambda_\alpha}+1, \ldots)\langle \mathbf{n}|(\mu^\beta \otimes 1)(1 \otimes \lambda_\alpha) + \cdots . \end{aligned} \tag{2.97}$$

In the third line we have shifted $\mathfrak{n}_{\lambda_\alpha}$ by 1 and interchanged the order of the factors $1 \otimes \lambda_\alpha$ and $\mu^\beta \otimes 1$. Now the action of $\mu^\beta \otimes 1$ on $\langle \mathbf{n}|$ is easily seen to produce the structure $1 \otimes \mu^\beta$ with an overall minus sign, together with a shift of $\mathfrak{n}_{\lambda_\beta}$ by minus one unit in $f(\mathbf{n})$ under the sum. As for the structure of the operator part, combined with the factor $(1 \otimes \lambda_\alpha)$ already produced, we get

$$(1 \otimes \mu^\beta)(1 \otimes \lambda_\alpha) = (1 \otimes \mu^\beta \lambda_\alpha) = (1 \otimes \lambda_\alpha \mu^\beta) = 1 \otimes J_\alpha{}^\beta, \tag{2.98}$$

where we have interchanged the order of μ^β and λ_α to get back $J_\alpha{}^\beta$ without producing any constant since we are considering the case with $\alpha \neq \beta$. Altogether we obtain the formula

$$\sum_{\mathbf{n}} f(n_{\lambda_\alpha}, n_{\lambda_\beta}, \ldots)\langle \mathbf{n}|(J_\alpha{}^\beta \otimes 1) = -\sum_{\mathbf{n}} f(n_{\lambda_\alpha}+1, n_{\lambda_\beta}-1, \ldots)\langle \mathbf{n}|(1 \otimes J_\alpha{}^\beta) \tag{2.99}$$

Thus the singlet condition demands

$$f(n_{\lambda_\alpha}, n_{\lambda_\beta}, \ldots) = f(n_{\lambda_\alpha}+1, n_{\lambda_\beta}-1, \ldots). \tag{2.100}$$

The general solution of this equation is

$$f(n_{\lambda_1}, n_{\lambda_2}, \ldots) = g(n_{\lambda_1}+n_{\lambda_2}, \ldots), \tag{2.101}$$

where g is an arbitrary function except that n_{λ_α}'s must appear as the sum $n_{\lambda_1}+n_{\lambda_2}$.

Repeating similar analyses for all the off-diagonal[12] generators of su$(2,2|4)$, one obtains the following list of singlet conditions.

For the bosonic generators, we get

(b1)	$\lambda_\alpha \mu^\beta$	$f(n_{\lambda_\alpha}, n_{\lambda_\beta}, \ldots) = f(n_{\lambda_\alpha}+1, n_{\lambda_\beta}-1, \ldots)$
(b2)	$\lambda_\alpha \tilde{\lambda}_{\dot\beta}$	$f(n_{\lambda_\alpha}, n_{\tilde{\lambda}_{\dot\beta}}, \ldots) = -f(n_{\lambda_\alpha}+1, n_{\tilde{\lambda}_{\dot\beta}}+1, \ldots)$
(b3)	$\tilde{\mu}^{\dot\alpha} \tilde{\lambda}_{\dot\beta}$	$f(n_{\tilde{\lambda}_{\dot\alpha}}, n_{\tilde{\lambda}_{\dot\beta}}, \ldots) = f(n_{\tilde{\lambda}_{\dot\alpha}}-1, n_{\tilde{\lambda}_{\dot\beta}}+1, \ldots)$
(b4)	$\tilde{\mu}^{\dot\alpha} \mu^\alpha$	$f(n_{\tilde{\lambda}_{\dot\alpha}}, n_{\lambda_\alpha}, \ldots) = -f(n_{\tilde{\lambda}_{\dot\alpha}}-1, n_{\lambda_\alpha}-1, \ldots)$
(b5)	$\bar{c}_i \bar{d}_j$	$f(n_{c_i}, n_{d_j}, \ldots) = -f(n_{c_i}+1, n_{d_j}+1, \ldots)$
(b6)	$\bar{c}_i c^j$	$f(n_{c_i}, n_{c_j}, \ldots) = f(n_{c_i}+1, n_{c_j}-1, \ldots)$
(b7)	$d^j c^i$	$f(n_{d_j}, n_{c_i}, \ldots) = -f(n_{d_j}-1, n_{c_i}-1, \ldots)$
(b8)	$d^j \bar{d}_k$	$f(n_{d_j}, n_{d_k}, \ldots) = f(n_{d_j}-1, n_{d_k}+1, \ldots)$

[12] "Off-diagonal" here means the generators like $\lambda_\alpha \mu^\beta$ with $\alpha \neq \beta$, etc. so that their (anti)commutators vanish. For them there is no difference between u$(2,2|4)$ and su$(2,2|4)$.

For the fermionic generators, the conditions are

$$
\begin{array}{lll}
\text{(f1)} & \lambda_\alpha c^i & f(n_{\lambda_\alpha}, n_{c_i}, \dots) = -f(n_{\lambda_\alpha}+1, n_{c_i}-1, \dots) \\
\text{(f2)} & \lambda_\alpha \bar{d}_j & f(n_{\lambda_\alpha}, n_{d_j}, \dots) = f(n_{\lambda_\alpha}+1, n_{d_j}+1, \dots) \\
\text{(f3)} & \tilde{\mu}^{\dot\alpha} c^i & f(n_{\tilde\lambda_{\dot\alpha}}, n_{c_i}, \dots) = f(n_{\tilde\lambda_{\dot\alpha}}-1, n_{c_i}-1, \dots) \\
\text{(f4)} & \tilde{\mu}^{\dot\alpha} \bar{d}_j & f(n_{\tilde\lambda_{\dot\alpha}}, n_{d_j}, \dots) = -f(n_{\tilde\lambda_{\dot\alpha}}-1, n_{d_j}+1, \dots) \\
\text{(f5)} & \bar{c}_i \mu^\alpha & f(n_{c_i}, n_{\lambda_\alpha}, \dots) = -f(n_{c_i}+1, n_{\lambda_\alpha}-1, \dots) \\
\text{(f6)} & \bar{c}_i \tilde\lambda_{\dot\alpha} & f(n_{c_i}, n_{\tilde\lambda_{\dot\alpha}}, \dots) = f(n_{c_i}+1, n_{\tilde\lambda_{\dot\alpha}}+1, \dots) \\
\text{(f7)} & d^j \mu^\alpha & f(n_{d_j}, n_{\lambda_\alpha}, \dots) = f(n_{d_j}-1, n_{\lambda_\alpha}+1, \dots) \\
\text{(f8)} & d^j \tilde\lambda_{\dot\alpha} & f(n_{d_j}, n_{\tilde\lambda_{\dot\alpha}}, \dots) = -f(n_{d_j}-1, n_{\tilde\lambda_{\dot\alpha}}+1, \dots)
\end{array}
$$

With the hint from the analysis of the bosonic (μ, λ) sector, it is actually easy to write down the most general solution satisfying these equations. The answer is

$$f(n_{\lambda_1}, n_{\lambda_2}, \dots) = (-1)^{n_{\tilde\lambda_{\dot 1}} + n_{\tilde\lambda_{\dot 2}} + n_{c_1} + n_{c_2}} h(C) \tag{2.102}$$

$$2C = (n_{\lambda_1} + n_{\lambda_2}) - (n_{\tilde\lambda_{\dot 1}} + n_{\tilde\lambda_{\dot 2}}) + (n_{c_1} + n_{c_2}) - (n_{d_1} + n_{d_2}) \tag{2.103}$$

where $h(x)$ is an arbitrary function of one argument. It is important to note that C is precisely the central charge of $\mathrm{u}(2,2|4)$. As such it can be set to a number in an irreducible representation. In particular, the fundamental SYM fields of our interest belong to the sector where $C = 0$ and $h(0)$ is just an overall constant, which we shall set to unity for simplicity.

Now we must examine the diagonal generators, such as $\lambda_\alpha \mu^\alpha$ and $\bar{c}_i c^i$, etc. Because an extra constant is produced upon interchanging the order of the oscillators, for example like $\lambda_\alpha \mu^\alpha = \mu^\alpha \lambda_\alpha - 2$, etc. in the process of the manipulation as in (2.98), in general the singlet condition is not satisfied. However, as we already stressed, for the diagonal generators which belong to $\mathrm{su}(2,2|4)$ and $\mathrm{psu}(2,2|4)$ such constants cancel. Therefore, the conditions we obtained for the function $f(\mathbf{n})$ do not change and the singlet projector for the physical SYM states is obtained as[13]

$$
\begin{aligned}
{}_{\mathrm{psu}}\langle \mathbf{1}_{12}| = \langle Z| \otimes \overline{\langle \bar{Z}|} \sum_{\mathbf{n} \geq 0,\, C=0} (-1)^{n_{\tilde\lambda_{\dot 1}} + n_{\tilde\lambda_{\dot 2}} + n_{c_1} + n_{c_2}} \\
\times \prod_{\alpha, \dot\beta, i, j} \frac{(\mu^\alpha \otimes \lambda_\alpha)^{n_{\lambda_\alpha}}}{n_{\lambda_\alpha}!} \frac{(\tilde\mu^{\dot\alpha} \otimes \tilde\lambda_{\dot\alpha})^{n_{\tilde\lambda_{\dot\alpha}}}}{n_{\tilde\lambda_{\dot\alpha}}!} \frac{(c^i \otimes \bar{c}_i)^{n_{c_i}}}{n_{c_i}!} \frac{(d^j \otimes \bar{d}_j)^{n_{d_j}}}{n_{d_j}!}
\end{aligned}
\tag{2.104}
$$

Because of the restriction $C = 0$ in the sum over the n_*'s, this expression *does not* quite take the form of an exponential. However, we can remove the restriction $C = 0$ in the sum when applying $\langle \mathbf{1}_{12}|$ to the physical SYM states, since the extra states with $C \neq 0$ produced are orthogonal to $C = 0$ states and do not contribute to the inner product with the physical states. Thus, with the $C = 0$ restriction removed, the singlet state above *can be* written as a simple exponential given by

$${}_{\mathrm{psu}}\langle \mathbf{1}_{12}| = \langle Z| \otimes \overline{\langle \bar{Z}|} \exp\left(\lambda_\alpha \otimes \mu^\alpha - \tilde\lambda_{\dot\alpha} \otimes \tilde\mu^{\dot\alpha} + \bar{c}_i \otimes c^i - \bar{d}_j \otimes d^j \right), \tag{2.105}$$

[13] If one takes a different value of C one obtains a singlet projector for that sector. Here we focus on the physical SYM fields for which $C = 0$.

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where in the exponent the sum is implied for the repeated indices. If one wishes to perform the Wick contraction in a manifestly symmetric fashion, one can use the form

$$\frac{1}{2}\left({}_{\rm psu}\langle\mathbf{1}_{12}| + {}_{\rm psu}\langle\tilde{\mathbf{1}}_{12}|\,\right)\,, \tag{2.106}$$

where

$${}_{\rm psu}\langle\tilde{\mathbf{1}}_{12}| = \overline{\langle\bar{Z}|}\otimes\langle Z|\exp\left(\mu^\alpha\otimes\lambda_\alpha - \tilde{\mu}^{\dot\alpha}\otimes\tilde{\lambda}_{\dot\alpha} + c^i\otimes\bar{c}_i - d^j\otimes\bar{d}_j\right)\,. \tag{2.107}$$

Hereafter, we shall suppress for simplicity the subscript psu and write $\langle\mathbf{1}_{12}|$ for ${}_{\rm psu}\langle\mathbf{1}_{12}|$.

2.2.2 Crossing relations for the oscillators

Before ending this subsection, let us make an important remark on the property of the singlet projector (2.105). Although we have constructed this state by demanding that it be singlet under the generators of $psu(2,2|4)$ satisfying (2.89), it is easy to see from the process of construction above that actually the singlet projector (2.105) effects the following "crossing relations" for the *individual oscillators*:

$$\langle\mathbf{1}_{12}|(\bar{\zeta}^A\otimes 1) = \langle\mathbf{1}_{12}|(1\otimes\bar{\zeta}^A)\,, \tag{2.108}$$

$$\langle\mathbf{1}_{12}|(\zeta_A\otimes 1) = -\langle\mathbf{1}_{12}|(1\otimes\zeta_A)\,. \tag{2.109}$$

Clearly these relations themselves have no group theoretical meaning and appear to be stronger than the singlet condition. It is remarkable that yet they follow from the requirement of the singlet condition and will be quite useful in the computation of the correlation function, as we shall see in the next subsection.

2.3 Wick contraction of the basic fields using the singlet projector

Let us now show that the Wick contraction of the basic fields of the $\mathcal{N}=4$ super Yang-Mills theory can be computed quite easily by using the singlet projector constructed in the previous subsection. This can be identified as the method of Ward identity already introduced in [33]. However, as we use the D-scheme from the outset, our method is much more direct and simpler, without the need of rather complicated conversion operator U.

Consider first the scalar field $\phi_{ab}(x)$ belonging to the 6-dimensional anti-symmetric representation of SU(4), which corresponds to the state $\bar{\xi}_{[a}\bar{\xi}_{b]}e^{iP\cdot x}|0\rangle$. Then the Wick contraction of two such fields $\overbracket{\phi_{ab}(x)\phi_{cd}}(y)$ can be computed as $\langle\mathbf{1}_{12}|(\bar{\xi}_{[a}\bar{\xi}_{b]}e^{iP\cdot x}|0\rangle\otimes\bar{\xi}_{[c}\bar{\xi}_{d]}e^{iP\cdot y}|0\rangle)$. Since the singlet structure for the SU(4) part gets extracted as the unique factor ϵ_{abcd}, we obtain

$$\overbracket{\phi_{ab}(x)\phi_{cd}}(y)\propto\epsilon_{abcd}I(x,y)\,, \tag{2.110}$$

where

$$I(x,y)\equiv\langle\mathbf{1}_{12}|(e^{iP\cdot x}|0\rangle\otimes e^{iP\cdot y}|0\rangle)\,. \tag{2.111}$$

The function $I(x,y)$ will be seen below to be the basic building block for the contractions of all the super Yang-Mills fields and can be easily fixed by the singlet conditions[14] with $J^A{}_B$

[14] As we shall see below, the singlet conditions produce Ward identities.

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taken to be translation and the dilatation generators in the following way. First, applying the singlet condition (2.89) taking $J^A{}_B$ to be the translation generator, we have

$$\begin{aligned}0 &= \langle \mathbf{1}_{12}|(iP_\mu e^{iP\cdot x}|0\rangle \otimes e^{iP\cdot y}|0\rangle) + \langle \mathbf{1}_{12}|(e^{iP\cdot x}|0\rangle \otimes iP_\mu e^{iP\cdot y}|0\rangle) \\ &= \left(\frac{\partial}{\partial x^\mu} + \frac{\partial}{\partial y^\mu}\right) I(x,y) \end{aligned} \tag{2.112}$$

This gives $I(x,y) = I(x-y)$. Next, we use the dilatation operator given by $D = (i/2)(\lambda_\alpha \mu^\alpha + \tilde{\lambda}_{\dot\alpha}\tilde{\mu}^{\dot\alpha} + 2)$. Since $P \cdot x$ can be written as $\lambda_\alpha \tilde{\lambda}_{\dot\alpha} x^{\dot\alpha\alpha}$, the action of $\lambda_\alpha \mu^\alpha$ in D on $e^{iP\cdot x}|0\rangle$ gives

$$\lambda_\alpha \mu^\alpha e^{iP\cdot x}|0\rangle = i\lambda_\alpha \tilde{\lambda}_{\dot\alpha} x^{\dot\alpha\alpha} e^{i\lambda_\alpha\tilde{\lambda}_{\dot\alpha}x^{\dot\alpha\alpha}}|0\rangle = iP\cdot x e^{iP\cdot x}|0\rangle = x^\mu \frac{\partial}{\partial x^\mu} e^{iP\cdot x}|0\rangle\,. \tag{2.113}$$

Evidently, the action of $\tilde{\lambda}_{\dot\alpha}\tilde{\mu}^{\dot\alpha}$ on $e^{iP\cdot x}|0\rangle$ gives exactly the same contribution. In a similar manner, the contribution from the D acting on $e^{iP\cdot y}|0\rangle$ in the singlet condition relation produces the same result with x^μ replaced by y^μ. Altogether, the singlet condition with $J^A{}_B = D$ yields

$$\left(x^\mu \frac{\partial}{\partial x^\mu} + y^\mu \frac{\partial}{\partial y^\mu} + 2\right) I(x-y) = 0\,. \tag{2.114}$$

The solution is obviously

$$I(x-y) \propto \frac{1}{(x-y)^2}\,. \tag{2.115}$$

Let us now describe how the contraction of the fundamental fermions, i.e. $\overbrace{\psi_{\alpha a}(x)\bar{\psi}^b_{\dot\alpha}}(y)$ can be done using the singlet projector. The singlet part for the R-symmetry obviously gives δ^b_a and hence we have

$$\overbrace{\psi_{\alpha a}(x)\bar{\psi}^b_{\dot\alpha}}(y) \propto \delta_a{}^b \langle \mathbf{1}_{12}|(e^{iP\cdot x}\lambda_\alpha|0\rangle_D \otimes e^{iP\cdot y}\tilde{\lambda}_{\dot\alpha}|0\rangle_D)\,. \tag{2.116}$$

In this case, we may use the crossing relation (2.109) for the oscillators to rewrite the r.h.s. as

$$\langle \mathbf{1}_{12}|(e^{iP\cdot x}\lambda_\alpha|0\rangle_D \otimes e^{iP\cdot y}\tilde{\lambda}_{\dot\alpha}|0\rangle_D) = \langle \mathbf{1}_{12}|(e^{iP\cdot x}|0\rangle_D \otimes e^{iP\cdot y}\lambda_\alpha\tilde{\lambda}_{\dot\alpha}|0\rangle_D) = -i\frac{\partial}{\partial y^{\dot\alpha\alpha}} I(x-y) \tag{2.117}$$

Therefore, up to an overall normalization, we obtain

$$\overbrace{\psi_{\alpha a}(x)\bar{\psi}^b_{\dot\alpha}}(y) \propto 2i\delta^b_a \frac{(x-y)_{\alpha\dot\alpha}}{|x-y|^4} \tag{2.118}$$

Likewise, the Wick contraction for the self-dual field strength can be computed, again using the crossing relations for the oscillators, as

$$\begin{aligned}\overbrace{F_{\alpha\beta}(x)\bar{F}_{\dot\alpha\dot\beta}}(y) &\propto \langle \mathbf{1}_{12}|(e^{iP\cdot x}\lambda_\alpha\lambda_\beta|0\rangle_D \otimes e^{iP\cdot y}\tilde{\lambda}_{\dot\alpha}\tilde{\lambda}_{\dot\beta}|0\rangle_D) \\ &= -\frac{1}{2}\Big(\langle \mathbf{1}_{12}|(e^{iPx}\lambda_\alpha\tilde{\lambda}_{\dot\alpha}|0\rangle_D \otimes e^{iPy}\lambda_\beta\tilde{\lambda}_{\dot\beta}|0\rangle_D) \\ &\quad + \langle \mathbf{1}_{12}|(e^{iPx}\lambda_\alpha\tilde{\lambda}_{\dot\beta}|0\rangle_D \otimes e^{iPy}\lambda_\beta\tilde{\lambda}_{\dot\alpha}|0\rangle_D)\Big)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left(\frac{\partial}{\partial x^{\dot{\alpha}\alpha}}\frac{\partial}{\partial y^{\dot{\beta}\beta}} + \frac{\partial}{\partial x^{\dot{\beta}\alpha}}\frac{\partial}{\partial y^{\dot{\alpha}\beta}}\right) I(x-y) \\
&= -6\frac{(x-y)_{(\alpha\dot{\alpha}}(x-y)_{\beta)\dot{\beta}}}{|x-y|^6}\,.
\end{aligned}
\tag{2.119}
$$

Normalizations of these two point functions depend of course on the choice of the normalization of the individual fields but once we fix one of them, then the rest can be determined by supersymmetry.

With the demonstrations above, we wish to emphasize that our method of using the conformally covariant D-scheme is quite simple and useful in that the properties of the singlet projector can be directly and effectively utilized.

3 Monodromy relations for correlation functions in psu(2,2|4) spin chain system

Having constructed the singlet projector in the conformally covariant basis, we shall now generalize the so-called monodromy relations for the correlation functions developed in our previous work [29] for the SU(2) sector to the full psu(2,2|4) sector. Here one must first note the following new features. In the case of the SU(2) sector, the structure of the auxiliary Hilbert space is unequivocally chosen to be identical to that of the quantum Hilbert space, both two dimensional, describing the up and down "spin" states. On the other hand, for psu(2,2|4) there are two appropriate choices for the auxiliary space. To see this, we should recall the properties of the general R-matrix, to be denoted by $\mathbb{R}_{ij}(u)$, from which the monodromy matrix is constructed. It is a linear map acting on the tensor product of two vector spaces $V_i \otimes V_j$, i.e. $\mathbb{R}_{ij} \in \mathrm{End}(V_i \otimes V_j)$, and satisfying the following Yang-Baxter equation:

$$
\mathbb{R}_{12}(u_1-u_2)\mathbb{R}_{13}(u_1)\mathbb{R}_{23}(u_2) = \mathbb{R}_{23}(u_2)\mathbb{R}_{13}(u_1)\mathbb{R}_{12}(u_1-u_2)\,, \tag{3.1}
$$

where complex parameters u_i are the spectral parameters. From such $\mathbb{R}_{ij}(u)$ matrices, one constructs the monodromy matrix $\Omega(u) = \mathbb{R}_{a1}(u)\cdots\mathbb{R}_{a\ell}(u)$, where a here is the label for the auxiliary space V_a and the numbers 1 through ℓ denote the location of the site at which u(2,2|4) spin state resides to make up a spin chain.[15] Then, out of the monodromy matrix, one defines the transfer matrix $T_a(u)$ by taking the trace over the auxiliary space, namely $T(u) := \mathrm{Tr}_a\Omega(u)$. The prime importance of the Yang-Baxter equation (3.1) is that it ensures the commutativity of the transfer matrices at different spectral parameters, i.e. $[T(u), T(v)] = 0$, which in turn implies that the quantities obtained as the coefficients of the power expansion in the spectral parameter all commute. In particular, as one of such quantities can be identified with the Hamiltonian of the spin chain, all the coefficients can be interpreted as conserved charges. This is at the heart of the integrability.

Now in the case of the u(2,2|4) spin chain, while the quantum Hilbert space is taken to be the Fock space $\mathcal{V}$ constructed by the oscillators introduced in the previous section,

[15] For the discussion of concepts requiring the Yang-Baxter equation, we must consider u(2,2|4), but not psu(2,2|4), as it is the R-matrix associated with the former which satisfies the Yang-Baxter equation. We shall give more detailed discussion on this point later.

there are two natural choices for the auxiliary space V_a, which should form a representation of u(2,2|4) or its complexified version gl(4|4).[16] One is the fundamental representation of gl(4|4), i.e. $V_a = \mathbb{C}^{4|4}$ and the other is the choice $V_a = \mathcal{V}$, i.e. the auxiliary space being the same as the infinite dimensional quantum space in structure. We shall call the corresponding R-matrix as "fundamental" for the former case and "harmonic" for the latter choice.

For the former case, the monodromy matrix is finite dimensional and its components are operators acting on the quantum space. These components satisfy the exchange relations (or Yang-Baxter algebra) coming from the Yang-Baxter equations, and are quite powerful in diagonalizing the transfer matrix in the context of algebraic Bethe ansatz. It should be noted that a similar finite dimensional monodromy matrix can be defined classically in the strong coupling regime using the flat connections of the string sigma model and can be used to determine the semi-classical spectrum [51]. Further, beyond the spectral problem, the monodromy relation of this type has its counterpart in the computation of the three-point functions in the strong coupling regime [27–30] as the triviality of the total monodromy of the form $\Omega_1\Omega_2\Omega_3 = 1$, where Ω_i is the local monodromy produced around the i-th vertex operator in the so-called auxiliary linear problem. As explained in [27, 30], this seemingly weak relation is disguisingly powerful, as it captures the important global information governing the three-point functions.

Such monodromy relations for the fundamental R-matrix for gl(4|4) can be derived through a procedure similar to the one for the SU(2) case worked out in detail in our previous work [29] and has been discussed in [32]. Besides the purpose of completeness, we shall re-derive these relations below since we shall use the definition of the Lax operator, slightly different from the one used in [32], which is more natural in connection with the strong coupling counterpart.

Next let us briefly describe the characteristics of the monodromy relations we shall derive for the harmonic R-matrix, which are completely new. In this case, the monodromy matrix is no longer finite dimensional since the auxiliary space V_a is the same as the infinite dimensional quantum spin-chain Hilbert space $\mathcal{V}$. One of the virtues of considering such a harmonic R-matrix is that, just as in the case of the SU(2) Heisenberg spin chain, the construction of the conserved charges including the Hamiltonian is much easier, since due to the identical structure of V_a and $\mathcal{V}$ the R-matrix at specific value of the spectral parameter serves as the permutation operator P_{an}. Such an operator is known to be extremely useful in extracting the Hamiltonian (i.e. the dilatation operator). Because of this and other features, the harmonic R-matrix and the related quantities have already found an interesting applications in the computation of the scattering amplitudes from the point of view of integrability [39–46] and are expected to be useful in the realm of the correlation functions as well.

In any case, since the monodromy matrix, constructed out of either "fundamental" or "harmonic" R-matrices, is a generating function of an infinite number of conserved charges, the monodromy relations can be regarded as a collection of "Ward identities"

[16]The most of the discussion to follow is insensitive to whether we consider u(2,2|4) or its complexified version gl(4|4). Thus, when the description is easier with the complexified version, we shall use gl(4|4) in place of u(2,2|4).

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associated with such higher charges, which should characterize the important properties of the correlation functions.

Remarks on the relevance of of u(2,2|4) for the monodromy relation and psu(2,2|4) for the singlet projector. Before we begin the discussion of the monodromy relations, let us give some important clarifying remarks on the relevance of the different super algebras for the two topics we discuss in this work and the role of their oscillator representation.

- The monodromy relation, to be discussed below, is deeply rooted in the integrability of the theory and hence it is crucial that the relevant R-matrix and the Lax matrix must satisfy the Yang-Baxter equations and the RLL=LLR equations. A method has long been known [35–38] that one can construct such an R-matrix and a Lax matrix from a suitable Lie super algebra. In the present case, one can do so for u(2,2|4) algebra but not for su(2,2|4) or psu(2,2|4). This is a general mathematical statement and has nothing to do with a particular oscillator representation nor with the super Yang-Mills theory. However, when one makes use of the singleton oscillator representation, then one can easily construct the states which form the fundamental Field strength multiplet of SYM theory and the R-matrix and the monodromy matrix can be constructed in terms of the generators bilinear in the oscillators. Although the basic SYM fields carry a special value of the central charge $C = 0$ and the global symmetry of the $\mathcal{N} = 4$ SYM theory is psu(2,2|4), still when we discuss the monodromy relations for the correlation functions for the composite operators made up of these SYM fields, the generators and the related quantities to be used must be those of u(2,2|4).

- On the other hand, when we use the singlet projector to perform the Wick contractions efficiently in the computation of the correlation functions, the projector is a singlet for su(2,2|4) and psu(2,2|4). This notion has nothing to do with the integrability. In fact a singlet projector for u(2,2|4) does not exist at least in the oscillator representation utilized and this point gives a subtle effect in the crossing relation, to be discussed in the next subsection.

Thus, in the monodromy relations for the correlation functions, two different superalgebras are playing their respective role. The monodromy matrices to be inserted are associated with u(2,2|4), while the singlet projector which works as an elegant device in forming the correlation function for the physical SYM fields is valid for psu(2,2|4).

3.1 Basic monodromy relation in the case of fundamental R-matrix

Let us begin with the case of the monodromy relations with the use of the fundamental R-matrix. We shall first give the definitions and conventions for the fundamental R-matrix and the associated Lax matrix, which are slightly different from the ones used in [32], and then discuss the two important relations, namely the crossing relations and the inversion relations, which will lead immediately to the monodromy relations of interest.

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3.1.1 Fundamental R-matrix and Lax operator

Consider the fundamental R-matrix, for which the quantum space is $\mathcal{V}$ and the auxiliary space is taken to be $\mathbb{C}^{4|4}$. This kind of R-matrix is often called the Lax operator and will be denoted by $L_{an}(u)$, where a and n refer, respectively, to the auxiliary space and the position on the spin chain. It satisfies the important relation called RLL=LLR relation

$$R_{12}(u_1 - u_2)L_{a_1 n}(u_1)L_{a_2 n}(u_2) = L_{a_2 n}(u_2)L_{a_1 n}(u_1)R_{12}(u_1 - u_2)\,, \tag{3.2}$$

which follows from the basic Yang-Baxter equation (3.1) by setting $V_1 = V_2 = \mathbb{C}^{4|4}$ and $V_3 = \mathcal{V}_n$, where n is the position of the spin. The R-matrix $R_{12}(u)$ appearing in this equation acts on the tensor product of two copies of the auxiliary space $V_1 \otimes V_2$ and, besides the RLL=LLR equation, it also satisfies the original Yang-Baxter equation denoted as RRR=RRR equation:

$$R_{12}(u_1 - u_2)R_{13}(u_1)R_{23}(u_2) = R_{23}(u_2)R_{13}(u_1)R_{12}(u_1 - u_2)\,. \tag{3.3}$$

The solution of the above RRR relation turns out to be of the form

$$R_{ij}(u) = u + \eta(-1)^{|B|}E_i{}^A{}_B \otimes E_j{}^B{}_A\,, \qquad (E^A{}_B)^C_D \equiv \delta^A_C\delta^B_D\,, \tag{3.4}$$

where η is an arbitrary complex parameter[17] and $E_i{}^A{}_B$ is the fundamental representation of gl(4|4) acting non-trivially on $V_i \cong \mathbb{C}^{4|4}$. To check that the R-matrix above actually satisfies the Yang-Baxter it is useful to note that the operator $\Pi_{ij} := (-1)^{|B|}E_i{}^A{}_B \otimes E_j{}^B{}_A$ serves as the graded permutation operator.[18] For example, $\Pi_{12}(a \otimes b \otimes c) = (-1)^{|a||b|}(b \otimes a \otimes c)$, $\Pi_{13}(a \otimes b \otimes c) = (-1)^{|a|(|b|+|c|)+|b||c|}(c \otimes b \otimes a)$ and so on. Then, the Lax operator satisfying (3.3) is given by

$$L_{a_i n}(u) = u + \eta(-1)^{|B|}E_i{}^A{}_B \otimes J_n{}^A{}_B\,, \tag{3.7}$$

where $J_n{}^A{}_B$'s are the generators of gl(4|4) defined on the n-th site of the spin chain. It is tedious but straightforward to show that the Lax operator indeed satisfies the RLL=LLR relation, by explicitly computing the both sides. In performing this calculation, one should remember that there are no grading relations between the auxiliary space and quantum spaces, which are two independent spaces. Explicitly, this means

$$(E^A{}_B \otimes J^A{}_B)(E^C{}_D \otimes J^C{}_D) = (E^A{}_B E^C{}_D) \otimes (J^A{}_B J^C{}_D) = E^A{}_D \otimes (J^A{}_B J^B{}_D)\,, \tag{3.8}$$

where we have used $E^A{}_B E^C{}_D = \delta^B_C E^A{}_D$. Differently put, the definition of the product of the Lax operators is not as supermatrices but as usual matrices. Although the choice for

[17] Although the Yang-Baxter equation holds for arbitrary η, we will later set $\eta = \pm i$ for our interest.

[18] To prove this, we should pay attention to the non-trivial gradings between two auxiliary spaces

$$(E^A{}_B \otimes E^C{}_D)(a \otimes b) = (-1)^{a(|C|+|D|)}(E^A{}_B a) \otimes (E^C{}_D b) \quad a \otimes b \in \mathbb{C}^{4|4} \otimes \mathbb{C}^{4|4}\,, \tag{3.5}$$

$$(E^A{}_B \otimes E^C{}_D)(E^E{}_F \otimes E^G{}_H) = (-1)^{(|C|+|D|)(|E|+|F|)}(E^A{}_B E^E{}_F) \otimes (E^C{}_D E^G{}_H)\,. \tag{3.6}$$

.

the mutual grading between these two spaces is a matter of convention,[19] our choice is a natural one from the point of view of connecting to the strong coupling regime. This is simply because the monodromy matrix at strong coupling is defined by the path ordered exponential of the integral of the flat connection and the multiplication rule for such matrices is the ordinary one. With this convention, the explicit form of the Lax operator is given in terms of the superconformal generators by

$$(L(u))^A{}_B = u\delta^A{}_B + \eta(-1)^{|B|} J^A{}_B = \left(\begin{array}{cc|c} u+\eta Y_\alpha{}^\beta & i\eta P_{\alpha\dot\beta} & -\eta Q^b_\alpha \\ i\eta K^{\dot\alpha\beta} & u+\eta Y^{\dot\alpha}{}_{\dot\beta} & -i\eta \bar S^{\dot\alpha b} \\ \hline \eta S^\beta_a & i\eta \bar Q_{\dot\beta a} & u-\eta W^b_a \end{array}\right)_{AB} . \tag{3.9}$$

As usual the monodromy matrix is defined as the product of the Lax operators on each site going around the spin chain of length ℓ:

$$\Omega_a(u) := L_{a1}(u)\cdots L_{a\ell}(u)\,. \tag{3.10}$$

The monodromy matrix so defined satisfies the following relation, since each Lax operator satisfies the RLL=LLR relation:

$$R_{12}(u_1-u_2)\Omega_{a_1}(u_1)\Omega_{a_2}(u_2) = \Omega_{a_2}(u_2)\Omega_{a_1}(u_1)R_{12}(u_1-u_2)\,. \tag{3.11}$$

If we write out the above equation for each component, we obtain the so-called Yang-Baxter exchange algebra. In the rest of this subsection, when there is no confusion we drop the indices for the auxiliary space for simplicity.

3.1.2 Monodromy relation

Let us now derive the generic monodromy relation. This can be achieved by proving the following two important relations for the Lax operators, called the crossing relation and the inversion relation. They are respectively of the form

$$\text{(C)}: \quad \langle \mathbf{1}_{12}| L^{(1)}_n(u) = -\langle \mathbf{1}_{12}| L^{(2)}_{\ell-n+1}(\eta-u)\,, \tag{3.12}$$

$$\text{(I)}: \quad L^{(i)}_n(u) L^{(i)}_n(v) = u(\eta-u)\,, \quad (u+v=\eta)\,, \tag{3.13}$$

where the superscript (i) on $L^{(i)}_n$ denotes the i-th spin chain. The crossing relation (C) connects the Lax operator defined on the n-th site of a spin chain called 1 to that defined on the $\ell-n+1$-th site of another spin chain called 2. To get the feeling for the crossing relation, it suffices to recall that the singlet projector $\langle \mathbf{1}_{12}|$ effects the Wick contraction between a field at the n-th site of one spin chain and a field at the $\ell-n+1$-th site of another chain. Actually, it is easy to prove it more precisely from the defining property of the singlet $\langle \mathbf{1}_{12}|$. As it was already emphasized in section 2.2, the operator $\langle \mathbf{1}_{12}|$ is a singlet projector for su(2,2|4) or psu(2,2|4) but *not* for u(2,2|4) which is of our concern

[19] In [32], the authors adopt the convention where non-trivial gradings between the auxiliary space and the quantum space exist. Namely, $(E^A{}_B \otimes J^A{}_B)(E^C{}_D \otimes J^C{}_D) = (-1)^{(|A|+|B|)(|C|+|D|)}(E^A{}_B E^C{}_D)\otimes(J^A{}_B J^C{}_D)$. Because of this, the definition of the Lax operator they use, i.e. $L(u) := u - i/2 - i(-1)^{|A|}E^A{}_B \otimes J^B{}_A$, is slightly different from ours.

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here. So the operator $\langle \mathbf{1}_{12}|$ transforms the generator $J^A{}_B$ of u(2,2|4) acting on the first spin chain into the operator $-J^A_B - (-1)^{|A|}\delta^A_B$ acting on the second spin chain, where the constant piece $-(-1)^{|A|}\delta^A_B$ comes from the (anti-)commutator term. Applying this to the Lax operator $(L(u))^A_B = u\delta^A_B + \eta(-1)^{|B|}J^A_B$, one sees that the constant term shifts the spectral parameter by η and we get the crossing relation as shown in (3.12).

The proof of the inversion relation (I), which says that the Lax operator can be inverted for a specific value of the spectral parameter, is slightly more involved. The product of two Lax operators gives

$$(L(u)L(v))^A_B = uv\delta^A_B + \eta(u+v)(-1)^{|B|}J^A_B + \eta^2(-1)^{|B|+|C|}J^A_C J^C_B\,. \tag{3.14}$$

First look at the last term quadratic in the generators. In general this cannot be simplified further. However, as we are specializing in the oscillator representation, we can write $J^A_B = \bar{\zeta}^A\zeta_B$ using the oscillators satisfying $[\zeta_A, \bar{\zeta}^B] = \delta_A{}^B$, as described in (2.44). Therefore we can reduce the product of the generator in the following way:

$$(-1)^{|C|}J^A_C J^C_B = \bar{\zeta}^A(\bar{\zeta}^C\zeta_C)\zeta_B = (2C-1)J^A_B\,. \tag{3.15}$$

For the first equality, we have used the fact that the constant term $(-1)^{|C|}\delta^C_C$, which appears from the commutation relation, vanishes. Further, since $[\bar{\zeta}^C\zeta_C, \bar{\zeta}^A] = 1$ and the central charge is given by $2C = \mathrm{tr}J = \bar{\zeta}^C\zeta_C$, we obtain the result above. Hence the product of the Lax operators (3.14) is simplified to

$$(L(u)L(v))^A_B = uv\delta^A_B + \eta(u+v+\eta(2C-1))(-1)^{|B|}J^A_B\,. \tag{3.16}$$

Now as we repeatedly emphasized, for the Yang-Mills fields of our interest we can set $C = 0$ and hence the coefficient in front of $(-1)^{|B|}J^A{}_B$ becomes $\eta(u+v-\eta)$. Therefore when $u + v = \eta$, the r.h.s. of (3.16) becomes $uv\delta^A_B = u(\eta-u)\delta^A_B$, which is precisely the r.h.s. of the inversion equation (3.13).

We are now ready to present the generic form of the monodromy relation, which takes the form

$$\langle \mathbf{1}_{12}|\Omega^{(1)}(u)\Omega^{(2)}(u) = \langle \mathbf{1}_{12}|F_\ell(u)\,, \tag{3.17}$$

where $F_\ell(u)$ is some function of u, to be given shortly. Both sides of this relation should be understood as acting on a tensor product of states on two Hilbert spaces of the form $|\mathcal{O}_1\rangle\otimes|\mathcal{O}_2\rangle$. To show (3.17), we first prove the following relation with the use of the crossing relation (3.12):

$$\langle \mathbf{1}_{12}|\Omega^{(1)}(u) = (-1)^\ell \langle \mathbf{1}_{12}|\overleftarrow{\Omega}^{(2)}(\eta-u)\,, \tag{3.18}$$

$$\overleftarrow{\Omega}^{(2)}(u) := L^{(2)}_\ell(u)\cdots L^{(2)}_1(u)\,. \tag{3.19}$$

Focus first on the l.h.s. of (3.18) and consider moving the Lax operator $L^{(1)}_n(u)$ at the n-th site in $\Omega^{(1)}$ to the left towards $\langle \mathbf{1}_{12}|$. Since the components of the Lax operators on different sites commute in the graded sense, namely

$$(L_n(u))_{AB}(L_m(v))_{CD} = (-1)^{(|A|+|B|)(|C|+|D|)}(L_m(v))_{CD}(L_n(u))_{AB}\,, \tag{3.20}$$

we can move $L_n^{(1)}(u)$ all the way to the left and hit $\langle \mathbf{1}_{12}|$ like

$$\langle \mathbf{1}_{12}| \ldots (L_n^{(1)}(u))_{AB} \cdots = (-1)^{(|A|+|B|)(\ldots)} \langle \mathbf{1}_{12}| (L_n^{(1)}(u))_{AB} \ldots \cdots . \tag{3.21}$$

We can now use the crossing relation (C) to replace the Lax operator $(L_n^{(1)}(u))_{AB}$ with $(L_{\ell-n+1}^{(2)}(\eta-u))_{AB}$ and move it back again to the original position. In this process, the sign factors which appear through the exchange of operators exactly cancel with those produced in the previous process and we get

$$\langle \mathbf{1}_{12}| \cdots L_n^{(1)}(u) \cdots = -\langle \mathbf{1}_{12}| \cdots L_{\ell-n+1}^{(2)}(\eta-u) \cdots . \tag{3.22}$$

Repeating this to all the Lax operators making up the monodromy matrices, we immediately get (3.18). Now apply $\Omega^{(2)}(u)$ to the both sides of (3.18) and use the relation $\overleftarrow{\Omega}^{(2)}(\eta-u)\Omega^{(2)}(u) \propto 1$, with the overall factor which is readily computable using the inversion relation (I). In this way we obtain (3.17) with the function $F_\ell(u)$ given by $F_\ell(u) = (u(u-\eta))^\ell$. Now if we apply (3.17) explicitly to the state $|\psi_1\rangle \otimes |\psi_2\rangle$, we obtain the more explicit monodromy relation for the two-point function

$$(-1)^{(|C|+|B|)|\psi_1|} \Big\langle (\Omega^{(1)}(u))_{AB}|\psi_1\rangle \, , (\Omega^{(2)}(u))_{BC}|\psi_2\rangle \Big\rangle = F_\ell(u)\delta_{AC} \left\langle |\psi_1\rangle \, , |\psi_2\rangle \right\rangle , \tag{3.23}$$

$$F_\ell(u) = (u(u-\eta))^\ell \, . \tag{3.24}$$

Here $\langle \, , \rangle$ denotes the pairing with the singlet, which gives the Wick contraction between two operators. The sign in front arises when we pass the monodromy through the first state $|\psi_1\rangle$.

At this point, it is of importance to remark that we obtain the usual Ward identities at the leading order in the expansion of the above equation around $u = \infty$. This is a direct consequence of the su(2,2|4) invariance of the singlet projector.

Once the monodromy relation is obtained for the two-point functions, the one for the three-point functions can be obtained easily, just as was shown explicitly for the SU(2) sector in [31]. The only differences from that case are the form of the prefactor function $F_{123}(u)$ and some sign factors due to the superalgebra nature of psu(2,2|4). Thus the monodromy relation for the three-point function takes the form

$$\begin{aligned} &\Big\langle (\Omega(u))_{AB}|\psi_1\rangle, (-1)^{|\psi_1|(|B|+|C|)}(\Omega(u))_{BC}|\psi_2\rangle, (-1)^{(|\psi_1|+|\psi_2|)(|C|+|D|)}(\Omega(u))_{CD}|\psi_3\rangle \Big\rangle \\ &= F_{123}(u)\delta_{AD} \left\langle |\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle \right\rangle , \qquad F_{123}(u) = (u(u-\eta))^{\ell_1+\ell_2+\ell_3} \, . \end{aligned} \tag{3.25}$$

3.2 Basic monodromy relation in the case of harmonic R-matrix

We shall now discuss another important version of the R-matrix, called the harmonic R-matrix, to be denoted by the bold letter $\mathbf{R}$. We shall derive the inversion and the crossing relations for it and finally prove the relations for the correlation functions obtained with the insertion of monodromy matrices constructed out of the harmonic R-matrices.

The term "harmonic" stems from the form of the Hamiltonian (or dilatation) density first derived in [49], which can be expressed as[20] $H_{12} = h(J_{12})$, where the function

[20]The subscript 12 signifies that the Hamiltonian is restricted to two fields 1 and 2.

$h(j) = \sum_{k=1}^{j} 1/k$ is the so-called harmonic number and J_{12}^2 is the quadratic Casimir operator. This Hamiltonian is intimately related to the one in the SL(2) subsector and in that context was derived also as the logarithmic derivative of the R-matrix, just as in the case of the SU(2) Heisenberg spin chain.

The harmonic R-matrix is recently applied in the context of the scattering amplitudes for the $\mathcal{N} = 4$ SYM theory, as a tool to construct the building blocks for the deformed Grassmannian formulas characterized as Yangian invariants [39–46]. The spectral parameter can be naturally introduced in the deformed formulas and turned out to serve as a regulator for the IR divergences.

Since this type of R-matrix is less well-known, we shall first give a brief review of the basic facts on the harmonic R-matrix following [40] and then using these properties derive the crossing and inversion relations, which are essential, as in the case of the fundamental R-matrix, in obtaining the monodromy relations we seek.

3.2.1 Review of the harmonic R-matrix

The harmonic R-matrix $\mathbf{R}_{12}$ acting on the tensor product of two copies of the Fock space $\mathcal{V}_1 \otimes \mathcal{V}_2$ should satisfy the following RLL=LLR relation

$$\mathbf{R}_{12}(u_1 - u_2) L_1(u_1) L_2(u_2) = L_2(u_2) L_1(u_1) \mathbf{R}_{12}(u_1 - u_2) . \tag{3.26}$$

This is obtained from the general formula (3.1) by setting $V_1 = \mathcal{V}_1$, $V_2 = \mathcal{V}_2$, $V_3 = \mathbb{C}^{4|4}$ and replacing $\mathbb{R}_{i3}(u_i)$ with the Lax operator $L_i(u_i) = u_i + \eta(-1)^{|B|} E^A{}_B \otimes J_i{}^A{}_B$. Renaming $u = u_1 - u_2$, $u_2 = \tilde{u}$ and expanding the above equation in powers of $\tilde{u}$, one can show that the harmonic R-matrix satisfies the following two types of equations:

$$(i) \quad [\mathbf{R}_{12}(u), J_1{}^A{}_B + J_2{}^A{}_B] = 0 , \tag{3.27}$$

$$(ii) \quad (-1)^{|C|}\eta(\mathbf{R}_{12}(u) J_1{}^A{}_C J_2{}^C{}_B - J_2{}^A{}_C J_1{}^C{}_B \mathbf{R}_{12}(u)) - u(J_2{}^A{}_B \mathbf{R}_{12}(u) - \mathbf{R}_{12}(u) J_2{}^A{}_B) = 0 . \tag{3.28}$$

The first expresses the invariance of the harmonic R-matrix under gl(4|4), while the second implies the invariance under the level 1 generators of the Yangian algebra. They together ensure the full Yangian invariance of the harmonic R-matrix. As it can be explicitly verified after constructing $\mathbf{R}_{12}$ explicitly, the product $\mathbf{R}_{12}(u)\mathbf{R}_{12}(-u)$ is proportional to the unit operator $\mathbf{1}_{12}$ but the overall normalization can be arbitrary, since the equations (i) and (ii) above are both linear in $\mathbf{R}$. Therefore one can impose the following unitarity condition, or inversion relation, to fix the overall scale:

$$\mathbf{R}_{12}(u)\mathbf{R}_{12}(-u) = \mathbf{1}_{12} . \tag{3.29}$$

Conversely, the solution satisfying (3.27)–(3.29) is unique, as we demonstrate later.

To actually find the form of $\mathbf{R}_{12}(u)$, we will first solve the equation (3.27). For this purpose, it is convenient to introduce the following notations for sets of oscillators:

$$\bar{\alpha}^{\mathsf{A}} = \begin{pmatrix} \lambda_\alpha \\ \bar{c}_i \end{pmatrix} , \quad \alpha_{\mathsf{A}} = \begin{pmatrix} \mu^\alpha \\ c^i \end{pmatrix} , \tag{3.30}$$

$$\bar{\beta}^{\dot{\mathsf{A}}} = \begin{pmatrix} \tilde{\lambda}_{\dot{\alpha}} \\ \bar{d}_i \end{pmatrix}, \quad \beta_{\dot{\mathsf{A}}} = \begin{pmatrix} \tilde{\mu}^{\dot{\alpha}} \\ d^i \end{pmatrix}. \tag{3.31}$$

Notice that $\alpha_{\mathsf{A}}|Z\rangle = \beta_{\dot{\mathsf{A}}}|Z\rangle = 0$. These oscillators transform covariantly under the subalgebras $\mathrm{gl}(2|2) \oplus \mathrm{gl}(2|2) \subset \mathrm{gl}(4|4)$, whose bosonic parts are given by the Lorentz and $\mathrm{su}(2)_L \oplus \mathrm{su}(2)_R$ R-symmetry subalgebras. In other words, the indices $\mathsf{A}, \mathsf{B}, \ldots$ are associated with the (anti-) fundamental representation of one $\mathrm{gl}(2|2)$ and the indices $\dot{\mathsf{A}}, \dot{\mathsf{B}}, \ldots$ describe the (anti-) fundamental representation of the other $\mathrm{gl}(2|2)$. Accordingly, the $\mathrm{gl}(4|4)$ generators are decomposed into diagonal parts and the off-diagonal parts with respect to these two $\mathrm{gl}(2|2)$ subalgebras in the following form:

$$J^A{}_B \longrightarrow \begin{pmatrix} J^{\mathsf{A}}{}_{\mathsf{B}} & J^{\mathsf{A}}{}_{\dot{\mathsf{B}}} \\ J^{\dot{\mathsf{A}}}{}_{\mathsf{B}} & J^{\dot{\mathsf{A}}}{}_{\dot{\mathsf{B}}} \end{pmatrix}. \tag{3.32}$$

The explicit form of these generators in terms of α, β-oscillators are given in [40], but we shall not write them down here.

Now using the oscillators above, one introduces the following basis of linear operators acting on $\mathcal{V}_1 \otimes \mathcal{V}_2$, which will be useful for solving the condition (3.27):

$$\begin{aligned} \mathbf{Hop}^{(12)}_{k,l,m,n} &= : \frac{(\bar{\alpha}_2 \alpha^1)^k}{k!} \frac{(\bar{\beta}^2 \beta_1)^l}{l!} \frac{(\bar{\alpha}_1 \alpha^2)^m}{m!} \frac{(\bar{\beta}^1 \beta_2)^n}{n!} : \\ &= \frac{1}{k!l!m!n!} \bar{\alpha}_2^{\mathsf{A}_1} \cdots \bar{\alpha}_2^{\mathsf{A}_k} \bar{\beta}^2_{\dot{\mathsf{A}}_1} \cdots \bar{\beta}^2_{\dot{\mathsf{A}}_l} \bar{\alpha}_1^{\mathsf{B}_1} \cdots \bar{\alpha}_1^{\mathsf{B}_m} \bar{\beta}^1_{\dot{\mathsf{B}}_1} \cdots \bar{\beta}^1_{\dot{\mathsf{B}}_n} \\ &\quad \cdot \beta_2^{\dot{\mathsf{B}}_n} \cdots \beta_2^{\dot{\mathsf{B}}_1} \alpha^2_{\mathsf{B}_m} \cdots \alpha^2_{\mathsf{B}_1} \beta_1^{\dot{\mathsf{A}}_l} \cdots \beta_1^{\dot{\mathsf{A}}_1} \alpha^1_{\mathsf{A}_k} \cdots \alpha^1_{\mathsf{A}_1}. \end{aligned} \tag{3.33}$$

Here the symbol $: * :$ in the first line denotes the normal ordering of the oscillators. The name **Hop** stems from the following properties of this operator. Its action transforms $k+l$ oscillators with label 1 to those with label 2 and $m+n$ oscillators with label 2 to those with label 1. Thus it effects a kind of hopping operation. Note that these operators are manifestly invariant under the diagonal part of (3.32), namely, $J^{\mathsf{A}}{}_{\mathsf{B}}$ and $J^{\dot{\mathsf{A}}}{}_{\dot{\mathsf{B}}}$, as all the relevant indices in (3.33) are contracted. Therefore, the general solution $\mathbf{R}_{12}(u)$ of (3.27) should be obtained as a linear combination of $\mathbf{Hop}^{(12)}_{k,l,m,n}$ of the form

$$\mathbf{R}_{12}(u) = \sum_{k,l,m,n} \mathcal{A}^{(\mathbf{N})}_{k,l,m,n}(u) \mathbf{Hop}^{(12)}_{k,l,m,n}, \tag{3.34}$$

where $\mathbf{N}$ stands for the total number operator defined by

$$\mathbf{N} = \mathbf{N}^{(1)} + \mathbf{N}^{(2)}, \quad \mathbf{N}^{(i)} = \mathbf{N}^{(i)}_\alpha + \mathbf{N}^{(i)}_\beta = \bar{\alpha}_i^{\mathsf{A}} \alpha^i_{\mathsf{A}} + \bar{\beta}^i_{\dot{\mathsf{A}}} \beta_i^{\dot{\mathsf{A}}}. \tag{3.35}$$

Note that the coefficients $\mathcal{A}^{(\mathbf{N})}_{k,l,m,n}(u)$ can depend in general on the spectral parameter u, the total number operator $\mathbf{N}$ and the central charge,[21] since they all commute with the diagonal generators. As explained in detail in [40], the invariance under the remaining

[21] Notice that the central charge is given by $C^{(i)} = \mathbf{N}^{(i)}_\alpha - \mathbf{N}^{(i)}_\beta$. Since we are interested in the representation in which the central charge vanishes, we neglect dependence on this combination.

off-diagonal generators $J^{\mathsf{A}}{}_{\dot{\mathsf{B}}}, J^{\dot{\mathsf{A}}}{}_{\mathsf{B}}$ together with the invariance under the level-one Yangian generators (3.28) uniquely fix the coefficients up to an overall coefficient $\rho(u)$ as

$$\begin{aligned} \mathcal{A}^{(\mathbf{N})}_{k,l,m,n} &= a_{k,l,m,n}\mathcal{A}^{(\mathbf{N})}_{I}\,, \quad a_{k,l,m,n} := \delta_{k+n,l+m}(-1)^{(k+l)(m+n)}\,, \\ \mathcal{A}^{(\mathbf{N})}_{I}(u) &= \rho(u)(-1)^{I+\frac{\mathbf{N}}{2}}\mathcal{B}(I,u+\mathbf{N}/2)\,, \quad \mathcal{B}(x,y) := \frac{\Gamma(x+1)}{\Gamma(x-y+1)\Gamma(y+1)}\,. \end{aligned} \tag{3.36}$$

Here $I := \frac{k+l+m+n}{2}$ is an integer since $k+n = l+m$, and $\mathcal{B}(x,y)$ is a natural generalization of the binomial coefficient, whose arguments can be complex. Notice that it satisfies $\mathcal{B}(x,y) = \mathcal{B}(x,x-y)$ by definition. As already mentioned before, the overall coefficient $\rho(u)$ is determined so that the unitarity condition (3.29) is satisfied. Explicit calculation gives

$$\rho(u) = \Gamma(u+1)\Gamma(1-u)\,. \tag{3.37}$$

An important characteristic of the harmonic R-matrix, for which the quantum and the auxiliary spaces are identical just as for the SU(2) Heisenberg spin chain, is that the R-matrix at $u = 0$ yields precisely the permutation operator. Because of this fact, through the well-known manipulation, the Hamiltonian can be extracted from the R-matrix simply as a logarithmic derivative. This is summarized as[22]

$$\mathbf{P}_{12} = \mathbf{R}_{12}(0)\,, \tag{3.38}$$

$$\mathbf{H}_{12} = \frac{d}{du}\ln \mathbf{R}_{12}(u)|_{u=0}\,. \tag{3.39}$$

3.2.2 Monodromy relation

Having reviewed the basic facts on the harmonic R-matrix and displayed its explicit form in terms of the oscillators, we now discuss the monodromy relations involving such R-matrices. As in the case of the fundamental R-matrix, the basic ingredients for the derivation is (i) the inversion relation and (ii) the crossing relation.

The inversion relation is already given in (3.29), together with the computation of the factor $\rho(u)$, shown in (3.37), needed for the normalization.

As for the crossing relation, its form for the harmonic R-matrix turned out to be similar to (but not identical with) the one for the Lax matrix for the case of the fundamental R-matrix shown in (3.12) and is given by

$$\langle \mathbf{1}_{12}|\mathbf{R}^{(1)}_{an}(u) = \langle \mathbf{1}_{12}|\mathbf{R}^{(2)}_{a\ell-n+1}(-u)\,. \tag{3.40}$$

Once this is verified, the crossing relation for the product of harmonic R-matrices is easily given by

$$\langle \mathbf{1}_{12}|\mathbf{R}^{(1)}_{a1}(u)\cdots\mathbf{R}^{(1)}_{a\ell}(u) = \langle \mathbf{1}_{12}|\mathbf{R}^{(2)}_{a\ell}(-u)\cdots\mathbf{R}^{(2)}_{a1}(-u)\,. \tag{3.41}$$

Now define the monodromy matrix as

$$\Omega^{(i)}_{\boldsymbol{mn}}(u) := \langle \boldsymbol{m}|\mathbf{R}^{(i)}_{a1}(u)\cdots\mathbf{R}^{(i)}_{a\ell}(u)|\boldsymbol{n}\rangle_a\,, \tag{3.42}$$

[22]Equivalently, the expansion of the R-matirx around $u = 0$ is of the form $\mathbf{R}_{12}(u) = \mathbf{P}_{12}(1 + u\mathbf{H}_{12} + \cdots)$.

where $\{|\boldsymbol{n}\rangle\}$ is a complete set of states in the auxiliary space satisfying $1 = \sum_{\boldsymbol{n}} |\boldsymbol{n}\rangle\langle\boldsymbol{n}|$. Note that the components of the monodromy matrix take values in operators acting on the quantum space. The crossing relation for them follow immediately from (3.41) by taking the matrix element between the states $\langle\boldsymbol{m}|$ and $|\boldsymbol{n}\rangle$ and is expressed as

$$\langle \mathbf{1}_{12}|\Omega^{(1)}_{\boldsymbol{mn}}(u) = \langle \mathbf{1}_{12}|\overleftarrow{\Omega}^{(2)}_{\boldsymbol{mn}}(-u)\,. \tag{3.43}$$

Now contract this relation with $\overleftarrow{\Omega}^{(2)}_{\boldsymbol{nl}}(-u)$ and sum over $\boldsymbol{n}$. Then, since the completeness of the states $\{|\boldsymbol{n}\rangle\}$ in the auxiliary space implies $\overleftarrow{\Omega}^{(2)}_{\boldsymbol{kl}}(-u)\Omega^{(2)}_{\boldsymbol{lm}}(u) = \delta_{\boldsymbol{km}}$, we obtain the basic monodromy relation

$$\sum_{n}\langle \mathbf{1}_{12}|\Omega^{(1)}_{\boldsymbol{mn}}(u)\Omega^{(2)}_{\boldsymbol{nl}}(u) = \langle \mathbf{1}_{12}|\delta_{\boldsymbol{ml}}\,. \tag{3.44}$$

As an example for the use of this relation, contract both sides with $|\mathcal{O}_1\rangle \otimes |\mathcal{O}_2\rangle$. We then obtain the monodromy relation for a two-point function of the form

$$\sum_{\boldsymbol{l}} \left\langle \Omega^{(1)}_{\boldsymbol{kl}}(u)|\mathcal{O}_1\rangle\,, \Omega^{(2)}_{\boldsymbol{lm}}(u)|\mathcal{O}_2\rangle \right\rangle = \delta_{\boldsymbol{km}}\left\langle |\mathcal{O}_1\rangle\,, |\mathcal{O}_2\rangle\right\rangle\,, \tag{3.45}$$

where $\langle\,,\rangle$ denotes the Wick contraction via the singlet projector.

Just as in the case of the fundamental R-matrix, the derivation of the monodormy relation for the three-point functions is straightforward. In fact, the prefactor function in this case is trivial and the result takes the simple form:

$$\sum_{\boldsymbol{l},\boldsymbol{m}} \left\langle \Omega^{(1)}_{\boldsymbol{kl}}(u)|\mathcal{O}_1\rangle\,, \Omega^{(2)}_{\boldsymbol{lm}}(u)|\mathcal{O}_2\rangle, \Omega^{(3)}_{\boldsymbol{mn}}(u)|\mathcal{O}_3\rangle \right\rangle = \delta_{\boldsymbol{kn}}\left\langle |\mathcal{O}_1\rangle\,, |\mathcal{O}_2\rangle, |\mathcal{O}_3\rangle\right\rangle\,. \tag{3.46}$$

The reason for the absence of the sign factors in contrast to the case of the fundamental R-matrix (3.25) is because all the components of the harmonic R-matrix $\langle\boldsymbol{m}|\mathbf{R}(u)|\boldsymbol{n}\rangle$ are bosonic: they are composed of even number of oscillators, as seen from the definitions (3.33)–(3.36).

Let us now describe how one can prove the basic crossing relation (3.40). As shown in (3.34), the harmonic R-matrix is made up of the hopping operators (3.33) and the coefficients (3.36). Therefore we need to derive the crossing relations for these two quantities. Since the manipulations are somewhat involved, we relegate the details to appendix C and only sketch the procedures and some relevant intermediate results below.

First, using the crossing relations for the oscillators, it is easy to derive the crossing relation for an arbitrary function of the number operators. The result reads

$$\langle \mathbf{1}_{12}|f(\mathbf{N}^{(a)} + \mathbf{N}^{(1)}) = \langle \mathbf{1}_{12}|f(\mathbf{N}^{(a)} - \mathbf{N}^{(2)})\,. \tag{3.47}$$

In particular, the coefficient $\mathcal{A}_I^{(\mathbf{N}^{(a)}+\mathbf{N}^{(1)})}$ becomes $\mathcal{A}_I^{(\mathbf{N}^{(a)}-\mathbf{N}^{(2)})}$ under such crossing.

Next one can show that the crossing relation for the hopping operator takes the form

$$\langle \mathbf{1}_{12}|\mathbf{Hop}^{(a1)}_{k,l,m,n} = \langle \mathbf{1}_{12}|\mathcal{C}\circ \mathbf{Hop}^{(a1)}_{k,l,m,n} \tag{3.48}$$

$$:= (-1)^{l+m}\langle \mathbf{1}_{12}| \sum_{p=0}^{\min(k,m)} \sum_{q=0}^{\min(l,n)} \mathcal{B}(\mathbf{N}_\alpha^{(a)} - m + p, p)\mathcal{B}(\mathbf{N}_\beta^{(a)} - n + q, q)\mathbf{Hop}^{(a2)}_{k-p,l-q,m-p,n-q}\,,$$

where $\mathcal{C}\circ\mathbf{Hop}^{(a1)}_{k,l,m,n}$ denotes the crossed **Hop** operator and the function $\mathcal{B}(x,y)$ was defined in (3.36).

Combining these crossing operations, we find that

$$\langle \mathbf{1}_{12}|\mathbf{R}^{(1)}(u) = \langle \mathbf{1}_{12}| \sum_{k,l,m,n}\sum_{p,q} a_{k,l,m,n}\mathcal{A}_I^{(\mathbf{N}^{(a)}-\mathbf{N}^{(2)}+k+l-m-n)}\mathcal{C}\circ\mathbf{Hop}^{(a1)}_{k,l,m,n} \tag{3.49}$$

$$= \langle \mathbf{1}_{12}| \sum_{k,l,m,n} a_{k,l,m,n}\tilde{\mathcal{A}}_I^{(\mathbf{N})}\mathbf{Hop}^{(a2)}_{k,l,m,n}\,, \tag{3.50}$$

where

$$\tilde{\mathcal{A}}_I^{(\mathbf{N})}(u) = \sum_{p,q}^{\infty}(-1)^I\mathcal{B}(\mathbf{N}_\alpha^{(a)} - m, p)\mathcal{B}(\mathbf{N}_\beta^{(a)} - n, q)\mathcal{A}_{I+p+q}^{(2I-\mathbf{N}+2\mathbf{M})}\,. \tag{3.51}$$

In the above expression of $\tilde{\mathcal{A}}_I^{(\mathbf{N})}(u)$, we have renamed $(k-p, l-q, m-p, n-q) \to (k, l, m, n)$ and defined $\mathbf{M} := \mathbf{N}^{(a)} - m - n$. Note that, under this change of labels, $a_{k,l,m,n} = \delta_{k+n,l+m}(-1)^{(k+l)(m+n)}$ becomes $a_{k,l,m,n}(-1)^{p+q}$ and $I = \frac{k+l+m+n}{2}$ changes into $I + p + q$. Using the binomial identity $\mathcal{B}(\alpha+\beta, k) = \sum_{j=0}^{k}\mathcal{B}(\alpha, k-j)\mathcal{B}(\beta, j)$,[23] the summation over p, q in $\tilde{\mathcal{A}}_I^{(\mathbf{N})}$ can be converted into a simpler expression

$$\begin{aligned}\tilde{\mathcal{A}}_I^{(\mathbf{N})}(u) &= \sum_{r=0}^{\infty}(-1)^I\left(\sum_{p=0}^{r}\mathcal{B}(\mathbf{N}_\alpha^{(a)} - m, p)\mathcal{B}(\mathbf{N}_\beta^{(a)} - n, r-p)\right)\mathcal{A}_{I+r}^{(2I-\mathbf{N}+2\mathbf{M})}(u)\\ &= \sum_{r=0}^{\infty}(-1)^I\mathcal{B}(\mathbf{M}, r)\mathcal{A}_{I+r}^{(2I-\mathbf{N}+2\mathbf{M})}(u)\,.\end{aligned} \tag{3.52}$$

In appendix C, we will show that this sum giving $\tilde{\mathcal{A}}_I^{(\mathbf{N})}(u)$ can be evaluated and leads to the desired equality

$$\tilde{\mathcal{A}}_I^{(\mathbf{N})}(u) = \sum_{r=0}^{\infty}(-1)^I\mathcal{B}(\mathbf{M}, r)\mathcal{A}_{I+r}^{(2I-\mathbf{N}+2\mathbf{M})} = \mathcal{A}_I^{(\mathbf{N})}(-u)\,. \tag{3.53}$$

Putting this result into (3.50) and summing over k, l, m, n, we find that the r.h.s. of (3.50) becomes $\langle \mathbf{1}_{12}|\mathbf{R}^{(2)}(-u)$ and this proves the crossing relation (3.40).

4 Reduction of monodromy relation to subsectors

In the previous section, we have derived the monodromy relation for the full psu(2,2|4) sector both in the case of the fundamental R-matrix and of the harmonic R-matrix. The former can be obtained by a rather straightforward generalization of the SU(2) case discussed in our previous work [31] and we tried to give a slightly more detailed exposition

[23]This is a direct consequence from the binomial theorem, namely, $(1+x)^\alpha = \sum_{k=0}^{\infty}\mathcal{B}(\alpha, k)x^k$. The formula readily follows by considering $(1+x)^{\alpha+\beta} = (1+x)^\alpha \cdot (1+x)^\beta$.

compared with the result already given in [32]. On the other hand, the case for the harmonic R-matrix is new. Although its derivation turned out to be substantially more involved than the case of the fundamental R-matrix, the symmetric set up for which the quantum and the auxiliary spaces carry identical representation of psu(2,2|4) can be of particular value, as was already indicated in the application to the scattering amplitude. Also the fact that the Hamiltonian can be obtained simply as the logarithmic derivative of the harmonic R-matrix should find useful applications.

The most important original purpose for formulating the monodromy relations, however, is their possible use as the set of powerful equations which govern, from the integrability perspective, the structures of the correlation functions. Although this idea has not yet been studied explicitly, to perform such an analysis it is natural to begin with the simplest set-ups, namely the cases of important tractable subsectors of the full theory. In what follows, we shall consider the compact SU(2) sector and the non-compact SL(2) sector as typical examples, and derive the monodromy relations for them from the point of view of a systematic reduction of the general psu(2,2|4) case.

4.1 Reduction to the SU(2) subsector

4.1.1 Embedding of $\mathbf{su(2)_L \oplus su(2)_R}$ in $\mathbf{u(2,2|4)}$

In the case of the SU(2) sector, the monodromy relation was already obtained in our previous work [29]. Therefore the purpose here is to re-derive it through the reduction of the psu(2,2|4) case. To do this, the first step is to identify the generators of $\mathrm{SU(2)_L}$ and $\mathrm{SU(2)_R}$, given in (2.71) and (2.72) in terms of those of u(2,2|4) shown in (2.47). This is simple since $\mathrm{SU(2)_L \times SU(2)_R}$ is contained entirely in the R-symmetry group SU(4), and hence the relevant u(2,2|4) generators are ${W_a}^b = \bar{\xi}_a \xi^b = {R_a}^b + \frac{1}{2}\delta_a^b B$ given in (2.50). Explicitly, we have

$$J_3^L = \frac{1}{2}({W_3}^3 - {W_1}^1)\,, \quad J_+^L = -{W_3}^1\,, \quad J_-^L = -{W_1}^3\,, \tag{4.1}$$

$$J_3^R = \frac{1}{2}({W_4}^4 - {W_2}^2)\,, \quad J_+^R = -{W_4}^2\,, \quad J_-^R = -{W_2}^4\,. \tag{4.2}$$

It will be important to recognize that the following combinations, B_L and B_R, of diagonal ${W_a}^a$ generators act as central charges for the group $\mathrm{SU(2)_L \times SU(2)_R}$:

$$B_L = \frac{1}{2}({W_3}^3 + {W_1}^1)\,, \qquad B_R = \frac{1}{2}({W_4}^4 + {W_2}^2)\,. \tag{4.3}$$

For example, B_L together with J_i^L form $\mathrm{U(2)_L = U(1) \times SU(2)_L}$ of which B_L is the U(1) part. Thus, $\left[B_L, J_i^L\right] = 0$. Obviously B_L commutes with $\mathrm{SU(2)_R}$. The argument for B_R is entirely similar. Being the central charges, they take definite values in an irreducible representation, which in our case of interest is the spin $(\frac{1}{2})_L \times (\frac{1}{2})_R$ representation. Evaluating B_L and B_R on any of the states in this representation, say $|Z\rangle = \bar{\xi}_3\bar{\xi}_4|0\rangle$, it is easy to obtain $B_L = B_R = \frac{1}{2}$.

With this in mind, let us write down the $\mathrm{SU(2)_L}$ Lax operators as embedded in that of gl(4|4), given in (3.9). We get

$$L_{\mathrm{SU(2)}_L}(u) = \begin{pmatrix} u + \frac{i}{2}({W_3}^3 - {W_1}^1) & -i{W_1}^3 \\ -i{W_3}^1 & u - \frac{i}{2}({W_3}^3 - {W_1}^1) \end{pmatrix} \tag{4.4}$$

Using the relation $B_L = \frac{1}{2} = \frac{1}{2}(W_3{}^3 + W_1{}^1)$, i.e. $W_3{}^3 + W_1{}^1 = 1$, this can be re-written as

$$L_{\mathrm{SU}(2)_L}(u) = \begin{pmatrix} u + \frac{i}{2} - iW_1{}^1 & -iW_1{}^3 \\ -iW_3{}^1 & u + \frac{i}{2} - iW_3{}^3 \end{pmatrix} \tag{4.5}$$

Let us now recall the general form of the Lax operator for gl(4|4), which is $L(u)^A{}_B = u\delta^A_B + \eta(-1)^{|B|}J^A{}_B$, and its more refined form in terms of the superconformal generators given in (3.9). The part relevant for the SU(2) sector is in the lower diagonal corner given by $u\delta^b_a - \eta W_a{}^b$. For the $\mathrm{SU}(2)_{\mathrm{L}}$, we can identify the indices for a and b to be 1 and 3. In the entirely similar manner, the Lax operator for the $\mathrm{SU}(2)_{\mathrm{R}}$ sector is obtained from the one for the $\mathrm{SU}(2)_{\mathrm{L}}$ sector by substitution of the indices $1 \to 2$ and $3 \to 4$. Then, it is easy to see that the Lax operators for the $\mathrm{SU}(2)_{\mathrm{L}}$ and $\mathrm{SU}(2)_{\mathrm{R}}$ sectors are obtained from the gl(4|4) Lax operator by taking into account the shift $u \to u + i/2$ in the form

$$L_{\mathrm{SU}(2)_L}(u)_{ab} = L(u+i/2)_{ab}\,, \qquad \text{with } \eta = i\,, \quad \{a,b\} = \{1,3\}\,, \tag{4.6}$$

$$L_{\mathrm{SU}(2)_R}(u)_{\bar{a}\bar{b}} = L(u+i/2)_{\bar{a}\bar{b}}\,, \qquad \text{with } \eta = i\,, \quad \{\bar{a},\bar{b}\} = \{2,4\}\,. \tag{4.7}$$

It is important to note that the occurrence of the shift of the spectral parameter is due to the emergence of the extra central charges when a group is restricted to its subgroup and hence is a rather general phenomenon.

4.1.2 Inversion relation

Let us now derive the inversion relation. As in the previous discussion, we shall concentrate on the $\mathrm{SU}(2)_{\mathrm{L}}$ part. The object to consider is the product of two u(2,2|4) Lax operators given in (3.14), with the indices A, B taken to be the SU(2) indices[24] a, b. This gives

$$(L(u)L(v))_{\mathrm{ab}} = (L(u))_{\mathrm{ac}}(L(v))_{\mathrm{cb}} + \eta^2(-1)^\gamma(-1)^{|\mathrm{b}|}J^{\mathrm{a}}{}_\gamma J^\gamma{}_{\mathrm{b}}\,, \tag{4.8}$$

where γ runs over the indices of the auxiliary space other than the SU(2) indices $\mathrm{a}, \mathrm{b}, \ldots$. Using the oscillator representation, the second terms can be simplified as

$$\begin{aligned}(-1)^\gamma J^{\mathrm{a}}{}_\gamma J^\gamma{}_{\mathrm{b}} &= \bar{\zeta}^{\mathrm{a}}(\mu^\alpha\lambda_\alpha - \tilde{\lambda}_{\dot{\alpha}}\tilde{\mu}^{\dot{\alpha}} - \xi^{\bar{d}}\bar{\xi}_{\bar{d}})\zeta_{\mathrm{b}} \\ &= J^{\mathrm{a}}{}_{\mathrm{b}}(2Z_L + 2Z_R + 2B_R)\,,\end{aligned} \tag{4.9}$$

where

$$Z_L := \frac{1}{2}(\lambda_1\mu^1 - \tilde{\lambda}_{\dot{1}}\tilde{\mu}^{\dot{1}}) = \frac{1}{2}(J_1{}^1 + J_{\dot{1}}{}^{\dot{1}})\,, \tag{4.10}$$

$$Z_R := \frac{1}{2}(\lambda_2\mu^2 - \tilde{\lambda}_{\dot{2}}\tilde{\mu}^{\dot{2}}) = \frac{1}{2}(J_2{}^2 + J_{\dot{2}}{}^{\dot{2}})\,, \tag{4.11}$$

and B_R was already defined in (4.3). In the SU(2) sector the quantities Z_L and Z_R vanish[25] and $B_R = 1/2$, as was already explained. Therefore the factor $2Z_L + 2Z_R + 2B_R$ is simply unity and we simply obtain

$$(-1)^\gamma J^{\mathrm{a}}{}_\gamma J^\gamma{}_{\mathrm{b}} = J^{\mathrm{a}}{}_{\mathrm{b}}\,. \tag{4.12}$$

[24] Here and *until* the end of subsection 4.1, we shall use roman letters a, b, etc. to denote the genuine SU(2) indices, which take the values 1 and 2, in order to distinguish them from the italic letters a, b, etc., which are SU(4) indices taking values from 1 to 4.

[25] Actually, as it will be explained in the next subsection, the quantities Z_L and Z_R are central charges for $\mathrm{SL}(2)_{\mathrm{L}} \times \mathrm{SL}(2)_{\mathrm{R}}$.

Substituting this back into (4.9) and rewriting the spectral parameters in order to express the result in terms of the Lax operator for $\mathrm{SU}(2)_{\mathrm{L}}$, we obtain the relation

$$
\begin{aligned}
(L(u)L(v))_{\mathrm{ab}} &= (L(u))_{\mathrm{ac}}(L(v))_{\mathrm{cb}} - \eta^2 J^{\mathrm{a}}{}_{\mathrm{b}} \\
&= (L(u+\eta/2))_{\mathrm{ac}}(L(v+\eta/2))_{\mathrm{cb}} - ((u+\eta/2)(v+\eta/2) - uv)\delta_{\mathrm{ab}} \\
&= (L_{\mathrm{SU}(2)}(u))_{\mathrm{ac}}(L_{\mathrm{SU}(2)}(v))_{\mathrm{cb}} - ((u+i/2)(v+i/2) - uv)\delta_{\mathrm{ab}}\,.
\end{aligned} \tag{4.13}
$$

Now from the inversion relation for the Lax operator for u(2,2|4) given in (3.13), when $u+v=\eta=i$, the left hand side of the above equation becomes $uv\delta_{ab}$, and thus we obtain the inversion relation for the $\mathrm{SU}(2)_{\mathrm{L}}$ Lax operator to be of the form

$$
(L_{\mathrm{SU}(2)}(u)L_{\mathrm{SU}(2)}(v))_{\mathrm{ab}} = f_{\mathrm{SU}}(u,v)\delta_{\mathrm{ab}}\,, \quad u+v=i\,, \tag{4.14}
$$

$$
f_{\mathrm{SU}}(u,v) = uv - \frac{3}{4} \tag{4.15}
$$

To compare with the result of [31], we must substitute $u \to -u + \frac{i}{2}$ and $v \to u + \frac{i}{2}$. Then the relation above takes the form

$$
(L_{\mathrm{SU}(2)}(-u+(i/2))L_{\mathrm{SU}(2)}(u+(i/2)))_{\mathrm{ab}} = -(u^2+1)\delta_{\mathrm{ab}}\,, \tag{4.16}
$$

which agrees with the equation (5.1) of [31] (with the inhomogeneity parameter θ set to zero).

From this inversion relation, one can easily obtain the monodromy relation, as already described in [29]. So we shall omit this derivation for the SU(2) sector. Instead, in the next subsection, we shall present a derivation of the monodromy relation for the SL(2) sector *directly* from that for the psu(2,2|4) sector. The result is new and the method can of course be applied to the SU(2) case as well to provide an alternative derivation of a known result given in [31].

4.2 Reduction to the SL(2) subsector

Having been warmed up with the reduction to the simplest SU(2) sector, we now perform the reduction to the SL(2) subsector to derive the explicit form of its monodromy relation, which is new.

4.2.1 Embedding of $\mathrm{sl}(2)_{\mathrm{L}} \oplus \mathrm{sl}(2)_{\mathrm{R}}$ in u(2,2|4) and a derivation of monodromy relation

Let us begin by clarifying how the generators of $\mathrm{SL}(2)_{\mathrm{L}} \times \mathrm{SL}(2)_{\mathrm{R}}$ are embedded in u(2,2|4). First consider the simple "light-cone" combinations of operators given by

$$
p_+ = \frac{1}{2}(P_0 - P_3)\,, \quad k^+ = -\frac{1}{2}(K_0 + K_3)\,, \quad d_+ = \frac{i}{2}(M_{03} - D)\,, \tag{4.17}
$$

$$
p_- = \frac{1}{2}(P_0 + P_3)\,, \quad k^- = \frac{1}{2}(-K_0 + K_3)\,, \quad d_- = -\frac{i}{2}(M_{03} + D)\,. \tag{4.18}
$$

They satisfy the following simple set of commutation relations:

$$
[d_\pm, p_\pm] = p_\pm\,, \quad [d_\pm, k^\pm] = -k^\pm\,, \quad [k^\pm, p_\pm] = 2d_\pm\,. \tag{4.19}
$$

This shows that the generators of the $\mathrm{SL}(2)_L\times\mathrm{SL}(2)_R$ can be taken as

$$\mathrm{SL}(2)_L: \quad S_- = -ip_+\,, \quad S_+ = -ik^+\,, \quad S_0 = -d_+\,, \tag{4.20}$$
$$\mathrm{SL}(2)_R: \quad \tilde{S}_- = -ip_-\,, \quad \tilde{S}_+ = -ik^-\,, \quad \tilde{S}_0 = -d_-\,. \tag{4.21}$$

In this notation, the commutation relations are

$$[S_0, S_\pm] = \pm S_\pm\,, \qquad [S_+, S_-] = 2S_0\,, \tag{4.22}$$
$$\left[\tilde{S}_0, \tilde{S}_\pm\right] = \pm\tilde{S}_\pm\,, \qquad \left[\tilde{S}_+, \tilde{S}_-\right] = 2\tilde{S}_0\,. \tag{4.23}$$

Now from the definition of spinor notations (2.26), (2.27) and the form of the $\mathrm{u}(2,2|4)$ generators $J^A_{\ B}$ given in (2.47), one finds that, for example, the generators $\{S_0, S_\pm\}$ of $\mathrm{SL(2)_L}$ are embedded in $\mathrm{u}(2,2|4)$ as[26]

$$J_1{}^1 = M_1{}^1 - \frac{i}{2}D - \frac{1}{2}B = \frac{i}{2}(M_{03} - iM_{12}) - \frac{i}{2}(D - iB) = -S_0 + \frac{1}{2}M_{12} - \frac{1}{2}B \tag{4.24}$$
$$J^{\dot{1}}{}_{\dot{1}} = M^{\dot{1}}_{\dot{1}} + \frac{i}{2}D - \frac{1}{2}B = \frac{i}{2}(-M_{03} - iM_{12}) + \frac{i}{2}(D + iB) = S_0 + \frac{1}{2}M_{12} - \frac{1}{2}B\,, \tag{4.25}$$
$$J_{1\dot{1}} = iP_{1\dot{1}} = ip_+ = -S_-\,, \quad J^{\dot{1}1} = iK^{\dot{1}1} = ik^+ = -S_+\,. \tag{4.26}$$

At this point we note that, just as the extra central charges B_L and B_R appeared for the subgroup $\mathrm{SU(2)_L\times SU(2)_R}$, the quantities Z_L and Z_R previously defined in (4.10) and (4.11) behave as central charges for $\mathrm{SL}(2)_L\times\mathrm{SL}(2)_R$. This will be important below.

Now in what follows, let us concentrate on the $\mathrm{SL}(2)_L$ subsector, where the composite operators are constructed by multiple actions of the covariant derivatives along the light-cone direction as

$$\begin{aligned}\mathcal{O}(x) &= \frac{1}{n_1!n_2!\ldots}\mathrm{Tr}(\mathcal{D}^{n_1}_{1\dot{1}}Z\mathcal{D}^{n_2}_{1\dot{1}}Z\cdots) \\ &\mapsto \exp(i(x^+p_+ + x^-p_-))\frac{(\lambda_1\tilde{\lambda}_{\dot{1}})^{n_1}}{n_1!}|Z\rangle\otimes\frac{(\lambda_1\tilde{\lambda}_{\dot{1}})^{n_2}}{n_2!}|Z\rangle\otimes\cdots\,.\end{aligned} \tag{4.27}$$

Here, $|Z\rangle$ is the scalar state $\bar{\xi}_3\bar{\xi}_4|0\rangle$ and each factor in the total tensor product signifies the operator at different positions of the spin chain. Note that on this type of states the rotation operator M_{12} vanishes and the hypercharge operator $B = \frac{1}{2}\bar{\xi}_a\xi^a$ takes the definite value 1. Furthermore, one can easily check that $Z_L = Z_R = 0$ by acting them on such a state. All these relations is consistent with the vanishing of the central charge $C = Z_L + Z_R + B - 1$ for the physical SYM operators.

With these properties, the relations (4.24) and (4.25) simplify and we can easily embed the $\mathrm{SL(2)_L}$ Lax operator into that of u(2,2|4) in the following fashion:

$$L_{\mathrm{SL(2)}}(u) := \begin{pmatrix} u + iS_0 & iS_- \\ iS_+ & u - iS_0 \end{pmatrix} = \begin{pmatrix} u - \frac{i}{2} - iJ_1{}^1 & -iJ_{1\dot{1}} \\ -iJ^{\dot{1}1} & u - \frac{i}{2} + iJ^{\dot{1}}{}_{\dot{1}} \end{pmatrix}\,. \tag{4.28}$$

[26] 1 and $\dot{1}$ are the indices for $\mathrm{SL}(2,\mathbb{C})\times\overline{\mathrm{SL}(2,\mathbb{C})}$.

Here again the shift of $u \to u - \frac{i}{2}$ is due to the effect of the central charge, as in the case of SU(2). Now comparing this with the form of the , we readily find that the $\mathrm{SL}(2)_{\mathrm{L}}$ Lax operator is embedded in the u(2,2|4) Lax operator as[27]

$$(L_{\mathrm{SL}(2)}(u))_{\mathrm{ab}} = (L(u - i/2))_{\mathrm{ab}}\,, \qquad (\mathrm{a}, \mathrm{b} = 1, \dot{1})\,, \qquad \eta = -i\,. \tag{4.29}$$

(Just as we did in the discussion of the SU(2) sector in the previous subsection, we shall hereafter use the roman letters a, b, etc. to denote the Lorentz spinor indices $\{\mathrm{a}, \mathrm{b}\} = \{1, \dot{1}\}$ for the $\mathrm{SL}(2)_{\mathrm{L}}$ sector, in order to distinguish them from the SU(4) indices a, b, etc, which run over 1 to 4.)

To derive the monodromy relation, we first need to find the inversion relation for the Lax operator. Just as in the SU(2) case, the product of the $\mathrm{SL}(2)_{\mathrm{L}}$ part of the u(2,2|4) Lax operators give

$$(L(u)L(v))_{\mathrm{ab}} = (L(u))_{\mathrm{ac}}(L(v))_{\mathrm{cb}} + \eta^2(-1)^{|\gamma|} J^{\mathrm{a}}{}_{\gamma} J^{\gamma}{}_{\mathrm{b}} \tag{4.30}$$

where γ runs over the all indices of the auxiliary except for those of SL(2), namely $\mathrm{a}, \mathrm{b} = 1, \dot{1}$. Note at this stage the sign of the second term is opposite to the one for SU(2) case given in (4.8). Now as before we can simplify this term quadratic in the generators in the following manner:

$$\begin{aligned}(-1)^{|\gamma|} J^{\mathrm{a}}{}_{\gamma} J^{\gamma}{}_{\mathrm{b}} &= \begin{pmatrix} \lambda_1 \\ \tilde{\mu}^1 \end{pmatrix}^{\mathrm{a}} (\mu^2\lambda_2 - \tilde{\lambda}_{\dot{2}}\tilde{\mu}^{\dot{2}} - \xi^i\bar{\xi}_i)(\mu^1 \;\; - \tilde{\lambda}_{\dot{1}})_{\mathrm{b}} \\ &= J^{\mathrm{a}}_{\;\mathrm{b}}(2Z_R + 2B - 3) \\ &= -J^{\mathrm{a}}_{\;\mathrm{b}}\,,\end{aligned} \tag{4.31}$$

where we used $Z_R = 0, B = 1$. Thus we get an extra minus sign from this manipulation and hence obtains the same sign for the second term as for the SU(2) case. Now we make an appropriate shift of the spectral parameter and get

$$(L(u)L(v))_{\mathrm{ab}} = (L(u))_{\mathrm{ac}}(L(v))_{\mathrm{cb}} - \eta^2 J^{\mathrm{a}}_{\;\mathrm{b}} \tag{4.32}$$

$$= (L(u - \eta/2))_{\mathrm{ac}}(L(v - \eta/2))_{\mathrm{cb}} + \frac{\eta}{2}(u + v - \eta)\delta_{\mathrm{ab}} + \frac{\eta^2}{4}\delta_{\mathrm{ab}} \tag{4.33}$$

$$= (L_{\mathrm{SL}(2)})(u))_{\mathrm{ac}}(L_{\mathrm{SL}(2)}(v))_{\mathrm{cb}} + \frac{\eta}{2}(u + v - \eta)\delta_{\mathrm{ab}} + \frac{\eta^2}{4}\delta_{\mathrm{ab}}\,,\,. \tag{4.34}$$

Thus if we set $u + v = \eta = -i$, then since the left hand side of the above equation becomes $uv\delta_{\mathrm{ab}}$, we get

$$(L_{\mathrm{SL}(2)})(u))_{\mathrm{ac}}(L_{\mathrm{SL}(2)}(v))_{\mathrm{cb}} = f_{\mathrm{SL}}(u, v)\delta_{\mathrm{ab}}\,, \qquad u + v = -i \tag{4.35}$$

$$f_{\mathrm{SL}}(u, v) = uv + \frac{1}{4}\,. \tag{4.36}$$

[27]Similarly, the Lax operator for $\mathrm{SL}(2)_R$ part is embedded as $(L_{\widetilde{\mathrm{SL}}(2)}(u))_{\mathrm{ab}} = (L(u - i/2))_{\mathrm{ab}}$ $(\mathrm{a}, \mathrm{b} = 2, \dot{2})$ with $\eta = -i$. In this equation, a and b refer to the indices of 2×2 matrix, which are part of the $\mathrm{SL}(2, \mathbb{C}) \times \overline{\mathrm{SL}(2, \mathbb{C})}$ indices.

Note that if we compare (4.36) with (4.15), we can recognize that the function $f(u,v)$ for the SU(2) and the SL(2) cases can be written in a unified manner as

$$f(u,v) = uv - \mathbb{S}^2 = \begin{cases} uv - s(s+1) & \text{for SU(2)} \\ uv - s(s-1) & \text{for SL(2)} \end{cases} , \qquad s = \frac{1}{2} , \tag{4.37}$$

where $\mathbb{S}^2$ is the Casimir operator for the respective group.

4.2.2 Method of direct reduction from psu(2,2|4) monodromy relation

We now wish to demonstrate that we can derive the monodromy relation for the two-point function in the SL(2) sector *more directly* from that for psu(2,2|4), with the judicious use of the product relation (4.34) for the Lax operators.

The monodromy relation for psu(2,2|4) is given in (3.23). To reduce it to the SL(2) sector, we set the indices A and C to be those of SL(2), say a and c which take values in $\{1, \dot{1}\}$, and for convenience make a shift of the spectral parameter $u \to u - i/2$. Hereafter we shall employ the notation $f^-(u) \equiv f(u - i/2)$ for such a shift for any function $f(u)$. Then from (3.23) we get

$$\langle \Omega_\ell^-(u)_{\mathrm{a}B} |\psi_1\rangle , \Omega_\ell^-(u)_{B\mathrm{c}} |\psi_2\rangle\rangle = F_\ell^-(u) \delta_{\mathrm{ac}} \langle |\psi_1\rangle, |\psi_2\rangle\rangle , \tag{4.38}$$

or more succinctly, before taking the inner product with $|\psi_1\rangle \otimes |\psi_2\rangle$,

$$\langle \mathbf{1}_{12} | \Omega_\ell^{(1)-}(u)_{\mathrm{a}B} \Omega_\ell^{(2)-}(u)_{B\mathrm{c}} = F_\ell^-(u) \delta_{\mathrm{ac}} \langle \mathbf{1}_{12} | . \tag{4.39}$$

What we wish to derive from it is the relation involving the SL(2) monodormy matrices with only the SL(2) indices $\{1, \dot{1}\}$. However, obviously this reduction is non-trivial since the general gl(4|4) indices A, B which occur for a neighboring product of two Lax operators $(LL)_{AB}$ in $\Omega_{\mathrm{a}B}$ may take all possible gl(4|4) values.

This difficulty can be overcome by noting that the formula for the crossing relation for the individual u(2, 2|4) Lax matrices is quite simple and that a certain product of the u(2, 2|4) generators which appear in the product of two Lax matrices can be reduced to a single generator, as was demonstrated already in (4.31). To make use of these properties, we first focus on the leftmost and the rightmost Lax operators forming the two monodromy matrices and make the split

$$\Omega_\ell^{(1)-}(u)_{\mathrm{a}B} = L_1^{(1)-}(u)_{\mathrm{a}D} \Omega_{\ell-1}^{(1)-}(u)_{DB} , \tag{4.40}$$

$$\Omega_\ell^{(2)-}(u)_{B\mathrm{c}} = \Omega_{\ell-1}^{(2)-}(u)_{BE} L_\ell^{(2)-}(u)_{E\mathrm{c}} , \tag{4.41}$$

Then, the l.h.s. of (4.39) becomes

$$\langle \mathbf{1}_{12} | \Omega^{(1)-}{}_\ell(u)_{\mathrm{a}B} \Omega^{(2)-}{}_\ell(u)_{B\mathrm{c}} = \langle \mathbf{1}_{12} | L_1^{(1)-}(u)_{\mathrm{a}D} \Omega^{(1)-}{}_{\ell-1}(u)_{DB} \Omega^{(2)-}{}_{\ell-1}(u)_{BE} L_\ell^{(2)-}(u)_{E\mathrm{c}} . \tag{4.42}$$

As the operators at different positions all commute, we can move $\Omega^{(1)-}{}_{\ell-1}\Omega^{(2)-}{}_{\ell-1}$ in the middle to the left until they hit $\langle \mathbf{1}_{12}|$. Then we can use the monodromy relation (4.38) for the case of length $\ell - 1$ and write (4.42) as

$$\langle \mathbf{1}_{12} | \Omega^{(1)-}{}_\ell(u)_{\mathrm{a}B} \Omega^{(2)-}{}_\ell(u)_{B\mathrm{c}} = F_{\ell-1}^- \langle \mathbf{1}_{12} | L_1^{(1)-}(u)_{\mathrm{a}B} L_\ell^{(2)-}(u)_{B\mathrm{c}} . \tag{4.43}$$

Now substitute the definition $L(u)^A{}_B = u\delta^A_B + \eta(-1)^{|B|}J^A{}_B$ for the Lax operators on the r.h.s. and expand. This gives

$$\langle \mathbf{1}_{12}|L_1^{(1)-}(u)_{\mathrm{a}B}L_1^{(2)-}(u)_{B\mathrm{c}} = \langle \mathbf{1}_{12}|L_1^{(1)-}(u)_{\mathrm{ab}}L_\ell^{(2)-}(u)_{\mathrm{bc}} + \langle \mathbf{1}_{12}|L_1^{(1)-}(u)_{\mathrm{a}\beta}L_\ell^{(2)-}(u)_{\beta\mathrm{c}}\,, \tag{4.44}$$

where β stands for indices other than those of SL(2). Further the second term on the r.h.s. can be written out explicitly as

$$L_1^{(1)-}(u)_{\mathrm{a}\beta}L_\ell^{(2)-}(u)_{\beta\mathrm{c}} = \eta^2(-1)^{|\beta|}J_1^{(1)\mathrm{a}}{}_\beta J_\ell^{(2)\beta}{}_{\mathrm{c}}\,, \tag{4.45}$$

where we used $(-1)^{|\mathrm{c}|} = 1$ since c is a bosonic index. Apply both sides now to the singlet projector $\langle \mathbf{1}_{12}|$ and use the crossing relation for the generator $\langle \mathbf{1}_{12}|J_1^{(1)\mathrm{a}}{}_\beta = -\langle \mathbf{1}_{12}|J_\ell^{(2)\mathrm{a}}{}_\beta$, which is valid since $\mathrm{a} \neq \beta$. Then using the identity (4.31) to the r.h.s. , (4.45) becomes $\langle \mathbf{1}_{12}|L_1^{(1)-}(u)_{\mathrm{a}\beta}L_\ell^{(2)-}(u)_{\beta\mathrm{c}} = -\eta^2\langle \mathbf{1}_{12}|J_\ell^{(2)\mathrm{a}}{}_{\mathrm{c}}$ and the equation (4.43) can be simplified to

$$\langle \mathbf{1}_{12}|\Omega^{(1)-}{}_\ell(u)_{\mathrm{a}B}\Omega^{(2)-}{}_\ell(u)_{B\mathrm{c}} = F^-_{\ell-1}\langle \mathbf{1}_{12}|\left(L_1^{(1)-}(u)_{\mathrm{ab}}L_\ell^{(2)-}(u)_{\mathrm{bc}} - \eta^2 J_\ell^{(2)\mathrm{a}}{}_{\mathrm{c}}\right)\,. \tag{4.46}$$

Note that on the r.h.s. all the indices have turned into SL(2) indices.

Now we make a slight trick to split the last term of the r.h.s. into identical halves[28] as $J_\ell^{(2)\mathrm{a}}{}_{\mathrm{c}} = \frac{1}{2}J_\ell^{(2)\mathrm{a}}{}_{\mathrm{c}} + \frac{1}{2}J_\ell^{(2)\mathrm{a}}{}_{\mathrm{c}}$ and then hit just one half to $\langle \mathbf{1}_{12}|$ to change it into $J_\ell^{(1)\mathrm{a}}{}_{\mathrm{c}}$ acting on the spin chain 1. Since the generator here is that of u(2, 2|4), in doing so there appears an extra constant term $\propto \delta^{\mathrm{a}}_{\mathrm{c}}$ coming from the commutator of the oscillators forming this generator. Then we get

$$\langle \mathbf{1}_{12}|J_\ell^{(2)\mathrm{a}}{}_{\mathrm{c}} = \frac{1}{2}\langle \mathbf{1}_{12}|J_\ell^{(2)\mathrm{a}}{}_{\mathrm{c}} - \frac{1}{2}\langle \mathbf{1}_{12}|\left(J_1^{(1)a}{}_c + \delta^{\mathrm{a}}_{\mathrm{c}}\right) \tag{4.47}$$

We shall now show that the terms linear in the generators appearing on the r.h.s. can be absorbed by a judicious shifts of the spectral parameters of the expression $L_1^{(1)-}(u)_{\mathrm{ab}}L_\ell^{(2)-}(u)_{\mathrm{bc}}$, which is the first term on the r.h.s. of (4.44). In fact, one can easily check that (4.44) can be re-expressed as

$$\langle \mathbf{1}_{12}|L_1^-(u+\eta/2)_{\mathrm{ab}}L_\ell^-(u-\eta/2)_{\mathrm{bc}} + f_1^-(\eta)\delta_{\mathrm{ac}}\langle \mathbf{1}_{12}|\,, \tag{4.48}$$

where the factor $f_1(\eta)$ is given by $f_1(\eta) = -\eta^2/4$. Thus combining with (4.43), the l.h.s. of the original psu(2, 2|4) monodromy relation (4.39), namely $\langle \mathbf{1}_{12}|\Omega_\ell^{(1)-}(u)_{\mathrm{a}B}\Omega_\ell^{(2)-}(u)_{B\mathrm{c}}$, becomes

$$\langle \mathbf{1}_{12}|\left(F^-_{\ell-1}\delta_{\mathrm{bd}}L_1^-(u+\eta/2)_{ab}L_\ell^-(u-\eta/2)_{\mathrm{dc}} + f_1(u)F^-_{\ell-1}\delta_{\mathrm{ac}}\right)\,. \tag{4.49}$$

As the final step, we now rewrite the quantity $\langle \mathbf{1}_{12}|F^-_{\ell-1}\delta_{\mathrm{bd}}$ in the first term by using the fundamental monodromy relation (4.39) with ℓ replaced by $\ell - 1$. This process is a reversal of the splitting procedure we started out with and inserts the products of the monodromy matrices defined on the r.h.s. of the splitting equations (4.40) and (4.41). Then, the equation above becomes

$$\langle \mathbf{1}_{12}|L_1^-(u+\eta/2)_{ab}(\Omega^{(1)-}_{\ell-1})_{\mathrm{b}C}(\Omega^{(2)-}_{\ell-1})_{C\mathrm{d}}L_\ell^-(u-\eta/2)_{\mathrm{dc}}$$

[28] If one wishes, one can actually use a more general split with coefficients α and β satisfying $\alpha + \beta = 1$ and follow the same logic to be described below for the $\frac{1}{2} + \frac{1}{2}$ split. This will lead to more general forms of the SL(2) monodromy relations. Below we shall only describe the simplest split for the sake of clarity.

$$+\langle \mathbf{1}_{12}|f_1(u)F^-_{\ell-1}\delta_{\rm ac}\,. \tag{4.50}$$

On the other hand, each Lax operator on the r.h.s. of the relation (4.43) can be interpreted as a special monodromy matrix of length one and hence we can use the monodromy relation to write the r.h.s. as

$$F^-_{\ell-1}\langle \mathbf{1}_{12}|L^{(1)-}_1(u)_{{\rm a}B}L^{(2)-}_\ell(u)_{B{\rm c}} = F^-_{\ell-1}F^-_1\delta_{\rm ac}\langle \mathbf{1}_{12}| \tag{4.51}$$

Thus equating (4.50) and (4.51) and rearranging, we obtain an important relation

$$\langle \mathbf{1}_{12}|L^-_1(u+\eta/2)_{\rm ab}(\Omega^{(1)-}_{\ell-1})_{{\rm b}C}(\Omega^{(2)-}_{\ell-1})_{C{\rm d}}L^-_\ell(u-\eta/2)_{\rm dc} = g_1(u)F^-_{\ell-1}\langle\delta|_{\rm ac}\mathbf{1}_{12}\,, \tag{4.52}$$

where the function $g_1(u)$ is given by

$$g_1(u) = F^-_1(u) - f^-_1(u) = u^2\,. \tag{4.53}$$

The point to be noted here is that the part containing the unrestricted indices in the above relation is $(\Omega^{(1)-}_{\ell-1})_{{\rm b}C}(\Omega^{(2)-}_{\ell-1})_{C{\rm d}}$, namely the monodromy matrices of length $\ell-1$, shorter by one unit from the original ℓ.

It should now be clear that we can perform this reduction process repeatedly until all the indices become those of SL(2) only. Then, taking η to be $-i$ and identifying the SL(2) Lax operator as

$$L_{\rm SL(2)}(u)_{\rm ab} \equiv L^-(u)_{\rm ab}\,, \tag{4.54}$$

upon acting on the state $|\psi_1\rangle\otimes|\psi_2\rangle$ we obtain the monodromy relation for the genuine SL(2) monodromy matrices inserted as

$$\langle(\Omega_{\rm SL(2)}(u-i/2))_{\rm ab}|\psi_1\rangle, (\Omega_{\rm SL(2)}(u+i/2))_{\rm bc}|\psi_2\rangle\rangle = u^{2\ell}\delta_{\rm ac}\langle|\psi_1\rangle,|\psi_2\rangle\rangle\,. \tag{4.55}$$

This completes the direct derivation of the SL(2) monodromy relation from that of psu(2, 2|4) relations.

5 Discussions

In this paper, we studied the tree-level three-point functions in the entire psu(2, 2|4) sector of $\mathcal{N}=4$ super Yang-Mills theory from a group theoretic and integrability-based point of view. We in particular developed the manifestly conformally invariant construction of the singlet-projection operator and used it to express the Wick contraction. Unlike the preceding works [32, 33], our construction doesn't necessitate the "U-operator" which intertwines two schemes of representations of the superconformal algebra. This property greatly simplifies the analysis and allowed us to derive the monodromy relation for the harmonic R-matrix, as well as for the usual fundamental R-matrix.

The simplicity and the manifest conformal covariance of our construction will surely be of help when analyzing the weak-coupling three-point functions using integrability. So far, such analysis was performed thoroughly only for a particular class of three-point functions in su(2) [16, 31], sl(2) [52], su(3) [53] and su(1|1) [54] sectors. In the forthcoming paper [55], we will use our formalism to study more general three-point functions in the sl(2) sector, which involve more than one non-BPS operators. It would be an interesting future problem to study other sectors, in particular higher-rank sectors based on our construction.

It would also be interesting to study the loop correction in our formulation. For this purpose, a more detailed analysis of the harmonic R-matrix may be useful since the harmonic R-matrix is intimately related to the local conserved charges including the one-loop Hamiltonian. Another avenue of research is to explore the relation with the scattering amplitudes [39–46]. Also in that case, the harmonic R-matrix and the monodromy relation played an important role. It would be interesting if one could make a more direct connection.

Lastly, it would be important to understand the relations with the recently-proposed non-perturbative frameworks for the string vertex [56] and the three-point functions [57]. Understanding such a non-perturbative framework from the perturbative-gauge-theory point of view will lead to deeper understanding of the AdS/CFT correspondence.

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A Commutation relations for u(2,2|4)

In this appendix, all the explicit forms of the commutation relations for the superconformal generators are listed in the D-scheme basis. First, the algebra for the bosonic generators, namely, SO(2,4) and SU(4) generators are given by

$$[M_\alpha{}^\beta, M_\gamma{}^\delta] = \delta_\gamma{}^\beta M_\alpha{}^\delta - \delta_\alpha{}^\delta M_\gamma{}^\beta\,, \qquad [\bar{M}^{\dot\alpha}{}_{\dot\beta}, \bar{M}^{\dot\gamma}{}_{\dot\delta}] = \delta^{\dot\gamma}{}_{\dot\beta}\bar{M}^{\dot\alpha}{}_{\dot\delta} - \delta^{\dot\alpha}{}_{\dot\delta}\bar{M}^{\dot\gamma}{}_{\dot\beta}\,, \tag{A.1}$$

$$[M_\alpha{}^\beta, P_{\gamma\dot\delta}] = \delta_\gamma^\beta P_{\alpha\dot\delta} - \frac{1}{2}\delta_\alpha{}^\beta P_{\gamma\dot\delta}\,, \qquad [\bar{M}^{\dot\alpha}{}_{\dot\beta}, P_{\gamma\dot\delta}] = -\delta_{\dot\delta}^{\dot\alpha} P_{\gamma\dot\beta} + \frac{1}{2}\delta_{\dot\beta}^{\dot\alpha} P_{\gamma\dot\delta}\,, \tag{A.2}$$

$$[M_\alpha{}^\beta, K^{\dot\gamma\delta}] = -\delta_\alpha^\delta K^{\dot\gamma\beta} + \frac{1}{2}\delta_\alpha{}^\beta K^{\dot\gamma\delta}\,, \qquad [\bar{M}^{\dot\alpha}{}_{\dot\beta}, K^{\dot\gamma\delta}] = \delta_{\dot\beta}^{\dot\gamma} K^{\dot\alpha\delta} - \frac{1}{2}\delta^{\dot\alpha}{}_{\dot\beta} K^{\dot\gamma\delta}\,, \tag{A.3}$$

$$[D, P_{\alpha\dot\beta}] = iP_{\alpha\dot\beta}\,,\ [D, K^{\dot\alpha\beta}] = -iK^{\dot\alpha\beta}\,, \qquad [D, M_\alpha{}^\beta] = [D, \bar{M}^{\dot\alpha}{}_{\dot\beta}] = 0\,, \tag{A.4}$$

$$[P_{\alpha\dot\beta}, K^{\dot\gamma\delta}] = \delta_\alpha^\delta \bar{M}^{\dot\gamma}{}_{\dot\beta} - \delta_{\dot\beta}^{\dot\gamma} M_\alpha{}^\delta + i\delta_\alpha^\delta\delta_{\dot\beta}^{\dot\gamma} D\,, \tag{A.5}$$

$$[R_a{}^b, R_c{}^d] = \delta_c{}^b R_a{}^d - \delta_a{}^d R_c{}^b\,. \tag{A.6}$$

The commutators between the fermionic generators and the conformal generators D, P, K are

$$[D, Q_\alpha^a] = \frac{i}{2}Q_\alpha^a\,,\ [D, Q_{\dot\alpha a}] = \frac{i}{2}Q_{\dot\alpha a}\,,\ [D, S_a^\alpha] = -\frac{i}{2}S_a^\alpha\,,\ [D, \bar{S}^{\dot\alpha a}] = -\frac{i}{2}\bar{S}^{\dot\alpha a}\,, \tag{A.7}$$

$$[P_{\alpha\dot{\beta}}, S^\gamma_a] = -\delta^\gamma_\alpha \bar{Q}_{\dot{\beta}a}\,, \quad [P_{\alpha\dot{\beta}}, \bar{S}^{\dot{\gamma}a}] = -\delta^{\dot{\gamma}}_{\dot{\beta}} Q^a_\alpha\,, \tag{A.8}$$

$$[K^{\dot{\alpha}\beta}, Q^a_\gamma] = \delta^\beta_\gamma \bar{S}^{\dot{\alpha}a}\,, \quad [K^{\dot{\alpha}\beta}, \bar{Q}_{\dot{\gamma}a}] = \delta^{\dot{\alpha}}_{\dot{\gamma}} S^\beta_a\,. \tag{A.9}$$

Under the action of the Lorentz generators and the R-symmetry generators, the fermionic generators transform as follows

$$[M_\alpha{}^\beta, Q^a_\gamma] = \delta^\beta_\gamma Q^a_\alpha - \frac{1}{2}\delta_\alpha{}^\beta Q^a_\gamma\,, \qquad [\bar{M}^{\dot{\alpha}}{}_{\dot{\beta}}, \bar{Q}_{\dot{\gamma}a}] = -\delta^{\dot{\alpha}}_{\dot{\gamma}} \bar{Q}_{\dot{\beta}a} + \frac{1}{2}\delta^{\dot{\alpha}}{}_{\dot{\beta}} \bar{Q}_{\dot{\gamma}a}\,, \tag{A.10}$$

$$[M_\alpha{}^\beta, S^{\gamma a}] = -\delta^\gamma_\alpha S^{\beta a} + \frac{1}{2}\delta_\alpha{}^\beta S^{\gamma a}\,, \qquad [\bar{M}^{\dot{\alpha}}{}_{\dot{\beta}}, \bar{S}^{\dot{\gamma}a}] = \delta^{\dot{\gamma}}_{\dot{\beta}} \bar{S}^{\dot{\alpha}a} - \frac{1}{2}\delta^{\dot{\alpha}}{}_{\dot{\beta}} \bar{S}^{\dot{\gamma}a}\,, \tag{A.11}$$

$$[R_a{}^b, Q^c_\alpha] = -\delta_a{}^c Q^b_\alpha + \frac{1}{4}\delta_a{}^b Q^a_\alpha\,, \qquad [R_a{}^b, \bar{Q}_{\dot{\alpha}c}] = \delta_c{}^b \bar{Q}_{\dot{\alpha}a} - \frac{1}{4}\delta_a{}^b \bar{Q}_{\dot{\alpha}c}\,, \tag{A.12}$$

$$[R_a{}^b, S^\alpha_c] = \delta_c{}^b S^\alpha_a - \frac{1}{4}\delta_a{}^b S^\alpha_c\,, \qquad [R_a{}^b, \bar{S}^{\dot{\alpha}c}] = -\delta_a{}^c \bar{S}^{\dot{\alpha}b} + \frac{1}{4}\delta_a{}^b \bar{S}^{\dot{\alpha}c}\,. \tag{A.13}$$

The anti-commutators for the fermionic generators are

$$\{Q^a_\alpha, \bar{Q}_{\dot{\beta}b}\} = \delta^a_b P_{\alpha\dot{\beta}}\,, \quad \{\bar{S}^{\dot{\alpha}a}, S^\beta_b\} = \delta^a_b K^{\dot{\alpha}\beta}\,, \tag{A.14}$$

$$\{Q^a_\alpha, S^\beta_b\} = \delta^a_b M_\alpha{}^\beta - \frac{i}{2}\delta^a_b \delta_\alpha{}^\beta (D + iC) + \delta^\beta_\alpha R_b{}^a\,, \tag{A.15}$$

$$\{\bar{S}^{\dot{\alpha}a}, \bar{Q}_{\dot{\beta}b}\} = -\delta^a_b \bar{M}^{\dot{\alpha}}{}_{\dot{\beta}} - \frac{i}{2}\delta^a_b \delta^{\dot{\alpha}}{}_{\dot{\beta}} (D - iC) - \delta^{\dot{\alpha}}_{\dot{\beta}} R_b{}^a\,. \tag{A.16}$$

Notice that the central charge C appears in the anti-commutators of supercharges and superconformal charges. When we impose the condition of supertracelessness for the generators, we obtain su(2,2|4). If we further drop the central charges on the r.h.s. of the anti-commutators, we get psu(2,2|4). Of course the central charge commutes with all the generators and the hypercharge essentially counts the fermion number $F(J)$ of the generator J:

$$[C, J] = 0\,, \quad [B, J] = \frac{1}{2}F(J)J\,. \tag{A.17}$$

The only generators carrying non-vanishing fermion numbers are the supercharges and the superconformal charges. Their fermion numbers are $F(\bar{Q}_{\dot{\alpha}a}) = F(S^\alpha_a) = 1$ and $F(Q^a_\alpha) = F(\bar{S}^{\dot{\alpha}a}) = -1$.

B Comment on the relation to the singlet state for the SU(2) sector of the previous paper

The exponential form of the singlet projector for $\mathrm{u}(2,2|4)$ constructed in section 2.2 looks rather different from the simple non-exponential form given in our previous work [31] for the $\mathrm{SU}(2)$ subsector. If we write it explicitly in terms of the scalar states in this subsector, it is given by[29]

$$|\mathbf{1}_{12}\rangle_{\mathrm{SO}(4)} = |Z\rangle \otimes |\bar{Z}\rangle - |X\rangle \otimes |(-\bar{X})\rangle - |(-\bar{X})\rangle \otimes |X\rangle + |\bar{Z}\rangle \otimes |Z\rangle\,. \tag{B.1}$$

[29]Here, since we are only concerned with the SU(2) sector where only the SU(4) oscillators are relevant, we shall denote $\overline{|Z\rangle}$ by $|\bar{Z}\rangle$ for simplicity .

In this appendix, we briefly explain how this form is indeed obtained from the exponential form.

The SU(2) sector is only a part of the large Hilbert space in which the exponential state belongs. Also, in our previous paper, we were only considering the spin 1/2 representation for $\mathrm{SU}(2)_L$ and $\mathrm{SU}(2)_R$. Thus to get our singlet formula (B.1) from the exponential form, we must project out such a sector from the full exponential projector.

It turns out that to do this appropriately, we must first write out the exponential state for the full $\mathrm{SU}(4) \simeq \mathrm{SO}(6)$ sector which are generated by the fermionic oscillators only. This is given by

$$|\mathbf{1}_{12}\rangle_{\mathrm{SO}(6)} = e^A |Z\rangle \otimes |\bar{Z}\rangle \tag{B.2}$$
$$A = \bar{c}_1 \otimes c^1 + \bar{c}_2 \otimes c^2 - \bar{d}_1 \otimes d^1 - \bar{d}_2 \otimes d^2 \tag{B.3}$$

Since, for each Hilbert space component, A consists of four different fermionic oscillators, the expansion of e^A stops at order A^4. The terms coming from the odd powers of A, i.e. A and A^3, are fermionic. When we take inner product with scalar states, they do not contribute. Thus we only need to look at terms of order $1, A^2$ and A^4.

(i) At order 1, we simply get $|Z\rangle \otimes |\bar{Z}\rangle$. (ii) The next simplest contribution comes from A^4. Writing this out explicitly, we get

$$\frac{1}{4!} A^4 |Z\rangle \otimes |\bar{Z}\rangle = (\bar{c}_1 \bar{d}_1)(\bar{c}_2 \bar{d}_2) |Z\rangle \otimes (d^1 c^1)(d^2 c^2) |\bar{Z}\rangle \,. \tag{B.4}$$

To see the meaning of this expression clearly, it is instructive to write down the generators of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ in terms of these fermionic oscillators. They are given by

$$J_3^L = \frac{1}{2}(d^1 \bar{d}_1 - \bar{c}_1 c^1) \,, \qquad J_+^L = d_1 c_1 \,, \qquad J_-^L = \bar{c}_1 \bar{d}_1 \,, \tag{B.5}$$
$$J_3^R = \frac{1}{2}(d^2 \bar{d}_2 - \bar{c}_2 c^2) \,, \qquad J_+^R = d_2 c_2 \,, \qquad J_-^R = \bar{c}_2 \bar{d}_2 \,. \tag{B.6}$$

From this we see that the r.h.s. of (B.4) can be written as $J_-^L J_-^R |Z\rangle \otimes J_+^L J_+^R |\bar{Z}\rangle$. The action of these lowering and raising operators turn $|Z\rangle$ into $|\bar{Z}\rangle$ and $|\bar{Z}\rangle$ into $|Z\rangle$, so that we get the simple result

$$\frac{1}{4!} A^4 |Z\rangle \otimes |\bar{Z}\rangle = |\bar{Z}\rangle \otimes |Z\rangle \,. \tag{B.7}$$

(iii) Finally consider the A^2 terms. This produces 6 terms of various structures. To see which terms are relevant to the SO(4) sector, it is useful to look at the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ quantum numbers of the oscillators:

$$c^1 = \left(\frac{1}{2}, 0\right), \quad \bar{c}_1 = \left(-\frac{1}{2}, 0\right), \quad c^2 = \left(0, \frac{1}{2}\right), \quad \bar{c}_2 = \left(0, -\frac{1}{2}\right) \tag{B.8}$$
$$d^1 = \left(\frac{1}{2}, 0\right), \quad \bar{d}_1 = \left(-\frac{1}{2}, 0\right), \quad d^2 = \left(0, \frac{1}{2}\right), \quad \bar{d}_2 = \left(0, -\frac{1}{2}\right) \tag{B.9}$$

Then, we can classify the 6 terms produced at order A^2 by their quantum numbers as follows:

$$\bar{c}_1 \bar{c}_2 |Z\rangle \otimes c^1 c^2 |\bar{Z}\rangle : \quad (0,0) \otimes (0,0) \tag{B.10}$$

$$-\bar{c}_1\bar{d}_1|Z\rangle\otimes c^1d^1|\bar{Z}\rangle: \quad -\Big(-\frac{1}{2},\frac{1}{2}\Big)\otimes\Big(\frac{1}{2},-\frac{1}{2}\Big)\simeq -|-\bar{X}\rangle\otimes|X\rangle \tag{B.11}$$

$$-\bar{c}_1\bar{d}_2|Z\rangle\otimes c^1d^2|\bar{Z}\rangle: \quad -(0,0)\otimes(0,0) \tag{B.12}$$

$$-\bar{c}_2\bar{d}_1|Z\rangle\otimes c^2d^1|\bar{Z}\rangle: \quad -(0,0)\otimes(0,0) \tag{B.13}$$

$$-\bar{c}_2\bar{d}_2|Z\rangle\otimes c^2d^2|\bar{Z}\rangle: \quad -\Big(\frac{1}{2},-\frac{1}{2}\Big)\otimes\Big(-\frac{1}{2},\frac{1}{2}\Big)\simeq -|X\rangle\otimes|-\bar{X}\rangle \tag{B.14}$$

$$\bar{d}_1\bar{d}_2|Z\rangle\otimes d^1d^2|\bar{Z}\rangle: \quad (0,0)\otimes(0,0) \tag{B.15}$$

The four terms with the quantum numbers $(0,0)\otimes(0,0)$ are orthogonal to the SO(4) scalar states of our interest and hence can be ignored in the singlet projector for the SU(2) sector. Thus, collecting the relevant states, we find

$$\begin{aligned}|\mathbf{1}_{12}\rangle_{\mathrm{SO}(6)} &= e^{A}|Z\rangle\otimes|\bar{Z}\rangle\\ &\ni |\mathbf{1}_{12}\rangle_{\mathrm{SO}(4)} = |Z\rangle\otimes|\bar{Z}\rangle-|X\rangle\otimes|(-\bar{X})\rangle-|(-\bar{X})\rangle\otimes|X\rangle+|\bar{Z}\rangle\otimes|Z\rangle,\end{aligned} \tag{B.16}$$

which is precisely the singlet state (B.1) constructed in our previous work.

C Some details for the derivation of the crossing relation for the harmonic R-matrix

In this appendix, we provide some details of the derivation of the intermediate formulas which are needed for the proof of the crossing relation for the harmonic R-matrix.

Proof of the formula (3.47). To prove the formula (3.47) for the crossing of the number operators for the quantum and the auxiliary spaces, we should first recall the crossing property of the oscillators given in (3.30), (3.31):

$$\langle\mathbf{1}_{12}|\bar{\alpha}_{(1)}=\langle\mathbf{1}_{12}|\bar{\alpha}_{(2)}\,, \qquad \langle\mathbf{1}_{12}|\alpha^{(1)}=-\langle\mathbf{1}_{12}|\alpha^{(2)}\,, \tag{C.1}$$

$$\langle\mathbf{1}_{12}|\bar{\beta}_{(1)}=-\langle\mathbf{1}_{12}|\bar{\beta}_{(2)}\,, \qquad \langle\mathbf{1}_{12}|\beta^{(1)}=\langle\mathbf{1}_{12}|\beta^{(2)}\,. \tag{C.2}$$

Here, the subscripts (1),(2) label the two different spin chains corresponding to two operators. For simplicity we have suppressed the indices for the $\mathrm{gl}(2|2)\oplus\mathrm{gl}(2|2)$ and the labels for the different sites in the spin chain. Now from these relations, we immediately see that, under crossing, the number operator for the quantum space $\mathbf{N}^{(1)}$ transforms as $\mathbf{N}^{(1)}\to-\mathbf{N}^{(2)}$,[30] while the number operator for the auxiliary space $\mathbf{N}^{(a)}$ does not change.

Proof of the formula (3.48). This formula can be understood in the following way. For simplicity, we concentrate on the oscillators $\bar{\alpha},\alpha$. We first transform the creation operators $\bar{\alpha}_1^{\mathsf{A}}$ by crossing and get

$$\langle\mathbf{1}_{12}|\frac{1}{k!l!m!n!}\bar{\alpha}_2^{\mathsf{A}_1}\cdots\bar{\alpha}_2^{\mathsf{A}_k}\cdots\bar{\alpha}_a^{\mathsf{B}_1}\cdots\bar{\alpha}_a^{\mathsf{B}_m}\cdots\alpha^1_{\mathsf{B}_m}\cdots\alpha^1_{\mathsf{B}_1}\cdots\alpha^a_{\mathsf{A}_k}\cdots\alpha^a_{\mathsf{A}_1}\,. \tag{C.3}$$

[30] To be precise, $\mathbf{N}_\alpha^{(1)}$ transforms as $\mathbf{N}_\alpha^{(1)}\to-\mathbf{N}_\alpha^{(2)}-(-1)^{|\mathsf{A}|}\delta^{\mathsf{A}}_{\mathsf{A}}$. However, the constant term vanishes as the signs are opposite for the bosonic and fermionic oscillators and hence they exactly cancel with each other in the present case. This is also true for $\mathbf{N}_\beta^{(1)}$.

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Then, we wish to move the annihilation operators $\alpha^1_{\mathsf{A}_1}$ next to the singlet projector in order to use the crossing formula for them. This can be easily done, but after the crossing, we need to move them back to the original position, which in turn generates extra terms since α^2_{A}'s do not commute with the creation operators $\bar{\alpha}^{\mathsf{A}}_2$. As a result, the Kronecker delta $\delta^{\mathsf{A}}_{\mathsf{B}}$ appears, which contracts the indices for the oscillators of the auxiliary space $\bar{\alpha}^{\mathsf{B}}_a, \alpha^a_{\mathsf{A}}$. In this way we find that the number operator of the auxiliary space $\mathbf{N}^{(a)}_\alpha$ is inserted in the middle of the oscillators. Now when we move the number operator to the left most position, the number operator is shifted by a constant due to the presence of the creation operators on the way. This gives the expression of the form

$$\begin{aligned}&(\mathbf{N}^{(a)}_\alpha - m + 1)\cdots(\mathbf{N}^{(a)}_\alpha - m + p) \\ &\times \bar{\alpha}^{\mathsf{A}_1}_2 \cdots \bar{\alpha}^{\mathsf{A}_{k-p}}_2 \cdots \bar{\alpha}^{\mathsf{B}_1}_a \cdots \bar{\alpha}^{\mathsf{B}_{m-p}}_a \cdots \alpha^2_{\mathsf{B}_{m-p}} \cdots \alpha^2_{\mathsf{B}_1} \cdots \alpha^a_{\mathsf{A}_{k-p}} \cdots \alpha^a_{\mathsf{A}_1} \,.\end{aligned} \tag{C.4}$$

By carefully treating the numerical coefficients and performing the same calculation for the oscillators $\bar{\beta}, \beta$, we find that the crossing formula for the hopping operator is given by (3.48).

Explanation of (3.49). Let us make cautionary remarks for using the crossing relations for the coefficients and the hopping operator already obtained to derive the crossing relation for the harmonic R-matrix given in (3.49). This has to do with the effects due to the order of crossing. Although the hopping operators preserve the total number of oscillators of the quantum space and as well as of the auxiliary space and commute with the number operator $\mathbf{N}$, the expression $\mathbf{Hop}^{(a2)}_{k,l,m,n}$ which appears *after* the crossing no longer commutes with the total number operator of the form $\mathbf{N}^{(a)} + \mathbf{N}^{(1)}$. In fact, since the hopping operator $\mathbf{Hop}^{(a2)}_{k,l,m,n}$ moves $k + l$ oscillators from the auxiliary space to the quantum space and moves $m + n$ oscillators from the quantum space to the auxiliary space, the following exchange relation holds:

$$\mathbf{Hop}^{(a2)}_{k,l,m,n} f(\mathbf{N}^{(a)} + \mathbf{N}^{(1)}) = f(\mathbf{N}^{(a)} + \mathbf{N}^{(1)} + k + l - m - n)\mathbf{Hop}^{(a2)}_{k,l,m,n} \,. \tag{C.5}$$

This effect has to be duly taken into account. More specifically, we first move the hopping operator $\mathbf{Hop}^{(a1)}_{k,l,m,n}$ to the left all the way until it hits the singlet projector. This operation does not shift the number operator as the labels for the quantum space are different and they commute with each other. Now, upon hitting the singlet state we use the crossing relation to convert it to $\mathbf{Hop}^{(a2)}_{k,l,m,n}$ and then we try to move it back to the original position. In this process we come across the shift for the number operator as in (C.5).[31] After this procedure we make the crossing of the coefficients as $\mathcal{A}^{(\mathbf{N})}_I \to \mathcal{A}^{(\mathbf{N}^{(a)} - \mathbf{N}^{(2)})}_I$. In this way, we obtain the relation (3.49).

Proof of the relation (3.53). Finally, we shall provide a proof of the relation (3.53). Let us recall that the definition for the coefficient $\mathcal{A}^{(\mathbf{N})}_I$ is given in terms of the generalized binomial (3.36). Hence, we have

$$\tilde{\mathcal{A}}^{(\mathbf{N})}_I = \rho(u)(-1)^{I + \frac{\mathbf{N}}{2} + \mathbf{M}} \sum_{r=0}^{\infty} (-1)^r \mathcal{B}(\mathbf{M}, r) \mathcal{B}\left(I + r, r - \mathbf{M} - u + \frac{\mathbf{N}}{2}\right), \tag{C.6}$$

[31] Actually, we need to exchange $\mathbf{Hop}^{(a2)}_{k-p,l-q,m-p,n-q}$ through the coefficient $\mathcal{A}^{(\mathbf{N})}_I$. But this produces the same shift as in (C.5).

where we have used the identity $\mathcal{B}(x,y)=\mathcal{B}(x,x-y)$. From the above expression, it turns out that the proof for the relation (3.53) is equivalent to verify the relation

$$\sum_{r=0}^{\infty}(-1)^r\mathcal{B}(\mathbf{M},r)\mathcal{B}\Big(I+r,r-\mathbf{M}+-u+\frac{\mathbf{N}}{2}\Big)=(-1)^{\mathbf{M}}\mathcal{B}\Big(I,-u+\frac{\mathbf{N}}{2}\Big)\,. \tag{C.7}$$

For this purpose, we will consider the following more general formula

$$\sum_{r=0}^{\infty}(-1)^r\mathcal{B}(\gamma,r)\mathcal{B}(\alpha+r,r-\gamma+\beta)=\frac{\sin\pi(\beta-\gamma)}{\sin\pi\beta}\mathcal{B}(\alpha,\beta)\,, \tag{C.8}$$

where α,γ are arbitrary complex numbers and we assume β to be generally a non-integer complex number. Once we can justify this relation, we easily obtain the relation we need by setting $\alpha=I$, $\beta=-u+\frac{\mathbf{N}}{2}$ and $\gamma=\mathbf{M}$. Using the definition for the generalized binomial and the well-known identity for the gamma function $\Gamma(x)\Gamma(1-x)=\frac{1}{\sin\pi x}$, the left hand side becomes

$$\begin{aligned}(\text{L.H.S})&=\sum_{r=0}^{\infty}(-1)^r\frac{\Gamma(\gamma+1)}{\Gamma(r+1)\Gamma(\gamma-r+1)}\frac{\Gamma(\alpha+r+1)}{\Gamma(r-\gamma+\beta+1)\Gamma(\alpha+\gamma-\beta+1)}\\&=\frac{\Gamma(\gamma+1)\Gamma(\alpha+1)}{\Gamma(\alpha+\gamma-\beta+1)}\sum_{r=0}^{\infty}(-1)^r\frac{\Gamma(\alpha+r+1)}{\Gamma(r+1)\Gamma(\alpha+1)}\frac{\Gamma(\gamma-r-\beta)\sin\pi(r-\gamma+\beta+1)}{\Gamma(\gamma-r+1)}\\&=\frac{\sin\pi(\beta-\gamma)}{\sin\pi\beta}\frac{\Gamma(\gamma+1)\Gamma(\alpha+1)}{\Gamma(\alpha+\gamma-\beta+1)\Gamma(\beta+1)}\sum_{r=0}^{\infty}\mathcal{B}(\alpha+r,r)\mathcal{B}(\gamma-r-\beta-1,\gamma-r)\,.\end{aligned} \tag{C.9}$$

The summation over the products of binomials turns out to be equal to $\mathcal{B}(\alpha-\beta+\gamma,\gamma)$ since the following identity holds for arbitrary complex numbers a,b,c

$$\mathcal{B}(a+b+c-1,c)=\sum_{k=0}^{\infty}\mathcal{B}(a+k-1,k)\mathcal{B}(b+c-k-1,c-k)\,. \tag{C.10}$$

When c is any positive integer, the above relation immediately follows from

$$\frac{1}{(1+x)^a}=\sum_{k=0}^{\infty}(-1)^k\mathcal{B}(a+k-1,k)x^k\,,\quad\frac{1}{(1+x)^{a+b}}=\frac{1}{(1+x)^a}\cdot\frac{1}{(1+x)^b}\,. \tag{C.11}$$

As the both sides of (C.10) are analytic functions of c, it turns out that the relation holds for arbitrary complex c by analytic continuation. By using this identity with $a=\alpha+1,b=-\beta,c=\gamma$, we find

$$(\text{L.H.S})=\frac{\sin\pi(\beta-\gamma)}{\sin\pi\beta}\frac{\Gamma(\gamma+1)\Gamma(\alpha+1)}{\Gamma(\alpha+\gamma-\beta+1)\Gamma(\beta+1)}\mathcal{B}(\alpha-\beta+\gamma,\gamma)=\frac{\sin\pi(\beta-\gamma)}{\sin\pi\beta}\mathcal{B}(\alpha,\beta)\,. \tag{C.12}$$

Therefore, we have shown (C.8), which completes the proof.

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