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M-theory on non-Kähler eight-manifolds

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Abstract: We show that M-theory admits a class of supersymmetric eight-dimensional compactification background solutions, equipped with an internal complex pure spinor, more general than the Calabi-Yau one. Building-up on this result, we obtain a a particular class of supersymmetric M-theory eight-dimensional non-geometric compactification backgrounds with external three-dimensional Minkowski space-time, proving that the global space of the non-geometric compactification is again a differentiable manifold, although with very different geometric and topological properties respect to the corresponding standard M-theory compactification background: it is a compact complex manifold admitting a Kähler covering with deck transformations acting by holomorphic homotheties with respect to the Kähler metric. We show that this class of non-geometric compactifications evade the Maldacena-Nuñez no-go theorem by means of a mechanism originally developed by Mario García-Fernández and the author for Heterotic Supergravity, and thus do not require l_P-corrections to allow for a nontrivial warp factor or four-form flux. We obtain an explicit compactification background on a complex Hopf four-fold that solves all the equations of motion of the theory, including the warp factor equation of motion. We also show that this class of non-geometric compactifications are equipped with a holomorphic principal torus fibration over a projective Kähler base as well as a codimension-one foliation with nearly-parallel G_2-leaves, making thus contact with the work of M. Babalic and C. Lazaroiu on the foliation structure of the most general M-theory supersymmetric compactifications.

Keywords: Flux compactifications, M-Theory, F-Theory, Differential and Algebraic Geometry

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1 Introduction and summary of results

Supersymmetry has been linked in many different and profound ways to geometry since its discovery in the seventies, see for example [1–5] for more information and further references. In particular, supersymmetric solutions to Supergravity theories are closely linked to spinorial geometry, since they consist of manifolds equipped with spinors constant respect to a particular connection, whose specific form depends on the Supergravity theory under consideration [6, 7]. The global existence of spinors and the other Supergravity fields usually constrains the global geometry of the manifold. However, the final resolution of the Supergravity equations of motion usually resorts to the use of adapted coordinates to the problem at a local patch of the manifold. Once we have solved the Supergravity equations of motion, a really hard problem by itself, we have to face another difficulty: in order to fully understand the solution, we need to extract as much information as possible about the global geometry of the manifold just from the existence of some explicit tensors and spinors, which we only know at a local patch. In other words, we want to know which manifolds are compatible with a particular set of tensors and spinors whose form is only known locally.

In fact, this is not a new problem in Theoretical Physics or Differential Geometry. It was already encountered soon after the discovery of General Relativity. Solving General Relativity's equations of motion[1] usually means solving the metric at a local patch of a

[1]In contrast with what it is usually implied in the mathematical literature, General Relativity's equations are in general not just the requirement of Ricci-flatness of the Levi-Civita connection.

manifold which is not known a priori. In order to find which would be the physically meaningful manifold compatible with a locally defined metric, physicists back then created a procedure, by now textbook material [8], to obtain the *maximally analytic extension* of a given local patch endowed with a locally defined metric. In doing so for a *simple* solution, namely the Schwarzschild black-hole, one can find for example that the corresponding manifold can indeed be covered by a single system of coordinates and it is thus homeomorphic to an open set in $\mathbb{R}^4$. This procedure has been carried out in other popular solutions of General Relativity, for example the Reissner-Nordström and the Kerr black holes, which are relatively simple solutions compared to the kind of solutions that one obtains in Supergravity, where finding the maximally analytic extension associated to a local solution is more difficult due to their complexity.

Still, for supersymmetric solutions of Supergravity some information about the global geometry of the manifold can be obtained simply from the analysis of the existence of constant spinors: for example it may be possible to show that the manifold is equipped with various geometric structures, like Killing vectors or complex, Kähler, Hyperkähler, Quaternionic... appropriately defined structures. This already constrains the problem to a relatively specific class of manifolds. However, in performing such analysis sometimes there are involved various kinds of subtle choices, which, if modified, would yield a different global solution, a different manifold which however is locally indistinguishable from the unmodified one, since they exactly carry the same structure at the local level.

The first thing we are going to do in this note is to precisely modify one condition that had been implicitly assumed so far in String-Theory warped compactifications [9]: we are going to consider that the warp-factor is not a globally defined function on the compact manifold, but only, given a good open covering, locally defined on each open set. In order to do this consistently we will keep in mind that the physical fields of the theory must remain as well-defined tensors on the manifold, as it is required from physical considerations. The warp factor will turn out to be globally described as a section of an appropriate line bundle.

We are going to apply the previous modification to M-theory compactifications to three-dimensional Minkowski space-time preserving $\mathcal{N} = 2$ supersymmetries. M-theory compactifications to three dimensions preserving different amounts of supersymmetry have been extensively studied in the literature [10–17]. In references [16, 17] a very rigorous and complete study of the geometry of the internal eight-dimensional manifold has been carried out using the theory of codimension-one foliations, which turns out to be the right mathematical tool to characterize it, as suggested in [18].

Coming back to the case of compactifications to three-dimensional Minkowski space-time preserving $\mathcal{N} = 2$ supersymmetries, the analysis of the seminal reference [10] concludes that, among other things, the internal eight-dimensional manifold is a Calabi-Yau four-fold, although the physical metric is not the Ricci-flat metric but conformally related to it. This class of M-theory compactifications is very important for F-theory [19] applications, since compactifications of F-theory are in fact defined through them by assuming that the internal manifold is an elliptically fibered Calabi-Yau manifold, see [20] for a review and further references.

By assuming that the warp factor is not a global function, we will be able to generalize the result of reference [10]: we will find that the internal manifold must be a locally con-

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formally Kähler manifold [21–23] locally equipped with a preferred Calabi-Yau structure. Evidently, standard Calabi-Yau manifolds are a particular case inside this class. Let us say that this note is of course not the first attempt to obtain admissible F-theory backgrounds beyond the Calabi-Yau result; see references [24, 25] for applications of Spin(7)-manifolds to F-theory compactifications.

It turns out that the solution that is obtained by assuming that the warp factor is not globally defined belongs to a *simple* class of non-geometric compactification backgrounds, and this is the approach that we will use in section 4. By non-geometric solution we mean here a global solution obtained by patching up local solutions to the equations of motion by means of local diffeomorphisms, gauge transformations, and global symmetries of the equations of motion, namely U-dualities. Notice that the term *non-geometric* is somewhat misleading since, although there is no guarantee that the global of a non-geometric solution is a smooth differentiable manifold, it will be for sure a well-defined mathematical object, with well-defined topological an geometric properties. We will use anyway the term *non-geometric* since it is widely used in the literature.

Non-geometric compactification backgrounds have been intensively studied in the literature from different points of view, see for example [26–29] for more details and further references. References [30, 31] consider compactifications that are non-geometric from the Heterotic point of view and that become geometric compactifications via duality with F-theory. References [32–35] contain a very interesting approach, named there *G-theory*, along the main idea of this work: among other things, they provide a very detailed construction of non-perturbative vacua by gluing local solutions to the equations of motion using different types of U-dualities. When performing such a non-trivial global patching, it is usually very difficult to obtain precise results about the topological and geometric properties of the global space of the compactification. This is partly due to the fact that the symmetries of the local equations of motion involved in the global patching can be relatively involved. That is why here we will consider the arguably simplest non-geometric global patching of local solutions to the equations of motion of eleven-dimensional Supergravity on a warped compactification background to three-dimensional Minkowski space-time. In exchange, we will be able to fully characterize the topology and the geometry of the global space.

More precisely, we will consider local solutions to the eleven-dimensional Supergravity equations of motion and we will globally patch them using local diffeomorphisms, gauge transformations and the *trombone symmetry* of the warp factor, which simply consists on rescalings of the warp factor by a real constant. Therefore, the global symmetry of the equations of motion that we will use to patch the local solutions is the simplest one. The idea is to consider the simplest non-geometric scenario in order to be able to fully characterize topologically as well as geometrically the global space of the compactification, something of utmost importance in order to understand the moduli space of a non-geometric compactification space. Hence, we hope this compactification background will hep to understand the nature of non-geometric compactification spaces, starting from the simplest case. In fact, we will be able to show that the global space of the compactification is a differentiable manifold, but with topological and geometric properties drastically different from the corresponding standard geometric compactification backgrounds.

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Let us be more precise. In this letter we will prove, among other things, that:

- The non-geometric compactification space M is a particular class of compact complex manifolds admitting a Kähler covering with deck transformations acting by holomorphic homotheties with respect to the Kähler metric. In other words, M is a particular type of locally conformally Kähler manifold. Therefore, M admits a Käler covering $\tilde{M}$ with Kähler form $\tilde{\omega}$, fitting in the following short sequence:

$$\Gamma \to \tilde{M} \to M\,. \tag{1.1}$$

  The non-geometric warp factor is encoded in the geometry of M in an elegant way. Given a $2d$-dimensional locally conformally Kähler manifold (M, ω, θ) with Kähler form ω and closed Lee-form θ, let L the trivializable flat line bundle associated to the representation $A \to |\det A|^{\frac{1}{d}}, A \in Gl(2d, \mathbb{R})$, with a flat connection $\nabla_\theta \equiv d + \theta$. The line bundle L is usually called the weight bundle of M and its holonomy coincides with the character $\chi\colon \pi_1(M) \to \mathbb{R}^+$. The image of χ is called the monodromy group of M. The warp factor is given by a flat connection of L which, after choosing a trivialization, is given by a closed one-form on M. If M is simply-connected its holonomy is trivial and then M becomes a Kähler manifold and the compactification becomes geometric.

- The non-geometry of the solution is then associated to the space being non-simply-connected. If we take M to be simply connected, then M becomes a Kähler manifold and we obtain a standard geometric solution.

- We obtain an explicit solution, preserving locally $\mathcal{N} = 2$ supersymmetry, on a complex Hopf four-fold that solves all the local equations of motion of the theory, including the equation of motion for the wrap factor. We explicitly write the local metric, flux and warp factor.

- The previous solution evades the Maldacena-Nuñez theorem by means of a mechanism originally developed by Mario García-Fernández and the author for Heterotic Supergravity, and thus there are non-geometric solutions with non-zero warp factor and flux without the need of higher derivative corrections.

- The explicit solution on the complex Hopf four-fold is equipped with a holomorphic elliptic fibration over a Kähler base. This points out to a possible application of this backgrounds to F-theory compactifications.

- The explicit solution on the complex Hopf four-fold admits a codimension-one foliation equipped with a nearly-parallel G_2 structure on the leaves. Then, the solution, even non-geometric, preserves the structure of the most general geometric compactification background of eleven-dimensional Supergravity on an eight-manifold, studied in references [16, 17, 36, 37].

In addition, the moduli space of locally conformally Kähler manifolds is usually very restricted, so compactification on this backgrounds may partially evade the moduli stabilization problem, present in many String Theory compactifications.

To summarize, we think this kind of non-geometric backgrounds is simple enough to be manageable, in particular it is possible to study their global topological and geometric properties, yet it is an honest non-trivial non-geometric compactification background. Therefore it might be a good starting point to a systematic rigorous study of non-geometric Supergravity backgrounds. This letter is a first small step in that direction.

The consequences of compactifying M-theory on a locally conformally Kähler manifold instead of a Calabi-Yau four-fold are manifold since the former is not Ricci-flat in a compatible way and has different topology than the latter. This deserves further study. In particular we think that it would be interesting to obtain, if possible, the effective action of a M-theory compactification on a non-Calabi-Yau locally conformally Kähler manifold.

The plan of this paper goes as follows. In section 2 we review, following [10], the analysis of M-theory compactifications to three-dimensional Minkowski space-time preserving $\mathcal{N} = 2$ supersymmetries, pointing out in a precise way the well-known *issue* of imposing at the same time the classical Killing spinor equations and the l_P-corrected equations of motion, an issue that is not present in the non-geometric setting since the Maldacena-Nuñez no-go theorem does no hold and thus there is no need of considering l_P-corrections in order to have non-trivial solutions. In section 3 we modify the procedure explained in section 2 by considering a warp-factor which is not a globally defined function on the internal manifold. In section 4 we reinterpret the previous construction as a non-geometric compactification background. In section 5 we construct the non-geometric compactification and we obtain an explicit solution to all the equations of motion, studying some of its properties. In particular, we show that it is equipped with a holomorphic torus fibration over a projective Kähler base and with a codimension-one foliation with nearly-parallel G_2-leaves.

2 M-theory compactifications on eight-manifolds

In this note we are interested in a particular class of non-geometric M-theory compactification backgrounds to $\mathcal{N} = 2$ three-dimensional Minkowski space-time. These type of non-geometric compactifications will be introduced in section 4. In this section we will consider standard M-theory supersymmetric solutions, in order to motivate how the non-geometric version of these solutions may be useful in evading some of the *issues* present in the standard M-theory supersymmetric compactification case, such as the Maldacena-Nuñez no-go theorem [38]. The effective, low-energy, description of M-theory [39] is believed to be given by eleven-dimensional $\mathcal{N} = 1$ Supergravity [40], which we will formulate on an eleven-dimensional, oriented, spinnable, differentiable manifold[2] M. We will denote by $S \to M$ the corresponding spinor bundle, which is a bundle of $Cl(1, 10)$ Clifford modules.

[2]By differentiable manifold we mean a Hausdorff, second-countable, topological space equipped with a *differentiable structure*.

At each point $p \in M$ we thus have that S_p is a thirty two real, symplectic, $Cl(1,10)$ Clifford module,[3] with symplectic form ω.

The field content of eleven-dimensional Supergravity is given by a Lorentzian metric g, a closed four-form $\mathsf{G} \in \Omega^4_{cl}(\mathcal{M})$ and a Majorana gravitino $\Psi \in \Gamma\left(S \otimes \Lambda^1(\mathcal{M})\right)$. We will focus only on bosonic solutions $(M, \mathsf{g}, \mathsf{G})$ of the theory, so we will truncate the gravitino. The classical bosonic equations of motion are given by:

$$E_0 = \mathrm{Ric} - \frac{1}{2}\mathsf{G} \circ \mathsf{G} + \frac{1}{6}\mathsf{g}\,|\mathsf{G}|^2 = 0\,, \qquad F_0 = d * \mathsf{G} + \mathsf{G} \wedge \mathsf{G} = 0\,, \tag{2.1}$$

where

$$(\mathsf{G} \circ \mathsf{G})(v, v) = |\iota_v \mathsf{G}|^2\,, \qquad v \in \mathfrak{X}(M)\,, \tag{2.2}$$

is a symmetric $(2,0)$ tensor. Eleven-dimensional Supergravity supersymmetric bosonic solutions, and in particular supersymmetric compactification backgrounds, are defined as being solutions of eleven-dimensional Supergravity admitting at least one real spinor $\epsilon \in \Gamma(S)$ such that:

$$D\epsilon = 0\,, \tag{2.3}$$

where D is the Supergravity connection acting on the bundle of Clifford-modules S. It is given by:

$$D_v \epsilon \equiv \nabla_v \epsilon + \frac{1}{6}\iota_v \mathsf{G} \cdot \epsilon + \frac{1}{12} v^\flat \wedge \mathsf{G} \cdot \epsilon\,. \tag{2.4}$$

Here ∇ is the spin connection induced from the Levi-Civita connection on the tangent bundle and $\cdot$ denotes the Clifford action of forms on sections of S.

A supersymmetric configuration $(M, \mathsf{g}, \mathsf{G})$, namely a manifold admitting a D-constant spinor, does not necessarily solves the eleven-dimensional Supergravity equations of motion, but it is in some sense not far from being a solution, since the integrability condition of (2.3) can be written in terms of the equations of motion of the theory. The integrability condition of (2.3) can be found to be:

$$\iota_v\, E \cdot \epsilon - \frac{1}{6 \cdot 3!} v^\flat \wedge (*F) \cdot \epsilon + \frac{1}{3!}\iota_v (*F) \cdot \epsilon = 0\,, \tag{2.5}$$

where E_0 denotes the Einstein equation and F_0 denotes the Maxwell equation of eleven-dimensional Supergravity, see (2.1). Supersymmetric solutions of eleven-dimensional Supergravity can be divided in two classes, the time-like class and the null class, see references [41, 42], where the classification of supersymmetric solutions of eleven-dimensional Supergravity was obtained. The time-like class is given by the supersymmetric solutions that satisfy:

$$g(\xi^\flat, \xi^\flat) > 0\,, \qquad \xi(v) = \omega(\epsilon, v \cdot \epsilon)\,, \tag{2.6}$$

where ξ is the one-form associated to ϵ. Null supersymmetric solutions on the other hand, are those that satisfy $g(v, v) = 0$. For time-like configurations, it can be shown that if the Maxwell equation is satisfied, then the Einstein equations follow from the integrability

[3]There are two $Cl(1,10)$ Clifford modules, which can be distinguished by the action of the volume form of $Cl(1,10)$.

condition of the Killing spinor equation (2.3), see reference [41]. In other words, the Einstein equations follow from supersymmetry and the Maxwell equations. Hence, as it is well known in the literature, supersymmetry is closely related to the equations of motion but it does no *always* imply them. Supersymmetric compactification backgrounds are indeed time-like supersymmetric solutions of eleven-dimensional Supergravity.

Compactification backgrounds of eleven-dimensional Supergravity are subject to the Maldacena-Nuñez no-go theorem [38], which we state here for completeness, applied to eleven-dimensional Supergravity.

Theorem 2.1. *Every warped compactification of eleven-dimensional Supergravity on a closed manifold necessarily has constant warp factor and zero four-form flux G.*

Therefore it would seem that if we want to define F-theory compactifications through eleven-dimensional Supergravity compactifications on an eight-dimensional manifold we will end-up having only the *trivial* flux-less solution. The standard way to evade the Maldacena-Nuñez theorem is to include in the theory higher-derivative corrections and/or negative-tension objects. Since it is not clear whether negative-tension objects exist in M-theory, the strategy of reference [10] was to include the particular higher-derivative correction to eleven-dimensional Supergravity which was known at the time and which gives a negative contribution to the energy-momentum tensor of the theory. This correction was computed for the first time in the one obtained in reference [44]. By means of M/F-Theory duality, higher-derivative corrections to M-theory and negative-tension objects in String Theory are dual manifestations of the same phenomena [43].[4] The only dimension-full parameter in eleven-dimensional Supergravity is the Planck-length l_P and the higher-derivative corrections of M-theory arise in an expansion in powers of this constant over the relevant length-scale of the the problem under consideration. For example, the higher-derivative term considered in [10] is a l_P^6-correction. For simplicity from now on we will refer to the higher-derivative corrections of M-theory as *l_P-corrections*.

The correction to the Killing spinor equation (2.3) corresponding to the correction considered in [10] is not known, so the analysis performed in [10] uses the classical Killing spinor equations and at the same time imposes l_P-corrected equations of motion. This immediately runs into a possible inconsistency, since classical supersymmetry is consistent with the classical equations of motion through the integrability condition of the Killing spinor equation, so imposing l_P-corrected equations of motion on a classical supersymmetric configuration leads to extra constraints that make the problem over determined. The possible inconsistency can be computed explicitly. Let E and F denote the l_P-corrected Einstein and Maxwell equations of motion. They can be written as:

$$E = E_0 + E_1 \,, \qquad F = F_0 + F_1 \,, \tag{2.7}$$

where E_1 and F_1 denote the corresponding corrections to the classical equations of motion E_0 and F_0, and which include the appropriate l_P factors. Now, in order to study the consistency of imposing the l_P-corrected equations of motion E and F as well as classical

[4]We thank JHEP's referee for explaining this point.

supersymmetry, we only have to assume that we indeed have a solution of l_P-corrected equations of motion and compute what is the extra-constraint that appears when imposing the integrability condition of the classical Killing spinor equation. The result is, for every $v \in \mathfrak{X}(M)$, given by:

$$\iota_v E_1 \cdot \epsilon - \frac{1}{6 \cdot 3!} v^\flat \wedge (*F_1) \cdot \epsilon + \frac{1}{3!} \iota_v (*F_1) \cdot \epsilon = 0 \,. \tag{2.8}$$

Therefore, if we want a solution of the classical Killing spinor equation to be a solution of the l_P-corrected equations of motion, the constraint (2.8) must be necessarily satisfied.

The outcome of the analysis of reference [10] is that classical supersymmetry imposes the manifold to be a Calabi-Yau four-fold, although the physical metric does not correspond to the Ricci-flat Calabi-Yau metric. Strictly speaking then, if we want to have a solution of the l_P-corrected equations of motion, not every such Calabi-Yau is an admissible compactification background: only those satisfying equation (2.8), if any, should be considered as honest solutions of the equations of motion. Let us be more explicit for the case of [10]. In reference [10] the equations of motion of classical eleven-dimensional Supergravity were modified by the only known l_P-correction at the time, obtained in reference [44], and which only affects the equation of motion for G. Hence, $E_1 = 0$ and F_1 is given by:

$$F_1 = \beta X_8 = \beta \left(p_1^2 - p_2 \right) \,, \tag{2.9}$$

where p_1 and p_2 are respectively the first and second Pontryagin classes of M, and β is an appropriate constant. Plugging equation (2.9) into equation (2.8) we obtain the explicit constraint that the Calabi-Yau four-folds coming out of the supersymmetry analysis of [10] have to satisfy in order to be an honest solution of the corrected equations of motion:

$$\left(-v^\flat \wedge (*X_8) + 6 \, \iota_v (*X_8) \right) \cdot \epsilon = 0 \,. \tag{2.10}$$

Hence, and again strictly speaking, equation (2.10) constrains the class of admissible F-theory compactification manifolds. Admissible in the sense of honestly solving the equations of motion of l_P-corrected eleven-dimensional Supergravity and at the same time satisfying the classical Killing spinor equation of eleven-dimensional Supergravity. Of course, this *problem* is well-known to experts on the field, but unfortunately, as long as the eleven-dimensional Supergravity l_P-corrected Killing spinor equation is not known, it seems not possible to solve it in a completely rigorous way. Important steps in this direction have been made in references [45–47], where a thoroughly and consistent analysis of M-theory compactifications in the presence of l_P-corrections has been made, and even an educated guess for the l_P-corrected Killing spinor equation has been proposed. Remarkably enough, the integrability condition of the l_P-corrected proposal for the Killing spinor equation is compatible with the l_P-corrected equations of motion, which definitely suggests that if the educated guess is not already the correct l_P-corrected Killing spinor equation, it cannot be far from being it. One of the main conclusions of [46] is that even when one consistently takes into account l_P-corrections, the internal manifold of the compactification is still topologically a Calabi-Yau four-fold. This strongly suggests that the conclusion of reference [10] is solid after properly taking into account l_P-corrections.

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A possible, temporary, solution to the problem of imposing classical supersymmetry and l_P-corrected equations of motion, would be to consider only the elliptically fibered Calabi-Yau four-folds, if any, that satisfy the constraint (2.8). This way we would be sure that we are dealing with honest solutions to l_P-corrected eleven-dimensional Supergravity and at the same time it would single out a preferred class of eliptically fibered Calabi-Yau manifolds.

In this letter we are going to propose a simple class of twisted compactifications that directly evades the Maldacena-Nuñez theorem at the classical level and admits an interpretation as non-geometric compactification backgrounds. Therefore, no l_P-corrections are needed to obtain non-trivial solutions, and thus no inconsistency arises, since there exist closed manifolds with non-trivial flux and warp factor that solve the equations of motion of the theory at the classical level. We don't want to imply with this that l_P-corrections are not relevant: they certainly are of utmost importance in order to understand String/M-theory backgrounds. However, we think that it may be a good idea to understand first non-geometric backgrounds without corrections, namely the *zero-order* solution, before considering l_P-corrections to non-geometric backgrounds. The non-geometric solutions presented in this letter thus constitute the zero order non-geometric solution, which happens to be non-trivial, in the sense that it allows for non-trivial flux and warp-facor, in contrast to what happens in the geometric case. Let us stress though that ideally the ultimate goal would be to include and understand l_P-corrections for geometric as well as for non-geometric compactification backgrounds.

2.1 $\mathcal{N} = 2$ compactifications

In this section we briefly review the standard analysis, following the seminal reference [10], of supersymmetric M-theory compactifications to three-dimensional Minkowski space-time preserving $\mathcal{N} = 2$ supersymmetry. We will consider the space-time to be an eleven-dimensional oriented spin manifold M. The supersymmetry condition corresponds to the vanishing of the Rarita-Schwinger supersymmetry transformation:

$$\delta_\epsilon \Psi = D\epsilon = 0\,, \tag{2.11}$$

where $\epsilon \in \Gamma(S)$ is the spinor generating the supersymmetry transformation and $D\colon \Gamma(S) \to \Gamma(T^*M \otimes S)$ is the eleven-dimensional Supergravity Clifford-valued connection given in terms of g and G. For M-theory compactifications we consider the space-time to be a topologically trivial product of three-dimensional Minkowski space $\mathbb{R}_{1,2}$ and an eight-dimensional Riemannian, compact, spin manifold M_8

$$M = \mathbb{R}_{1,2} \times M_8\,. \tag{2.12}$$

The metric and the four-form are taken to be given by

$$\begin{aligned} \mathsf{g} &= \Delta^2\, \delta_{1,2} + g \\ \mathsf{G} &= \mathrm{Vol} \wedge \xi + G\,, \end{aligned} \tag{2.13}$$

where $\Delta \in C^\infty(M_8)$ is a function, $\delta_{1,2}$ and Vol are the Minkowski metric and the volume form in $\mathbb{R}_{1,2}$, g is the Riemannian metric in M_8, and $G \in \Omega^4(M_8)$ is a closed four-form in the internal space. Finally, the supersymmetry spinor is decomposed as

$$\epsilon = \chi_1 \otimes \eta_1 + \chi_2 \otimes \eta_2 \,, \qquad \chi_1, \chi_2 \in \Gamma(S_{1,2}) \,, \quad \eta_1, \eta_2 \in \Gamma(S_8) \,, \tag{2.14}$$

where $S_{1,2}$ is the rank-two real spinor bundle over $\mathbb{R}_{1,2}$ and S_8 is the real, positive-chirality, rank-eight spinor bundle over M_8. We can form a complex pure spinor η as $\eta = \eta_1 + i\eta_2 \in \Gamma(S_8^{\mathbb{C}})$, where $S_8^{\mathbb{C}}$ is the complex, positive-chirality, spin bundle over M_8. Imposing the previous structure on M, together with supersymmetry condition (2.11), imposes restrictions on the flux G and constrains (M_8, g) at the topological as well as the differentiable level [10]:

- M_8 is equipped with a SU(4)-structure induced by η_1 and η_2, which we assume everywhere independent and non-vanishing. The topological obstruction for the existence of nowhere vanishing real spinor, or in other words, the existence of a Spin(7)-structure is given by
$$p_1^2 - 4p_2 + 8\chi(M_8) = 0 \,, \tag{2.15}$$
where p_1^2 and p_2 are the integrated P_1^2 and P_2 Poyntriagin classes, and $\chi(M_8)$ is the Euler characteristic of M_8.

- M_8 is equipped with a globally defined almost complex structure J, a real non-degenerate (1,1)-form $\omega = g \cdot J$ and a (4,0)-form Ω constructed as bilinears of η. The quadruplet
$$\{g, J, \omega, \Omega\} \tag{2.16}$$
makes M_8 into an almost hermitean manifold with topologically trivial canonical bundle.

- Let us make the following conformal transformation
$$\tilde{g} = \Delta g \,, \qquad \tilde{\eta} = \Delta^{-\frac{1}{2}} \eta \,, \tag{2.17}$$
which implies
$$\tilde{J} = J \,, \qquad \tilde{\omega} = \Delta\omega \,, \qquad \tilde{\Omega} = \Delta^2\Omega \,. \tag{2.18}$$
The usefulness of this conformal transformation comes from the fact that the transformed spinors are constant with respect to the transformed connection, namely
$$\tilde{\nabla}\tilde{\eta} = 0 \,, \tag{2.19}$$
where $\tilde{\nabla}$ is the Levi-Civita connection associated to $\tilde{g}$. Equation (2.19) automatically implies that M_8 has SU(4)-holonomy and thus M_8 is a Calabi-Yau four-fold. In particular we have
$$\tilde{\nabla}\tilde{J} = 0 \,, \qquad \tilde{\nabla}\tilde{\omega} = 0 \,, \qquad \tilde{\nabla}\tilde{\Omega} = 0 \,. \tag{2.20}$$
We can also see that M_8 is a Calabi-Yau four-fold as follows, which might be more natural from the algebraic-geometry point of view: $\left(\tilde{g}, \tilde{\omega}, \tilde{J}\right)$ is the compatible triplet

of a complex structure $\tilde{J}$, a symplectic structure $\tilde{\omega}$ and a Riemannian metric $\tilde{g}$ making M_8 into a Kähler manifold. Since $\tilde{\Omega}$ is an holomorphic (4,0)-form, the canonical bundle is holomorphically trivial, which together with the Kähler property of M_8, implies that it is a Calabi-Yau four-fold.

- The one-form ξ is given by de derivative of the warp factor Δ as follows

$$\xi = d\left(\Delta^3\right) , \tag{2.21}$$

and the four-form G is subject to the constraint

$$\iota_v G \cdot \eta = 0 \, , \qquad v \in \mathfrak{X}(M_8) \, . \tag{2.22}$$

Once we know that M_8 is a Calabi-Yau four-fold, equation (2.22) can be solved by taking G to be $(2,2)$ and primitive.

From the previous analysis we conclude that if we take M_8 to be a Calabi-Yau manifold, $G \in H^{(2,2)}(M_8)$ and primitive and ξ as in equation (2.21), we solve the supersymmetry conditions (2.11) and we obtain a supersymmetric compactification background of eleven-dimensional Supergravity to three-dimensional Minkowski space. Note that the physical metric g is conformally related to the Ricci-flat metric $\tilde{g}$, and by Yau's theorem we know that this is the unique Ricci-flat metric in its Kähler class, and thus it is, strictly speaking, the Calabi-Yau metric of M_8.

3 Global patching of the local supersymmetry conditions

In this section we are going to slightly generalize the set-up reviewed 2 by considering a situation where the conformal transformation (2.17) cannot be performed globally, but only locally. We will be still satisfying the eleven-dimensional Supergravity supersymmetry conditions, which are local, but globally we will be able to construct a manifold that is not necessarily a Calabi-Yau four-fold but of a more general type.

As we did in section 2, we will consider the space-time to be a topologically trivial product of three-dimensional Minkowski space $\mathbb{R}_{1,2}$ and an eight-dimensional Riemannian, compact spin manifold M_8

$$M = \mathbb{R}_{1,2} \times M_8 \, . \tag{3.1}$$

The supersymmetry spinor is also decomposed exactly as it was done in section 2, namely

$$\epsilon = \chi_1 \otimes \eta_1 + \chi_2 \otimes \eta_2 \, , \qquad \chi_1, \chi_2 \in \Gamma(S_{1,2}) \, , \quad \eta_1, \eta_2 \in \Gamma(S_8) \, , \tag{3.2}$$

Hence, as it happened in section 2, M_8 is equipped with two everywhere independent and non-vanishing Majorana-Weyl spinors, which implies again that the structure group of M_8 can be reduced to SU(4). Therefore M_8 still has to satisfy the obstruction (2.15).

Let $\{U_a\}_{a \in I}$ be a good open covering of M_8 and let us equip every open set U_a with a function $\Delta_a \in C^{\infty}(M_a)$ and a closed one form $\xi_a \in \Omega^1(U_a)$, so we can consider the triplet

$\{U_a, \Delta_a, \xi_a\}_{a\in I}$ on M_8. We will assume that the Lorentzian metric g and the four-form G can be written, for every open set $U_a \subset M$, as follows:

$$\begin{aligned} \mathsf{g}|_{U_a} &= \Delta_a^2\, \delta_{1,2} + g|_{U_a}\,, \\ \mathsf{G}|_{U_a} &= \mathrm{Vol}\wedge \xi_a + G|_{U_a}\,, \end{aligned} \tag{3.3}$$

where g is a Riemannian metric in M_8 and G is a closed four-form in M_8. In order to keep a clean exposition, we are not explicitly writing the atlas that we are using for $\mathbb{R}_{1,2}$, which, for each U_a consists of an open set which we take to be the whole $\mathbb{R}_{1,2}$ and its corresponding coordinate system ϕ_a. More precisely, the atlas that we are considering for the topologically trivial product $M = \mathbb{R}_{1,2}\times M_8$ is the following:

$$\mathcal{A} = \{V_a\times U_a, \phi_a\times\psi_a\}_{a\in I}\ , \tag{3.4}$$

where $V_a = \mathbb{R}_{1,2}$ for every $a\in I$, ϕ_a are the coordinates in V_a and ψ_a are the corresponding local coordinates in U_a. The atlas $\mathcal{A}$ is obviously not the simplest atlas for M, but anyway it is an admissible atlas which gives M the structure of a differentiable product manifold. We will see in a moment that the consistency of the procedure requires very specific changes of coordinates $\phi_a\circ\phi_b^{-1}\colon \mathbb{R}^3\to\mathbb{R}^3$, $U_a\cap U_b\neq\emptyset$. The one-form ξ is given again by equation (2.21), only this time the result is valid locally in U_a:

$$\xi_a = d\left(\Delta_a^3\right)\,. \tag{3.5}$$

Now, in order for the physical fields (g,G) to be well defined, they must be tensors on M. This is equivalent to, given any another open set U_b such that $U_a\cap U_b\neq\emptyset$, the following condition in $U_a\cap U_b$:

$$\begin{aligned} \Delta_a^2\,\delta_{1,2} + g|_{U_a\cap U_b} &= \Delta_b^2\,\delta_{1,2} + g|_{U_a\cap U_b}\,, \\ \mathrm{Vol}\wedge\xi_a + G|_{U_a\cap U_b} &= \mathrm{Vol}\wedge\xi_b + G|_{U_a\cap U_b}\,. \end{aligned} \tag{3.6}$$

Equation (3.6) is equivalent to:

$$\begin{aligned} \Delta_a^2\,\delta_{1,2} &= \Delta_b^2\,\delta_{1,2}\,, \\ \mathrm{Vol}\wedge\xi_a &= \mathrm{Vol}\wedge\xi_b\,. \end{aligned} \tag{3.7}$$

in $U_a\cap U_b$, up to of course a change of coordinates, which in turn is reflected as a symmetry of the equations of motion. Therefore, we must define the difference between Δ_a and Δ_b in $U_a\cap U_b$ to be such that it can be absorbed by means of a coordinate transformation in $\mathbb{R}_{1,2}$. The only possibility is:

$$\Delta_a = \lambda_{ab}\Delta_b\,, \tag{3.8}$$

in $U_a\cap U_b$, where $\lambda_{ab}\colon U_a\cap U_b\to\mathbb{R}$ is a constant function. Indeed, the multiplicative factor (3.8) can be absorbed by means of the following change of coordinates in $\mathbb{R}_{1,2}$:

$$\begin{aligned} \phi_a\circ\phi_b^{-1}\colon \mathbb{R}^3 &\to \mathbb{R}^3\,, \\ x &\mapsto \lambda_{ab}^{-1}x\,, \end{aligned} \tag{3.9}$$

which is of course a diffeomorphism. It can be easily seen that

$$\lambda_{ba} = \lambda_{ab}^{-1}\,, \qquad \lambda_{ab}\lambda_{bc}\lambda_{ca} = 1\,, \tag{3.10}$$

where the second equation holds in $U_a \cap U_b \cap U_c \neq \emptyset$. Therefore, the following data

$$\{M_8, U_a, \lambda_{ab}, \mathbb{R}\}\,, \tag{3.11}$$

defines a flat line bundle $L \to M_8$ over M_8 with connection that descends to a well defined closed one form φ in M_8, namely $[\varphi] \in H^1(M_8)$. Using L we can write the families $\{\Delta_a\}_{a\in I}$ and $\{\xi_a\}_{a\in I}$ as

$$\Delta \in C^\infty(M, L) \simeq \Gamma(L)\,, \qquad \xi \in \Omega^1(M, L^3)\,, \tag{3.12}$$

or in other words, in terms of a section of the line bundle L and a one-form taking values in L^3.

3.1 The global geometry of M_8

As it happened in section 2, M_8 is equipped with a globally defined almost complex structure J, a real non-degenerate (1,1)-form $\omega = g\cdot J$ and a (4,0)-form Ω, where J and Ω are constructed as a bilinears from η. The quadruplet

$$\{g, J, \omega, \Omega\} \tag{3.13}$$

makes M_8 into an almost hermitean manifold with topologically trivial canonical bundle.

The crucial difference from the situation that we encountered in section 2 is that the conformal transformation (2.17) cannot be performed globally. Therefore, we cannot perform the conformal transformation that *transforms* the quadruplet $\{g, J, \omega, \Omega\}$ into a Calabi-Yau structure in M_8, which thus cannot be taken to be a Calabi-Yau four-fold; in particular, the supersymmetry complex spinor is not constant respect to any Levi-Civita connection associated to a metric in the conformal class of the physical metric. We can however perform the conformal transformation locally on ever open set U_a, and thus we define

$$\tilde{g}_a = \Delta_a g|_{U_a}\,, \qquad \tilde{\eta}_a = \Delta_a^{-\frac{1}{2}}\eta|_{U_a}\,, \tag{3.14}$$

where now $\tilde{g}_a$ and $\tilde{\eta}_a$ are locally defined on U_a. The local conformal transformation (3.14) implies, again locally in U_a, that

$$\tilde{J}_a = J|_{U_a}\,, \qquad \tilde{\omega}_a = \Delta_a\omega|_{U_a}\,, \qquad \tilde{\Omega}_a = \Delta_a^2\Omega|_{U_a}\,. \tag{3.15}$$

Notice that J is invariant and thus its conformal transformed is a well defined tensor on M_8. An alternative characterization of these locally defined objects is through globally defined tensors taking values on the corresponding powers of the flat line bundle L, namely

$$\tilde{g} \in \Gamma(S^2T^*, L)\,, \quad \tilde{\eta} \in \Gamma(S_8^{\mathbb{C}}, L^{\frac{1}{2}})\,, \quad \tilde{\omega} \in \Omega^2(M_8, L)\,, \quad \tilde{\Omega} \in \Omega^4(M_8, L^2)\,. \tag{3.16}$$

Once we go to the locally transformed spinor and metric, we have that

$$\tilde{\nabla}^a\tilde{\eta}_a = 0\,, \tag{3.17}$$

where $\tilde{\nabla}^a$ is the Levi-Civita connection associated to $\tilde{g}_a$ in U_a. Equation (3.17) automatically implies, again locally, in U_a, that

$$\tilde{\nabla}^a \tilde{J}_a = 0\,, \qquad \tilde{\nabla}^a \tilde{\omega}_a = 0\,, \qquad \tilde{\nabla}^a \tilde{\Omega}_a = 0\,. \tag{3.18}$$

Hence, we can think of $\left\{\tilde{g}, \tilde{\omega}_a, \tilde{J}_a, \tilde{\Omega}_a\right\}$, as a sort of preferred local Calabi-Yau structure in U_a, which however does not extend globally to M_8. We can withal obtain globally defined differential conditions in M_8 which, as we will see later, implies that the geometry of M_8 belongs to a particular class of locally conformally Kähler manifolds. Notice that $\tilde{J}$ is a well-defined almost-complex structure; nonetheless it is not covariantly constant since the Levi-Civita connection in (3.18) is only defined locally in U_a, as $\tilde{g}_a$ is only locally defined in U_a. In spite of this, we can prove the following:

Proposition 3.1. *M_8 is an Hermitian manifold with Hermitian structure (g, J).*

Proof. Let N denote the Nijenhuis tensor associated to J. Then, on every open set $U_a \subset M_8$ we can locally write N as follows

$$N|_{U_a}(u,v) = (\tilde{\nabla}^a_u J)(Jv) - (\tilde{\nabla}^a_v J)(Ju) + (\tilde{\nabla}^a_{Ju} J)(v) - (\tilde{\nabla}^a_{Jv} J)(u)\,, \quad u, v \in \mathfrak{X}(M_8)\,, \tag{3.19}$$

and thus $N|_{U_a} = 0$ since J is covariantly constant respect to the locally defined Levi-Civita connection $\tilde{\nabla}_a$. Since this can be performed in every open set of the covering $\{U_a\}_{a\in I}$ of M, we conclude that $N = 0$ and hence J is a complex structure. Since the metric g is compatible with J, (M_8, g, J) is an Hermitian manifold. □

Hence, we conclude that M_8 is a complex Hermitian manifold. There is another global condition that we can extract from (3.18) and which will further restrict the global geometry of M_8. Equation (3.18) implies that on every open set U_a we can find a function, namely Δ_a, such that

$$d(\Delta_a \omega)|_{U_a} = 0\,. \tag{3.20}$$

The key point now is that the de-Rahm differential does not depend on the locally-defined Levi-Civita connection $\tilde{\nabla}^a$ and therefore we can actually extend equation (3.20) to an equivalent, well-defined, global condition in M_8. Equation (3.20) is equivalent to

$$d\omega|_{U_a} + d\log\,\Delta_a \wedge \omega|_{U_a} = 0\,. \tag{3.21}$$

Given another open set U_b such that $U_a \cap U_b \neq \emptyset$ we have that $\log \Delta_a = \log \Delta_b + \log \lambda_{ab}$ at the intersection and therefore $d\log \Delta_a = d\log \Delta_b$. Hence, there is a well-defined closed one-form $\varphi \in \Omega^1(M_8)$ such that

$$d\omega = \varphi \wedge \omega\,, \tag{3.22}$$

which is defined, in every open set U_a, as

$$\varphi|_{U_a} = d\log \Delta_a\,. \tag{3.23}$$

Therefore ω is a locally conformal symplectic structure [48] on M_8 and thus we have proven the following result:

Theorem 3.2. *Let M_8 be a compact,* SU(4)*-structure locally conformally Kähler manifold with locally conformally Ricci-flat Kähler metric and locally conformally parallel* (4, 0)*-form. Then, M_8 is an admissible supersymmetric internal space for a supersymmetric compactification of eleven-dimensional Supergravity to three-dimensional Minkowski space-time preserving $\mathcal{N} = 2$ supersymmetry.*

The closed one-form $[\varphi] \in H^1(M_8)$, which is usually called the Lee-form, is precisely a flat connection in $L \to M_8$. Alternatively, one can define the φ-*twisted* differential $d_\varphi = d - \varphi$ whose corresponding cohomology $H^*_\varphi(M_8)$ is isomorphic to $H^*(M_8, \mathcal{F}_\varphi)$, the cohomology of M_8 with values in the sheaf of local d_φ-closed functions. Very good references to learn about locally conformally Kähler geometry are the book [49] and the review [50].

3.2 Solving the G-form flux

In order to fully satisfy supersymmetry, we have to impose on the four-form G the constraint

$$\iota_v G \cdot \eta = 0\,, \qquad v \in \mathfrak{X}(M_8)\,. \tag{3.24}$$

In the Calabi-Yau case, this constraint was solved by taking G to be $(2, 2)$ and primitive. In our case M_8 is not a Calabi-Yau manifold but it is a Hermitian manifold and hence it is equipped with a complex structure J and a compatible metric g. This turns out to be enough, as we will see now, to conclude that indeed the same conditions, namely G to be $(2, 2)$ and primitive, solve equation (3.24) in our case.

First of all, since we will use this fact later, notice that taking into account that η has positive chirality then equation (3.24) implies that G is self-dual in M_8. Using the Clifford algebra $Cl(8, \mathbb{R})$ relations together with the expresion of g as bilinear of η, it can be shown that [10]:

$$\Gamma_{\bar{a}}\eta = \Gamma^{a}\eta = 0\,, \tag{3.25}$$

where $\{\Gamma_a\}$, $a = 1, \dots 8$ are the gamma matrices generating $Cl(8, \mathbb{R})$ and the bar denotes an antiholomorphic index. Then, equation (3.24) is equivalent to

$$G_{m\bar{a}\bar{b}\bar{c}}\Gamma^{\bar{a}\bar{b}\bar{c}}\eta + 3G_{m\bar{a}\bar{b}\bar{c}}\Gamma^{\bar{a}\bar{b}c}\eta = 0\,. \tag{3.26}$$

The vanishing of the first term in equation (3.26) is equivalent to

$$G_{m\bar{a}\bar{b}\bar{c}}\eta = 0\,, \tag{3.27}$$

which implies

$$G^{4,0} = G^{3,1} = G^{1,3} = G^{0,4} = 0\,. \tag{3.28}$$

The vanishing of the second term in equation (3.26) is equivalent to

$$G_{a\bar{b}c\bar{d}}\,g^{c\bar{d}} = 0\,. \tag{3.29}$$

Taking now into account that G is self-dual, we can rewrite equation (3.29) as

$$G \wedge J = 0\,, \tag{3.30}$$

and hence we finally conclude that G is primitive and $G \in H^{(2,2)}(M_8)$.

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3.3 The tadpole-cancellation condition

In order to allow for a non-zero G-flux in M_8, we have to consider l_P-corrections to eleven-dimensional Supergravity, due to the well-known no-go theorem of reference [38]. We will perform the calculation in this section in order to illustrate that although $\{\xi_a\}_{a\in I}$ is not a well-defined one-form in M_8, due to the fact that G is an *honest* tensor in M, the calculation can be carried out, and since M_8 is topologically Spin(7), we obtain the same result as in the standard case. The relevant correction for our purposes is given by [44]

$$\delta S = -T_{M2} \int_M C_3 \wedge X_8 \,, \tag{3.31}$$

where $G_4 = dC_3$ and X_8 is an eight-form given by

$$X_8 = \frac{1}{(2\pi)^4} \left(\frac{1}{192} \mathrm{tr}\, R^4 - \frac{1}{768} \left(\mathrm{tr}\, R^2\right)^2 \right) . \tag{3.32}$$

The corrected equation of motion for the four-form G adapted to the compactification background and written on M_8 reads

$$\frac{3}{2} d * \varphi = -\frac{1}{2} G \wedge G + \beta X_8 \,, \tag{3.33}$$

where X_8 can be rewritten in terms of the first and second Pontryagin forms of the internal manifold [51]

$$X_8 = \frac{1}{192} \left(P_1^2 - 4P_2\right) \,, \tag{3.34}$$

and β is an appropriate constant that we will not need explicitly. Notice that φ is a one-form locally given by the derivative of the corresponding local warp factor but it cannot be written globally as the derivative of a function, yet it is a well defined closed one-form in M_8. Asuming that M_8 is closed, we integrate equation (3.33) to obtain

$$\frac{1}{2\beta} \int_{M_8} G \wedge G = \int_{M_8} X_8 \,. \tag{3.35}$$

Using now that M_8 has a SU(4)-structure and in particular it satisfies equation (2.15), we obtain

$$\frac{1}{2\beta} \int_{M_8} G \wedge G = \frac{\chi(M_8)}{24} \,, \tag{3.36}$$

a result that was to be expected since it only depends on M_8 being equipped with a Spin(7)-structure.

4 A class of non-geometric M-theory compactification backgrounds

In section 3 we have proposed a *twist* in the standard gluing of the local equations of motion of eleven-dimensional Supergravity on eight-manifolds, by means of the use of a particular atlas on the space-time manifold. In this section we are going to adopt a different point of view, proposing a slightly modified construction, which highlights the interpretation of such twisted supersymmetric compactification backgrounds as non-geometric compactification backgrounds. As a result, we will obtain that the total space of the non-geometric solution is still a manifold, although necessarily non-simply-connected, and that the Supergravity fields become tensors taking values on a particular line bundle.

Remark 4.1. *The idea is to consider the local analysis of reference [10] and patch it globally in a non-trivial way by using not only local diffeomorphisms but also the trombone symmetry of the warp factor. We will see that when performing this non-trivial patching the global space is still a manifold, but with very different geometric properties and topology from the standard solution of reference [10].*

The starting point is the standard one for compactification spaces. We will assume that the space-time manifold $\mathcal{M}$ can be written as a topologically trivial direct product

$$\mathcal{M} = \mathbb{R}_{1,2} \times \mathcal{M}_8 \,, \tag{4.1}$$

where $\mathbb{R}_{1,2}$ is three-dimensional Minkowski space-time and $\mathcal{M}_8$ is an eight-dimensional, Riemannian, compact, oriented, spinnable manifold. According to the product structure (4.1) of the space-time manifold $\mathcal{M}$, the tangent bundle splits as follows[5]

$$T\mathcal{M} = \mathbb{R}_{1,2} \oplus T\mathcal{M}_8 \,. \tag{4.2}$$

Let $\mathcal{U} = \{U_a\}_{a\in I}$ be a good open covering of M_8. Then:

$$\mathcal{V} = \mathbb{R}_{1,2} \times \mathcal{U} = \{V_a = \mathbb{R}_{1,2} \times U_a\}_{a\in I} \,, \tag{4.3}$$

is a good open covering of M. We define in M a family $\mathsf{g} = \{\mathsf{g}_a\}_{a\in I}$ of local the Lorentzian metrics, where g_a is a locally defined metric on $\mathbb{R}_{1,2} \times U_a$, given by:

$$\mathsf{g}_a = \Delta_a^2 \eta_{1,2} \times g_8|_{U_a} \,, \tag{4.4}$$

where g_8 is a Riemannian metric on $\mathcal{M}_8$ and $\Delta_a \in C^\infty(U_a)$. Similarly, we define in M a family $\mathsf{G} = \{\mathsf{G}_a\}_{a\in I}$ of local closed four-forms, where G_a is a locally defined closed four-form on $\mathbb{R}_{1,2} \times U_a$, given by:

$$\mathsf{G}_a = \mathrm{Vol}_{1,2} \wedge \xi_a + G|_{U_a} \,, \qquad \xi_a \in \Omega^1(U_a) \,, \quad G \in \Omega^4_{cl}(\mathcal{M}_8) \,, \tag{4.5}$$

where Vol is the standard volume form of Minkowski space. The idea now is to impose, for every $a \in I$, that each $(\mathsf{g}_a, \mathsf{G}_a)$ solves the local equations of motion of eleven-dimensional Supergravity. Then, we will patch this solutions globally by using not only local diffeomorphisms but also a particular global symmetry of the equations of motion. As we will see in a moment, the global geometry of M will depend on the specific patching used for the family of local solutions. More precisely, for each $a \in I$ of the good open cover $\mathcal{U} = \{U_a\}_{a\in I}$ of M_8, let us denote by:

$$Sol_a = (g_8|_{U_a}, G|_{U_a}, \Delta_a, \xi_a) \,, \qquad \Delta_a \in C^\infty(U_a), \qquad \xi_a \in \Omega^1_{cl}(U_a) \,, \tag{4.6}$$

[5] We omit the pull-backs of the canonical projections.

a local solution to the equations of motion of the theory, in the compactification background explained above. Notice that, in contrast to Δ_a and ξ_a, which are defined only locally, $g_8|_{U_a}$ and $G|_{U_a}$ are just the restriction of the globally defined tensors g_8 and G to U_a, so they are well-defined globally. Now, a standard compactification would construct a global solution to the equations of motion by patching globally the family of local solutions $\{Sol_a\}_{a\in I}$ using just local diffeomorphisms. This way we would obtain a globally well-defined metric g and four-form G on M. On the contrary, a non-geometric compactification is characterized by patching-up local solutions by using not only local diffeomorphisms but also symmetries of the equations of motion.

What we did in section 3 was to patch up the global solution using local diffeomorphisms and also a particular symmetry of the equations of motion: the *trombone symmetry* of the warp factor, consisting on rescalings of the warp factor by a constant. In section 3 we used a very particular atlas in order to obtain that the Supergravity fields are tensors. We will drop here that condition and we will adopt the natural point of view of a non-geometric compactification: the global Supergravity fields obtained by the non-trivial patching of the local solutions may not be tensors but objects of a more general type. In our case we will obtain that the Supergravity fields are tensors valued on a particular line bundle L.

Hence, the kind of compactification backgrounds described in section 3 can be interpreted as being non-geometric, although of a simple type, namely the symmetry used to patch-up the solution globally is a simple rescaling of the warp factor. Remarkably enough, the global space of the compactification is still a manifold, something that is not guaranteed for more general non-geometric compactifications. Let us do then the global patching explicitly. Given the good open cover $\mathcal{U}$, for each $U_a \in \mathcal{U}$ we have a locally defined warp factor $\Delta_a \in C^\infty(U_a)$. As we have said, two local warp factors Δ_a and Δ_b, $a, b \in I$ are related by a rescaling of the warp factor on the non-empty intersection $U_a \cap U_b \neq \{\emptyset\}$ of U_a and U_b. Then we have:

$$\Delta_a = \beta_{ab}\Delta_b\,, \qquad \beta_{ab} \in \mathbb{R}^*\,, \tag{4.7}$$

which as we have said is a symmetry of the equations of motion, as it is required to obtain a global solution. Equation (4.7) implies that:

$$\alpha_{ab} = \beta_{ba}^{-1}\,, \qquad \beta_{ab}\beta_{bc}\beta_{ca} = 1\,, \tag{4.8}$$

where the last equation holds on the triple non-empty triple intersection $U_a \cap U_b \cap U_c \neq \{\emptyset\}$. Hence:

$$L = (\{U_a\}\,, \beta_{ab}, \mathbb{R})\ , \tag{4.9}$$

defines a real line bundle L over M. The warp factor it is thus globally given by a section $\Delta \in \Gamma(L)$. Although Δ is not a globally defined function on M_8, it does define a globally defined closed one form $\varphi \in \Omega^1(M_8)$, given on every open set $U_a \in \mathcal{U}$ by:

$$\varphi|_{U_a} = d\log\,\Delta_a\,. \tag{4.10}$$

Hence, we have obtained what is the global structure of the warp factor: after trivializing L, it is it is given as closed one-form by a connection on L.

We have to patch-up now the local solutions $\{Sol_a\}_{a\in I}$ of the the theory. We are not interested in patching up the most general local compactification background, but only the $\mathcal{N}=2$ supersymmetric compactification backgrounds of reference [10]. Therefore, each solution Sol_a, $a\in I$, will be a local solution of the type presented in [10], namely conformal to a Calabi-Yau four-fold. Therefore, from reference [10] we obtain that:

$$Sol_a = (g_8|_{U_a}, G|_{U_a}, \Delta_a, \xi_a)\,, \qquad \Delta_a \in C^{\infty}(U_a), \qquad \xi_a \in \Omega^1_{cl}(U_a)\,, \tag{4.11}$$

is equipped with a local SU(4)-structure $(J_a, \omega_a, \Omega_a)$ satisfying:

$$\nabla_a J_a = 0\,, \qquad \nabla_a \omega_a = 0\,, \qquad \nabla_a \Omega_a = 0\,, \tag{4.12}$$

where J_a is a local complex structure, ω_a is a local symplectic structure, Ω_a is a local $(4,0)$-form and ∇_a is the locally-defined Levi-Civita connection associated to $g_a = \Delta_a g_8|_{U_a}$. In other words, $(J_a, \omega_a, \Omega_a)$ is a local integrable SU(4)-structure. In addition:

$$\xi_a = d(\Delta_a^3)\,, \quad d * d\log\Delta_a + G|_{U_a}\wedge G|_{U_a} = 0\,, \quad \omega_a \wedge G|_{U_a} = 0\,, \quad G\in\Omega^{2,2}(U_a)\,, \tag{4.13}$$

where $G|_{U_a}$ is $(2,2)$ with respect to J_a.

Remark 4.2. *As we explained in section (2), supersymmetric compactification backgrounds are time-like supersymmetric solutions, it is enough to satisfy the Maxwell equations for* G *in order to satisfy all the equations of motion.*

Using now that the global patching is performed by means of only local diffeomorphisms and the trombone symmetry, together with the results of reference [10], we obtain that, for each $a\in I$, the local SU(4)-structure (J_a,ω_a,Ω_a) can be written as:

$$g_{8\,a} = \Delta_a g_8|_{U_a}\,, \qquad J_a = J|_{U_a}\,, \qquad \omega_a = \Delta_a\omega|_{U_a}\,, \qquad \Omega_a = \Delta_a^2\Omega|_{U_a}\,. \tag{4.14}$$

where (g_8, J, ω, Ω) is a global SU(4)-structure on M_8, namely J is an almost-complex structure, ω is the fundamental two-form and Ω is the $(4,0)$. In order to fully characterize the non-geometric compactification background, we have to obtain the geometry of M_8 from the local supersymmetry conditions (4.12), (4.13) and (4.14).

Proposition 4.3. *Equations* (4.12)*,* (4.13) *and* (4.14) *are equivalent to* (M_8, J, g_8) *being an Hermitian manifold with integrable almost-complex structure* J *which is equipped with a* SU(4)*-structure* (J,ω,Ω) *such that:*

$$d\omega = \varphi\wedge\omega\,, \qquad \nabla_a\Omega_a = 0\,. \tag{4.15}$$

and in addition

$$\xi_a = d(\Delta_a^3)\,, \qquad d * d\varphi + G\wedge G = 0\,, \qquad \omega\wedge G = 0\,, \qquad G\in\Omega^{2,2}(M_8)\,, \tag{4.16}$$

Therefore, M_8 *is a locally conformally Kähler manifold with Lee form* φ *and and locally conformally parallel* $(4,0)$*-form* Ω*.*

Proof. From (4.14) we see that the local complex structures $\{J_a\}_{a\in I}$ patch up to a well-defined almost-complex structure J in M_8. Writing the Nijenhuis tensor of J as:

$$N|_{U_a}(u,v) = (\tilde{\nabla}^a_u J)(Jv) - (\tilde{\nabla}^a_v J)(Ju) + (\tilde{\nabla}^a_{Ju} J)(v) - (\tilde{\nabla}^a_{Jv} J)(u)\,, \quad u,v\in\mathfrak{X}(M_8)\,, \tag{4.17}$$

we obtain that $N|_{U_a} = 0$ for every $U_a \in \mathcal{U}$, and thus J is integrable and (M_8, g_8, J) is a Hermitian manifold. In addition, G is globally $(2,2)$ in M_8. Since ω_a is, for each $a \in I$, a rescaling of $\omega|_{U_a}$ we obtain that $\omega_a \wedge G|_{U_a} = 0$ and $G \in \Omega^{2,2}(U_a)$ are equivalent to:

$$\omega\wedge G = 0\,, \qquad G\in\Omega^{2,2}(M_8)\,. \tag{4.18}$$

Using now that $\varphi|_{U_a} = d\log\Delta_a$, we obtain that the global form of the equation of motion for the warp factor is:

$$d*\varphi + G\wedge G = 0\,. \tag{4.19}$$

Since J_a is a complex structure, we obtain that the condition $\nabla_a\omega_a$ is equivalent to $d\omega_a = 0$, which in turn is equivalent to:

$$d\omega = \varphi\wedge\omega\,. \tag{4.20}$$

□

Using now proposition 4.3, we have then proven the following theorem:

Theorem 4.4. *Let M_8 be an eight-dimensional compact manifold equipped with a* SU(4)*-structure (J,ω,Ω) such that J is integrable, ω is a locally conformally Kähler structure with Lee-form φ and Ω is locally conformally parallel. Then, (M_8, J, ω, Ω) is a non-geometric admissible M-theory compactification background to three-dimensional Minkowski space-time provided that there exists a closed four-form $G\in\Omega^4(M_8)$ such that:*

$$\omega\wedge G = 0\,, \qquad G\in\Omega^{2,2}(M_8)\,, \tag{4.21}$$

and a solution to the equation of motion:

$$d*\varphi + G\wedge G = 0\,. \tag{4.22}$$

of the warp factor exists.

The non-geometric background that we have obtained is very different from the standard Calabi-Yau compactification background, as a result of the non-trivial global patching. The topology of both manifolds is completely different. Hence, we should expect the effective theories of the compactifications to be completely different too. In the next section we will indeed provide an explicit example that solves the equations of motion for ξ and G, giving thus a counterexample to the Maldacena-Nuñez no-go theorem. The Sueprgravity fields are no longer global tensors, but tensors taking values on the line bundle L. In fact, we have:

$$\mathsf{g}\in\Gamma(S^2T^*M_8, L)\,, \qquad \xi\in\Omega^1(M_8, L)\,. \tag{4.23}$$

To summarize, we have found a simple class of non-geometric M-theory backgrounds in which the total space is again a manifold and which:

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- Need an underlying non-simply connected topological manifold.
- Evade the Maldacena-Nuñez no-go theorem.

These are properties that are expected to be present in non-geometric backgrounds. It is because of the second feature that we will be able to construct an explicit eight-dimensional non-geometric background which evades the Maldacena-Nuñez no-go theorem and thus evades any possible inconsistency coming from introducing l_P-corrections in the equations of motion but not in the classical Killing spinor equations.

5 Locally conformally Kähler manifolds

We have obtained that the supersymmetric conditions on an eleven-dimensional Supergravity compactification to three-dimensional Minkowski space-time, locally preserving $\mathcal{N} = 2$ Supersymmetry, allow for locally Ricci-flat, SU(4)-structure, locally-conformal Kähler manifolds as internal spaces. It is first convenient to introduce the following definition:

Definition 5.1. *A n-complex dimensional locally conformal Calabi-Yau manifold is* $\mathrm{SU}(n)$*-structure locally-conformal Kähler manifold with locally Ricci-flat Hermitian metric and locally conformally parallel* $(n, 0)$*-form.*

Hence, the kind of SU(4)-structure locally-conformal Kähler manifolds that we have obtained as admissible M-theory compactification backgrounds are precisely locally conformal Calabi-Yau manifolds, which motivates the definition. These are not necessarily Calabi-Yau four-folds (which would be a special subclass) and thus it is worth characterizing their geometry. First of all let us summarize the main properties of a generic compact locally conformal Calabi-Yau manifolds:

1. M is a compact Hermitian manifold. In other words, it is a complex manifold with a Riemannian metric g compatible with the complex structure J of the manifold.
2. M is equipped with non-degenerate two-form ω constructed from$J$ and g, which is not closed but satisfies
$$d\omega = \varphi \wedge \omega\,. \tag{5.1}$$
Then M is a particular case of almost-Kähler manifold.
3. Although ω is not closed, locally one can always transform it such that the locally transformed two-form is closed. Therefore M is a particular case of locally conformally symplectic manifold [48].
4. The Riemannian metric g is not Ricci-flat. Despite of this, locally one can find a Ricci-flat metric locally conformal to g.
5. There is a globally defined complex spinor which is not constant respect to the Levi-Civita connection associated to g. However, we can make a conformal transformation on the spinor such that it becomes locally constant respect to the Levi-Civita connection associated to the locally transformed metric.

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6. M is equipped with a SU(n)-structure, or in other words, it has zero first Chern class in $\mathbb{Z}$. However, the canonical bundle is not holomorphically trivial, as the $(n, 0)$-form that topologically trivializes it is not holomorphic, but only locally conformally parallel.

7. M is not projective, in contrast to the Calabi-Yau case. This seemingly technical detail is important, since for example, algebraic-geometry tools are very much used in order to study F-theory on elliptically-fibered Calabi-Yau four-folds.

There are in the literature several definitions of Calabi-Yau manifolds, not always equivalent. For definiteness, and in order to compare compact Calabi-Yau manifolds with compact locally conformal Calabi-Yau manifolds, we will use the following two equivalent definitions

- A compact Calabi-Yau manifold is a compact manifold of real dimension $2n$ with holonomy contained in SU(n).

- A compact Calabi-Yau manifold is a compact Kähler manifold with holomorphically trivial canonical bundle.

From the previous definitions we see that a locally conformal Calabi-Yau manifold fails to be Calabi-Yau by only two conditions, namely they are not Kähler and they do not have an holomorphic $(n, 0)$-form, although they are equipped with a $(n, 0)$-form topologically trivializing the canonical bundle. The deviation from Calabi-Yau can be measured by φ, namely, M is Calabi-Yau if and only if $[\varphi]$ is the zero class in de Rahm cohomology. Hence, we have obtained the following result:

Corollary 5.2. *A simply-connected locally conformal Calabi-Yau manifolds is a Calabi-Yau manifold.*

Contrary to what happens with compact locally irreducible Calabi-Yau manifolds, compact locally conformally Calabi-Yau manifolds can have continuous isometries. Let us consider the case of a generic locally conformally Kähler manifold M: it is equipped with two canonical vector fields v and u given by

$$g(w, u) = \varphi(Jw)\,, \qquad g(w, v) = \varphi(Jw)\,, \qquad \forall w \in \mathfrak{X}(M)\,. \tag{5.2}$$

Then, the following result holds [52]:

Proposition 5.3. *The canonical vector field u is a Killing vector field on M if and only if is an infinitesimal automorphism of J, and in this case on has $[u, v] = 0$.*

Therefore we see that if u is a Killing vector field, then u and v commute and thus they are the infinitesimal generators of a $\mathbb{R} \times \mathbb{R}$-action on M. This is a nice starting point to end-up having a torus action and therefore a principal torus bundle on M, as explained in proposition 6.4 of [18], where the necessary and sufficient conditions for u and v to define a principal torus bundle where obtained.

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Now that we know that locally conformal Calabi-Yau manifolds are not necessarily Calabi-Yau, an explicit example of a non-Calabi-Yau locally conformal Calabi-Yau manifold is in order. A general a locally conformally Kähler manifold M can be written has follows [52]:

$$M = \tilde{M}/G\,, \tag{5.3}$$

where $\tilde{M}$ is a simply connected Kähler manifold, and G is a covering transformation group whose elements are conformal for the respective Kähler metric on $\tilde{M}$. This restricts the class of manifolds we can consider, but it is not enough to specify a manageable class. Fortunately, it turns out that there is a class of locally conformally Kähler manifolds that has been completely characterized, namely those whose local Kähler metric is flat, thanks to the following proposition [52]:

Theorem 5.4. *Let M be a compact locally conformally Kähler-flat manifold of complex dimension n. Then the universal covering space of M is $\mathbb{C}^n\backslash\{0\}$, and up to a global conformal change of the metric, M is a generalized Hopf manifold with the canonical metric. Every such manifold M has the same Betti numbers as the Hopf manifold H^n of the same complex dimension n.*

A generalized Hopf manifold is a locally conformally Kähler manifold such that its Lee-form is a parallel form. Among the generalized Hopf manifolds are of course the classical Hopf manifolds. Four-dimensional complex Hopf manifolds are equipped with a SU(4)-structure and in fact it is an example of a non-trivial compact locally conformal Calabi-Yau manifold. In particular the metric of a Hopf manifold is not only locally Ricci-flat but locally flat.

Let us explore then the geometry of compact complex Hopf manifolds, since they provide us with a non-trivial example of locally conformal Calabi-Yau manifolds.

5.1 An explicit solution on a complex Hopf manifold

A complex Hopf manifold $\mathbb{C}H^m_\alpha$ of complex dimension m is the quotient of $\mathbb{C}^m\backslash\{0\}$ by the free action of the infinite cyclic group $\mathfrak{S}_\alpha$ generated by $z \to \alpha z$, where $\alpha \in \mathbb{C}^*$ and $0 < |\alpha| < 1$. In other words, it is $\mathbb{C}^m\backslash\{0\}$ quotiented by the free action of $\mathbb{Z}$, where $\mathbb{Z}$, with generator α acting by holomorphic contractions. The group $\mathfrak{S}_\alpha$ acts freely on $\mathbb{C}^m\backslash\{0\}$ as a properly discontinuous group of complex analytic transformations of $\mathbb{C}^m\backslash\{0\}$. Hence, the quotient space:

$$\mathbb{C}H^m_\alpha = \left(\mathbb{C}^m\backslash\{0\}\right)/\mathfrak{S}_\alpha\,, \tag{5.4}$$

is a complex m-fold. It can be shown that complex m-dimensional Hopf manifolds $\mathbb{C}H^m_\alpha$ are diffeomorphic to $S^1 \times S^{2m-1}$. As a result:

$$b^1\left(\mathbb{C}H^m_\alpha\right) = b^{2m-1}\left(\mathbb{C}H^m_\alpha\right) = 1\,, \tag{5.5}$$

namely the first betti number is odd and hence $\mathbb{C}H^m_\lambda$ does not admit a Kähler metric. Notice the standard Kähler structure on $\mathbb{C}^m\backslash\{0\}$ does not descend to $\mathbb{C}H^m_\alpha$ since it is not

$\mathfrak{S}_\alpha$. It admits however locally conformally Kähler structure. To prove this, let us take $\mathbb{C}^m \setminus \{0\}$ equipped with the following metric and (1,1)-form:

$$g_0 = \frac{dz^t \otimes d\bar{z}}{\bar{z}^t z}\,, \qquad \omega_0 = i\frac{dz^t \wedge d\bar{z}}{\bar{z}^t z}\,. \tag{5.6}$$

The (1,1)-form ω_0 is not closed but it satisfies:

$$d\omega_0 = \varphi_0 \wedge \omega_0\,, \tag{5.7}$$

where:

$$\varphi_0 = \frac{z^t d\bar{z} + \bar{z}^t dz}{\bar{z}^t z}\,, \tag{5.8}$$

Since g_0, ω_0 and φ_0 are invariant under $\mathfrak{S}_\alpha$ transformations, they descend to a well defined metric g and (1,1)-form ω in $\mathbb{C}H^m_\alpha$, with corresponding Lee-form φ. In $\mathbb{C}^m \setminus \{0\}$ we have that φ_0 is exact, since $\varphi_0 = d \log z^t \bar{z}$. This should be expected, as (g_0, ω_0) is globally conformal to the standard Kähler structure on $\mathbb{C}^m \setminus \{0\}$. However, φ is not exact in $\mathbb{C}H^m_\alpha$, since there $\log z^t \bar{z}$ is not well-defined there. Let (U_a, z_a) be a coordinate chart in $\mathbb{C}H^m_\alpha$.

The non-geometric solution. Let us take now $m = 4$, and $\alpha = \bar{\alpha}$. Then $\mathbb{C}H^4_\alpha$ is an eight-dimensional manifold of the type just described. In particular, it is equipped with a locally conformally Kähler structure (g, ω) induced by the quotient of the (g_0, ω_0) given in equation (5.6). When α is real we can define in addition another globally defined (4,0)-form, induced by the following form on $\mathbb{C}^m \setminus \{0\}$:

$$\Omega_0 = \frac{dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4}{|z|^4}\,. \tag{5.9}$$

Now, since α is real, Ω_0 is $\mathfrak{S}_\alpha$ invariant and therefore it induces a globally defined (4,0)-form Ω on $\mathbb{C}H^4_\alpha$ satisfying:

$$\nabla_a \Omega_a = 0\,, \tag{5.10}$$

and in particular:

$$d\Omega = 2\varphi \wedge \Omega\,. \tag{5.11}$$

Therefore (g, ω, Ω) is precisely a locally conformally Calabi-Yau structure on $\mathbb{C}H^4_\alpha$, which is what was required by supersymmetry, see theorem 4.4. Therefore, in order to obtain a full non-geometric solution, we just have to solve the equation of motion for $\mathsf{G} = \{\mathsf{G}_a\}_{a\in I}$. Notice that, as we explained in section 4, local supersymmetry imposes:

$$\xi_a = d(\Delta^3_a)\,, \tag{5.12}$$

and that the only equation of motion that remains to be solved is the equation of motion for the warp factor, namely

$$d * \varphi + G \wedge G = 0\,. \tag{5.13}$$

In order to solve it, we are going to take $G = 0$. Notice that this does not trivialize the flux G since there is still a part with *one leg* on M_8. Taking $G = 0$ we obtain that the equation of motion for the warp factor reduces to:

$$d * \varphi = 0\,. \tag{5.14}$$

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Since φ is already closed, this means that φ must be harmonic, in order to solve equation (5.14). It turns out that φ is indeed harmonic; which, since it is already closed, is the same as requiring g_8 to be the Gauduchon metric. Therefore:

$$Sol = \left(\mathbb{C}H^4_\alpha, g_8, \omega, \Omega, \varphi\right) , \tag{5.15}$$

is a compact *non-geometric* solution of eleven-dimensional Supergravity with non-trivial flux and warp-factor. From a different point of view, one can see that Sol is locally conformal to flat space equipped with the standard Calabi-Yau structure and therefore it trivially solves the supersymmetry equations. Globally however the geometry is very different and that in turn allows for the existence of a non-trivial flux and warp factor. We could say then that the non-trivial warp-factor and flux *are supported* by the *non-geometry* of the solution.

Remark 5.5. *In the standard compactification scenario, where instead of φ we have the derivative of the warp factor, say df, where now f is a globally defined function on M_8, the equation of motion of the warp factor becomes, after setting G equals to zero:*

$$\Delta f = 0 . \tag{5.16}$$

Since M_8 is closed then f must be constant. In our non-geometric case however, we get a harmonic one-form φ, so as long as the first betti number of M_8 is bigger or equal than one, we are guaranteed to have at least one non-trivial solution.

For completeness, let us write locally the warp factor and flux in local coordinates: let (U_a, z_a) be a local chart of $\mathbb{C}H^4_\alpha$. Then we have that:

$$\varphi|_{U_a} = d\left(\log z^t_a \bar{z}_a + c'_a\right) , \qquad c'_a \in \mathbb{R} , \tag{5.17}$$

and thus the warp factor of eleven-dimensional Supergravity compactified on CH^4_λ is, at every coordinate chart (U_a, z_a), given by:

$$\Delta_a = c_a z^t_a \bar{z}_a , \qquad c_a \in \mathbb{R}^* . \tag{5.18}$$

Therefore, locally the four-form flux is given by:

$$\mathsf{G}|_{U_a} = \mathrm{Vol}_{1,2} \wedge d(c_a z^t_a \bar{z}_a)^3 . \tag{5.19}$$

Remark 5.6. *In section 3, a very particular atlas was used in order to make $\{\mathsf{G}_a\}_{a\in I}$ a globally defined tensor on M. However, from the point of view of a non-geometric compactification, we do not need to perform such an artificial construction. For non-geometric compactifications the global objects that locally correspond to the fields of the theory are not expected to be standard tensors. In this case G can be understood as a four-form taking values on a real line bundle L:*

$$\mathsf{G} = \mathrm{Vol} \wedge \xi , \qquad \xi \in \Omega^1(M_8; L^3) . \tag{5.20}$$

The real line bundle L twists G from being a standard four-form and this is the result of the non-trivial global patching of the solution.

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The solution $Sol = \left(\mathbb{C}H^4_\alpha, g_8, \omega, \Omega, \varphi\right)$ that we have obtained, although the simplest of its kind, has very interesting properties, some of them shared also by more general locally conformally Kähler manifolds. In particular, it is equipped with a holomorphic torus fibration and a transversely orientable, codimension one, real foliation with a G_2-structure on the leaves. Therefore, Sol has the geometric properties found in [16, 17] for the most general $\mathcal{N} = 1$ supersymmetric compactification of eleven-dimensional Supergravity to three-dimensional Minkowski space-time. This will be the subject of the next section.

5.2 Foliations and principal torus fibrations on Vaisman manifolds

Let (M, ω) be a Vaisman manifold, namely (M, ω) is a locally conformally Kähler manifold with a parallel Lee-form θ. Since θ is parallel, if it is non-zero at one point, it is non-zero at every point. Notice that the Hopf manifold that we found in section 5.1 to satisfy the local equations of motion of eleven-dimensional Supergravity is a particular example of Hopf manifold. A Vaisman manifold (M, ω) is equipped with four canonical foliations, which are defined on (M, ω) by means of the Lee-form θ and the complex structure J of M as follows [49, 53]:

- (M, ω) is equipped with a completely integrable and regular codimension-one distribution $F \subset TM$, given by $\theta = 0$. We will denote by $\mathcal{F}$ the corresponding foliation, which is totally geodesic.

- (M, ω) is equipped with a completely integrable and regular dimension one distribution $D \subset TM$ given by the vector field $v = \theta^\sharp$. We will denote by $\mathcal{D}$ the corresponding foliation, which is a geodesic foliation.

- (M, ω) is equipped with a completely integrable and regular dimension one distribution $D^\perp \subset TM$ given by the vector field $w = J \cdot v$. We will denote by $\mathcal{D}^\perp$ the corresponding foliation. Notice that the distribution $D^\perp$ is perpendicular to D, and hence the symbol used.

- (M, ω) is equipped with a completely integrable and regular dimension two distribution $T = D^\perp \oplus D \subset TM$. We will denote by $\mathcal{T}$ the corresponding foliation. The foliation $\mathcal{T}$ is a complex analytic foliation whose leaves are parallelizable complex analytic manifolds of complex dimension one. The leaves are totally geodesic, locally Euclidean submanifolds of M and the foliation is Riemannian.

If the foliation $\mathcal{T}$ is regular, as it happens for the solution $Sol = \left(\mathbb{C}H^4_\alpha, g_8, \omega, \Omega, \varphi\right)$ that we found in section 5.1, then the following result holds [49, 53].

Theorem 5.7. *If the foliation $\mathcal{T}$ on a compact Vaisman manifold (M, ω) is regular then:*

- *The leaves are totally geodesic flat torii.*

- *The leaf space $\mathcal{M} = M/\mathcal{F}$ is a compact Kähler manifold.*

- *The projection π is a locally trivial fibre bundle.*

Therefore, compact Vaisman manifolds with regular foliation $\mathcal{T}$ are equipped with a non-trivial torus principal bundle over a Kähler manifold, in the line of the suggestion made in [18]. This is interesting, because for F-theory applications one needs the eight-dimensional compactification manifold to admit a elliptic fibration, which must be singular to be non-trivial since the compactification space is an irreducible Calabi-Yau four-fold, over a Kähler base. In our case the fibration can be non-trivial yet non-singular, and that is indeed the case of the solution of section 5.1. The interpretation, if any, of such non-singular and non-trivial fibrations in the context of F-theory remains unclear. This of course does not mean that there are no locally conformally Kähler four-folds admitting singular elliptic fibrations; this is currently an open problem.

On the other hand, a compact Vaisman manifold admits a topological Spin(7)-structure, and in particular it is spin, as a consequence of having all the Chern numbers equal to zero. This Spin(7)-structure induces a G_2-structure on the leafs of the canonical foliation $\mathcal{F}$. If we restrict to the class of Hopf manifolds inside the class of Vaisman manifolds, then we have a very explicit result about the G_2-structure present in the leaves. Notice that the solution of section 5.1 is a Hopf manifold, so the following result applies [49, 53].

Proposition 5.8. *Let (M,ω) be a compact $2m$-dimensional Hopf manifold. Then, $\mathcal{F}$ is a totally geodesic foliation of $(2m-1)$-dimensional spheres defined through the diffeomorphism $M \simeq S^1 \times S^{2m-1}$.*

Let us apply proposition 5.8 to the $m = 8$ case. Then the foliation $\mathcal{F}$ is by seven-dimensional spheres S^7. But a seven-dimensional sphere S^7 is equipped with a nearly parallel G_2-structure $\phi \in$
$Omega^3(S^7)$, which satisfies:

$$d\phi = \tau_0 * \phi\,, \qquad d * \phi = 0\,, \qquad d\tau_0 = 0\,. \tag{5.21}$$

Let denote by $\tau_0 \in \Omega^0(S^7), \tau_1 \in \Omega^1(S^7), \tau_2 \in \Omega^2_{14}(S^7)$ and $\tau_2 \in \Omega^3_{27}(S^7)$ the torsion classes of the G_2 structure ϕ. Then, the G_2-structure ϕ satisfies $\tau_2 = 0$ and it is therefore a particular case of the general characterization found in references [16, 17, 36, 37] for the most general eleven-dimensional Supergravity supersymmetric compactification background to three-dimensions. It is rewarding to see that although we are considering non-geometric compactification backgrounds, the foliation structure of the most general geometric supersymmetric compactification background is preserved, which also indirectly shows that compactifying in this class of non-geometric compactification background should be possible in principle.

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