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Classical Virasoro irregular conformal block II

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Abstract: We present a new systematic way to evaluate the classical limit of the Virasoro irregular conformal block for arbitrary rank n based on the irregular partition function. In addition, we prove that the classical irregular conformal block has the exponential form as suggested by A. Zamolodchikov and Al. Zamolodchikov for the regular case. We provide an explicit calculation for the rank 2 case in detail.

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Contents

1 Introduction

In our recent work [1] which will be called I in the following, we studied the classical limit (NS limit) [2] of Virasoro irregular conformal block (ICB) using the irregular matrix model (IMM) [3]. IMM is a β-deformed one matrix model with a logarithmic as well as a finite number of inverse power potentials. The finite number n of the inverse powers is called the rank of irregular model. It is demonstrated that the classical ICB of rank n can be obtained by using the generalized Mathieu equation. This equation is equivalent to the loop equation of IMM and is solved on a unit circle with the Floquet exponent for the rank 1. However, this method is not easy to generalize to the case with arbitrary rank n. In this paper, we present a new systematic way to find ICB based on the loop equation.

In section 2 we present that the classical irregular conformal block $\mathcal{F}_\Delta^{(m:n)}$, the inner product between irregular modules of rank m and n has the exponential form:

$$\mathcal{F}_\Delta^{(m:n)} \overset{g_s \to 0}{\sim} \exp\left\{\frac{1}{g_s^2} f_\delta\right\}, \tag{1.1}$$

where $g_s \to 0$ corresponds to the classical limit and f_δ is finite in this limit. This is suggested in [4, 5] for the regular conformal block and extended to the ICB in [6]. We demonstrate this exponential behavior for the regular and irregular case using the conformal property of the loop equation. In section 3 we present a new systematic way to analyze the classical limit of IMM and give a non-trivial example for the rank 2 partition function $Z_{(0:2)}$. Further in section 4 we evaluate the partition function $Z_{(m:n)}$ based on the same method, which is sufficient to construct ICB. Section 5 is the conclusion and the appendix deals with the normalization of the partition function and technical details.

2 Classical form of conformal block

2.1 Setup of the formalism

Regular matrix model [7, 8] related with the regular conformal block is defined as the β-deformed Penner-type matrix model

$$Z_\beta = \int \left(\prod_{I=1}^{N} d\lambda_I\right) \prod_{I<J} (\lambda_I - \lambda_J)^{2\beta} e^{\frac{\sqrt{\beta}}{g}\sum_I V(\lambda_I)}. \tag{2.1}$$

$V(\lambda)$ is the Penner-type potential

$$V(z) = \sum_{a=0}^{K} \hat{\alpha}_a \log(z - z_a)\,. \tag{2.2}$$

This potential is obtained from the correlation of $K+2$ primary vertex operators (lying at $0, z_1, \cdots, z_K, \infty$) and screening operators (lying at z). β is related with the Virasoro screening charge $b = i\sqrt{\beta}$ and the Virasoro charge α of the primary operator is rescaled as $\hat{\alpha} = g_s \alpha$. We introduce the small expansion parameter g which is related with $g = i g_s/2$ so that $\sqrt{\beta}/g = -2b/g_s$.

Classical limit is obtained as $g_s \to 0$ so that $\hat{\alpha}_a$ is finite. On the other hand, b and g_s are related with the Ω deformation parameter $\epsilon_1 = g_s b$ and $\epsilon_2 = g_s/b$ of the Nekrasov partition function [9–11] according to AGT conjecture [12]. Therefore, classical limit is achieved either $\epsilon_2 \to 0$ but ϵ_1 finite which is the Nekrasov-Shatashvili (NS) limit or its dual ($\epsilon_1 \to 0$ but ϵ_2 finite). The two pictures are equivalent since Liouville theory has $b \to 1/b$ duality.

IMM is obtained in [13, 14] from the colliding limit [15, 16]. IMM has the same form of (2.1) but the potential is different,

$$V(z) = \hat{c}_0 \log z - \sum_{k=1}^{n} \left(\frac{\hat{c}_k}{k z^k}\right) + \sum_{\ell=1}^{m} \left(\frac{\hat{c}_{-\ell}\, z^\ell}{\ell}\right), \tag{2.3}$$

where $m + n = K$ and $(m+1)$ primary operators are put to ∞, and $(n+1)$ operators to 0. The coefficients $\hat{c}_k$ and $\hat{c}_{-\ell}$ are given in terms of the moments; $\hat{c}_k = \sum_{r=1}^{n} \hat{\alpha}_r (z_r)^k$ with $k \geq 0$ and $\hat{c}_{-\ell} = \sum_{a=1}^{m} \hat{\alpha}_a (z_a)^{-\ell}$ with $\ell > 0$. The irregular partition function with the potential (2.3) will be denoted as $Z_{(m;n)}(\hat{c}_0; \{\hat{c}_k\}, \{\hat{c}_{-\ell}\})$ in the following.

The matrix model (regular or irregular) has the loop equation which presents the symmetric property

$$4W(z)^2 + 4V'(z)W(z) + 2g_s Q W'(z) - g_s^2 W(z,z) = f(z)\,, \tag{2.4}$$

where $Q = b + 1/b$ is the background charge. $W(z)$ and $W(z,z)$ are the one and two point resolvents, defined as $W(z) = g\sqrt{\beta}\langle \sum_I \frac{1}{z-\lambda_I} \rangle_{\text{conn}}$ and $W(z,w) = \beta\langle \sum_I \frac{1}{(z-\lambda_I)(w-\lambda_I)} \rangle_{\text{conn}}$, respectively. The bracket $\langle O \cdots \rangle_{\text{conn}}$ denotes the connected part of the expectation value with respect to the matrix model (2.1). $f(z)$ is the expectation value determined by the potential $V(z)$, $f(z) = 4g\sqrt{\beta}\langle \sum_I \frac{V'(z)-V'(\lambda_I)}{z-\lambda_I} \rangle_{\text{conn}}$.

At the classical limit, the resolvents defined above remain finite [17]. Therefore, the loop equation (2.4) is simplified as

$$x(z)^2 + \epsilon x'(z) + U(z) = 0\,, \tag{2.5}$$

where $x(z) = 2W(z) + V'(z)$ and $U(z) = -\big(V'(z)\big)^2 - \epsilon V''(z) - f(z)$. $\epsilon = g_s Q$ is a finite parameter at the classical/NS limit. It should be noted that (2.5) is manifestly invariant for both limits (NS and its dual). This loop equation turns into a second order differential equation (Shrödinger-like equation) if one defines $\Psi(z) = \exp\big(\frac{1}{\epsilon}\int^z x(z')dz'\big)$:

$$\left(\epsilon^2 \frac{\partial^2}{\partial z^2} + U(z)\right)\Psi(z) = 0\,. \tag{2.6}$$

On the other hand, one may conveniently investigate the conformal block using a degenerate primary operator. In [1, 18], an expectation value $P(z) \equiv \langle \prod_I (z-\lambda_I)\rangle$ is introduced in relation with the degenerate operator. It should be noted that $P(z)$ is a polynomial of degree N,

$$P(z) = P_0 + P_1 z + P_2 z^2 + \cdots + P_{N-1} z^{N-1} + P_N z^N\,, \tag{2.7}$$

where N is the number of integration variables in (2.1) and P_N is normalized to be 1.

The wave function $\Psi(z)$ in (2.6) is closely related with $P(z)$. This can be seen if one notes that at the classical limit, one has [1]

$$\log\left(\frac{P(z)}{P(z_0)}\right) = \frac{2}{\epsilon}\int_{z_0}^{z} dz'\, W(z')\,. \tag{2.8}$$

Taking derivatives one has $W(z) = \frac{\epsilon}{2}\big(\log P(z)\big)' = \frac{\epsilon}{2}\frac{P'(z)}{P(z)}$ and therefore, the wave-function $\Psi(z)$ is given as

$$\Psi(z) = P(z)\exp\left(\frac{1}{\epsilon}\int^z V'(z')dz'\right). \tag{2.9}$$

This shows that the polynomial function satisfies the second order differential equation

$$\epsilon^2 P''(z) + 2\epsilon V'(z)P'(z) = f(z)P(z)\,, \tag{2.10}$$

which can be check from (2.5) or equivalently from (2.6). We will use this equation to investigate the partition function and conformal block.

2.2 Classical irregular conformal block

Let us investigate the exponential behavior of the classical conformal block. Note that (1.1) is equivalent to that

$$\lim_{g_s\to 0} g_s^2 \log \mathcal{F}_\Delta^{(m:n)} \to \text{finite}\,. \tag{2.11}$$

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In this section we concentrate on the case of ICB. Regular conformal block is commented in section 2.3.

The explicit form of ICB is given in tems of IMM $Z_{(m;n)}$ [3]:

$$\mathcal{F}_{\Delta}^{(m:n)}\big(\{\hat{c}_{-\ell}:\hat{c}_k\}\big)=\frac{e^{\zeta_{(m:n)}}Z_{(m:n)}\big(\hat{c}_0;\{\hat{c}_k\},\{\hat{c}_{-\ell}\}\big)}{Z_{(0:n)}\big(\hat{c}_0;\{\hat{c}_k\}\big)Z_{(0:m)}\big(\hat{c}_\infty;\{\hat{c}_{-\ell}\}\big)}\,, \tag{2.12}$$

where c_0 is fixed by the neutrality condition $c_0+c_\infty+Nb=Q$ with N the number of inserted screening operators. ICB has an extra factor $e^{\zeta_{(m:n)}}$ which comes from the limiting procedure $z_a\to\infty$ and $z_b\to 0$. Explicitly $\zeta_{(m:n)}=\hat{\zeta}_{(m:n)}/g_s^2$ where $\hat{\zeta}_{(m:n)}=\sum_k^{\min(m,n)}2\hat{c}_k\hat{c}_{-k}/k$.

One can confirm the exponential behavior (1.1) using the expression of ICB. We need to confirm the classical behavior

$$\lim_{g_s\to 0}g_s^2\big\{\zeta_{(m:n)}+\log Z_{(m:n)}-\log Z_{(0:n)}-\log Z_{(0:m)}\big\}\to\text{finite}\,. \tag{2.13}$$

It is easy to show that the first term is finite since it is given as

$$\lim_{g_s\to 0}g_s^2\,\zeta_{(m:n)}=\hat{\zeta}_{(m:n)}\,. \tag{2.14}$$

The contribution of $Z_{(0:n)}$ can be evaluated using $f(z)$. Note that $f(z)$ has a finite number of inverse powers of z: $f(z)=\sum_{k=0}^{n-1}d_k z^{-(k+2)}$. Therefore, if one expands eq. (2.10) in powers of z, one finds the equation has the terms running from z^{N-2} to z^{-n-1}. This provides $N+n$ number of equations. Since there are $N+n$ unknown variables: P_0, P_1, $\ldots$, P_{N-1} and d_0, d_1, $\ldots$, d_{n-1}, one can solve the equations to find d_k as a function of $\hat{c}_k$'s, which are finite at the classical limit $g_s\to 0$. Once the solution of d_k is found, one can find the partition function $Z_{(0:n)}$ using the differential equation [13]

$$-\,g_s^2\,v_k(\log Z_{(0:n)})=d_k\quad\text{for }0\le k\le n-1\,, \tag{2.15}$$

where v_k is the differential operator related with the Virasoro generator representation:

$$v_{k\ge 0}=\sum_{\ell>0}\ell\,\hat{c}_{\ell+k}\frac{\partial}{\partial\hat{c}_\ell}\,. \tag{2.16}$$

Here we use the convention $\hat{c}_\ell=0$ when c_ℓ does not belong to $\{\hat{c}_0,\cdots,\hat{c}_n\}$. Once d_k is known, one may rearrange (2.15) to put

$$-\,g_s^2\frac{\partial}{\partial\hat{c}_\ell}\log Z_{(0:n)}=F_\ell\big(\{\hat{c}_k\}\big)\quad\text{for }1\le\ell\le n\,. \tag{2.17}$$

Since $Z_{(0:n)}$ only depends on $\hat{c}_0$, $\hat{c}_1$, $\ldots$, $\hat{c}_n$, (2.17) is sufficient to determine $Z_{(0:n)}$ completely, up to the normalization factor $N_{(0:n)}$ which is independent of $\hat{c}_{\ell>0}$:

$$-\,g_s^2\log\left(\frac{Z_{(0:n)}}{N_{(0:n)}}\right)=H_{(0:n)}\big(\hat{c}_0,\{\hat{c}_k\}\big)\,, \tag{2.18}$$

with finite $H_{(0:n)}$ at the classical/NS limit.

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In a similar way, one has $f(z) = \sum_{k=-m}^{n-1} d_k z^{-(k+2)}$ for $Z_{(m:n)}$. By identifying each coefficient of z^ℓ in (2.10), there are $N+m+n$ equations, running from z^{N+m-2} to z^{-n-1}. The number of unknown variables are also $N+m+n$: P_0, P_1, …, P_{N-1} and d_{-m}, …, d_{-1}, d_0, d_1, …, d_{n-1}. Thus solutions of d_k exist as functions of $\hat{c}_k$. Furthermore this coefficients allows to find $Z_{(m:n)}$ through the differential equation [14]

$$\begin{aligned} -g_s^2\, v_k(\log Z_{(m:n)}) &= d_k && \text{for } 0 \le k \le n-1\,, \\ -g_s^2\, u_k(\log Z_{(m:n)}) &= d_{-k} - 2\epsilon N \hat{c}_{-k} && \text{for } 1 \le k < m-1\,, \end{aligned} \tag{2.19}$$

where u_k is the differential operator corresponding to $\hat{c}_{-\ell}$

$$u_{k>0} = \sum_{\ell>0} \ell\, \hat{c}_{-\ell-k} \frac{\partial}{\partial \hat{c}_{-\ell}}\,. \tag{2.20}$$

The solution is found similar to (2.18),

$$-\,g_s^2 \log\left(\frac{Z_{(m:n)}}{N_{(m:n)}}\right) = H_{(m:n)}\big(\hat{c}_0, \{\hat{c}_k\}, \{\hat{c}_{-\ell}\}\big)\,, \tag{2.21}$$

with finite $H_{(m:n)}$ at the classical/NS limit.

In addition, the conformal block (2.12) is defined as 1 if $\hat{c}_k = \hat{c}_{-\ell} = 0$ for $k, \ell > 0$. Therefore, the conformal block is independent of the normalization. (In appendix A we present how the normalization behaves at the classical/NS limit). Collecting all the terms, one has the classical ICB in the form of

$$\lim_{g_s\to 0} g_s^2 \log \mathcal{F}_\Delta^{(m:n)} = \hat{\zeta}_{(m:n)} - H_{(m:n)} + H_{(0:n)} + H_{(0:m)} \tag{2.22}$$

which is finite and thus, (1.1) is proved. In the following sections, we present explicit form of $H_{(0:n)}$ and $H_{(m:n)}$ which is indeed finite.

2.3 Classical regular conformal block

We may demonstrate that (1.1) holds for the classical regular conformal block too. For the regular case, we still have (2.10), but with $V'(z) = \sum_{a=0}^{K} \frac{\hat{\alpha}_a}{z-z_a}$ and

$$f(z) = \sum_{a=0}^{K} \frac{d_a}{z-z_a}\,, \qquad d_a = -g_s^2 \frac{\partial \log Z_\beta}{\partial z_a}\,. \tag{2.23}$$

We start with the equation of $P(z)$ in (2.10). If one takes the residue of (2.10) around each z_a, one obtains $K+1$ equations for $a = 0, 1, \ldots, K$:

$$2\epsilon \hat{\alpha}_a P'(z_a) = d_a P(z_a)\,. \tag{2.24}$$

Equivalently,

$$2\epsilon \hat{\alpha}_a \frac{\partial \log P(z_a)}{\partial z_a} = -g_s^2 \frac{\partial \log Z_\beta}{\partial z_a}\,. \tag{2.25}$$

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Thus, one has

$$Z_\beta = N_\beta \prod_{a=0}^{n} P(z_a)^{-2\epsilon\hat{\alpha}_a/g_s^2}, \tag{2.26}$$

where N_β is the normalization factor independent on z_a and is discussed in the appendix. One may also normalize the conformal block so that the z_a-independent factor as 1.

$P(z)$ is given as the solution of (2.10). If the solution exists, the solution should be finite. There is no singularity forbidding $z \to z_a$ since $P(z)$ is the polynomial with degree N. Thus, one may conclude for the regular conformal block $Z_\beta \overset{g_s\to 0}{\sim} \exp\{\zeta/g_s^2\}$ where

$$\zeta = -2\epsilon \sum_{a=0}^{n} \hat{\alpha}_a \ln P(z_a)\,. \tag{2.27}$$

3 Explicit evaluation of the partition function $Z_{(0:n)}$

One may find the explicit form of the partition function using (2.10). In this section we present how to obtain the partition function in a systematic way.

To obtain $Z_{(0:n)}$, we compare each coefficient of order z^l in (2.10). For the power z^{N-2-k} with $0 \le k \le N+n-1$, one has

$$P_N d_k + P_{N-1}d_{k-1} + \cdots + P_{N-k}d_0 = \epsilon^2\,(N-k)(N-k-1)P_{N-k} + 2\epsilon \sum_{l=0}^{k} \big(\hat{c}_{k-l}(N-l)P_{N-l}\big)\,. \tag{3.1}$$

Here we use the notation that P_a vanishes when $a<0$ or $a>N$. The highest power z^{N-2} shows that

$$d_0 = \epsilon^2\,N(N-1) + 2\epsilon\hat{c}_0 N\,, \tag{3.2}$$

which is independent of $\hat{c}_{k>0}$. Finding $d_{k>0}$ needs algebraic manipulation.

We present the case rank 2 ($n=2$) explicitly, which has d_1 only. First note that the partition function is given in terms of differential equation (2.15),

$$-g_s^2\left(\hat{c}_1\frac{\partial}{\partial\hat{c}_1} + 2\hat{c}_2\frac{\partial}{\partial\hat{c}_2}\right)\log Z_{(0:2)} = d_0\,, \tag{3.3}$$

$$-g_s^2\hat{c}_2\frac{\partial}{\partial\hat{c}_1}\log Z_{(0:2)} = d_1\,. \tag{3.4}$$

Eq. (3.3) is solved to get

$$g_s^2 \log Z_{(0:2)} = -\frac{d_0}{2}\log\hat{c}_2 + h(\tau)\,, \tag{3.5}$$

where $h(\tau)$ is any function of $\tau \equiv \hat{c}_2/\hat{c}_1^2$ which satisfies automatically $v_0\big(h(\tau)\big) = 0$. It is noted that the right hand side of eq. (3.5) is equivalent to $H_{(0:2)}$ given in (2.18) up to normalization. Eq. (3.4) requires $h(\tau)$ to satisfy

$$d_1 = 2g_s^2\,\hat{c}_1\tau^2\frac{\partial\log Z}{\partial\tau} = 2\hat{c}_1\tau^2 h'(\tau)\,. \tag{3.6}$$

This hints that $\tilde{d}_1 = d_1/\hat{c}_1$ should be a function of τ only and one has $h'(\tau) = \frac{\tilde{d}_1}{2\tau^2}$ which can be solved as

$$h(\tau) = \frac{1}{2}\int^{\tau} d\tau\, \tilde{d}_1/\tau^2. \tag{3.7}$$

Therefore it is enough to find $\tilde{d}_1$ as a function τ. As described in appendix B we find

$$\begin{aligned}\tilde{d}_1 &= 2\epsilon N + \tau a_N + \tau^2 \frac{a_N(a_N - a_{N-1})}{2\epsilon} \\ &+ \tau^3 \frac{(a_N - a_{N-1})^2 - a_{N-1}(a_N - a_{N-1})/2 + a_N(a_N - a_{N-1})}{(2\epsilon)^2} + \mathcal{O}(\tau^4)\,.\end{aligned} \tag{3.8}$$

Once $\tilde{d}_1$ is known, one can put $h(\tau)$ in (3.7) as

$$h(\tau) = \frac{1}{2}\left(-\frac{\tilde{d}_1^{(0)}}{\tau} + \tilde{d}_1^{(1)} \ln\tau + \sum_{\ell\geq 2} \frac{\tilde{d}_1^{(\ell)}}{\ell - 1}\tau^{\ell-1}\right), \tag{3.9}$$

where we neglect the τ-independent term which will be absorbed into the normalization $N_{(0:2)}$. This provides the explicit partition function of rank 2:

$$Z_{(0:2)} = N_{(0:2)} (\hat{c}_2)^{-\frac{\epsilon^2 N(N-1) + 2\epsilon\hat{c}_0 N}{2g_s^2}} \left(\frac{\hat{c}_2}{\hat{c}_1^2}\right)^{\frac{-\epsilon N(\epsilon(N-1)+\hat{c}_0)}{g_s^2}} e^{-\frac{\epsilon N \hat{c}_1^2}{g_s^2 \hat{c}_2} + \frac{1}{g_s^2}\mathcal{O}\left(\frac{\hat{c}_2}{\hat{c}_1^2}\right)}. \tag{3.10}$$

$N_{(0:2)}$ is the normalization factor independent of τ. This procedure demonstrates that finding $Z_{(0:n)}$ with $n > 2$ is straight-forward. On the other hand, it is to be noted that $Z_{(0:n)}$ has no filling fraction except N. This shows that $Z_{(0:n)}$ provides the solution of the one-cut case. In addition, it will be nice to find $P(z)$ and d_k in a more compact form.

4 Explicit evaluation of the partition function $Z_{(m:n)}$

In this section we evaluate $Z_{(m:n)}$. Its potential derivative is given as $V'(z) = \sum_{k=-m}^{n} \frac{\hat{c}_k}{z^{k+1}}$ and therefore, $f(z) = \sum_k d_k/z^{2+k}$ where k runs from $-m$ to $n-1$.

Power expansion of eq. (2.10) provides $N+m+n$ equations corresponding to $N+m+n$ variables. Explicitly, for the power of z^{N-k-2} with $-m \leq k \leq N+n-1$ one has the algebraic equation

$$2\epsilon(N-k+n)\hat{c}_n P_{N-k+n} + \sum_{s=-m}^{n-1}\Big(\big(2\epsilon(N-k+s)\hat{c}_s - d_s\big)P_{N-k+s}\Big) + \epsilon^2(N-k)(N-k-1)P_{N-k} = 0\,. \tag{4.1}$$

We use the same convention in the previous section: $P_N = 1$ and P_a vanishes when $a < 0$ or $a > N$. In addition, the coefficient $\hat{c}_\ell = 0$ when $\ell \geq n+1$ or $\ell < -m$.

One has the simple relation for the highest power z^{N+m-2} $(k = -m)$

$$d_{-m} = 2\epsilon N \hat{c}_{-m}\,, \tag{4.2}$$

and for the lowest power $z^{-(n+1)}$ $(k = N+n-1)$

$$P_0 d_{n-1} = 2\epsilon \hat{c}_n P_1\,. \tag{4.3}$$

Let us consider the case $(m,n)=(2,2)$ for concreteness. In this situation we need d_{-1}, d_0 and d_1 to find the partition function $Z_{(2:2)}$. The flow equations in (2.19) read

$$-g_s^2\hat{c}_{-2}\frac{\partial}{\partial\hat{c}_{-1}}\log Z_{(2:2)} = d_{-1} - 2\epsilon N\hat{c}_{-1}\,, \tag{4.4}$$

$$-g_s^2\left(\hat{c}_1\frac{\partial}{\partial\hat{c}_1} + 2\hat{c}_2\frac{\partial}{\partial\hat{c}_2}\right)\log Z_{(2:2)} = d_0\,, \tag{4.5}$$

$$-g_s^2\hat{c}_2\frac{\partial}{\partial\hat{c}_1}\log Z_{(2:2)} = d_1\,. \tag{4.6}$$

Appendix C shows that

$$d_{-1} = 2\epsilon\hat{c}_{-1}N - 2\epsilon N\hat{c}_{-2}\eta + \mathcal{O}(\eta^2)\,, \tag{4.7}$$

$$d_0 = 2\epsilon\hat{c}_0 N + \epsilon^2 N(N-1) - 2\epsilon N\hat{c}_{-1}\eta + \mathcal{O}(\eta^2)\,, \tag{4.8}$$

$$d_1 = 2\epsilon\hat{c}_1 N - 2\epsilon N\big(\epsilon_1(N-1)+\hat{c}_0\big)\eta \tag{4.9}$$

$$+\,2\left[\epsilon\hat{c}_{-1}N\eta^2 - \frac{\epsilon}{\hat{c}_1}N\big(\epsilon(N-1)+\hat{c}_0\big)\big(3\epsilon(N-1)+2\hat{c}_0\big)\right]\eta^2 + \mathcal{O}(\eta^3)\,, \tag{4.10}$$

where $\eta = \hat{c}_2/\hat{c}_1$. Using the linear combination (4.5) $-\frac{1}{\eta}\times$(4.6), we have

$$\begin{aligned}-g_s^2\left(\hat{c}_2\frac{\partial}{\partial\hat{c}_2}\right)\log Z_{(2:2)} &= 2\epsilon\hat{c}_0 N + \frac{3}{2}\epsilon^2 N(N-1) - \epsilon N\frac{\hat{c}_1}{\eta} \\ &\quad - \left[\epsilon\hat{c}_{-1}N + \big(\epsilon(N-1)+\hat{c}_0\big)\big(3\epsilon(N-1)+2\hat{c}_0\big)\frac{\epsilon N}{2\hat{c}_1}\right]\eta + \mathcal{O}(\eta^2)\,.\end{aligned} \tag{4.11}$$

Since (4.4), (4.6) and (4.11) are just simple derivative equations for $\hat{c}_{-1}$, $\hat{c}_1$ and $\hat{c}_2$, we can easily find $H_{(2:2)}$ given in (2.18).

$$\begin{aligned}H_{(2:2)} &= -2\epsilon N\big(\epsilon(N-1)+\hat{c}_0\big)\log\hat{c}_1 + \left(2\epsilon\hat{c}_0 N + \frac{3}{2}\epsilon^2 N(N-1)\right)\log\hat{c}_2 + \epsilon N\frac{\hat{c}_1^2}{\hat{c}_2} \\ &\quad - \left(2\epsilon N\hat{c}_{-1} - \frac{\epsilon}{2\hat{c}_1}N\big(\epsilon(N-1)+\hat{c}_0\big)\big(3\epsilon(N-1)+2\hat{c}_0\big)\right)\frac{\hat{c}_2}{\hat{c}_1} + \mathcal{O}(\eta^2)\,.\end{aligned} \tag{4.12}$$

Thus, one has the partition function

$$\begin{aligned}Z_{(2:2)} &= N_{(2:2)}\times(\hat{c}_1)^{\frac{2\epsilon N\left(\epsilon(N-1)+\hat{c}_0\right)}{g_s^2}}(\hat{c}_2)^{-\frac{2\epsilon\hat{c}_0 N+\frac{3}{2}\epsilon^2 N(N-1)}{g_s^2}} \\ &\quad\times e^{\frac{1}{g_s^2}\left\{2\epsilon N\hat{c}_{-1}\frac{\hat{c}_2}{\hat{c}_1} - \epsilon N\frac{\hat{c}_1^2}{\hat{c}_2} - \frac{\epsilon}{2}N\left(\epsilon(N-1)+\hat{c}_0\right)\left(3\epsilon(N-1)+2\hat{c}_0\right)\frac{\hat{c}_2}{\hat{c}_1^2} + \mathcal{O}(\eta^2)\right\}}\,.\end{aligned} \tag{4.13}$$

Here, $N_{(2:2)}$ is the normalization factor[1]. $\mathcal{O}(\eta^k)$ are polynomials of η, and they satisfy the following conditions:

1. They can be determined completely by the group of equations (C.5) with given N, using the perturbation method;
2. They are independent of g_s, i.e., they are finite because all the coefficients in the group of equations are independent of g_s.

In this way, (4.12) leads directly to the fact $\lim_{g_s\to 0} g_s^2\log(Z_{(2:2)}/N_{(2:2)}) \to$ finite.

[1] The normalization factor can be a function of $\hat{c}_0$ and $\hat{c}_{-2}$ since their derivatives are not given by the flow equations. However, there is no evidence for this parameter to be divergent.

5 Conclusion

Using the second order differential equation (2.10) of the polynomial $P(z)$, we find a straightforward method to calculate classical ICB, by assuming a hierarchical ordering in $\hat{c}_k$ so that P_k can be treated perturbatively. Compare to known methods, this new approach is efficient since $P(z)$ is a polynomial with a finite degree N, which leads to a finite number of equations with exact solutions. This property allows us to give a rigorous proof for the classical behavior of conformal blocks. Besides, the classical limit for Nekrasov partition function is proposed in [2] as $\mathcal{Z}_{Nek} \overset{\epsilon_2\to 0}{\sim} \exp\left\{\frac{1}{\epsilon_2}\mathcal{W}_{Nek}\right\}$, with $\mathcal{W}_{Nek}$ finite. This is naturally equivalent to classical ICB's exponential behavior, through the connection of AGT conjecture. We also expect that similar discussions can be applied to W-symmetry in future.

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A Normalization

In the text, we skip the normalization factor $N_{(0:n)}$ when the conformal block $\mathcal{F}_\Delta^{(m:n)}$ is considered since normalization does not contribute. It is noted that the duality $b \to 1/b$ holds for $\mathcal{F}_\Delta^{(m:n)}$. The duality is obvious since the loop equation (2.4) is manifestly dual (Q is dual in $b \to 1/b$).

On the other hand, the partition function (2.1) does not look invariant. However, as $g_s \to 0$ and $b \to \infty$ (NS limit), the partition function allows the perturbation expansion, and the normalization can be taken with $\hat{c}_{k\neq 0} \to 0$,

$$N_{(0:n)}(\hat{c}_0) = \int \left(\prod_{I=1}^{N} d\lambda_I\right) \prod_{I<J} (\lambda_I - \lambda_J)^{2\beta} e^{\frac{\sqrt{\beta}}{g}\sum_I \hat{c}_0 \log\lambda_I}. \tag{A.1}$$

This is given in terms of the Selberg integral [19]

$$\begin{aligned} S_N(\alpha,\delta,\beta) &\equiv \int_0^1 \cdots \int_0^1 \prod_{i=1}^{N} t_i^{\alpha-1}(1-t_i)^{\delta-1} \prod_{1\le i<j\le n} |t_i - t_j|^{2\beta}\, t_1\cdots t_n \\ &= \prod_{j=0}^{N-1} \frac{\Gamma(\alpha+j\beta)\Gamma(\delta+j\beta)\Gamma\big(1+(j+1)\beta\big)}{\Gamma\big(\alpha+\delta+(N+j-1)\beta\big)\Gamma(1+\beta)}\,, \end{aligned} \tag{A.2}$$

with $\alpha = 1 + \sqrt{\beta}\hat{c}_0/g$ and $\delta = 1$, one has[2]

$$N_{(0:n)}(\hat{c}_0) = \prod_{j=0}^{N-1} \frac{\Gamma(1+A\beta+j\beta)\Gamma(1+j\beta)\Gamma\big(1+(j+1)\beta\big)}{\Gamma\big(2+A\beta+(N+j-1)\beta\big)\Gamma(1+\beta)}\,, \tag{A.3}$$

where $A = 2\hat{c}_0/\epsilon$.

[2]There is some ambiguity in the integration range for IMM, which could be adjusted by rescaling λ_I, and does not effect our result shown here.

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Obversely $\lim_{g_s\to 0} g_s^2 \log N_{(0:n)}$ is equivalent to $\lim_{\beta\to\infty}(\log N_{(0:n)})/\beta$. Then using the property of Gamma function $\log\Gamma(a+Z) = z\log(z) - z + \mathcal{O}\big(\log(z)\big)$ when z is large and a is small, and making use of

$$\sum_{j=0}^{N-1}\big\{(A+j)+j+(j+1)\big\} = \sum_{j=0}^{N-1}\big\{(A+N+j-1)+1\big\}, \tag{A.4}$$

we find

$$\begin{aligned}\lim_{\beta\to\infty}\frac{\log N_{(0:n)}(\hat{c}_0)}{\beta} &= \lim_{\beta\to\infty}\sum_{j=0}^{N-1}(A+j)\log(A\beta+j\beta)+j\log(j\beta)+(j+1)\log\big((j+1)\beta\big)\\ &\qquad -(A+N+j-1)\log\big(A\beta+(N+j-1)\beta\big)-\log(\beta)\\ &= \sum_{j=0}^{N-1}(A+j)\log(A+j)+j\log(j)+(j+1)\log\big((j+1)\big)\\ &\qquad -(A+N+j-1)\log\big(A+(N+j-1)\big)\\ &= \text{const}\,. \end{aligned}\tag{A.5}$$

This means $\lim_{g_s\to 0} g_s^2\log N_{(0:n)}(\hat{c}_0) = \text{const}$.

For the regular case, N_β is the normalization factor independent on z_a, which means it can be achieved by setting all the $z_a = 0$. Actually

$$\begin{aligned} N_\beta &= \int\left(\prod_{I=1}^{N} d\lambda_I\right)\prod_{I<J}(\lambda_I-\lambda_J)^{2\beta}e^{\frac{\sqrt{\beta}}{g}\sum_I\sum_{a=0}^{n}\hat{\alpha}_a\log\lambda_I}\\ &= \int\left(\prod_{I=1}^{N} d\lambda_I\right)\prod_{I<J}(\lambda_I-\lambda_J)^{2\beta}e^{\frac{\sqrt{\beta}}{g}\sum_I\hat{c}_0\log\lambda_I}\\ &= N_{(0:n)}(\hat{c}_0)\,. \end{aligned}\tag{A.6}$$

According to the previous discussion we know $\lim_{g_s\to 0} g_s^2\log N_\beta = \text{const}$.

B Method to obtain d_1

The eq. (3.1) for rank 2 can be written as

$$(d_1 - 2\epsilon\hat{c}_1 t)P_t = \tilde{a}_t P_{t-1} + 2\epsilon\hat{c}_2(t+1)P_{t+1}\,, \tag{B.1}$$

where we put $N-k+1=t$ and $\tilde{a}_t = \big(\epsilon^2(t-2)+2\epsilon\hat{c}_0\big)(t-1)-d_0$. One can simplify this if one puts $r_t = P_t/P_{t-1}$,

$$d_1 = 2\epsilon\hat{c}_1 t + 2\epsilon\hat{c}_2(t+1)r_{t+1} + \tilde{a}_t/r_t\,, \tag{B.2}$$

where t runs from 0 to N. From the definition, one has $r_{N+1}=0$ and $r_0=\infty$.

When $t = 0$, one finds $d_1 = 2\epsilon\hat{c}_2\, r_1$ in a very compact notation. However, explicit form of r_1 as the function of $\hat{c}_1$, $\hat{c}_2$ is not easy to put. One way to find d_1 is to use perturbation. One may rescale $r_t = \hat{c}_1(N+1-t)\xi_t/(\hat{c}_2 t)$ in (B.1) to get

$$d_1/\hat{c}_1 = 2\epsilon\big(t + (N-t)\xi_{t+1}\big) + \tau\, a_t/\xi_t\,, \tag{B.3}$$

where $a_t = \tilde{a}_t\, t/(N+1-t)$ and $\tau = \hat{c}_2/\hat{c}_1^2$ as defined in (3.5). Eq. (B.3) shows that $d_1/\hat{c}_1$ is indeed a function of τ.

To the lowest order in τ, $\xi_t = 1$ and $d_1/\hat{c}_1 = 2\epsilon N$. Therefore, one may find $d_1/\hat{c}_1$ and ξ_t in powers of τ

$$\tilde{d}_1 := d_1/\hat{c}_1 = \sum_{\ell\geq 0} \tilde{d}_1^{(\ell)}\tau^\ell, \qquad \xi_t = \sum_{\ell\geq 0} \xi_t^{(\ell)}\tau^\ell, \tag{B.4}$$

where $\tilde{d}_1^{(0)} = 2\epsilon N$ and $\xi_t^{(0)} = 1$.

In addition, the solution of $\tilde{d}_1$ is t-independent. Therefore, the perturbative expansion is more facilitated if the equation is set into the form

$$\tilde{d}_1 = \tilde{d}_1^{(0)} + \tau a_N + \big[2\epsilon\hat{c}_1(N-t)(\xi_{k+1}-1) + \eta(a_t - a_N)\big] + \tau a_t(1-\xi_t)/\xi_t\,, \tag{B.5}$$

where decomposition $1/\xi_t = 1 + (1-\xi_t)/\xi_t$ is used to put $1/\xi_t$ perturbatively in t. To make $\tilde{d}_1$ t-independent, one has $\tilde{d}_1^{(1)} = a_N$ and the term in the squared bracket need to vanish at the order η,

$$2\epsilon\hat{c}_1(N-t)\xi_{t+1}^{(1)} - a_t - a_N = 0\,, \tag{B.6}$$

which fixes the $\xi_t^{(1)}$. In this way one can find $\tilde{d}_1$ order by oder

$$\begin{aligned}\tilde{d}_1 = 2\epsilon N + \tau a_N &+ \tau^2\frac{a_N(a_N - a_{N-1})}{2\epsilon}\\ &+ \tau^3\frac{(a_N - a_{N-1})^2 - a_{N-1}(a_N - a_{N-1})/2 + a_N(a_N - a_{N-1})}{(2\epsilon)^2} + \mathcal{O}(\tau^4)\,,\end{aligned} \tag{B.7}$$

which provides the explicit $\tilde{d}_1^{(\ell)}$ with $\ell = 0, 1, 2, 3$.

C Method to obtain d_{-1}, d_0 and d_1

For the case $(m,n) = (2,2)$, explicitly expanding eq. (2.10) in powers of z, we have

$$z^N:\quad d_{-2} = 2\epsilon N\hat{c}_{-2}\,, \tag{C.1}$$

$$z^{N-1}:\quad d_{-1} = 2\epsilon N\hat{c}_{-1} - 2\epsilon\hat{c}_{-2}P_{N-1}\,, \tag{C.2}$$

$$z^{N-2}:\quad d_0 = 2\epsilon N\hat{c}_0 + \epsilon^2 N(N-1) + \big(2\epsilon(N-1)\hat{c}_{-1} - d_{-1}\big)P_{N-1} - 4\epsilon\hat{c}_{-2}P_{N-2}\,, \tag{C.3}$$

$$\begin{aligned}z^{N-3}:\quad d_1 = 2\epsilon\hat{c}_1 N &+ \big(2\epsilon(N-1)\hat{c}_0 + \epsilon^2(N-1)(N-2) - d_0\big)P_{N-1}\,,\\ &+ \big(2\epsilon(N-2)\hat{c}_{-1} - d_{-1}\big)P_{N-2} - 6\epsilon\hat{c}_{-2}P_{N-3}\,.\end{aligned} \tag{C.4}$$

And in general for the power of z^{N-k-2} with $-1 \leq k \leq N+1$ we have

$$\begin{aligned}&2\epsilon(N-k+2)\hat{c}_2P_{N-k+2} + \big(2\epsilon(N-k+1)\hat{c}_1 - d_1\big)P_{N-k+1}\\ &\qquad + \big(2\epsilon(N-k)\hat{c}_0 + \epsilon^2(N-k)(N-k-1) - d_0\big)P_{N-k}\\ &+ \big(2\epsilon(N-k-1)\hat{c}_{-1} - d_{-1}\big)P_{N-k-1} - 2\epsilon(k+2)\hat{c}_{-2}P_{N-k-2} = 0\,.\end{aligned} \tag{C.5}$$

To find the d_i's, one may use perturbation in (C.5). First note that $P_{N-1} = -\big\langle \sum_I \lambda_I \big\rangle$ while $P_0 = \big\langle \prod_I (-\lambda_I) \big\rangle$. Therefore, P_{N-k} can grow as k-powers of the expectation values. One may assume $P_{N-t} = \mathcal{O}(\eta^t)$ with $|\eta| \ll 1$. Indeed, the equation (C.5) has the solution to the lowest order,

$$\begin{aligned}
z^{N-1}: &\quad d_{-1} = 2\epsilon\hat{c}_{-1}N + \mathcal{O}(\eta)\,, && \text{(C.6)}\\
z^{N-2}: &\quad d_0 = 2\epsilon\hat{c}_0 N + {\epsilon_1}^2\, N(N-1) + \mathcal{O}(\eta)\,, && \text{(C.7)}\\
z^{N-3}: &\quad d_1 = 2\epsilon\hat{c}_1 N + \mathcal{O}(\eta)\,, && \text{(C.8)}\\
z^{N-4}: &\quad P_{N-1} = \frac{\hat{c}_2}{\hat{c}_1} N + \mathcal{O}(\eta^2)\,, && \text{(C.9)}\\
z^{N-5}: &\quad P_{N-2} = \frac{N-1}{2}\frac{\hat{c}_2}{\hat{c}_1} P_{N-1} + \mathcal{O}(\eta^3)\,, && \text{(C.10)}\\
z^{N-t-3}: &\quad P_{N-t} = \frac{N-t+1}{t}\frac{\hat{c}_2}{\hat{c}_1} P_{N-t+1} + \mathcal{O}(\eta^{t+1})\,. && \text{(C.11)}
\end{aligned}$$

Obviously given the condition $|\hat{c}_2/\hat{c}_1| \ll 1$, while keeping c_1, c_0, c_{-1} and c_{-2} in the same order, we can choose $\eta \equiv \hat{c}_2/\hat{c}_1$, consistent with $P_{N-t} = \mathcal{O}(\eta^t)$.

Next order is given as follows:

$$\begin{aligned}
z^{N-1}: &\quad d_{-1}^{(1)} = -2\epsilon\hat{c}_{-2}N\,,\\
z^{N-2}: &\quad d_0^{(1)} = -2\epsilon\hat{c}_{-1}N\,,\\
z^{N-3}: &\quad d_1^{(1)} = -2\epsilon N\big(\epsilon(N-1) + \hat{c}_0\big)\,. && \text{(C.12)}
\end{aligned}$$

To calculate the partition function up to $\mathcal{O}(\eta)$, for a technical reason which will be clear soon below, we need the η^2 expansion of d_1, which could be obtained from the third order perturbation:

$$z^{N-3}: \quad d_1^{(2)} = 2\epsilon\hat{c}_{-1}N - \frac{\epsilon}{\hat{c}_1} N\big(\epsilon(N-1)+\hat{c}_0\big)\big(3\epsilon(N-1)+2\hat{c}_0\big) \qquad \text{(C.13)}$$

so that

$$\begin{aligned}
d_{-1} &= 2\epsilon\hat{c}_{-1}N - 2\epsilon N\hat{c}_{-2}\eta + \mathcal{O}(\eta^2)\,, && \text{(C.14)}\\
d_0 &= 2\epsilon\hat{c}_0 N + \epsilon^2\, N(N-1) - 2\epsilon N\hat{c}_{-1}\eta + \mathcal{O}(\eta^2)\,, && \text{(C.15)}\\
d_1 &= 2\epsilon\hat{c}_1 N - 2\epsilon N\big(\epsilon(N-1)+\hat{c}_0\big)\eta + d_1^{(2)}\eta^2 + \mathcal{O}(\eta^3)\,. && \text{(C.16)}
\end{aligned}$$

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