

Rigid 4D $\mathcal{N} = 2$ supersymmetric backgrounds and actions

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ABSTRACT: We classify all $\mathcal{N} = 2$ rigid supersymmetric backgrounds in four dimensions with both Lorentzian and Euclidean signature that preserve eight real supercharges, up to discrete identifications. Among the backgrounds we find specific warpings of $S^3 \times \mathbb{R}$ and $\text{AdS}_3 \times \mathbb{R}$, $\text{AdS}_2 \times S^2$ and $H^2 \times S^2$ with generic radii, and some more exotic geometries. We provide the generic two-derivative rigid vector and hypermultiplet actions and analyze the conditions imposed on the special Kähler and hyperkähler target spaces.

KEYWORDS: Supersymmetric gauge theory, Extended Supersymmetry, Superspaces

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1 Introduction

In the last few years, there has been a great deal of interest in supersymmetric field theories on rigid curved backgrounds, beginning with the seminal work of Pestun [1]. These efforts have exploited the principle of supersymmetric localization to evaluate path integrals and compute certain supersymmetric observables in various rigid backgrounds.

A systematic approach to addressing such curved spaces at the component level was initiated by Festuccia and Seiberg [2]. Taking the point of view that a rigid supersymmetric theory could be understood as a supergravity theory with the metric and other bosonic components frozen to some background values, they investigated the conditions required in both $4D \mathcal{N} = 1$ old minimal and new minimal supergravities so that four linear independent rigid supersymmetries existed. Other aspects, such as the weaker requirements imposed by fewer supercharges in both signatures, were addressed in later work [3–9]. In the case of extended $\mathcal{N} = 2$ supersymmetry in four dimensions, conditions required for a single supercharge were

analyzed by Gupta and Murthy [10] and by Klare and Zaffaroni [11]. The analysis of [11] determined the main geometric criterion in either Euclidean or Lorentzian signature: the spacetime must admit a conformal Killing vector. In the presence of such a vector, one supercharge may be kept by turning on background values for the R -symmetry gauge fields.

Our goal in this paper is to perform a complementary analysis to that of [10, 11]. First, we derive the conditions imposed by requiring *full* $\mathcal{N} = 2$ supersymmetry — eight linearly independent Killing spinors — in both Lorentzian and Euclidean signatures and classify the possible smooth backgrounds up to discrete identifications. Second, we construct the general vector and hypermultiplet actions on such spaces and find the conditions on the allowed target spaces.

Our analysis, some of which appeared in a different context in [12], leads to several interesting possibilities. In addition to the geometries allowed in the $\mathcal{N} = 1$ case [2] — AdS_4 , $\mathbb{R} \times S^3$, $\text{AdS}_3 \times \mathbb{R}$, and an Hpp-wave arising as a Penrose limit of the last two cases — we also find several backgrounds that support $\mathcal{N} = 2$ SUSY with no admissible truncation to $\mathcal{N} = 1$. The options in Lorentzian signature are summarized in table 1 of section 3. In brief, they involve the cases familiar from $\mathcal{N} = 1$ — with the S^3 , AdS_3 and Hpp-wave admitting two alternative $\mathcal{N} = 2$ supersymmetry algebras with differing R -symmetry groups — and several cases requiring extended supersymmetry:

- a squashed $\mathbb{R} \times S^3$,
- a timelike stretched, spacelike squashed, or null warped $\text{AdS}_3 \times \mathbb{R}$,
- a warped $S^3 \times \mathbb{R}$ where the S^1 fiber of S^3 is either timelike or null,
- $\text{AdS}_2 \times S^2$ with generic radii, with two different SUSY realizations for each choice of radii,

as well as an $\text{Heis}_3 \times \mathbb{R}$ space and Hpp-wave variants where the background fields become null. (Some of these correspond to Penrose limits of other cases.) Each of the resulting supersymmetry algebras can be identified as a massive deformation of the super-Poincaré algebra, and indeed each possesses a supercoset structure permitting the straightforward construction of each of the Killing spinors, which we compute explicitly.¹

The options in Euclidean signature are summarized in table 2 of section 6. In addition to S^4 , H^4 , a two-sheeted $H^3 \times \mathbb{R}$, and $S^3 \times \mathbb{R}$, we find several geometries where extended supersymmetry plays a major role:

- a squashed or stretched $S^3 \times \mathbb{R}$,
- a $\text{Heis}_3 \times \mathbb{R}$ group manifold,

¹It should be emphasized that the relation between extended supersymmetry and at least some of these spaces is already known. For example, the $\mathcal{N} = 2$ supersymmetry algebra on one of the unsquashed $\mathbb{R} \times S^3$ cases was discussed in [13]. More well-known is the case of $\text{AdS}_2 \times S^2$, related to the one-parameter superalgebra $D(2, 1; \alpha)$ [14]; this latter case includes for $\alpha = -1$ the Bertotti-Robinson spacetime relevant for the near horizon geometry of BPS black holes.

- a warped $H^3 \times \mathbb{R}$, where the hyperboloid corresponds to AdS_3 spacetime with a Euclidean metric
- $H^2 \times S^2$ with generic radii and two different SUSY realizations.

Aside from these, we find the possibility of flat Euclidean spaces where the left-handed (or right-handed) supercharges are deformed. These include as particular cases the full BPS limits of the Euclidean Ω -background (corresponding to $\epsilon_1 = \pm\epsilon_2$).

Because the spacetimes we discuss retain eight rigid supercharges, it is possible to construct rigid $\mathcal{N} = 2$ superspaces for each. In fact, this will be the principle guiding their classification. We follow the approach laid down by Kuzenko et al. in [15–17], which applies the analysis of general (conformal) isometries of curved superspaces [18] to geometries where full supersymmetry is maintained. The presence of a rigid superspace enables the explicit construction of the component Lagrangians for vector and hypermultiplets just as in a Minkowski background. We present these actions in their general form, applicable to any of the rigid $\mathcal{N} = 2$ backgrounds, and find the constraints on the special Kähler and hyperkähler target spaces imposed by rigid supersymmetry. We also give the constraints on the supersymmetric moduli spaces in these backgrounds and comment on how they differ from the flat case.

The paper is laid out as follows. In section 2, we motivate and discuss the general rigid Lorentzian $\mathcal{N} = 2$ supersymmetry algebra. The rigid backgrounds allowed by this algebra are analyzed in section 3, while the corresponding vector multiplet and hypermultiplet actions are given in section 4. In particular, we give the $\mathcal{N} = 2^*$ action in a general rigid Lorentzian background. In sections 5 and 6 we give the Euclidean supersymmetry algebra and the possible rigid backgrounds. The corresponding Euclidean actions are discussed in section 7.

There are four technical appendices. Our conventions are discussed in appendix A. The general action principles in rigid superspace are summarized in appendix B. Explicit expressions for the geometric data of the rigid backgrounds including Killing spinors, vielbeins, and background fields are provided in appendices C and D.

2 The general rigid Lorentzian supersymmetry algebra

We begin this section by describing the construction of the general rigid Lorentzian supersymmetry algebra, which arises by freezing one of the $\mathcal{N} = 2$ supergravities. As discussed in the introduction, any off-shell supergravity corresponds to conformal supergravity coupled to some compensating multiplet whose lowest component plays the role of the Planck mass.

It helps to review the $\mathcal{N} = 1$ case. As is well-known, in $\mathcal{N} = 1$ supergravity the simplest compensators are a chiral multiplet or a linear multiplet, leading respectively to old and new minimal Poincaré supergravity both with $12 + 12$ degrees of freedom. Other options include a complex linear multiplet (giving $20 + 20$), an unconstrained real superfield (giving

16 + 16), or an unconstrained complex superfield (giving 24 + 24). Each option eliminates the dilatation and S -supersymmetry and several also eliminate the R -symmetry.²

The $\mathcal{N} = 1$ conformal Killing spinor equation is given as

$$(\delta_Q + \delta_S)\psi_{m\alpha} = 2\mathcal{D}_m\xi_\alpha + 2i(\sigma_m\bar{\eta})_\alpha = 0 . \quad (2.1)$$

ξ_α and η_α are respectively the local Q and S -supersymmetry parameters, \mathcal{D}_m carries the R -symmetry, dilatation, and spin connections, and any solution ξ of this equation is called a conformal Killing spinor. Let Ω be a nowhere vanishing conformal compensator of Weyl weight two, so that the physical Weyl-invariant metric is Ωg_{mn} . The lowest fermion ψ_α of Ω plays the role of an S -supersymmetry compensator. The S -invariant gravitino is $\psi_{m\alpha} + \frac{i}{2}(\sigma_m\bar{\psi})_\alpha$. Taking the S -supersymmetry gauge where $\psi_\alpha = 0$, the Weyl gauge $\Omega = 1$, and the K -gauge where the dilatation connection vanishes, then one finds

$$(\delta_Q + \delta_S)\psi_\alpha = 4\xi_\alpha\bar{R} - 2G^b(\sigma_b\bar{\xi})_\alpha - 4\eta_\alpha \quad (2.2)$$

where \bar{R} and G_a correspond to auxiliary components of Ω at the θ^2 level, normalized to match the conventions of [19]. By solving for η , one finds the Killing spinor equation of U(1) supergravity,

$$\mathcal{D}_m\xi_\alpha = -i(\sigma_m\bar{\xi})_\alpha R - \frac{i}{2}G^b(\sigma_m\bar{\sigma}_b\xi)_\alpha . \quad (2.3)$$

Old minimal supergravity arises from choosing $\Omega = \Phi\bar{\Phi}$ for a chiral compensator Φ : then the $U(1)_R$ connection is fixed to G_a after imposing the Weyl-U(1) gauge $\Phi = 1$. Conversely, if we choose $\Omega = L$ for a linear multiplet compensator, R vanishes and G_a is related to the dual field strength of the two-form potential within L . This is new minimal supergravity. However, it is possible to work purely with U(1) supergravity, which simultaneously encompasses both minimal possibilities while allowing more general supergeometries.

A corresponding story holds for $\mathcal{N} = 2$ supergravity. The conformal Killing spinor equation is³

$$(\delta_Q + \delta_S)\psi_{m\alpha i} = 2\mathcal{D}_m\xi_{\alpha i} - iW_{mn}^-(\sigma^n\bar{\xi}_i)_\alpha + 2i(\sigma_m\bar{\eta}_i)_\alpha = 0 . \quad (2.4)$$

Introducing a generic real compensator Ω of dimension two and performing the analogous conformal gauge-fixings results in⁴

$$\mathcal{D}_m\xi_{\alpha i} = \frac{i}{2}\bar{S}_{ij}(\sigma_m\bar{\xi}^j)_\alpha - \frac{i}{2}(Y_{mn}^+ - W_{mn}^-)(\sigma^n\bar{\xi}_i)_\alpha - 2iG^n(\sigma_{nm}\xi_i)_\alpha + G^{nj}(\sigma_n\bar{\sigma}_m\xi_j)_\alpha . \quad (2.5)$$

We have introduced \bar{S}_{ij} , G_a , $G_a^i{}_j$, and Y_{ab}^+ corresponding to θ^2 components of the real compensator Ω .

The unconstrained superfield Ω corresponds to a “maximal” supergravity with 152 + 152 degrees of freedom: we refer to this option as U(2) supergravity because it retains the

²In matter-coupled $\mathcal{N} = 1$ supergravities, the most natural description involves a composite compensating multiplet corresponding to the Kähler cone from which the Hodge-Kähler manifold descends.

³We have relabelled the auxiliary field T_{ab}^\pm as $4W_{ab}^\pm$.

⁴To simplify the resulting supergravity algebra, we have redefined the $U(1)_R$ and $SU(2)_R$ connections as $A_m \rightarrow A_m + G_m$ and $\mathcal{V}_m^i{}_j \rightarrow \mathcal{V}_m^i{}_j + 2G_m^i{}_j$.

full R -symmetry structure group. The obvious “minimal” choices are a vector multiplet or a tensor multiplet, both leading to a $32 + 32$ supergravity multiplet. The lowest component X of a vector multiplet carries $U(1)_R$ charge, breaking the R -symmetry $U(2)_R$ to $SU(2)_R$. Its graviphoton field strength F_{ab} and pseudoreal auxiliary field Y^{ij} generate Y_{ab}^\pm and $S^{ij} = \bar{S}^{ij}$, while the auxiliary $G_a^i{}_j$ vanishes and G_a is related (as with the $\mathcal{N} = 1$ chiral multiplet) to the Higgsed $U(1)_R$ connection. The tensor multiplet is a bit more interesting. Its lowest component L^{ij} fixes dilatations and breaks $SU(2)_R$ to $SO(2)_R$. Its three-form field strength H_{abc} and complex auxiliary scalar F contribute respectively to G_a^{ij} and S^{ij} via $G_a^{ij} \sim \epsilon_{abcd} H^{bcd} L^{ij}$ and $S^{ij} \sim F L^{ij}$, while Y_{ab}^\pm and G_a vanish. However, it will be more convenient for us to remain with the generic $U(2)$ supergravity.

Now if we assume that the Killing spinors $\xi_{\alpha i}$ and $\bar{\xi}^{\dot{\alpha} i}$ are linearly independent at each point in spacetime, we can in principle derive integrability conditions that impose constraints on the background fields of the supergravity multiplet consistent with the existence of eight supercharges. In addition, there are also covariant fermions which must be invariant under supersymmetry, leading to constraints on the other auxiliary fields. Such a procedure was actually followed in [12].

Following [15], we will analyze the problem directly in curved superspace. The supergeometry corresponding to an unconstrained real compensator is the $U(2)$ superspace of [20] (we follow similar conventions as [21] and [22]). It involves a supermanifold $\mathcal{M}^{4|8}$ with local coordinates $z^M = (x^m, \theta^\mu, \bar{\theta}_{\dot{\mu}})$ consisting of four bosonic and eight Grassmann coordinates. It is equipped with a non-degenerate supervielbein E_M^A and local Lorentz, $SU(2)_R$ and $U(1)_R$ connections Ω_M^{ab} , $\mathcal{V}_M^i{}_j$ and \mathcal{A}_M , respectively. The covariant derivative $\mathcal{D}_A = (\mathcal{D}_\alpha^i, \bar{\mathcal{D}}^{\dot{\alpha} i}, \mathcal{D}_a)$ is defined as

$$\mathcal{D}_A = E_A^M \left(\partial_M - \frac{1}{2} \Omega_M^{ab} M_{ab} - \mathcal{V}_M^i{}_j I^j{}_i - \mathcal{A}_M \mathbb{A} \right). \tag{2.6}$$

The symbols M_{ab} , $I^i{}_j$ and \mathbb{A} denote respectively the Lorentz, $SU(2)_R$, and $U(1)_R$ generators, whose action is given by

$$\begin{aligned} [M_{ab}, \mathcal{D}_c] &= 2\eta_{c[a} \mathcal{D}_{b]}, & [M_{ab}, \mathcal{D}_\alpha^i] &= -(\sigma_{ab})_\alpha{}^\beta \mathcal{D}_\beta^i, \\ [I^i{}_j, \mathcal{D}_\alpha^k] &= \delta_j^k \mathcal{D}_\alpha^i - \frac{1}{2} \delta_j^i \mathcal{D}_\alpha^k, & [\mathbb{A}, \mathcal{D}_\alpha^i] &= -i \mathcal{D}_\alpha^i. \end{aligned} \tag{2.7}$$

We denote by w the R -charge of fields and operators, e.g. $w(\mathcal{D}_\alpha^i) = -1$. The graded algebra of covariant derivatives involves torsion and curvature tensors,

$$[\mathcal{D}_A, \mathcal{D}_B] = -T_{AB}{}^C \mathcal{D}_C - \frac{1}{2} R_{AB}{}^{cd} M_{cd} - R(\mathcal{V})_{AB}{}^i{}_j I^j{}_i - R(A)_{AB} \mathbb{A}. \tag{2.8}$$

For the case of $U(2)$ supergravity, these tensors are given by the superfields S^{ij} , G_a , $G_a^i{}_j$, Y_{ab}^- , and W_{ab}^- and their covariant derivatives.

Recall that the algebra of spinor derivatives \mathcal{D}_α^i and $\bar{\mathcal{D}}^{\dot{\alpha} i}$ encodes the structure of $U(2)$ supergravity. Any component bosonic or fermionic field ϕ is the lowest component of some superfield Φ , denoted $\phi = \Phi|$. The action of supersymmetry on ϕ derives from a covariant Lie derivative of Φ ,

$$\delta_Q \phi \equiv \delta \Phi| = \xi^\alpha{}_i (\mathcal{D}_\alpha^i \Phi)| + \bar{\xi}^{\dot{\alpha} i} (\bar{\mathcal{D}}^{\dot{\alpha} i} \Phi)| \tag{2.9}$$

with $\xi_{\alpha i}$ and $\bar{\xi}^{\dot{\alpha} i}$ the local supergravity parameters. The local supersymmetry algebra is equivalent to the spinor derivative algebra of (2.8), which is explicitly given by

$$\begin{aligned} \{\mathcal{D}_\alpha^i, \mathcal{D}_\beta^j\} &= 4S^{ij}M_{\alpha\beta} + \epsilon^{ij}\epsilon_{\alpha\beta}(Y^{cd-} - W^{cd+})M_{cd} + 2\epsilon^{ij}\epsilon_{\alpha\beta}S^k{}_l I^l{}_k - 4Y_{\alpha\beta}I^{ij}, \\ \{\bar{\mathcal{D}}^{\dot{\alpha} i}, \bar{\mathcal{D}}^{\dot{\beta} j}\} &= 4\bar{S}_{ij}\bar{M}^{\dot{\alpha}\dot{\beta}} - \epsilon_{ij}\epsilon^{\dot{\alpha}\dot{\beta}}(Y^{cd+} - W^{cd-})M_{cd} - 2\epsilon_{ij}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{S}^k{}_l I^l{}_k - 4\bar{Y}^{\dot{\alpha}\dot{\beta}}I_{ij}, \\ \{\mathcal{D}_\alpha^i, \bar{\mathcal{D}}^{\dot{\beta} j}\} &= -2i\delta_j^i\mathcal{D}_{\alpha\dot{\beta}} + 2(\sigma^{cd}\sigma^b + \sigma^b\bar{\sigma}^{cd})_{\alpha\dot{\beta}}(\delta_j^i G_b + iG_b{}^i{}_j)M_{cd} - 8G_{\alpha\dot{\beta}}I^i{}_j + 2G_{\alpha\dot{\beta}}{}^i{}_j\mathbb{A}. \end{aligned} \quad (2.10)$$

We remind the reader that our spinor conventions are summarized in appendix A. From the algebra, one may read off the R -charges

$$w(S^{ij}) = w(Y_{\alpha\beta}) = w(\bar{W}_{\dot{\alpha}\dot{\beta}}) = -2, \quad w(G_a) = w(G_a{}^i{}_j) = 0. \quad (2.11)$$

The reality properties of the superfields and spinor derivatives are

$$\begin{aligned} (W_{ab}^-)^* &= W_{ab}^+, & (Y_{ab}^-)^* &= Y_{ab}^+, & (S^{ij})^* &= \bar{S}_{ij}, & (G_a)^* &= G_a, \\ (G_a{}^i{}_j)^* &= -G_a{}^j{}_i, & (\mathcal{D}_\alpha^i)^* &= \bar{\mathcal{D}}_{\dot{\alpha}i}. \end{aligned} \quad (2.12)$$

We are interested in fermionic isometries of a fixed background manifold, that is, local covariant diffeomorphisms that leave its supervielbein, connections and the associated torsion and curvatures invariant. If we restrict to a bosonic background (i.e. all background fermions set to zero), to any such fermionic isometry is associated a Killing spinor satisfying (2.5). In a fully supersymmetric background, however, we do not need to solve this equation explicitly. Following [16], we observe that if eight linearly independent Killing spinors ξ exist, then requiring $\delta_Q\phi$ to vanish for any background field ϕ implies the background superfield Φ must be annihilated by the spinor derivatives. In particular, we have

$$\mathcal{D}_\alpha^i W_{ab}^- = \mathcal{D}_\alpha^i Y_{ab}^- = \dots = 0. \quad (2.13)$$

The integrability conditions

$$\{\mathcal{D}_\alpha^i, \mathcal{D}_\beta^j\}W_{ab}^- = \{\mathcal{D}_\alpha^i, \mathcal{D}_\beta^j\}Y_{ab}^- = \dots = 0, \quad (2.14)$$

together with closure of the algebra of covariant derivatives imply integrability of the Killing spinor equation and invariance under δ_Q of the background fermionic fields. It is thus sufficient to classify the solutions to these conditions.

Several of the equations (2.14) imply that products of various pairs of fields must vanish:

$$\begin{aligned} G_a{}^{ij}Y_{bc}^\pm &= G_a{}^{ij}W_{bc}^\pm = G_a{}^{ij}S^{kl} = G_a{}^{ij}\bar{S}_{kl} = G_a{}^{ij}G_b = 0, \\ S^{ij}G_a &= \bar{S}_{ij}G_a = 0, & S^{ij}Y_{ab}^\pm &= \bar{S}_{ij}Y_{ab}^\pm = 0, & S^{ij}W_{ab}^- &= \bar{S}_{ij}W_{ab}^+ = 0. \end{aligned} \quad (2.15)$$

In Lorentzian signature, the last condition is strengthened to $S^{ij}W_{ab}^\pm = \bar{S}_{ij}W_{ab}^\pm = 0$. From these relations we may identify four broad cases:

- (I) $S^{ij} \neq 0$, all other fields vanishing;
- (II) $G_a{}^i{}_j \neq 0$, all other fields vanishing;

(III) $G_a \neq 0$, perhaps with Y_{ab}^\pm and/or W_{ab}^\pm nonzero;

(IV) Y_{ab}^\pm and/or W_{ab}^\pm nonzero, but all other fields vanishing.

Additional integrability conditions depend on which case we are in. For (I), these amount to $S^{ij} \propto \bar{S}^{ij}$ and $\mathcal{D}_a S^{ij} = 0$. For (II), we find $G_{[a}^{ij} G_b]^{kl} = 0$ and $\mathcal{D}_a G_b^{ij} = 0$, implying that G_a^{ij} can be decomposed as a product of a covariantly constant vector and an $SU(2)$ tensor. For (III) and (IV), the additional conditions are

$$\begin{aligned} G^b W_{ba}^\pm &= G^b Y_{ba}^\mp, & Y_{ab}^\pm &\propto W_{ab}^\pm, & Y^{ab-} W_{ab}^- &= Y^{ab+} W_{ab}^+, \\ \mathcal{D}_a G_b &= 0, & (\mathcal{D}_a + \epsilon_a^{bcd} G_b M_{cd}) W_{ef}^\pm &= 0, & (\mathcal{D}_a + \epsilon_a^{bcd} G_b M_{cd}) Y_{ef}^\pm &= 0. \end{aligned} \quad (2.16)$$

The last equations above imply that W_{ab}^\pm and Y_{ab}^\pm are covariantly constant with respect to a torsionful spin connection, $\tilde{\omega}_{mab} = \omega_{mab} - \epsilon_{mabc} G^c$.

We can now construct the entire algebra of covariant derivatives for the general $\mathcal{N} = 2$ rigid superspace, treating all cases simultaneously. It is convenient to combine Y_{ab}^- and W_{ab}^+ into a single complex two-form \mathcal{Z}_{ab} with R -charge $w = -2$,

$$\mathcal{Z}_{ab} := Y_{ab}^- - W_{ab}^+, \quad \bar{\mathcal{Z}}_{ab} := Y_{ab}^+ - W_{ab}^-. \quad (2.17)$$

The full superspace algebra can then be compactly written as

$$\begin{aligned} \{\mathcal{D}_\alpha^i, \mathcal{D}_\beta^j\} &= 4S^{ij} M_{\alpha\beta} + \epsilon^{ij} \epsilon_{\alpha\beta} \mathcal{Z}^{cd} M_{cd} + 2\epsilon^{ij} \epsilon_{\alpha\beta} S^k{}_l I^l{}_k - 4\mathcal{Z}_{\alpha\beta} I^{ij}, \\ \{\bar{\mathcal{D}}^{\dot{\alpha}}{}_i, \bar{\mathcal{D}}^{\dot{\beta}}{}_j\} &= 4\bar{S}_{ij} \bar{M}^{\dot{\alpha}\dot{\beta}} - \epsilon_{ij} \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\mathcal{Z}}^{cd} M_{cd} - 2\epsilon_{ij} \epsilon^{\dot{\alpha}\dot{\beta}} \bar{S}^k{}_l I^l{}_k - 4\bar{\mathcal{Z}}^{\dot{\alpha}\dot{\beta}} I_{ij}, \\ \{\mathcal{D}_\alpha^i, \bar{\mathcal{D}}_{\dot{\beta}j}\} &= -2i \delta_j^i (\sigma^a)_{\alpha\dot{\beta}} \mathcal{D}_a - 2i (\sigma_a)_{\alpha\dot{\beta}} \epsilon^{abcd} (\delta_j^i G_b + i G_b^i{}_j) M_{cd} - 8G_{\alpha\dot{\beta}} I^i{}_j + 2G_{\alpha\dot{\beta}}^i{}_j \mathbb{A}, \\ [\mathcal{D}_a, \mathcal{D}_\beta^j] &= \frac{i}{2} (\sigma_a)_{\beta\dot{\gamma}} S^{jk} \bar{\mathcal{D}}^{\dot{\gamma}}{}_k - \frac{i}{2} \mathcal{Z}_{ab} (\sigma^b)_{\beta\dot{\gamma}} \bar{\mathcal{D}}^{\dot{\gamma}j} - 2i G^b (\sigma_{ba})_{\beta\dot{\gamma}} \mathcal{D}_\gamma^j - G_b^j{}_k (\sigma_a \bar{\sigma}^b)_{\beta\dot{\gamma}} \mathcal{D}_\gamma^k, \\ [\mathcal{D}_a, \bar{\mathcal{D}}^{\dot{\beta}}{}_j] &= \frac{i}{2} (\bar{\sigma}_a)^{\dot{\beta}\gamma} \bar{S}_{jk} \mathcal{D}_\gamma^k + \frac{i}{2} \bar{\mathcal{Z}}_{ab} (\bar{\sigma}^b)^{\dot{\beta}\gamma} \mathcal{D}_{\gamma j} + 2i G^b (\bar{\sigma}_{ba})^{\dot{\beta}\gamma} \bar{\mathcal{D}}^{\dot{\gamma}}{}_j + G_b^k{}_j (\bar{\sigma}_a \sigma^b)^{\dot{\beta}\gamma} \bar{\mathcal{D}}^{\dot{\gamma}}{}_k, \\ [\mathcal{D}_a, \mathcal{D}_b] &= -\frac{1}{2} R_{ab}{}^{cd} M_{cd}. \end{aligned} \quad (2.18)$$

The Riemann tensor is explicitly determined to be

$$\begin{aligned} R_{ab}{}^{cd} &= -\frac{1}{2} (\mathcal{Z}_{ab} \bar{\mathcal{Z}}^{cd} + \bar{\mathcal{Z}}_{ab} \mathcal{Z}^{cd}) + 8G^2 \delta_a^{[c} \delta_b^{d]} - 16G_{[a} G^{[c} \delta_b^{d]} \\ &\quad + 4G_{ij}^f G_f^{ij} \delta_a^{[c} \delta_b^{d]} - 8G_{[a}^{ij} G_{ij}^{[c} \delta_b^{d]} + S^{ij} \bar{S}_{ij} \delta_a^{[c} \delta_b^{d]}. \end{aligned} \quad (2.19)$$

One can show that the spacetime is conformally flat when either $Y_{ab}^- \equiv \mathcal{Z}_{ab}^-$ or $W_{ab}^- \equiv -\bar{\mathcal{Z}}_{ab}^-$ vanishes, and superconformally flat when $W_{ab}^- = 0$.

In terms of \mathcal{Z}_{ab} , the integrability conditions read

$$G^a \mathcal{Z}_{ab} = 0, \quad \epsilon^{abcd} \mathcal{Z}_{ab} \bar{\mathcal{Z}}_{cd} = 0, \quad \mathcal{Z}_{ab}^\pm \propto \bar{\mathcal{Z}}_{ab}^\pm, \quad (\mathcal{D}_a + \epsilon_a^{bcd} G_b M_{cd}) \mathcal{Z}_{ef} = 0. \quad (2.20)$$

It follows that \mathcal{Z}_{ab} is a closed complex two-form, so it possesses a complex locally-defined one-form potential

$$\mathcal{Z}_{(2)} = dC_{(1)}. \quad (2.21)$$

However, the dual of \mathcal{Z}_{ab} is not closed unless $G_a = 0$. In contrast, the dual three-forms of G_a and $G_a{}^i{}_j$ are always closed. We denote these by $H_{abc} = \epsilon_{abcd} G^d$ and $H_{abc}{}^i{}_j = \epsilon_{abcd} G^{di}{}_j$ and introduce their two-form potentials

$$H_{(3)} = dB_{(2)}, \quad H_{(3)}{}^i{}_j = dB_{(2)}{}^i{}_j. \quad (2.22)$$

These potentials will play a role in the vector and hypermultiplet actions.

3 Lorentzian backgrounds

3.1 General comments

Let us make a few modifications to the supersymmetry algebra (2.18). If one introduces a redefined vector derivative

$$\tilde{\mathcal{D}}_a := \mathcal{D}_a + \epsilon_a{}^{bcd} G_b M_{cd}, \quad (3.1)$$

corresponding to a spin connection with G -dependent torsion, then the algebra of covariant derivatives becomes

$$\begin{aligned} \{\mathcal{D}_\alpha{}^i, \mathcal{D}_\beta{}^j\} &= 4S^{ij} M_{\alpha\beta} + \epsilon^{ij}{}_{\alpha\beta} \mathcal{Z}^{cd} M_{cd} + 2\epsilon^{ij}{}_{\alpha\beta} S^k{}_l I^l{}_k - 4\mathcal{Z}_{\alpha\beta} I^{ij}, \\ \{\bar{\mathcal{D}}^{\dot{\alpha}}{}_i, \bar{\mathcal{D}}^{\dot{\beta}}{}_j\} &= 4\bar{S}_{ij} \bar{M}^{\dot{\alpha}\dot{\beta}} - \epsilon_{ij}{}^{\dot{\alpha}\dot{\beta}} \bar{\mathcal{Z}}^{cd} M_{cd} - 2\epsilon_{ij}{}^{\dot{\alpha}\dot{\beta}} \bar{S}^k{}_l I^l{}_k - 4\bar{\mathcal{Z}}^{\dot{\alpha}\dot{\beta}} I_{ij}, \\ \{\mathcal{D}_\alpha{}^i, \bar{\mathcal{D}}^{\dot{\beta}}{}_j\} &= -2i \delta_j^i (\sigma^a)_{\alpha\dot{\beta}} \tilde{\mathcal{D}}_a + 2(\sigma_a)_{\alpha\dot{\beta}} \epsilon^{abcd} G_b{}^i{}_j M_{cd} - 8G_{\alpha\dot{\beta}} I^i{}_j + 2G_{\alpha\dot{\beta}}{}^i{}_j \mathbb{A}, \\ [\tilde{\mathcal{D}}_a, \mathcal{D}_\beta{}^j] &= \frac{i}{2} (\sigma_a)_{\beta\dot{\gamma}} S^{jk} \bar{\mathcal{D}}^{\dot{\gamma}}{}_k - \frac{i}{2} \mathcal{Z}_{ab} (\sigma^b)_{\beta\dot{\gamma}} \bar{\mathcal{D}}^{\dot{\gamma}}{}_j - 4i G^b (\sigma_{ba})_{\beta\dot{\gamma}} \mathcal{D}_\gamma{}^j - G_b{}^j{}_k (\sigma_a \bar{\sigma}^b)_{\beta\dot{\gamma}} \mathcal{D}_\gamma{}^k, \\ [\tilde{\mathcal{D}}_a, \bar{\mathcal{D}}^{\dot{\beta}}{}_j] &= \frac{i}{2} (\bar{\sigma}_a)^{\dot{\beta}\dot{\gamma}} \bar{S}_{jk} \mathcal{D}_\gamma{}^k + \frac{i}{2} \bar{\mathcal{Z}}_{ab} (\bar{\sigma}^b)^{\dot{\beta}\dot{\gamma}} \mathcal{D}_\gamma{}^j + 4i G^b (\bar{\sigma}_{ba})^{\dot{\beta}\dot{\gamma}} \bar{\mathcal{D}}^{\dot{\gamma}}{}_j + G_b{}^k{}_j (\bar{\sigma}_a \sigma^b)^{\dot{\beta}\dot{\gamma}} \bar{\mathcal{D}}^{\dot{\gamma}}{}_k, \\ [\tilde{\mathcal{D}}_a, \tilde{\mathcal{D}}_b] &= -\tilde{T}_{ab}{}^c \tilde{\mathcal{D}}_c - \frac{1}{2} \tilde{R}_{ab}{}^{cd} M_{cd}, \end{aligned} \quad (3.2)$$

where the torsion and Lorentz curvature tensors are given by

$$\begin{aligned} \tilde{T}_{ab}{}^c &= -4 \epsilon_{ab}{}^{cd} G_d, \\ \tilde{R}_{ab}{}^{cd} &= -\frac{1}{2} (\mathcal{Z}_{ab} \bar{\mathcal{Z}}^{cd} + \bar{\mathcal{Z}}_{ab} \mathcal{Z}^{cd}) + 4 G_{ij}^f G_f{}^{ij} \delta_a{}^{[c} \delta_b{}^{d]} - 8 G_{[a}{}^{ij} G_{ij}{}^{c]} \delta_b{}^{d]} + S^{ij} \bar{S}_{ij} \delta_a{}^{[c} \delta_b{}^{d]}. \end{aligned} \quad (3.3)$$

The utility of this redefinition lies in the fact that the background superfields are now each covariantly constant with respect to $\tilde{\mathcal{D}}_a$, $\mathcal{D}_\alpha{}^i$ and $\bar{\mathcal{D}}^{\dot{\alpha}}{}_i$. This eliminates any algebraic obstruction to taking the background superfields (with tangent space indices) to be actually constant.⁵ Let us summarize the gauge choices for each case.

- We choose $S^{ij} = \bar{S}^{ij} = \mu v^{ij}$ with μ real and v^{ij} pseudoreal, $(v^{ij})^* = v_{ij}$, and normalized, $v^{ij} v_{jk} = \delta_k^i$. Because the superalgebra possesses only Lorentz and $\text{SO}(2) \subset \text{SU}(2)$ R -symmetry, there is no local obstruction to eliminating the $\text{U}(1)_R$ connection and aligning the $\text{SU}(2)_R$ connection along v^{ij} , consistent with $\mathcal{D}_A S^{ij} = 0$.

⁵For example, G^2 is constant so by a suitable Lorentz transformation, G^a can be taken constant when it is timelike, spacelike, or null but never vanishing. The case where G^a is null and vanishes somewhere is precluded because $\mathcal{D}_b G_a = 0$ implies the spacetime is not smooth.

- We take $G_a^{ij} = g_a v^i v^j$, with v^{ij} pseudoreal. The real constant g_a may be timelike, spacelike, or null and partially breaks Lorentz symmetry. The $SU(2)_R$ connection is again aligned along $v^i v^j$.
- We choose G_a constant and timelike, spacelike, or null. The $U(1)_R$ connection is pure gauge and can be taken to vanish. A non-zero \mathcal{Z} can be turned on in several configurations and thanks to the constraint $G^a \mathcal{Z}_{ab} = 0$ we can adopt a gauge where it is purely constant. The residual global $U(1)_R$ is broken by \mathcal{Z} .
- If only \mathcal{Z} is non-vanishing, the same discussion as above holds, but \mathcal{Z} is less constrained.

In each of these backgrounds, the algebra of covariant derivatives implies that at least some of the R -symmetry curvature tensors $R(V)_{AB}{}^{ij}$ and $R(A)_{AB}$ are non-vanishing, and so the superspace connections $\mathcal{V}_M{}^i{}_j$ and \mathcal{A}_M cannot all vanish. However, the vanishing of the component curvatures $R(V)_{ab}{}^{ij}$ and $R(A)_{ab}$ implies that the component connections $\mathcal{V}_m{}^i{}_j$ and A_m are always (at least locally) pure gauge. We will make use of this property in our constructions below by always choosing a gauge where the component $SU(2)_R$ connection vanishes. However, in a few of the cases, the $U(1)_R$ connection will possess a non-trivial holonomy.

We summarize in table 1 the resulting consistent Lorentzian backgrounds.

3.2 The supercoset construction

The full BPS geometries are at least locally isomorphic to supercoset spaces, generated by an algebra of bosonic and fermionic isometries that is isomorphic to the algebra of covariant derivatives. The cosets are obtained by quotienting out the Lorentz and R -symmetries remaining after gauge fixing the background fields to constant values. If we take these supercoset spaces as global solutions, existence of eight Killing spinors is automatically guaranteed by transitivity of the supergroup action on the background, and explicit expressions are straightforward to derive. Of course, solutions that are only locally isomorphic to supercoset spaces may be available, too, for instance in terms of orbifolds of homogeneous spaces.

Let us briefly review how the supercoset space construction works in practice and how it can be used to obtain explicit expressions for the metric and Killing spinors in particular (See for instance [23, 24]). We can consider the reciprocal superalgebra of formal generators $P_a, Q_\alpha{}^i, \bar{Q}^{\dot{\alpha}}{}_i$ satisfying the same commutation relations as $\tilde{\mathcal{D}}_a, \mathcal{D}_\alpha{}^i, \bar{\mathcal{D}}^{\dot{\alpha}}{}_i$. A full supergroup is generated by $P_a, Q_\alpha{}^i$, and $\bar{Q}^{\dot{\alpha}}{}_i$ together with any residual Lorentz and R -symmetries. Quotienting by these, we obtain a supercoset space. Representatives \mathbb{L} can be defined by translation and supersymmetry generators:

$$\mathbb{L} = \exp \left(x^a P_a + \theta^\alpha{}_i Q_\alpha{}^i + \bar{\theta}^{\dot{\alpha}}{}_i \bar{Q}^{\dot{\alpha}}{}_i \right), \tag{3.4}$$

although in practice we will make different coordinate and local gauge choices on a case by case basis. Under a supergroup transformation g , the coset representatives transform as

$$g\mathbb{L}(x, \theta, \bar{\theta}) = \mathbb{L}(x', \theta', \bar{\theta}')h(x, \theta, \bar{\theta}, g), \tag{3.5}$$

Background fields	Geometry
$S^{ij} \neq 0$	AdS_4
$G_a^i{}_j$ timelike	$\mathbb{R} \times S^3$
$G_a^i{}_j$ null	plane wave
$G_a^i{}_j$ spacelike	$\text{AdS}_3 \times \mathbb{R}$
G_a timelike	$\mathbb{R} \times S^3$
\mathcal{Z}_a^b elliptic	$\mathbb{R} \times S^3$ squashed
G_a null	plane wave
\mathcal{Z}_a^b elliptic	‘lightlike’ $S^3 \times \mathbb{R}$
\mathcal{Z}_a^b parabolic	plane wave
G_a spacelike	$\text{AdS}_3 \times \mathbb{R}$
\mathcal{Z}_a^b elliptic, $0 < \mathcal{Z} ^2 < 32 G^2$	timelike stretched $\text{AdS}_3 \times \mathbb{R}$
$ \mathcal{Z} ^2 = 32 G^2$	$\text{Heis}_3 \times \mathbb{R}$
$ \mathcal{Z} ^2 > 32 G^2$	warped ‘Lorentzian’ $S^3 \times \mathbb{R}$
\mathcal{Z}_a^b parabolic	null warped $\text{AdS}_3 \times \mathbb{R}$
\mathcal{Z}_a^b hyperbolic	spacelike squashed $\text{AdS}_3 \times \mathbb{R}$
$\mathcal{Z}_a^b \neq 0$ but $G_a = 0$	
\mathcal{Z}_a^b elliptic	$\mathbb{R}^{1,1} \times S^2$
\mathcal{Z}_a^b hyperbolic	$\text{AdS}_2 \times \mathbb{R}^2$
$\mathcal{Z}_a^b \sim$ elliptic + hyperbolic	$\text{AdS}_2 \times S^2$
\mathcal{Z}_a^b parabolic	plane wave

Table 1. Consistent Lorentzian backgrounds. As \mathcal{Z}_a^b can be decomposed as a complex linear combination of Lorentz generators, we distinguish its values in terms of orbits of the latter.

with $h(x, \theta, \bar{\theta}, g)$ a local Lorentz and R -symmetry transformation. In particular, P_a, Q_{α}^i , and $\bar{Q}^{\dot{\alpha}}_i$ generate part of the (left) isometry group. Since we have assumed absence of fermionic background fields, we can always restrict to the bosonic submanifold $\theta = \bar{\theta} = 0$, which is itself a coset space with a natural choice of representatives $L(x) \equiv \mathbb{L}|$.

The Cartan-Maurer form $\mathbb{L}^{-1}d\mathbb{L}$ can then be expanded in the superalgebra generators. The coefficients of the translation and supersymmetry generators will be chosen as the (super)vielbein, and the remaining terms give rise to spin and R -symmetry connections. Constructing covariant derivatives from these objects then reproduces (3.2).

The advantage of this procedure is that we are allowed to compute the Cartan-Maurer form in any convenient (faithful) representation of the superalgebra, which means we can calculate explicit expressions for any relevant object with ease. In particular, Killing spinors for a supercoset space are computed similarly to Killing vectors. One constructs a super-

symmetric isometry δ_Q in terms of eight constant spinors $\epsilon_{\alpha i}$ and $\bar{\epsilon}^{\dot{\alpha} i}$ via

$$\delta_Q = \mathbb{L}^{-1}(\epsilon_i Q^i + \bar{\epsilon}^i \bar{Q}_i) \mathbb{L} = \xi_i Q^i + \bar{\xi}^i \bar{Q}_i + \xi^a P_a + \frac{1}{2} \lambda^{ab} M_{ab} + \lambda^i_j I^j_i + \lambda \mathbb{A} . \quad (3.6)$$

The local parameters $\xi^A = (\xi^\alpha_i, \bar{\xi}^{\dot{\alpha} i}, \xi^a)$, λ^{ab} , λ^i_j , λ depend on x , θ , and any background fields and are parametrized linearly in terms of $\epsilon_{\alpha i}$ and $\bar{\epsilon}^{\dot{\alpha} i}$. The operation δ_Q acts on superfields as a combination of covariant superdiffeomorphisms and local gauge transformations,

$$\delta_Q = \xi^A \mathcal{D}_A + \frac{1}{2} \lambda^{ab} M_{ab} + \lambda^i_j I^j_i + \lambda \mathbb{A} . \quad (3.7)$$

The condition that δ_Q generates a superisometry is equivalent to $[\delta_Q, \mathcal{D}_A] = 0$; this leads to the Killing supervector condition (see [18] for a general discussion). As a consequence of the supercoset structure, this is satisfied automatically.

In practice, however, one only cares about how supersymmetry manifests on the bosonic manifold. Because of the absence of background fermions in the algebra, at $\theta = 0$ we have $\xi^a = \lambda^i_j = \lambda^{ab} = \lambda = 0$ and

$$\delta_Q = L^{-1}(\epsilon_i Q^i + \bar{\epsilon}^i \bar{Q}_i) L = \xi_i Q^i + \bar{\xi}^i \bar{Q}_i . \quad (3.8)$$

The functions $\xi_{\alpha i}(x)$ and $\bar{\xi}^{\dot{\alpha} i}(x)$ are the Killing spinors. Because (four-component) supercharges transform in some representation $R[\cdot]$ of the bosonic isometries,

$$\xi(x) = \epsilon R[L(x)] , \quad (3.9)$$

where ξ and ϵ are given by their appropriate (four-component) Majorana conjugates. The representation $R[L]$ can be deduced from the vector-spinor commutators of (3.2). Notice that in general it also includes a non-trivial action on R -symmetry indices.

In the next sections we provide an exhaustive classification of the supercoset spaces that give consistent global $\mathcal{N} = 2$ supersymmetric rigid backgrounds in Lorentzian signature.

3.3 AdS₄ spacetime

When $S^{ij} \equiv \mu v^{ij}$ is the only non-vanishing background field, we recover the $\mathcal{N} = 2$ AdS₄ superspace $\text{AdS}^{4|8} = \text{OSp}(2|4)/\text{SO}(3,1) \times \text{SO}(2)$. The bosonic body of the superalgebra is $\text{SO}(3,2) \times \text{SO}(2)$, corresponding to the product of the AdS₄ algebra and the residual R -symmetry $\text{SO}(2) \subset \text{SU}(2)$ generated by S^{ij} . The properties of this superalgebra and its Killing spinors were discussed long ago in [25] (see [26] and references therein for a superspace discussion), so we will not dwell on it here, except to remind that it possesses an $\mathcal{N} = 1$ subalgebra, which is evident upon performing a global $\text{SU}(2)_R$ transformation to adopt the gauge $v^{ij} = -\delta^{ij}$.

3.4 Round and squashed spatial three-spheres ($\mathbb{R} \times S^3$)

As a first non-trivial case we consider backgrounds locally isomorphic to $\mathbb{R} \times S^3$, where \mathbb{R} corresponds to the direction of a timelike Killing vector and the metric on the S^3 is static. We will show that full $\mathcal{N} = 2$ supersymmetry is compatible with a specific squashing of the S^3 . This case shall serve as a pedagogical example.

3.4.1 The round $\mathbb{R} \times S^3$ and $SU(2|2)$ supersymmetry

For the moment we assume that the S^3 possesses its round metric so that its full $SU(2) \times SU(2)$ isometry group is intact. This space admits two possible realizations of $\mathcal{N} = 2$ supersymmetry, corresponding to extending the isometry group $SU(2) \times SU(2)$ either to $SU(2|2) \times SU(2)$ or to $SU(2|1) \times SU(2|1)$.

The first realization of $\mathcal{N} = 2$ supersymmetry on the round S^3 involves the superalgebra $SU(2|2)$. This corresponds to turning on a timelike G^a only. We can adopt the Lorentz gauge $G^a = (-g, 0, 0, 0)$ for constant g , then the bosonic part of the superalgebra contains the generators $P_a = (P_0, P_I)$ with $I = 1, 2, 3$ and commutation relations

$$[P_0, P_I] = 0, \quad [P_I, P_J] = 4g \varepsilon_{IJK} P_K. \quad (3.10)$$

In addition, there are residual Lorentz generators M_{IJ} acting as

$$[M_{IJ}, P_K] = \delta_{IK} P_J - \delta_{JK} P_I. \quad (3.11)$$

Finally, there are the $SU(2)_R$ generators I_{ij} . It will be convenient to introduce the (dimensionless) $SU(2)$ generators

$$T_I \equiv \frac{1}{4g} P_I, \quad [T_I, T_J] = \varepsilon_{IJK} T_K. \quad (3.12)$$

It is apparent that the spacetime is diffeomorphic to $\mathbb{R} \times SU(2)$, the \mathbb{R} direction being generated by P_0 and the S^3 by T_I .

We can construct all relevant objects on this background exploiting its group manifold structure. Introducing a generic element L of $\mathbb{R} \times SU(2)$, the Maurer-Cartan form reads

$$\Omega = L^{-1} dL = dt P_0 + e^I P_I = dt P_0 + E^I T_I. \quad (3.13)$$

The E^I are a dimensionless left-invariant vielbein related to the physical vielbein $e^I = \frac{1}{4g} E^I$. They obey the structure equations $dE^I = -\frac{1}{2} \varepsilon_{IJK} E^J \wedge E^K$. Choosing the explicit parametrization $L = e^{tP_0} e^{\phi T_3} e^{\theta T_2} e^{\omega T_3}$ leads to the canonical choice of Euler angle coordinates on the S^3 ,

$$E^1 = -\sin \theta \cos \omega d\phi + \sin \omega d\theta, \quad E^2 = \sin \theta \sin \omega d\phi + \cos \omega d\theta, \quad E^3 = d\omega + \cos \theta d\phi, \quad (3.14)$$

with a round metric

$$ds^2 = -dt^2 + \frac{1}{16g^2} [(d\omega + \cos \theta d\phi)^2 + d\theta^2 + \sin^2 \theta d\phi^2]. \quad (3.15)$$

We choose standard ranges so that $\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$ and $\omega \in [0, 4\pi)$ with ω periodic with period 4π so that the element L covers all of $\mathbb{R} \times SU(2)$ exactly once. It will be useful to decompose L in terms of embedding coordinates for S^3 of unit radius. Defining

$$X = \cos \frac{\theta}{2} \cos \frac{\omega + \phi}{2}, \quad Y = \sin \frac{\theta}{2} \cos \frac{\omega - \phi}{2}, \quad V = \sin \frac{\theta}{2} \sin \frac{\omega - \phi}{2}, \quad W = \cos \frac{\theta}{2} \sin \frac{\omega + \phi}{2}, \quad (3.16)$$

we can write $L = e^{tP_0}[X + 2(VT_1 + YT_2 + WT_3)]$ in any spinorial representation of $SU(2)$, where the generators are constructed from a Clifford algebra and obey $T_I T_J = -\frac{1}{4}\delta_{IJ} + \frac{1}{2}\epsilon_{IJK}T_K$.

The group manifold construction extends to a full supercoset construction. From (3.2) and (3.12) we identify the superalgebra to be a central extension of $SU(2|2)$, on which the residual Lorentz group acts as external automorphisms and the central charge is P_0 . We stress that this central charge is not an internal symmetry, but rather the timelike isometry of this spacetime. Quotienting out the R -symmetry $SU(2)$ we arrive at the supermanifold⁶

$$\frac{SU(2|2)_{(P_0)}}{SU(2)_R} . \tag{3.17}$$

We can now construct the Killing spinors. The bosonic part of the supercoset (3.17) has numerator $\mathbb{R} \times SU(2) \times SU(2)_R$. The generators Q_α^i transform in the $(\mathbf{2}, \bar{\mathbf{2}})_0$ representation; this is evident by noting $[T_I, Q_\alpha^i] = -\frac{1}{2}\epsilon_{IJK}(\sigma_{JK})_\alpha^\beta Q_\beta^i$ and similarly for $\bar{Q}^{\dot{\beta}}_j$. Hence we find the appropriate representation to construct the Killing spinors:

$$R[T_I] = \frac{1}{2}\epsilon_{IJK}(\sigma_{JK})_\alpha^\beta \delta^i_j \quad R[P_0] = 0 . \tag{3.18}$$

They can be written directly in embedding coordinates (3.16), and they are given by

$$\begin{aligned} \xi_{\alpha i} &= [X - 2(W\sigma^{12} + V\sigma^{23} + Y\sigma^{31})]_\alpha^\beta \epsilon_{\beta i} \\ \bar{\xi}^{\dot{\alpha}}_i &= [X - 2(W\bar{\sigma}^{12} + V\bar{\sigma}^{23} + Y\bar{\sigma}^{31})]^{\dot{\alpha}}_{\dot{\beta}} \bar{\epsilon}^{\dot{\beta}}_i . \end{aligned} \tag{3.19}$$

These are just the well-known Killing spinors on S^3 (see for instance [27] for a general construction of Killing spinors on spheres). The R -symmetry index is untouched by the Killing spinor equation in this case. Equivalently, the superalgebra possesses an obvious $\mathcal{N} = 1$ subalgebra. Notice that the R -symmetry connections are vanishing, and all Killing spinors are time-independent.⁷ By a local Lorentz transformation we can actually take the Killing spinors to be entirely constant. This corresponds to generating the supercoset using the right isometries of S^3 .

3.4.2 The squashed S^3

The background we have just discussed has full superisometry group $SU(2|2)_{(P_0)} \times SU(2)$. The latter factor is generated by linear combinations of P_I and M_{IJ} and corresponds to the right isometries of S^3 . Since they do not affect the supercharges, we may ask if it is possible to deform the S^3 geometry in a way that breaks only its right isometries, obtaining a squashed $\mathbb{R} \times S^3$ with full $\mathcal{N} = 2$ supersymmetry.

If we try to break all the right isometries, the requirement of a smooth limit to a flat space supersymmetry algebra uniquely fixes the normalizations of the P_I generators to match those of the previous section. We hence end up back to the round S^3 geometry.

⁶In principle, we could also include the $SU(2)$ Lorentz symmetries in the numerator, but they would be factored out in the denominator.

⁷We have redefined the R -symmetry connections with respect to conformal supergravity, as described in footnote 4. Our statement is unambiguous in the sense that a theory coupled to this background need not have (full) $U(2)_R$ symmetry.

The only possibility is therefore to break the right $SU(2)$ to $U(1)$. In fact, such a background is realized by turning on \mathcal{Z} along the S^3 . The resulting supercoset space is generated by $SU(2|2)$ with the addition of a second central charge. The P_I generators turn out to be a linear combination of the left S^3 isometries and this central charge, which results in a change in the normalizations of the physical vielbein that gives rise to a squashed S^3 . Explicitly, we have $G^a = (-g, 0, 0, 0)$ and gauge fix $\mathcal{Z}_{ab} = 4\lambda\delta_{ab}^{12}$ so that $|\mathcal{Z}|^2 \equiv \mathcal{Z}_{ab}\bar{\mathcal{Z}}^{ab} = 8\lambda^2$. The bosonic part of the supercoset space is now generated by P_a and M_{12} , in particular

$$[P_1, P_2] = 4gP_3 + 4\lambda^2M_{12}, \quad [P_3, P_1] = 4gP_2, \quad [P_2, P_3] = 4gP_1. \quad (3.20)$$

The time translation P_0 commutes with all generators. We define $v = \left(1 + \frac{\lambda^2}{4g^2}\right)^{-1}$ and introduce dimensionless generators T_I satisfying the $SU(2)$ algebra as before, $[T_I, T_J] = \epsilon_{IJK}T_K$, as well as a second central charge U :

$$T_{1,2} \equiv \frac{\sqrt{v}}{4g}P_{1,2}, \quad T_3 \equiv v\left(\frac{1}{4g}P_3 + \frac{\lambda^2}{4g^2}M_{12}\right), \quad U \equiv \frac{1}{4g}P_3 - M_{12}, \quad (3.21)$$

The bosonic background is still topologically $\mathbb{R} \times S^3$, as we can use P_0 and T_I to generate it. However, since P_I do not close onto themselves it is more convenient to consider the whole isometry group and quotient out the only residual Lorentz generator M_{12} , leading to the coset space

$$\mathbb{R} \times \frac{SU(2) \times U(1)_U}{U(1)_M}. \quad (3.22)$$

The form of the quotient already suggests the nature of the squashing: regarding S^3 as the Hopf fibration of an S^1 over $S^2 \simeq SU(2)/U(1)_M$, we rescale the metric along the fiber as determined by the ratio of T_3 and U in $U(1)_M$. The Cartan-Maurer form can be expanded as earlier, recalling that the physical vielbein is given by the coefficients of P_a so that

$$e^0 = dt, \quad e^3 = v\frac{1}{4g}E^3, \quad e^{1,2} = \sqrt{v}\frac{1}{4g}E^{1,2}, \quad (3.23)$$

in terms of the ‘round’ vielbein E^I defined by $\Omega = dt P_0 + E^I T_I$. We can use the same coset representative as in the unsquashed case. As anticipated, the metric on the S^3 is squashed due to the different normalization of the physical vielbein along the fiber S^1 . For our choice of coordinates we obtain

$$ds^2 = -dt^2 + \frac{v}{16g^2} [d\theta^2 + \sin^2\theta d\omega^2 + v(d\phi + \cos\theta d\omega)^2]. \quad (3.24)$$

Notice that $0 < v \leq 1$, so that we are only allowed to squash, but not stretch, along the fiber. This constraint is a consequence of $\mathcal{N} = 2$ supersymmetry in Lorentzian signature and will be relaxed in the Euclidean case.

Computing Killing spinors using the supercoset construction proceeds as before, except now the supergroup possesses two central charges, P_0 and U ; we denote it $SU(2|2)_{(U,P_0)}$. The supercoset space is now

$$\frac{SU(2|2)_{(U,P_0)}}{SU(2)_R \times U(1)_M}. \quad (3.25)$$

The supersymmetry generators mix chirality under the action of P_a , so it will be convenient to introduce a four-component spinor and associated Killing spinors:

$$Q_{\hat{\alpha}^i} = \begin{pmatrix} Q_{\alpha^i} \\ \epsilon^{ij} \bar{Q}_{\hat{\alpha}^j} \end{pmatrix}, \quad \xi^{\hat{\alpha}^i} = \left(\xi^{\alpha^i}, -\epsilon_{ij} \bar{\xi}^{\hat{\alpha}^j} \right). \quad (3.26)$$

Then the bosonic generators are represented as

$$R[P_a] = -2i G^b \gamma_5 \gamma_{ba} + \frac{1}{4} (\mathcal{Z}_{ab}(1 + \gamma_5) - \bar{\mathcal{Z}}_{ab}(1 - \gamma_5)) \gamma^b, \quad R[M_{12}] = -\frac{1}{2} \gamma_{12} \quad (3.27)$$

and the Killing spinors are $\xi^{\hat{\alpha}^i} = \epsilon^{\hat{\beta}^i} R[L]_{\hat{\beta}^i}^{\hat{\alpha}^i}$. Notice that we can still write $R[L] = R[X + 2(VT_1 + YT_2 + WT_3)]$ with embedding coordinates defined in (3.16) for the round S^3 . Using standard Weyl notation the Killing spinors can then be expressed as

$$\xi_{\alpha^i} = [X\mathbb{I}_2 - 2(W\sigma^{12} + \sqrt{v}V\sigma^{23} + \sqrt{v}Y\sigma^{31})]_{\alpha^{\beta}} \epsilon_{\beta i} + i\sqrt{v} \frac{\lambda}{2g} (Y\sigma^1 - V\sigma^2)_{\alpha\hat{\beta}} \bar{\epsilon}^{\hat{\beta}^i}, \quad (3.28)$$

with $\bar{\epsilon}^{\hat{\alpha}^i}$ given by complex conjugation. When λ is non-vanishing, there is non-trivial mixing between the different R -symmetry components of the Killing spinors. This implies that there is no truncation to an $\mathcal{N} = 1$ subalgebra for the case of a squashed S^3 . The right isometries of S^3 being partially broken, we cannot make the Killing spinors constant by a local gauge transformation. However, notice that they are still time-independent with vanishing R -symmetry connections.

For later use, we will need explicit expressions for the potentials of the background fields. The \mathcal{Z} two-form admits a globally defined potential $C_{(1)}$, given by

$$C_{(1)} = -\frac{1}{2} \frac{\lambda}{4g^2 + \lambda^2} (\cos \theta d\phi + d\omega) = -\frac{\lambda}{2g} e^3. \quad (3.29)$$

Associated with G^a we have its dual two-form potential

$$B_{(2)} = \frac{v^2}{64g^2} \times \begin{cases} (\cos \theta - 1) d\omega d\phi & \theta \in [0, \pi/2], \\ (\cos \theta + 1) d\omega d\phi & \theta \in [\pi/2, \pi]. \end{cases} \quad (3.30)$$

3.4.3 $SU(2|1) \times SU(2|1)$ supersymmetry

As anticipated, there is a second way to realize $\mathcal{N} = 2$ supersymmetry on $\mathbb{R} \times S^3$. This is associated with a nonvanishing $G_a{}^i{}_j$ field, breaking the $SU(2)_R$ symmetry and possibly turning on a non-trivial axial $U(1)_R$ connection. Let us choose $G_a{}^i{}_j \equiv ig_a(\sigma_3)^i{}_j$ with $g^a = (-g, 0, 0, 0)$. Noting that $(\sigma_3)^i{}_j = (-)^{i+1} \delta^i{}_j$, the fermionic part of the superalgebra is

$$\{Q_{\alpha^i}, Q_{\beta^j}\} = \{\bar{Q}^{\hat{\alpha}^i}, \bar{Q}^{\hat{\beta}^j}\} = 0, \quad \{Q_{\alpha^i}, \bar{Q}_{\hat{\beta}^j}\} = -2i \delta^i{}_j \Delta_{\alpha\hat{\beta}}^{(i)} \quad (3.31)$$

where the eight generators $\Delta_a^{(i)}$ with $a = 0, \dots, 3$ and $i = \underline{1}, \underline{2}$ can be written

$$\Delta_a^{(i)} \equiv P_a + (-)^i g (\epsilon_{0abc} M^{bc} + \delta_a^0 \mathbb{A}). \quad (3.32)$$

We see that the rest of the superalgebra similarly decomposes,

$$\begin{aligned} [\Delta_a^{(i)}, Q_\alpha^j] &= (-)^{i+1} \delta^{ij} 2i g \left(\delta_a^0 \delta_\alpha^\beta - 2(\sigma_{0a})_\alpha^\beta \right) Q_\beta^i, \\ [\Delta_a^{(i)}, \Delta_b^{(j)}] &= (-)^i \delta^{ij} 4g \epsilon_{0ab}{}^c \Delta_c^{(i)}. \end{aligned} \tag{3.33}$$

This superalgebra is just $SU(2|1) \times SU(2|1)$ with each copy labeled by i . Up to normalizations, each supergroup is generated by bosonic elements $\Delta_a^{(i)}$ corresponding to $SU(2) \times U(1)$ and odd elements Q_α^i and $\bar{Q}^{\dot{\alpha}i}$. The temporal generator P_0 and the $U(1)_R$ generator \mathbb{A} correspond respectively to the antidiagonal and diagonal combinations of the two $U(1)$ factors $\Delta_0^{(i)}$. The surviving $SO(3)$ generators of the Lorentz group correspond to the diagonal $SU(2)$ generated by $-\Delta_I^{(1)} + \Delta_I^{(2)}$.

In order to construct the spatial part of the coset space, it is sufficient to take group elements of either $SU(2)$ factor as coset representatives. We will choose the second $SU(2)$, so that

$$T_I \equiv \frac{1}{4g} \Delta_I^{(2)}, \quad [T_I, T_J] = \epsilon_{IJK} T_K, \quad [\Delta_0^{(2)}, T_I] = 0, \tag{3.34}$$

and construct the coset representatives as in the previous sections, but using $\Delta_0^{(2)}$ rather than P_0 for the time direction for reasons that will be apparent soon. This introduces a $U(1)_R$ transformation that can be trivially quotiented away at the bosonic level, but will become relevant in the supercoset construction. We obtain the same round vielbein and metric as in the previous sections.

The situation with the Killing spinors is quite interesting. Let us denote them $\hat{\xi}$ in this section. The odd elements of the supergroup transform in the representation

$$(\mathbf{2}, \mathbf{1})_{+1/2,0} + (\bar{\mathbf{2}}, \mathbf{1})_{-1/2,0} + (\mathbf{1}, \mathbf{2})_{0,+1/2} + (\mathbf{1}, \bar{\mathbf{2}})_{0,-1/2} \tag{3.35}$$

of $U(2)_{(1)} \times U(2)_{(2)}$, with the natural Majorana condition giving eight supercharges. The same representations hold for the Killing spinors, so if we generate the coset representatives using only $U(2)_{(2)}$ generators, the Weyl spinors $\hat{\xi}_\pm$ and their complex conjugates will be entirely constant. We will shortly prove this explicitly by construction, but it helps to motivate this first by taking a look at the Killing spinor equations,

$$(\mathcal{D}_a + g\epsilon_{0acd}\sigma^{cd} + ig\delta_a^0)\hat{\xi}_\pm = 0, \quad (\mathcal{D}_a - g\epsilon_{0acd}\sigma^{cd} - ig\delta_a^0)\hat{\xi}_\pm = 0. \tag{3.36}$$

The background field $G_a{}^i{}_j$ couples asymmetrically to the two spinors. From the Maurer-Cartan form we find that the $U(1)_R$ connection is given by

$$A = g e^0 = g dt, \tag{3.37}$$

therefore the $U(1)$ connection within \mathcal{D}_a cancels the additional ig factor in the $\hat{\xi}_\pm$ equation. Similarly, we find that our choice of vielbein has led to a spin connection within \mathcal{D}_a that cancels the additional $g\epsilon_{0acd}\sigma^{cd}$ term.

We can now see this explicitly from the coset construction. The generators T_I and $\Delta_0^{(2)}$ commute with Q_α^1 , implying that the associated Killing spinors $\hat{\xi}$ are constants,

$$\hat{\xi}_{\alpha 1} = \epsilon_{\alpha 1}, \quad \bar{\hat{\xi}}^{\dot{\alpha} 1} = \epsilon^{\dot{\alpha} 1}. \tag{3.38}$$

However, the second set of Killing spinors are non-trivial. Noting that now the time generator $\Delta_0^{(2)}$ is non-trivially represented, we obtain Killing spinors

$$\hat{\xi}_{\alpha\bar{2}} = e^{2igt}\xi_{\alpha\bar{2}}, \quad \bar{\hat{\xi}}^{\dot{\alpha}\bar{2}} = e^{-2igt}\bar{\xi}^{\dot{\alpha}\bar{2}}, \quad (3.39)$$

with $\xi_{\alpha\bar{2}}, \bar{\xi}^{\dot{\alpha}\bar{2}}$ defined in (3.19). It is easy to see that the t -dependence in the second set of Killing spinors can be shuffled into the first set via a $U(1)_R$ gauge transformation, and the same is true for the additional Lorentz factors.

Despite the fact that we are working in Lorentzian signature, it is instructive to see what happens if we compactify the time direction. Periodic Killing spinors are allowed only for $t \simeq t + n\pi/g$ for integer n . A non-trivial Wilson line for A is generated for odd n . For even n , the gauge field could be turned off by a gauge transformation, leaving $\hat{\xi}_{\underline{1}}$ and $\hat{\xi}_{\underline{2}}$ both with t -dependent factors e^{-igt} and e^{+igt} , respectively. These conditions are similar to the findings of [2, 5] in $\mathcal{N} = 1$. Analogous conditions arise for the other backgrounds generated by G^i_j .

The dual potential associated with G^i_j is simply $B_{(2)}^i_j = B_{(2)}(i\sigma_3)^i_j$, with $B_{(2)}$ given by (3.30).

3.5 Warped AdS₃ spaces (wAdS₃ × ℝ)

Most of our discussion of the three-sphere can be repeated for backgrounds locally isomorphic to AdS₃ × ℝ, so we defer most of the details to appendix C and discuss mainly the differences. These geometries are sourced by spacelike G or G^i_j and give rise to SU(1,1|2) and SU(1,1|1)² superalgebras respectively. In the former case, the AdS₃ space can be warped by \mathcal{Z} . Without warping the two supercoset spaces are

$$\frac{SU(1,1|2)}{SU(2)_R} \quad \text{and} \quad \frac{SU(1,1|1) \times SU(1,1|1)}{U(1)_R}. \quad (3.40)$$

The former case includes a central charge isometry generating the flat ℝ direction. In the latter case, as with the SU(2|1)² sphere, the spacetime can be entirely generated by the even part of one SU(1,1|1) factor, leaving half of the Killing spinors entirely constant. Also in this case, if we compactify the flat direction to a circle we find similar restrictions on its radius, and a non-trivial connection for the axial U(1)_R might be required in order to have globally defined Killing spinors. No such restriction is necessary for SU(1,1|2) supersymmetry, where Killing spinors can be made entirely constant by a choice of local Lorentz gauge and for vanishing R -symmetry connections, analogously to the SU(2|2) sphere.

In each of these situations, SU(1,1) generators T_I with $I = 0, 1, 2$ can be defined and we can pick as group or coset representative

$$L = e^{\phi T_2} e^{\rho T_1} e^{\tau T_0}. \quad (3.41)$$

We can regard AdS₃ ≃ SU(1,1) either as a timelike S^1 fibered over an H^2 or as an H^1 fibered over AdS₂. These two possible fibers correspond to the rightmost and leftmost factors of L respectively. Warped AdS₃ comes in different kinds depending on which fiber

we deform, which is reflected in the orientation of \mathcal{Z} along the spacetime. The unwarped metric reads

$$ds^2 = \frac{1}{16g^2}(-d\tau^2 + d\phi^2 + d\rho^2 - 2d\tau d\phi \sinh \rho) + dz^2, \quad (3.42)$$

where g^2 is the norm of the background vector. Two convenient orthonormal frames can be constructed from the left- and right-invariant Cartan-Maurer forms on AdS_3 , defined as $\Omega = L^{-1}dL$ and $\Omega' = Ld(L^{-1})$: they read

$$\begin{aligned} E^0 &= d\tau + \sinh \rho d\phi, & E'^0 &= -\cosh \phi \cosh \rho d\tau - \sinh \phi d\rho, \\ E^1 &= \cos \tau d\rho + \sin \tau \cosh \rho d\phi, & E'^1 &= -\cosh \phi d\rho - \sinh \phi \cosh \rho d\tau, \\ E^2 &= -\sin \tau d\rho + \cos \tau \cosh \rho d\phi, & E'^2 &= -d\phi + \sinh \rho d\tau. \end{aligned} \quad (3.43)$$

Both sets satisfy the same structure equations $2dE^I = -\eta^{IJ}\epsilon_{JKL}E^K \wedge E^L$ with $\eta_{IJ} = \text{diag}(-1, 1, 1)$. Each choice privileges a different Hopf fibration of AdS_3 . The coordinates we use can be related to embedding coordinates defining AdS_3 as the surface $X^2 - Y^2 - V^2 - W^2 = 1$, parameterized as

$$\begin{aligned} X &= \cos \frac{\tau}{2} \cosh \frac{\rho}{2} \cosh \frac{\phi}{2} - \sin \frac{\tau}{2} \sinh \frac{\rho}{2} \sinh \frac{\phi}{2}, \\ Y &= \cos \frac{\tau}{2} \sinh \frac{\rho}{2} \sinh \frac{\phi}{2} + \sin \frac{\tau}{2} \cosh \frac{\rho}{2} \cosh \frac{\phi}{2}, \\ V &= \cos \frac{\tau}{2} \sinh \frac{\rho}{2} \cosh \frac{\phi}{2} + \sin \frac{\tau}{2} \cosh \frac{\rho}{2} \sinh \frac{\phi}{2}, \\ W &= \cos \frac{\tau}{2} \cosh \frac{\rho}{2} \sinh \frac{\phi}{2} - \sin \frac{\tau}{2} \sinh \frac{\rho}{2} \cosh \frac{\phi}{2}. \end{aligned} \quad (3.44)$$

The τ coordinate has periodicity 4π for global AdS_3 .

3.5.1 Timelike stretched $\text{AdS}_3 \times \mathbb{R}$

Let us now focus on $\text{SU}(1, 1|2)$ supersymmetry and introduce warping. This is allowed by breaking the right $\text{SU}(1, 1)$ isometries of AdS_3 to a one-dimensional subgroup determined by \mathcal{Z} . The first case we consider corresponds to a timelike stretching of AdS_3 , obtained by turning on \mathcal{Z} along the H^2 base of the fibration $S^1 \hookrightarrow \text{AdS}_3 \rightarrow H^2$. Defining $G^2 = g^2$, $|\mathcal{Z}|^2 = 8\lambda^2$, we impose $\lambda^2 < 4g^2$. We will later discuss the geometries obtained for larger λ . The supercoset space is

$$\frac{\text{SU}(1, 1|2)_{(U, G \cdot P)}}{\text{U}(1)_M \times \text{SU}(2)_R} \quad (3.45)$$

analogously to the squashed $\mathbb{R} \times S^3$ case, the denominator $\text{U}(1)_M$ mixes the compact isometry of $\text{SU}(1, 1)$ with a central charge U , generating the warping. The other central charge $G \cdot P = G^a P_a$ corresponds to the flat \mathbb{R} direction (we can gauge fix it to P_3 for definiteness). The resulting metric reads

$$ds^2 = \frac{v}{16g^2}[-v(d\tau + \sinh \rho d\phi)^2 + d\rho^2 + \cosh^2 \rho d\phi^2] + dz^2, \quad (3.46)$$

with warping parameter $v \equiv 1/(1 - \frac{\lambda^2}{4g^2}) \geq 1$. As a bosonic background, timelike ‘squashed’ AdS_3 with $0 < v < 1$ would be also possible. Supersymmetry restricts us to timelike ‘stretching’ only. It is known that for any value of the stretching this space contains closed timelike curves. In particular, $v = 2$ corresponds to Gödel spacetime [28, 29].

3.5.2 Spacelike squashed $\text{AdS}_3 \times \mathbb{R}$

Now let us take \mathcal{Z} along an AdS_2 subspace of AdS_3 and define $|\mathcal{Z}|^2 = -8\lambda^2$. The right isometries are broken to the $\text{SO}(1, 1)$ that preserves \mathcal{Z} and the supercoset reads

$$\frac{\text{SU}(1, 1|2)_{(U, G \cdot P)}}{\text{SO}(1, 1)_M \times \text{SU}(2)_R}. \quad (3.47)$$

This time the spacetime is squashed along the non-compact fiber over AdS_2 , with metric

$$ds^2 = \frac{v}{16g^2} [-\cosh^2 \rho d\tau^2 + d\rho^2 + v(d\phi + \sinh \rho d\tau)^2] + dz^2, \quad (3.48)$$

with $0 < v \equiv 1/(1 + \frac{\lambda^2}{4g^2}) \leq 1$. In this case supersymmetry only allows for squashing rather than stretching.

3.5.3 Lightlike warped $\text{AdS}_3 \times \mathbb{R}$

In this final case, we take \mathcal{Z} along a null surface determined for instance as the coset space AdS_3/N_- , the denominator being an (everywhere) null isometry. We use the notation N_T for the monoparametric subgroup of a parabolic generator T . Unsurprisingly, the supercoset space becomes

$$\frac{\text{SU}(1, 1|2)_{(U, G \cdot P)}}{N_M \times \text{SU}(2)_R}, \quad (3.49)$$

where both U and the residual Lorentz generator are null. In our usual set of global coordinates the metric reads

$$ds^2 = \frac{1}{16g^2} [-d\tau^2 + d\phi^2 + d\rho^2 - 2\sinh \rho d\tau d\phi - \frac{\lambda^2}{4g^2} e^{2\phi} (d\rho + \cosh \rho d\tau)^2] + dz^2. \quad (3.50)$$

A shift in ϕ , which corresponds to one of the isometries of AdS_3 broken by \mathcal{Z} , can absorb the absolute value of the squashing parameter. The sign of the warping, however, is fixed by supersymmetry. It can be convenient to rewrite this metric in Poincaré coordinates:

$$ds^2 = \frac{1}{4g^2} \left(\frac{dr^2}{r^2} + \frac{dx_+ dx_-}{r^2} - \frac{\lambda^2}{4g^2} \frac{dx_+^2}{r^4} \right) + dz^2. \quad (3.51)$$

3.6 $\text{AdS}_2 \times S^2$ spacetimes and $\text{D}(2, 1; \alpha)$

Another rich and interesting spacetime geometry is $\text{AdS}_2 \times S^2$. These geometries are sourced by a complex \mathcal{Z} flux alone, excluding the case in which it is entirely supported on a null hypersurface. Then \mathcal{Z} is generally a complex linear combination of two real forms wrapping a timelike and a spacelike hypersurface respectively. These forms source the $\text{AdS}_2 \times S^2$ background, and when either vanishes the associated factor in the geometry is flattened. The $\mathcal{N} = 2$ supersymmetric extensions of $\text{AdS}_2 \times S^2$ are real forms of the $\text{D}(2, 1; \alpha)$ Lie superalgebras, where α is basically the ratio of the radii of AdS_2 and S^2 [14].

It can be useful to see how the $\text{D}(2, 1; \alpha)$ superalgebra arises. Let us gauge-fix the $\text{U}(1)_R$ and local Lorentz gauge so that

$$\mathcal{Z}_{ab} = 2i\lambda_+ \delta_{ab}^{12} - 2\lambda_- \delta_{ab}^{03}, \quad \lambda_{\pm} \in \mathbb{R}. \quad (3.52)$$

The Riemann tensor then reads $R_{ab}{}^{cd} = 4(\lambda_-^2 \delta_{ab}^{03} \delta_{03}^{cd} - \lambda_+^2 \delta_{ab}^{12} \delta_{12}^{cd})$, with curvature radii $1/|\lambda_-|$, $1/|\lambda_+|$ for AdS₂ and S^2 respectively. The isometry algebra is

$$[P_a, P_b] = -2\lambda_-^2 \delta_{ab}^{03} M_{03} + 2\lambda_+^2 \delta_{ab}^{12} M_{12}, \quad (3.53)$$

corresponding to $SU(1,1) \times SU(2)$. For definiteness, an appropriate choice of coordinates gives us the explicit metric

$$ds^2 = \frac{1}{\lambda_-^2} (-d\tau^2 \cosh \rho + d\rho^2) + \frac{1}{\lambda_+^2} (d\theta^2 + \sin^2 \theta d\phi^2). \quad (3.54)$$

The supercharges transform in the $(\mathbf{2}, \mathbf{2}, \mathbf{2})$ of $SU(1,1) \times SU(2) \times SU(2)_R$. This can be made explicit by a similarity transformation on the four-component spinor $Q_{\hat{\alpha}}{}^i$ of (3.26):

$$Q_{\tilde{a}\tilde{\alpha}}{}^i \equiv S_{\tilde{a}\tilde{\alpha}}{}^{\hat{\alpha}} Q_{\hat{\alpha}}{}^i, \quad S \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad (3.55)$$

where \tilde{a} , $\tilde{\alpha}$ are fundamental indices of $SU(1,1)$ and $SU(2)$ respectively. The new set of gamma matrices reads

$$\tilde{\gamma}_0 = i\sigma_1 \otimes \sigma_3, \quad \tilde{\gamma}_1 = -\mathbb{I}_2 \otimes \sigma_2, \quad \tilde{\gamma}_2 = \mathbb{I}_2 \otimes \sigma_1, \quad \tilde{\gamma}_3 = \sigma_2 \otimes \sigma_3, \quad \tilde{\gamma}_5 = \sigma_3 \otimes \sigma_3. \quad (3.56)$$

After some algebra, we obtain the commutation relations

$$\begin{aligned} \{Q_{\tilde{a}\tilde{\alpha}}{}^i, Q_{\tilde{b}\tilde{\beta}}{}^j\} &= 2(\lambda_+ + \lambda_-) \epsilon_{\tilde{a}\tilde{b}} \epsilon_{\tilde{\alpha}\tilde{\beta}} I_{ij} - 2\lambda_+ \epsilon_{\tilde{a}\tilde{b}} T_{\tilde{\alpha}\tilde{\beta}} \epsilon_{ij} - 2\lambda_- T_{\tilde{a}\tilde{b}} \epsilon_{\tilde{\alpha}\tilde{\beta}} \epsilon_{ij}, \\ [T_{\tilde{a}\tilde{b}}, Q_{\tilde{c}\tilde{\alpha}}{}^i] &= \epsilon_{\tilde{c}(\tilde{a}} Q_{\tilde{b})\tilde{\alpha}}{}^i, \quad [T_{\tilde{\alpha}\tilde{\beta}}, Q_{\tilde{a}}{}^{\tilde{\gamma}}{}^i] = \epsilon_{\tilde{\gamma}(\tilde{\alpha}} Q_{\tilde{a}|\tilde{\beta})}{}^i, \end{aligned} \quad (3.57)$$

where the conventions for all ϵ symbols are the same as for standard spinor indices and we have defined

$$T_{\tilde{a}\tilde{b}} \equiv \begin{pmatrix} \frac{i}{\lambda_-} (P_0 + P_3) & M_{03} \\ M_{03} & \frac{i}{\lambda_-} (-P_0 + P_3) \end{pmatrix}, \quad T_{\tilde{\alpha}\tilde{\beta}} \equiv \begin{pmatrix} \frac{1}{\lambda_+} (P_2 + iP_1) & iM_{12} \\ iM_{12} & \frac{1}{\lambda_+} (P_2 - iP_1) \end{pmatrix}, \quad (3.58)$$

so that they all have a commutator with the supercharges analogous to I_{ij} . The reality conditions on the generators are

$$(Q_{\tilde{a}\tilde{\alpha}}{}^i)^* = (\sigma_3)_{\tilde{a}}{}^{\tilde{b}} \epsilon^{\tilde{\alpha}\tilde{\beta}} Q_{\tilde{b}\tilde{\beta}}{}^i, \quad (T_{\tilde{a}\tilde{b}})^* = -(\sigma_3)_{\tilde{a}}{}^{\tilde{c}} (\sigma_3)_{\tilde{b}}{}^{\tilde{d}} T_{\tilde{c}\tilde{d}}, \quad (T_{\tilde{\alpha}\tilde{\beta}})^* = \epsilon^{\tilde{\alpha}\tilde{\gamma}} \epsilon^{\tilde{\beta}\tilde{\delta}} T_{\tilde{\gamma}\tilde{\delta}}. \quad (3.59)$$

The (anti)commutation relations we have just uncovered are the ones of the real form of $D(2,1;\alpha)$ with Lie subalgebra $SU(1,1) \times SU(2) \times SU(2)_R$. The parameter α can be defined as $\alpha \equiv \lambda_+/\lambda_-$ for $\lambda_- \neq 0$. We will slightly abuse the notation and include the case $\lambda_- = 0, \lambda_+ \neq 0$ as $\alpha = \infty$. There are four special subcases of the general algebra (see the discussion in [14]):⁸

⁸We use the same symbol for the real and complex form of $D(2,1;\alpha)$. Other real forms exist, but we will not encounter them. Because the algebra is real, α must be real as well.

- $D(2, 1; 1) \simeq \text{OSp}(4^*|2)$ corresponds to a superspace built on $\text{AdS}_2 \times S^2$ with equal radii. The space is superconformally flat, as reflected by the vanishing of the tensor $W_{ab}^+ = -Z_{ab}^+$. It can be embedded in the superconformal group $\text{SU}(2, 2|2)$.
- $D(2, 1; 0) \simeq \text{SU}(1, 1|2)$, where the Lie subalgebra reduces to $\text{SU}(1, 1) \times \text{SU}(2)_R$ and the space is $\text{AdS}_2 \times \mathbb{R}^2$. The original $\text{SU}(2)$ isometries of the sphere contract to $\text{ISO}(2)$, the compact generator acting as an external automorphism.
- $D(2, 1; -1) \simeq \text{SU}(1, 1|2)$, where this time the $\text{SU}(2)$ factor corresponds to the S^2 isometries and $\text{AdS}_2 \times S^2$ have the same radii. The $\text{SU}(2)_R$ group acts as external automorphisms. This is the superalgebra obtained as the near horizon limit of supersymmetric black holes in $\mathcal{N} = 2$, $D = 4$ supergravity.
- $D(2, 1; \infty) \simeq \text{SU}(2|2)$, where AdS_2 is flattened to $\mathbb{R}^{1,1}$ and $\text{SU}(1, 1)$ is contracted to $\text{ISO}(1, 1)$, an $\text{SO}(1, 1)$ subgroup acting as an external automorphism.

All other values of α correspond to $\text{AdS}_2 \times S^2$ with different radii.⁹ Notice that for each choice of radii, there are two distinct supersymmetry algebras depending on the sign of α . The supercosets built from these algebras can be encoded in the general expression

$$\frac{D(2, 1; \alpha)}{\text{SO}(1, 1) \times \text{U}(1) \times \text{SU}(2)_R}, \tag{3.60}$$

with the exception of the cases $\alpha = 0, -1, \infty$, where the numerator includes respectively the $\text{ISO}(2)$, $\text{SU}(2)_R$ or $\text{ISO}(1, 1)$ external automorphisms as a semidirect product. The coset construction allows us to define Killing spinors with ease as usual, see appendix C.

3.7 Other geometries

3.7.1 Warped Lorentzian $S^3 \times \mathbb{R}$

Our analysis of rigid backgrounds also gives rise to more exotic solutions beyond the spaces discussed so far. A first interesting class of less conventional geometries is given by $\text{SU}(2)$ group manifolds with non-Euclidean metrics. Regarding $S^3 \sim \text{SU}(2)$ as a Hopf fibration, these geometries correspond to taking the fiber circle to be either timelike or lightlike.

The first case we analyze is an S^3 with Lorentzian metric. Such a manifold is easily defined using the standard left-invariant one-forms of $\text{SU}(2)$ (3.14) and treating the one associated with the fiber as timelike. This space corresponds to a radial section of Taub-NUT [30].

The background is sourced by the same field configuration as timelike stretched $\text{AdS}_3 \times \mathbb{R}$, i.e. a spacelike G and a Z two-form along a timelike hypersurface. However, we now take $\lambda^2 > 4g^2$. This induces a change of topology from timelike stretched AdS_3 to a compact space with $\text{SU}(2) \times \text{U}(1)$ isometries. The supercoset is formally identical to the standard squashed $\mathbb{R} \times S^3$:

$$\frac{\text{SU}(2|2)_{(U, G \cdot P)}}{\text{SU}(2)_R \times \text{U}(1)_M}, \tag{3.61}$$

⁹There are two other isomorphisms. First, $\alpha = -2$ gives rise to an $\text{OSp}(4^*|2)$ algebra, however it exchanges the roles of the spatial $\text{SU}(2)$ with $\text{SU}(2)_R$. As a consequence, this case is not superconformally flat, as is clear from the fact that T_{ab}^+ is non-vanishing. Second, $\alpha = -1/2$ gives rise to an $\text{OSp}(4|2)$ algebra.

though now the deformed S^1 fiber is regarded as the timelike direction. The metric reads

$$ds^2 = \frac{v}{4g^2} [-v(d\omega + \cos\theta d\phi)^2 + d\theta^2 + \sin^2\theta d\phi^2] + dz^2, \quad v \equiv \left(\frac{\lambda^2}{4g^2} - 1 \right)^{-1} > 0. \quad (3.62)$$

Such a geometry admits only $SU(2) \times U(1)$ as an isometry group, regardless of the value of v , because two of the (right) isometries are broken by the choice of Lorentzian signature.

3.7.2 Lightlike $S^3 \times \mathbb{R}$

In 4D it is also possible to treat the Hopf fiber of S^3 as a lightlike direction, obtaining a metric of the form

$$ds^2 = \frac{1}{4\lambda^2} (2du(d\omega + \cos\theta d\phi) + d\theta^2 + \sin^2\theta d\phi^2). \quad (3.63)$$

The resulting space is (locally) isomorphic to a ‘lightlike’ $S^3 \times \mathbb{R}$. Such a space arises from our analysis if we take G null (but everywhere non-vanishing) and \mathcal{Z} spacelike with $|\mathcal{Z}|^2 \equiv 8\lambda^2$. Setting $\mathcal{Z} = 0$ yields a different geometry which is not topologically a sphere. The supercoset space is again (3.61). Killing spinors are constructed in appendix C. A full analysis of the global properties of this space goes beyond the scope of this paper, but we note that a global lightlike $S^3 \times \mathbb{R}$ has closed null curves but no closed timelike curves.

3.7.3 ‘Overstretched’ AdS_3 ($Heis_3 \times \mathbb{R}$)

There is a threshold case between timelike stretched $AdS_3 \times \mathbb{R}$ and the Lorentzian $S^3 \times \mathbb{R}$, obtained for spacelike G and spacelike \mathcal{Z} with $\lambda^2 = 4g^2$. It corresponds to a non-semisimple contraction of the $SU(1, 1|2)$ and $SU(2|2)$ superalgebra with $Heis_3 \times U(1)_M \times \mathbb{R}_{G,P}$ bosonic isometries. The geometry is in fact the group manifold $Heis_3 \times \mathbb{R}$ where the central charge corresponds to the timelike isometry. The metric is

$$ds^2 = -(dt + 2g x dy - 2g y dx)^2 + dx^2 + dy^2 + dz^2. \quad (3.64)$$

This metric can also be interpreted as a rotating relativistic fluid.

3.7.4 Plane waves

Finally, there is a class of plane wave geometries admitting full $\mathcal{N} = 2$ supersymmetry, generalizing the supergravity solution of Kowalski-Glikman [31].¹⁰ Some of these can be obtained as Penrose limits of the geometries encountered so far, following e.g. [34, 35], but it is more convenient (and more general) to study these backgrounds independently.

Not surprisingly, plane waves are obtained for null G or G^i_j , giving rise to the same spacetime with different realizations of $\mathcal{N} = 2$ supersymmetry. Reflecting their origin as Penrose limits, these superalgebras are contractions of those encountered for $S^3 \times \mathbb{R}$ and $AdS_3 \times \mathbb{R}$. At the bosonic level we can just restrict to a null, everywhere non-vanishing G . Then the most general plane wave geometry is obtained turning on also \mathcal{Z} along some null hypersurface parallel to G .

¹⁰Also see [32, 33] for similar solutions of 11d and Type IIB supergravity.

Let us show how the bosonic background arises. We take $G^a = \frac{1}{\sqrt{2}}(g, 0, 0, g) = g\delta_+^a$ and $\mathcal{Z}_{ab} = 2\sqrt{2}\lambda_+\delta_{ab}^{-1} - 2\sqrt{2}i\lambda_-\delta_{ab}^{-2}$, where we can fix $\lambda_+ \geq 0$, $\lambda_- \in \mathbb{R}$. The cases $g = 0$ and $\lambda_{\pm} = 0$ are included in our analysis as all objects depend smoothly on these parameters. When $\lambda_{\pm} = 0$ the same bosonic background can be realized by G^i_j . Defining $P_{\pm} \equiv \frac{1}{\sqrt{2}}(P_3 \pm P_0)$, the non-vanishing commutators of the bosonic isometry algebra read

$$\begin{aligned} [P_a, P_b] &= -4g\epsilon_{+ab}{}^c P_c + 4\lambda_-^2 \delta_{ab}^{-2} M_{+2} + 4\lambda_+^2 \delta_{ab}^{-1} M_{+1}, & (\epsilon_{+-12} = +1) \\ [M_{+d}, P_-] &= P_d, & [M_{+d}, P_d] = -P_+, & (d = 1, 2). \end{aligned} \quad (3.65)$$

This algebra is the semi-direct product of P_- and a five-dimensional Heisenberg algebra (with central charge P_+). The former generates the null direction complementary to the three-dimensional null space defined by G . The residual Lorentz algebra contains the two null generators M_{+1} , M_{+2} . Then the general plane wave that we find is a coset space

$$\frac{\mathbb{R}_{P_-} \times \text{Heis}_5}{N_{M_{+1}, M_{+2}}^2}. \quad (3.66)$$

When $\mathcal{Z} = 0$ this simplifies to the group manifold $\mathbb{R}_{P_-} \times \text{Heis}_3$. We obtain the metric

$$ds^2 = 2du dv + 4g(ydx - xdy)du - 2(\lambda_-^2 x^2 + \lambda_+^2 y^2)du^2 + dx^2 + dy^2. \quad (3.67)$$

Another possibility is to use Brinkmann coordinates, which we obtain via

$$x \rightarrow \hat{x} = x \cos 2gu + y \sin 2gu, \quad y \rightarrow \hat{y} = -x \sin 2gu + y \cos 2gu. \quad (3.68)$$

The metric then takes the standard plane wave form

$$ds^2 = 2du dv + A_{mn}(gu)\hat{x}^m \hat{x}^n du^2 + \delta_{mn} d\hat{x}^m d\hat{x}^n, \quad m = 1, 2, \quad (3.69)$$

$$A(gu) = -(4g^2 + \lambda_+^2 + \lambda_-^2)\mathbb{I}_2 + (\lambda_+^2 - \lambda_-^2) \begin{pmatrix} \cos 4gu & -\sin 4gu \\ -\sin 4gu & -\cos 4gu \end{pmatrix}. \quad (3.70)$$

4 Rigid supersymmetric actions

Each of the spaces we have discussed admits eight Killing spinors and is described by the same rigid $\mathcal{N} = 2$ supersymmetry algebra (2.18), parametrized by the background fields S^{ij} , \mathcal{Z}_{ab} , G_a , and $G_a^i_j$. This suggests it should be possible to construct in a unified way the supersymmetric vector multiplet and hypermultiplet actions for these spaces simultaneously: one should simply insert the relevant values for the metric and the other background fields. That will be the goal of this section. We describe first off-shell vector multiplets and on-shell hypermultiplets in these geometries. After this, we discuss how some of these actions could be derived directly from conformal supergravity by taking a certain rigid limit. Finally, as an explicit example, we describe the $\mathcal{N} = 2^*$ action in a rigid background.

4.1 Vector multiplets

We will begin our discussion of vector multiplets with the abelian case, where the situation is comparably simpler, before gauging the isometries of the special Kähler manifold.

Abelian vector multiplets

An abelian $\mathcal{N} = 2$ vector multiplet consists of a complex scalar X^I , a Weyl fermion $\lambda_{\alpha I}$, a pseudoreal auxiliary triplet Y^{ijI} , and a real connection A_m^I . In Lorentzian signature,

$$(X^I)^* = \bar{X}^I, \quad (\lambda_{\alpha I})^* = \bar{\lambda}_{\dot{\alpha}}{}^{iI}, \quad (Y^{ijI})^* = Y_{ij}{}^I. \quad (4.1)$$

In rigid superspace, these components are contained within a complex superfield \mathcal{X}^I which is chiral, $\bar{\mathcal{D}}_{\dot{\alpha}i}\mathcal{X}^I = 0$, and obeys a superspace Bianchi identity [36, 37]

$$(\mathcal{D}^{ij} + 4S^{ij})\mathcal{X}^I = (\bar{\mathcal{D}}^{ij} + 4\bar{S}^{ij})\bar{\mathcal{X}}^I. \quad (4.2)$$

The component fields are defined using the superfield via

$$\begin{aligned} X^I &:= \mathcal{X}^I|, & \bar{X}^I &:= \bar{\mathcal{X}}^I|, \\ \lambda_{\alpha I} &:= -\mathcal{D}_{\alpha i}\mathcal{X}^I|, & \bar{\lambda}^{\dot{\alpha}iI} &:= \bar{\mathcal{D}}^{\dot{\alpha}i}\bar{\mathcal{X}}^I|, \\ Y^{ijI} &:= -\frac{1}{2}(\mathcal{D}^{ij} + 4S^{ij})\mathcal{X}^I|, \\ F_{ab}{}^I &= \frac{1}{4}(\sigma_{ab})^{\beta\alpha}\mathcal{D}_{\beta\alpha}\mathcal{X}^I| + \frac{1}{4}(\bar{\sigma}_{ab})^{\dot{\beta}\dot{\alpha}}\bar{\mathcal{D}}_{\dot{\beta}\dot{\alpha}}\bar{\mathcal{X}}^I| - \mathcal{Z}_{ab}X^I - \bar{\mathcal{Z}}_{ab}\bar{X}^I, \end{aligned} \quad (4.3)$$

where the vertical bar denotes taking the $\theta = 0$ projection.

As a consequence of the superspace Bianchi identity, $F_{mn}{}^I$ is a field strength for a one-form A_m^I . Both of these can be lifted to superspace forms \mathcal{A}^I and \mathcal{F}^I . One must constrain the tangent space components $\mathcal{F}_{AB}{}^I$ so that it is entirely determined in terms of the superfield \mathcal{X}^I :

$$\begin{aligned} \mathcal{F}_{\alpha\beta}{}^{ijI} &= -4\epsilon_{\alpha\beta}\epsilon^{ij}\bar{\mathcal{X}}^I, & \mathcal{F}_i{}^{\dot{\alpha}\beta I} &= 4\epsilon^{\dot{\alpha}\beta}\epsilon_{ij}\mathcal{X}^I, \\ \mathcal{F}_{\alpha\beta\dot{\beta}}{}^iI &= 2i\epsilon_{\alpha\beta}\bar{\mathcal{D}}_{\dot{\beta}}^i\bar{\mathcal{X}}^I, & \mathcal{F}_{\dot{\alpha}i\beta\dot{\beta}}{}^I &= 2i\epsilon_{\dot{\alpha}\dot{\beta}}\mathcal{D}_{\beta i}\mathcal{X}^I, \\ \mathcal{F}_{ab}{}^I &= \frac{1}{4}(\sigma_{ab})^{\beta\alpha}\mathcal{D}_{\beta\alpha}\mathcal{X}^I + \frac{1}{4}(\bar{\sigma}_{ab})^{\dot{\beta}\dot{\alpha}}\bar{\mathcal{D}}_{\dot{\beta}\dot{\alpha}}\bar{\mathcal{X}}^I - \mathcal{Z}_{ab}\mathcal{X}^I - \bar{\mathcal{Z}}_{ab}\bar{\mathcal{X}}^I. \end{aligned} \quad (4.4)$$

Because the vector multiplet includes its auxiliary field, the supersymmetry algebra closes off-shell. The variations of the component fields are

$$\delta X^I = -\xi_i^\alpha \lambda_\alpha^i{}^I, \quad \delta \bar{X}^I = \bar{\xi}_{\dot{\alpha}}^i \bar{\lambda}_{\dot{\alpha}}^i{}^I, \quad (4.5a)$$

$$\begin{aligned} \delta \lambda_{\alpha I} &= (F_{ab}{}^I + \mathcal{Z}_{ab}X^I + \bar{\mathcal{Z}}_{ab}\bar{X}^I)(\sigma^{ab}\xi_i)_\alpha + (Y_{ij}{}^I + 2S_{ij}X^I)\xi_\alpha^j \\ &\quad - 2i\mathcal{D}_a X^I(\sigma^a\bar{\xi}_i)_\alpha + 4iG_{a ij}X^I(\sigma^a\bar{\xi}^j)_\alpha, \end{aligned} \quad (4.5b)$$

$$\begin{aligned} \delta \bar{\lambda}^{\dot{\alpha}iI} &= (F_{ab}{}^I + \mathcal{Z}_{ab}X^I + \bar{\mathcal{Z}}_{ab}\bar{X}^I)(\bar{\sigma}^{ab}\xi^i)^{\dot{\alpha}} - (Y^{ijI} + 2\bar{S}^{ij}\bar{X}^I)\bar{\xi}^{\dot{\alpha}j} \\ &\quad + 2i\mathcal{D}_a \bar{X}^I(\bar{\sigma}^a\xi^i)^{\dot{\alpha}} + 4iG_a{}^{ij}\bar{X}^I(\bar{\sigma}^a\xi_j)^{\dot{\alpha}}, \end{aligned} \quad (4.5c)$$

$$\begin{aligned} \delta Y_{ij}{}^I &= 2i\xi_{(i}\mathcal{D}_{j)}\bar{\lambda}_{j)}^I - 4iG_{ak(i}\xi_j)\sigma^a\bar{\lambda}^{kI} - 2G_a\xi_{(i}\sigma^a\bar{\lambda}_{j)}^I \\ &\quad - 2i\bar{\xi}_{(i}\mathcal{D}_{j)}\lambda_{j)}^I + 4iG_{ak(i}\bar{\xi}_j)\bar{\sigma}^a\lambda^{kI} - 2G_a\bar{\xi}_{(i}\bar{\sigma}^a\lambda_{j)}^I, \end{aligned} \quad (4.5d)$$

$$\delta A_m^I = i(\xi_j\sigma_m\bar{\lambda}^j{}^I) + i(\bar{\xi}^j\bar{\sigma}_m\lambda_j^I). \quad (4.5e)$$

Abelian vector multiplet action

The vector multiplet actions are straightforward to construct in superspace. They are given by a chiral superspace integral of a holomorphic prepotential function $F(\mathcal{X}^I)$,

$$-i \int d^4x d^4\theta \mathcal{E} F(\mathcal{X}) + i \int d^4x d^4\bar{\theta} \bar{\mathcal{E}} \bar{F}(\bar{\mathcal{X}}). \quad (4.6)$$

In a Minkowski background, or indeed in any rigid $\mathcal{N} = 2$ background with $G_a^{ij} = 0$, the prepotential needs to satisfy no further restrictions. However, in a rigid background with nonvanishing G_a^{ij} , the $U(1)_R$ symmetry forces the prepotential to be homogeneous of weight two — that is, it must be a superconformal model.

Deriving the component action of (4.6) is a straightforward application of superspace techniques (see e.g. [38]). As in the rigid Minkowski case, the sigma model is an affine special Kähler manifold (see e.g. [39]) with Kähler potential K and metric g_{IJ} :

$$K = iX^I \bar{F}_I - i\bar{X}^I F_I, \quad g_{IJ} = -iF_{IJ} + i\bar{F}_{IJ}. \quad (4.7)$$

The leading bosonic kinetic terms are

$$\mathcal{L} = -g_{IJ} \mathcal{D}_a X^I \mathcal{D}^a \bar{X}^J + \frac{i}{4} F_{IJ} F_{ab}^{-I} F^{ab-J} - \frac{i}{4} \bar{F}_{IJ} F_{ab}^{+I} F^{ab+J} + \dots \quad (4.8)$$

We will give the full Lagrangian, including gauged isometries, in due course, but for now there are some important features to discuss in the ungauged case. Foremost is that the action admits the same group of duality transformations as in flat space. These can be seen easily in the superspace description, where the superfield equations of motion imply that the chiral superfield $F_I(\mathcal{X})$ also obeys the constraint (4.2). The duality transformations belong to the inhomogeneous symplectic group $\text{ISp}(2n, \mathbb{R})$,

$$\begin{pmatrix} \mathcal{X}^I \\ F_I \end{pmatrix} \longrightarrow \begin{pmatrix} U^I{}_J & Z^{IJ} \\ W_{IJ} & V_I{}^J \end{pmatrix} \begin{pmatrix} \mathcal{X}^J \\ F_J \end{pmatrix} + \begin{pmatrix} C^I \\ C_I \end{pmatrix} \quad (4.9)$$

where the matrix is an element of $\text{Sp}(2n, \mathbb{R})$ and C^I and C_I are constant complex numbers.¹¹ When S^{ij} is nonzero, C^I must obey the extra constraint $S^{ij} C^I = \bar{S}^{ij} \bar{C}^I$, and similarly for C_I , as a consequence of (4.2). One may decompose the C^I and C_I in terms of real parameters $U^I{}_{(j)}$ and $W_{I(j)}$ with $j = 1, 2$, so that

$$C^I = U^I{}_{(1)} + iU^I{}_{(2)}, \quad C_I = W_{I(1)} + iW_{I(2)}. \quad (4.10)$$

This suggestive decomposition reflects the presence of so-called background vector multiplets in the rigid supersymmetric geometries we have been discussing.

¹¹One can also consider a global $U(1)_R$ transformation as in the Minkowski case, but it will not play a role in what follows.

Background vector multiplets and central charges

There is a close relationship between background vector multiplets and the possibility of extending the supersymmetry algebra by a complex internal central charge Z . Provided that $G_a^{ij} = 0$, we can deform the algebra with the extra terms

$$\{\mathcal{D}_\alpha^i, \mathcal{D}_\beta^j\} \sim 4\epsilon_{\alpha\beta}\epsilon^{ij}\bar{Z}, \quad \{\bar{\mathcal{D}}^{\dot{\alpha}}_i, \bar{\mathcal{D}}^{\dot{\beta}}_j\} \sim -4\epsilon^{\dot{\alpha}\dot{\beta}}\epsilon_{ij}Z, \quad [\mathcal{D}_a, \mathcal{D}_b] \sim \mathcal{Z}_{ab}Z + \bar{\mathcal{Z}}_{ab}\bar{Z}. \quad (4.11)$$

The covariant derivatives \mathcal{D}_M now carry a complex central charge connection, with the background field \mathcal{Z}_{ab} playing the role of the bosonic field strength for Z . These observations can be clarified if we write the complex central charge in terms of two real central charges, $Z = Z_{(1)} + iZ_{(2)}$, and interpret the algebra of covariant derivatives above as

$$[\mathcal{D}_A, \mathcal{D}_B] \sim -\mathcal{F}_{AB}^{(i)}Z_{(i)} \quad (4.12)$$

where $\mathcal{F}_{MN}^{(i)}$ is the field strength for two real abelian one-forms $\mathcal{A}_M^{(i)}$. Comparing to (4.4), one can see that the field strengths $\mathcal{F}_{AB}^{(i)}$ are associated with constant vector multiplet superfields $\mathcal{X}^{(1)} = 1$ and $\mathcal{X}^{(2)} = i$. In particular, the background vectors possess non-vanishing field strengths with one-form potentials (see (4.4)),

$$\begin{aligned} F_{ab}^{(1)} &= -2 \operatorname{Re} \mathcal{Z}_{ab}, & F_{ab}^{(2)} &= 2 \operatorname{Im} \mathcal{Z}_{ab}, \\ A_m^{(1)} &= -2 \operatorname{Re} C_m, & A_m^{(2)} &= 2 \operatorname{Im} C_m. \end{aligned} \quad (4.13)$$

More generally, we can choose $\mathcal{X}^{(1)}$ to possess any phase and take $\mathcal{X}^{(2)} = i\mathcal{X}^{(1)}$. A subtlety emerges when S^{ij} is non-vanishing: then (4.2) implies $S^{jk}\mathcal{X}^{(i)} = \bar{S}^{jk}\bar{\mathcal{X}}^{(i)}$, and so only one independent choice of $\mathcal{X}^{(i)}$ is possible. For pseudoreal S^{ij} , we take $\mathcal{X}^{(1)} = 1$ and drop $\mathcal{X}^{(2)}$.

These observations readily admit a simple explanation of the inhomogeneous symplectic transformation (4.9). Rewriting that transformation as

$$\begin{pmatrix} \mathcal{X}^I \\ F_I \end{pmatrix} \rightarrow \begin{pmatrix} U^I{}_J & Z^{IJ} \\ W_{IJ} & V_I{}^J \end{pmatrix} \begin{pmatrix} \mathcal{X}^J \\ F_J \end{pmatrix} + \begin{pmatrix} U^I{}_{(j)}\mathcal{X}^{(j)} \\ W_{I(j)}\mathcal{X}^{(j)} \end{pmatrix} \quad (4.14)$$

one finds that it can be embedded into $\operatorname{Sp}(2n+4, \mathbb{R})$ acting on the $2n+4$ vector $(\mathcal{X}^{\hat{I}}, F_{\hat{I}})$ where $\mathcal{X}^{\hat{I}} = (\mathcal{X}^I, \mathcal{X}^{(i)})$. The prepotential $F(\mathcal{X}^{\hat{I}})$ should be taken as a homogeneous prepotential with two constant vector multiplets $\mathcal{X}^{(i)}$.¹² Because the background multiplets are never placed on-shell, $F_{(i)}$ does not obey (4.2) and must not mix into the other multiplets: this zeroes out the entries $Z^{I(j)}$ and $V_I{}^{(j)}$ of the $\operatorname{Sp}(2n+4, \mathbb{R})$ matrix, recovering (4.14). Imposing in addition the invariance of the constant $\mathcal{X}^{(i)}$ determines the other entries.

Non-abelian (gauged) vector multiplets

Until now we have been dealing with abelian (ungauged) vector multiplets. Before discussing the action in detail, we should allow for the possibility of gauging isometries of the

¹²When both constant vector multiplets are present, the lift of the original inhomogeneous $F(\mathcal{X}^{\hat{I}})$ to the homogeneous $F(\mathcal{X}^{\hat{I}})$ is not unique.

special Kähler manifold. It is sufficient to discuss the group of isometries on the one-forms \mathcal{A}^I . Taking into account the additional background one-forms $\mathcal{A}^{(i)}$, we are led to consider

$$\begin{aligned}\delta\mathcal{A}^I &= d\Lambda^I + \mathcal{A}^J(\Lambda^K f_{KJ}^I + \Lambda^{(k)} f_{(k)J}^I) + \mathcal{A}^{(j)}(\Lambda^K f_{K(j)}^I + \Lambda^{(k)} f_{(k)(j)}^I), \\ \delta\mathcal{A}^{(i)} &= d\Lambda^{(i)}.\end{aligned}\quad (4.15)$$

Collectively these can be written $\delta\mathcal{A}^{\hat{I}} = d\Lambda^{\hat{I}} + \mathcal{A}^{\hat{J}}\Lambda^{\hat{K}}f_{\hat{K}\hat{J}}^{\hat{I}}$. The Lie algebra here is

$$[T_I, T_J] = f_{IJ}{}^K T_K, \quad [Z_{(i)}, T_J] = f_{(i)J}{}^K T_K, \quad [Z_{(i)}, Z_{(j)}] = f_{(i)(j)}{}^K T_K. \quad (4.16)$$

In the gauged case, $Z_{(i)}$ no longer commute with the gauge generators but we will still occasionally refer to them as central charges. The superfields \mathcal{X}^I and $\mathcal{X}^{(i)}$ transform as

$$\delta\mathcal{X}^I = \mathcal{X}^J(\Lambda^K f_{KJ}^I + \Lambda^{(k)} f_{(k)J}^I) + \mathcal{X}^{(j)}(\Lambda^K f_{K(j)}^I + \Lambda^{(k)} f_{(k)(j)}^I), \quad \delta\mathcal{X}^{(i)} = 0. \quad (4.17)$$

Collectively these can be written as $\delta\mathcal{X}^{\hat{I}} = \mathcal{X}^{\hat{J}}\Lambda^{\hat{K}}f_{\hat{K}\hat{J}}^{\hat{I}}$. Comparing with (4.14), one can see that this gauges a subgroup of $\text{ISP}(2n, \mathbb{R})$ with infinitesimal elements

$$u^I{}_J = \Lambda^K f_{KJ}^I + \Lambda^{(k)} f_{(k)J}^I, \quad u^I{}_{(j)} = \Lambda^K f_{K(j)}^I + \Lambda^{(k)} f_{(k)(j)}^I. \quad (4.18)$$

The full embedding of the gauge group may also involve $w_{I\hat{J}}$, as we will discuss shortly.

Now defining the superspace covariant derivatives to carry the connections \mathcal{A}^I and $\mathcal{A}^{(i)}$, the SUSY transformations of the vector multiplets \mathcal{X}^I can be determined; they differ only slightly from (4.5). It is convenient to introduce the Killing vector J_J^I defined by

$$\delta X^I = \Lambda^{\hat{J}} J_{\hat{J}}^I, \quad J_{\hat{J}}^I = -X^{\hat{K}} f_{\hat{K}\hat{J}}^I, \quad (4.19)$$

Including a uniform coupling constant g to track the gauging terms, one finds

$$\delta X^I = -\xi_\alpha^i \lambda_\alpha^i{}^I, \quad \delta \bar{X}^I = \bar{\xi}_{\dot{\alpha}}^i \bar{\lambda}_{\dot{\alpha}}^i{}^I, \quad (4.20a)$$

$$\begin{aligned}\delta \lambda_{\alpha i}{}^I &= (F_{ab}{}^I + \mathcal{Z}_{ab} X^I + \bar{\mathcal{Z}}_{ab} \bar{X}^I)(\sigma^{ab} \xi_i)_\alpha + (Y_{ij}{}^I + 2S_{ij} X^I) \xi_\alpha^j \\ &\quad - 2i \mathcal{D}_a X^I (\sigma^a \bar{\xi}_i)_\alpha + 4i G_{a ij} X^I (\sigma^a \bar{\xi}^j)_\alpha - 2g \bar{X}^{\hat{J}} J_{\hat{J}}^I \xi_{\alpha i},\end{aligned}\quad (4.20b)$$

$$\begin{aligned}\delta \bar{\lambda}^{\dot{\alpha} i}{}^I &= (F_{ab}{}^I + \mathcal{Z}_{ab} X^I + \bar{\mathcal{Z}}_{ab} \bar{X}^I)(\bar{\sigma}^{ab} \bar{\xi}^i)^{\dot{\alpha}} - (Y^{ij}{}^I + 2\bar{S}^{ij} \bar{X}^I) \bar{\xi}^{\dot{\alpha} j} \\ &\quad + 2i \mathcal{D}_a \bar{X}^I (\bar{\sigma}^a \xi^i)^{\dot{\alpha}} + 4i G_a{}^{ij} \bar{X}^I (\bar{\sigma}^a \xi_j)^{\dot{\alpha}} - 2g X^{\hat{J}} \bar{J}_{\hat{J}}^I \bar{\xi}^{\dot{\alpha} i},\end{aligned}\quad (4.20c)$$

$$\begin{aligned}\delta Y_{ij}{}^I &= 2i \xi_{(i} \mathcal{D} \bar{\lambda}_{j)}^I - 4i G_{ak(i} \xi_{j)} \sigma^a \bar{\lambda}^{kI} - 2G_a \xi_{(i} \sigma^a \bar{\lambda}_{j)}^I \\ &\quad - 2i \bar{\xi}_{(i} \mathcal{D} \lambda_{j)}^I + 4i G_{ak(i} \bar{\xi}_{j)} \bar{\sigma}^a \lambda^{kI} - 2G_a \bar{\xi}_{(i} \bar{\sigma}^a \lambda_{j)}^I \\ &\quad - 4g \xi_{(i} \lambda_{j)}^J \bar{J}_{J}^I - 4g \bar{\xi}_{(i} \bar{\lambda}_{j)}^J J_{J}^I,\end{aligned}\quad (4.20d)$$

$$\delta A_m{}^I = i(\xi_j \sigma_m \bar{\lambda}^{jI}) + i(\bar{\xi}^j \bar{\sigma}_m \lambda_j^I). \quad (4.20e)$$

The vector derivative carries the full set of connections, i.e.

$$\mathcal{D}_a X^I = e_a{}^m \left(\partial_m X^I - 2i A_m X^I + g A_m^{\hat{J}} X^{\hat{K}} f_{\hat{K}\hat{J}}^I \right), \quad (4.21)$$

with the contributions from the constants $X^{(k)}$ gauging inhomogeneous transformations. Similarly, the field strengths $F_{mn}{}^I$ are now given by

$$F_{mn}{}^I = 2\partial_{[m} A_{n]}^I + g A_m^{\hat{J}} A_n^{\hat{K}} f_{\hat{K}\hat{J}}^I \quad (4.22)$$

Vector multiplet action with gauged isometries

Now we will give the final form of the action, including gauged isometries. Assuming the prepotential F is gauge-invariant, we calculate the component reduction of (4.6):

$$\begin{aligned}
 \mathcal{L} = & -g_{IJ}\mathcal{D}_a X^I \mathcal{D}^a \bar{X}^J - \frac{i}{4}g_{IJ}\lambda_j^I \hat{\mathcal{D}}\bar{\lambda}^{jJ} - \frac{i}{4}g_{IJ}\bar{\lambda}^{jJ} \hat{\mathcal{D}}\lambda_j^I \\
 & + \frac{i}{4}F_{IJ}F_{ab}^{-I}F^{ab-J} - \frac{i}{4}\bar{F}_{IJ}F_{ab}^{+I}F^{ab+J} + \frac{1}{8}g_{IJ}Y^{ijI}Y_{ij}^J \\
 & + 4G^a \left(F_I \mathcal{D}_a \bar{X}^I + \bar{F}_I \mathcal{D}_a X^I \right) + Y_{ij}^I \mathcal{O}_I^{ij} + F_{ab}^I \mathcal{O}_I^{ab} + \mathcal{L}_{\text{pot}} + \mathcal{L}_{\text{ferm}} . \quad (4.23)
 \end{aligned}$$

We have collected the moment couplings into \mathcal{O}_I^{ab} and the terms linear in the auxiliary field into \mathcal{O}_I^{ij} . Additional terms contributing to the scalar potential are given in \mathcal{L}_{pot} and additional fermionic terms appear in $\mathcal{L}_{\text{ferm}}$. Before giving these expressions, let us comment on the leading terms of (4.23). The covariant derivative of the scalars is given in (4.21), whereas the covariant derivative of the gauginos includes the special Kähler connection,

$$\hat{\mathcal{D}}_a \lambda_{\alpha j}^I = \mathcal{D}_a \lambda_{\alpha j}^I + ig^{IJ} F_{JKL} \mathcal{D}_a X^K \lambda_{\alpha j}^L . \quad (4.24)$$

The terms involving the field strengths can be rewritten as usual as

$$-\frac{1}{8}g_{IJ}F_{ab}^I F^{abJ} - \frac{1}{8}(F_{IJ} + \bar{F}_{IJ})F_{ab}^I \tilde{F}^{abJ} \quad (4.25)$$

where $F_{IJ} + \bar{F}_{IJ}$ describes a generalized θ -term. The expression involving G^a in (4.23) can be rewritten up to a total derivative using its dual two-form potential as

$$\begin{aligned}
 4G^a (F_I \mathcal{D}_a \bar{X}^I + \bar{F}_I \mathcal{D}_a X^I) &= \epsilon^{mnpq} B_{mn} \left(2i g_{IJ} \mathcal{D}_p X^I \mathcal{D}_q \bar{X}^J - g F_{pq}^I D_{\hat{I}} \right) \\
 &= 2i \epsilon^{mnpq} B_{mn} \partial_p X^I \partial_q \bar{X}^J g_{IJ} + \frac{2}{3}g \epsilon^{mnpq} H_{mnp} A_q^{\hat{I}} D_{\hat{I}} . \quad (4.26)
 \end{aligned}$$

This involves a generalized BF coupling between the background two-form B and the special Kähler two-form $-i g_{IJ} dX^I \wedge d\bar{X}^J$. Note that we have dropped the $U(1)_R$ connection A_m above because when G^a is non-zero this connection is always taken to vanish.

The remaining terms in the Lagrangian involve

$$\begin{aligned}
 \mathcal{O}_I^{ab} = & \frac{1}{4}\epsilon^{abcd} \mathcal{Z}_{cd} \left(F_I - \frac{1}{2}(F_{IJ} + \bar{F}_{IJ})X^J \right) + \frac{1}{4}\epsilon^{abcd} \bar{\mathcal{Z}}_{cd} \left(\bar{F}_I - \frac{1}{2}(F_{IJ} + \bar{F}_{IJ})\bar{X}^J \right) \\
 & - \frac{1}{4}g_{IJ}(X^J \mathcal{Z}^{ab} + \bar{X}^J \bar{\mathcal{Z}}^{ab}) - \frac{i}{8}F_{IJK}\lambda^{kJ}\sigma^{ab}\lambda_k^K + \frac{i}{8}\bar{F}_{IJK}\bar{\lambda}^{kJ}\bar{\sigma}^{ab}\bar{\lambda}_k^K , \quad (4.27)
 \end{aligned}$$

$$\mathcal{O}_I^{ij} = \frac{i}{2}S^{ij}(F_I - F_{IJ}X^J) - \frac{i}{2}\bar{S}^{ij}(\bar{F}_I - \bar{F}_{IJ}\bar{X}^J) + \frac{i}{8}F_{IJK}(\lambda^{iJ}\lambda^{jK}) - \frac{i}{8}\bar{F}_{IJK}(\bar{\lambda}^{iJ}\bar{\lambda}^{jK}) , \quad (4.28)$$

$$\begin{aligned}
 \mathcal{L}_{\text{pot}} = & -\frac{1}{8}g_{IJ}(X^I \mathcal{Z}_{ab} + \bar{X}^I \bar{\mathcal{Z}}_{ab})(X^J \mathcal{Z}^{ab} + \bar{X}^J \bar{\mathcal{Z}}^{ab}) \\
 & - \frac{1}{8}(F_{IJ} + \bar{F}_{IJ}) \left(X^I \mathcal{Z}_{ab} + \bar{X}^I \bar{\mathcal{Z}}^{ab} \right) \left(X^J \tilde{\mathcal{Z}}^{ab} + \bar{X}^J \tilde{\bar{\mathcal{Z}}}^{ab} \right) \\
 & + \frac{1}{2}\mathcal{Z}_{ab}\tilde{\mathcal{Z}}^{ab}(F_I X^I - F) + \frac{1}{2}\bar{\mathcal{Z}}_{ab}\tilde{\bar{\mathcal{Z}}}^{ab}(\bar{F}_I \bar{X}^I - \bar{F})
 \end{aligned}$$

$$\begin{aligned}
 & + iS^{ij}S_{ij}\left(2F_I X^I - \frac{1}{2}X^I X^J F_{IJ} - 3F\right) - i\bar{S}^{ij}\bar{S}_{ij}\left(2\bar{F}_I \bar{X}^I - \frac{1}{2}\bar{F}_{IJ}\bar{X}^I \bar{X}^J - 3\bar{F}\right) \\
 & + S^{ij}\bar{S}_{ij}K + 2G^{aj}G_{a\bar{i}j}K - g^2 g^{IJ}D_I D_J, \tag{4.29}
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{L}_{\text{ferm}} = & g_{IJ}\lambda_i^I \sigma^a \bar{\lambda}^{jJ}\left(\frac{1}{2}\delta_j^i G_a + iG_a{}^i{}_j\right) + \frac{i}{4}(\lambda_i^I \lambda_j^J)S^{ij}F_{IJK}X^K - \frac{i}{4}(\bar{\lambda}^{iI}\bar{\lambda}^{jJ})\bar{S}_{ij}\bar{F}_{IJK}\bar{X}^K \\
 & - \frac{i}{8}(\lambda^{kI}\sigma^{ab}\lambda_k^J)(\bar{Z}_{ab}\bar{X}^K + Z_{ab}X^K)F_{IJK} + \frac{i}{8}(\bar{\lambda}^{kI}\bar{\sigma}^{ab}\bar{\lambda}_k^J)(Z_{ab}X^K + \bar{Z}_{ab}\bar{X}^K)\bar{F}_{IJK} \\
 & - \frac{i}{48}F_{IJKL}(\lambda^{iI}\lambda^{jJ})(\lambda_i^K \lambda_j^L) + \frac{i}{48}\bar{F}_{IJKL}(\bar{\lambda}^{iI}\bar{\lambda}^{jJ})(\bar{\lambda}_i^K \bar{\lambda}_j^L) \\
 & - \frac{1}{2}g\lambda^{kI}\lambda_k^J g_{IK}\bar{J}^K - \frac{1}{2}g\bar{\lambda}^{kI}\bar{\lambda}_k^J g_{IK}J^K. \tag{4.30}
 \end{aligned}$$

We have utilized the Killing potentials (or moment maps) for J_I^J given by

$$D_{\hat{I}} = f_{\hat{I}j}{}^K (X^{\hat{J}}\bar{F}_K + \bar{X}^{\hat{J}}F_K). \tag{4.31}$$

The moment maps D_I associated with the dynamical vector multiplets may also be written

$$D_I = -ig_{IJ}X^{\hat{K}}\bar{X}^{\hat{L}}f_{\hat{K}\hat{L}}{}^J. \tag{4.32}$$

Suppose now that the prepotential F is not gauge invariant but transforms as

$$\delta F = \frac{1}{2}\Lambda^{\hat{K}}C_{\hat{K},\hat{I}\hat{J}}X^{\hat{I}}X^{\hat{J}}, \tag{4.33}$$

with real $C_{\hat{K},\hat{I}\hat{J}}$. This transformation lies in a subgroup of (4.14) with infinitesimal $w_{I\hat{J}} = \Lambda^{\hat{K}}C_{\hat{K},\hat{I}\hat{J}}$. Provided we write the BF coupling as in (4.26), the Lagrangian fails to be gauge invariant only due to the generalized θ term involving $F_{IJ} + \bar{F}_{IJ}$ in (4.25), the moment couplings involving $\epsilon^{abcd}Z_{cd}$ in (4.27), and the potential terms involving products like $Z_{ab}Z_{cd}\epsilon^{abcd}$ in (4.29). These have a natural interpretation as generalized θ terms involving the background vector multiplets. As in the Minkowski case, gauge invariance can be restored by adding the Chern-Simons like term [40, 41]

$$\mathcal{L}_{\text{CS-like}} = -\frac{2}{3}g\epsilon^{mnpq}C_{\hat{K},\hat{I}\hat{J}}A_m{}^{\hat{K}}A_n{}^{\hat{I}}(\partial_p A_q{}^{\hat{J}} - \frac{3}{8}gf_{\hat{M}\hat{N}}{}^{\hat{J}}A_p{}^{\hat{M}}A_q{}^{\hat{N}}), \tag{4.34}$$

which involves both the physical connections $A_m{}^I$ and the background connections $A_m{}^{(i)}$. In addition, one must modify the Killing potentials in the various expressions above to

$$D_{\hat{I}} = f_{\hat{I}j}{}^K (X^{\hat{J}}\bar{F}_K + \bar{X}^{\hat{J}}F_K) - C_{\hat{I},\hat{J}\hat{K}}X^{\hat{J}}\bar{X}^{\hat{K}}, \tag{4.35}$$

with (4.32) still holding. These modifications also restore supersymmetry.

BPS conditions for the vector multiplet

From the supersymmetry transformations (4.20), we may characterize the moduli space of supersymmetric configurations for a vector multiplet in a generic rigid background. Recall that in Minkowski spacetime a supersymmetric configuration for a vector multiplet is given by constant scalar X^I and vanishing fermions, field strengths, and auxiliary fields. In a

generic rigid $\mathcal{N} = 2$ background characterized by the background fields S^{ij} , \mathcal{Z}_{ab} , G_a , and G_a^{ij} , the situation differs.

Let us first assume that $G_a^{ij} = 0$ and that we have eliminated the spacetime $U(1)_R$ connection. Requiring $\delta X^I = 0$ for eight linearly independent supercharges implies that the fermions $\lambda_{\alpha i}^I$ vanish. Requiring $\delta \lambda_{\alpha i}^I = 0$ leads to the additional constraints

$$\begin{aligned} Y_{ij}^I &= -2S_{ij}X^I = -2\bar{S}_{ij}\bar{X}^I, & \mathcal{D}_a X^I &= 0, \\ F_{ab}^I &= -\mathcal{Z}_{ab}X^I - \bar{\mathcal{Z}}_{ab}\bar{X}^I, & X^J \bar{X}^{\hat{K}} f_{\hat{K}J}^I &= 0. \end{aligned} \tag{4.36}$$

The first condition fixes the auxiliary field and relates the phase of X^I to that of S^{ij} (provided S^{ij} is non-vanishing). The condition on F_{ab}^I is a BPS attractor equation in a fully supersymmetric background; equivalently, given a field strength (which must be related to \mathcal{Z}_{ab}) it fixes the values of the scalars X^I .¹³ The conditions on the right force X^I to be covariantly constant and constrain the VEVs of the scalars when non-abelian couplings are present. Note that the usual result $[X^I T_I, \bar{X}^J T_J] = X^I \bar{X}^J f_{IJ}^K T_K = 0$ is deformed by the background vector multiplets.

If instead we have a background with $G_a^{ij} \neq 0$, the situation is drastically simpler. We find that X^I must vanish and so the entire multiplet vanishes. This is a consequence of the non-trivial R -symmetry appearing in the supersymmetry algebra: since the superfield \mathcal{X}^I carries R -charge, it must completely vanish.

4.2 Hypermultiplets

Hypermultiplets are on-shell representations of the supersymmetry algebra consisting of $4n$ real scalars ϕ^μ with $\mu = 1, \dots, 4n$ and $2n$ chiral fermions ζ_a^a , with $a = 1, \dots, 2n$, obeying $(\zeta_a^a)^* = \bar{\zeta}_{\bar{a}}^{\bar{a}}$. We follow the conventions of [47]. As in a Minkowski background, the scalars parametrize the target space of a hyperkähler manifold with metric $g_{\mu\nu}$ and three covariantly constant complex structures $(\mathcal{J}_A)^\mu{}_\nu$, obeying the quaternion algebra

$$\mathcal{J}_A \mathcal{J}_B = -\delta_{AB} + \epsilon_{ABC} \mathcal{J}_C. \tag{4.37}$$

Introducing $\mathcal{J}^i{}_j = \frac{i}{2}(\sigma_A)^i{}_j \mathcal{J}_A$ where σ_A are the three Pauli matrices, we can construct three hyperkähler two-forms $(\Omega_{ij})_{\mu\nu} = \epsilon_{ik} g_{\mu\rho} (\mathcal{J}^k{}_j)^\rho{}_\nu$.

There are three important classes of isometries. The first are the triholomorphic isometries J^μ , which obey $\mathcal{L}_J g_{\mu\nu} = \mathcal{L}_J (\Omega_{ij})_{\mu\nu} = 0$. These may be gauged by vector multiplets (including the background ones), in which case we denote them by $J_{\hat{f}}^\mu$. Associated with each J^μ is a moment map (or Killing potential) D^{ij} obeying

$$\nabla_\mu D_{\hat{f}}^{ij} = -(\Omega^{ij})_{\mu\nu} J_{\hat{f}}^\nu. \tag{4.38}$$

This defines the moment map up to a constant Fayet-Iliopoulos term.

The second class of isometry, denoted by V^μ , rotates the complex structures,

$$\mathcal{L}_V g_{\mu\nu} = 0, \quad \mathcal{L}_V (\Omega_{ij})_{\mu\nu} = -2v^k ({}_{(i} \Omega_{j)k})_{\mu\nu}, \quad \mathcal{L}_V \mathcal{J}_A = -2\epsilon_{ABC} v_B \mathcal{J}_C, \tag{4.39}$$

¹³This result generalizes the attractor equation found in the near horizon limit of BPS black holes [42–46], where the supersymmetry algebra is given by the supergroup $D(2, 1; -1)$ with $\mathcal{Z}_{ab} = -W_{ab}^+ = -\frac{1}{4}T_{ab}^+$.

where v^{ij} is a symmetric pseudoreal normalized SU(2) triplet, $v^{ij}v_{jk} = \delta_k^i$, equivalent to a normalized real SO(3) vector, $v^A = -\frac{i}{2}(\sigma_A)^i_j v^j$. Because V^μ is holomorphic with respect to the complex structure $v^A \mathcal{J}_A$, there is a corresponding moment map, which we denote \mathcal{K} , for this specific complex structure. We normalize it so that

$$\nabla_\mu \mathcal{K} = v_{ij}(\Omega^{ij})_{\mu\nu} V^\nu = v^A (\mathcal{J}_A)_\mu{}^\nu V_\nu . \tag{4.40}$$

\mathcal{K} is also a Kähler potential for the metric $g_{\mu\nu}$ with respect to any complex structure perpendicular to $v^A \mathcal{J}_A$. That is for $w^A \mathcal{J}_A$ with $w^A v_A = 0$, one can show that

$$g_{\mu\nu} = \frac{1}{2}(\delta_\mu{}^\rho \delta_\nu{}^\sigma + w^A w^B \mathcal{J}_{A\mu}{}^\rho \mathcal{J}_{B\nu}{}^\sigma) \nabla_\rho \nabla_\sigma \mathcal{K} . \tag{4.41}$$

This isometry is relevant whenever $S^{ij} = \mu v^{ij}$ is non-vanishing, which implies the presence of an SO(2) subgroup of SU(2)_R in the supersymmetry algebra; this isometry is manifested on the target space as V^μ .¹⁴ This requirement was observed in [48, 49].

The final case of interest is when the target space is a hyperkähler cone. Then there is a homothetic conformal Killing vector χ^μ , obeying $\nabla_\mu \chi^\nu = \delta_\mu{}^\nu$, and the target space has a globally defined hyperkähler potential $\chi = \frac{1}{2} \chi^\mu \chi_\mu$. One can construct a family of isometries that rotate the complex structures in any direction: they are given by $V_A^\mu = -(\mathcal{J}_A)^\mu{}_\nu \chi^\nu$. The isometries generated by χ^μ and V_A^μ are the target space realization of the superconformal dilatation and SU(2)_R generators, and so hyperkähler cones are precisely those target spaces that may be coupled directly to conformal supergravity. (The hypermultiplets are inert under U(1)_R.) In fact, the presence of the SU(2)_R isometries is sufficient to deduce the presence of the dilatation isometry on the target space. For the cases with $\mathcal{Z}_{ab}^- = Y_{ab}^-$ and/or G_a nonzero, the supersymmetry algebra generates an arbitrary SU(2)_R element, and so these superalgebras require a hyperkähler cone for the hypermultiplets.

Supersymmetry transformations

There is one additional feature necessary to describe the hypermultiplet supersymmetry transformations: the structure group is Sp(n) × Sp(1). That is, one can introduce a target space vielbein $f_{\mu i}{}^a$ and its inverse $f_a{}^{i\mu}$ (see [50–54]) where $a = 1, \dots, 2n$, obeying

$$\begin{aligned} f_{\mu i}{}^a f_a{}^{i\nu} &= \delta_\mu{}^\nu , & f_a{}^{i\mu} f_{\mu j}{}^b &= \delta_a{}^b \delta^i_j , & f_{\mu i}{}^a &= -\epsilon_{ij} \omega^{ab} g_{\mu\nu} f_b{}^{j\nu} , \\ g_{\mu\nu} &= \epsilon^{ij} \omega_{ab} f_{\mu i}{}^a f_{\nu j}{}^b , & (\mathcal{J}_A)^\mu{}_\nu &= i f_{\nu i}{}^a (\sigma_A)^i_j f_a{}^{j\mu} , & (\Omega^{ij})_{\mu\nu} &= f_\mu{}^a (f_\nu{}^{bj}) \omega_{ab} , \end{aligned} \tag{4.42}$$

where ω_{ab} is an antisymmetric matrix with ω^{ab} obeying $\omega^{ab} \omega_{bc} = -\delta_c^a$. One can introduce the complex conjugate of $f_{\mu i}{}^a$, given by $(f_{\mu i}{}^a)^* = f_\mu{}^{i\bar{a}}$, so that

$$g_{\mu\nu} = f_{\mu i}{}^a f_\nu{}^{i\bar{b}} g_{a\bar{b}} \tag{4.43}$$

in terms of an Sp(n) metric $g_{a\bar{b}}$. This implies that $f_\mu{}^{i\bar{a}} = \epsilon^{ij} g^{\bar{a}a} \omega_{ab} f_{\mu j}{}^b$. If the Sp(n) indices are chosen to be flat tangent space indices, then one can choose $g_{a\bar{b}} = \delta_{a\bar{b}}$ and take ω_{ab} to

¹⁴Even though $G_a{}^{ij}$ admits a similar decomposition, it does not generate an R -symmetry transformation in the SUSY algebra and has no effect on the target space geometry.

be the canonical antisymmetric tensor of $\text{Sp}(n)$. Following [53, 54], we will instead keep a non-trivial $\text{Sp}(n)$ metric and a covariantly constant ω_{ab} . Any vector V^μ can be related to an $\text{Sp}(n) \times \text{Sp}(1)$ vector $V_i^a = V^\mu f_{\mu i}^a$, and similarly for tensors. The hyperkähler Riemann tensor is valued in $\text{Sp}(n)$ alone,

$$R_a^i b^j c^k d^l := f_a^i{}^\mu f_b^j{}^\nu f_c^k{}^\rho f_d^l{}^\sigma R_{\mu\nu\rho\sigma} = R_{abcd} \epsilon^{ij} \epsilon^{kl} \quad (4.44)$$

where R_{abcd} is totally symmetric. One can always take the $\text{Sp}(1)$ connection to vanish, and then the $\text{Sp}(n)$ connection $\Gamma_{\nu b}^a$ is determined by requiring $f_{\mu i}^a$ to be covariantly constant.

The supersymmetry transformations of the hypermultiplet fields are

$$\begin{aligned} \delta\phi^\mu &= \xi_i \zeta^b f_b^i{}^\mu + \bar{\xi}^i \bar{\zeta}^b f_{\bar{b}i}{}^\mu, \\ \delta\zeta_\alpha^a &= \left(2i \mathcal{D}_{\alpha\beta} \phi^\mu - 4G_{\alpha\beta} \chi^\mu\right) f_{\mu i}^a \bar{\xi}^{\beta i} + \left(2\mu V^\mu + 4g \bar{X}^{\hat{I}} J_{\hat{I}}^\mu\right) f_{\mu i}^a \epsilon^{ij} \xi_{\alpha j} \\ &\quad + 2Y_\alpha{}^\beta \chi^\mu f_{\mu i}^a \epsilon^{ij} \xi_{\beta j} - \Gamma_{\mu b}^a \delta\phi^\mu \zeta_\alpha^b, \\ \delta\bar{\zeta}^{\dot{\alpha}\bar{a}} &= \left(2i \mathcal{D}^{\dot{\alpha}\beta} \phi^\mu + 4G^{\dot{\alpha}\beta} \chi^\mu\right) f_{\mu i}^{\dot{\alpha}\bar{a}} \xi_{\beta i} - \left(2\bar{\mu} V^\mu + 4g X^{\hat{I}} J_{\hat{I}}^\mu\right) f_{\mu i}^{\dot{\alpha}\bar{a}} \epsilon_{ij} \bar{\xi}^{\dot{\alpha} j} \\ &\quad - 2\bar{Y}^{\dot{\alpha}}{}_\beta \chi^\mu f_{\mu i}^{\dot{\alpha}\bar{a}} \epsilon_{ij} \bar{\xi}^{\beta j} - \Gamma_{\mu\bar{b}}^{\dot{\alpha}} \delta\phi^\mu \bar{\zeta}^{\dot{\alpha}\bar{b}}. \end{aligned} \quad (4.45)$$

This is an on-shell supersymmetry algebra only. The component Lagrangian is

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2} \mathcal{D}^m \phi^\mu \mathcal{D}_m \phi^\nu g_{\mu\nu} - \frac{i}{4} g_{b\bar{b}} (\zeta^{\alpha b} \overleftrightarrow{\mathcal{D}}_{\alpha\dot{\alpha}} \bar{\zeta}^{\dot{\alpha}\bar{b}}) + \frac{1}{16} \zeta^a \zeta^b \bar{\zeta}^{\dot{a}} \bar{\zeta}^{\dot{b}} R_{a\dot{a}b\dot{b}} \\ &\quad + g \zeta^a \lambda_i^{\hat{I}} J_{Ia}^i + g \bar{\zeta}^{\dot{a}} \bar{\lambda}^{\dot{I}i} J_{I\dot{a}i} - 2g^2 X^{\hat{I}} \bar{X}^{\hat{J}} J_{\hat{I}}^\mu J_{\hat{J}}^\nu g_{\mu\nu} + g Y_{ij}^{\hat{I}} D_I^{ij} \\ &\quad - \frac{g}{4} \zeta^a \zeta^b X^{\hat{I}} \nabla_\mu J_{\hat{I}\nu} f_{aj}{}^\mu f_b{}^{j\nu} - \frac{g}{4} \bar{\zeta}^{\dot{a}} \bar{\zeta}^{\dot{b}} \bar{X}^{\hat{I}} \nabla_\mu J_{\hat{I}\nu} f_{\dot{a}j}{}^\mu f_{\dot{b}}{}^{j\nu} \\ &\quad + \mathcal{L}_S + \mathcal{L}_{G_{ij}} + \mathcal{L}_{ZG}. \end{aligned} \quad (4.46)$$

We have written the Riemann curvature term using $R_{a\dot{a}b\dot{b}} := \omega_b^c \omega_{\dot{c}}^d R_{abcd}$. This and other terms in the first three lines are straightforward covariantizations of the general gauged hyperkähler sigma model in a Minkowski background. The covariant derivatives are

$$\begin{aligned} \mathcal{D}_m \phi^\mu &= \partial_m \phi^\mu - A_m^{\hat{I}} J_{\hat{I}}^\mu, \\ \hat{\mathcal{D}}_m \zeta_\alpha^b &= \partial_m \zeta_\alpha^b + \frac{1}{2} \omega_m^{ab} (\sigma_{ab})_\alpha{}^\beta \zeta_\beta^b + i A_m \zeta_\alpha^b \\ &\quad - \frac{1}{2} A_m^{\hat{I}} \zeta_\alpha^c f_c{}^{j\mu} \nabla_\mu J_{\hat{I}}^\nu f_{\nu j}{}^b + \mathcal{D}_m \phi^\mu \zeta_\alpha^c \Gamma_{\mu c}^b. \end{aligned} \quad (4.47)$$

The remaining terms have been set apart corresponding to their dependence on the background fields. The terms in \mathcal{L}_S are present only when S^{ij} is non-vanishing:

$$\begin{aligned} \mathcal{L}_S &= -\frac{1}{8} \bar{\mu} \zeta^a \zeta^b f_{aj}{}^\mu f_b{}^{j\nu} \nabla_\mu V_\nu - \frac{1}{8} \mu \bar{\zeta}^{\dot{a}} \bar{\zeta}^{\dot{b}} f_{\dot{a}j}{}^\mu f_{\dot{b}}{}^{j\nu} \nabla_\mu V_\nu \\ &\quad - \frac{1}{2} |\mu|^2 V^\mu V_\mu + 3|\mu|^2 \mathcal{K} - g\mu X^{\hat{I}} J_{\hat{I}}^\mu V_\mu - g\bar{\mu} \bar{X}^{\hat{I}} J_{\hat{I}}^\mu V_\mu \\ &\quad - 2g\mu X^{\hat{I}} D_I^{ij} v_{ij} - 2g\bar{\mu} \bar{X}^{\hat{I}} D_I^{ij} v_{ij} - 6g\mu X^{(k)} D_{(k)}^{ij} v_{ij}. \end{aligned} \quad (4.48)$$

Here we left S^{ij} complex, i.e. $S^{ij} = \mu v^{ij}$ and $\bar{S}^{ij} = \bar{\mu} v^{ij}$, to distinguish which terms arise from \bar{S}^{ij} . This distinction will be important in the Euclidean case. Recall that when

$S^{ij} \neq 0$, the target space must possess an isometry V^μ that rotates the complex structures. In this case, the triholomorphic isometries associated with frozen vector multiplets may be absorbed into a redefinition of V^μ [55].

The terms involving G_a^{ij} are simplest when written in terms of its dual two-form,

$$\begin{aligned} \mathcal{L}_{G_{ij}} &= \epsilon^{mnpq} B_{mnij} \left(\mathcal{D}_p \phi^\mu \mathcal{D}_q \phi^\nu \Omega_{\mu\nu}{}^{ij} + g F_{pq}{}^{\hat{I}} D_{\hat{I}}{}^{ij} \right) \\ &= \epsilon^{mnpq} B_{mnij} \partial_p \phi^\mu \partial_q \phi^\nu \Omega_{\mu\nu}{}^{ij} - \frac{2}{3} g \epsilon^{mnpq} H_{mnp}{}^{ij} A_q{}^{\hat{I}} D_{\hat{I}}{}^{ij}, \end{aligned} \quad (4.49)$$

with equality holding up to total derivatives. The first term is just the spacetime pullback of the hyperkähler two-forms. This expression may be interpreted as the hyperkähler analogue of the special Kähler couplings involving G^a (4.26). It should be emphasized that when G_a^{ij} is present, no constraint is placed on the hyperkähler target space.

Finally, we give the remaining terms involving \mathcal{Z}_{ab} or G_a . We give them both in terms of \mathcal{Z}_{ab} and in terms of $Y_{ab}^- = \mathcal{Z}_{ab}^-$ and $W_{ab}^+ = -\mathcal{Z}_{ab}^+$:

$$\begin{aligned} \mathcal{L}_{ZG} &= \frac{1}{8} (W_{ab}^- Y^{ab-} + W_{ab}^+ Y^{ab+}) \chi + 4G^a G_a \chi - \frac{1}{4} \zeta^{\alpha a} \zeta^{\beta b} W_{\alpha\beta} \omega_{ab} - \frac{1}{4} \bar{\zeta}^{\dot{\alpha} \bar{a}} \bar{\zeta}^{\dot{\beta} \bar{b}} \bar{W}_{\dot{\alpha}\dot{\beta}} \omega_{\bar{a}\bar{b}} \\ &= -\frac{1}{8} \mathcal{Z}_{ab} \bar{\mathcal{Z}}^{ab} \chi + 4G^a G_a \chi + \frac{1}{4} \zeta^{\alpha a} \zeta^{\beta b} \bar{\mathcal{Z}}_{\alpha\beta} \omega_{ab} + \frac{1}{4} \bar{\zeta}^{\dot{\alpha} \bar{a}} \bar{\zeta}^{\dot{\beta} \bar{b}} \mathcal{Z}_{\dot{\alpha}\dot{\beta}} \omega_{\bar{a}\bar{b}}. \end{aligned} \quad (4.50)$$

The hyperkähler potential χ appears only when the target space must be superconformal, that is when either G^a or $Y_{ab}^- = \mathcal{Z}_{ab}^-$ is nonzero.

BPS conditions for hypermultiplets

As with the vector multiplets, it is easy to find the conditions for full supersymmetry:

$$D_a \phi^\mu = 0, \quad \mu V^\mu = -2g \bar{X}^{\hat{I}} J_{\hat{I}}{}^\mu, \quad G_a \chi^\mu = Y_{ab}^\pm \chi^\mu = 0. \quad (4.51)$$

The first condition is simple enough to understand: covariant constancy of the hypermultiplet scalars. The second condition leads to a complicated alignment criterion between the hyperscalars and the vector scalars that does not always admit a solution.¹⁵ The last condition implies that when G_a or Y_{ab}^\pm are nonzero, in which case the target space is a cone, the scalars must lie at the origin of the cone where χ^μ vanishes.

The off-shell origin of on-shell hypermultiplets

We have given the on-shell hypermultiplet SUSY transformations and action without derivation. It turns out they can be derived directly from the off-shell formulation for hypermultiplets given in curved projective superspace [21, 37, 56]. Let us briefly sketch this topic using the conventions of [57].

An off-shell hypermultiplet is described by a complex arctic superfield Υ^+ living on $\mathcal{M}^{4|8} \times \text{SU}(2)$ where $\mathcal{M}^{4|8}$ is the original $\mathcal{N} = 2$ superspace and $\text{SU}(2)$ is an auxiliary manifold parametrizing the infinite number of auxiliary fields, with coordinates v^{i+} and

¹⁵For instance, take Minkowski spacetime ($\mu = 0$) and target space \mathbb{R}^4 and gauge a constant shift symmetry with a central charge. Then $X^{(i)} J_{(i)}{}^\mu$ is everywhere non-vanishing.

$v_i^- = (v^{i+})^*$ obeying $v^{i+}v_i^- = 1$. Actually, only the space $\mathbb{C}P^1 = \text{SU}(2)/\text{U}(1)$ plays a role, and the charge on the superfield Υ^+ denotes its weight under $\text{U}(1)$. A general hypermultiplet Lagrangian is described in flat superspace by an action

$$S = -\frac{1}{2\pi} \oint_{\mathcal{C}} v_i^+ dv^{i+} \int d^4x d^4\theta^+ \mathcal{F}^{++}(\Upsilon^+, \check{\Upsilon}^+, v^{i+}) \quad (4.52)$$

where \mathcal{F}^{++} is an arbitrary charge-two function of its arguments: the arctic superfield Υ^+ , its antarctic conjugate $\check{\Upsilon}^+$, and the auxiliary coordinates v^{i+} . The auxiliary integral is over a contour \mathcal{C} in $\mathbb{C}P^1$. When no further restriction is imposed on \mathcal{F}^{++} , the target space is a generic hyperkähler manifold upon eliminating the auxiliary fields.

When coupled to conformal supergravity, the action generalizes to

$$S = -\frac{1}{2\pi} \oint_{\mathcal{C}} d\tau \int d^4x d^4\theta^+ \mathcal{E}^{--} \mathcal{F}^{++}(\Upsilon^+, \check{\Upsilon}^+) \quad (4.53)$$

where \mathcal{E}^{--} is an appropriate superspace measure, including the generalization of $v_i^+ \partial_\tau v^{i+}$, where τ parametrizes the contour \mathcal{C} . The Lagrangian \mathcal{F}^{++} is now superconformal, possessing no explicit dependence on v^{i+} . The component reduction of this class was discussed in [47], where it was shown how to recover the Lagrangian of a hyperkähler cone coupled to conformal supergravity [53].

For the rigid supergeometries of interest in this paper, there are three cases to consider:

- (1) The most restrictive case is when Y_{ab}^\pm and/or G_a are turned on; then the SUSY algebra generates the full $\text{SU}(2)_R$ group. This requires that the superspace Lagrangian \mathcal{F}^{++} is covariant under $\text{SU}(2)$ diffeomorphisms, so it cannot depend explicitly on the coordinates v^{i+} . It takes the same form as (4.53) and upon reduction to components leads to a hyperkähler cone.
- (2) The next case is when $S^{ij} = \mu v^{ij}$, corresponding to AdS_4 in Lorentzian signature. The most general action is of the form [26]

$$S = -\frac{1}{2\pi} \oint_{\mathcal{C}} d\tau \int d^4x d^4\theta^+ \mathcal{E}^{--} \mathcal{F}^{++}(\Upsilon^+, \check{\Upsilon}^+, v^{++}), \quad v^{++} = v^{ij} v_i^+ v_j^+ . \quad (4.54)$$

Here only an $\text{SO}(2)_R$ subgroup of $\text{SU}(2)_R$ must be preserved. At the component level, this leads to a hyperkähler target space with an $\text{SO}(2)_R$ isometry that rotates the complex structures. These actions have already been discussed extensively in four dimensions [48, 49, 55]; similar results hold in five [58, 59] and three dimensions [60].

- (3) The final case involves background field configurations with only W_{ab}^\pm or G_a^{ij} . Because no $\text{SU}(2)_R$ symmetry survives in the SUSY algebra, the Lagrangian need not respect $\text{SU}(2)$ diffeomorphisms and may depend arbitrary on the coordinates v^{i+} ,

$$S = -\frac{1}{2\pi} \oint_{\mathcal{C}} d\tau \int d^4x d^4\theta^+ \mathcal{E}^{--} \mathcal{F}^{++}(\Upsilon^+, \check{\Upsilon}^+, v^{i+}) . \quad (4.55)$$

This leads, as in a flat background, to an unconstrained hyperkähler target space.

The procedure of reducing the various cases above to the explicit on-shell actions is a straightforward application of the covariant techniques of [47]. Because it is rather cumbersome, we do not give the derivation explicitly. A more roundabout derivation will be sketched for cases (1) and (2) in the next section.

4.3 Conformal supergravity and the origin of rigid actions

In the preceding sections, we have emphasized the origin of the vector multiplet and hypermultiplet actions within a purely rigid supersymmetric framework, sketching their derivation from rigid superspace. It is instructive to briefly discuss how to reproduce the above results within the context of conformal supergravity and existing component actions.¹⁶

This approach is easiest to understand when applied to the vector multiplets. When frozen vector multiplets $X^{(i)}$ are included, the prepotentials $F(X^I)$ may be lifted (albeit non-uniquely) to homogeneous conformal prepotentials $F(X^I, X^{(i)})$. The general action coupling vector multiplets to conformal supergravity was given in [40]. Moving from the conformal framework to the rigid supersymmetric framework involves freezing the vector multiplets $X^{(i)}$ to constant values (a Weyl-U(1) gauge-fixing), turning off all background fermions including the gauginos $\lambda^{(i)}$ (a choice of S -gauge), and setting to zero the dilatation connection (a conformal gauge-fixing). The auxiliary field D , the Ricci scalar, and the auxiliary fields T_{ab}^\pm are fixed as

$$\begin{aligned}
 D &= \frac{1}{12} \mathcal{Z}_{ab} \bar{\mathcal{Z}}^{ab}, & \frac{1}{4} T_{ab}^- &\equiv W_{ab}^- = -\bar{\mathcal{Z}}_{ab}^-, & \frac{1}{4} T_{ab}^+ &\equiv W_{ab}^+ = -\mathcal{Z}_{ab}^+, \\
 \mathcal{R} &= -\mathcal{Z}_{ab} \bar{\mathcal{Z}}^{ab} + 6S_{ij} \bar{S}^{ij} + 24G^2 + 12G^{aij} G_{aij},
 \end{aligned}
 \tag{4.56}$$

and one must redefine the R -symmetry connections, as discussed in footnote 4. The field strengths and auxiliary fields of the frozen vector multiplets are

$$F_{ab}^{(i)} = -\mathcal{Z}_{ab} X^{(i)} - \bar{\mathcal{Z}}_{ab} \bar{X}^{(i)}, \quad Y_{ij}^{(k)} = -2S_{ij} X^{(k)} = -2\bar{S}_{ij} \bar{X}^{(k)}. \tag{4.57}$$

For the hypermultiplets, the problem is more subtle. If the hyperkähler target space we seek is a cone, we may directly couple it to conformal supergravity, apply the identifications (4.56) for the Weyl multiplet, and redefine the $U(1)_R$ and $SU(2)_R$ connections. However, in the case that the target space is not a cone, one must introduce additional hypermultiplet compensators and identify the appropriate rigid limit.

For the class (2) given in (4.54), the object $v^{++} = v^{ij} v_i^+ v_j^+$ may be identified as a frozen tensor multiplet $L^{++} = L^{ij} v_i^+ v_j^+$. In the component setting, one can take a hyperkähler cone with an abelian isometry, dualize the hypermultiplets associated with the isometry into a tensor multiplet (and reintroduce its auxiliary fields), and then freeze the multiplet to a rigid configuration. A residual $SO(2)_R$ isometry will survive, as required.

For the class (3) given in (4.55), one may identify v^{i+} as frozen values of an arctic multiplet Υ_0^+ and its conjugate $\check{\Upsilon}_0^+$. However, these multiplets must be frozen *prior to the elimination of the auxiliary fields* — they are *not* on-shell multiplets — and so it is unclear

¹⁶For a complete and pedagogical review of conformal supergravity-matter systems, we refer the reader to the recent textbook [61].

to us how the superspace procedure is related to taking the rigid $4n$ -dimensional limit of a $4n + 4$ -dimensional hyperkähler cone. Thankfully, there are no terms in the action unique to this class — all expressions in the rigid hypermultiplet action can be compared with the rigid Minkowski case, the hyperkähler cone action coupled to conformal supergravity [53], or the class (2) sketched above.

A related question is whether the actions and their corresponding supersymmetry algebras can be derived *dynamically* from some supergravity-matter action. We will return to this issue in the final section.

4.4 A simple example: the $\mathcal{N} = 2^*$ action

As a simple application, we will give the $\mathcal{N} = 2^*$ Lagrangian in a rigid background. The matter content consists of n vector multiplets X^I transforming in some compact non-Abelian gauge group of dimension n with metric $g_{IJ} = \delta_{IJ}$, coupled to $4n$ hypermultiplets transforming in the adjoint. We set the θ term to zero for simplicity. In a Minkowski background, the on-shell field content is an $\mathcal{N} = 4$ multiplet; the $\mathcal{N} = 2^*$ theory arises after giving the hypermultiplet a mass by coupling it to a background vector multiplet.

Let us sketch here the relevant geometric data for the hyperkähler manifold. We choose a complex basis for the bosons so that the third complex structure is diagonal,

$$\phi^\mu = (A^I, B_I, \bar{A}_I, \bar{B}^I), \quad (\mathcal{J}_3)^\mu{}_\nu = \text{diag}(i\delta^I{}_J, i\delta_I{}^J, -i\delta^I{}_J, -i\delta_I{}^J). \quad (4.58)$$

Because $g_{IJ} = \delta_{IJ}$, the complex fields A^I and B_I transform in the same (adjoint) representation. The three hyperkähler two-forms are

$$\Omega_{1\bar{1}} = dA^I \wedge dB_I, \quad \Omega_{2\bar{2}} = d\bar{A}_I \wedge d\bar{B}^I, \quad \Omega_{1\bar{2}} = \frac{1}{2}dA^I \wedge d\bar{A}_I + \frac{1}{2}dB_I \wedge d\bar{B}^I. \quad (4.59)$$

The target space is a cone, so $\chi = A^I \bar{A}_I + B_I \bar{B}^I$ is the hyperkähler potential for all complex structures. The fermions are

$$\zeta^a = (\psi^I, \rho_I), \quad \bar{\zeta}^{\bar{a}} = (\bar{\psi}_I, \bar{\rho}^I) \quad (4.60)$$

and the target space vielbein $f_{\mu i}{}^a$ can be identified from $\phi^\mu f_{\mu \underline{1}}{}^a = (A^I, B_I)$ and $\phi^\mu f_{\mu \underline{2}}{}^a = (-\bar{B}^I, \bar{A}_I)$. We charge the hypermultiplets under a $U(1)$ associated with the background vector multiplet $X^{(1)} = 1$ so that A and ψ have charge $+e$ while B and ρ have charge $-e$.

In a Minkowski background, the Lagrangian is

$$\begin{aligned} \mathcal{L} = & -\mathcal{D}_m \bar{A}_I \mathcal{D}^m A^I - \mathcal{D}_m \bar{B}^I \mathcal{D}^m B_I - \mathcal{D}_m \bar{X}^I \mathcal{D}^m X^I \\ & - \frac{i}{4} \psi^I \overleftrightarrow{\mathcal{D}} \bar{\psi}_I - \frac{i}{4} \rho_I \overleftrightarrow{\mathcal{D}} \bar{\rho}^I - \frac{i}{4} \lambda_j^I \overleftrightarrow{\mathcal{D}} \bar{\lambda}^{jI} - \frac{1}{8} F_{ab}{}^I F^{abI} \\ & + \frac{1}{2} g X^{\dot{i}jI} \lambda_i^J \lambda_j^K f_{IJK} + \frac{1}{2} g X_{\dot{i}j}{}^I \bar{\lambda}^{iJ} \bar{\lambda}^{jK} f_{IJK} - \frac{1}{2} g^2 \text{Tr}([X_{\dot{i}j}, X_{\dot{k}l}][X^{\dot{i}j}, X^{\dot{k}l}]) \\ & - 2e^2 (A^I \bar{A}_I + B_I \bar{B}^I) - 2ie(\psi^I \rho_I) + 2ie(\bar{\psi}_I \bar{\rho}^I), \end{aligned} \quad (4.61)$$

after integrating out the auxiliary field $Y_{ij}{}^I$. The $U(1)$ charge corresponds to a hypermultiplet mass $e\sqrt{2}$, and in the massless limit, we recover $\mathcal{N} = 4$ SYM. For that reason, we

have grouped some terms together into an $SU(4)$ covariant form,

$$X^{\mathbf{ij}} = (X_{\mathbf{ij}})^* = \begin{pmatrix} 0 & \bar{X} & -\bar{A} & \bar{B} \\ -\bar{X} & 0 & B & A \\ \bar{A} & -B & 0 & X \\ -\bar{B} & -A & -X & 0 \end{pmatrix}, \quad \lambda_{\mathbf{i}} = (\lambda_i, \psi, \rho) = (\bar{\lambda}^{\mathbf{i}})^*, \quad (4.62)$$

with the index \mathbf{i} labelling the $\mathbf{4}$ of $SU(4)$. We normalize the trace so that

$$\text{Tr}\left([X_{\mathbf{ij}}, X_{\mathbf{kl}}][X^{\mathbf{ij}}, X^{\mathbf{kl}}]\right) = X_{\mathbf{ij}}^I X_{\mathbf{kl}}^J f_{IJ}^K X^{\mathbf{ij}L} X^{\mathbf{kl}M} f_{LM}^K. \quad (4.63)$$

When $S^{ij} = \mu v^{ij}$ is non-vanishing, one additional piece of information is required: the form of the Killing vector V^μ . We choose $v^{ij} = -\delta^{ij}$ so that $V^\mu = (\bar{B}^I, -\bar{A}_I, B_I, -A^I)$; the identification is simple because the target space is a cone. Introducing all of the required couplings leads to

$$\begin{aligned} \mathcal{L} = & -\mathcal{D}_m \bar{A}_I \mathcal{D}^m A^I - \mathcal{D}_m \bar{B}^I \mathcal{D}^m B_I - \mathcal{D}_m \bar{X}^I \mathcal{D}^m X^I \\ & - \frac{i}{4} \psi^I \overleftrightarrow{\mathcal{D}} \bar{\psi}_I - \frac{i}{4} \rho_I \overleftrightarrow{\mathcal{D}} \bar{\rho}^I - \frac{i}{4} \lambda_j^I \overleftrightarrow{\mathcal{D}} \bar{\lambda}^{jI} - \frac{1}{8} F_{ab}^I F^{abI} \\ & + \frac{1}{2} F_{ab}^I (W^{ab+} X^I + W^{ab-} \bar{X}^I) + \mathcal{L}_{BF} + \mathcal{L}_{\text{pot}} + \mathcal{L}_{\text{ferm}}. \end{aligned} \quad (4.64)$$

The potential terms are given by

$$\begin{aligned} \mathcal{L}_{\text{pot}} = & 2(|\mu|^2 - e^2)(A^I \bar{A}_I + B_I \bar{B}^I) + 2|\mu|^2 X^I \bar{X}^I + 2i \mu e (A^I B_I - \bar{A}_I \bar{B}^I) \\ & - \frac{1}{4} Z_{ab} \bar{Z}^{ab} \left(X^I \bar{X}^I + \frac{1}{2} A^I \bar{A}_I + \frac{1}{2} B_I \bar{B}^I \right) - \frac{1}{4} (W_{ab}^+)^2 X^I X^I - \frac{1}{4} (W_{ab}^-)^2 \bar{X}^I \bar{X}^I \\ & + 2G_{a ij} G^{a ij} X^I \bar{X}^I + 4G^2 (A^I \bar{A}_I + B_I \bar{B}^I) \\ & - \frac{1}{2} g^2 \text{Tr}\left([X_{\mathbf{ij}}, X_{\mathbf{kl}}][X^{\mathbf{ij}}, X^{\mathbf{kl}}]\right). \end{aligned} \quad (4.65)$$

Because we chose $X^{(1)} = 1$, μ must be real. The fermionic couplings are

$$\begin{aligned} \mathcal{L}_{\text{ferm}} = & (\lambda_i^I \sigma^a \bar{\lambda}^{jI}) \left(\frac{1}{2} \delta_j^i G_a + i G_a^i{}_j \right) - 2ie(\psi^I \rho_I) + 2ie(\bar{\psi}_I \bar{\rho}^I) \\ & + \frac{1}{2} \psi^{\alpha I} \rho^\beta{}_I \bar{Z}_{\alpha\beta} + \frac{1}{2} \bar{\psi}_{\dot{\alpha} I} \bar{\rho}_{\dot{\beta}}{}^I Z^{\dot{\alpha}\dot{\beta}} \\ & + \frac{1}{2} g X^{\mathbf{ij}I} \lambda_i^J \lambda_j^K f_{IJK} + \frac{1}{2} g X_{\mathbf{ij}}^I \bar{\lambda}^{iJ} \bar{\lambda}^{jK} f_{IJK}. \end{aligned} \quad (4.66)$$

Finally, the generalized BF terms are

$$\begin{aligned} \mathcal{L}_{BF} = & 2i \epsilon^{mnpq} B_{mn} \partial_p X^I \partial_q \bar{X}^I + \epsilon^{mnpq} B_{mn}{}^{ij} \partial_p \phi^\mu \partial_q \phi^\nu \Omega_{\mu\nu ij} \\ & + \frac{2}{3} g \epsilon^{mnpq} H_{mnp} A_q^I D_I - \frac{2}{3} g \epsilon^{mnpq} H_{mnp}{}^{ij} A_q^I D_I{}_{ij}, \end{aligned} \quad (4.67)$$

where the special Kähler and hyperkähler moment maps are

$$\begin{aligned} D_I = & -i X^J \bar{X}^K f_{JKI}, & D_{I\bar{1}\bar{2}} = & \frac{1}{2} (A^J \bar{A}^K + B^J \bar{B}^K) f_{JKI}, \\ D_{I\bar{1}\bar{1}} = & A^J B^K f_{JKI}, & D_{I\bar{2}\bar{2}} = & \bar{A}^J \bar{B}^K f_{JKI}. \end{aligned} \quad (4.68)$$

5 The general Euclidean supersymmetry algebra

Now we turn to the case of Euclidean SUSY. As in the Lorentzian case, we seek first a general rigid Euclidean superalgebra admitting eight rigid supersymmetries. A straightforward way to construct such an algebra is via analytic continuation from the Lorentzian case; the approach we follow resembles that chosen by [11].

This can briefly be described as follows. First complexify the Lorentzian superalgebra (2.18), relaxing the reality conditions on all of the operators and fields. The algebra still closes because the Bianchi identity is holomorphic in these quantities. Next, one makes a Wick rotation on all vector quantities V , taking $V_0 \rightarrow iV_4$ and $V^0 \rightarrow -iV^4$, and similarly for any tensors. The algebra retains the same form provided we take $\sigma^0 = -i\sigma^4$ and $\bar{\sigma}^0 = -i\bar{\sigma}^4$ and replace η_{ab} with the Euclidean δ_{ab} . We also exchange the Lorentzian ϵ_{abcd} (with $\epsilon_{0123} = 1$) for $i\epsilon_{abcd}$ (with $\epsilon_{1234} = 1$). Now one must impose a reality condition so that the momentum generators are Hermitian, but how exactly (if at all) to impose this condition on the supercharges is an interesting question.

This is an old topic in the literature and one can identify two schools of thought. A real Lorentzian supersymmetry algebra maps naturally under Wick rotation to a *reflection positive* Euclidean supersymmetry algebra. This means that a Euclidean SUSY algebra (or action) arising directly from analytic continuation does not need to be real in the conventional sense, but rather real in the sense of Osterwalder and Schrader. This agrees with the approach taken by Nicolai [62]. The second approach, originally proposed by Zumino [63], involves maintaining reality of the SUSY algebra and associated actions by choosing Majorana supercharges, but this is possible only for $\mathcal{N} \geq 2$. Real Euclidean actions and algebras naturally continue to CT -even (but potentially complex) Lorentzian actions and algebras. As noted in [64], the first case automatically gives the correct Green's functions under Wick rotation, while the latter case arises via timelike dimensional reduction from 5D. Because the first possibility is just a Wick rotation of the Lorentzian case (and so should offer no new features), we will focus on the second possibility exclusively.

A real Euclidean superalgebra requires a real Euclidean superspace.¹⁷ We take

$$(\mathcal{D}_a)^* = \mathcal{D}_a, \quad (\mathcal{D}_\alpha^i)^* = -\mathcal{D}^{\alpha i}, \quad (\bar{\mathcal{D}}^{\dot{\alpha}i})^* = -\bar{\mathcal{D}}_{\dot{\alpha}i}, \quad (5.1)$$

so that the Killing spinors are symplectic Majorana-Weyl,

$$(\xi_\alpha^i)^* = \xi^\alpha_i, \quad (\bar{\xi}^{\dot{\alpha}i})^* = \bar{\xi}_{\dot{\alpha}i}. \quad (5.2)$$

Consistency with the flat space Euclidean supersymmetry algebra implies that the R -symmetry group $U(1)$ must map to the non-compact group $SO(1,1)$ corresponding to chiral dilatations [63]. We account for this by exchanging the generator \mathbb{A} for $i\mathbb{U}$ where

$$[\mathbb{U}, \mathcal{D}_\alpha^i] = -\mathcal{D}_\alpha^i, \quad [\mathbb{U}, \bar{\mathcal{D}}^{\dot{\alpha}i}] = +\bar{\mathcal{D}}^{\dot{\alpha}i}. \quad (5.3)$$

We correspondingly continue the $U(1)_R$ connection to an $SO(1,1)_R$ connection. The $SU(2)_R$ generator and connection are unchanged. Note that we keep the notation $\bar{\xi}^{\dot{\alpha}i}$ and $\bar{\mathcal{D}}^{\dot{\alpha}i}$ even though these are not the complex conjugates of the unbarred quantities.

¹⁷Euclidean superspaces for both real and holomorphic SUSY were introduced in [65].

After these modifications, the rigid Euclidean superspace algebra is

$$\begin{aligned}
 \{\mathcal{D}_\alpha^i, \mathcal{D}_\beta^j\} &= 4S^{ij}M_{\alpha\beta} + \epsilon^{ij}\epsilon_{\alpha\beta}Z^{cd}M_{cd} + 2\epsilon^{ij}\epsilon_{\alpha\beta}S^k{}_l I^l{}_k - 4Z_{\alpha\beta}I^{ij}, \\
 \{\bar{\mathcal{D}}^{\dot{\alpha}}{}_i, \bar{\mathcal{D}}^{\dot{\beta}}{}_j\} &= 4\bar{S}_{ij}\bar{M}^{\dot{\alpha}\dot{\beta}} - \epsilon_{ij}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{Z}^{cd}M_{cd} - 2\epsilon_{ij}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{S}^k{}_l I^l{}_k - 4\bar{Z}^{\dot{\alpha}\dot{\beta}}I_{ij}, \\
 \{\mathcal{D}_\alpha^i, \bar{\mathcal{D}}^{\dot{\beta}}{}_j\} &= -2i\delta_j^i(\sigma_a)_{\alpha\dot{\beta}}\mathcal{D}_a + 2(\sigma_a)_{\alpha\dot{\beta}}\epsilon^{abcd}(\delta_j^i G_b + iG_b{}^i{}_j)M_{cd} - 8G_{\alpha\dot{\beta}}I^i{}_j + 2iG_{\alpha\dot{\beta}}{}^i{}_j\mathbb{U}, \\
 [\mathcal{D}_a, \mathcal{D}_\beta^j] &= \frac{i}{2}(\sigma_a)_{\beta\dot{\gamma}}S^{jk}\bar{\mathcal{D}}^{\dot{\gamma}}{}_k - \frac{i}{2}Z_{ab}(\sigma^b)_{\beta\dot{\gamma}}\bar{\mathcal{D}}^{\dot{\gamma}}{}_j - 2iG^b(\sigma_{ba})_{\beta\dot{\gamma}}\mathcal{D}_\gamma^j - G_b{}^j{}_k(\sigma_a\bar{\sigma}^b)_{\beta\dot{\gamma}}\mathcal{D}_\gamma^k, \\
 [\mathcal{D}_a, \bar{\mathcal{D}}^{\dot{\beta}}{}_j] &= \frac{i}{2}(\bar{\sigma}_a)^{\dot{\beta}\gamma}\bar{S}_{jk}\mathcal{D}_\gamma^k + \frac{i}{2}\bar{Z}_{ab}(\bar{\sigma}^b)^{\dot{\beta}\gamma}\mathcal{D}_\gamma^j + 2iG^b(\bar{\sigma}_{ba})^{\dot{\beta}\gamma}\bar{\mathcal{D}}^{\dot{\gamma}}{}_j + G_b{}^k{}_j(\bar{\sigma}_a\sigma^b)^{\dot{\beta}\gamma}\bar{\mathcal{D}}^{\dot{\gamma}}{}_k, \\
 [\mathcal{D}_a, \mathcal{D}_b] &= -\frac{1}{2}R_{ab}{}^{cd}M_{cd}.
 \end{aligned} \tag{5.4}$$

The Riemann tensor is explicitly determined to be

$$\begin{aligned}
 R_{ab}{}^{cd} &= -\frac{1}{2}(Z_{ab}\bar{Z}^{cd} + \bar{Z}_{ab}Z^{cd}) + 8G^2\delta_a{}^{[c}\delta_b{}^{d]} - 16G_{[a}G^{[c}\delta_b{}^{d]} \\
 &\quad + 4G_{ij}^f G_f{}^{ij}\delta_a{}^{[c}\delta_b{}^{d]} - 8G_{[a}^{ij}G_{ij}{}^{c}]\delta_b{}^{d]} + S^{ij}\bar{S}_{ij}\delta_a{}^{[c}\delta_b{}^{d]}.
 \end{aligned} \tag{5.5}$$

It will be useful to retain the same decomposition (2.17) for Z_{ab} . The space is conformally flat when both $Z_{ab}^-\bar{Z}_{cd}^-$ and $Z_{ab}^+\bar{Z}_{cd}^+$ vanish and superconformally flat when $Z_{ab}^+ = \bar{Z}_{ab}^- = 0$.

The Killing spinor equations are

$$\begin{aligned}
 \mathcal{D}_b\xi_\alpha^i &= -2iG^c(\sigma_{cb}\xi^i)_\alpha - G^{ci}{}_j(\sigma_c\bar{\sigma}_b\xi^j)_\alpha - \frac{i}{2}\bar{S}^{ij}(\sigma_b\bar{\xi}_j)_\alpha - \frac{i}{2}\bar{Z}_{bc}(\sigma^c\bar{\xi}^i)_\alpha, \\
 \mathcal{D}_b\bar{\xi}^{\dot{\alpha}i} &= +2iG^c(\bar{\sigma}_{cb}\bar{\xi}^i)^{\dot{\alpha}} - G^{ci}{}_j(\bar{\sigma}_c\sigma_b\bar{\xi}^j)^{\dot{\alpha}} + \frac{i}{2}S^{ij}(\bar{\sigma}_b\xi_j)^{\dot{\alpha}} + \frac{i}{2}Z_{bc}(\bar{\sigma}^c\xi^i)^{\dot{\alpha}}.
 \end{aligned} \tag{5.6}$$

Because ξ_α^i and $\bar{\xi}^{\dot{\alpha}i}$ are symplectic Majorana-Weyl, these two equations are independent. The reality conditions on the curvature fields are

$$(Z_{ab})^* = -Z_{ab}, \quad (S^{ij})^* = -S_{ij}, \quad (G_a)^* = -G_a, \quad (G_a{}^{ij})^* = G_{a ij}, \tag{5.7}$$

and similarly for \bar{Z} and \bar{S}_{ij} . We emphasize that Z_{ab} , S^{ij} and G_a are (pseudo)imaginary in Euclidean signature. Note that the $\text{SO}(1,1)$ weights of the various fields are given by

$$w(S^{ij}) = w(Z_{ab}) = -2, \quad w(G_a) = w(G_a{}^i{}_j) = 0. \tag{5.8}$$

An important feature of Euclidean signature is that the barred and unbarred fields, e.g. Z_{ab} and \bar{Z}_{ab} , are *completely independent*.

6 Euclidean backgrounds

We may again introduce torsion by redefining the spin connection as

$$\tilde{\mathcal{D}}_a := \mathcal{D}_a + i\varepsilon_a{}^{bcd}G_b M_{cd}. \tag{6.1}$$

Because G_a is imaginary in Euclidean signature, this modification leaves the spin connection real. The modified superspace algebra is then

$$\{\mathcal{D}_\alpha^i, \mathcal{D}_\beta^j\} = 4S^{ij}M_{\alpha\beta} + \epsilon^{ij}\epsilon_{\alpha\beta}Z^{cd}M_{cd} + 2\epsilon^{ij}\epsilon_{\alpha\beta}S^k{}_l I^l{}_k - 4Z_{\alpha\beta}I^{ij},$$

$$\begin{aligned}
 \{\bar{\mathcal{D}}^{\dot{\alpha}i}, \bar{\mathcal{D}}^{\dot{\beta}j}\} &= 4\bar{S}_{ij}\bar{M}^{\dot{\alpha}\dot{\beta}} - \epsilon_{ij}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{Z}^{cd}M_{cd} - 2\epsilon_{ij}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{S}^k{}_l I^l{}_k - 4\bar{Z}^{\dot{\alpha}\dot{\beta}}I_{ij}, \\
 \{\mathcal{D}_\alpha{}^i, \bar{\mathcal{D}}_{\dot{\beta}j}\} &= -2i\delta_j^i(\sigma_\alpha)_{\dot{\alpha}\dot{\beta}}\tilde{\mathcal{D}}_a + 2i(\sigma_\alpha)_{\dot{\alpha}\dot{\beta}}\epsilon^{abcd}G_b{}^i{}_j M_{cd} - 8G_{\alpha\dot{\beta}}I^i{}_j + 2iG_{\alpha\dot{\beta}}{}^i{}_j\mathbb{U}, \\
 [\tilde{\mathcal{D}}_a, \mathcal{D}_\beta{}^j] &= \frac{i}{2}(\sigma_a)_{\beta\dot{\gamma}}S^{jk}\bar{\mathcal{D}}^{\dot{\gamma}k} - \frac{i}{2}\bar{Z}_{ab}(\sigma^b)_{\beta\dot{\gamma}}\bar{\mathcal{D}}^{\dot{\gamma}j} - 4iG^b(\sigma_{ba})_{\beta\dot{\gamma}}\mathcal{D}_\gamma{}^j - G_b{}^j{}_k(\sigma_a\bar{\sigma}^b)_{\beta\dot{\gamma}}\mathcal{D}_\gamma{}^k, \\
 [\tilde{\mathcal{D}}_a, \bar{\mathcal{D}}^{\dot{\beta}j}] &= \frac{i}{2}(\bar{\sigma}_a)^{\dot{\beta}\dot{\gamma}}\bar{S}_{jk}\mathcal{D}_\gamma{}^k + \frac{i}{2}\bar{Z}_{ab}(\bar{\sigma}^b)^{\dot{\beta}\dot{\gamma}}\mathcal{D}_{\gamma j} + 4iG^b(\bar{\sigma}_{ba})^{\dot{\beta}\dot{\gamma}}\bar{\mathcal{D}}^{\dot{\gamma}j} + G_b{}^k{}_j(\bar{\sigma}_a\sigma^b)^{\dot{\beta}\dot{\gamma}}\bar{\mathcal{D}}^{\dot{\gamma}k}, \\
 [\tilde{\mathcal{D}}_a, \tilde{\mathcal{D}}_b] &= -\tilde{T}_{ab}{}^c\tilde{\mathcal{D}}_c - \frac{1}{2}\tilde{R}_{ab}{}^{cd}M_{cd}, \tag{6.2}
 \end{aligned}$$

where the torsion and Lorentz curvature tensors are given by

$$\begin{aligned}
 \tilde{T}_{ab}{}^c &= -4i\epsilon_{ab}{}^{cd}G_d, \\
 \tilde{R}_{ab}{}^{cd} &= -\frac{1}{2}(\mathcal{Z}_{ab}\bar{Z}^{cd} + \bar{\mathcal{Z}}_{ab}\mathcal{Z}^{cd}) + 4G_{ij}^f G_f{}^{ij}\delta_a{}^{[c}\delta_b{}^{d]} - 8G_{[a}{}^{ij}G_{ij]}{}^{[c}\delta_b{}^{d]} + S^{ij}\bar{S}_{ij}\delta_a{}^{[c}\delta_b{}^{d]}. \tag{6.3}
 \end{aligned}$$

Now we can analyze the integrability conditions for all of these background fields exactly as in Lorentzian signature, leading again to (2.15). These now lead to *five* possibilities:

- (I) S^{ij} and/or \bar{S}_{ij} nonzero, all other fields vanishing;
- (II) $G_a{}^i{}_j \neq 0$, all other fields vanishing;
- (III) $G_a \neq 0$, perhaps with some of $\mathcal{Z}_{ab}^\pm, \bar{\mathcal{Z}}_{ab}^\pm$ nonzero;
- (IV) \mathcal{Z}_{ab}^\pm and/or $\bar{\mathcal{Z}}_{ab}^\pm$ nonzero, but all other fields vanishing;
- (V) S^{ij} and $\mathcal{Z}_{ab}^+ = -W_{ab}^+$ nonzero, but all other fields vanishing.

The fifth case was not possible in Lorentzian signature. In the Euclidean case, both S^{ij} and $\mathcal{Z}_{ab}^+ = -W_{ab}^+$ are pseudoimaginary and not the complex conjugates of \bar{S}_{ij} and $\bar{\mathcal{Z}}_{ab}^- = -W_{ab}^-$, and so it is possible to keep one set while discarding the other. This leads to a SUSY algebra where only the left or right-handed generators are deformed. Actually, the fact that the right-handed and left-handed SUSY generators are no longer related by complex conjugation leads to somewhat different possibilities in the other cases as well. Such variants correspond to full supersymmetric relatives of the Ω background. The analysis of the R -symmetry connections is analogous to the Lorentzian case.

Now the additional integrability conditions are identical to the Lorentzian case,

$$\begin{aligned}
 S^{ij} &\propto \bar{S}^{ij}, & G_{[a}{}^{ij}G_{b]}{}^{kl} &= 0, & G^a\mathcal{Z}_{ab} &= 0, & \epsilon^{abcd}\mathcal{Z}_{ab}\bar{\mathcal{Z}}_{cd} &= 0, \\
 \mathcal{Z}_{ab}^\pm &\propto \bar{\mathcal{Z}}_{ab}^\pm, & \tilde{\mathcal{D}}_a G_b &= \tilde{\mathcal{D}}_a G_b{}^{ij} = 0, & \tilde{\mathcal{D}}_a \mathcal{Z}_{bc} &= \tilde{\mathcal{D}}_a \bar{\mathcal{Z}}_{bc} = 0. \tag{6.4}
 \end{aligned}$$

As before, \mathcal{Z}_{ab} is a closed complex two-form, but its dual is not closed unless $G_a = 0$. We summarize in table 2 the resulting consistent Euclidean backgrounds.

6.1 S^4 and H^4

The first cases we consider are associated to non-vanishing S^{ij} and \bar{S}^{ij} . For definiteness, let us gauge-fix them to $S^{ij} = i\mu\delta^{ij}$ and $\bar{S}^{ij} = i\bar{\mu}\delta^{ij}$ for real $\mu, \bar{\mu}$. Depending on the relative

Background fields	Geometry
$S^{ij} \neq 0$ and $\bar{S}^{ij} \neq 0$	S^4 and H^4
$G_a^i{}_j \neq 0$	Two-sheeted $H^3 \times \mathbb{R}$
$G_a \neq 0$	$S^3 \times \mathbb{R}$
$\mathcal{Z} \cdot \bar{\mathcal{Z}} > 32 G^2$	Warped $S^3 \times \mathbb{R}$
$\mathcal{Z} \cdot \bar{\mathcal{Z}} = 32 G^2$	Heis ₃ $\times \mathbb{R}$
$\mathcal{Z} \cdot \bar{\mathcal{Z}} < 32 G^2$	Warped one-sheeted $H^3 \times \mathbb{R}$
$\mathcal{Z}_a{}^b \neq 0$ but $G_a = 0$	$H^2 \times S^2$, $\mathbb{R}^2 \times S^2$ and $H^2 \times \mathbb{R}^2$
$S^{ij} \neq 0$, $\mathcal{Z} \neq 0$, $S^{ij} \mathcal{Z}^- = 0$	Flat space (deformed susy)

Table 2. Consistent Euclidean backgrounds.

sign of μ and $\bar{\mu}$, two different superalgebras arise with the associated supercoset spaces:

$$\frac{\text{OSp}(2|4)}{\text{SO}(4) \times \text{SO}(2)} \quad \text{and} \quad \frac{\text{OSp}(2|2, 2)}{\text{SO}(4) \times \text{SO}(2)}, \quad (6.5)$$

with the latter obtained for $\bar{\mu} = -\mu$ and corresponding to the Wick rotation of the AdS₄ spacetime and gives the hyperboloid $\text{SO}(4, 1)/\text{SO}(4)$, noting the isomorphism of the algebras $\text{Sp}(2, 2) \simeq \text{SO}(4, 1)$. The $\text{OSp}(2|4)$ case corresponds to a round S^4 geometry. As usual, further details and explicit expressions for the Killing spinors are given in appendix D.

6.2 Squashed and stretched $S^3 \times \mathbb{R}$ and $S^3 \times S^1$

The (squashed) $\mathbb{R} \times S^3$ geometry generated by G_a and \mathcal{Z}_{ab} in Lorentzian signature can be Wick rotated to Euclidean and the background fields satisfy appropriate reality conditions. In fact, in Euclidean signature we have more freedom in deforming the geometry of the three-sphere and the associated supersymmetry algebra, basically because of the independence of \mathcal{Z} and $\bar{\mathcal{Z}}$.

In analogy with the Lorentzian case, non-vanishing G gives rise to an $S^3 \times \mathbb{R}$ geometry with a (centrally extended) $\text{SU}(2|2) \times \text{SU}(2)$ supersymmetry algebra. The flat direction is generated by the central charge, hence there is no obstruction to compactifying it to S^1 . Turning on \mathcal{Z} and $\bar{\mathcal{Z}}$ fluxes along S^3 and defining $\mathcal{Z}^2 \equiv -8\lambda^2$, $\bar{\mathcal{Z}}^2 \equiv -8\tilde{\lambda}^2$ and $G^2 = -g^2$, we obtain a warped S^3 geometry with metric

$$ds^2 = \frac{v}{16g^2} [d\theta^2 + \sin^2 \theta d\omega^2 + v(d\phi + \cos \theta d\omega)^2] + dz^2 \quad (6.6)$$

and warping parameter $v \equiv 1/(1 + \frac{\lambda\tilde{\lambda}}{4g^2})$, provided $\lambda\tilde{\lambda} > -4g^2$. Notice that independence of \mathcal{Z} and $\bar{\mathcal{Z}}$ permits to source both squashing and stretching of the S^3 . The superspace is again (3.25) with the substitution $P_0 \rightarrow -iG \cdot P$.

When both \mathcal{Z} and $\bar{\mathcal{Z}}$ are non-vanishing, we can set $|\lambda| = |\tilde{\lambda}|$ by an $\text{SO}(1, 1)_R$ gauge choice. An interesting possibility is to set $\bar{\mathcal{Z}} = 0$ and keep $\mathcal{Z} \neq 0$ (or vice versa). In this case there is no squashing of the S^3 geometry, but the supersymmetry algebra is deformed

in the sense that the isometry $SU(2)$ group of the sphere does not preserve chirality of the supercharges. This fact also breaks the residual Lorentz symmetry to the $U(1)$ stabilizing \mathcal{Z} , and the superalgebra is $SU(2|2)$ with two central charges just as in the squashed case.

In full analogy with the discussion in Lorentzian signature, the Killing spinors we find do not depend on the Euclidean time specified by $G \cdot P$, and the R -symmetry connections are vanishing.¹⁸

The requirement for real supersymmetry in Euclidean signature excludes the possibility of an $SU(2|1)^2$ realization of $S^3 \times \mathbb{R}$. This is reflected in the different reality conditions (5.7) for G_a and $G_a^i{}_j$.¹⁹

6.3 Warped one-sheeted $H^3 \times \mathbb{R}$

In the previous section we imposed $\lambda\tilde{\lambda} > -4g^2$ in order to obtain an S^3 geometry. If we consider now $\lambda\tilde{\lambda} < -4g^2$, the resulting manifold is $AdS_3 \times \mathbb{R}$ with a warped Euclidean signature metric. This is basically the converse of what happens for the S^3 with Lorentzian metric in section 3.7.1. The choice of Euclidean signature automatically breaks the isometry group from the natural $SU(1,1)^2$ to $SU(1,1) \times U(1)$. In fact, depending on $\lambda, \tilde{\lambda}$ the geometry is warped along the circular fiber of the Hopf fibration $S^1 \hookrightarrow H^3 \rightarrow H^2$ (this is what we would call timelike warped AdS_3 if the metric were Lorentzian). The supercoset structure of this space is the same as for timelike stretched AdS_3 , cfr. (3.45), the difference being the choice of signature and the fact that we are now allowed to both stretch and squash along the circular fiber. In the same coordinates we use for AdS_3 , we have the metric

$$ds^2 = \frac{v}{16g^2} [v(d\tau + \cosh \rho d\phi)^2 + d\rho^2 + \sinh^2 \rho d\phi^2] + dz^2, \quad v \equiv -\left(1 + \frac{\lambda\tilde{\lambda}}{4g^2}\right)^{-1} > 0. \quad (6.7)$$

6.4 The $Heis_3 \times \mathbb{R}$ limit

We shall now consider the threshold case between S^3 and H^3 , obtained by setting $\lambda\tilde{\lambda} = -4g^2$. In complete analogy with section 3.7.3, the supersymmetry algebra is a non-semisimple contraction of the (centrally extended) $SU(2|2)$ superalgebra of $S^3 \times \mathbb{R}$ containing $Heis_3 \times U(1) \times \mathbb{R}$ as spacetime isometries. The independent factor is generated by $G \cdot P$ as usual, and the central charge of $Heis_3$ is a linear combination of a translation and Lorentz generator. We arrive at the metric

$$ds^2 = (dw + 2g y dx - 2g x dy)^2 + dx^2 + dy^2 + dz^2. \quad (6.8)$$

6.5 The two-sheeted $H^3 \times \mathbb{R}$

Another hyperbolic background arises from $G_a^i{}_j$. Euclidean reality conditions do not allow for $SU(2|1)^2$ or $SU(1,1|1)^2$ superalgebras. What turns out to be allowed instead is

¹⁸We point out again that these connections have been redefined in footnote 4. Our statement is unambiguous in the sense that a theory coupled to this background need not have (full) R -symmetry, even when we compactify to $S^3 \times S^1$.

¹⁹Of course, breaking the reality condition on the fermions one could consider the Wick rotation of the Lorentzian background. Compactification of the flat direction would then always break half the supersymmetries.

the superalgebra $SL(2|1, \mathbb{C})$. Its bosonic part is isomorphic to $SO(3, 1) \times U(1) \times SO(1, 1)_R$. We have therefore the supercoset space

$$\frac{SL(2|1, \mathbb{C})}{SU(2) \times SO(1, 1)_R}, \tag{6.9}$$

which covers one sheet of the hyperboloid $X^2 - Y^2 - V^2 - W^2 = 1$. We may take

$$X = \cosh \rho, \quad Y = \sinh \rho \cos \theta, \quad V = \sinh \rho \sin \theta \cos \phi, \quad W = \sinh \rho \sin \theta \sin \phi. \tag{6.10}$$

These coordinates have range and periodicity $\rho \in [0, +\infty)$, $\theta \in [0, \pi]$ and $\phi \simeq \phi + 2\pi$. The metric is the standard

$$ds^2 = \frac{1}{4g^2} (d\rho^2 + \sinh^2 \rho (d\theta^2 + \sin^2 \theta d\phi^2)) + dz^2. \tag{6.11}$$

Let us comment on the properties of the Killing spinors: the supercharges Q_{α^1} and Q_{α^2} have opposite charges under $G \cdot P$, and the same holds for the antichiral ones. This is reflected in the Killing spinors depending on z with factors $e^{\pm igz}$ (see (D.22)). If we take the universal cover \mathbb{R} for the z direction, these Killing spinors are globally defined. However, compactifying to $H^3 \times S^1$ we find periodic Killing spinors only for $z \simeq z + \frac{2\pi k}{g}$, $k \in \mathbb{Z}$.

6.6 $H^2 \times S^2$ and $D(2, 1; \alpha)$

The appropriate Wick rotation of $AdS_2 \times S^2$ gives an $H^2 \times S^2$ space, where H^2 is (one sheet of) the two-sheeted hyperboloid. The discussion follows again lines similar to the Lorentzian case. The background is determined by \mathcal{Z} and $\bar{\mathcal{Z}}$, breaking the Lorentz symmetry to $U(1) \times U(1)$. One obtains the same real form of $D(2, 1; \alpha)$, and the supercoset space is

$$\frac{D(2, 1; \alpha)}{SU(2)_R \times U(1)^2}, \tag{6.12}$$

with $\alpha \equiv \lambda_+/\lambda_-$. The radii of curvature of H^2 and S^2 are $1/|\lambda_-|$ and $1/|\lambda_+|$ respectively. Let us specify $\mathcal{Z}_{ab} = 2i(\lambda_+ \delta_{ab}^{12} - \lambda_- \delta_{ab}^{34})$ and $\bar{\mathcal{Z}}_{ab} = -2i(\lambda_+ \delta_{ab}^{12} + \lambda_- \delta_{ab}^{34})$. The only differences with the $AdS_2 \times S^2$ case are that now the $SU(1, 1)$ generators read

$$T_{\tilde{a}\tilde{b}} \equiv \begin{pmatrix} \frac{1}{\lambda_-} (iP_3 - P_4) & -iM_{34} \\ -iM_{34} & \frac{1}{\lambda_-} (iP_3 + P_4) \end{pmatrix}, \tag{6.13}$$

and the reality conditions now are

$$(Q_{\tilde{a}\tilde{\alpha}i})^* = (\sigma_1)_{\tilde{a}}^{\tilde{b}} \epsilon^{\tilde{\alpha}\tilde{\beta}} Q_{\tilde{b}\tilde{\beta}}^i, \quad (T_{\tilde{a}\tilde{b}})^* = -(\sigma_1)_{\tilde{a}}^{\tilde{c}} (\sigma_1)_{\tilde{b}}^{\tilde{d}} T_{\tilde{c}\tilde{d}}, \quad (T_{\tilde{\alpha}\tilde{\beta}})^* = \epsilon^{\tilde{\alpha}\tilde{\gamma}} \epsilon^{\tilde{\beta}\tilde{\delta}} T_{\tilde{\gamma}\tilde{\delta}}. \tag{6.14}$$

The analysis of the superalgebra isomorphisms for special values of α is identical to $AdS_2 \times S^2$ up to an obvious difference for $\alpha = \infty$, which now corresponds to $SU(2|2) \rtimes ISO(2)$ associated with an $\mathbb{R}^2 \times S^2$ geometry. Choosing polar coordinates ω , ρ on H^2 , we obtain the metric

$$ds^2 = \frac{1}{\lambda_-^2} (d\rho^2 + \sinh^2 \rho d\omega^2) + \frac{1}{\lambda_+^2} (d\theta^2 + \sin^2 \theta d\phi^2). \tag{6.15}$$

6.7 Deformed supersymmetry in flat space

The facts that in Euclidean signature the reality conditions on spinors do not mix chiralities and the self- and anti self-dual parts of a field strength are independent imply the possibility of deforming only the chiral or the anti-chiral part of the supersymmetry algebra. In our notation, this corresponds to turning on S^{ij} and \mathcal{Z} only. As the Riemann tensor vanishes, these backgrounds correspond to flat Euclidean space with a deformed supersymmetry algebra.

Closure of the supersymmetry algebra imposes the constraint $S^{ij}\mathcal{Z}^- = 0$, so that there are essentially two possibilities: either we turn on a generic \mathcal{Z} , or we turn on a self-dual \mathcal{Z} and S^{ij} . Because this constraint does not arise as an integrability condition for the Killing spinor equations (5.6), we can give a common solution:

$$\xi_\alpha^i = \epsilon_\alpha^i, \quad \bar{\xi}^{\dot{\alpha}}_i = \bar{\epsilon}^{\dot{\alpha}}_i - \frac{i}{2}x^a(S_{ij}\delta_{ab} - \epsilon_{ij}\mathcal{Z}_{ab})\bar{\sigma}^{b\dot{\alpha}\alpha}\epsilon_\alpha^j \quad (S^{ij}\mathcal{Z}_{ab}^- = 0). \quad (6.16)$$

The self-dual part of \mathcal{Z} is proportional to the T^+ tensor of the Weyl multiplet. We identify the background obtained by $\mathcal{Z}^+ \neq 0$ alone as the $\epsilon_1 = -\epsilon_2$ limit of the Ω -background in flat space (see e.g. [11, 66]). A generic Ω -deformation on flat space breaks half of the supersymmetry, so it cannot arise in our analysis.

7 Rigid Euclidean supersymmetric actions

7.1 Vector multiplets

The structure of $\mathcal{N} = 2$ Euclidean vector multiplets has been discussed elsewhere in a number of places (see e.g. [10, 11, 66–69]). Of particular relevance is the work of Cortes et al. which constructs vector multiplets in Euclidean signature via a timelike reduction from 5D [64]. Because their construction naturally leads to a real Euclidean superalgebra, the approach we sketch below will naturally match theirs up to conventions.

It is simplest to motivate the reality conditions of the vector multiplet from superspace. If we straightforwardly analytically continue all of the \mathcal{F}_{AB}^I field strengths, requiring that the new ones be real in Euclidean signature, we discover that \mathcal{X}^I and $\bar{\mathcal{X}}^I$ are imaginary superfields. It will be convenient to make the formal replacements

$$\mathcal{X}^I = i\mathcal{X}_+^I, \quad \bar{\mathcal{X}}^I = -i\mathcal{X}_-^I, \quad (7.1)$$

where \mathcal{X}_+^I and \mathcal{X}_-^I are real chiral and antichiral superfields obeying the Bianchi identity

$$(\mathcal{D}^{ij} + 4S^{ij})\mathcal{X}_+^I = -(\bar{\mathcal{D}}^{ij} + 4\bar{S}^{ij})\mathcal{X}_-^I. \quad (7.2)$$

We define now the covariant components of \mathcal{X}_\pm^I as

$$\lambda_\alpha^i = -i\mathcal{D}_\alpha^i\mathcal{X}_+^I, \quad \bar{\lambda}^{\dot{\alpha}i} = -i\bar{\mathcal{D}}^{\dot{\alpha}i}\mathcal{X}_-^I, \quad Y^{ijI} := -\frac{i}{2}(\mathcal{D}^{ij} + 4S^{ij})\mathcal{X}_+^I \quad (7.3)$$

so that they formally do not change upon continuation to Euclidean signature. The fermions are now symplectic Majorana-Weyl but the auxiliary field is still pseudoreal,

$$(\lambda_\alpha^i)^* = \lambda^{\alpha i}, \quad (\bar{\lambda}^{\dot{\alpha}i})^* = \lambda_{\dot{\alpha}i}, \quad (Y^{ijI})^* = Y_{ij}^I. \quad (7.4)$$

The component two-form field strength is

$$F_{ab}{}^I = \frac{i}{4}(\sigma_{ab})^{\beta\alpha}\mathcal{D}_{\beta\alpha}\mathcal{X}_+^I - \frac{i}{4}(\bar{\sigma}_{ab})^{\dot{\beta}\dot{\alpha}}\bar{\mathcal{D}}_{\dot{\beta}\dot{\alpha}}\mathcal{X}_-^I - i\mathcal{Z}_{ab}\mathcal{X}_+^I + i\bar{\mathcal{Z}}_{ab}\mathcal{X}_-^I. \quad (7.5)$$

We follow the notation \mathcal{X}_{\pm}^I of [64], which denotes the chirality of the associated gaugino. Note that \mathcal{X}_+^I contains the anti self-dual field strength F_{ab}^{-I} .

The supersymmetry transformations are easy to find: one simply makes the replacements $X \rightarrow iX_+$ and $\bar{X} \rightarrow -iX_-$ everywhere, leading to

$$\delta X_+^I = i\xi_i^\alpha \lambda_\alpha^i{}^I, \quad \delta X_-^I = i\bar{\xi}_{\dot{\alpha}}^i \bar{\lambda}_i^{\dot{\alpha}I}, \quad (7.6a)$$

$$\begin{aligned} \delta\lambda_{\alpha i}{}^I &= (F_{ab}{}^I + i\mathcal{Z}_{ab}X_+^I - i\bar{\mathcal{Z}}_{ab}X_-^I)(\sigma^{ab}\xi_i)_\alpha + (Y_{ij}{}^I + 2iS_{ij}X_+^I)\xi_\alpha^j \\ &\quad + 2\mathcal{D}_a X_+^I(\sigma^a\bar{\xi}_i)_\alpha - 4G_{a ij}X_+^I(\sigma^a\bar{\xi}^j)_\alpha - 2gX_+^J\bar{J}_{+j}{}^I\xi_{\alpha i}, \end{aligned} \quad (7.6b)$$

$$\begin{aligned} \delta\bar{\lambda}^{\dot{\alpha}i}{}^I &= (F_{ab}{}^I + i\mathcal{Z}_{ab}X_+^I - i\bar{\mathcal{Z}}_{ab}X_-^I)(\bar{\sigma}^{ab}\bar{\xi}^i)^{\dot{\alpha}} - (Y^{ij}{}^I - 2i\bar{S}^{ij}X_-^I)\bar{\xi}^{\dot{\alpha}j} \\ &\quad + 2\mathcal{D}_a\bar{X}_-^I(\bar{\sigma}^a\xi^i)^{\dot{\alpha}} + 4G_a{}^{ij}\bar{X}_-^I(\bar{\sigma}^a\xi_j)^{\dot{\alpha}} - 2gX_+^J\bar{J}_{-j}{}^I\bar{\xi}^{\dot{\alpha}i}, \end{aligned} \quad (7.6c)$$

$$\begin{aligned} \delta Y_{ij}{}^I &= 2i\xi_{(i}\mathcal{D}\bar{\lambda}_{j)}^I - 4iG_{ak(i}\xi_{j)}\sigma^a\bar{\lambda}^{kI} - 2G_a\xi_{(i}\sigma^a\bar{\lambda}_{j)}^I \\ &\quad - 2i\bar{\xi}_{(i}\mathcal{D}\lambda_{j)}^I + 4iG_{ak(i}\bar{\xi}_{j)}\bar{\sigma}^a\lambda^{kI} - 2G_a\bar{\xi}_{(i}\bar{\sigma}^a\lambda_{j)}^I \\ &\quad + 4ig\xi_{(i}\lambda_{j)}^J\bar{J}_{-J}{}^I - 4ig\bar{\xi}_{(i}\bar{\lambda}_{j)}^J J_{+J}{}^I, \end{aligned} \quad (7.6d)$$

$$\delta A_m{}^I = i(\xi_j\sigma_m\bar{\lambda}^j{}^I) + i(\bar{\xi}^j\bar{\sigma}_m\lambda_j^I). \quad (7.6e)$$

We normalize the vector multiplet action in superspace as

$$- \int d^4x d^4\theta \mathcal{E} F(X_+) + \int d^4x d^4\bar{\theta} \bar{\mathcal{E}} \bar{F}(X_-) \quad (7.7)$$

where $F = F(X_+)$ and $\bar{F} = \bar{F}(X_-)$ are both real functions, i.e. $F(X_+)^* = F(X_+)$. These can be derived from the Lorentzian case by formally replacing $F \rightarrow -iF$ and $\bar{F} \rightarrow -i\bar{F}$ and simultaneously replacing their arguments by (7.1). This amounts to

$$\begin{aligned} X^I &\rightarrow iX_+^I, & \bar{X}^I &\rightarrow -iX_-^I, & F &\rightarrow -iF, & \bar{F} &\rightarrow -i\bar{F}, \\ F_I &\rightarrow -F_I, & \bar{F}_I &\rightarrow \bar{F}_I, & F_{IJ} &\rightarrow iF_{IJ}, & \bar{F}_{IJ} &\rightarrow i\bar{F}_{IJ}. \end{aligned} \quad (7.8)$$

Now the special Kähler potential and metric are given by $K = X_-^I F_I - X_+^I \bar{F}_I$ and $g_{IJ} = F_{IJ} - \bar{F}_{IJ}$. The target space geometry is actually a special para-Kähler manifold given in terms of adapted coordinates. Our conventions here differ somewhat from [64].

Consistent with the modifications (7.8), we should take

$$J_I{}^J \rightarrow iJ_{+I}{}^J, \quad \bar{J}_I{}^J \rightarrow -iJ_{-I}{}^J, \quad D_I \rightarrow iD_I^{(E)}, \quad C_{\hat{I},\hat{J}\hat{K}} \rightarrow iC_{\hat{I},\hat{J}\hat{K}} \quad (7.9)$$

where the real Euclidean moment map $D_I^{(E)}$ is given by

$$D_{\hat{I}}^{(E)} = f_{\hat{I}\hat{J}}{}^K (X^{\hat{J}}\bar{F}_K + \bar{X}^{\hat{J}}F_K) - C_{\hat{I},\hat{J}\hat{K}} X^{\hat{J}}\bar{X}^{\hat{K}}. \quad (7.10)$$

The symplectic vectors are not just the straightforward Wick rotations of the Lorentzian case. An additional factor of i is needed for the dual field strengths to account

for exchanging the Lorentzian ϵ_{abcd} for the Euclidean ε_{abcd} . Our choice of conventions above account for this so that the symplectic vectors take the same form, (X_+^I, F_I) and (X_-^I, \bar{F}_I) . Duality transformations are still described by $\text{ISp}(2n, \mathbb{R})$ but now the inhomogeneous terms are real numbers R_\pm^I and R_I^\pm . This reflects the presence of two real background vector multiplets $X_\pm^{(i)}$ that can be introduced in each case, corresponding to the *two real central charges* possible in the Euclidean supersymmetry algebra. A conventional choice for the background vector multiplets is

$$X_+^{(1)} = X_-^{(1)} = 1, \quad X_+^{(2)} = -X_-^{(2)} = 1, \quad (7.11)$$

but sometimes a different choice is necessary. As before, the presence of various background fields affects the geometry and the possibilities for the frozen multiplets. If G_a^{ij} is present so that an $\text{SO}(1, 1)_R$ symmetry is maintained, the frozen multiplets must be absent and the action must be superconformal, with the prepotentials both homogeneous of degree two. If S^{ij} and/or \bar{S}^{ij} is present, the background multiplets must obey $S^{ij} X_+^{(i)} = -\bar{S}^{ij} X_-^{(i)}$ so only one choice of background vector multiplets is possible. Generally if both S^{ij} and \bar{S}^{ij} are non-vanishing, we can choose either $S^{ij} = \bar{S}^{ij}$ or $S^{ij} = -\bar{S}^{ij}$ via an $\text{SO}(1, 1)_R$ gauge choice; this allows either $X_\pm^{(2)}$ or $X_\pm^{(1)}$ given above but not both. In the degenerate case $S^{ij} \neq 0$ but $\bar{S}^{ij} = 0$, we must have $X_+^{(1)} = X_+^{(2)} = 0$, and only one of $X_-^{(1)}$ or $X_-^{(2)}$ is needed. This situation has no analogue in the Lorentzian theory.

We will not explicitly give the Euclidean action, but it is straightforward to write down by applying the necessary changes to the Lorentzian action.

7.2 Hypermultiplets

Unlike the vector multiplets, hypermultiplets in Euclidean signature can be defined without any alteration of the target space. This can be understood in the framework of [64] by dimensionally reducing the 5D hypermultiplet action along a timelike circle. The reality properties of the hyperkähler metric and its associated vielbeins $f_{\mu i}^a$ are unchanged. Requiring the same supersymmetry conditions for $\delta\phi^\mu$ as in the first line of (4.45) leads to the following reality conditions for the fermions:

$$(\zeta_\alpha^b)^* = \omega^{\bar{b}c} \zeta^{\alpha c} = g^{\bar{b}b} \omega_{bc} \zeta^{\alpha c}, \quad (\bar{\zeta}^{\dot{\alpha} b})^* = g^{b\bar{b}} \omega_{\bar{b}\bar{c}} \bar{\zeta}^{\dot{\alpha} \bar{c}} = \omega^{bc} g_{c\bar{c}} \bar{\zeta}^{\dot{\alpha} \bar{c}}. \quad (7.12)$$

It would be reasonable to denote the antichiral spinor as $\bar{\zeta}^{\dot{\alpha} b}$ by contracting with a factor involving the $\text{Sp}(n)$ metric and symplectic two-form but we will avoid doing so to keep our formulae as similar to the Lorentzian case as possible.

Now it is quite easy to convert the Lorentzian SUSY rules and action to Euclidean signature. The only issue to keep in mind is that we must now take

$$S^{ij} = i\mu v^{ij}, \quad \bar{S}^{ij} = i\tilde{\mu} v^{ij}, \quad (7.13)$$

with μ and $\tilde{\mu}$ in principle different. This means we must analytically continue $\mu \rightarrow i\mu$ and $\tilde{\mu} \rightarrow i\tilde{\mu}$ from the Lorentzian formulae. The SUSY transformations become

$$\delta\phi^\mu = \xi_i \zeta^b f_b^{i\mu} + \bar{\xi}^i \bar{\zeta}^{\bar{b}} f_{\bar{b}i}^{\bar{\mu}},$$

$$\begin{aligned}
\delta\zeta_\alpha^a &= \left(2i\mathcal{D}_{\alpha\dot{\beta}}\phi^\mu - 4G_{\alpha\dot{\beta}}\chi^\mu\right)f_{\mu i}{}^a\bar{\xi}^{\dot{\beta}i} + \left(2i\mu V^\mu - 4igX_-^{\hat{I}}J_{\hat{I}}^\mu\right)f_{\mu i}{}^a\epsilon^{ij}\xi_{\alpha j} \\
&\quad + 2Y_\alpha{}^\beta\chi^\mu f_{\mu i}{}^a\epsilon^{ij}\xi_{\beta j} - \Gamma_{\mu\dot{b}}{}^a\delta\phi^\mu\zeta_\alpha^{\dot{b}}, \\
\delta\bar{\zeta}^{\dot{\alpha}\bar{a}} &= \left(2i\mathcal{D}^{\dot{\alpha}\beta}\phi^\mu + 4G^{\dot{\alpha}\beta}\chi^\mu\right)f_{\mu}{}^{i\bar{a}}\xi_{\beta i} - \left(2i\tilde{\mu}V^\mu + 4igX_+^{\hat{I}}J_{\hat{I}}^\mu\right)f_{\mu}{}^{i\bar{a}}\epsilon_{ij}\bar{\xi}^{\dot{\alpha}j} \\
&\quad - 2\bar{Y}^{\dot{\alpha}}{}_{\dot{\beta}}\chi^\mu f_{\mu}{}^{i\bar{a}}\epsilon_{ij}\bar{\xi}^{\dot{\beta}j} - \Gamma_{\mu\bar{b}}{}^{\bar{a}}\delta\phi^\mu\bar{\zeta}^{\dot{\alpha}\bar{b}}.
\end{aligned} \tag{7.14}$$

We do not explicitly give the action in Euclidean signature, but it is easy to work out by making the appropriate replacements in the Lorentzian action.

8 Discussion and conclusions

In this paper we classified the rigid backgrounds and actions that admit full (real) $\mathcal{N} = 2$ supersymmetry in Lorentzian and Euclidean signatures. The principle guiding this classification has been the identification of the supercoset spaces which arise in curved superspace when the requirement of full supersymmetry is imposed. It is worth noting that similar results as those in section 2 were found for full $\mathcal{N} = 1$ Lorentzian supersymmetry in 5D [16]. Our analysis regards the supercoset spaces as global manifolds. Quotients by discrete isometries are allowed if they preserve the supercharges. For example, in an appropriate Lorentz gauge it is straightforward to see that the $SU(2|2)$ realizations of $S^3 \times \mathbb{R}$ admit quotienting by discrete right isometries. This includes in particular lens spaces S^3/\mathbb{Z}_p , where the quotient acts freely on the Hopf fiber, and warping is also allowed. We leave to future work a full analysis of all discrete quotients for general backgrounds.

One interesting issue that we have not addressed is the dynamical origin of the rigid supersymmetric backgrounds. In particular, which of them solve Einstein's equations of some 4D supergravity? Since any such theory is on-shell equivalent to conformal supergravity coupled to two compensators — a vector multiplet and hypermultiplet — one can look for simultaneous solutions of the BPS conditions (4.36) and (4.51). The equation of motion of the Weyl multiplet auxiliary D relates the hyperkähler potential and the special Kähler potential to the Planck scale; because both potentials require non-vanishing VEVs for the vector and hyperkähler scalars, one must choose $G_a = Y_{ab}^\pm = G_a{}^{ij} = 0$ in either signature. This reproduces the well known fact [70] that the only fully supersymmetric dynamical vacua arising in 4D Lorentzian supergravity are AdS_4 with $S^{ij} \neq 0$ and $AdS_2 \times S^2$ (or its Penrose limit) with $W_{ab}^\pm = \frac{1}{4}T_{ab}^\pm \neq 0$. Analogous statements hold for (real) Euclidean supergravity where we find only S^4 , H^4 , or $H^2 \times S^2$. However, this does not mean the other backgrounds are unphysical: they might arise from a higher derivative theory similarly to what happens in three dimensions [17], or they could arise via dimensional reduction from higher dimensions. For example, the general class of $AdS_2 \times S^2$ with unequal radii can arise from a higher dimensional supergravity theory with an $AdS_2 \times S^2 \times S^2$ factor (see e.g. [71, 72]).²⁰ The latter possesses the supersymmetry algebra $D(2, 1; \alpha)$; upon reduction, the isometry group of the internal S^2 becomes the 4D R -symmetry group. It would be interesting to understand better possible uplifts of the other cases to higher dimensions.

²⁰We thank Dmitri Sorokin for this observation and for bringing these references to our attention.

Another interesting feature of many of the backgrounds is the presence of a single timelike or spacelike \mathbb{R} or S^1 factor. When the Killing spinors are independent of this dimension, such as with the squashed S^3 , one recovers $\mathcal{N} = 4$ Killing spinors after a timelike or spacelike dimensional reduction to three dimensions.

We should emphasize again that we have restricted the Euclidean backgrounds to those admitting real supercharges. This excluded such cases as $S^2 \times S^2$ recently discussed in [67–69] and reflects the well-known fact that the $D(2, 1; \alpha)$ superalgebra possesses no real form with bosonic part $SU(2) \times SU(2) \times SU(2)$ [73, 74].²¹ Nevertheless, one may still exploit our results to investigate such cases by relaxing the requirement of real supercharges.

For instance, one might allow a superisometry algebra where supercharges and bosonic isometries appear with complex coefficients. A nice example is offered by $S^2 \times S^2$; this arises by analytically continuing $\lambda_- \rightarrow i\lambda_-$ and $\rho \rightarrow i\rho$ in the $H^2 \times S^2$ supercoset space, leading to a complex form of $D(2, 1; \alpha)$ possessing imaginary α . One finds eight complex Killing spinors corresponding to the analytic continuation of (D.27). These correspond to the (untwisted) Killing spinors discussed in [67, 68]. The actions we have found in section 7 hold with all of the matter fields now understood as complex quantities.

An interesting intermediate case between pure reality and pure complexity in Euclidean signature would be to demand that some bosonic fields, such as the vector multiplet connection $A_\mu{}^I$, remain real under repeated application of supersymmetry. This would impose only that the bosonic isometries generated in the superalgebra possess real coefficients. In the interesting case of $S^2 \times S^2$, one can show that this is impossible. In particular, there is no non-trivial subalgebra where the supercharges generate real non-vanishing bosonic isometries. In other words, any choice of supercharges either generates bosonic isometries with complex coefficients, or the supercharges are purely nilpotent.²²

Finally, in both Lorentzian and Euclidean cases, we have found a modified set of full supersymmetry conditions for the vector and hypermultiplets. It would be interesting to understand what role these may play in the analysis of quantum field theories on these curved manifolds, especially in light of the results of [66] on ellipsoids.

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²¹Note that one can obtain four real supercharges on $S^2 \times S^2$ via an equivariant twist [67, 69].

²²Alternatively, one might consider $S^2 \times S^2$ with split signature, $\eta_{ab} = (-1, -1, +1, +1)$. Now the supergravity R -symmetry group becomes $SL(2) \times SO(1, 1)$. The superalgebra on $S^2 \times S^2$ becomes a real form of $D(2, 1; \alpha)$ with bosonic group $SU(2) \times SU(2) \times SL(2)$.

A Conventions

A.1 Lorentzian signature

We employ conventions similar to [75] and [19], with minor modifications. Undotted Greek indices α, β, \dots denote left-handed spinors and dotted Greek indices $\dot{\alpha}, \dot{\beta}, \dots$ denote right-handed spinors. These are raised and lowered using $\epsilon_{\alpha\beta}$ and $\epsilon_{\dot{\alpha}\dot{\beta}}$, obeying $\epsilon^{12} = \epsilon_{\dot{2}\dot{1}} = 1$. We denote $SU(2)_R$ indices by i, j, k, \dots with $i = \underline{1}, \underline{2}$ and raise and lower them with ϵ_{ij} and ϵ^{ij} as with spinor indices.

The sigma matrices $(\sigma^a)_{\alpha\dot{\alpha}}$ are defined as

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (\text{A.1})$$

and the antisymmetric symbol ϵ_{abcd} is normalized as $\epsilon_{0123} = +1$ and $\epsilon^{0123} = -1$. The conjugate sigma matrix $(\bar{\sigma}^a)^{\dot{\alpha}\alpha}$ is given by

$$(\bar{\sigma}^a)^{\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}} \epsilon^{\alpha\beta} (\sigma^a)_{\beta\dot{\beta}}, \quad \bar{\sigma}^0 = \sigma^0, \quad \bar{\sigma}^{1,2,3} = -\sigma^{1,2,3}. \quad (\text{A.2})$$

The product of σ^a with $\bar{\sigma}^b$ is

$$\sigma^a \bar{\sigma}^b = -\eta^{ab} + 2\sigma^{ab}, \quad \bar{\sigma}^a \sigma^b = -\eta^{ab} + 2\bar{\sigma}^{ab}. \quad (\text{A.3})$$

The tensors $(\sigma^{ab})_{\alpha}{}^{\beta}$ and $(\bar{\sigma}^{ab})^{\dot{\alpha}}{}_{\dot{\beta}}$ are anti-selfdual and selfdual, respectively,

$$\frac{1}{2}\epsilon_{abcd}\sigma^{cd} = -i\sigma_{ab}, \quad \frac{1}{2}\epsilon_{abcd}\bar{\sigma}^{cd} = +i\bar{\sigma}_{ab}. \quad (\text{A.4})$$

The product of three sigma matrices is

$$\begin{aligned} \sigma^a \bar{\sigma}^b \sigma^c &= -\eta^{ab}\sigma^c + \eta^{ca}\sigma^b - \eta^{bc}\sigma^a + i\epsilon^{abcd}\sigma_d, \\ \bar{\sigma}^a \sigma^b \bar{\sigma}^c &= -\eta^{ab}\bar{\sigma}^c + \eta^{ca}\bar{\sigma}^b - \eta^{bc}\bar{\sigma}^a - i\epsilon^{abcd}\bar{\sigma}_d. \end{aligned} \quad (\text{A.5})$$

We will also use the four-component gamma matrices and charge conjugation matrix:

$$\gamma^a = \begin{pmatrix} 0 & i(\sigma^a)_{\alpha\dot{\beta}} \\ i(\bar{\sigma}^a)^{\dot{\alpha}\beta} & 0 \end{pmatrix}, \quad (\gamma^a)^\dagger = \gamma_a, \quad \{\gamma^a, \gamma^b\} = 2\eta^{ab}, \quad C = \begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & \epsilon_{\dot{\alpha}\dot{\beta}} \end{pmatrix}. \quad (\text{A.6})$$

Associated with any vector V^a is the bispinor $V_{\alpha\dot{\beta}} = V^a(\sigma_a)_{\alpha\dot{\beta}}$. Given a tensor F_{ab} , we define the dual \tilde{F}_{ab} and selfdual (antiselfdual) components F_{ab}^\pm by

$$F_{ab} = F_{ab}^- + F_{ab}^+, \quad \tilde{F}_{ab} = \frac{1}{2}\epsilon_{abcd}F^{cd} = -iF_{ab}^- + iF_{ab}^+, \quad F_{ab}^\pm = \frac{1}{2}(F_{ab} \mp i\tilde{F}_{ab}). \quad (\text{A.7})$$

The selfdual (anti-selfdual) components are related to symmetric bispinors $F_{\dot{\alpha}\dot{\beta}} (F_{\alpha\beta})$ by

$$F_{ab}^- = -(\sigma_{ab})^{\alpha\beta} F_{\alpha\beta} = (\sigma_{ab})_{\alpha}{}^{\beta} F_{\beta}{}^{\alpha}, \quad F_{ab}^+ = -(\bar{\sigma}_{ab})^{\dot{\alpha}\dot{\beta}} F_{\dot{\alpha}\dot{\beta}} = (\bar{\sigma}_{ab})^{\dot{\alpha}}{}_{\dot{\beta}} F_{\dot{\beta}}{}^{\dot{\alpha}}. \quad (\text{A.8})$$

Complex conjugation exchanges dotted for undotted spinors but does not change their positions. That is, $(\psi_\alpha)^* = \bar{\psi}_{\dot{\alpha}}$, $(\psi^\alpha)^* = \bar{\psi}^{\dot{\alpha}}$ and $((\sigma^a)_{\alpha\dot{\beta}})^* = (\sigma^a)_{\beta\dot{\alpha}}$. As usual, complex

conjugation transposes Grassmann quantities, so that $(\psi_\alpha \rho_\beta)^* = \bar{\rho}_{\dot{\beta}} \bar{\psi}_{\dot{\alpha}} = -\bar{\psi}_{\dot{\alpha}} \bar{\rho}_{\dot{\beta}}$. If ψ and ρ are operators with a non-trivial anticommutator, one must interpret this statement as $(\psi_\alpha \rho_\beta)^* = -\bar{\psi}_{\dot{\alpha}} \bar{\rho}_{\dot{\beta}}$. If V_a is a real vector, then $(V_{\alpha\dot{\beta}})^* = V_{\beta\dot{\alpha}}$, and if F_{ab} is a real tensor, then $(F_{\alpha\beta})^* = -F_{\dot{\alpha}\dot{\beta}}$. Killing spinors $\xi_{\alpha i}$ and $\bar{\xi}^{\dot{\alpha} i}$ obey

$$(\xi_{\alpha i})^* = \bar{\xi}_{\dot{\alpha}}^i, \quad (\bar{\xi}^{\dot{\alpha} i})^* = \xi_{\alpha i}. \quad (\text{A.9})$$

Because $(\epsilon^{ij})^* = -\epsilon_{ij}$ one has $(\xi_{\alpha}^i)^* = -\bar{\xi}_{\dot{\alpha} i}$ and $(\bar{\xi}^{\dot{\alpha} i})^* = -\xi^{\alpha i}$. For the covariant derivatives of superspace, one finds

$$(\mathcal{D}_{\alpha\dot{\beta}})^* = \mathcal{D}_{\beta\dot{\alpha}}, \quad (\mathcal{D}_{\alpha}^i)^* = \bar{\mathcal{D}}_{\dot{\alpha} i}, \quad (\bar{\mathcal{D}}^{\dot{\alpha} i})^* = \mathcal{D}^{\alpha i}. \quad (\text{A.10})$$

A.2 Euclidean signature

Our Euclidean conventions amount to taking $V^4 = iV^0$ for all vector and tensor-valued objects. In particular,

$$(\sigma^4)_{\alpha\dot{\beta}} = i(\sigma^0)_{\alpha\dot{\beta}} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad (\bar{\sigma}^4)^{\dot{\alpha}\beta} = i(\bar{\sigma}^0)^{\dot{\alpha}\beta} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}. \quad (\text{A.11})$$

We keep all the other σ matrices and $\epsilon_{\alpha\beta}$ unchanged. The Euclidean metric is $\eta_{ab} = \delta_{ab}$. We trade the antisymmetric symbol $\epsilon_{0123} = +1$ for the Euclidean $\epsilon_{1234} = +1$, which amounts to exchanging ϵ_{abcd} for $i\epsilon_{abcd}$. Now one finds

$$\begin{aligned} \sigma^a \bar{\sigma}^b &= -\delta^{ab} + 2\sigma^{ab}, & \bar{\sigma}^a \sigma^b &= -\delta^{ab} + 2\bar{\sigma}^{ab}, \\ \frac{1}{2}\epsilon_{abcd} \sigma^{cd} &= -\sigma_{ab}, & \frac{1}{2}\epsilon_{abcd} \bar{\sigma}^{cd} &= +\bar{\sigma}_{ab}, \\ \sigma^a \bar{\sigma}^b \sigma^c &= -\eta^{ab} \sigma^c + \eta^{ca} \sigma^b - \eta^{bc} \sigma^a - \epsilon^{abcd} \sigma_d, \\ \bar{\sigma}^a \sigma^b \bar{\sigma}^c &= -\eta^{ab} \bar{\sigma}^c + \eta^{ca} \bar{\sigma}^b - \eta^{bc} \bar{\sigma}^a + \epsilon^{abcd} \bar{\sigma}_d. \end{aligned} \quad (\text{A.12})$$

The four-component γ^a and charge conjugation matrix are

$$\gamma^a = \begin{pmatrix} 0 & i(\sigma^a)_{\alpha\dot{\beta}} \\ i(\bar{\sigma}^a)^{\dot{\alpha}\beta} & 0 \end{pmatrix}, \quad (\gamma^a)^\dagger = \gamma^a, \quad \{\gamma^a, \gamma^b\} = 2\delta^{ab}, \quad C = \begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & \epsilon_{\dot{\alpha}\dot{\beta}} \end{pmatrix}. \quad (\text{A.13})$$

The self-dual and anti-self-dual components of F_{ab} are

$$F_{ab} = F_{ab}^- + F_{ab}^+, \quad \tilde{F}_{ab} = \frac{1}{2}\epsilon_{abcd} F^{cd} = -F_{ab}^- + F_{ab}^+, \quad F_{ab}^\pm = \frac{1}{2}(F_{ab} \pm \tilde{F}_{ab}). \quad (\text{A.14})$$

Under complex conjugation undotted indices remain undotted but are raised or lowered,

$$((\sigma^a)_{\alpha\dot{\beta}})^* = -(\bar{\sigma}^a)^{\dot{\beta}\alpha}, \quad ((\sigma^{ab})_{\alpha\beta})^* = -(\sigma^{ab})_{\beta\alpha}, \quad ((\sigma^{ab})_{\alpha\beta})^* = (\sigma^{ab})^{\alpha\beta}, \quad (\text{A.15})$$

so if V_a and F_{ab} are real, then

$$(V_{\alpha\dot{\beta}})^* = -V^{\dot{\beta}\alpha}, \quad (F_{\alpha\beta})^* = F^{\alpha\beta}. \quad (\text{A.16})$$

The Killing spinors are chosen to be symplectic Majorana-Weyl,

$$(\xi_{\alpha}^i)^* = \xi_{\alpha}^i, \quad (\bar{\xi}^{\dot{\alpha} i})^* = \bar{\xi}_{\dot{\alpha} i}. \quad (\text{A.17})$$

Keeping in mind that $(\epsilon^{\alpha\beta})^* = -\epsilon_{\alpha\beta}$, one can see that $(\xi^{\alpha i})^* = -\xi_{\alpha i}$, so the positions of the indices must be observed. These conditions imply that

$$(\mathcal{D}_{\alpha}^i)^* = -\mathcal{D}_{\alpha}^i, \quad (\bar{\mathcal{D}}^{\dot{\alpha} i})^* = -\bar{\mathcal{D}}^{\dot{\alpha} i}. \quad (\text{A.18})$$

B General action principle in rigid superspace

For the rigid superspace geometry discussed in this paper, there are a few general formulae that will be relevant for relating superspace actions to component ones. We emphasize that these expressions are valid only for the rigid supergeometries discussed. In particular, they assume the covariant constancy of the various superfields S^{ij} , $Y_{\alpha\beta}$, $W_{\alpha\beta}$, etc.

First, let us relate a chiral superspace action to a component one. We take

$$S = \int d^4x d^4\theta \mathcal{E} \mathcal{L}_c \tag{B.1}$$

where \mathcal{L}_c is a covariantly chiral superfield, $\bar{\mathcal{D}}_i^{\dot{\alpha}} \mathcal{L}_c = 0$, and \mathcal{E} is the appropriate superspace measure constructed from the superdeterminant (Berezinian) of the chiral part of the superspace vielbein. The component Lagrangian constructed from (B.1) is

$$\begin{aligned} \mathcal{L} = & \frac{1}{96} \mathcal{D}^{ij} \mathcal{D}_{ij} \mathcal{L}_c - \frac{1}{96} \mathcal{D}^{\alpha\beta} \mathcal{D}_{\alpha\beta} \mathcal{L}_c + \frac{2}{3} S_{ij} \mathcal{D}^{ij} \mathcal{L}_c - \frac{1}{3} Y^{\alpha\beta} \mathcal{D}_{\alpha\beta} \mathcal{L}_c \\ & + \left(3S^{ij} S_{ij} - Y^{\alpha\beta} Y_{\alpha\beta} + \bar{W}_{\dot{\alpha}\dot{\beta}} \bar{W}^{\dot{\alpha}\dot{\beta}} \right) \mathcal{L}_c \end{aligned} \tag{B.2}$$

where we have defined $\mathcal{D}^{ij} := \mathcal{D}^{\gamma(i} \mathcal{D}_{\gamma}^{j)}$ and $\mathcal{D}_{\alpha\beta} := \mathcal{D}_{(\alpha}^k \mathcal{D}_{\beta)k}$ and the projection to $\theta = 0$ is assumed. This is a special case of the chiral action presented in [76].

Next is the action relating projective superspace actions to component actions. We use the projective superspace action principle adapted to the rigid superspace geometries discussed in this paper,

$$S = -\frac{1}{2\pi} \oint_{\mathcal{C}} d\tau \int d^4x d^4\theta^+ \mathcal{E}^{--} \mathcal{L}^{++} = -\frac{1}{2\pi} \oint_{\mathcal{C}} v_i^+ dv^{i+} \int d^4x e \mathcal{L}^{--}, \tag{B.3}$$

with the $SL(2, \mathbb{C})$ representation of projective superspace [21, 37, 56] so that only a single contour integral is needed (see the discussion in [57]). The component Lagrangian is

$$\begin{aligned} \mathcal{L}^{--} = & \frac{1}{16} (\mathcal{D}^-)^2 (\bar{\mathcal{D}}^-)^2 \mathcal{L}^{++} + \frac{i}{2} G^{\dot{\alpha}\alpha--} [\mathcal{D}_{\alpha}^-, \bar{\mathcal{D}}_{\dot{\alpha}}^-] \mathcal{L}^{++} \\ & + \frac{3}{4} S^{--} (\bar{\mathcal{D}}^-)^2 \mathcal{L}^{++} + \frac{3}{4} \bar{S}^{--} (\mathcal{D}^-)^2 \mathcal{L}^{++} + 9\bar{S}^{--} S^{--} \mathcal{L}^{++} \\ & - \frac{i}{4} \mathcal{V}_m^{--} (\bar{\sigma}^m)^{\dot{\alpha}\alpha} [\mathcal{D}_{\alpha}^-, \bar{\mathcal{D}}_{\dot{\alpha}}^-] \mathcal{L}^{++} - 12\mathcal{V}_a^{--} G^{a--} \mathcal{L}^{++}. \end{aligned} \tag{B.4}$$

This expression is real under the modified complex conjugation of projective superspace. One must still perform the contour integral to arrive at a standard Lagrangian in x -space.

For completeness, we include also the relation between full superspace and chiral superspace actions, although the former generically involve higher derivative interactions and play no role in this paper:

$$\int d^4x d^4\theta d^4\bar{\theta} E \mathcal{L} = \int d^4x d^4\theta \mathcal{E} \left(\frac{1}{48} \bar{\mathcal{D}}^{ij} \bar{\mathcal{D}}_{ij} \mathcal{L} + \frac{1}{12} \bar{S}_{ij} \bar{\mathcal{D}}^{ij} \mathcal{L} + \frac{1}{4} \bar{Y}_{\dot{\alpha}\dot{\beta}} \bar{\mathcal{D}}^{\dot{\alpha}\dot{\beta}} \mathcal{L} \right). \tag{B.5}$$

C Details of Lorentzian backgrounds

C.1 Warped AdS₃ spaces (wAdS₃ × ℝ)

We gauge fix $G^a = (0, 0, 0, g)$ for constant g . First consider the case without warping, where no other background fields are turned on. Then the bosonic part of the superalgebra involves only the generators $P_a = (P_I, P_3)$ with $I = 0, 1, 2$,

$$[P_I, P_3] = 0, \quad [P_I, P_J] = 4g \varepsilon_{IJK} \eta^{KL} P_L, \quad \eta = \text{diag}(-1, 1, 1). \quad (\text{C.1})$$

The dimensionless generators used to construct the group manifold are just $T_I \equiv \frac{1}{4g} P_I$. We choose the explicit parameterization which leads to a global set of coordinates for AdS₃ with $\tau \in [0, 4\pi)$:

$$L = e^{\phi T_2} e^{\rho T_1} e^{\tau T_0} e^{z P_3}, \quad (\text{C.2})$$

Both choices of orthonormal frame specified in (3.43) can be used to construct the physical vielbein and arrive at the metric (3.42).

In the spinor representation (3.27) we can decompose L in terms of the embedding coordinates (D.19):

$$L = X + 2(Y T_0 + V T_1 + W T_2), \quad L^{-1} = X - 2(Y T_0 + V T_1 + W T_2). \quad (\text{C.3})$$

In absence of warping, using $e^I = E^I/4g$ as vielbein, the Killing spinors read

$$\xi_{\alpha i} = [X - 2(Y \sigma^{12} + W \sigma^{01} + V \sigma^{20})]_{\alpha}{}^{\beta} \epsilon_{\beta i}, \quad (\text{C.4})$$

while if we choose $e^I = E^I/4g$ we must flip the sign of Y , W and V (i.e. we exchange L with L^{-1}).

Timelike stretched AdS₃ × ℝ. Keeping $G^a = (0, 0, 0, g)$, we turn on $\mathcal{Z}_{ab} = 4\lambda \delta_{ab}^{12}$ and require $|\lambda| < 2|g|$. The appropriate choice of dimensionless generators is now

$$T_0 \equiv v \left(\frac{1}{4g} P_0 - \frac{\lambda^2}{4g^2} M_{12} \right), \quad T_{1,2} \equiv \sqrt{v} \frac{1}{4g} P_{1,2}, \quad U \equiv \frac{1}{4g} P_0 - M_{12}, \quad (\text{C.5})$$

with $v = \left(1 - \frac{\lambda^2}{4g^2} \right)^{-1} \geq 1$. We keep the same choice of coset representatives. The left-invariant vielbein reads

$$e^0 = v \frac{1}{4g} E^0, \quad e^{1,2} = \frac{\sqrt{v}}{4g} E^{1,2}, \quad e^3 = dz. \quad (\text{C.6})$$

The final metric is provided in equation (3.46).

The Killing spinors are still given by the spinorial representation (3.27) of L , which now gives

$$\xi_{\alpha i} = [X - 2Y \sigma^{12} - 2\sqrt{v}(W \sigma^{01} + V \sigma^{20})]_{\alpha}{}^{\beta} \epsilon_{\beta i} + i\sqrt{v} \frac{\lambda}{2g} (W \sigma^1 - V \sigma^2)_{\alpha \dot{\beta}} \bar{\epsilon}^{\dot{\beta}}{}_i. \quad (\text{C.7})$$

Finally, the potentials for \mathcal{Z} and $*G$ are

$$C_{(1)} = \frac{\lambda}{2g} e^0, \quad B_{(2)} = \frac{v^2}{64g^2} \sinh \rho d\phi d\tau. \quad (\text{C.8})$$

Spacelike squashed AdS₃ × ℝ. This time we take $\mathcal{Z}_{ab} = 4\lambda\delta_{ab}^{01}$. We define $v = \left(1 + \frac{\lambda^2}{4g^2}\right)^{-1}$, $0 < v \leq 1$ and

$$T_{0,1} \equiv \frac{\sqrt{v}}{4g} P_{0,1}, \quad T_2 \equiv v \left(\frac{1}{4g} P_2 - \frac{\lambda^2}{4g^2} M_{01} \right), \quad U \equiv \frac{1}{4g} P_2 + M_{01}. \quad (\text{C.9})$$

The residual Lorentz generator as well as U are non-compact. It proves convenient to choose the right-invariant forms E^I (3.43) to obtain the physical vielbein

$$e^{0,1} = \frac{\sqrt{v}}{4g} E^{0,1}, \quad e^2 = \frac{v}{4g} E'^2. \quad (\text{C.10})$$

The metric takes the form (3.48).

The Killing spinors are computed as usual:

$$\xi_{\alpha i} = [X + 2(W\sigma^{01} + \sqrt{v}Y\sigma^{12} + \sqrt{v}V\sigma^{20})]_{\alpha\beta} \epsilon_{\beta i} - i\sqrt{v}\frac{\lambda}{2g}(V\sigma^0 - Y\sigma^1)_{\alpha\dot{\beta}} \bar{\epsilon}^{\dot{\beta}}_i. \quad (\text{C.11})$$

The background potentials for \mathcal{Z} and $*G$ are now respectively

$$C_{(1)} = -\frac{\lambda}{2g} e^2, \quad B_{(2)} = -\frac{v^2}{64g^2} \sinh \rho \, d\phi \, d\tau. \quad (\text{C.12})$$

Null warped AdS₃ × ℝ. We take \mathcal{Z} to be null and fix it to $\mathcal{Z}_{ab} = 4\lambda(\delta_{ab}^{02} + \delta_{ab}^{12})$ with $\lambda > 0$. The usual procedure yields the SU(1, 1) generators

$$\begin{aligned} T_0 &= \left(1 + \frac{v}{2}\right) \frac{1}{4g} P_0 - \frac{v}{8g} P_1 + v(M_{02} - M_{12}), & U &= M_{12} - M_{02} + \frac{1}{4g}(P_1 - P_0), \\ T_1 &= \left(1 - \frac{v}{2}\right) \frac{1}{4g} P_1 + \frac{v}{8g} P_0 + v(M_{02} - M_{12}), & T_2 &= \frac{1}{4g} P_2. \end{aligned} \quad (\text{C.13})$$

where this time $v = \frac{\lambda^2}{4g^2} \geq 0$. Using again the one forms E'^I for convenience, we construct the physical vielbein giving rise to the metric (3.50)

$$e^0 = \frac{1}{4g} \left(\left(1 + \frac{v}{2}\right) E'^0 + \frac{v}{2} E'^1 \right), \quad e^1 = \frac{1}{4g} \left(\left(1 - \frac{v}{2}\right) E'^1 - \frac{v}{2} E'^0 \right), \quad e^2 = \frac{1}{4g} E'^2. \quad (\text{C.14})$$

We also find the Killing spinors:

$$\begin{aligned} \xi_{\alpha i} &= [X + 2(W\sigma^{01} + Y\sigma^{12} + V\sigma^{20}) - v(Y + V)(\sigma^{12} - \sigma^{20})]_{\alpha\beta} \epsilon_{\beta i} \\ &\quad - i\frac{\lambda}{2g} [W(\sigma^0 + \sigma^1) - (Y + V)\sigma^2]_{\alpha\dot{\beta}} \bar{\epsilon}^{\dot{\beta}}_i. \end{aligned} \quad (\text{C.15})$$

As usual we provide the potentials for \mathcal{Z} and $*G$:

$$C_{(1)} = \frac{\lambda}{8g^2} (E'^0 + E'^1), \quad B_{(2)} = -\frac{1}{64g^2} \sinh \rho \, d\phi \, d\tau. \quad (\text{C.16})$$

If we introduce Poincaré coordinates as in equation (3.51), the appropriate coset representative, dimensionless vielbein and embedding coordinates read

$$\begin{aligned}
 L &= e^{\sqrt{2}x_- T_-} r^{-2T_2} e^{\sqrt{2}x_+ T_+}, \quad T_{\pm} = \frac{1}{\sqrt{2}}(T_1 \pm T_0), \\
 E^0 &= dx_- + \frac{2x_-}{r} dr - \frac{x_-^2 + 1}{r^2} dx_+, \quad E^1 = -dx_- + \frac{2x_-}{r} dr + \frac{x_-^2 - 1}{r^2} dx_+, \\
 E'^2 &= \frac{2dr}{r} + \frac{2x_- dx_+}{r^2}, \\
 X &= \frac{r^2 + 1 + x_- x_+}{2r}, \quad Y = \frac{x_+ - x_-}{2r}, \quad V = \frac{x_+ + x_-}{2r}, \quad W = \frac{r^2 - 1 + x_- x_+}{2r}.
 \end{aligned} \tag{C.17}$$

AdS₃ × ℝ and SU(1, 1|1)² supersymmetry. The second realization of $\mathcal{N} = 2$ supersymmetry on a ‘round’ AdS₃ × ℝ is supported by $G_a^i{}_j$, which we gauge-fix to $G_a^i{}_j = g_a(i\sigma_3)^i{}_j$, $g_a = (0, 0, 0, g)$. Analogously to the S^3 case we define

$$\Delta_a^{(i)} \equiv P_a + (-)^i g(\delta_a^3 \mathbb{A} + \epsilon_{a3}{}^{cd} M_{cd}) \tag{C.18}$$

which gives us the non-vanishing (anti)commutators of SU(1, 1|1)²

$$\begin{aligned}
 \{Q_\alpha^i, \bar{Q}_{\beta j}\} &= -2i\delta_j^i \Delta_{\alpha\beta}^{(i)}, \\
 [\Delta_a^{(i)}, Q_\alpha^j] &= (-)^{i+1} \delta^{ij} 2g \left[i\delta_a^3 \delta_\alpha^\beta + \epsilon_{a3cd}(\sigma^{cd})_\alpha^\beta \right] Q_\beta^i, \\
 [\Delta_a^{(i)}, \Delta_b^{(j)}] &= (-)^i \delta^{ij} 4g\epsilon_{ab}{}^c{}_3 \Delta_c^{(i)}.
 \end{aligned} \tag{C.19}$$

We will choose $T_I \equiv \frac{1}{4g} \Delta_I^{(2)}$ and generate the flat direction using $\Delta_3^{(2)}$. We can take coordinates and (left invariant) vielbein as in section 3.5, so that the U(1)_R connection turns out to be $A = g e^3 = g dz$ and we arrive at the Killing spinors $\hat{\xi}$

$$\hat{\xi}_{\alpha\bar{1}} = \epsilon_{\alpha\bar{1}}, \quad \hat{\xi}_{\alpha\bar{2}} = e^{2igz} \xi_{\alpha\bar{2}}, \tag{C.20}$$

where $\xi_{\alpha\bar{2}}$ is defined in (C.4). If the spatial direction generated by P_3 is compact, the choice of U(1)_R connection is not necessarily pure gauge and might correspond to a Wilson line along the circle. For the Killing spinors $\hat{\xi}_{\bar{2}}$ to be well-defined the radius of the z circle must be a multiple of $n\pi/g$, and the U(1)_R connection is non-trivial for odd n . The potential for $*G^i{}_j$ is $i\sigma_3$ times the potential $B_{(2)}$ of (C.12).

C.2 AdS₂ × S² spacetimes and D(2, 1; α)

Let us gauge-fix $\mathcal{Z}_{ab} = -2\lambda_- \delta_{ab}^{03} + 2i\lambda_+ \delta_{ab}^{12}$, with λ_{\pm} real. We define coset representatives using dimensionless coordinates τ, ρ on AdS₂ and ϕ, θ on S²

$$L \equiv e^{\tau \frac{1}{\lambda_-} P_0} e^{\rho \frac{1}{\lambda_-} P_3} e^{\phi M_{12}} e^{\theta \frac{1}{\lambda_+} P_2} e^{-\phi M_{12}}, \tag{C.21}$$

from which we obtain a vielbein associated with the metric (3.54)

$$e^0 = \frac{1}{\lambda_-} \cosh \rho d\tau, \quad e^1 = -\frac{1}{\lambda_+} (\sin \phi d\theta + \cos \phi \sin \theta d\phi)$$

$$e^3 = \frac{1}{\lambda_-} d\rho, \quad e^2 = \frac{1}{\lambda_+} (\cos \phi d\theta - \sin \phi \sin \theta d\phi). \quad (C.22)$$

The choice of Lorentz gauge in L is convenient to guarantee that it is well-behaved at the north pole of S^2 .

$$L(\tau, \rho, \phi, \theta) \rightarrow L^{(\text{south})}(\tau, \rho, \phi, \theta) \equiv L(\tau, \rho, \phi, \theta - \pi) e^{\pi \frac{1}{\lambda_+} P_2} = L(\tau, \rho, \phi, \theta) e^{2\phi M_{12}}. \quad (C.23)$$

As evidenced by the last equality, this is not a change of coordinates, but rather just a change in local Lorentz gauge.²³ The associated vielbein along the S^2 is

$$e^1_{(\text{south})} = \frac{1}{\lambda_+} (\sin \phi d\theta - \cos \phi \sin \theta d\phi), \quad e^2_{(\text{south})} = \frac{1}{\lambda_+} (\cos \phi d\theta + \sin \phi \sin \theta d\phi). \quad (C.24)$$

In the northern gauge we can write Killing spinors in terms of the matrices

$$\begin{aligned} A &= \cosh \frac{\rho}{2} \left(\cos \frac{\theta}{2} \cos \frac{\tau}{2} + 2 \sin \frac{\theta}{2} \sin \frac{\tau}{2} (\sin \phi \sigma^{01} - \cos \phi \sigma^{02}) \right) \\ &\quad + 2i \sinh \frac{\rho}{2} \left(\cos \frac{\theta}{2} \sin \frac{\tau}{2} \sigma^{12} - \sin \frac{\theta}{2} \cos \frac{\tau}{2} (\cos \phi \sigma^{01} + \sin \phi \sigma^{02}) \right), \\ B &= -i \sinh \frac{\rho}{2} \left(\cos \frac{\theta}{2} \cos \frac{\tau}{2} + \sin \frac{\theta}{2} \sin \frac{\tau}{2} (\sin \phi \sigma^1 - \cos \phi \sigma^2) \right) \\ &\quad + \cosh \frac{\rho}{2} \left(i \cos \frac{\theta}{2} \sin \frac{\tau}{2} \sigma^3 + \sin \frac{\theta}{2} \cos \frac{\tau}{2} (\cos \phi \sigma^1 + \sin \phi \sigma^2) \right), \end{aligned} \quad (C.25)$$

so that the Killing spinors of $\text{AdS}_2 \times S^2$ are simply

$$\xi_{\alpha i} = A_{\alpha}^{\beta} \epsilon_{\beta i} + B_{\alpha \beta} \bar{\epsilon}^{\beta}_i. \quad (C.26)$$

Notice that these Killing spinors differ from [27], because our choice of Lorentz gauge makes them periodic in ϕ and well-behaved at the north pole of S^2 . For each Killing spinor we can compute another expression that differs by a local Lorentz gauge transformation and is single-valued at the south pole. This is the transformation described in (D.2) and on the spinors it corresponds to the substitution

$$\xi_{(i)\hat{\alpha}}(\tau, \rho, \phi, \theta) \longrightarrow \xi_{(i)\hat{\alpha}}^{(\text{south})} \equiv (i\gamma_1)_{\hat{\alpha}}^{\hat{\beta}} \xi_{(i)\hat{\beta}}(\tau, \rho, \phi, \theta - \pi). \quad (C.27)$$

For the spacetimes $\mathbb{R}^{1,1} \times S^2$ and $\text{AdS}_2 \times \mathbb{R}^2$, the isometry generators associated with the flat directions become trivially represented in the spinorial representation and the Killing spinors are obtained setting to zero the corresponding coordinates (τ, ρ and ϕ, θ respectively) in the expression for $\text{AdS}_2 \times S^2$.

The background complex two-form \mathcal{Z} has potential

$$C_{(1)} = \frac{1}{\lambda_-} \sinh \rho d\tau - \frac{i}{\lambda_+} (\cos \theta \pm 1) d\phi, \quad (C.28)$$

the plus/minus sign corresponding to the northern and southern patches of S^2 .

²³The isotropy group of the south pole is the same of the north pole and the transformation $\exp \frac{\pi}{\lambda_+} P_2$ induces the appropriate automorphism that gives rise to a single-valued gauge at the south pole. This way to induce the change of gauge easily generalizes to higher dimensional spheres.

C.3 Lorentzian $S^3 \times \mathbb{R}$

We take $G^a = (0, 0, 0, g)$ and $\mathcal{Z}_{ab} = 4\lambda\delta_{ab}^{12}$, imposing now $\lambda^2 > 4g^2$. We introduce generators T_0, T_1, T_2 , satisfying the $SU(2)$ algebra, defining $v = (\frac{\lambda^2}{4g^2} - 1)^{-1} > 0$ and

$$T_0 \equiv v \left(-\frac{1}{4g}P_0 + \frac{\lambda^2}{4g^2}M_{12} \right), \quad T_{1,2} \equiv \sqrt{v}\frac{1}{4g}P_{1,2}, \quad U \equiv \frac{1}{4g}P_0 - M_{23}, \quad (C.29)$$

with U commuting with everything. We can choose the same group representative L as for the standard S^3 , trade T_3 there for T_0 here and write in the spinorial representation

$$L = e^{\phi T_0} e^{\theta T_2} e^{\omega T_0} = X + 2(YT_0 + VT_1 + WT_2), \quad (C.30)$$

where we have also introduced the embedding coordinates (3.16). We can also recycle the left-invariant dimensionless vielbein in (3.14) and identify the physical vielbein

$$e^0 = \frac{v}{4g}E^3, \quad e^{1,2} = \frac{\sqrt{v}}{4g}E^{1,2}, \quad e^3 = dz. \quad (C.31)$$

The metric is given in equation (3.62).

The Killing spinors are

$$\xi_{\alpha i} = [X\mathbb{I}_2 - 2(W\sigma^{12} + \sqrt{v}V\sigma^{20} + \sqrt{v}Y\sigma^{01})]_{\alpha}^{\beta} \epsilon_{\beta i} + i\sqrt{v}\frac{\lambda}{2g}(Y\sigma^1 - V\sigma^2)_{\alpha\dot{\beta}} \bar{\epsilon}^{\dot{\beta}}_i. \quad (C.32)$$

The background potentials are $C_{(1)} = -\frac{\lambda}{2g}e^0$ and $B_{(2)}$ given in (3.30).

C.4 Lightlike $S^3 \times \mathbb{R}$

We choose $G^a = \frac{1}{\sqrt{2}}(g, 0, 0, g)$ and $\mathcal{Z}_{ab} = 4\lambda\delta_{ab}^{12}$. The absolute value of g has no physical relevance as it can be rescaled by a Lorentz boost. The appropriate choice of dimensionless generators turns out to be

$$T_{1,2} \equiv \frac{1}{2\lambda}P_{1,2}, \quad T_3 \equiv M_{12} - \frac{g}{\lambda^2}P_+, \quad U \equiv \frac{1}{4g}P_- + M_{12}, \quad (C.33)$$

with $P_{\pm} = \frac{1}{\sqrt{2}}(P_3 \pm P_0)$. Together with P_+ , these generators form $SU(2) \times U(1)_U \times U(1)_{P_+}$. We can choose the usual coordinates and expressions (3.14) for the $SU(2)$ manifold, generating the fourth direction by $\exp(u\frac{1}{4g}P_-)$. We then read off the physical vielbein giving rise to the metric (3.63)

$$e^- = \frac{1}{4g}du, \quad e^+ = \frac{g}{\lambda^2}E^3, \quad e^{1,2} = \frac{1}{2\lambda}E^{1,2}. \quad (C.34)$$

Killing spinors are computed in terms of the embedding coordinates (3.16):

$$\xi_{\alpha i} = [X\mathbb{I}_2 - 2(\frac{g}{\lambda}(V + iY)(\sigma^{20} - i\sigma^{01}))]_{\alpha}^{\beta} \epsilon_{\beta i} + i(Y\sigma^1 - V\sigma^2)_{\alpha\dot{\beta}} \bar{\epsilon}^{\dot{\beta}}_i. \quad (C.35)$$

The background forms are analogous to the other spheres:

$$C_{(1)} = -\frac{\lambda}{2g}e^+, \quad B_{(2)} = \frac{1}{16\lambda^2}(\cos\theta \pm 1) d\phi du. \quad (C.36)$$

C.5 ‘Overstretched’ AdS₃

There is a threshold case between timelike stretched AdS₃ × ℝ and the Lorentzian sphere. We take $G^a = (0, 0, 0, g)$ and $\mathcal{Z} = 4\lambda\delta_{ab}^{12}$, and choose the specific value $\lambda = 2g$. The commutation relations read

$$\begin{aligned} [P_0, P_1] &= 4gP_2, & [M_{12}, P_1] &= P_2, \\ [P_0, P_2] &= -4gP_1, & [M_{12}, P_2] &= -P_1. \\ [P_1, P_2] &= -4gP_0 + 4gM_{12} \equiv -4gH, \end{aligned} \tag{C.37}$$

The isometry generators P_1 and P_2 close on a central charge, and we obtain the algebra of Heis₃ × U(1)_M × U(1)_{P₃} or, alternatively, ISO(2)_(H) × U(1)_{P₃}, H being a central extension of ISO(2). We are left with a group manifold Heis₃ × U(1)_{P₃}, of which we parameterize a generic element as $L = e^{tH} e^{xP_1+yP_2} e^{zP_3}$ and easily arrive at the vielbein

$$e^0 = dt + 2g(xdy - ydx), \quad e^1 = dx, \quad e^2 = dy, \quad e^3 = dz \tag{C.38}$$

and the metric (3.64). The Killing spinors are:

$$\xi_{\alpha i} = [\mathbb{I}_2 - 4gx\sigma^{20} - 4gy\sigma^{01}]_{\alpha}^{\beta} \epsilon_{\beta i} - 2ig(x\sigma^2 - y\sigma^1)_{\alpha\dot{\beta}} \bar{\epsilon}^{\dot{\beta}}_i. \tag{C.39}$$

The background potentials are $C_{(1)} = 2g(xdy - ydx)$ and $B_{(2)} = \frac{1}{4}C_{(1)} \wedge dt$.

C.6 Plane waves

We have $G^a = \frac{1}{\sqrt{2}}(g, 0, 0, g) = g\delta_+^a$ and $\mathcal{Z}_{ab} = 2\sqrt{2}\lambda_+\delta_{ab}^{-1} - 2\sqrt{2}i\lambda_-\delta_{ab}^{-2}$ as in the main text. The generic isometry algebra is $\mathbb{R}_{P_-} \times \text{Heis}_5$. The generator P_- is an elliptic generator of the superalgebra, despite the fact that it corresponds to a null direction in spacetime. Its orbit on Heis₅ is not necessarily closed, depending on the values of λ_{\pm} . We pick the coset representative

$$L \equiv e^{vP_+} e^{uP_-} e^{xP_1+yP_2} e^{4gyM_{+1}-4gxM_{+2}}. \tag{C.40}$$

This choice allows us to take advantage of the solvability of the Heisenberg algebra to compute the Cartan-Maurer form explicitly: we obtain the vielbein giving rise to the metric (3.67)

$$e^+ = dv + 2g(ydx - xdy) - (\lambda_-^2 x^2 + \lambda_+^2 y^2)du, \quad e^- = du, \quad e^1 = dx, \quad e^2 = dy. \tag{C.41}$$

Note that we can also switch to Brinkmann coordinates as shown in equation (3.69).

Expressions for the Killing spinors can be derived as usual computing $R[L]$ from (3.27). We write them as

$$\xi_{\alpha i} = A_{\alpha}^{\beta} \epsilon_{\beta i} + B_{\alpha\dot{\beta}} \bar{\epsilon}^{\dot{\beta}}_i, \tag{C.42}$$

$$A = \begin{pmatrix} \cos k_+ u - \frac{2ig}{k_+} \sin k_+ u & 0 \\ -\frac{\lambda_+ + \lambda_-}{\sqrt{2}k_+} (\lambda_+ x + i\lambda_- y) \sin k_+ u & \cos k_- u + \frac{2ig}{k_-} \sin k_- u \end{pmatrix}, \tag{C.43}$$

$$B = \begin{pmatrix} 0 & -i \frac{\lambda_+ + \lambda_-}{\sqrt{2}k_+} \sin k_+ u \\ i \frac{\lambda_- - \lambda_+}{\sqrt{2}k_-} \sin k_- u & (\lambda_- y - i\lambda_+ x) \left(\cos k_+ u + \frac{2ig}{k_+} \sin k_+ u \right) \end{pmatrix}. \quad (\text{C.44})$$

We have also defined $k_{\pm} = (4g^2 + (\lambda_+ \pm \lambda_-)^2/2)^{1/2}$.

The background potentials for \mathcal{Z} and $*G$ are $C_{(1)} = (\lambda_+ e^2 + i\lambda_- e^1)/2\sqrt{2}g$ and $B_{(2)} = \frac{1}{4}e^- \wedge e^+$.

The above geometry for $\lambda_+ = \lambda_- = 0$ admits a second realization of supersymmetry obtained by trading G_a for G_a^{ij} . Using the same light-cone coordinates as above, we can choose $G_a^{ij} = g\delta_a^- i(\sigma_3)^{ij}$, and introduce the two commuting sets of isometry generators $\Delta_a^{(i)}$:

$$\Delta_a^{(i)} \equiv P_a + (-)^i g(\delta_a^- \mathbb{A} - \epsilon_{+a}{}^{cd} M_{cd}). \quad (\text{C.45})$$

The superalgebra is easily computed and corresponds to two copies of the contraction of $SU(2|1)$, containing $U(1)_{P_-} \times \text{Heis}_3$ as bosonic subalgebra. The central charges $\Delta_+^{(i)}$ of Heis_3^2 extend to central charges of the full superalgebra. Following the analogy with the previous cases based on G_a^{ij} , it is not surprising that generating the coset space using $T_a \equiv \frac{1}{4g}\Delta_a^{(2)}$ we induce a choice of spin and $U(1)_R$ connections such that half of the Killing spinors are entirely constant. The other half is $\hat{\xi}_{\alpha 2} = e^{2igu}\xi_{\alpha 2}$, with $\xi_{\alpha 2}$ defined in (C.42) for $\mathcal{Z} = \lambda_{\pm} = 0$.

D Details of Euclidean backgrounds

D.1 S^4 and H^4

We can set $S^{ij} = i\mu\delta^{ij}$ and $\bar{S}^{ij} = i\bar{\mu}\delta^{ij}$ for real $\mu, \bar{\mu}$. Whenever they are both non-vanishing we can use the $SO(1,1)$ R -symmetry to set $|\mu| = |\bar{\mu}|$.

For $\mu = \bar{\mu}$ the geometry is S^4 . The standard sphere line element can be obtained by the coset representative

$$L \equiv e^{\omega M_{12}} e^{\phi M_{23}} e^{\rho M_{34}} e^{\frac{\theta}{\mu} P_4} e^{-\rho M_{34}} e^{-\phi M_{23}} e^{-\omega M_{12}}, \quad (\text{D.1})$$

where the specific choice of local Lorentz gauge renders L well-behaved at the north pole. In order to define single-valued objects at the south pole we perform a change of Lorentz gauge analogous to that discussed around (C.23):

$$L(\omega, \phi, \rho, \theta) \longrightarrow L^{(\text{south})} \equiv L(\omega, \phi, \rho, \theta - \pi) e^{\frac{\pi}{\mu} P_4}. \quad (\text{D.2})$$

In practice the effect is to flip the sign of ρ in the rightmost exponential of (D.1). In the northern gauge (D.1) can be rewritten as $L = \exp\left(\frac{\theta}{\mu}\tilde{x}^a P_a\right)$ with

$$\tilde{x}_1 = -\sin \rho \sin \phi \sin \omega, \quad \tilde{x}_2 = \sin \rho \sin \phi \cos \omega, \quad \tilde{x}_3 = -\sin \rho \cos \phi, \quad \tilde{x}_4 = \cos \rho. \quad (\text{D.3})$$

The vielbein then can be written in the compact form

$$e^a = \frac{1}{\mu}(\tilde{x}^a d\theta + \sin \theta d\tilde{x}^a), \quad \text{northern patch}$$

$$e^{a\neq 4} = -\frac{1}{\mu}(\tilde{x}^a d\theta - \sin\theta d\tilde{x}^a), \quad e^4 = \frac{1}{\mu}(\tilde{x}^4 d\theta - \sin\theta d\tilde{x}^4), \quad \text{southern patch.} \quad (\text{D.4})$$

The metric is given by the standard line element

$$ds^2 = \frac{1}{\mu^2} \left[d\theta^2 + \sin^2\theta (d\rho^2 + \sin^2\rho (d\phi^2 + \sin^2\phi d\omega^2)) \right]. \quad (\text{D.5})$$

In this case we use four-component Killing spinors $\xi_{(i)\hat{\alpha}} = (\xi_{\alpha i}, \bar{\xi}^{\hat{\alpha}i})$ and a similar form for a constant spinor $\epsilon_{(i)\hat{\alpha}}$. We then compute

$$\xi_{(i)\hat{\alpha}} = \left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2} \tilde{x}^a \gamma_a \gamma_5 \right)_{\hat{\alpha}}^{\hat{\beta}} \epsilon_{(i)\hat{\beta}}. \quad (\text{D.6})$$

These spinors are periodic in ϕ and well-behaved at the north pole.²⁴ The above change of Lorentz gauge for the southern patch corresponds to the substitution

$$\xi_{(i)\hat{\alpha}}(\omega, \phi, \rho, \theta) \longrightarrow \xi_{(i)\hat{\alpha}}^{(\text{south})} \equiv -(\gamma_4 \gamma_5)_{\hat{\alpha}}^{\hat{\beta}} \xi_{(i)\hat{\beta}}(\omega, \phi, \rho, \theta - \pi). \quad (\text{D.7})$$

Setting now $\bar{\mu} = -\mu$ we find the surface $-X^2 + Y^2 + Z^2 + V^2 + W^2 = -1$. We content ourselves with the description of one connected component. We can use the same expression for the coset representative as above, though now P_4 is a noncompact generator and the vielbein and metric read

$$e^a = \frac{1}{\mu}(\tilde{x}^a d\theta + \sinh\theta d\tilde{x}^a),$$

$$ds^2 = \frac{1}{\mu^2} \left[d\theta^2 + \sinh^2\theta (d\rho^2 + \sin^2\rho (d\phi^2 + \sin^2\phi d\omega^2)) \right]. \quad (\text{D.8})$$

with the same \tilde{x}_a . The Killing spinors are

$$\xi_{(i)\hat{\alpha}} = \left(\cosh\frac{\theta}{2} + \sinh\frac{\theta}{2} \tilde{x}^a \gamma_a \right)_{\hat{\alpha}}^{\hat{\beta}} \epsilon_{(i)\hat{\beta}}. \quad (\text{D.9})$$

D.2 Squashed $S^3 \times \mathbb{R}$ and $S^3 \times S^1$

Let us fix $G^a = (0, 0, 0, -ig)$, $g \in \mathbb{R}$ and $\mathcal{Z}_{ab} = 4i\lambda\delta_{ab}^{12}$, $\bar{\mathcal{Z}}_{ab} = -4i\tilde{\lambda}\delta_{ab}^{12}$ with real $\lambda, \tilde{\lambda}$. Whenever they are both non-vanishing, we are free to set $|\lambda| = |\tilde{\lambda}|$. Keeping independent $\lambda, \tilde{\lambda}$ for the time being, we still find an S^3 geometry if $\lambda\tilde{\lambda} > -4g^2$. Defining the squashing parameter $v = \left(1 + \frac{\lambda\tilde{\lambda}}{4g^2}\right)^{-1}$, we can use the coset representative

$$L = e^{x_4 P_4} e^{\phi T_3} e^{\theta T_2} e^{\omega T_3} \quad (\text{D.10})$$

analogous to section 3.4.1 and reuse the expressions (3.14), (3.21), (3.23) for the generators and vielbein. In terms of the embedding coordinates (3.16), the Killing spinors are

$$\xi_{\alpha i} = [X\mathbb{I}_2 - 2(W\sigma^{12} + \sqrt{v}V\sigma^{23} + \sqrt{v}Y\sigma^{31})]_{\alpha}^{\beta} \epsilon_{\beta i} + \frac{\sqrt{v}}{2g} (\lambda Y\sigma^1 - \tilde{\lambda} V\sigma^2)_{\alpha\hat{\beta}} \bar{\epsilon}^{\hat{\beta}i},$$

²⁴They coincide with those of [1] up to a coordinate transformation and those of [27] up to a local Lorentz transformation.

$$\bar{\xi}^{\dot{\alpha}}_i = [X\mathbb{I}_2 - 2(W\bar{\sigma}^{12} + \sqrt{v}V\bar{\sigma}^{23} + \sqrt{v}Y\bar{\sigma}^{31})]^{\dot{\alpha}}_{\dot{\beta}}\bar{\epsilon}^{\dot{\beta}}_i + \frac{\sqrt{v}}{2g}(\lambda Y\bar{\sigma}^1 - \tilde{\lambda}V\bar{\sigma}^2)^{\dot{\alpha}\beta}\epsilon_{\beta i}. \quad (\text{D.11})$$

The potentials for \mathcal{Z} , $\bar{\mathcal{Z}}$ and $*G$ are respectively

$$C_{(1)} = -i\frac{\lambda}{2g}e^3, \quad \bar{C}_{(1)} = i\frac{\tilde{\lambda}}{2g}e^3, \quad B_{(2)} = i\frac{v^2}{64g^2}(\cos\theta \pm 1)d\phi d\omega. \quad (\text{D.12})$$

The two signs in $B_{(2)}$ are associated with the northern and southern patches.

D.3 The one-sheeted $H^3 \times \mathbb{R}$

We consider now $\lambda\tilde{\lambda} < -4g^2$, define $v = -(1 + \frac{\lambda\tilde{\lambda}}{4g^2})^{-1} > 0$ and choose

$$T_0 = -v\left(\frac{1}{4g}P_3 + \frac{\lambda\tilde{\lambda}}{4g^2}M_{12}\right), \quad T_{1,2} = \sqrt{v}\frac{1}{4g}P_{1,2}, \quad U \equiv \frac{1}{4g}P_3 - M_{12}, \quad (\text{D.13})$$

so that the algebra is formally the same as section 3.5. Borrowing results from there, we arrive at the physical vielbein $e^{1,2} = \frac{\sqrt{v}}{4g}E^{1,2}$, $e^3 = -\frac{v}{4g}E^0$, $e^4 = dx_4$ and the metric (6.7). The Killing spinors are (in terms of (3.44))

$$\begin{aligned} \xi_{\alpha i} &= [X - 2Y\sigma^{12} - 2\sqrt{v}(W\sigma^{23} + V\sigma^{31})]_{\alpha}^{\beta}\epsilon_{\beta i} + \sqrt{v}\frac{\tilde{\lambda}}{2g}(W\sigma^1 - V\sigma^2)_{\alpha\dot{\beta}}\bar{\epsilon}^{\dot{\beta}}_i, \\ \bar{\xi}^{\dot{\alpha}}_i &= [X - 2Y\bar{\sigma}^{12} - 2\sqrt{v}(W\bar{\sigma}^{23} + V\bar{\sigma}^{31})]^{\dot{\alpha}}_{\dot{\beta}}\bar{\epsilon}^{\dot{\beta}}_i + \sqrt{v}\frac{\lambda}{2g}(W\bar{\sigma}^1 - V\bar{\sigma}^2)^{\dot{\alpha}\beta}\epsilon_{\beta i}. \end{aligned} \quad (\text{D.14})$$

The potentials for \mathcal{Z} , $\bar{\mathcal{Z}}$ and $*G$ are respectively

$$C_{(1)} = -i\frac{\lambda}{2g}e^3, \quad \bar{C}_{(1)} = i\frac{\tilde{\lambda}}{2g}e^3, \quad B_{(2)} = i\frac{v^2}{64g^2}\sinh\rho d\phi d\tau. \quad (\text{D.15})$$

D.4 The Heis₃ $\times \mathbb{R}$ limit

We shall now set $\lambda\tilde{\lambda} = -4g^2$. A flat direction is generated by P_4 as usual, while $[P_1, P_2] = \frac{1}{4g}P_3 - M_{12} \equiv H$ identifies the central charge H of Heis₃. Using $L = e^{xP_1+yP_2}e^{wH}e^{zP_4}$ we arrive at the vielbein

$$e^1 = dx, \quad e^2 = dy, \quad e^3 = dw + 2g(ydx - xdy), \quad e^4 = dz \quad (\text{D.16})$$

and the metric (6.8). Gauge-fixing $\lambda = -\tilde{\lambda} = 2g$, the Killing spinors are

$$\begin{aligned} \xi_{\alpha i} &= [\mathbb{I}_2 - 4g x\sigma^{23} - 4g y\sigma^{31}]_{\alpha}^{\beta}\epsilon_{\beta i} + 2g(x\sigma^2 - y\sigma^1)_{\alpha\dot{\beta}}\bar{\epsilon}^{\dot{\beta}}_i, \\ \bar{\xi}^{\dot{\alpha}}_i &= [\mathbb{I}_2 - 4g x\bar{\sigma}^{23} - 4g y\bar{\sigma}^{31}]^{\dot{\alpha}}_{\dot{\beta}}\bar{\epsilon}^{\dot{\beta}}_i - 2g(x\bar{\sigma}^2 - y\bar{\sigma}^1)^{\dot{\alpha}\beta}\epsilon_{\beta i}. \end{aligned} \quad (\text{D.17})$$

The background forms \mathcal{Z} , $\bar{\mathcal{Z}}$ and $*G$ are easily integrated to potentials

$$C_{(1)} = i\lambda(xdy - ydx), \quad \bar{C}_{(1)} = -i\tilde{\lambda}(xdy - ydx), \quad B_{(2)} = -\frac{ig}{2}(xdy - ydx)dw. \quad (\text{D.18})$$

D.5 The two-sheeted $H^3 \times \mathbb{R}$

We turn on and gauge-fix $G_a^i{}_j = ig\delta_a^4(\sigma_3)^i{}_j$. The structure of the superalgebra is then most evident if we define $\Delta_a^{(i)} \equiv P_a + (-)^{i+1}ig(\epsilon_{abc4}M^{bc} + \delta_a^4\mathbb{U})$, which together with their complex conjugates generate $SL(2, \mathbb{C}) \times GL(1, \mathbb{C})$. We choose as coset representatives

$$L \equiv e^{zP_4} e^{\phi M_{12}} e^{\theta M_{31}} e^{\frac{\rho}{2g}P_3} e^{-\theta M_{31}} e^{-\phi M_{12}}, \quad (\text{D.19})$$

which represents polar coordinates along one sheet of the hyperboloid in a convenient gauge. We can use embedding coordinates satisfying $X^2 - Y^2 - V^2 - W^2 = 1$:

$$X = \cosh \rho, \quad Y = \sinh \rho \cos \theta, \quad V = \sinh \rho \sin \theta \cos \phi, \quad W = \sinh \rho \sin \theta \sin \phi. \quad (\text{D.20})$$

The ranges and periodicities are $\rho \in [0, +\infty)$, $\theta \in [0, \pi]$ and $\phi \simeq \phi + 2\pi$. The vielbein is

$$\begin{aligned} e^1 &= \frac{1}{2g} (\cos \phi \sin \theta d\rho + \sinh \rho d(\cos \phi \sin \theta)), \\ e^2 &= \frac{1}{2g} (\sin \phi \sin \theta d\rho + \sinh \rho d(\sin \phi \sin \theta)), \\ e^3 &= \frac{1}{2g} (\cos \theta d\rho - \sinh \rho \sin \theta d\theta), \quad e^4 = dz \end{aligned} \quad (\text{D.21})$$

and the metric is (6.11).

The supercharges $Q_{\alpha,1}$ and $Q_{\alpha,2}$ have opposite charges under P_a , and the same holds for the antichiral ones. Hence, we find:

$$\begin{aligned} \xi_{\alpha 1} &= e^{-igz} \frac{1}{\sqrt{1+X}} [1 + X + 2(Yi\sigma^{12} + Vi\sigma^{23} + Wi\sigma^{31})]_{\alpha}^{\beta} \epsilon_{\beta 1}, \\ \xi_{\alpha 2} &= e^{+igz} \frac{1}{\sqrt{1+X}} [1 + X - 2(Yi\sigma^{12} + Vi\sigma^{23} + Wi\sigma^{31})]_{\alpha}^{\beta} \epsilon_{\beta 2}, \\ \bar{\xi}^{\dot{\alpha}}_1 &= e^{-igz} \frac{1}{\sqrt{1+X}} [1 + X - 2(Yi\bar{\sigma}^{12} + Vi\bar{\sigma}^{23} + Wi\bar{\sigma}^{31})]_{\dot{\beta}}^{\dot{\alpha}} \bar{\epsilon}^{\dot{\beta}}_1, \\ \bar{\xi}^{\dot{\alpha}}_2 &= e^{+igz} \frac{1}{\sqrt{1+X}} [1 + X + 2(Yi\bar{\sigma}^{12} + Vi\bar{\sigma}^{23} + Wi\bar{\sigma}^{31})]_{\dot{\beta}}^{\dot{\alpha}} \bar{\epsilon}^{\dot{\beta}}_2. \end{aligned} \quad (\text{D.22})$$

The potential for $*G^i{}_j$ is

$$B_{(2)}^i{}_j = \frac{1}{8g^2} (1 - \cos \theta) \sinh^2 \rho d\rho d\phi (i\sigma_3)^i{}_j. \quad (\text{D.23})$$

D.6 $H^2 \times S^2$ and $D(2, 1; \alpha)$

We take $\mathcal{Z}_{ab} = 2i(\lambda_+\delta_{ab}^{12} - \lambda_-\delta_{ab}^{34})$ and $\bar{\mathcal{Z}}_{ab} = -2i(\lambda_+\delta_{ab}^{12} + \lambda_-\delta_{ab}^{34})$. The resulting superalgebra is the same as in section 3.6 with the substitutions $P_0 = iP_4$, $M_{03} = -iM_{34}$. Choosing polar coordinates ω , ρ on H^2 , we pick the coset representative

$$L = e^{\omega M_{34}} e^{\rho \frac{1}{\lambda_-} P_3} e^{-\omega M_{34}} e^{\phi M_{12}} e^{\theta \frac{1}{\lambda_+} P_2} e^{-\phi M_{12}}. \quad (\text{D.24})$$

To have a single-valued representative at the south pole of S^2 it is sufficient to perform the same change of gauge as in (C.23). We obtain the metric (6.15) from the vielbein

$$e^1 = \frac{1}{\lambda_+} (-\sin \phi d\theta - \sin \theta \cos \phi d\phi), \quad e^2 = \frac{1}{\lambda_+} (\cos \phi d\theta - \sin \theta \sin \phi d\phi),$$

$$e^3 = \frac{1}{\lambda_-} (\cos \omega d\rho - \sinh \rho \sin \omega d\omega), \quad e^4 = \frac{1}{\lambda_-} (\sin \omega d\rho + \sinh \rho \cos \omega d\omega). \quad (\text{D.25})$$

In the northern patch of S^2 we can write Killing spinors in terms of the matrices

$$\begin{aligned} A &= \cos \frac{\theta}{2} \cosh \frac{\rho}{2} + 2 \sin \frac{\theta}{2} \sinh \frac{\rho}{2} (\cos(\omega - \phi) \sigma^{23} + \sin(\omega - \phi) \sigma^{31}), \\ B &= \sin \frac{\theta}{2} \cosh \frac{\rho}{2} (\cos \phi \sigma^1 + \sin \phi \sigma^2) - \cos \frac{\theta}{2} \sinh \frac{\rho}{2} (i \cos \omega - \sin \omega \sigma^3), \\ \bar{A} &= \cos \frac{\theta}{2} \cosh \frac{\rho}{2} - 2 \sin \frac{\theta}{2} \sinh \frac{\rho}{2} (\cos(\omega + \phi) \bar{\sigma}^{23} + \sin(\omega + \phi) \bar{\sigma}^{31}), \\ \bar{B} &= \sin \frac{\theta}{2} \cosh \frac{\rho}{2} (\cos \phi \bar{\sigma}^1 + \sin \phi \bar{\sigma}^2) + \cos \frac{\theta}{2} \sinh \frac{\rho}{2} (i \cos \omega - \sin \omega \bar{\sigma}^3), \end{aligned} \quad (\text{D.26})$$

with

$$\xi_{\alpha i} = A_{\alpha}^{\beta} \epsilon_{\beta i} + B_{\alpha \dot{\beta}} \bar{\epsilon}^{\dot{\beta}}_i, \quad \bar{\xi}^{\dot{\alpha}}_i = \bar{A}^{\dot{\alpha}}_{\dot{\beta}} \bar{\epsilon}^{\dot{\beta}}_i + \bar{B}^{\dot{\alpha} \beta} \epsilon_{\beta i}. \quad (\text{D.27})$$

The change of Lorentz gauge for the southern patch of S^2 corresponds to the substitution

$$\xi_{(i)\hat{\alpha}}(\omega, \rho, \phi, \theta) \longrightarrow \xi_{(i)\hat{\alpha}}^{(\text{south})} \equiv (i\gamma_1)_{\hat{\alpha}}^{\hat{\beta}} \xi_{(i)\hat{\beta}}(\omega, \rho, \phi, \theta - \pi). \quad (\text{D.28})$$

The potentials for \mathcal{Z} , $\bar{\mathcal{Z}}$ are respectively

$$\begin{aligned} C_{(1)} &= -\frac{i}{\lambda_+} (\cos \theta \pm 1) d\phi - \frac{i}{\lambda_-} \cosh \rho d\omega, \\ \bar{C}_{(1)} &= \frac{i}{\lambda_+} (\cos \theta \pm 1) d\phi - \frac{i}{\lambda_-} \cosh \rho d\omega. \end{aligned} \quad (\text{D.29})$$

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