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New formulation of the type IIB superstring action in $\mathrm{AdS}_5 \times S^5$

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Abstract: Previous studies of the type IIB superstring in an $\mathrm{AdS}_5 \times S^5$ background are based on a description of the superspace geometry as the quotient space $\mathrm{PSU}(2,2|4)/\mathrm{SO}(4,1) \times \mathrm{SO}(5)$. This paper develops an alternative approach in which the Grassmann coordinates provide a nonlinear realization of $\mathrm{PSU}(2,2|4)$ based on the quotient space $\mathrm{PSU}(2,2|4)/\mathrm{SU}(2,2) \times \mathrm{SU}(4)$, and the bosonic coordinates are described as a submanifold of $\mathrm{SU}(2,2) \times \mathrm{SU}(4)$. This formulation keeps all bosonic symmetries manifest, and it provides the complete dependence on the Grassmann coordinates in terms of simple analytic expressions. It is used to construct the superstring world-sheet action in a form in which the $\mathrm{PSU}(2,2|4)$ symmetry is manifest and kappa symmetry can be established. This formulation might have some advantages compared to previous ones, but this remains to be demonstrated.

Keywords: Superstrings and Heterotic Strings, AdS-CFT Correspondence, Superstring Vacua

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1 Introduction

The conjectured duality [1] between type IIB superstring theory [2] in a maximally supersymmetric $\mathrm{AdS}_5 \times S^5$ background, with N units of self-dual five-form flux, and four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory [3], with a $\mathrm{U}(N)$ gauge group, has been studied extensively. This is a precisely defined conjecture, because the $\mathrm{AdS}_5 \times S^5$ background is an exact solution of type IIB superstring theory [4]. Most studies have focused on the large-N limit for fixed 't Hooft parameter $\lambda = g_{YM}^2 N$. (See [5] and references therein.) This limit corresponds to the planar approximation to the field theory [6] and the classical (or leading genus) approximation to the string theory. The planar approximation to the field theory is an integrable four-dimensional theory, with an infinite-dimensional Yangian symmetry generated by the superconformal group $\mathrm{PSU}(2, 2|4)$ and a dual conformal group. Its perturbative expansion parameter is proportional to λ.

The isometry supergroup of the $\text{AdS}_5 \times S^5$ solution of type IIB superstring theory is also PSU$(2, 2|4)$. Its bosonic subgroup is $\text{SU}(4) \times \text{SU}(2, 2)$, where $\text{SU}(4) = \text{Spin}(6)$ and $\text{SU}(2, 2) = \text{Spin}(4, 2)$. This supergroup has 32 fermionic generators, which we will refer to as supersymmetries. This is the maximum number possible and the same number as the flat-space solution, which corresponds to the large-radius (or large-λ) limit of the $\text{AdS}_5 \times S^5$ solution. The string theory, for the background in question, is described by an interacting two-dimensional world-sheet theory, whose perturbative expansion parameter is proportional to $1/\sqrt{\lambda}$. This theory is also integrable.

Even though the planar $\mathcal{N} = 4$ super Yang-Mills theory and the leading-genus $\text{AdS}_5 \times S^5$ superstring action are both integrable, both of them are also very challenging to study. Nonetheless, a lot of progress has been achieved, providing convincing evidence in support of the duality, thanks to an enormous effort by many very clever people. The goal of the present work is to derive a new formulation of the superstring world-sheet theory. It will turn out to be equivalent to the previous formulation by Metsaev and Tseytlin [7] and others [8, 9].[1] However, it has some attractive features that might make it more useful.

In recent work the author has studied the bosonic truncation of the world-volume action of a probe D3-brane embedded in this background and made certain conjectures concerning an interpretation of this action that should hold when the fermionic degrees of freedom are incorporated [11, 12]. This provided the motivation for developing a convenient formalism for adding the fermions in which all of the symmetries can be easily understood. While that is the motivation, the present work does not require the reader to be familiar with those papers, nor does it depend on the correctness of their conjectures, which have aroused considerable skepticism. This paper, which is about superspace geometry and the superstring action, does not make any bold conjectures, and therefore it should be noncontroversial.

String world-sheet actions have much in common with WZW models for groups, supergroups, cosets, etc. though there a few differences. One difference is that they are invariant under reparametrization of the world-sheet coordinates. One way of implementing this is to couple the sigma model to two-dimensional gravity. If one chooses a conformally flat gauge, the action reduces to the usual two-dimensional Minkowski space form, but it is supplemented by Virasoro constraints. Residual symmetries in this gauge allow further gauge fixing, the main example being light-cone gauge. In addition to the local reparametrization invariance, superstring actions also have local fermionic symmetries, called kappa symmetry. They are rather subtle, and they play a crucial role. One of the goals of this paper is to give a clear explanation of how kappa symmetry is realized.

In constructing chiral sigma models for homogeneous spaces that have an isometry group G, but are not group manifolds, the standard approach is to formulate them as coset theories. Thus, for example, a theory on a sphere S^n is formulated as an $\text{SO}(n+1)/\text{SO}(n)$ coset theory. The formulas that describe symmetric spaces as $M = G/H$ coset theories are well-known [13, 14]. They involve a construction that incorporates global G symmetry and local H symmetry. This is the standard thing to do, and so it is not surprising that this is the approach that was utilized in [7] to construct the superstring world-sheet action for

[1]For a recent review, see [10].

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$AdS_5 \times S^5$. In this case the coset in question is $PSU(2,2|4)/SO(4,1) \times SO(5)$. This paper describes an alternative procedure in which the Grassmann coordinates provide a nonlinear realization of $PSU(2,2|4)$ based on the quotient space $PSU(2,2|4)/SU(2,2) \times SU(4)$, and the bosonic coordinates are described as a submanifold of $SU(2,2) \times SU(4)$.

The description of S^2 as a subspace of SU(2) is a very simple analog of the procedure that will be used. (It is relevant to the discussion of $AdS_2 \times S^2$, which is analogous to $AdS_5 \times S^5$.) The group SU(2) consists of 2×2 unitary unimodular matrices, and the group manifold is S^3. An S^2 can be embedded in this group manifold in many different ways. The one that is most relevant for our purposes is the subspace consisting of all symmetric SU(2) matrices. This subspace can be expressed in terms of Pauli matrices in the form $\sigma_2 \vec{\sigma} \cdot \hat{x}$, where $\hat{x}$ is a unit-length 3-vector. The action of a group element g on an element of this sphere, represented by a symmetric matrix g_0, is $g_0 \to g^T g_0 g$. This is a point on the same S^2, since $g^T g_0 g$ is also a symmetric SU(2) matrix. The isometry group of S^2 is actually SO(3), because the group elements g and $-g$ describe the same map. Clearly, a specific subspace of SU(2) has been selected, in a way that does not depend on any arbitrary choices, to describe S^2. This paper applies a similar procedure to the description of S^5 and AdS_5 as subspaces of SU(4) and SU(2,2), respectively. In particular, the description of S^5 in terms of antisymmetric SU(4) matrices is discussed in detail in appendix A.

This formulation of the superspace geometry that enters in the construction of the superstring action makes it possible to keep all of the bosonic symmetries manifest throughout the analysis,[2] and many formulas, including the superstring action itself, have manifest $PSU(2,2|4)$ symmetry. Also, the complete dependence on the Grassmann coordinates for all relevant quantities is given by simple tractable analytic expressions. So far, we have just rederived results that have been known for a long time, but the hope is that this reformulation of the world-sheet theory will be helpful for obtaining new results.

2 The bosonic truncation

Before confronting superspace geometry, let us briefly review the bosonic structure of $AdS_5 \times S^5$, which has the isometry $SO(4,2) \times SO(6)$. The generators of SO(6), denoted J^{ab}, where $a, b = 1, 2, \ldots 6$, can be viewed as generators of rotations of $\mathbb{R}^6$ about the origin. They also generate the isometries of a unit-radius S^5 centered about the origin ($\hat{z} \cdot \hat{z} = \sum_1^6 (z^a)^2 = 1$). Similarly, J^{mn} generates isometries of a unit-radius AdS_5 embedded in $\mathbb{R}^{4,2}$ by the equation

$$\hat{y} \cdot \hat{y} = \sum_{m,n=0}^{5} \eta_{mn} y^m y^n = -(y^0)^2 + (y^1)^2 + (y^2)^2 + (y^3)^2 + (y^4)^2 - (y^5)^2 = -1. \tag{2.1}$$

This equation describes the Poincaré patch of AdS_5, which is all that we are concerned with in this work. The two algebras are distinguished by the choice of indices (a, b, c, d or m, n, p, q).

[2] A previous attempt to make the SU(4) symmetry manifest is described in [15].

We prefer to write the unit-radius $\mathrm{AdS}_5 \times S^5$ metric in a form in which all of the isometries are manifest. There are various ways to achieve this. One option is

$$ds^2 = d\hat{z} \cdot d\hat{z} + d\hat{y} \cdot d\hat{y}, \tag{2.2}$$

where $\hat{z}$ and $\hat{y}$ are understood to satisfy the constraints described above. This is the description that will be utilized in most of this paper.

Lie-algebra-valued connection one-forms associated to the SO(6) symmetry of S^5 are easily constructed in terms of the unit six-vector z^a. (We do not display hats to avoid clutter.) The one-form is

$$\Omega_0^{ab} = 2(z^a dz^b - z^b dz^a). \tag{2.3}$$

The subscript 0 is used to refer to the bosonic truncation. The normalization is chosen to ensure that this is a flat connection, i.e., its two-form curvature is

$$d\Omega_0^{ab} + \Omega_0^{ac} \wedge \eta_{cd}\, \Omega_0^{db} = 0. \tag{2.4}$$

This is easily verified using $z^a\, \eta_{ab}\, dz^b = 0$, which is a consequence of $z^2 = z^a\, \eta_{ab}\, z^b = 1$. In the case of SO(6), the metric η is just a 6×6 unit matrix, which we denote I_6. Similarly, there is a Lie-algebra-valued flat connection

$$\tilde{\Omega}_0^{mn} = -2(y^m dy^n - y^n dy^m) \tag{2.5}$$

associated to the SO(4,2) symmetry of AdS_5. In the SO(4,2) case $\eta = I_{4,2}$, which has diagonal components $(1,1,1,1,-1,-1)$. Recall that for this choice $y^2 = -1$, which is why $\tilde{\Omega}_0^{mn}$ requires an extra minus sign to ensure flatness.

The bosonic truncation of the superstring action can be expressed entirely in terms of the induced world-volume metric,

$$G_{\alpha\beta} = \partial_\alpha \hat{z} \cdot \partial_\beta \hat{z} + \partial_\alpha \hat{y} \cdot \partial_\beta \hat{y}, \tag{2.6}$$

where it is understood that y and z are functions of the world-sheet coordinates σ^α, $\alpha = 0, 1$. The action is

$$S = -\frac{R^2}{2\pi\alpha'} \int d^2\sigma \sqrt{-G}, \tag{2.7}$$

where $G = \det G_{\alpha\beta}$ and α' is the usual string theory Regge slope parameter, which (for $\hbar = c = 1$) has dimensions of length squared. A standard rewriting of such a metric involves introducing an auxiliary world-sheet metric field $h_{\alpha\beta}$. Then the action can be recast as

$$S = -\frac{R^2}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} h^{\alpha\beta} G_{\alpha\beta}. \tag{2.8}$$

This form has a Weyl symmetry given by an arbitrary local rescaling of $h_{\alpha\beta}$. The simplest way to understand the equivalence of the two forms of S is to note that the $h_{\alpha\beta}$ classical equation of motion is solved by $h_{\alpha\beta} = \lambda G_{\alpha\beta}$, i.e., they are conformally equivalent. The conformal factor cancels out classically. For a critical string theory, without conformal anomaly, it should also cancel quantum mechanically. The bosonic truncation described

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here is not critical, but the complete theory with the Grassmann coordinates included should be. The complete superstring world-sheet action includes a Wess-Zumino term that vanishes for the bosonic truncation.

When the fermionic degrees of freedom are included, the dual CFT is $\mathcal{N} = 4$ super Yang-Mills theory with a $U(N)$ gauge group. N is related to a five-form flux, which does not appear in the string world-sheet action. (It does appear in the D3-brane action.) The gauge theory has a dimensionless 't Hooft parameter $\lambda = g_{YM}^2 N$. AdS/CFT duality gives the identification

$$g_s = \frac{g_{YM}^2}{4\pi}, \tag{2.9}$$

where g_s is the string coupling constant (determined by the vev of the dilaton field). The radius R of the S^5 and the AdS_5 is introduced by replacing the unit-radius metric ds^2 by $R^2 ds^2$. Then, utilizing the AdS/CFT identification

$$R^2 = \alpha' \sqrt{\lambda}, \tag{2.10}$$

one obtains

$$S = -\frac{\sqrt{\lambda}}{2\pi} \int d^2\sigma \sqrt{-G}. \tag{2.11}$$

In the large N limit, taken at fixed λ, the CFT is described by the planar approximation, and the string theory is described by the classical approximation, i.e., leading order in the world-sheet genus expansion. Even so, the two-dimensional world-sheet theory must be treated as a quantum theory, with a perturbation expansion in powers of $1/\sqrt{\lambda} \sim l_s/R$, in order to determine the string spectrum and tree amplitudes. Flat ten-dimensional spacetime is the leading approximation in this expansion. The dual planar CFT, on the other hand, has a perturbation expansion in powers of λ.

Let us introduce null world-sheet coordinates $\sigma^\pm = \sigma^1 \pm \sigma^0$. It is often convenient to choose a conformally flat gauge. This means using the two diffeomorphism symmetries to set

$$G_{++} = G_{--} = 0. \tag{2.12}$$

Then the action simplifies to

$$S = -\frac{\sqrt{\lambda}}{2\pi} \int d^2\sigma G_{+-}, \tag{2.13}$$

which is supplemented by the Virasoro constraints $G_{++} = G_{--} = 0$.[3] In the geometry at hand, we have

$$G_{+-} = \partial_+ \hat{z} \cdot \partial_- \hat{z} + \partial_+ \hat{y} \cdot \partial_- \hat{y}. \tag{2.14}$$

This can then be varied to give equations of motion. Taking account of the constraints $\hat{z} \cdot \hat{z} = 1$ and $\hat{y} \cdot \hat{y} = -1$, we obtain

$$(\eta^{ab} - z^a z^b)\partial_+\partial_- z_b = 0 \quad \text{and} \quad (\eta^{mn} + y^m y^n)\partial_+\partial_- y_n = 0. \tag{2.15}$$

[3] When the world-sheet theory is quantized, G_{++} and G_{--} become operators that need to be treated with care. In any case, the bosonic truncation of the world-sheet theory is inconsistent beyond the classical approximation due to a conformal anomaly.

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Conservation of the SO(6) and SO(4, 2) Noether currents implies that

$$\partial_\alpha(z^a\partial^A z^b - z^b\partial^A z^a) = 0 \quad \text{and} \quad \partial_\alpha(y^m\partial^A y^n - y^n\partial^A y^m) = 0, \tag{2.16}$$

which are equivalent to the equations of motion. Expressed more elegantly,

$$d\star\Omega_0^{ab} = 0 \quad \text{and} \quad d\star\tilde{\Omega}_0^{mn} = 0. \tag{2.17}$$

The bosonic connections Ω_0 and $\tilde{\Omega}_0$ are simultaneously conserved and flat when the equations of motion are taken into account. These conditions allow one to construct a one-parameter family of flat connections, whose existence is the key to classical integrability of the world-sheet theory [16]. In the remainder of this manuscript we will add Grassmann coordinates and construct the complete superstring action with PSU(2, 2|4) symmetry. Since this will be a "critical" string theory (without conformal anomaly), its integrability is expected to be valid for the quantum theory, i.e., taking full account of the dependence on λ, but only at leading order in the genus expansion.

3 Supersymmetrization

Our goal is to add fermionic (Grassmann) coordinates θ to the metric of the preceding section so as to make it invariant under PSU(2, 2|4). In addition to bosonic one-forms Ω^{ab} and $\tilde{\Omega}^{mn}$, whose bosonic truncations are Ω_0^{ab} and $\tilde{\Omega}_0^{mn}$ described in section 2, we also require a fermionic one-form Ψ, which is dual to the fermionic supersymmetry generators of the superalgebra. Ψ and $\Psi^\dagger$ should encode 32 fermionic one-forms, which transform under SU(4) × SU(2, 2) as $(\mathbf{4}, \bar{\mathbf{4}}) + (\bar{\mathbf{4}}, \mathbf{4})$.

Let us recast the connections Ω and $\tilde{\Omega}$ in spinor notation. The construction for SU(4) requires 4×4 analogs of Pauli matrices, or Dirac matrices, denoted Σ^a, which are described in appendix A. In the notation described there, we define

$$\Omega^\alpha{}_\beta = \frac{1}{4}(\Sigma_{ab})^\alpha{}_\beta\Omega^{ab}. \tag{3.1}$$

Also in the notation described in appendix A, there is an identical-looking formula for SU(2, 2),

$$\tilde{\Omega}^\mu{}_\nu = \frac{1}{4}(\Sigma_{mn})^\mu{}_\nu\tilde{\Omega}^{mn}. \tag{3.2}$$

Infinitesimal parameters of SU(4) and SU(2, 2) transformations are described in spinor notation by matrices $\omega^\alpha{}_\beta$ and $\tilde{\omega}^\mu{}_\nu$ in an analogous manner.

The fermionic one-form, transforming as $(\mathbf{4}, \bar{\mathbf{4}})$, is written $\Psi^\alpha{}_\mu$. Its hermitian conjugate, which transforms as $(\bar{\mathbf{4}}, \mathbf{4})$ is written $(\Psi^\dagger)^\mu{}_\alpha$. Spinor indices can be lowered or contracted using the 4×4 invariant tensors $\eta_{\alpha\bar{\beta}}$ and $\eta_{\mu\bar{\nu}}$. $\eta_{\alpha\bar{\beta}}$ is just the unit matrix I_4, and $\eta_{\mu\bar{\nu}}$ is the SU(2, 2) metric $I_{2,2}$. Thus, for example, $\Psi^{\alpha\bar{\mu}} = \Psi^\alpha{}_\nu\eta^{\nu\bar{\mu}}$. By always using matrices with unbarred indices we avoid the need to ever display η matrices explicitly. The price one pays for this is that expressions that are called adjoints, such as $\Psi^\dagger$, are not conventional adjoints, since they contain additional η factors. However, this "adjoint" is still an involution, since the square of an η is a unit matrix. In this notation, it makes sense to call the matrix $\tilde{\Omega}$ antihermitian despite the indefinite signature of SU(2, 2).

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3.1 Supermatrices

Since it is convenient to represent supergroups using supermatrices, let us review a few basic facts and our conventions. There are various conventions in the literature, and we shall introduce yet another one. We write an 8×8 supermatrix in terms of 4×4 blocks as follows

$$M = \begin{pmatrix} a & \tau b \\ \tau c & d \end{pmatrix}, \tag{3.3}$$

where a and d are Grassmann even and b and c are Grassmann odd. a is the SU(4) block and b is the SU(2, 2) block. This formula contains the phase

$$\tau = e^{-i\pi/4}, \tag{3.4}$$

which satisfies $\tau^2 = -i$. By introducing factors of τ in this way various formulas have a more symmetrical appearance than is the case for other conventions.

The "superadjoint" is defined by

$$M^\dagger = \begin{pmatrix} a^\dagger & -\tau c^\dagger \\ -\tau b^\dagger & d^\dagger \end{pmatrix}. \tag{3.5}$$

This definition, which reduces to the usual one for the diagonal blocks, is chosen to ensure the identity

$$(M_1 M_2)^\dagger = M_2^\dagger M_1^\dagger. \tag{3.6}$$

By definition, a unitary supermatrix satisfies $MM^\dagger = I$ and an antihermitian supermatrix satisfies $M + M^\dagger = 0$. The "supertrace" is defined by

$$\mathrm{str} M = \mathrm{tr}\, a - \mathrm{tr}\, d. \tag{3.7}$$

The main virtue of this definition is that the familiar identity $\mathrm{tr}(a_1 a_2) = \mathrm{tr}(a_2 a_1)$ generalizes to

$$\mathrm{str}(M_1 M_2) = \mathrm{str}(M_2 M_1). \tag{3.8}$$

Our main concern in this work is the supergroup PSU(2, 2|4). The corresponding superalgebra is best described in terms of matrices belonging to the superalgebra $\mathfrak{su}(2,2|4)$. This algebra consists of antihermitian supermatrices with vanishing supertrace. (It is implicit here that one takes appropriate account of the indefinite signature of $\mathfrak{su}(2,2)$.) Given this algebra, one defines the $\mathfrak{psu}(2,2|4)$ algebra to consist of $\mathfrak{su}(2,2|4)$ matrices modded out by the equivalence relation $M \sim M + \lambda I$, where I denotes the unit supermatrix.

The literature contains a definition of the "supertranspose", which is utilized to define a $\mathbb{Z}_4$ grading of the superalgebra [17]. This grading is supposed to be responsible for the integrability of the string world-sheet theory as well as the simple structure of its Wess-Zumino term. Be that as it may, we will be able to reproduce all of these key results without explicit reference to the supertranspose or a $\mathbb{Z}_4$ grading. As long as we discuss supermatrices with diagonal and off-diagonal blocks separately, we do not go astray by using the usual definition of a transpose. The usual rule $(M_1 M_2)^T = M_2^T M_1^T$ holds if M_1 and M_2 only have diagonal blocks. If both of them only have off-diagonal blocks, then $(M_1 M_2)^T = -M_2^T M_1^T$.

3.2 Nonlinear realization of the superalgebra

Superspace is described by the bosonic spacetime coordinates y^m and z^a, satisfying $z^2 = 1$ and $y^2 = -1$, introduced in section 2, and Grassmann coordinates $\theta^\alpha{}_\mu$. The θ coordinates are 16 complex Grassmann numbers that transform under $\mathrm{SU}(4) \times \mathrm{SU}(2,2)$ as $(\mathbf{4}, \bar{\mathbf{4}})$, like the one-form Ψ discussed above. It will be extremely helpful to think of θ as a 4×4 matrix rather than as a 16-component spinor. The two points of view are equivalent, of course, but matrix notation will lead to much more elegant formulas. If all matrix multiplications were done from one side, an awkward tensor product notation would be required. Using matrix notation, we will obtain simple analytic expressions describing the full θ dependence of all quantities that are required to formulate the superstring action.

One clue to understanding the $\mathrm{PSU}(2,2|4)$ symmetry of the $\mathrm{AdS}_5 \times S^5$ geometry is its relationship to the super-Poincaré symmetry algebra of flat ten-dimensional superspace, which corresponds to the large-radius limit. The large-radius limit preserves all 32 fermionic symmetries, but it only accounts for 30 of the 55 bosonic symmetries of the Poincaré algebra in 10 dimensions. The 25 rotations and Lorentz transformations that relate the $\mathbb{R}^{4,1}$ piece of the geometry that descends from AdS_5 to the $\mathbb{R}^5$ piece that descends from S^5 are additional "accidental" symmetries of the limit.

A little emphasized feature of the superspace description of the flat-space geometry is that the entire super-Poincaré algebra closes on the Grassmann coordinates θ. A possible reason for this lack of emphasis may be that the ten spacetime translations act trivially, i.e., they leave θ invariant. We will demonstrate here that the entire $\mathfrak{psu}(2,2|4)$ superalgebra closes on the fermionic coordinates $\theta^\alpha{}_\mu$ even though the radius is finite. In this case all of the symmetries transform θ nontrivially, and none of the transformations of θ give rise to expressions involving the y or z coordinates. This means that the Grassmann coordinates provide a nonlinear realization of the superalgebra. Conceptually, this is similar to the way the supersymmetry of a field theory in flat spacetime is realized nonlinearly on a spinor field (the Goldstino) [20]. In fact, the algebra for the two problems is quite similar. The nonlinear Lagrangian for the Goldstino field was generalized to anti de Sitter space in [21]. However, that work is not directly relevant, since the goal of the present work is to describe world-sheet fields and not ten-dimensional target-space fields. The latter may deserve further consideration in the future.

The infinitesimal bosonic symmetry transformations of θ are relatively trivial; they are "manifest" in the sense that they are determined by the types of spinor indices that appear. In matrix notation,

$$\delta\theta^\alpha{}_\mu = (\omega\theta - \theta\tilde{\omega})^\alpha{}_\mu. \tag{3.9}$$

The infinitesimal parameters $\omega^\alpha{}_\beta$ and $\tilde{\omega}^\mu{}_\nu$ take values in the $\mathfrak{su}(4)$ and $\mathfrak{su}(2,2)$ Lie algebras, respectively. Thus, they are anti-hermitian (in the sense discussed earlier) and traceless.

Let us now consider infinitesimal supersymmetry transformations of θ. In the case of flat space this is just $\delta\theta = \varepsilon$, where ε is an infinitesimal constant matrix of complex

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Grassmann parameters. In the case of unit radius it is a bit more interesting:[4]

$$\delta\theta^{\alpha}{}_{\mu} = \varepsilon^{\alpha}{}_{\mu} + i(\theta\varepsilon^{\dagger}\theta)^{\alpha}{}_{\mu}. \tag{3.10}$$

The hermitian conjugate equation is then

$$\delta(\theta^{\dagger})^{\mu}{}_{\alpha} = (\varepsilon^{\dagger})^{\mu}{}_{\alpha} + i(\theta^{\dagger}\varepsilon\theta^{\dagger})^{\mu}{}_{\alpha}. \tag{3.11}$$

We have displayed the spinor indices, but the more compact formulas $\delta\theta = \varepsilon + i\theta\varepsilon^{\dagger}\theta$ and $\delta\theta^{\dagger} = \varepsilon^{\dagger} + i\theta^{\dagger}\varepsilon\theta^{\dagger}$ are completely unambiguous. In our notation, the quantities

$$u = i\theta\theta^{\dagger} \quad \text{and} \quad \tilde{u} = i\theta^{\dagger}\theta \tag{3.12}$$

are both hermitian. The way we think about the adjoint of a product of two Grassmann numbers involves moving one of them past the other, which contributes a minus sign.

In our conventions all coordinates (θ, y, z) are dimensionless, since they pertain to unit radius $(R = 1)$. If we were to give them the usual dimensions, by absorbing appropriate powers of R, then the second term in $\delta\theta$ would contain a coefficient $1/R$. This makes it clear that it vanishes in the large-radius limit.

It is a beautiful exercise to compute the commutator of two of these supersymmetry transformations,

$$\begin{aligned}
[\delta_1, \delta_2]\theta &= \delta_1(\varepsilon_2 + i\theta\varepsilon_2^{\dagger}\theta) - (1 \leftrightarrow 2) \\
&= i(\varepsilon_1 + i\theta\varepsilon_1^{\dagger}\theta)\varepsilon_2^{\dagger}\theta + i\theta\varepsilon_2^{\dagger}(\varepsilon_1 + i\theta\varepsilon_1^{\dagger}\theta) - (1 \leftrightarrow 2) \\
&= \omega_{12}\theta - \theta\tilde{\omega}_{12},
\end{aligned} \tag{3.13}$$

where ω_{12} and $\tilde{\omega}_{12}$ are

$$(\omega_{12})^{\alpha}{}_{\beta} = i(\varepsilon_1\varepsilon_2^{\dagger} - \varepsilon_2\varepsilon_1^{\dagger})^{\alpha}{}_{\beta} - \text{trace}, \tag{3.14}$$

$$(\tilde{\omega}_{12})^{\mu}{}_{\nu} = i(\varepsilon_1^{\dagger}\varepsilon_2 - \varepsilon_2^{\dagger}\varepsilon_1)^{\mu}{}_{\nu} - \text{trace}. \tag{3.15}$$

These are antihermitian, as required, since $(\varepsilon_1\varepsilon_2^{\dagger})^{\dagger} = -\varepsilon_2\varepsilon_1^{\dagger}$. Traces are subtracted in order that they are Lie-algebra valued. This is possible due to the fact that the two trace terms give canceling contributions to eq. (3.13). This commutator is exactly what the superalgebra requires it to be, demonstrating that $\mathfrak{psu}(2,2)$ is nonlinearly realized entirely in terms of the Grassmann coordinates.

The transformation rule in eq. (3.10) is not a unique choice. The nonuniqueness corresponds to the possibility of redefining θ by introducing $\theta' = \theta + ic_1\theta\theta^{\dagger}\theta + \ldots$. Then, the transformation rule would be modified accordingly. One could even incorporate y and z in a redefinition, which would be truly perverse. The choice that we have made is clearly the simplest and most natural one, so it will be used in the remainder of this work.

It is possible to construct elements of the supergroup, represented by unitary supermatrices, which are constructed entirely out of the Grassmann coordinates. For this purpose,

[4] This formula has appeared previously in [18, 19].

let us consider the supermatrix[5]

$$\Gamma = \begin{pmatrix} I & \tau\theta \\ \tau\theta^\dagger & I \end{pmatrix} \begin{pmatrix} f^{-1} & 0 \\ 0 & \tilde{f}^{-1} \end{pmatrix} = \begin{pmatrix} f^{-1} & 0 \\ 0 & \tilde{f}^{-1} \end{pmatrix} \begin{pmatrix} I & \tau\theta \\ \tau\theta^\dagger & I \end{pmatrix}. \tag{3.16}$$

In this formula f denotes a real analytic function of $u = i\theta\theta^\dagger$ and $\tilde{f}$ denotes the same function with argument $\tilde{u} = i\theta^\dagger\theta$. These functions are actually polynomials of degree 16 or less, since higher powers necessarily vanish. The equality of the two ways of writing Γ is a consequence of the identities

$$f\theta = \theta\tilde{f} \quad \text{and} \quad \theta^\dagger f = \tilde{f}\theta^\dagger. \tag{3.17}$$

The choice of the function f is determined by requiring that Γ is superunitary, i.e., $\Gamma^\dagger\Gamma = I$, using the definition of the superadjoint given in eq. (3.5).

$$\Gamma^\dagger = \begin{pmatrix} I & -\tau\theta \\ -\tau\theta^\dagger & I \end{pmatrix} \begin{pmatrix} f^{-1} & 0 \\ 0 & \tilde{f}^{-1} \end{pmatrix} = \begin{pmatrix} f^{-1} & 0 \\ 0 & \tilde{f}^{-1} \end{pmatrix} \begin{pmatrix} I & -\tau\theta \\ -\tau\theta^\dagger & I \end{pmatrix}. \tag{3.18}$$

Since

$$\begin{pmatrix} I & \tau\theta \\ \tau\theta^\dagger & I \end{pmatrix} \begin{pmatrix} I & -\tau\theta \\ -\tau\theta^\dagger & I \end{pmatrix} = \begin{pmatrix} I+u & 0 \\ 0 & I+\tilde{u} \end{pmatrix}, \tag{3.19}$$

it is clear that the correct choices for f and $\tilde{f}$ to satisfy $\Gamma^\dagger\Gamma = I$ are the hermitian matrices

$$f = \sqrt{I+u} = I + \frac{1}{2}u + \ldots \quad \text{and} \quad \tilde{f} = \sqrt{I+\tilde{u}} = I + \frac{1}{2}\tilde{u} + \ldots. \tag{3.20}$$

For this choice Γ can be regarded to be an element of the supergroup.

3.3 Grassmann-valued connections

Various one-forms that can be regarded as connections associated to the superalgebra will arise in the course of this work. Here we utilize the nonlinear realization that we just found to construct ones that only involve the Grassmann coordinates. The y and z coordinates will need to be incorporated later, and then new connections will be defined.

Consider the super-Lie-algebra-valued one-form

$$A = \Gamma^{-1}d\Gamma = \begin{pmatrix} K & \tau\Psi \\ \tau\Psi^\dagger & \tilde{K} \end{pmatrix}. \tag{3.21}$$

This supermatrix is super-antihermitian, as required. (As usual, the requisite η matrices to take account of the indefinite signature of $\mathfrak{su}(2,2)$ are implicit.) Explicit calculation gives

$$K = -df f^{-1} + i\theta\Psi^\dagger = f^{-1}df - i\Psi\theta^\dagger, \tag{3.22}$$

and

$$\tilde{K} = -d\tilde{f}\tilde{f}^{-1} + i\theta^\dagger\Psi = \tilde{f}^{-1}d\tilde{f} - i\Psi^\dagger\theta, \tag{3.23}$$

[5]This description was suggested by W. Siegel, who brought his related work to our attention [22–24].

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where

$$\Psi = f^{-1} d\theta \tilde{f}^{-1} \quad \text{and} \quad \Psi^\dagger = \tilde{f}^{-1} d\theta^\dagger f^{-1}. \tag{3.24}$$

We prefer to not subtract the trace parts of K and $\tilde{K}$, which would be required to make them elements of $\mathfrak{su}(4)$ and $\mathfrak{su}(2,2)$, respectively. Since we define $\mathfrak{psu}(2,2|4)$ as a quotient space of $\mathfrak{su}(2,2|4)$, it is sufficient for our purposes that $\mathrm{tr}K = \mathrm{tr}\tilde{K}$, which implies that $\mathrm{str}A = 0$. This ensures that the traces could be removed, as in the case of $\omega_{12}\theta - \theta\tilde{\omega}_{12}$, which was discussed earlier.

The fact that A is "pure gauge" implies that it is a flat connection, i.e.,

$$dA + A \wedge A = 0. \tag{3.25}$$

In terms of 4×4 blocks the zero-curvature equations are

$$dK + K \wedge K - i\Psi \wedge \Psi^\dagger = 0, \qquad d\tilde{K} + \tilde{K} \wedge \tilde{K} - i\Psi^\dagger \wedge \Psi = 0, \tag{3.26}$$

$$d\Psi + K \wedge \Psi + \Psi \wedge \tilde{K} = 0, \qquad d\Psi^\dagger + \tilde{K} \wedge \Psi^\dagger + \Psi^\dagger \wedge K = 0. \tag{3.27}$$

These equations have the same structure as the Maurer-Cartan equations of the superalgebra.

Since we know how θ transforms under an infinitesimal supersymmetry transformation, we can compute how A transforms. The result is

$$\delta_\varepsilon A = -d \begin{pmatrix} M & 0 \\ 0 & \tilde{M} \end{pmatrix} - \left[A, \begin{pmatrix} M & 0 \\ 0 & \tilde{M} \end{pmatrix} \right], \tag{3.28}$$

where

$$M = (\delta_\varepsilon f - i f \varepsilon \theta^\dagger) f^{-1} = -f^{-1}(\delta_\varepsilon f - i\theta\varepsilon^\dagger f) \tag{3.29}$$

and

$$\tilde{M} = (\delta_\varepsilon \tilde{f} - i\tilde{f}\varepsilon^\dagger\theta)\tilde{f}^{-1} = -\tilde{f}^{-1}(\delta_\varepsilon \tilde{f} - i\theta^\dagger\varepsilon\tilde{f}). \tag{3.30}$$

As in the case of K, these expressions are antihermitian. Also, $\mathrm{tr}M = \mathrm{tr}\tilde{M}$, so that the supertrace vanishes. It is worth noting that a global supersymmetry transformation is implemented by means of specific (ε-dependent) local $\mathfrak{su}(4)$ and $\mathfrak{su}(2,2)$ transformations. This supports interpreting the nonlinear realization of $\mathrm{PSU}(2,2|4)$ in terms of θ as a coset construction[6]

$$\mathrm{PSU}(2,2|4)/\mathrm{SU}(4) \times \mathrm{SU}(2,2). \tag{3.31}$$

In terms of components

$$\delta_\varepsilon K = -(dM + [K, M]) \quad \text{and} \quad \delta_\varepsilon \tilde{K} = -(d\tilde{M} + [\tilde{K}, \tilde{M}]), \tag{3.32}$$

$$\delta_\varepsilon \Psi = M\Psi - \Psi\tilde{M} \quad \text{and} \quad \delta_\varepsilon \Psi^\dagger = \tilde{M}\Psi^\dagger - \Psi^\dagger M. \tag{3.33}$$

The coset-space interpretation of Γ can be explored further by considering $\delta_\varepsilon\Gamma$. A straightforward calculation gives

$$\delta_\varepsilon \Gamma = \begin{pmatrix} M & 0 \\ 0 & \tilde{M} \end{pmatrix} \Gamma + \Gamma \begin{pmatrix} 0 & \tau\varepsilon \\ \tau\varepsilon^\dagger & 0 \end{pmatrix}. \tag{3.34}$$

[6]This was pointed out by E. Witten.

This shows that under a supersymmetry transformation Γ is multiplied on the left by a local $\mathfrak{su}(4)\times\mathfrak{su}(2,2)$ transformation and on the right by a global supersymmetry transformation, just as one would expect in a coset construction.

3.4 Inclusion of bosonic coordinates

The formulas that have been described in this section so far describe the supermanifold geometry for fixed values of the bosonic coordinates y and z, which we sometimes refer to collectively as x. Our goal now is to describe the generalization that also allows the bosonic coordinates to vary. For this purpose, the first step is to recast y and z as 4×4 matrices denoted Y and Z. This is described in detail in appendix A. The result that is established there is that

$$Y^{\mu\nu} = y^m(\tilde{\Sigma}_m)^{\mu\nu} \quad \text{and} \quad Z^{\alpha\beta} = z^a(\Sigma_a)^{\alpha\beta} \tag{3.35}$$

are antisymmetric matrices belonging to the groups SU(2, 2) and SU(4), respectively. Thus, in our notation, $Y^T = -Y$, $Z^T = -Z$, $Y^{-1} = \eta Y^\dagger \eta$, $Z^{-1} = Z^\dagger$, and $\det Y = \det Z = 1$. These equations are consequences of the relations $y^2 = -1$ and $z^2 = 1$, as well as Clifford-algebra-like formulas for the Σ matrices. Thus, S^5 is described as a specific codimension 10 submanifold of the SU(4) group manifold, and AdS_5 is described as a specific codimension 10 submanifold of the SU(2, 2) group manifold.

The supersymmetry transformations of the bosonic coordinates are

$$\delta_\varepsilon Z = MZ + ZM^T \quad \text{and} \quad \delta_\varepsilon Y = \tilde{M}Y + Y\tilde{M}^T. \tag{3.36}$$

To prove that these are the correct formulas one should verify that the commutator of two such transformations gives the correct infinitesimal $\mathfrak{su}(4)$ and $\mathfrak{su}(2,2)$ transformations. This is achieved in the $\mathfrak{su}(4)$ case provided that

$$[M_2, M_1] + \delta_1 M_2 - \delta_2 M_1 = \omega_{12}, \tag{3.37}$$

where $M_i = M(\varepsilon_i)$ and ω_{12} is given in eq. (3.14). It is straightforward to verify that $M(\varepsilon)$, as given in eq. (3.29), satisfies this equation. The δY equation is established in the same way. Note that eq. (3.36) implies that the variations of Y and Z are θ dependent. This supports the previous claim that they should be considered to be even elements of the Grassmann algebra.

It is useful to define covariant derivatives

$$\Pi = DZ = dZ + KZ + ZK^T \quad \text{and} \quad \tilde{\Pi} = DY = dY + \tilde{K}Y + Y\tilde{K}^T, \tag{3.38}$$

which transform under supersymmetry transformations in the same way as Z and Y. This means that

$$\delta_\varepsilon \Pi = M\Pi + \Pi M^T \quad \text{and} \quad \delta_\varepsilon \tilde{\Pi} = \tilde{M}\tilde{\Pi} + \tilde{\Pi}\tilde{M}^T. \tag{3.39}$$

These equations imply that

$$\delta_\varepsilon(\Pi\,\Pi^\dagger) = [M, \Pi\,\Pi^\dagger]. \tag{3.40}$$

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and

$$\delta_\varepsilon(\tilde{\Pi}\,\tilde{\Pi}^\dagger) = [\tilde{M}, \tilde{\Pi}\,\tilde{\Pi}^\dagger]. \tag{3.41}$$

Therefore $\mathrm{tr}(\Pi\,\Pi^\dagger)$ and $\mathrm{tr}(\tilde{\Pi}\,\tilde{\Pi}^\dagger)$ are *separately* invariant under the entire supergroup! One way to deduce the correct linear combination to describe the supersymmetrized metric is by requiring that they give the correct flat-space ($R \to \infty$) limit. Alternatively, requiring the correct bosonic truncation also suffices. Later, we will discover that kappa symmetry of the superstring action also relates them. Any of these implies that the correct supersymmetrization of the metric $ds^2 = dz \cdot dz + dy \cdot dy$ is

$$ds^2 = \frac{1}{4}\left(\mathrm{tr}(\Pi\,\Pi^\dagger) - \mathrm{tr}(\tilde{\Pi}\,\tilde{\Pi}^\dagger)\right). \tag{3.42}$$

It is also useful to define the antihermitian connections

$$\Omega = Z\Pi^\dagger = -\Pi Z^{-1} = ZdZ^{-1} - K - ZK^T Z^{-1}. \tag{3.43}$$

and

$$\tilde{\Omega} = Y\tilde{\Pi}^\dagger = -\tilde{\Pi}Y^{-1} = YdY^{-1} - \tilde{K} - Y\tilde{K}^T Y^{-1}. \tag{3.44}$$

These connections satisfy

$$Z\Omega^T Z^{-1} = \Omega \quad \text{and} \quad Y\tilde{\Omega}^T Y^{-1} = \tilde{\Omega}. \tag{3.45}$$

Under supersymmetry transformations

$$\delta_\varepsilon \Omega = [M, \Omega] \quad \text{and} \quad \delta_\varepsilon \tilde{\Omega} = [\tilde{M}, \tilde{\Omega}]. \tag{3.46}$$

Since $\Omega^2 = -\Pi\,\Pi^\dagger$ and $\tilde{\Omega}^2 = -\tilde{\Pi}\,\tilde{\Pi}^\dagger$,

$$ds^2 = -\frac{1}{4}\left(\mathrm{tr}(\Omega^2) - \mathrm{tr}(\tilde{\Omega}^2)\right). \tag{3.47}$$

3.5 Majorana-Weyl matrices and Maurer-Cartan equations

In the flat spacetime limit, a fermionic matrix such as θ corresponds to a complex Weyl spinor, which (in the notation of [25]) satisfies an equation of the form $\Gamma_{11}\theta = \theta$. This spinor describes a reducible representation of the $\mathcal{N} = 2B$, $D = 10$ super-Poincaré group, and so it can be decomposed into a pair of Majorana-Weyl spinors $\theta = \theta_1 + i\theta_2$. In a Majorana representation of the Dirac algebra the MW spinors θ_1 and θ_2 each contain 16 real components. In the case of $\mathrm{PSU}(2,2|4)$ the group theory is different. The relevant representation of $\mathrm{SU}(2,2) \times \mathrm{SU}(4)$ is still reducible, $(\mathbf{4}, \bar{\mathbf{4}}) + (\bar{\mathbf{4}}, \mathbf{4})$, but it does not make group-theoretic sense to extract the real and imaginary parts by adding and subtracting these two pieces. Fortunately, there is a construction that is group theoretically sensible and connects smoothly with the flat-space limit.

The transformations given previously imply that Ψ and

$$\Psi' = Z\Psi^\star Y^{-1} \tag{3.48}$$

transform in the same way under all $\mathrm{PSU}(2,2|4)$ transformations. To understand this definition one should follow the indices. The complex conjugate is $(\Psi^\alpha{}_\mu)^\star = (\Psi^\star)^{\bar{\alpha}}{}_{\bar{\mu}}$, but as usual we convert to unbarred indices, $(\Psi^\star)_\alpha{}^\mu$, using η matrices, i.e., $\Psi^\star \to \eta\Psi^\star\eta$. Then $(\Psi')^\alpha{}_\mu = Z^{\alpha\beta}(\Psi^\star)_\beta{}^\nu(Y^{-1})_{\nu\mu}$. Therefore it makes group-theoretic sense to define

$$\Psi_1 = \frac{1}{2}(\Psi + \Psi') \quad \text{and} \quad \Psi_2 = \frac{1}{2i}(\Psi - \Psi'). \tag{3.49}$$

Then $\Psi = \Psi_1 + i\Psi_2$ and $\Psi' = \Psi_1 - i\Psi_2$. We will refer to Ψ_1 and Ψ_2 as *Majorana-Weyl matrices*. A MW matrix, such as Ψ_1, satisfies the "reality" identities

$$Z^{-1}\Psi_1 Y = \Psi_1^\star \quad \text{and} \quad Z\Psi_1^\star Y^{-1} = \Psi_1. \tag{3.50}$$

What we have here is a generalization of complex conjugation given by

$$\rho \to \mu(\rho) = \rho' = Z\rho^\star Y^{-1}, \tag{3.51}$$

where $\rho^\alpha{}_\mu$ is an arbitrary fermionic matrix (not necessarily a one-form) that transforms under $\mathrm{SU}(2,2) \times \mathrm{SU}(4)$ transformations like Ψ or θ. Using the antisymmetry and unitarity of Y and Z it is easy to verify that μ is an involution, like complex conjugation, i.e., $\mu \circ \mu = I$, where I is the identity operator. Therefore,

$$\mu_\pm = \frac{1}{2}(I \pm \mu) \tag{3.52}$$

are a pair of orthogonal projection operators that separate ρ into two pieces, $\rho = \rho_1 + i\rho_2$. In the flat-space limit ρ_1 and ρ_2 correspond to conventional MW spinors.

Let us now define

$$A_1 = \begin{pmatrix} \Omega & 0 \\ 0 & \tilde{\Omega} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & \tau\Psi \\ \tau\Psi^\dagger & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & \tau\Psi' \\ \tau\Psi'^\dagger & 0 \end{pmatrix}, \tag{3.53}$$

and

$$X = \begin{pmatrix} Z & 0 \\ 0 & Y \end{pmatrix}, \tag{3.54}$$

These matrices have the important properties

$$XA_1^T X^{-1} = A_1, \quad XA_2^T X^{-1} = A_3, \quad XA_3^T X^{-1} = A_2. \tag{3.55}$$

Each of the A_i supermatrices transforms in the same way under a supersymmetry transformation:

$$\delta_\varepsilon A_i = \left[\begin{pmatrix} M & 0 \\ 0 & \tilde{M} \end{pmatrix}, A_i \right]. \tag{3.56}$$

We also define

$$J_i = \Gamma^{-1} A_i \Gamma. \tag{3.57}$$

Utilizing eq. (3.34), one can show that the transformation of these supermatrices under arbitrary infinitesimal $\mathfrak{psu}(2,2|4)$ transformations is given by

$$\delta_\Lambda J_i = [\Lambda, J_i], \tag{3.58}$$

where the infinitesimal parameters are given by

$$\Lambda = \begin{pmatrix} \omega & -\tau\varepsilon \\ -\tau\varepsilon^\dagger & \tilde{\omega} \end{pmatrix}. \tag{3.59}$$

These transformation rules are those that are required for the $\mathfrak{psu}(2,2|4)$ Noether currents of the superstring. Thus, one can reasonably expect that the Noether currents and the equations of motion are given by linear combinations of the three supermatrix one-forms J_i and their Hodge duals, which transform in the same manner.

It is straightforward to compute Maurer-Cartan (MC) equations for the J_i currents. They are

$$dJ_1 = -J_1 \wedge J_1 + J_2 \wedge J_2 + J_3 \wedge J_3 - J_1 \wedge J_2 - J_2 \wedge J_1, \tag{3.60}$$

$$dJ_2 = -2J_2 \wedge J_2, \tag{3.61}$$

$$dJ_3 = -(J_1 + J_2) \wedge J_3 - J_3 \wedge (J_1 + J_2). \tag{3.62}$$

The first and third equations imply that the two supercurrents

$$J_\pm = J_1 + J_2 \pm iJ_3 \tag{3.63}$$

are flat, while the second equation implies that $2J_2$ is flat.

3.6 Wess-Zumino terms

Let us consider the construction of differential forms that are closed and invariant under the entire supergroup. Three-forms of this type are required to construct the Wess-Zumino terms for strings. Type IIB superstring theory has an infinite $\mathrm{SL}(2,\mathbb{Z})$ multiplet of (p,q) strings, but we are primarily interested in the fundamental $(1,0)$ superstring here. The D3-brane world-volume action contains a Wess-Zumino term determined by a self-dual five-form.

The supertrace of an n-fold wedge product of a current J, $T_n = \mathrm{str}(J \wedge J \ldots \wedge J)$, vanishes for n even. The reason for this is that the cyclic identity of the supertrace, $\mathrm{str}(AB) = \mathrm{str}(BA)$, acquires an additional minus sign if A is a one-form and B is a differential form of odd degree. Next suppose that n is odd, so that T_n can be nonzero, and that J is a flat connection. In this case the exterior derivative dT_n is proportional to T_{n+1}, which is equal to zero. Therefore T_n is closed.

Let us now utilize this logic to construct a closed three-form based on the flat connections that we have found. The simple choice $\mathrm{str}(J_2 \wedge J_2 \wedge J_2)$ is closed, but it is also zero, since the product of the three factors has vanishing blocks on the diagonal. Therefore, let us consider instead

$$T_3 = \mathrm{str}(J_+ \wedge J_+ \wedge J_+). \tag{3.64}$$

This is complex, and therefore it encodes two real three-forms that are invariant and closed. The same two three-forms would be obtained if we used J_- instead of J_+. Now let us substitute $J_1 + J_2 + iJ_3$ for J_+. Doing this, and only keeping those terms that give diagonal blocks that could contribute to the supertrace, leaves

$$T_3 = \mathrm{str}(J_1 \wedge J_1 \wedge J_1) + 3\,\mathrm{str}(J_1 \wedge (J_2 + iJ_3) \wedge (J_2 + iJ_3)). \tag{3.65}$$

However, $\mathrm{str}(J_1 \wedge J_1 \wedge J_1)$ vanishes as a consequence of eq. (3.45). Therefore,

$$T_3 = 3(T_D + iT_F), \tag{3.66}$$

where T_F and T_D are two real three-forms

$$T_F = \mathrm{str}(J_1 \wedge (J_2 \wedge J_3 + J_3 \wedge J_2)). \tag{3.67}$$

and

$$T_D = \mathrm{str}(J_1 \wedge (J_2 \wedge J_2 - J_3 \wedge J_3)). \tag{3.68}$$

The notation is meant to indicate that T_F enters in the construction of the Wess-Zumino term of the fundamental string and T_D enters in the corresponding construction of the D-string. The fundamental string is the one of greatest interest from the point of view of AdS/CFT duality in the large-N limit. The closed three-form T_F is exact

$$T_F = d\,\mathrm{str}(J_2 \wedge J_3). \tag{3.69}$$

There is no comparably simple expression for T_D.

The derivation of T_F presented here could be turned around. Consider all invariant two-forms of the type $T_{ij} = \mathrm{str}(J_i \wedge J_j)$. Since J_i is a one-form, cyclic permutation gives $T_{ij} = -T_{ji}$. Furthermore, $T_{13} = T_{23} = 0$, because the expressions inside the supertrace contain no diagonal blocks. Thus, up to normalization, the only nonzero two-form of this type is T_{23}, which is the one required to construct the Wess-Zumino term for the fundamental superstring.

A self-dual five-form plays an important role in type IIB superstring theory. For the case of the $\mathrm{AdS}_5 \times S^5$ background it has a nonzero bosonic truncation unlike the three-forms described above. The bosonic part is proportional to the sum (or difference) of the volume form of AdS_5 and the volume form of S^5. The supersymmetric completion of this five-form is derived in appendix B. It determines the Wess-Zumino term for the D3-brane world-volume action in the $\mathrm{AdS}_5 \times S^5$ background.

4 The superstring world-sheet action

The world-volume actions of supersymmetric probe branes, including the fundamental superstring, are written as a sum of two terms. The first term, which we denote S_1, is of the Nambu-Goto/Volkov-Akulov/Dirac-Born-Infeld type.[7] The second term, which we denote S_2, is of the Wess-Zumino/Chern-Simons type. Each of these terms is required to have local reparametrization invariance. In the case of S_1 this requires a world-sheet metric, whereas the S_2 term is independent of the world-sheet metric. Also, the target superspace isometry, which in the present case is $\mathrm{PSU}(2,2|4)$, is realized as a global symmetry of S_1 and S_2 separately. Furthermore, there should be a local fermionic symmetry, called

[7] Nambu-Goto refers to a universal feature of the brane embedding. Born-Infeld refers to the inclusion of a world-volume U(1) gauge field, which is only present in the case of D-branes. Volkov-Akulov refers to the appearance of goldstino fields, which only appear in supersymmetric theories.

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kappa symmetry. Kappa symmetry implies that half of the Grassmann coordinates θ are gauge degrees of freedom that can be eliminated by a suitable gauge choice. Unlike all other symmetries, kappa symmetry is not a symmetry of S_1 and S_2 separately. Rather, it requires a conspiracy between them. Given S_1, a specific S_2, unique up to sign, is required. In the case of flat ten-dimensional spacetime, the action $S = S_1 + S_2$ turns out to be a free theory, a fact that can be made manifest in light-cone gauge. In this gauge the exact spectrum of the string is easily determined. In the case of an $\mathrm{AdS}_5 \times S^5$ background, the superstring world-sheet theory is not a free theory, but it is an integrable theory [26].

The construction of S_1 works exactly as explained for the bosonic truncation in section 2. The only change is that now $G_{\alpha\beta}$ is determined by the supersymmetrized target-space metric in eq. (3.47). It is

$$G_{\alpha\beta} = -\frac{1}{4}\left(\mathrm{tr}(\Omega_\alpha\Omega_\beta) - \mathrm{tr}(\tilde{\Omega}_\alpha\tilde{\Omega}_\beta)\right) = -\frac{1}{4}\mathrm{str}(A_{1\alpha}A_{1\beta}) = -\frac{1}{4}\mathrm{str}(J_{1\alpha}J_{1\beta}). \tag{4.1}$$

As explained in section 2,

$$S_1 = -\frac{\sqrt{\lambda}}{4\pi}\int d^2\sigma\sqrt{-h}h^{\alpha\beta}G_{\alpha\beta}. \tag{4.2}$$

The Noether procedure for constructing the conserved current associated with a global symmetry instructs us to consider a local infinitesimal transformation, which is not a symmetry. The variation of the action is then given by the derivative of the infinitesimal parameter times the conserved current. Since the superstring action consists of two terms, S_1 and S_2, which separately possess all of the global symmetries, each of the currents is the sum of two contributions, one from S_1 and one from S_2.

Let us begin by considering a local $\mathfrak{su}(4)$ transformation of S_1. Since

$$\Omega = ZdZ^{-1} - K - ZK^TZ^{-1}, \tag{4.3}$$

we begin by considering the variation of $K = f^{-1}df - i\Psi\theta^\dagger$ using $\delta_\omega Z = \omega Z + Z\omega^T$ and $\delta_\omega\theta = \omega\theta$. We only keep terms involving $d\omega$, since we know that all nonderivative dependence on ω will cancel. This is the meaning of δ' in the following. This gives

$$\delta'_\omega K = \zeta - d\omega, \tag{4.4}$$

where

$$\zeta = f^{-1}d\omega f^{-1}. \tag{4.5}$$

From this it then follows that

$$\delta'_\omega\Omega = -\zeta - Z\zeta^TZ^{-1}. \tag{4.6}$$

The one-form $\tilde{\Omega}$, which takes values in the $\mathfrak{su}(2,2)$ Lie algebra, also has a non-zero variation under a local $\mathfrak{su}(4)$ transformation, namely

$$\delta'_\omega\tilde{\Omega} = -i\theta^\dagger\zeta\theta - iY(\theta^\dagger\zeta\theta)^TY^{-1}. \tag{4.7}$$

Putting these together gives

$$\delta'_\omega \left(\mathrm{tr}(\Omega^2) - \mathrm{tr}(\tilde{\Omega}^2)\right) = -4\,\mathrm{tr}\left(\zeta[\Omega + i\theta\tilde{\Omega}\theta^\dagger]\right). \tag{4.8}$$

Thus, the S_1 contribution to the $\mathfrak{su}(4)$ Noether current is $f^{-1}(\Omega + i\theta\tilde{\Omega}\theta^\dagger)f^{-1}$. The normalization has been chosen so that the bosonic truncation is $\Omega_0 = ZdZ^{-1}$, which is a flat connection ($d\Omega_0 + \Omega_0 \wedge \Omega_0 = 0$). This $\mathfrak{su}(4)$ current is given by the upper left block of the supercurrent $J_1 = \Gamma^{-1}A_1\Gamma$ introduced in the previous section. Therefore, J_1 is the S_1 contribution to the $\mathfrak{psu}(2,2|4)$ Noether current.

The second term in the superstring world-sheet action, denoted S_2, should also be invariant under the entire $PSU(2,2|4)$ supergroup. Furthermore, there should exist an exact invariant three-form $H_3 = dB_2$, such that $S_2 = \int B_2$. Finally, the normalization of H_3 (or B_2) should be chosen such that $S = S_1 + S_2$ has a local fermionic symmetry, called kappa symmetry, which implies that half of the θ coordinates are gauge degrees of freedom of the world-sheet theory. The analysis presented in section 4 suggests that

$$S_2 = k\int \mathrm{str}(J_2 \wedge J_3), \tag{4.9}$$

where k is a normalization constant to be determined.

The S_2 contribution to the Noether currents is proportional to the dual of the supercurrent J_3. Since it is derived from a differential form, the expression that naturally arises is the dual of the desired current. Thus, anticipating the coefficient of J_3, the total Noether current is

$$J = J_1 + \star J_3, \tag{4.10}$$

where the Hodge dual is constructed using the metric $h_{\alpha\beta}$. The current conservation equation is then

$$d\star J = d\star J_1 + dJ_3 = 0. \tag{4.11}$$

This encodes some of the equations of motion. To derive the others, we need to understand kappa symmetry.

5 Theta variations and kappa symmetry

5.1 Additional equations of motion

There are additional equations of motion, beyond those given by conservation of the Noether currents. To derive them let us consider arbitrary variations of the Grassmann coordinates $\delta\theta$ (and $\delta\theta^\dagger$). This already determines the variation of A, in analogy with eq. (3.28), to be

$$\delta A = -d\begin{pmatrix} \mathcal{M} & \tau\rho \\ \tau\rho^\dagger & \tilde{\mathcal{M}} \end{pmatrix} - \left[A, \begin{pmatrix} \mathcal{M} & \tau\rho \\ \tau\rho^\dagger & \tilde{\mathcal{M}} \end{pmatrix}\right], \tag{5.1}$$

where

$$\mathcal{M} = -f^{-1}\delta f + i\rho\theta^\dagger = \delta f f^{-1} - i\theta\rho^\dagger \tag{5.2}$$

$$\tilde{\mathcal{M}} = -\tilde{f}^{-1}\delta\tilde{f} + i\rho^\dagger\theta = \delta\tilde{f}\tilde{f}^{-1} - i\theta^\dagger\rho, \tag{5.3}$$

and

$$\rho = f^{-1}\delta\theta\tilde{f}^{-1}. \tag{5.4}$$

In terms of blocks, eq. (5.1) implies that

$$\delta K = -d\mathcal{M} - [K,\mathcal{M}] + \Delta, \quad \text{and} \quad \delta\tilde{K} = -d\tilde{\mathcal{M}} - [\tilde{K},\tilde{\mathcal{M}}] + \tilde{\Delta}, \tag{5.5}$$

$$\delta\Psi = D\rho + \mathcal{M}\Psi - \Psi\tilde{\mathcal{M}}, \tag{5.6}$$

where

$$D\rho = d\rho + \rho K - \tilde{K}\rho, \tag{5.7}$$

$$\Delta = i(\rho\Psi^\dagger - \Psi\rho^\dagger), \quad \text{and} \quad \tilde{\Delta} = i(\rho^\dagger\Psi - \Psi^\dagger\rho). \tag{5.8}$$

Next let us decree that Y and Z are simultaneously varied, as in the case of the ε transformation formula in eq. (3.36), according to the rule

$$\delta Z = \mathcal{M}Z + Z\mathcal{M}^T \quad \text{and} \quad \delta Y = \tilde{\mathcal{M}}Y + Y\tilde{\mathcal{M}}^T. \tag{5.9}$$

The kappa variation of Ω is more complicated than in the ε case, because $\delta\theta$ is no longer expressed in terms of a constant parameter. This causes the formula to contain additional terms involving Δ

$$\delta\Omega = [\mathcal{M},\Omega] - \Delta - Z\Delta^T Z^{-1} \quad \text{and} \quad \delta\tilde{\Omega} = [\tilde{\mathcal{M}},\tilde{\Omega}] - \tilde{\Delta} - Y\tilde{\Delta}^T Y^{-1}. \tag{5.10}$$

The computation of δS_1 requires the variation of $\mathrm{tr}(\Omega^2) - \mathrm{tr}(\tilde{\Omega}^2)$. Using the formulas given above,

$$\delta\mathrm{tr}(\Omega^2) = -2\mathrm{tr}(\Omega(\Delta + Z\Delta^T Z^{-1})) = \delta\mathrm{tr}(\Omega^2) = -4\mathrm{tr}(\Omega\Delta). \tag{5.11}$$

The variation of S_1 now becomes

$$\delta S_1 = -\frac{\sqrt{\lambda}}{4\pi}\int d^2\sigma\sqrt{-h}h^{\alpha\beta}\delta G_{\alpha\beta}, \tag{5.12}$$

where $\delta G_{\alpha\beta}$ is determined by $\delta(ds^2) = \delta G_{\alpha\beta}d\sigma^\alpha d\sigma^\beta$, which is

$$\delta(ds^2) = \mathrm{tr}(\Omega\Delta) - \mathrm{tr}(\tilde{\Omega}\tilde{\Delta}) = i\,\mathrm{tr}\left([\rho\Psi^\dagger - \Psi\rho^\dagger]\Omega\right) - i\,\mathrm{tr}\left([\rho^\dagger\Psi - \Psi^\dagger\rho]\tilde{\Omega}\right). \tag{5.13}$$

Reexpressed in terms of supermatrices and differential forms

$$\delta S_1 = -\frac{\sqrt{\lambda}}{4\pi}\int \mathrm{str}(R[A_1\wedge\star A_2 + \star A_2\wedge A_1]), \tag{5.14}$$

where

$$R = \begin{pmatrix} 0 & \tau\rho \\ \tau\rho^\dagger & 0 \end{pmatrix}. \tag{5.15}$$

The next step is to compute

$$\delta S_2 = k\int \delta\,\mathrm{str}(J_2\wedge J_3). \tag{5.16}$$

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Using $J_i = \Gamma^{-1} A_i \Gamma$ and $A_3 = X A_2^T X^{-1}$,

$$\mathrm{str}(J_2 \wedge J_3) = \mathrm{str}(A_2 \wedge X A_2^T X^{-1}). \tag{5.17}$$

Defining

$$\mathcal{N} = \begin{pmatrix} \mathcal{M} & 0 \\ 0 & \tilde{\mathcal{M}} \end{pmatrix}, \tag{5.18}$$

the required variations are

$$\delta A_2 = [\mathcal{N}, A_2] + DR \tag{5.19}$$

and

$$\delta X = \mathcal{N} X + X \mathcal{N}^T. \tag{5.20}$$

Putting these facts and definitions together,

$$\delta \mathrm{str}(J_2 \wedge J_3) = 2\, \mathrm{str}(DR \wedge A_3). \tag{5.21}$$

The variation δS_1 does not involve derivatives of ρ, but the expression we have just found does contain them. Thus, if these two terms are to combine nicely, an integration by parts is required. The appropriate formula is

$$\mathrm{str}(DR \wedge A_3) = d\, \mathrm{str}(R \wedge A_3) - \mathrm{str}(R D A_3). \tag{5.22}$$

Using the identity

$$DA_3 = A_3 \wedge A_1 + A_1 \wedge A_3, \tag{5.23}$$

$$\delta\, \mathrm{str}(J_2 \wedge J_3) = -2\, \mathrm{str}(R[A_3 \wedge A_1 + A_1 \wedge A_3]) + 2d\, \mathrm{str}(R \wedge A_3). \tag{5.24}$$

Adjusting the normalization of the Wess-Zumino term by setting

$$k = \frac{\sqrt{\lambda}}{8\pi} \tag{5.25}$$

gives the variation

$$\delta S_2 = -\frac{\sqrt{\lambda}}{4\pi} \int \mathrm{str}(R[A_3 \wedge A_1 + A_1 \wedge A_3]). \tag{5.26}$$

Combining eqs. (5.14) and (5.26),

$$\delta S = -\frac{\sqrt{\lambda}}{4\pi} \int \mathrm{str}(R[(\star A_2 + A_3) \wedge A_1 + A_1 \wedge (\star A_2 + A_3)]). \tag{5.27}$$

This implies that

$$(\star A_2 + A_3) \wedge A_1 + A_1 \wedge (\star A_2 + A_3) = 0 \tag{5.28}$$

is an equation of motion. Equivalently,

$$(\star J_2 + J_3) \wedge J_1 + J_1 \wedge (\star J_2 + J_3) = 0 \tag{5.29}$$

or

$$\star J_1 \wedge J_2 + J_2 \wedge \star J_1 = J_1 \wedge J_3 + J_3 \wedge J_1. \tag{5.30}$$

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By taking the transpose of this equation and conjugating by X one deduces that

$$(\star J_3 + J_2) \wedge J_1 + J_1 \wedge (\star J_3 + J_2) = 0 \tag{5.31}$$

or

$$\star J_1 \wedge J_3 + J_3 \wedge \star J_1 = J_1 \wedge J_2 + J_2 \wedge J_1. \tag{5.32}$$

These equations, together with the conservation of the Noether current ($d(\star J_1 + J_3) = 0$) and the MC equations (3.60)–(3.62), are the ingredients required for the proof of integrability given in [26]. Specifically, in terms of a spectral parameter x,

$$J = c_1 J_1 + c_1' \star J_1 + c_2 J_2 + c_3 J_3 \tag{5.33}$$

is flat (i.e., $dJ + J \wedge J = 0$) for

$$c_1 = -\sinh^2 x, \quad c_1' = \pm \sinh x \cosh x, \quad c_2 = 1 \mp \cosh x, \quad c_3 = \sinh x. \tag{5.34}$$

The integrability of this theory was explored further in [27].

5.2 Kappa symmetry

Let us rewrite eq. (5.13) in terms of the MW matrices defined in section 3.5. Substituting $\rho = \rho_1 + i\rho_2$ and $\Psi = \Psi_1 + i\Psi_2$, where ρ_I and Ψ_I are MW matrices, and using the identity

$$\mathrm{tr}(\rho_I \Psi_J^\dagger \Omega) = \mathrm{tr}(\rho_I^\star \Psi_J^T \Omega^T) = -\mathrm{tr}(\Psi_J \rho_I^\dagger \Omega), \tag{5.35}$$

Eq. (5.13) takes the form

$$\delta(ds^2) = -2i \sum_{I=1}^{2} \mathrm{tr}(\Psi_I^\dagger [\Omega \rho_I - \rho_I \tilde{\Omega}]). \tag{5.36}$$

This result can be recast in a form that will prove convenient later, when we wish to combine δS_1 with δS_2. The key new ingredient is an identity derived in appendix C, which for MW matrices ρ_I is

$$\star (\Omega \rho_I - \rho_I \tilde{\Omega}) = \Omega \gamma(\rho_I) - \gamma(\rho_I) \tilde{\Omega}, \tag{5.37}$$

where the involution map $\gamma(\rho)$ is

$$\gamma(\rho) = \frac{\varepsilon^{\alpha\beta}}{2\sqrt{-G}} \left(\Pi_\alpha \Pi_\beta^\dagger \rho - 2\Pi_\alpha \rho^\star \tilde{\Pi}_\beta^\dagger + \rho \tilde{\Pi}_\alpha \tilde{\Pi}_\beta^\dagger \right). \tag{5.38}$$

Using this identity, the variation of S_1 can be recast in the form

$$\delta S_1 = i \frac{\sqrt{\lambda}}{2\pi} \int \sum_{I=1}^{2} \mathrm{tr}(\Psi_I^\dagger \wedge [\Omega \gamma(\rho_I) - \gamma(\rho_I) \tilde{\Omega}]). \tag{5.39}$$

Evaluating the supertrace, eq. (5.24) can be brought to the form

$$\delta\, \mathrm{str}(J_2 \wedge J_3) = -2i\, \mathrm{tr}(\Psi^\dagger \wedge [\Omega \rho' - \rho' \tilde{\Omega}]) + 2i\, \mathrm{tr}(\Psi'^\dagger \wedge [\Omega \rho - \rho \tilde{\Omega}]). \tag{5.40}$$

Recasting this variation in terms of MW matrices $\rho_1, \rho_2, \Psi_1, \Psi_2$, as in the analysis of $\delta(ds^2)$, yields

$$\delta\, \mathrm{str}(J_2 \wedge J_3) = 4i\, \mathrm{tr}\left(\Psi_1^\dagger \wedge [\Omega\rho_1 - \rho_1\tilde{\Omega}]\right) - 4i\, \mathrm{tr}\left(\Psi_2^\dagger \wedge [\Omega\rho_2 - \rho_2\tilde{\Omega}]\right), \tag{5.41}$$

and thus

$$\delta S_2 = i\frac{\sqrt{\lambda}}{2\pi}\int \mathrm{tr}\left(\Psi_1^\dagger \wedge [\Omega\rho_1 - \rho_1\tilde{\Omega}]\right) - \mathrm{tr}\left(\Psi_2^\dagger \wedge [\Omega\rho_2 - \rho_2\tilde{\Omega}]\right). \tag{5.42}$$

Combining eqs. (5.39) and (5.42) gives the variation of $S = S_1 + S_2$

$$\delta\, S = i\frac{\sqrt{\lambda}}{\pi}\int \mathrm{tr}\left(\Psi_1^\dagger \wedge [\Omega\gamma_+(\rho_1) - \gamma_+(\rho_1)\tilde{\Omega}]\right) - \mathrm{tr}\left(\Psi_2^\dagger \wedge [\Omega\gamma_-(\rho_2) - \gamma_-(\rho_2)\tilde{\Omega}]\right), \tag{5.43}$$

where we have introduced projection operators $\gamma_\pm = \frac{1}{2}(I \pm \gamma)$, so that

$$\gamma_\pm(\rho) = \frac{1}{2}(\rho \pm \gamma(\rho)). \tag{5.44}$$

Since $\gamma_+ \circ \gamma_- = \gamma_- \circ \gamma_+ = 0$, S is invariant for the choices

$$\rho_1 = \gamma_-(\kappa) \quad \text{and} \quad \rho_2 = \gamma_+(\kappa), \tag{5.45}$$

where κ is an arbitrary (local) MW matrix. Since θ describes 32 real fermionic coordinates, this means that half of them are gauge degrees of freedom, which can be eliminated by a gauge choice. Recalling that $\rho = \rho_1 + i\rho_2 = f^{-1}\delta\theta\tilde{f}^{-1}$, we see that under a kappa symmetry transformation

$$\delta_\kappa\theta = f\big(\gamma_-(\kappa) + i\gamma_+(\kappa)\big)\tilde{f}. \tag{5.46}$$

The bosonic coordinates Y and Z are varied at the same time in the way described in section 5.1. The superspace (x, θ) has $10 + 32$ dimensions. However, the local reparametrization and kappa symmetries imply that only $8 + 16$ of them induce independent dynamical degrees of freedom of the superstring.

In the case of the flat-space theory, there is a suitable gauge choice for the kappa symmetry, which in conjunction light-cone gauge turns the world-sheet theory into a free theory [28]. This is certainly not the case for the AdS $\times S^5$ background, though substantial simplification can be achieved. This has been discussed extensively beginning with [8, 29–31]. This important issue will not be pursued here.

6 Conclusion

The problem of describing the superspace geometry of the $\mathrm{AdS}_5 \times S^5$ solution of type IIB superstring theory and the dynamics of a fundamental superstring in this geometry has been reexamined from a somewhat new perspective. We began by presenting a nonlinear realization of the superspace isometry supergroup PSU(2, 2|4) in terms of Grassmann coordinates only. The resulting formulas were interpreted as corresponding to a PSU(2, 2|4)/SU(2, 2) × SU(4) coset construction. Following that we added the matrices

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$Z = \Sigma \cdot \hat{z}$ and $Y = \tilde{\Sigma} \cdot \hat{y}$ to describe the S^5 and AdS_5 coordinates, respectively. These matrices were interpreted as describing embeddings of S^5 inside SU(4) and AdS_5 inside SU(2, 2).

Next we constructed supermatrix one-forms J_1, J_2 and J_3 that transform linearly under infinitesimal global $\mathfrak{psu}(2,2|4)$ transformations, $\delta J_i = [\Lambda, J_i]$. In terms of these currents the superstring world-sheet action was shown to be proportional to $\int [\mathrm{str}(J_1 \wedge \star J_1) + \frac{1}{2}\mathrm{str}(J_2 \wedge J_3)]$. This is invariant under local kappa symmetry transformations, which were shown to arise from an interplay of three involutions. The conserved Noether current is $J_1 + \star J_3$. A one-parameter family of flat connections, required for the proof of integrability, was obtained. All of these results are in complete agreement with what others have found long ago.

So far, the main achievement of this work is to reproduce well-known results. However, the formulation described here has some attractive features that are not shared by previous ones. For one thing, the complete dependence of all quantities on the Grassmann coordinates is described by simple analytic expressions. Also, all formulas have manifest $\mathrm{SU}(4) \times \mathrm{SU}(2,2)$ symmetry, and many of them have manifest PSU(2, 2|4) symmetry. The utility of this formalism for obtaining new results remains to be demonstrated. There are two main directions to explore. One is to derive new facts about this theory. The other is to formulate (or reformulate) other theories in a similar way.

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A Matrices for SU(4) and SU(2, 2)

In order to give an economical superspace description of $\mathrm{AdS}_5 \times S^5$ and its PSU(2, 2|4) isometry, it is desirable to describe the bosonic coordinates and the bosonic subalgebra and in an appropriate way. It is well-known that the description of SU(2) is very conveniently carried out using the three 2×2 Pauli matrices σ^a. This appendix will construct 4×4 matrices, Σ^a and Σ^m, that are convenient for describing SU(4) and SU(2, 2).

In the case of SU(4), we wish to define six antisymmetric 4×4 matrices $(\Sigma^a)^{\alpha\beta}$ and their hermitian conjugates $(\Sigma^{a\dagger})^{\bar{\alpha}\bar{\beta}}$. These matrices are invariant tensors of SU(4) specifying how the six-vector representation couples to the antisymmetric Kronecker product of two four-dimensional representations $\mathbf{4} \times \mathbf{4}$ and $\bar{\mathbf{4}} \times \bar{\mathbf{4}}$, respectively. An essential difference from the case of SU(2) is that the $\mathbf{6}$ is not the adjoint representation of SU(4). (The latter arises in the Kronecker product $\mathbf{4} \times \bar{\mathbf{4}}$.) Another difference is that the $\mathbf{4}$ representation SU(4) is complex, whereas the $\mathbf{2}$ representation of SU(2) is pseudoreal. The invariant matrix $\eta_{\beta\bar{\beta}}$ is used to contract spinor indices in matrix products such as $\Sigma^a \eta \Sigma^{b\dagger}$. However, η is just the unit matrix I_4 in the case of SU(4), so we can omit it without causing confusion. In the case of SU(2, 2), the matrix η is not the unit matrix, so we will display it.

We use the matrices Σ^a and $\Sigma^{a\dagger}$ to define 4×4 matrices

$$Z = \vec{\Sigma} \cdot \hat{z} \quad \text{and} \quad Z^\dagger = \vec{\Sigma}^\dagger \cdot \hat{z}. \tag{A.1}$$

The six-vector $\hat{z}$ describes a unit five-sphere, so $\hat{z} \cdot \hat{z} = 1$. We can encode a specific choice of the six antisymmetric matrices $(\Sigma^a)^{\alpha\beta}$ by introducing three complex coordinates $u = z^1 + iz^2$, $v = z^3 + iz^4$, and $w = z^5 + iz^6$ and defining[8]

$$Z^{\alpha\beta} = \begin{pmatrix} 0 & u & v & w \\ -u & 0 & -\bar{w} & \bar{v} \\ -v & \bar{w} & 0 & -\bar{u} \\ -w & -\bar{v} & \bar{u} & 0 \end{pmatrix}. \tag{A.2}$$

It is easy to verify that this choice satisfies

$$ZZ^\dagger = Z^\dagger Z = I_4, \tag{A.3}$$

which implies that Z is a unitary matrix.

The formulas given above imply that the Σ matrices satisfy the equations

$$(\Sigma^a \Sigma^{b\dagger} + \Sigma^b \Sigma^{a\dagger})^\alpha{}_\beta = 2\delta^{ab}\delta^\alpha_\beta \tag{A.4}$$

and

$$(\Sigma^{a\dagger} \Sigma^b + \Sigma^{b\dagger} \Sigma^a)^{\bar{\alpha}}{}_{\bar{\beta}} = 2\delta^{ab}\delta^{\bar{\alpha}}_{\bar{\beta}}. \tag{A.5}$$

These imply, in particular, that

$$\mathrm{tr}(\Sigma^a \Sigma^{b\dagger} + \Sigma^b \Sigma^{a\dagger}) = 8\delta^{ab}. \tag{A.6}$$

One can also verify that

$$\frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}(\Sigma^a)^{\gamma\delta} = (\Sigma^{a\dagger})_{\alpha\beta}, \tag{A.7}$$

which implies that

$$\frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta} Z^{\gamma\delta} = Z^\dagger_{\alpha\beta}, \tag{A.8}$$

as expected. It is also interesting to note that

$$\det Z = (|u|^2 + |v|^2 + |w|^2)^2 = 1. \tag{A.9}$$

Thus, Z belongs to SU(4), which means that Z parametrizes S^5 as a subspace of SU(4). This is analogous to the equation $\sigma_2 \vec{\sigma} \cdot \hat{x}$, discussed in the introduction, which describes S^2 as a subspace of SU(2). The explicit formula for Z, given in eq. (A.2), is never utilized. The purpose of presenting it is to demonstrate the existence of matrices Σ^a such that $Z = \vec{\Sigma} \cdot \hat{z}$ is an SU(4) matrix.

In the introduction we interpreted the S^2 subspace of SU(2) as a conjugacy class of SU(2). Therefore it is natural to seek the corresponding interpretation of the S^5 subspace of SU(4). Since Z is an antisymmetric matrix, the appropriate equivalence relation is that two elements of SU(4), g_0 and g_0', are equivalent if and only if there exists an element $g \in$ SU(4) such that $g_0' = g^T g_0 g$. For this choice of equivalence relation, the space of antisymmetric SU(4) matrices forms an equivalence class, and the action of an arbitrary

[8] This matrix and the one called Y (below) have appeared previously in the $\mathrm{AdS}_5 \times S^5$ literature.

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group element g on an element g_0 in this class is $g_0 \to g_0' = g^T g_0 g$. The action of the center of SU(4), which is $\mathbb{Z}_4$, has a $\mathbb{Z}_2$ image. If g is i times the unit matrix, which is an element of the center, the map sends $g_0 \to -g_0$. So the isometry group is really SO(6), as it should be. There are actually two S^5's inside SU(4), which are distinguished by a change of sign in eq. (A.8). The map $Z \to iZ$ is a one-to-one map relating the two spheres.

In the case of SU(2, 2) and AdS_5 we should redefine two of the six Σ matrices given above by a factor of i in order to incorporate the indefinite signature of Spin(4, 2). Therefore we modify the SU(4) formulas accordingly and define $Y = \vec{\tilde{\Sigma}} \cdot \hat{y}$ by

$$Y^{\mu\nu} = \begin{pmatrix} 0 & iu & v & w \\ -iu & 0 & -\bar{w} & \bar{v} \\ -v & \bar{w} & 0 & -i\bar{u} \\ -w & -\bar{v} & i\bar{u} & 0 \end{pmatrix}. \tag{A.10}$$

Now, in the notation of eq. (1), we make the identifications $u = y^0 + iy^5$, $v = y^1 + iy^2$, and $w = y^3 + iy^4$. Since $-y^2 = |u|^2 - |v|^2 - |w|^2 = 1$ describes the Poincaré patch of AdS_5, we see that the determinant of Y is unity. Next we take account of the indefinite signature of SU(2, 2) by defining

$$\eta^{\mu\bar{\nu}} = \eta_{\mu\bar{\nu}} = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} = I_{2,2}. \tag{A.11}$$

Then, using this metric to contract spinor indices, one finds that

$$Y\eta Y^\dagger \eta = I_4, \tag{A.12}$$

where we use $-y^2 = |u|^2 - |v|^2 - |w|^2 = 1$ once again. This implies that Y is an element of SU(2, 2). Thus, just as in the compact case, we find that AdS_5 is represented as a subspace of the SU(2, 2) group manifold. Eq. (A.12) implies the algebra

$$(\Sigma^m \eta \Sigma^{n\dagger} \eta + \Sigma^n \eta \Sigma^{m\dagger} \eta)^\mu{}_\nu = -2\eta^{mn} \delta^\mu_\nu, \tag{A.13}$$

where η^{mn} is the SO(4, 2) metric.

The main text takes factors of η into account by only using "unbarred" indices, i.e., by defining

$$Y^\dagger_{\mu\nu} = \eta_{\mu\bar{\mu}} Y^{\dagger\bar{\mu}\bar{\nu}} \eta_{\bar{\nu}\nu}. \tag{A.14}$$

Then we can write $YY^{-1} = I$, even though Y is not unitary.

Other interesting quantities are the connection one-forms for $\mathfrak{su}(4)$ and $\mathfrak{su}(2,2)$. The former is given by

$$\Omega_0 = ZdZ^\dagger = -dZZ^\dagger, \tag{A.15}$$

This matrix is antihermitian and traceless, which implies that it belongs to the $\mathfrak{su}(4)$ Lie algebra. To eliminate any possible doubt about this, we have computed the matrix explicitly:

$$\Omega_0 = \begin{pmatrix} ud\bar{u} + vd\bar{v} + wd\bar{w} & wdv - vdw & udw - wdu & vdu - udv \\ \bar{v}d\bar{w} - \bar{w}d\bar{v} & ud\bar{u} + \bar{v}dv + \bar{w}dw & ud\bar{v} - \bar{v}du & ud\bar{w} - \bar{w}du \\ \bar{w}d\bar{u} - \bar{u}d\bar{w} & vd\bar{u} - \bar{u}dv & vd\bar{v} + \bar{w}dw + \bar{u}du & vd\bar{w} - \bar{w}dv \\ \bar{u}d\bar{v} - \bar{v}d\bar{u} & wd\bar{u} - \bar{u}dw & wd\bar{v} - \bar{v}dw & wd\bar{w} + \bar{v}dv + \bar{u}du \end{pmatrix}. \tag{A.16}$$

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Tracelessness is a consequence of $|u|^2 + |v|^2 + |w|^2 = 1$. This is a flat connection, since the two-form $d\Omega_0 + \Omega_0 \wedge \Omega_0$ vanishes. Similarly, the connection one-form

$$\tilde{\Omega}_0 = Y\eta dY^\dagger\eta = -dY\eta Y^\dagger\eta \tag{A.17}$$

belongs to the $\mathfrak{su}(2,2)$ Lie algebra, as it should. Moreover, $d\tilde{\Omega}_0 + \tilde{\Omega}_0 \wedge \tilde{\Omega}_0 = 0$, so it is also a flat connection.

To represent the Lie algebra of $\mathfrak{su}(4)$ we introduce the fifteen traceless antihermitian 4×4 matrices

$$(\Sigma^{ab})^\alpha{}_\beta = \frac{1}{2}(\Sigma^a\Sigma^{b\dagger} - \Sigma^b\Sigma^{a\dagger})^\alpha{}_\beta. \tag{A.18}$$

Similarly, for $\mathfrak{su}(2,2)$ we have

$$(\tilde{\Sigma}^{mn})^\mu{}_\nu = \frac{1}{2}(\tilde{\Sigma}^m\eta\tilde{\Sigma}^{n\dagger}\eta - \tilde{\Sigma}^n\eta\tilde{\Sigma}^{m\dagger}\eta)^\mu{}_\nu. \tag{A.19}$$

In this notation, the representations of the $\mathfrak{su}(4)$ and $\mathfrak{su}(2,2)$ Lie algebras are

$$\frac{1}{2}[\Sigma^{ab}, \Sigma^{cd}] = \delta^{bc}\Sigma^{ad} + \delta^{ad}\Sigma^{bc} - \delta^{ac}\Sigma^{bd} - \delta^{bd}\Sigma^{ac} \tag{A.20}$$

and

$$\frac{1}{2}[\tilde{\Sigma}^{mn}, \tilde{\Sigma}^{pq}] = \eta^{np}\tilde{\Sigma}^{mq} + \eta^{mq}\tilde{\Sigma}^{np} - \eta^{mp}\tilde{\Sigma}^{nq} - \eta^{nq}\tilde{\Sigma}^{mp}. \tag{A.21}$$

B The self-dual five-form

We could begin, as we did for the three-forms in section 4.3, by considering the closed and invariant expression

$$T_5 = \mathrm{str}(J_+ \wedge J_+ \wedge J_+ \wedge J_+ \wedge J_+) = \mathrm{str}(A_+ \wedge A_+ \wedge A_+ \wedge A_+ \wedge A_+), \tag{B.1}$$

where $A_+ = A_1 + A_2 + iA_3$. Making this substitution gives 3^5 terms, though many of them vanish and many others are related by symmetries. Rather than following that route, we begin by considering

$$T_{5a} = \mathrm{str}(A_1 \wedge A_1 \wedge A_1 \wedge A_1 \wedge A_1), \tag{B.2}$$

which is the term that contains the expected nonzero self-dual bosonic truncation. In contrast to $\mathrm{str}(A_1 \wedge A_1 \wedge A_1)$, it is not zero. The overall normalization, which is not our present concern, should include a factor of N, since the integral over the five-sphere determines the number of units of five-form flux. Therefore, the closed five-form is not exact, in contrast to the three-form that determines the Wess-Zumino term for the fundamental string.

The five-form T_{5a} is manifestly PSU$(2,2|4)$ invariant, but it is not closed. The plan is to identify additional invariant five-forms that need to be added to give a closed form. This is systematized by expanding in the number of fermi fields (A_2 and A_3). The computation uses the identity

$$DA_1 = -A_1 \wedge A_1 - \mathcal{F} - \mathcal{F}', \tag{B.3}$$

where

$$\mathcal{F} = A_2 \wedge A_2 \quad \text{and} \quad \mathcal{F}' = X\mathcal{F}^T X^\dagger = A_3 \wedge A_3. \tag{B.4}$$

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Other useful formulas are

$$D\mathcal{F} = 0 \tag{B.5}$$

and

$$D\mathcal{F}' = \mathcal{F}' \wedge A_1 - A_1 \wedge \mathcal{F}'. \tag{B.6}$$

Using the identity

$$\mathrm{str}([\mathcal{F} - \mathcal{F}'] \wedge A_1 \wedge A_1 \wedge A_1 \wedge A_1) = 0, \tag{B.7}$$

we find that

$$d\,T_{5a} = -10\,\mathrm{str}(\mathcal{F} \wedge A_1 \wedge A_1 \wedge A_1 \wedge A_1). \tag{B.8}$$

To cancel the terms of order Ψ^2, we now add

$$T_{5b} = -10\,\mathrm{str}(\mathcal{F} \wedge A_1 \wedge A_1 \wedge A_1). \tag{B.9}$$

Using

$$\mathrm{str}(\mathcal{F} \wedge \mathcal{F}' \wedge A_1 \wedge A_1) = \mathrm{str}(\mathcal{F}' \wedge \mathcal{F} \wedge A_1 \wedge A_1) = 0, \tag{B.10}$$

we obtain

$$d(T_{5a} + T_{5b}) = 20\,\mathrm{str}(\mathcal{F} \wedge \mathcal{F} \wedge A_1 \wedge A_1) - 10\,\mathrm{str}(\mathcal{F} \wedge A_1 \wedge \mathcal{F}' \wedge A_1). \tag{B.11}$$

The uncanceled expression is now of order Ψ^4, which is progress.

Finally, we add

$$T_{5c} = 20\,\mathrm{str}(\mathcal{F} \wedge \mathcal{F} \wedge A_1) - 10\,\mathrm{tr}(\mathcal{F} \wedge \mathcal{F}' \wedge A_1). \tag{B.12}$$

This leaves us with

$$d(T_{5a} + T_{5b} + T_{5c}) = 0. \tag{B.13}$$

In this final step, we have used the identities

$$\mathrm{str}(\mathcal{F} \wedge \mathcal{F} \wedge \mathcal{F}) = \mathrm{str}(\mathcal{F} \wedge \mathcal{F} \wedge \mathcal{F}') = \mathrm{str}(\mathcal{F} \wedge \mathcal{F}' \wedge \mathcal{F}') = 0. \tag{B.14}$$

Thus, aside from normalization, the closed invariant five-form

$$T_5 = T_{5a} + T_{5b} + T_{5c}. \tag{B.15}$$

is the desired supersymmetric completion of the bosonic expression

$$\mathrm{tr}(\Omega_0 \wedge \Omega_0 \wedge \Omega_0 \wedge \Omega_0 \wedge \Omega_0) - \mathrm{tr}(\tilde{\Omega}_0 \wedge \tilde{\Omega}_0 \wedge \tilde{\Omega}_0 \wedge \tilde{\Omega}_0 \wedge \tilde{\Omega}_0). \tag{B.16}$$

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C Kappa symmetry projection operators

In order to figure out how kappa symmetry should work in the current context, it is very helpful to review the flat-space limit first. The flat-space theory was worked out using two 32-component MW spinors θ_i of the same chirality [28]. We will summarize the results in the spinor notation of section 5.2 of [25], without explaining that notation here. For an appropriate normalization constant k, it was shown that the variations are

$$\delta S_1 = k \int d^2\sigma \sqrt{-G} G^{\alpha\beta} (\partial_\alpha \bar{\theta}_1 \Pi_\beta \rho_1 + \partial_\alpha \bar{\theta}_2 \Pi_\beta \rho_2) \tag{C.1}$$

where $\Pi_\alpha = \Gamma_\mu \Pi_\alpha^\mu$, $G_{\alpha\beta} = \eta_{\mu\nu} \Pi_\alpha^\mu \Pi_\beta^\nu$, and $\rho_i = \delta\theta_i$. Similarly,

$$\delta S_2 = k \int d^2\sigma \varepsilon^{\alpha\beta} (\partial_\alpha \bar{\theta}_1 \Pi_\beta \rho_1 - \partial_\alpha \bar{\theta}_2 \Pi_\beta \rho_2). \tag{C.2}$$

Despite notational differences, it should be plausible that these equations describe the flat-space limit of the results founds in section 5.1.

In this setting, the appropriate involution γ turned out to be $\gamma(\rho) = \gamma\rho$, where

$$\gamma = \frac{1}{2} \frac{\varepsilon^{\alpha\beta}}{\sqrt{-G}} \Pi_\alpha^\mu \Pi_\beta^\nu \Gamma_{\mu\nu}, \tag{C.3}$$

The formula $\gamma^2 = I$ is equivalent to

$$\frac{1}{2} \varepsilon^{\alpha\beta} \varepsilon^{\alpha'\beta'} \Pi_\alpha^\mu \Pi_\beta^\nu \Pi_{\alpha'}^\rho \Pi_{\beta'}^\lambda \{\Gamma_{\mu\nu}, \Gamma_{\rho\lambda}\} = -4GI. \tag{C.4}$$

To prove this, note that

$$\frac{1}{2} \{\Gamma_{\mu\nu}, \Gamma_{\rho\lambda}\} = (\eta_{\mu\lambda}\eta_{\nu\rho} - \eta_{\mu\rho}\eta_{\nu\lambda}) I + \Gamma_{\mu\nu\rho\lambda}, \tag{C.5}$$

but the last term does not contribute, because α, β, α', and β' only take two values.

Another useful identity, which is proved in a similar manner, is

$$\sqrt{-G} G^{\alpha\beta} \Pi_\beta \gamma = \varepsilon^{\alpha\beta} \Pi_\beta. \tag{C.6}$$

Multiplying on the right by γ and using $\gamma^2 = I$, it is also true that

$$\sqrt{-G} G^{\alpha\beta} \Pi_\beta = \varepsilon^{\alpha\beta} \Pi_\beta \gamma. \tag{C.7}$$

Substituting the latter identity into δS_1, one obtains

$$\delta S_1 + \delta S_2 = 2k \int d^2\sigma \varepsilon^{\alpha\beta} \Big(\partial_\alpha \bar{\theta}_1 \Pi_\beta \gamma_+ \rho_1 - \partial_\alpha \bar{\theta}_2 \Pi_\beta \gamma_- \rho_2 \Big), \tag{C.8}$$

where $\gamma_\pm = \frac{1}{2}(1 \pm \gamma)$ are projection operators. Thus, $\rho_1 = \delta\theta_1 = \gamma_- \kappa$ and $\rho_2 = \delta\theta_2 = \gamma_+ \kappa$ are 16 local symmetries. This means that half of the θ coordinates are gauge degrees of freedom of the string world-sheet theory.

The $\mathrm{AdS}_5 \times S^5$ case. Kappa symmetry works in an almost identical way for the $\mathrm{AdS}_5 \times S^5$ background geometry. The main challenge is to transcribe the flat-space formulas into the matrix notation used in this manuscript. The key equation is the defining equation of the involution γ. We claim that the correct counterpart of the operator γ defined by eq. (C.3) is

$$\gamma(\rho) = \frac{1}{2}\frac{\varepsilon^{\alpha\beta}}{\sqrt{-G}}\left(\Pi_\alpha\Pi_\beta^\dagger\rho - 2\Pi_\alpha\rho^\star\tilde{\Pi}_\beta^\dagger + \rho\tilde{\Pi}_\alpha\tilde{\Pi}_\beta^\dagger\right). \tag{C.9}$$

This formula is unique up to sign ambiguities that are related to discrete symmetries of the world-sheet theory. It can be rewritten in the equivalent form

$$\gamma(\rho) = -\frac{1}{2}\frac{\varepsilon^{\alpha\beta}}{\sqrt{-G}}\left(\Omega_\alpha\Omega_\beta\rho - 2\Omega_\alpha\rho'\tilde{\Omega}_\beta + \rho\tilde{\Omega}_\alpha\tilde{\Omega}_\beta\right). \tag{C.10}$$

If ρ is a MW matrix, then $\gamma(\rho)$ is also a MW matrix. This is proved by showing that $\mu\circ\gamma(\rho) = \gamma\circ\mu(\rho)$ using the definition $\mu(\rho) = Z\rho^\star Y^{-1}$ and the identities $Z^{-1}\Pi = -\Pi^\dagger Z$ and $Y^{-1}\tilde{\Pi} = -\tilde{\Pi}^\dagger Y$.

The proof that $\gamma\circ\gamma = I$ is interesting. Iterating the γ operation generates five types of terms, which schematically have the structure $\Pi^4\rho$, $\Pi^3\rho^\star\tilde{\Pi}$, $\Pi^2\rho\tilde{\Pi}^2$, $\Pi\rho^\star\tilde{\Pi}^3$, and $\rho\tilde{\Pi}^4$. To understand them, recall that we need to generate a factor of the determinant of $G_{\alpha\beta}$ to cancel the denominator. In the present problem $G_{\alpha\beta}$ is a sum of two terms

$$G_{\alpha\beta} = g_{\alpha\beta} + \tilde{g}_{\alpha\beta}, \tag{C.11}$$

where

$$g_{\alpha\beta} = \eta_{ab}\Pi_\alpha^a\Pi_\beta^b \quad \text{and} \quad \tilde{g}_{\alpha\beta} = \eta_{mn}\tilde{\Pi}_\alpha^m\tilde{\Pi}_\beta^n, \tag{C.12}$$

and the two η metrics have signature $(6,0)$ and $(4,2)$, respectively. The determinant of $G_{\alpha\beta}$ is the sum of three pieces: $\det g$, $\det\tilde{g}$, and terms that mix g and $\tilde{g}$. The claim is that the $\Pi^4\rho$ terms give the $\det g$ piece, the $\Pi^2\rho\tilde{\Pi}^2$ terms give the mixed pieces, and the $\rho\tilde{\Pi}^4$ terms give the $\det\tilde{g}$ piece. Furthermore, the $\Pi^3\rho^\star\tilde{\Pi}$ and $\Pi\rho^\star\tilde{\Pi}^3$ terms vanish (due to canceling contributions). It is straightforward to verify these assertions using the same sorts of manipulations as in the flat-space case for the matrices defined in appendix A. Having established that $\gamma\circ\gamma = I$, we can now define projection operators $\gamma_\pm$ by

$$\gamma_+(\rho) = \frac{1}{2}[\rho + \gamma(\rho)] \quad \text{and} \quad \gamma_-(\rho) = \frac{1}{2}[\rho + \gamma(\rho)]. \tag{C.13}$$

It should be emphasized that this discussion is valid for an arbitrary fermionic matrix ρ that transforms as $(\mathbf{4},\bar{\mathbf{4}})$ under $\mathrm{SU}(4)\times\mathrm{SU}(2,2)$. In the application that follows the formulas will be applied to the MW matrices ρ_1 and ρ_2.

The counterpart of eq. (C.6) is

$$\sqrt{-G}G^{\alpha\beta}(\Pi_\beta\gamma(\rho)^\star - \gamma(\rho)\tilde{\Pi}_\beta) = \varepsilon^{\alpha\beta}(\Pi_\beta\rho^\star - \rho\tilde{\Pi}_\beta). \tag{C.14}$$

This can be proved using eq. (C.9) and the identities

$$\Pi_\alpha\Pi_\beta^\dagger = g_{\alpha\beta}I - \frac{1}{2}\varepsilon_{\alpha\beta}\varepsilon^{\alpha'\beta'}\Pi'_\alpha\Pi^\dagger_{\beta'} \quad \text{and} \quad \tilde{\Pi}_\alpha\tilde{\Pi}_\beta^\dagger = -\tilde{g}_{\alpha\beta}I - \frac{1}{2}\varepsilon_{\alpha\beta}\varepsilon^{\alpha'\beta'}\tilde{\Pi}'_\alpha\tilde{\Pi}^\dagger_{\beta'}. \tag{C.15}$$

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The crucial minus sign in the second of these equations can be traced back to the fact that $y^2 = -1$. Equation (C.14) can be rewritten in the equivalent form

$$\sqrt{-G}G^{\alpha\beta}(\Omega_\beta\gamma(\rho') - \gamma(\rho)\tilde{\Omega}_\beta) = \varepsilon^{\alpha\beta}(\Omega_\beta\rho' - \rho\tilde{\Omega}_\beta). \tag{C.16}$$

Defining a pair of one-forms,

$$p = \Omega\gamma(\rho') - \gamma(\rho)\tilde{\Omega} \quad \text{and} \quad q = \Omega\rho' - \rho\tilde{\Omega}, \tag{C.17}$$

Eq. (C.16) can be recast in the more elegant form

$$p = \star q, \tag{C.18}$$

where the Hodge dual is defined using the induced metric $G_{\alpha\beta}$. This crucial identity, which is used to establish kappa symmetry in section 5.2, relates three involutions: $\star$, μ, and γ.

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