

Scattering equations and BCJ relations for gauge and gravitational amplitudes with massive scalar particles

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ABSTRACT: We generalize the scattering equations to include both massless and massive particles. We construct an expression for the tree-level n -point amplitude with $n-2$ gluons or gravitons and a pair of massive scalars in arbitrary spacetime dimension as a sum over the $(n-3)!$ solutions of the scattering equations, à la Cachazo, He, and Yuan. We derive the BCJ relations obeyed by these massive amplitudes.

KEYWORDS: Scattering Amplitudes, Duality in Gauge Field Theories, Field Theories in Higher Dimensions, QCD

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1 Introduction

Enormous interest has been generated over the last several years by the discovery of color-kinematic duality in gauge-theory amplitudes, in particular because it allows for the construction of gravitational amplitudes through the double-copy procedure [1–3]. In their initial work, Bern, Carrasco, and Johansson showed that the assumption of color-kinematic duality implies a previously unknown¹ set of relations among tree-level color-ordered n -gluon amplitudes [1]. These BCJ relations were subsequently proven using string-theory and field-theory techniques [6–10]. Although color-kinematic duality has not yet been proved at loop level, impressive evidence has been amassed to support the conjecture (see, e.g., refs. [2, 3, 11–14]). Recent work of Cachazo, He, and Yuan (CHY) has opened a new window on color-kinematic duality and the double-copy procedure by providing an alternative formula for tree-level gauge-theory and gravitational amplitudes in arbitrary spacetime dimension in terms of solutions of the scattering equations [15–18]. This work has generated much interest [19–38] and has been utilized in proofs [39–42] of a new soft graviton theorem [43].

Previous studies have focused on scattering amplitudes for massless particles transforming in the adjoint representation of the gauge group. In this paper, we explore color-kinematic duality of gauge-theory amplitudes involving massive particles and in representations other than the adjoint.² Specifically, we examine the class of amplitudes consisting

¹The existence of these relations was presaged in the early 1980’s in certain four-point amplitudes [4, 5].

²Recent work on color-kinematic duality for other representations includes refs. [23, 44–46].

of $(n - 2)$ gluons and a pair of massive particles ψ of arbitrary spin transforming in the fundamental representation of the gauge group. These amplitudes can be decomposed into $(n - 2)!$ color-ordered amplitudes

$$\mathcal{A}(1_\psi, 2, \dots, n - 1, n_{\bar{\psi}}) = \sum_{\gamma \in S_{n-2}} \mathbf{t}_{1\gamma n} A(1_\psi, \gamma(2), \dots, \gamma(n - 1), n_{\bar{\psi}}), \quad (1.1)$$

$$\mathbf{t}_{1\gamma n} = (T^{a_{\gamma(2)}} T^{a_{\gamma(3)}} \dots T^{a_{\gamma(n-1)}})^i_{i_n} \quad (1.2)$$

where T^a denote generators in the fundamental representation.

The BCJ relations that apply to tree-level color-ordered n -gluon amplitudes hinge on two key ingredients: color-kinematic duality, and the properties of the propagator matrix [47]. The rank of the propagator matrix fixes the number of BCJ relations, and the form of these relations is determined by the null eigenvectors of this matrix. In section 2 of this paper, we show that the propagator matrix for the amplitude (1.1) has the same rank as that for the n -gluon amplitude, and that its null eigenvectors imply that the color-ordered amplitudes obey

$$0 = \sum_{a=3}^n \left(-m_\psi^2 + \sum_{b=a}^n s_{2b} \right) A(1_\psi, 3, \dots, a - 1, 2, a, \dots, n_{\bar{\psi}}) \quad (1.3)$$

where m_ψ is the mass of ψ , provided that the amplitude satisfies color-kinematic duality. We check eq. (1.3) using various results in the literature, providing evidence for the assumption of color-kinematic duality for the class of amplitudes we are considering.

In this paper, we also propose a generalization of the scattering equations [15–18] to massive particles

$$\sum_{b \neq a} \frac{k_a \cdot k_b + \Delta_{ab}}{\sigma_a - \sigma_b} = 0, \quad \sigma_a \in \mathbb{CP}^1, \quad a = 1, \dots, n \quad (1.4)$$

where

$$\Delta_{ab} = \Delta_{ba}, \quad \sum_{b \neq a} \Delta_{ab} = m_a^2 \quad (1.5)$$

is imposed to guarantee $SL(2, \mathbb{C})$ invariance of the equations. We then use these scattering equations to construct amplitudes in a trio of theories:

- a double-color theory of massless and massive scalars belonging to the adjoint and fundamental representations of $U(N) \times U(\tilde{N})$,
- a gauge theory of gluons and massive scalars in the fundamental representation, and
- a gravitational theory of gravitons and massive scalars.

We present a CHY-type formula for the amplitudes of $(n - 2)$ massless and two massive scalars of the double-color theory in terms of a sum over solutions of the massive scattering equations (1.4). We establish its validity by proving that the double-partial amplitudes agree with the propagator matrix for the amplitudes (1.1) computed in section 2.

We present a CHY-type formula for the amplitudes of $(n - 2)$ gluons and two massive scalars in the fundamental representation, and check agreement with previously-known results. We explicitly show that the color-ordered amplitudes satisfy the BCJ relations (1.3). In general all solutions of the scattering equations contribute to each amplitude. In four dimensions and for massless scalars, however, we observe that only a subset of solutions contributes to the amplitude, the subset depending on the helicities of the gluons.

We also propose a CHY-type formula for amplitudes of $(n - 2)$ gravitons and two massive scalars, and verify agreement with the known four-point amplitude.

This paper is structured as follows: in section 2, we demonstrate the relationship between the propagator matrices for the n -gluon amplitude and for the amplitude for $(n - 2)$ gluons and two massive fundamentals. We use this to derive BCJ relations for the latter, assuming color-kinematic duality. In section 3, we generalize the scattering equations to massive external particles. In sections 4, 5, and 6, we construct the amplitudes in double-color, gauge, and gravity theories respectively. Section 7 contains conclusions and directions for further work.

2 BCJ relations for massless and massive amplitudes

The number and form of the BCJ relations among color-ordered amplitudes are completely fixed by the rank and null eigenvectors of the propagator matrix. In this section, we explicitly define this matrix, and determine the relationship between the propagator matrices for the n -gluon amplitude and the amplitude for $(n - 2)$ gluons and a pair of massive fundamentals. This allows us to derive the form of the BCJ relations among massive amplitudes, provided color-kinematic duality is satisfied.

2.1 BCJ relations for n -gluon amplitudes

We begin by reviewing the BCJ relations for the tree-level n -gluon amplitude in the spirit of ref. [47]. This amplitude can be expressed as a sum over the $(2n - 5)!!$ diagrams that can be assembled from cubic vertices [1]

$$\mathcal{A}(1, 2, \dots, n) = \sum_i \frac{c_i n_i}{d_i}. \tag{2.1}$$

We define the *backbone* of a diagram as the path from the first to the n th external line. All of the other external lines attach to the backbone either directly or via side branches. The subset of $(n - 2)!$ diagrams with no side branches (i.e. all external lines emerge directly from the backbone) we refer to as half-ladder diagrams.

Associated with each diagram i is a color factor c_i obtained by sewing together three-gluon³ vertices f^{abc} . Among these are the color factors $\mathbf{c}_{1\gamma n}$ associated with half-ladder diagrams

$$\mathbf{c}_{1\gamma n} \equiv \mathbf{c}_{1\gamma(2)\dots\gamma(n-1)n} \equiv \sum_{\mathbf{b}_1, \dots, \mathbf{b}_{n-3}} f^{\mathbf{a}_1 \mathbf{a}_{\gamma(2)} \mathbf{b}_1} f^{\mathbf{b}_1 \mathbf{a}_{\gamma(3)} \mathbf{b}_2} \dots f^{\mathbf{b}_{n-3} \mathbf{a}_{\gamma(n-1)} \mathbf{a}_n} \tag{2.2}$$

³The four-gluon vertex is expressed in terms of a linear combination of products of three-gluon factors $f^{abe} f^{cde}$, $f^{ace} f^{dbe}$, and $f^{ade} f^{bce}$, which is why the contribution from Feynman diagrams containing quartic vertices can be parceled out among several purely cubic color factors.

where γ denotes a permutation of $\{2, \dots, n-1\}$. An arbitrary color factor c_i can be written as a linear combination of half-ladder color factors by repeatedly applying the Jacobi identity $f^{abe}f^{cde} + f^{ace}f^{dbe} + f^{ade}f^{bce} = 0$ to the side branches, starting from the backbone and working outward, as described in ref. [48]. Thus, the set of $(n-2)!$ half-ladder color factors forms an independent basis for the color factors⁴

$$c_i = \sum_{\gamma \in S_{n-2}} M_{i,1\gamma n} \mathbf{c}_{1\gamma n} \tag{2.3}$$

and can be used to decompose the n -gluon amplitude (2.1) as [48, 49]

$$\mathcal{A}(1, 2, \dots, n) = \sum_{\gamma \in S_{n-2}} \mathbf{c}_{1\gamma n} A(1, \gamma(2), \dots, \gamma(n-1), n), \tag{2.4}$$

$$A(1, \gamma(2), \dots, \gamma(n-1), n) = \sum_i \frac{M_{i,1\gamma n} n_i}{d_i} \tag{2.5}$$

where $A(1, \gamma(2), \dots, \gamma(n-1), n)$ are the color-ordered amplitudes belonging to the Kleiss-Kuijff basis.⁵

The kinematic numerators n_i associated with each diagram i are functions of the momenta and polarizations of the external gluons. The hypothesis of color-kinematic duality is that the kinematic numerators n_i obey the same Jacobi relations as c_i , and thus can similarly be expressed in terms of $(n-2)!$ half-ladder numerators $\mathbf{n}_{1\gamma n}$:

$$n_i = \sum_{\gamma \in S_{n-2}} M_{i,1\gamma n} \mathbf{n}_{1\gamma n}. \tag{2.6}$$

Using eq. (2.6), the color-ordered amplitude (2.5) can be written

$$A(1, \gamma(2), \dots, \gamma(n-1), n) = \sum_{\delta \in S_{n-2}} m(1\gamma n|1\delta n) \mathbf{n}_{1\delta n} \tag{2.7}$$

where we define the propagator matrix

$$m(1\gamma n|1\delta n) = \sum_i \frac{M_{i,1\gamma n} M_{i,1\delta n}}{d_i} \tag{2.8}$$

as the sum (weighted by the denominators $1/d_i$ of the diagrams) over those cubic diagrams that contribute to both $\text{Tr}[1\gamma n]$ and $\text{Tr}[1\delta n]$.

Although not obvious *a priori*, the $(n-2)! \times (n-2)!$ propagator matrix $m(1\gamma n|1\delta n)$ has rank $(n-3)!$ as a consequence of momentum conservation [47]. In the scattering equation approach, $m(1\gamma n|1\delta n)$ can be interpreted as a *double-partial amplitude* in a theory of scalar particles transforming in the adjoint representation of the group $U(N) \times U(\tilde{N})$ [18]. The matrix of double-partial amplitudes can be expressed in terms of the $(n-3)!$ independent solutions of the scattering equations, which makes its reduced rank manifest.

⁴The coefficients $M_{i,\alpha}$ can be computed by using $f^{abc} = \text{Tr}(T^a[T^b, T^c])$ to decompose c_i into a linear combination of traces $\text{Tr}[\alpha] \equiv \text{Tr}(T^{a\alpha(1)} T^{a\alpha(2)} \dots T^{a\alpha(n)})$, and reading off the coefficients; see, e.g., ref. [38].

⁵All other color-ordered amplitudes are related to these by the Kleiss-Kuijff relations [48, 50].

The reduced rank of the propagator matrix implies that it possesses $(n-2)! - (n-3)!$ null eigenvectors. Consequently, the $(n-2)!$ Kleiss-Kuijf color-ordered amplitudes (2.7) obey an equal number of independent relations. All of these BCJ relations can be generated by the fundamental BCJ relation (and permutations thereof) [6, 8, 51]

$$0 = \sum_{a=3}^n \left(\sum_{b=a}^n s_{2b} \right) A(1, 3, \dots, a-1, 2, a, \dots, n) \tag{2.9}$$

where $s_{ab} \equiv (k_a + k_b)^2$ and k_a are the momenta of the external particles.

We close by emphasizing that the number and form of the BCJ relations are entirely determined by the propagator matrix, independent of the expressions for the kinematic numerators n_i (provided only that the latter obey color-kinematic duality).

2.2 BCJ relations for amplitudes with massive particles

Next we turn to tree-level amplitudes for $(n-2)$ gluons and a pair of massive fundamentals ψ of arbitrary spin. Again, we can express this as a sum over cubic diagrams

$$\mathcal{A}(1_\psi, 2, \dots, n-1, n_{\bar{\psi}}) = \sum_i \frac{c'_i n'_i}{d'_i} \tag{2.10}$$

where we decorate the color factors, kinematic numerators, and denominators with primes to distinguish them from the analogous quantities for n -gluon amplitudes. These diagrams are in one-to-one correspondence with the n -gluon diagrams, in which the backbone of each n -gluon diagram is replaced by a string of propagators of massive fundamentals.

The color factor c'_i associated with each new diagram is obtained by sewing together cubic⁶ ggg vertices f^{abc} and $\bar{\psi}g\psi$ vertices $(T^a)^i_j$. As in the case of n -gluon diagrams, each color factor c'_i can be reduced to a linear combination of half-ladder color factors $\mathbf{t}_{1\gamma n}$, defined in eq. (1.2), by repeatedly applying⁷

$$f^{abc} (T^c)^i_j = [T^a, T^b]^i_j \tag{2.11}$$

to any gluon propagator emerging from the backbone, until all the factors of f^{abc} are removed from c'_i . This process results in the decomposition

$$c'_i = \sum_{\gamma \in S_{n-2}} M_{i,1\gamma n} \mathbf{t}_{1\gamma n} \tag{2.12}$$

where the coefficients $M_{i,1\gamma n}$ are precisely the same as in the n -gluon case. The $\mathbf{t}_{1\gamma n}$ can thus be used to decompose the amplitude (2.10) into color-ordered amplitudes

$$\mathcal{A}(1_\psi, 2, \dots, n-1, n_{\bar{\psi}}) = \sum_{\gamma \in S_{n-2}} \mathbf{t}_{1\gamma n} A(1_\psi, \gamma(2), \dots, \gamma(n-1), n_{\bar{\psi}}), \tag{2.13}$$

$$A(1_\psi, \gamma(2), \dots, \gamma(n-1), n_{\bar{\psi}}) = \sum_i \frac{M_{i,1\gamma n} n'_i}{d'_i}. \tag{2.14}$$

⁶As before, four-gluon vertices can be parceled out among three separate pairs of cubic vertices. For spin-half ψ , there are no $\bar{\psi}g\psi$ vertices, whereas for spin-zero ψ , the $\bar{\psi}g\psi$ vertices are proportional to $\{T^a, T^b\}^i_j$ and so can be recast as a pair of $\bar{\psi}g\psi$ vertices.

⁷This was also used recently in ref. [46].

To discover whether the color-ordered amplitudes (2.14) satisfy relations analogous to those for n -gluon amplitudes, we must determine (a) whether color-kinematic duality continues to hold, and (b) whether the propagator matrix continues to possess null eigenvectors as a consequence of momentum conservation even when some of the particles are massive.

Four-point amplitudes of gluons and massive fundamental fields were examined in refs. [4, 5] and it was shown that the kinematic numerators n'_i obey algebraic relations analogous to c'_i for both spin-zero and spin-half fundamentals. We will therefore proceed to assume that this condition is satisfied for higher-point amplitudes to examine the consequences. In that case, we can write

$$n'_i = \sum_{\gamma \in S_{n-2}} M_{i,1\gamma n} \mathbf{n}'_{1\gamma n} \tag{2.15}$$

implying that

$$A(1_\psi, \gamma(2), \dots, \gamma(n-1), n_{\bar{\psi}}) = \sum_{\delta \in S_{n-2}} m'(1\gamma n|1\delta n) \mathbf{n}'_{1\delta n} \tag{2.16}$$

where

$$m'(1\gamma n|1\delta n) = \sum_i \frac{M_{i,1\gamma n} M_{i,1\delta n}}{d'_i} \tag{2.17}$$

The null eigenvectors of eq. (2.17) therefore determine the (BCJ) relations among the color-ordered amplitudes (2.16), provided that color-kinematic duality is satisfied.

Observe that the propagator matrix (2.17) is the same as that for the n -gluon amplitude (2.8) except that the denominators d'_i must be adjusted to account for the mass of ψ . The denominator of each diagram consists of a product of inverse propagators. Each inverse propagator belonging to a side branch is of the form $(\sum_{a \in S} k_a)^2$, where S is some subset of the gluon momenta $\{k_2, \dots, k_{n-1}\}$. Since $k_a^2 = 0$ for the gluons, this consists of a sum of terms $k_a \cdot k_b$ where $2 \leq a, b \leq n-1$, which are the same for d_i and d'_i . Each inverse propagator belonging to the backbone of fundamental fields is of the form

$$\left(k_n + \sum_{a \in S} k_a\right)^2 - m_\psi^2 = \left(\sum_{a \in S} k_a\right)^2 + \sum_{a \in S} 2k_a \cdot k_n \tag{2.18}$$

Thus, when d'_i is expressed in terms of $k_a \cdot k_b$ with $2 \leq a < b \leq n$ (eliminating k_1 if necessary by using momentum conservation $\sum_{a=1}^n k_a = 0$), the dependence on m_ψ disappears,⁸ so d'_i is identical to d_i . Consequently, propagator matrix $m'(1\gamma n|1\delta n)$ when expressed in terms of these same variables⁹ is identical to the n -gluon propagator matrix $m(1\gamma n|1\delta n)$.

As a result, we may obtain the BCJ relations for the amplitudes of $(n-2)$ gluons and two fundamentals in terms of those for the n -gluon amplitude. Expressed in terms of $k_a \cdot k_b$ with $2 \leq a < b \leq n$, the fundamental BCJ relation (2.9) for the n -gluon amplitude is

$$0 = \sum_{a=3}^n \left(\sum_{b=a}^n 2k_2 \cdot k_b \right) A(1, 3, \dots, a-1, 2, a, \dots, n) \tag{2.19}$$

⁸The remaining constraint among this set of variables $\sum_{2 \leq a < b \leq n} k_a \cdot k_b = 0$ is also independent of m_ψ .

⁹Naturally, one can alternatively express d'_i in terms of $k_a \cdot k_b$ with $1 \leq a < b \leq n-1$, eliminating k_n using momentum conservation.

Since the propagator matrix $m'(1\gamma n|1\delta n)$ in these variables has same form as $m(1\gamma n|1\delta n)$, so do their null eigenvectors, and therefore the fundamental BCJ relation for the massive amplitude is

$$0 = \sum_{a=3}^n \left(\sum_{b=a}^n 2k_2 \cdot k_b \right) A(1_\psi, 3, \dots, a-1, 2, a, \dots, n_{\bar{\psi}}). \quad (2.20)$$

When rewritten in terms of s_{ab} this becomes

$$0 = \sum_{a=3}^n \left(-m_\psi^2 + \sum_{b=a}^n s_{2b} \right) A(1_\psi, 3, \dots, a-1, 2, a, \dots, n_{\bar{\psi}}). \quad (2.21)$$

This is the fundamental BCJ relation obeyed by tree-level amplitudes with $(n-2)$ gluons and two massive fundamentals ψ , provided that color-kinematic duality is satisfied.

For $n=4$, eq. (2.21) becomes

$$(s_{12} - m_\psi^2)A(1_\psi, 2, 3, 4_{\bar{\psi}}) = (s_{13} - m_\psi^2)A(1_\psi, 3, 2, 4_{\bar{\psi}}) \quad (2.22)$$

which was established in refs. [4, 5] for both spin-zero and spin-half fundamentals. We have also verified eq. (2.21) for various five- and six-point amplitudes with massive scalars, using known results in four dimensions [52–54], e.g. eqs. (5.11)–(5.13) for $n=5$ and eq. (5.14) for $n=6$. This provides evidence for the assumption of color-kinematic duality for this class of amplitudes.

3 Scattering equations

The equations¹⁰

$$\sum_{b \neq a} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0, \quad \sigma_a \in \mathbb{CP}^1, \quad a = 1, \dots, n \quad (3.1)$$

dubbed *scattering equations* by Cachazo, He, and Yuan (CHY), have recently proven to play a key role in the structure of scattering amplitudes of massless particles [15–18]. The set of equations (3.1) is invariant under $SL(2, \mathbb{C})$ transformations

$$\sigma \longrightarrow \frac{A\sigma + B}{C\sigma + D}, \quad AD - BC = 1 \quad (3.2)$$

provided $\sum_{a=1}^n k_a = 0$ and $k_a^2 = 0$. Equivalently, only $n-3$ of the n equations (3.1) are independent. In refs. [16, 17], CHY presented novel expressions for tree-level scattering amplitudes of gluons and gravitons in terms of sums over the solutions of eq. (3.1).

In order to apply the CHY approach to the class of amplitudes considered in this paper, we generalize the scattering equations to the case where the external particles are massive, $k_a^2 = m_a^2$. (Dolan and Goddard have previously considered a generalization in which all external masses are equal [25, 32].) We propose modifying eq. (3.1) to

$$f_a = 0, \quad a = 1, \dots, n \quad (3.3)$$

¹⁰These equations have appeared previously in a string-theory context [55–57].

where

$$f_a \equiv \sum_{b \neq a} \frac{k_a \cdot k_b + \Delta_{ab}}{\sigma_a - \sigma_b}, \quad a = 1, \dots, n \quad (3.4)$$

with

$$\Delta_{ab} = \Delta_{ba}, \quad \sum_{b \neq a} \Delta_{ab} = m_a^2. \quad (3.5)$$

It is straightforward to verify that $f_a \rightarrow (C\sigma_a + D)^2 f_a$ under eq. (3.2), so that the modified scattering equations (3.3) remain invariant under $SL(2, \mathbb{C})$ transformations. Furthermore, the three linear combinations

$$\sum_{a=1}^n f_a, \quad \sum_{a=1}^n \sigma_a f_a, \quad \sum_{a=1}^n \sigma_a^2 f_a \quad (3.6)$$

vanish identically precisely when eq. (3.5) is satisfied, which implies that only $n - 3$ of the massive scattering equations (3.3) are independent.

Dolan and Goddard showed that when all external particles have equal mass m , the scattering equations are generalized to the form (3.3) with $\Delta_{ab} = \frac{1}{2}m^2(\delta_{a+1,b} + \delta_{a-1,b})$, which indeed satisfies eq. (3.5). They observe, however, that this choice imposes a specific ordering on the n particles, breaking the permutation invariance of the massless equations [25, 32].

For the case of the amplitudes considered in this paper, in which only two of the external particles are massive ($m_1^2 = m_n^2 = m_\psi^2$ and $m_a = 0$ for $a = 2, \dots, n - 1$), the constraints (3.5) are rather naturally satisfied by choosing $\Delta_{1n} = \Delta_{n1} = m_\psi^2$ with all other Δ_{ab} vanishing. Specifically, the massive scattering equations (3.3) become

$$\sum_{b=2}^{n-1} \frac{k_1 \cdot k_b}{\sigma_1 - \sigma_b} + \frac{k_1 \cdot k_n + m_\psi^2}{\sigma_1 - \sigma_n} = 0, \quad (3.7)$$

$$\sum_{b \neq a} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0, \quad \text{for } a = 2, \dots, n - 1 \quad (3.8)$$

$$\frac{k_n \cdot k_1 + m_\psi^2}{\sigma_n - \sigma_1} + \sum_{b=2}^{n-1} \frac{k_n \cdot k_b}{\sigma_n - \sigma_b} = 0. \quad (3.9)$$

Despite the presence of m_ψ^2 , these equations are essentially equivalent to the massless scattering equations. This can be seen by using momentum conservation to eliminate k_1 (alternatively, k_n). The equations, when expressed in terms of the variables $k_a \cdot k_b$ with $2 \leq a < b \leq n$ (alternatively, $1 \leq a < b \leq n - 1$), are identical to the massless scattering equations, and therefore have the same set of solutions.¹¹

We now use these equations to construct amplitudes for scalar, gauge, and gravitational theories with massive particles.

¹¹Again, we note the fact that the remaining constraint $\sum_{2 \leq a < b \leq n} k_a \cdot k_b = 0$ among this set of variables is independent of m_ψ .

4 Double-color amplitudes

In ref. [18], Cachazo, He, and Yuan presented a new formulation for the tree-level amplitudes of three interrelated theories in terms of the solutions of the massless scattering equations (3.1). Their unified formula computes the n -point amplitudes of colored scalars, of gluons, and of gravitons. The scalar theory is the simplest, and contains massless scalar particles $\phi^{aa'}$ in the adjoint of the color group $U(N) \times U(\tilde{N})$ with cubic interactions of the form

$$f^{abc} \tilde{f}^{a'b'c'} \phi_{aa'} \phi_{bb'} \phi_{cc'} \quad (4.1)$$

where f^{abc} and $\tilde{f}^{a'b'c'}$ are the structure constants of $U(N)$ and $U(\tilde{N})$. We will refer to this as the *double-color theory*. The partial amplitudes of this theory were shown in refs. [18, 25] to be equivalent to the propagator matrix (2.8).

In this section, we generalize the double-color theory to include also scalar particles $\psi^{ii'}$ in the (*fund*, *fund*) representation of the group $U(N) \times U(\tilde{N})$, with cubic couplings

$$(T^a)^i_j (\tilde{T}^{a'})^i_j \bar{\psi}_{ii'} \phi_{aa'} \psi^{jj'} \quad (4.2)$$

as well as a mass term

$$m_\psi^2 \bar{\psi}_{ii'} \psi^{ii'}. \quad (4.3)$$

The tree-level amplitude for $(n-2)$ ϕ fields and two ψ fields is given by the sum over all cubic diagrams

$$\mathcal{A}_{\text{scalar}}(1_\psi, 2_\phi, \dots, n-1_\phi, n_{\bar{\psi}}) = \sum_i \frac{c'_i \tilde{c}'_i}{d'_i} \quad (4.4)$$

where c'_i, \tilde{c}'_i are the color factors constructed from the cubic vertices (4.1) and (4.2), as discussed in section 2, and d'_i is the product of massless ϕ and massive ψ propagators. Equation (4.4) can be rewritten (again following the discussion in section 2) as

$$\mathcal{A}_{\text{scalar}}(1_\psi, 2_\phi, \dots, n-1_\phi, n_{\bar{\psi}}) = \sum_{\gamma, \delta \in S_{n-2}} \mathbf{t}_{1\gamma n} m'(1\gamma n | 1\delta n) \tilde{\mathbf{t}}_{1\delta n} \quad (4.5)$$

where $m'(1\gamma n | 1\delta n)$ is defined in eq. (2.17). In the context of the double-color theory, the $m'(1\gamma n | 1\delta n)$ play the role of double-partial amplitudes.

We propose that the generalization of the CHY formulation to the double-color amplitude with $n-2$ massless ϕ and two massive ψ fields is¹²

$$\mathcal{A}_{\text{scalar}}(1_\psi, 2_\phi, \dots, n-1_\phi, n_{\bar{\psi}}) = (-1)^{n-1} \int \frac{d^n \sigma}{\text{vol SL}(2, \mathbb{C})} \prod'_a \delta \left(\sum_{b \neq a} \frac{k_a \cdot k_b + \Delta_{ab}}{\sigma_a - \sigma_b} \right) C(\sigma) \tilde{C}(\sigma) \quad (4.6)$$

where $\Delta_{ab} = m_\psi^2 (\delta_{a,1} \delta_{b,n} + \delta_{a,n} \delta_{b,1})$ and

$$C(\sigma) = \sum_{\gamma \in S_{n-2}} \frac{\mathbf{t}_{1\gamma n}}{\sigma_{1,\gamma(2)} \cdots \sigma_{\gamma(n-1),n} \sigma_{n,1}}, \quad \tilde{C}(\sigma) = \sum_{\gamma \in S_{n-2}} \frac{\tilde{\mathbf{t}}_{1\gamma n}}{\sigma_{1,\gamma(2)} \cdots \sigma_{\gamma(n-1),n} \sigma_{n,1}} \quad (4.7)$$

¹²Our overall sign convention differs from ref. [18] in order that the double-partial amplitudes of the theory will be precisely equal to the propagator matrix (2.17).

with $\sigma_{ab} \equiv \sigma_a - \sigma_b$. Equation (4.6) differs in two respects from the corresponding expression in ref. [18] for the pure ϕ amplitudes: (a) $\mathbf{c}_{1\gamma n}$ is replaced by $\mathbf{t}_{1\gamma n}$ in the definition of $C(\sigma)$, and (b) the arguments of the delta functions are the massive rather than the massless scattering equations.

As explained in ref. [17], the delta functions completely localize the integral (4.6). Because of the linear dependence among the n scattering equations (3.3) the delta functions for three of them ($a = i, j$, and k) may be omitted. The expression appearing in eq. (4.6)

$$\prod'_a \delta \left(\sum_{b \neq a} \frac{k_a \cdot k_b + \Delta_{ab}}{\sigma_a - \sigma_b} \right) \equiv \sigma_{ij} \sigma_{jk} \sigma_{ki} \prod_{a \neq i, j, k} \delta \left(\sum_{b \neq a} \frac{k_a \cdot k_b + \Delta_{ab}}{\sigma_a - \sigma_b} \right) \quad (4.8)$$

is independent of the choice of i, j , and k . Furthermore, because the integrand in eq. (4.6) is $\text{SL}(2, \mathbb{C})$ -invariant, three of the σ_a (arbitrarily chosen as $a = p, q$, and r) can be fixed. Including the Faddeev-Popov Jacobian that results from this, the integral (4.6) evaluates to

$$\mathcal{A}_{\text{scalar}}(1_\psi, 2_\phi, \dots, n-1_\phi, n_{\bar{\psi}}) = (-1)^{n-1} \sum_{\{\sigma\} \in \text{solutions}} \frac{C(\sigma) \tilde{C}(\sigma)}{\det' \Phi(\sigma)} \quad (4.9)$$

where the sum is over the $(n-3)!$ solutions of the scattering equations (3.3) and

$$\det' \Phi \equiv \frac{|\Phi|_{pqr}^{ijk}}{(\sigma_{pq} \sigma_{qr} \sigma_{rp})(\sigma_{ij} \sigma_{jk} \sigma_{ki})}. \quad (4.10)$$

Here Φ is a $n \times n$ matrix with entries

$$\Phi_{ab} = \frac{2(k_a \cdot k_b + \Delta_{ab})}{(\sigma_a - \sigma_b)^2}, \quad a \neq b; \quad \Phi_{aa} = - \sum_{c \neq a} \frac{2(k_a \cdot k_c + \Delta_{ac})}{(\sigma_a - \sigma_c)^2}. \quad (4.11)$$

This matrix has rank $(n-3)$ since $\sum_{a \neq b} \Phi_{ab} = \sum_{a \neq b} \sigma_a \Phi_{ab} = \sum_{a \neq b} \sigma_a^2 \Phi_{ab} = 0$ when the scattering equations are satisfied. Φ_{pqr}^{ijk} is the nonsingular matrix obtained by removing rows i, j , and k , and columns p, q , and r , and $|\Phi|_{pqr}^{ijk}$ is its signed determinant. Then $\det' \Phi$ is independent of the choices of removed rows and columns.

Equation (4.9) implies that the massive double-partial amplitudes defined by eq. (4.5) are given by

$$m'(1\gamma n | 1\delta n) = \sum_{\{\sigma\} \in \text{solutions}} \frac{(-1)^{n-1}}{\det' \Phi} \frac{1}{(\sigma_{1,\gamma(2)} \cdots \sigma_{\gamma(n-1),n} \sigma_{n,1})(\sigma_{1,\delta(2)} \cdots \sigma_{\delta(n-1),n} \sigma_{n,1})}. \quad (4.12)$$

Each term in this sum is invariant under $\text{SL}(2, \mathbb{C})$, so we may use an $\text{SL}(2, \mathbb{C})$ transformation to set $\sigma_1 = 0$, $\sigma_2 = 1$, and $\sigma_n = \infty$ when evaluating eq. (4.12). To give a concrete example, for $n = 4$ we obtain

$$\begin{pmatrix} m'(1234|1234) & m'(1234|1324) \\ m'(1324|1234) & m'(1324|1324) \end{pmatrix} = \begin{pmatrix} \frac{1}{2k_2 \cdot k_3} + \frac{1}{2k_3 \cdot k_4} & -\frac{1}{2k_2 \cdot k_3} \\ -\frac{1}{2k_2 \cdot k_3} & \frac{1}{2k_2 \cdot k_3} + \frac{1}{2k_2 \cdot k_4} \end{pmatrix} \quad (4.13)$$

where in this case there is a single solution to the scattering equations $\sigma_3 = -k_2 \cdot k_4 / k_3 \cdot k_4$ on which $\det' \Phi \rightarrow 2(k_3 \cdot k_4)^3 / [\sigma_4^4 (k_2 \cdot k_3)(k_2 \cdot k_4)]$.

To show that eq. (4.6) correctly calculates the tree-level amplitude for $n - 2$ massless ϕ and two massive ψ fields, we must establish that eqs. (2.17) and (4.12) yield equivalent results for all the double-partial amplitudes. First observe that, when expressed in terms of $k_a \cdot k_b$ with $2 \leq a < b \leq n$, eq. (4.12) is independent of m_ψ , and furthermore has exactly the same form as the analogous quantity for the massless ϕ amplitude in ref. [18]. This is apparent from the example (4.13), and can be verified in general from the massive scattering equations and the definition of Φ . Moreover, as was shown in section 2, when expressed in terms of the same variables, the massive propagator matrix (2.17) is identical to the propagator matrix (2.17) for the n -gluon theory. The equivalence between the double-partial amplitudes of the massless ϕ amplitude and the propagator matrix for the n -gluon theory was previously established in ref. [18] (see also ref. [25]). Thus, eq. (4.6) is validated.

Next we turn to gauge-theory amplitudes involving massive scalar fields in the fundamental representation.

5 Gauge theory amplitudes

Cachazo, He, and Yuan have presented a formula for the tree-level n -gluon amplitude in arbitrary spacetime dimension in terms of a sum over solutions of the massless scattering equations [17, 18], which was subsequently proved in ref. [25]. In this section, we propose a CHY-type expression for the gauge-theory amplitude for $(n - 2)$ gluons and two massive scalars transforming in the fundamental representation, namely

$$\begin{aligned} \mathcal{A}(1_\psi, 2, \dots, n-1, n_{\bar{\psi}}) &= (-1)^{n-1} \int \frac{d^n \sigma}{\text{vol SL}(2, \mathbb{C})} \prod'_a \delta \left(\sum_{b \neq a} \frac{k_a \cdot k_b + \Delta_{ab}}{\sigma_{a,b}} \right) C(\sigma) E(\sigma) \\ &= (-1)^{n-1} \sum_{\{\sigma\} \in \text{solutions}} \frac{C(\sigma) E(\sigma)}{\det' \Phi(\sigma)} \end{aligned} \tag{5.1}$$

which is obtained from the double-color amplitude (4.6) presented in the previous section by simply replacing the factor $\tilde{C}(\sigma)$ with $E(\sigma)$. Using eq. (4.7), we find that the color-ordered amplitudes defined in eq. (2.13) are given by

$$A(1_\psi, \gamma(2), \dots, \gamma(n-1), n_{\bar{\psi}}) = \sum_{\{\sigma\} \in \text{solutions}} \frac{(-1)^{n-1}}{\det' \Phi} \frac{E(\sigma)}{\sigma_{1,\gamma(2)} \cdots \sigma_{\gamma(n-1),n} \sigma_{n,1}}. \tag{5.2}$$

By substituting eq. (4.12) into eq. (2.13) and comparing with eq. (5.2), we may deduce that $E(\sigma)$ is related to the kinematic numerators via

$$E(\sigma) = \sum_{\delta \in S_{n-2}} \frac{\mathbf{n}'_{1\delta n}}{\sigma_{1,\delta(2)} \cdots \sigma_{\delta(n-1),n} \sigma_{n,1}}. \tag{5.3}$$

The kinematic numerators depend on the momenta and polarizations of the external particles. We now present an explicit expression for $E(\sigma)$ in terms of the pfaffian of an antisymmetric matrix Ψ .

In refs. [17, 18], Ψ is a $2n \times 2n$ matrix, in which the first n entries correspond to the momenta of the gluons and the second n entries correspond to their polarizations. This matrix is singular, so to obtain a nonvanishing pfaffian it was necessary to remove two of the first n rows and columns. The choice of which rows/columns to remove was made arbitrarily and the result was shown to be independent of this choice. In our case, there is the rather natural choice of removing the first and n th rows and columns, which are singled out as the momenta of the two massive scalars. Furthermore, we must also remove the $(n+1)$ th and $(2n)$ th rows and columns, since the scalars have no polarizations. In our case, therefore, Ψ is an antisymmetric $(2n-4) \times (2n-4)$ matrix

$$\Psi = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix} \tag{5.4}$$

where A, B and C are the $(n-2) \times (n-2)$ submatrices

$$A_{ab} = \begin{cases} \frac{2(k_a \cdot k_b + \Delta_{ab})}{\sigma_a - \sigma_b}, & a \neq b; \\ 0, & a = b; \end{cases} \quad B_{ab} = \begin{cases} \frac{2(\epsilon_a \cdot \epsilon_b + \Delta_{ab})}{\sigma_a - \sigma_b}, & a \neq b; \\ 0, & a = b; \end{cases} \tag{5.5}$$

$$C_{ab} = \begin{cases} \frac{2(\epsilon_a \cdot k_b + \Delta_{ab})}{\sigma_a - \sigma_b}, & a \neq b; \\ -\sum_{c \neq a} \frac{2(\epsilon_a \cdot k_c + \Delta_{ac})}{\sigma_a - \sigma_c}, & a = b; \end{cases} \tag{5.6}$$

where the range of a and b is restricted to $2, \dots, n-1$. (The sum over c in C_{aa} , however, runs from 1 to n , omitting a .) Thus our Ψ is simply a truncated version of the Ψ defined in refs. [17, 18], modified to include masses.

One may verify that the pfaffian of Ψ is gauge-invariant (i.e., under $\epsilon_a \rightarrow \epsilon_a + k_a$) when evaluated on a solution of the scattering equations. We now define

$$E(\sigma) = \frac{1}{\sigma_{1n}^2} \text{Pf} \Psi \tag{5.7}$$

which implies that $E(\sigma) \rightarrow E(\sigma) \prod_{a=1}^n (C\sigma_a + D)^2$ under an $\text{SL}(2, \mathbb{C})$ transformation (3.2), the correct behavior to ensure that each term in eq. (5.2) is $\text{SL}(2, \mathbb{C})$ -invariant as well as gauge-invariant.

We now evaluate eq. (5.2) for specific values of n . It is convenient to define the $\text{SL}(2, \mathbb{C})$ -invariant expression

$$\hat{E}(\sigma) = \sigma_{12}\sigma_{23} \cdots \sigma_{n-1,n}\sigma_{n1} E(\sigma). \tag{5.8}$$

For example, for $n=4$, we obtain¹³

$$\hat{E}(\sigma) = -4 \left[\epsilon_2 \cdot k_1 \epsilon_3 \cdot k_4 - \frac{\sigma_{12}\sigma_{34}}{\sigma_{13}\sigma_{24}} \epsilon_2 \cdot k_4 \epsilon_3 \cdot k_1 + \frac{\sigma_{12}\sigma_{34}}{\sigma_{23}\sigma_{14}} k_2 \cdot k_3 \epsilon_2 \cdot \epsilon_3 \right], \quad n=4 \tag{5.9}$$

¹³after using $\sum_{b \neq a} \epsilon_a \cdot k_b = -\epsilon_a \cdot k_a = 0$

expressed in terms of $SL(2, \mathbb{C})$ -invariant cross ratios. Then, evaluating this on the single solution of the scattering equations $\sigma_1 = 0$, $\sigma_2 = 1$, $\sigma_3 = -k_2 \cdot k_4 / k_3 \cdot k_4$, and $\sigma_4 \rightarrow \infty$, we obtain

$$\begin{aligned}
 A(1_\psi, 2, 3, 4_{\bar{\psi}}) &= \frac{2k_2 \cdot k_4}{k_2 \cdot k_3} \left[\frac{\epsilon_2 \cdot k_1 \epsilon_3 \cdot k_4}{k_3 \cdot k_4} + \frac{\epsilon_2 \cdot k_4 \epsilon_3 \cdot k_1}{k_2 \cdot k_4} + \epsilon_2 \cdot \epsilon_3 \right] \\
 &= \frac{4(u - m_\psi^2)}{t} \left[\frac{\epsilon_2 \cdot k_1 \epsilon_3 \cdot k_4}{s - m_\psi^2} + \frac{\epsilon_2 \cdot k_4 \epsilon_3 \cdot k_1}{u - m_\psi^2} + \frac{1}{2} \epsilon_2 \cdot \epsilon_3 \right] \quad (5.10)
 \end{aligned}$$

which is in agreement with a direct Feynman diagram evaluation.

Various tree-level amplitudes for gluons and massive scalars have been calculated in four dimensions using recursive techniques [52–54]. We have numerically evaluated eq. (5.2) for $n = 5$ and $n = 6$ and have obtained agreement with these results (up to overall normalization). Specifically, we have verified that for $n = 5$ with various helicity configurations eq. (5.2) yields¹⁴

$$A(1_\psi, 2^+, 3^+, 4^+, 5_{\bar{\psi}}) = 2\sqrt{2} \left[\frac{m_\psi^2 (\langle 42 \rangle \langle 2 | k_1 | 2 \rangle + \langle 43 \rangle \langle 3 | k_1 | 2 \rangle)}{(2k_1 \cdot k_2) \langle 23 \rangle \langle 34 \rangle (2k_4 \cdot k_5)} \right], \quad (5.11)$$

$$\begin{aligned}
 A(1_\psi, 2^+, 3^+, 4^-, 5_{\bar{\psi}}) &= 2\sqrt{2} \left[-\frac{(\langle 4 | k_5 | 2 \rangle \langle 2 | k_1 | 2 \rangle + \langle 4 | k_5 | 3 \rangle \langle 3 | k_1 | 2 \rangle)^2}{(2k_1 \cdot k_2) \langle 23 \rangle \langle 34 \rangle (2k_4 \cdot k_5) (\langle 42 \rangle \langle 2 | k_1 | 2 \rangle + \langle 43 \rangle \langle 3 | k_1 | 2 \rangle)} \right. \\
 &\quad \left. + \frac{m_\psi^2 [23]^3}{(k_2 + k_3 + k_4)^2 [34] (\langle 42 \rangle \langle 2 | k_1 | 2 \rangle + \langle 43 \rangle \langle 3 | k_1 | 2 \rangle)} \right], \quad (5.12)
 \end{aligned}$$

$$\begin{aligned}
 A(1_\psi, 2^+, 3^-, 4^+, 5_{\bar{\psi}}) &= 2\sqrt{2} \left[-\frac{\langle 3 | k_1 | 2 \rangle^2 \langle 3 | k_5 | 4 \rangle^2}{(2k_1 \cdot k_2) \langle 23 \rangle \langle 34 \rangle (2k_4 \cdot k_5) (\langle 42 \rangle \langle 2 | k_1 | 2 \rangle + \langle 43 \rangle \langle 3 | k_1 | 2 \rangle)} \right. \\
 &\quad \left. + \frac{m_\psi^2 [24]^4}{(k_2 + k_3 + k_4)^2 [23] [34] (\langle 42 \rangle \langle 2 | k_1 | 2 \rangle + \langle 43 \rangle \langle 3 | k_1 | 2 \rangle)} \right]. \quad (5.13)
 \end{aligned}$$

For $n = 6$, we have verified that eq. (5.2) gives

$$A(1_\psi, 2^+, 3^+, 4^+, 5^+, 6_{\bar{\psi}}) = 4 \left[\frac{-m_\psi^2 [5 | k_6 (4 + 5) (2 + 3) k_1 | 2]}{(2k_1 \cdot k_2) (2k_1 \cdot k_2 + 2k_1 \cdot k_3 + 2k_2 \cdot k_3) (2k_5 \cdot k_6) \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle} \right], \quad (5.14)$$

and have also checked agreement with $A(1_\psi, 2^+, 3^+, 4^+, 5^-, 6_{\bar{\psi}})$ in ref. [53].

While generically all $(n - 3)!$ solutions of the scattering equation contribute to the amplitude (5.2), we have observed that, in the massless limit $m_\psi \rightarrow 0$ in four dimensions, only a subset of solutions contributes to any given amplitude. For amplitudes in which all gluons have the same helicity, the amplitude vanishes as $m_\psi \rightarrow 0$, and in fact the contribution from each solution of the scattering equations individually vanishes. For amplitudes in which all gluons but one have the same helicity (MHV amplitudes), only one

¹⁴Note the change in sign in the second term of $A(1_\psi, 2^+, 3^+, 4^-, 5_{\bar{\psi}})$ relative to ref. [53]. This correction was also noted in ref. [54].

of the solutions of the scattering equations contributes to the amplitude. More precisely, for the mostly-plus helicity amplitude, only the solution (cf. ref. [31])

$$\sigma_i = \frac{\langle i1 \rangle \langle 2n \rangle}{\langle in \rangle \langle 21 \rangle}, \quad i = 1, \dots, n \quad (5.15)$$

contributes, while for the mostly-minus helicity amplitude, only the solution

$$\sigma_i = \frac{[i1][2n]}{[in][21]}, \quad i = 1, \dots, n \quad (5.16)$$

contributes. For $n = 6$, neither of these solutions contributes to the amplitudes with two positive and two negative helicities, but all of the other four solutions do. We expect these patterns to continue to higher values of n .

5.1 BCJ relations from scattering equations

The BCJ relations follow from color-kinematic duality and the reduced rank of the propagator matrix. As shown in ref. [18], a gauge-theory amplitude that is written in the CHY form automatically satisfies these properties and consequently, the color-ordered amplitudes must satisfy BCJ relations. Here we will explicitly demonstrate that the CHY-type expression (5.2) for the color-ordered amplitudes for $(n-2)$ gluons and two massive scalars obeys the fundamental BCJ relation for massive amplitudes (1.3). Although this argument has essentially been given already in refs. [15, 58], we repeat it here to be self-contained.

The fundamental BCJ relation (2.20) can be recast as

$$0 = \sum_{b=3}^n k_2 \cdot k_b \sum_{a=3}^b A(1_\psi, 3, \dots, a-1, 2, a, \dots, n_{\bar{\psi}}). \quad (5.17)$$

We use eq. (5.2) to compute

$$\begin{aligned} & \sum_{a=3}^b A(1_\psi, 3, \dots, a-1, 2, a, \dots, n_{\bar{\psi}}) \\ &= \sum_{\{\sigma\} \in \text{solutions}} \frac{(-1)^{n-1}}{\det' \Phi} \frac{E(\sigma)}{\sigma_{13}\sigma_{34} \cdots \sigma_{n-1,n}\sigma_{n1}} \left[\frac{\sigma_{13}}{\sigma_{12}\sigma_{23}} + \sum_{a=4}^b \frac{\sigma_{a-1,a}}{\sigma_{a-1,2}\sigma_{2a}} \right] \\ &= \sum_{\{\sigma\} \in \text{solutions}} \frac{(-1)^{n-1}}{\det' \Phi} \frac{E(\sigma)}{\sigma_{13}\sigma_{34} \cdots \sigma_{n-1,n}\sigma_{n1}} \frac{\sigma_{1b}}{\sigma_{12}\sigma_{2b}}. \end{aligned} \quad (5.18)$$

Now consider

$$\begin{aligned} \sum_{b=3}^n k_2 \cdot k_b \frac{\sigma_{1b}}{\sigma_{12}\sigma_{2b}} &= \sum_{b \neq 2} k_2 \cdot k_b \frac{\sigma_{1b}}{\sigma_{12}\sigma_{2b}} = \sum_{b \neq 2} k_2 \cdot k_b \frac{\sigma_{12} + \sigma_{2b}}{\sigma_{12}\sigma_{2b}} \\ &= \sum_{b \neq 2} \frac{k_2 \cdot k_b}{\sigma_{2b}} + \frac{1}{\sigma_{12}} \sum_{b \neq 2} k_2 \cdot k_b = \sum_{b \neq 2} \frac{k_2 \cdot k_b}{\sigma_{2b}} \end{aligned} \quad (5.19)$$

where the last equality follows by momentum conservation and $m_2^2 = 0$. Then

$$\begin{aligned} & \sum_{b=3}^n k_2 \cdot k_b \sum_{a=3}^b A(1_\psi, 3, \dots, a-1, 2, a, \dots, n_{\bar{\psi}}) \\ &= \sum_{\{\sigma\} \in \text{solutions}} \frac{(-1)^{n-1}}{\det' \Phi} \frac{E(\sigma)}{\sigma_{13} \sigma_{34} \cdots \sigma_{n-1, n} \sigma_{n1}} \left(\sum_{b \neq 2} \frac{k_2 \cdot k_b}{\sigma_{2b}} \right) = 0 \end{aligned} \quad (5.20)$$

because the term in parentheses vanishes on any solution of the scattering equations (3.8).

6 Gravitational amplitudes

Cachazo, He, and Yuan have presented a formula for the tree-level n -graviton amplitude in arbitrary spacetime dimension in terms of a sum over solutions of the massless scattering equations [17, 18]. In this section, we propose an analogous expression for the amplitude for $(n-2)$ gravitons and two massive scalars in terms of solutions of the massive scattering equations, namely

$$\begin{aligned} \mathcal{A}_{\text{grav}}(1_\psi, 2, \dots, n-1, n_{\bar{\psi}}) &= (-1)^{n-1} \int \frac{d^n \sigma}{\text{vol SL}(2, \mathbb{C})} \prod'_a \delta \left(\sum_{b \neq a} \frac{k_a \cdot k_b + \Delta_{ab}}{\sigma_{a,b}} \right) E(\sigma) \tilde{E}(\sigma) \\ &= (-1)^{n-1} \sum_{\{\sigma\} \in \text{solutions}} \frac{E(\sigma) \tilde{E}(\sigma)}{\det' \Phi(\sigma)} \end{aligned} \quad (6.1)$$

which is obtained from the gauge-theory amplitude (5.1) presented in the previous section by replacing the factor $C(\sigma)$ with $\tilde{E}(\sigma)$. All the ingredients in this equation have already been defined in previous sections.

For $n = 4$, the two-graviton two-scalar scattering amplitude was computed long ago [59]; the calculation is rather tedious. In contrast, it is trivial to evaluate eq. (6.1) on the single solution of the scattering equations $\sigma_1 = 0$, $\sigma_2 = 1$, $\sigma_3 = -k_2 \cdot k_4 / k_3 \cdot k_4$, and $\sigma_4 \rightarrow \infty$ and use eq. (5.9) to obtain

$$\begin{aligned} \mathcal{A}_{\text{grav}}(1_\psi, 2, 3, 4_{\bar{\psi}}) &= -\frac{8k_2 \cdot k_4}{k_2 \cdot k_3} \frac{k_3 \cdot k_4}{k_3 \cdot k_4} \left[\frac{\epsilon_2 \cdot k_1}{k_3 \cdot k_4} \frac{\epsilon_3 \cdot k_4}{k_2 \cdot k_4} + \frac{\epsilon_2 \cdot k_4}{k_2 \cdot k_4} \frac{\epsilon_3 \cdot k_1}{k_3 \cdot k_4} + \epsilon_2 \cdot \epsilon_3 \right] \\ &\quad \times \left[\frac{\tilde{\epsilon}_2 \cdot k_1}{k_3 \cdot k_4} \frac{\tilde{\epsilon}_3 \cdot k_4}{k_2 \cdot k_4} + \frac{\tilde{\epsilon}_2 \cdot k_4}{k_2 \cdot k_4} \frac{\tilde{\epsilon}_3 \cdot k_1}{k_3 \cdot k_4} + \tilde{\epsilon}_2 \cdot \tilde{\epsilon}_3 \right]. \end{aligned} \quad (6.2)$$

This has the nice feature of yielding the known factorized form for the amplitude [60, 61].

We hope that the expression (6.1) may play a useful role in the current lively discussion of soft graviton theorems [39–43].

7 Conclusions

In this paper, we have examined color-kinematic duality for gauge theories with massive particles in representations other than the adjoint. We derived the form of the BCJ relations for tree-level amplitudes with $n - 2$ gluons and a pair of massive particles that are implied by color-kinematic duality.

We have also generalized the scattering equations to include both massless and massive particles, and have proposed CHY-type expressions for tree-level amplitudes in three interrelated theories in terms of the solutions to these equations:

$$\begin{aligned}
 \mathcal{A}_{\text{scalar}}(1_\psi, 2_\phi, \dots, n-1_\phi, n_{\bar{\psi}}) &= (-1)^{n-1} \sum_{\{\sigma\} \in \text{solutions}} \frac{C(\sigma)\tilde{C}(\sigma)}{\det' \Phi(\sigma)}, \\
 \mathcal{A}_{\text{gauge}}(1_\psi, 2, \dots, n-1, n_{\bar{\psi}}) &= (-1)^{n-1} \sum_{\{\sigma\} \in \text{solutions}} \frac{C(\sigma)E(\sigma)}{\det' \Phi(\sigma)}, \\
 \mathcal{A}_{\text{grav}}(1_\psi, 2, \dots, n-1, n_{\bar{\psi}}) &= (-1)^{n-1} \sum_{\{\sigma\} \in \text{solutions}} \frac{E(\sigma)\tilde{E}(\sigma)}{\det' \Phi(\sigma)}.
 \end{aligned} \tag{7.1}$$

Particles 2 through $n - 1$ are massless scalars, gluons, and gravitons respectively, and particles 1 and n are in each case massive scalars. The summand of each of these expressions is a product of factors: $C(\sigma)$ represents the color factor (4.7) and $E(\sigma)$ represents a polarization-dependent kinematic factor (5.3), which can be written in terms of the pfaffian of a matrix (5.4). Color-kinematic duality of the gauge theory and the double-copy prescription for gravity are completely manifest in these expressions.

When the scattering equations have only one solution (e.g. for four-point amplitudes), eq. (7.1) implies that the amplitude itself can be expressed as a product of factors. This neatly explains the factorization of various four-point amplitudes observed in gauge theory [4, 5] and gravity [60, 61]. Conversely, the fact that gauge-theory and gravity four-point amplitudes involving massive fermions also factorize strongly hints that expressions analogous to eq. (7.1) should also exist for fermions.

In cases where only one solution of the scattering equations contributes to the amplitude, e.g., MHV amplitudes in four dimensions with $m_\psi = 0$, eq. (7.1) again implies that the amplitude should factorize. This observation could lead to simpler expressions for this class of amplitudes.

Finally, an obvious and very important direction for future research is the generalization of the scattering equation approach to loop-level amplitudes.

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