't Hooft loops and integrability

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ABSTRACT: We consider the defect CFT defined by a 't Hooft line embedded in $\mathcal{N} = 4$ super Yang-Mills theory. By explicitly quantizing around the given background we exactly reproduce a prediction from S-duality for the correlators between the 't Hooft line and chiral primaries in the bulk and pave the way for higher loop analyses for non-protected operators. Furthermore, we demonstrate at the leading perturbative order that correlators between the 't Hooft line and non-protected bulk operators can be efficiently computed using integrability. As a byproduct we find new integrable overlaps in $\mathfrak{sl}(2)$ spin chains in different representations.

Contents

1   Introduction

A 't Hooft loop [1] is a typical disorder operator defined by singular boundary conditions for gauge fields, rather than by taking a product of field operators. In essence the 't Hooft loop is a trajectory of a point-like Dirac monopole. When embedded in $\mathcal{N} = 4$ super Yang Mills theory, the 't Hooft line constitutes a 1/2 BPS defect of co-dimension three and gives rise to a defect conformal field theory [2]. This defect CFT has been subject of extensive analyses building on S-duality and supersymmetric localization [3–5], but should be amenable to other exact tools such as the boundary conformal bootstrap and integrability.

The holographic dual of the 't Hooft line consists of a D1-brane appropriately embedded in the $AdS_5 \times S^5$ background [6], and the boundary conditions provided by this object for

the string sigma model belong to the list of integrable ones given in [7]. Furthermore, the boundary bootstrap equations for the present set-up were argued to immediately follow from those of the more studied co-dimension one, 1/2 BPS defect version of $\mathcal{N} = 4$ SYM, dual to the D3-D5 probe brane system [8]. Seemingly, the 't Hooft line set-up is the simplest defect CFT where all these exact methods come together.

The 't Hooft loop is related to the more studied Wilson loop by S-duality [9–11]. A number of exact results obtained for the Wilson loop using matrix model techniques can hence be directly translated to the 't Hooft loop. This holds for instance for the expectation value of the 't Hooft loop itself as well as its correlators with chiral primary operators in the bulk. We shall demonstrate that exact results for correlators with non-protected local bulk operators can be obtained from integrability. Our approach is for the moment perturbative and should be seen as a necessary preparatory step for a full non-perturbative solution evoking integrability bootstrap in the spirit of [12–15].

At the leading perturbative order the computation of correlators reduces to a combinatorial problem which can be handled efficiently using the tools of integrability [16, 17]. At the quantum level, the computations entail the study of beautiful examples of exactly solvable quantum mechanical problems, including Dirac's original monopole problem [18] and the problem of a spin one particle coupled to a scalar in a monopole potential. A first outcome of the perturbative analysis is a solid confirmation of the predictions made by S-duality and a number of exact expressions for the leading contribution to correlators between the 't Hooft line and non-protected operators in specific subsectors.

Our paper is organized as follows. We begin by introducing the 't Hooft line in section 2 and turn to discussing the implications of S-duality for its expectation value and its correlators with local, protected bulk operators in section 3. Section 4 is devoted to the quantization of the 't Hooft line and ends with the above mentioned confirmation of an S-duality prediction. A discussion of how to reproduce the S-duality prediction from supergravity and string theory has been relegated to an appendix. In section 5 we demonstrate the leading order integrability of the 't Hooft line. Finally, section 6 contains our conclusion.

2 Kinematics

The 't Hooft loop represents a trajectory of a point-like monopole, and is defined by the prescribed singularity of the gauge fields. For the static 't Hooft line [2]:

$$F_{ij} = \frac{B}{2}\,\varepsilon_{ijk}\,\frac{x_k}{r^3}, \tag{2.1}$$

$$\Phi_I = \frac{Bn_I}{2r}\,. \tag{2.2}$$

These should be regarded as boundary conditions for the fields in the path integral. For the straight 't Hooft line, x_i are the spatial coordinates, more generally they should be regarded as coordinates of the 3d hyperplane orthogonal to the loop at a given point. And the boundary conditions are imposed at $x_i \to 0$.
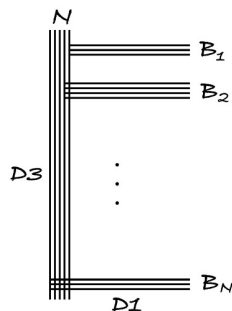
Figure 1. A 't Hooft line can be modeled by a stack of D1-branes ending on a stack of D3-branes. Because of the no-force condition the D1-branes are aligned in the transverse directions and are hence characterized by the same R-charge vector n_i, but can be split along the D3 world volume. An elementary object is thus a 't Hooft loop with one unit of magnetic charge.
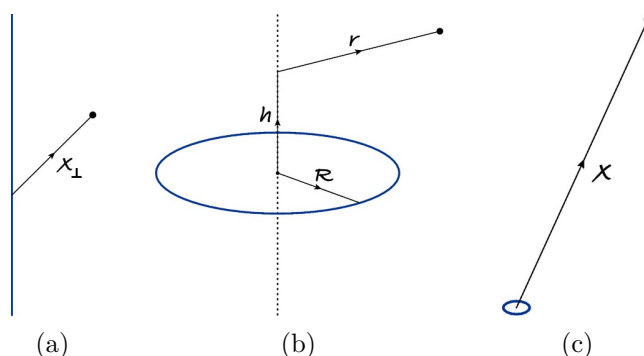

Figure 2. (a) A correlator between the 't Hooft line and a local operator. (b) The correlator with the circular loop in general position. (c) The OPE limit thereof.

The 't Hooft loop operator is characterized by the magnetic charge vector B and the R-charge orientation n_i, a unit vector in six dimensions. The magnetic charges are embedded in the Cartan algebra of the gauge group:

$$B = \text{diag}(B_1, \ldots, B_N), \tag{2.3}$$

and obey the Dirac quantization condition, that requires all B_i's to be integer. The brane configuration describing this setup is shown in figure 1. The D1-branes can be split along the worldvolume of the D3's at no energy cost, in this sense the generic 't Hooft line is a composite object consisting of elementary building blocks with unit magnetic charge. We will concentrate on the elementary 't Hooft line and will set

$$B = \text{diag}(1, 0, \ldots, 0), \tag{2.4}$$

that corresponds to a single D1-brane attached to one of the $D3$'s in the stack.

The straight 't Hooft line in $\mathcal{N} = 4$ SYM preserves scale invariance along with some supersymmetry[1] and thus defines a supersymmetric defect CFT. Local operators can be

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[1]The subalgebra of $\mathfrak{psu}(2,2|4)$ preserved by the 't Hooft line is $\mathfrak{osp}(4^*|4)$ [8], the same as for the circular Wilson loop [19].

confined to the defect or inserted in the bulk, for instance the correlator of the 't Hooft line with a local operator defines a one-point bulk correlation function:

$$\langle \mathcal{O}_a(x) \rangle_T \equiv \frac{\langle T(L)\mathcal{O}_a(x) \rangle}{\langle T(L) \rangle} \, . \tag{2.5}$$

Since the only scale in the problem is the distance from the line to the operator insertion, figure 2a, the scale symmetry alone fixes the one-point function up to a single constant:

$$\langle \mathcal{O}_a(x) \rangle_T = \frac{\mathbb{C}_a}{(2|x_\perp|)^{\Delta_a}} \, . \tag{2.6}$$

The constant depends on the operator normalization but once normalization is fixed carries dynamical information on both the defect and operator. We fix the normalization ambiguity by demanding the standard form of the two-point function far from the defect:

$$\left\langle \mathcal{O}_a^\dagger(x)\mathcal{O}_b(0) \right\rangle = \frac{\delta_{ab}}{|x|^{2\Delta_a}} \, . \tag{2.7}$$

An extra 2 in (2.6) is a rather standard convention, and is introduced for the following reason. An inversion map makes a circle out of a line, figure 2b. Applying inversion to (2.6) and reading off the geometric data from the result one finds the correlator of a circular loop with a local operator in general position:

$$\frac{\langle T(C)\mathcal{O}_a(x) \rangle}{\langle T(C) \rangle} = \frac{\mathbb{C}_a R^{\Delta_a}}{\left[h^2 + (r-R)^2\right]^{\frac{\Delta_a}{2}} \left[h^2 + (r+R)^2\right]^{\frac{\Delta_a}{2}}} \, . \tag{2.8}$$

Consistency with (2.6) can be checked by setting $r = R$ and sending $h \to 0$.

In the opposite limit, figure 2c,

$$\frac{\langle T(C)\mathcal{O}_a(x) \rangle}{\langle T(C) \rangle} \overset{x \to \infty}{\simeq} \frac{\mathbb{C}_a R^{\Delta_a}}{|x|^{2\Delta_a}} \, . \tag{2.9}$$

Notice a factor of two difference with (2.6). This formula shows that $\mathbb{C}_a$ is the OPE coefficient of the 't Hooft loop expanded in the basis of local operators [20]:

$$T(C) = \langle T(C) \rangle \left(1 + \sum_a \mathbb{C}_a R^{\Delta_a} \mathcal{O}_a(0) \right) . \tag{2.10}$$

The normalization conventions are such that $\mathbb{C}_\mathbf{1} = 1$ and all of the OPE coefficients are dimensionless. Those latter will be the focus of our discussion.

3 S-duality and protected operators

The expectation value of the circular Wilson loop and its correlators with chiral primary operators admit a matrix model representation [21–23] that can be derived from supersymmetric localization on S^4 [24]. An equivalent representation is through 2d Yang-Mills theory [25]. Either way, the resulting matrix model is Gaussian and the correlators can be computed exactly, at any value of the gauge coupling and for any N [22, 26]. The

correlators of the 't Hooft loop follow from S-duality which inverts the Yang-Mills coupling (not the 't Hooft coupling that we use throughout the paper):

$$\lambda \to \frac{16\pi^2 N^2}{\lambda}. \tag{3.1}$$

The S-duality and the large-N limit do not commute, so to reconstruct the 't Hooft loop we really need to know the Wilson loop at any finite N, not just in the 't Hooft limit.

3.1 't Hooft loop expectation value

The expectation value of the circular Wilson loop is known exactly at any value of the coupling [22]:

$$\langle W \rangle = e^{\frac{\lambda}{8N}} L_{N-1}^1 \left(-\frac{\lambda}{4N} \right), \tag{3.2}$$

where $L_n^m(x)$ are the Laguerre polynomials. Applying S-duality we get for the expectation value of the circular 't Hooft loop:

$$\langle T \rangle = e^{\frac{2\pi^2 N}{\lambda}} L_{N-1}^1 \left(-\frac{4\pi^2 N}{\lambda} \right). \tag{3.3}$$

The large-N limit for the 't Hooft and Wilson loops are quite different because the factors of N are reversed by S-duality. Yet, the 't Hooft loop in the planar limit appears to be related to the multiply-wound Wilson loop in the limit when the number of windings scales with N [27]. The k-wound Wilson loop is given by (3.2) with λ replaced by λk^2. If we now take the simultaneous limit of $N \to \infty$, $k \to \infty$ with $\kappa = \lambda k/N$ fixed the problem becomes formally equivalent to computing the 't Hooft loop in the conventional large-N limit with $\kappa = 4\pi/\sqrt{\lambda}$.

The multiply-wound Wilson loop is known to exponentiate [26–28] and the expectation value of the 't Hooft loop can be inferred from that result just by replacing κ with $4\pi/\sqrt{\lambda}$:

$$\langle T \rangle \overset{N \to \infty}{\simeq} e^{N F_T(\lambda)}, \tag{3.4}$$

$$F_T(\lambda) = \frac{2\pi}{\sqrt{\lambda}} \sqrt{1 + \frac{\pi^2}{\lambda}} + 2\ln\left(\sqrt{1 + \frac{\pi^2}{\lambda}} + \frac{\pi}{\sqrt{\lambda}} \right). \tag{3.5}$$

The next order in $1/N$ can be found in [29, 30]. Derivation of this result by a direct computation is very easy and is sketched below.

It follows from an integral representation of (3.3):

$$\langle T \rangle = \frac{\sqrt{\lambda N}}{2\pi N!} e^{-\frac{2\pi^2 N}{\lambda}} \int_0^\infty dt\, t^{N-\frac{1}{2}} e^{-t} I_1\left(4\pi \sqrt{\frac{Nt}{\lambda}} \right). \tag{3.6}$$

Changing variables $t = N\tau$, using the Stirling formula for $N!$ and taking the asymptotics of the Bessel function we get:

$$\langle T \rangle \simeq \frac{\lambda^{\frac{3}{4}}}{8\pi^{\frac{5}{2}}} \int_0^\infty d\tau\, \tau^{-\frac{3}{4}} e^{-N S(\tau)}, \tag{3.7}$$

where

$$S(\tau) = \tau - \ln \tau - 4\pi \sqrt{\frac{\tau}{\lambda}} + \frac{2\pi^2}{\lambda} - 1. \tag{3.8}$$

The main contribution to the integral comes from the saddle point at

$$\sqrt{\tau_*} = \sqrt{1 + \frac{\pi^2}{\lambda}} + \frac{\pi}{\sqrt{\lambda}}, \tag{3.9}$$

and in the leading approximation we get (3.4), (3.5).

The localization prediction is consistent with the semiclassical nature of the 't Hooft loop. At weak coupling:

$$F_T \overset{\lambda \to 0}{\simeq} \frac{2\pi^2}{\lambda} + \ln \frac{4\pi^2}{\lambda} + \dots \tag{3.10}$$

The first term is just the Yang-Mills action evaluated on the field of the monopole [3, 27], while the log-correction comes from the zero modes and the gauge group volume [3, 27].

On the strong-coupling side:

$$F_T \overset{\lambda \to \infty}{\simeq} \frac{4\pi}{\sqrt{\lambda}} + \dots \tag{3.11}$$

This can be compared to the action of the hemi-spherical D1-brane ending on the circular loop at the boundary of AdS_5. Indeed, the D1-brane tension is[2]

$$T_{D1} = \frac{2N}{\sqrt{\lambda}}, \tag{3.12}$$

and the regularized volume of the minimal surface equals $A = -2\pi$ [32, 33]. Thus,

$$\langle T \rangle \simeq e^{-T_{D1}A} = e^{\frac{4\pi N}{\sqrt{\lambda}}},$$

in agreement with the result obtained by localization.

3.2 Correlators with chiral primaries

Consider chiral primary operators

$$\mathcal{O}_L = \frac{1}{\sqrt{L}} \left(\frac{4\pi^2}{\lambda} \right)^{\frac{L}{2}} \operatorname{tr} Z^L. \tag{3.13}$$

The overall factor makes them unit-normalized. To make contact with localization we need to align the R-charge of the operator to that of the 't Hooft loop. For that matter, we assign the orientation vector $n_i = (1, 0, \dots, 0)$ to the 't Hooft loop and define the operator's polarization by $Z = \Phi_1 + i\Phi_2$.

For the Wilson loop with the same R-charge assignment the one-point function is known exactly [26]:

$$\langle W \mathcal{O}_L(x) \rangle = \frac{1}{\sqrt{L}} \left(\frac{\sqrt{\lambda}}{2NR} \right)^L e^{\frac{\lambda}{8N}} \sum_{k=1}^{L} L_{N-k}^L \left(-\frac{\lambda}{4N} \right). \tag{3.14}$$

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[2]This is the standard textbook formula [31] re-written in the AdS units, whereupon the string tension is defined by the dimensionless ratio: $\alpha'/R^2 = 1/\sqrt{\lambda}$, and the string coupling is traded for the gauge coupling: $g_s = 4\pi\lambda/N$.

This formula is literally correct if the operator is inserted in the centre of the circle, otherwise R should be replaced by the appropriately defined conformal distance from (2.8).

Applying S-duality we get for the correlator with the 't Hooft loop:

$$\langle T\mathcal{O}_L(x)\rangle = \frac{1}{\sqrt{L}} \left(\frac{2\pi}{\sqrt{\lambda}\,R}\right)^L e^{\frac{2\pi^2 N}{\lambda}} \sum_{k=1}^{L} L_{N-k}^L \left(-\frac{4\pi^2 N}{\lambda}\right). \tag{3.15}$$

The large-N limit can again be extracted from the integral representation:

$$\langle T\mathcal{O}_L(x)\rangle = \frac{1}{\sqrt{\lambda}\,N^{\frac{L}{2}}\,R^L}\, e^{-\frac{2\pi^2 N}{\lambda}} \int_0^\infty dt\, e^{-t} \sum_{k=1}^{L} \frac{t^{N-k+\frac{L}{2}}}{(N-k)!}\, I_L\left(4\pi\sqrt{\frac{tN}{\lambda}}\right). \tag{3.16}$$

Changing variables to $\tau = tN$ and taking the large-N limit of all the prefactors we get:

$$\langle T\mathcal{O}_L(x)\rangle \simeq \frac{\lambda^{\frac{1}{4}}}{4\pi^{\frac{3}{2}}\sqrt{L}\,R^L} \int_0^\infty d\tau \sum_{k=1}^{L} \tau^{\frac{L}{2}-k-\frac{1}{4}}\, e^{-NS(\tau)}, \tag{3.17}$$

with $S(\tau)$ given by (3.8). Evaluating the integral in the saddle-point approximation and normalizing by (3.7) we get:

$$\langle \mathcal{O}_a(x)\rangle_T = \frac{2\pi}{\sqrt{\lambda L}\,R^L} \sum_{k=1}^{L} \tau_*^{\frac{L}{2}-k+\frac{1}{2}} = \frac{2\pi}{\sqrt{\lambda L}\,R^L}\, \frac{\tau_*^{\frac{L}{2}} - \tau_*^{-\frac{L}{2}}}{\tau_*^{\frac{1}{2}} - \tau_*^{-\frac{1}{2}}}, \tag{3.18}$$

with τ_* from (3.9). Comparing to (2.9) we read off the OPE coefficient:

$$\mathbb{C}_L = \frac{1}{\sqrt{L}}\left[\left(\sqrt{1+\frac{\pi^2}{\lambda}} + \frac{\pi}{\sqrt{\lambda}}\right)^L - \left(\sqrt{1+\frac{\pi^2}{\lambda}} - \frac{\pi}{\sqrt{\lambda}}\right)^L\right]. \tag{3.19}$$

Foreseeing an integrability description we can interpret the first term as an asymptotic answer, to be dressed by Bethe-ansatz structures for non-protected operators, while the second term is naturally interpreted as a wrapping correction [34]. The two terms are indeed separated by L orders of perturbation theory as appropriate for diagrams "wrapping the cylinder".

We can make the argument sharper by introducing the Zhukowski variable

$$x(u) + \frac{1}{x(u)} = \frac{4\pi u}{\sqrt{\lambda}}, \tag{3.20}$$

and denoting

$$ix_a \equiv x\left(\frac{ai}{2}\right). \tag{3.21}$$

Then (3.19) can be concisely written as

$$\mathbb{C}_L = \frac{1}{\sqrt{L}}\left(x_1^L - \frac{1}{x_1^L}\right). \tag{3.22}$$

Structurally, this is very similar to the one-point function in the D3-D5 dCFT [13] but is much simpler. In that latter case the asymptotic part is a sum of k terms (k is a quantum
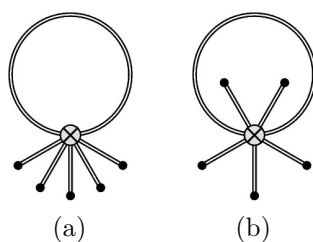
(a)   (b)

Figure 3. The one-loop correction to the one-point function. The dots denote the classical field (2.1), (2.2). The loop propagator is the Green's function in this background.

number of the defect) of the form x_a^L with a running between $1-k$ and $k-1$, and the wrapping corrections are a series in $1/x_a^L$. Each of the asymptotic terms corresponds to a particular bound state formed by a elementary magnons with the D5-brane. There is only one asymptotic term here and we expect that the string modes do not form bound states, so the scattering theory for the 't Hooft loop must be way simpler.

Expanding the one-point function at weak coupling, we find:

$$\mathbb{C}_L \simeq \frac{1}{\sqrt{L}} \left(\frac{4\pi^2}{\lambda} \right)^{\frac{L}{2}} \left(1 + \frac{g_{YM}^2(N-1)}{4\pi^2} L + \dots \right). \tag{3.23}$$

In the 't Hooft limit this formula is just Taylor series of (3.19), but written that way remains valid at any N, as follows from (3.15), (3.3) and the asymptotic form of the Laguerre polynomials:

$$L_n^s(x) = \frac{x^n}{n!} \left[1 + \frac{n(n+s)}{x} + \dots \right]. \tag{3.24}$$

The leading order is consistent with substituting the classical field (2.2) into (3.13).[3] The next order can likewise be reproduced by a quantum field theoretical calculation as we will demonstrate in section 4 by explicitly calculating the planar diagram in figure 3a and showing that the non-planar corrections in figure 3b vanish.

Planar corrections at the next L orders come entirely from the first term in (3.19). The simplicity of the answer suggests many cancellations and points to simple combinatorics of the higher-loop corrections. We can also compare the exact OPE coefficient with supergravity and string theory at strong coupling. This analysis is relegated to appendix A.

4 Quantizing 't Hooft lines

In this section we will set up the program for doing perturbative calculations in the monopole background, and as a first application we reproduce the one-loop result predicted by S-duality (3.23). Various aspects of the quantization of 't Hooft lines have been discussed in [3, 4]. The problem is similar to quantization of monopole operators in 3d [35, 36], which has been discussed in the gauge theory with sixteen supercharges (the dimensional reduction

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[3]It is important to recall that the classical field (2.2) describes an infinite 't Hooft line, figure 2a, correlator which has an extra factor of $1/2^\Delta$ compared to the circular model, see (2.6).

of $\mathcal{N} = 4$ SYM) [37]. Unfortunately we cannot use these results directly because the theory is not conformal in 3d and the mixing pattern of various field components is different as a result.

4.1 Expanding around the monopole background

We start from the $\mathcal{N} = 4$ SYM action in the following form

$$S = \frac{2}{g_{YM}^2} \int d^4x \operatorname{tr} \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} D_\mu \varphi_i D^\mu \varphi_i + \frac{1}{4} [\varphi_i, \varphi_j][\varphi_i, \varphi_j] + \text{fermions} \right], \quad (4.1)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu], \quad D_\mu \varphi_i = \partial_\mu \varphi_i - i[A_\mu, \varphi_i], \quad (4.2)$$

with $\mu, \nu = 0, 1, 2, 3$ and $i, j = 1, 2, \ldots, 6$, and we wish to expand around the classical field configuration given by (2.1), (2.2) and (2.4). Inserting

$$\varphi_i = \varphi_i^{cl} + \tilde{\varphi}_i, \quad A_\mu = A_\mu^{cl} + \tilde{A}_\mu, \quad (4.3)$$

and adding a gauge fixing term

$$S_{gf} = \frac{2}{g_{YM}^2} \int d^4x \operatorname{tr} \left[-\frac{1}{2} \left(\partial_\mu \tilde{A}^\mu + i[\tilde{A}^\mu, A_\mu^{cl}] + i[\tilde{\varphi}_i, \varphi_i^{cl}] \right)^2 \right], \quad (4.4)$$

we get for the resulting action[4]

$$
\begin{aligned}
S = \frac{2}{g_{YM}^2} \int d^4x \operatorname{tr} \Big\{ &-\frac{1}{2} \bar{D}_\nu A_\mu \bar{D}^\nu A^\mu - \frac{1}{2} \bar{D}_\nu \varphi_i \bar{D}^\nu \varphi_i \\
&+ i \bar{F}_{\mu\nu} [A^\mu, A^\nu] + 2i \, \partial_\mu \varphi_i^{cl} [A^\mu, \varphi_i] \\
&+ \frac{1}{2} [A_\mu, \varphi_i^{cl}][A^\mu, \varphi_i^{cl}] + \frac{1}{2} [\varphi_j, \varphi_i^{cl}][\varphi_j, \varphi_i^{cl}] \Big\},
\end{aligned}
\quad (4.5)
$$

where we have omitted the tildes on the quantum fields and where

$$\bar{D}_\nu = \partial_\nu - i[A_\nu^{cl}, \cdot]. \quad (4.6)$$

The symmetry breaking pattern of the classical fields implies the following block decomposition of the quantum fields

$$
A_\mu, \varphi_i =
\begin{array}{cc}
 1 N-1 \\
\begin{bmatrix}
\mathring{a} & \varnothing & \varnothing & \varnothing \\
\varnothing & \rule{0.4cm}{0.4pt} & \rule{0.4cm}{0.4pt} & \rule{0.4cm}{0.4pt} \\
\varnothing & \rule{0.4cm}{0.4pt} & \rule{0.4cm}{0.4pt} & \rule{0.4cm}{0.4pt} \\
\varnothing & \rule{0.4cm}{0.4pt} & \rule{0.4cm}{0.4pt} & \rule{0.4cm}{0.4pt}
\end{bmatrix}
\begin{array}{c}
1 \\
\\
N-1 \\
\\
\end{array}
\end{array}
\quad (4.7)
$$

Here the action for the field components of type æ as well as for the component å are unaffected by the presence of the defect whereas new quadratic terms appear for the fields of type ø.

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[4]Strictly speaking we should consider the full $\mathcal{N} = 4$ SYM and perform the gauge fixing by adding a BRST exact term as in [38, 39].

We now choose $n_i = (1, 0, 0, 0, 0, 0)$, and clearly we can set $A_0^{cl} = 0$, cf. eqs. (2.2), (2.1). Next, we can replace φ_i^{cl} with φ_1^{cl} in the expression (4.5) for the action, and we observe that the fields split into two groups, namely the simple fields $\{A_0, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6\}$ for which the quadratic terms are diagonal in flavour and the complicated fields $\{\varphi_1, A_1, A_2, A_3\}$ which mix between each other. A convenient gauge choice for the remaining components of A^{cl} [18] is[5]

$$A_\phi^{cl} = \frac{B}{2r} \frac{1 - \cos\theta}{\sin\theta}, \quad A_r^{cl} = A_\theta^{cl} = 0, \tag{4.8}$$

or expressed in Cartesian coordinates

$$A_x^{cl} = -\frac{By}{2r(r+z)}, \quad A_y^{cl} = \frac{Bx}{2r(r+z)}, \quad A_z^{cl} = 0. \tag{4.9}$$

4.2 Dirac's quantum mechanical monopole problem

Determining the propagators for the fields without flavour mixing amounts to solving Dirac's original quantum mechanical monopole problem [18]. The starting point is the quadratic term in the Lagrangian density for the ø-components of the simple fields which reads

$$\mathcal{L}_s = \varphi_{1j} \big(\partial^2 + 2i A_\nu^{cl} \partial_\nu - (A_\nu)^{cl} (A^\nu)^{cl} - (\varphi_1^{cl})^2 \big) \varphi_{j1}, \tag{4.10}$$

$$= \varphi_{j1}^\dagger \left(\partial_t^2 + (\partial^k + i A_{cl}^k)(\partial_k + i A_k^{cl}) - \frac{B}{4r^2} \right) \varphi_{j1}, \quad j \neq 1, \tag{4.11}$$

where it is understood that the classical fields are replaced by their non-vanishing 11-component and where we have introduced the notation

$$\varphi_{1j}^\dagger = (\varphi^\dagger)_{1j} = \varphi_{j1}. \tag{4.12}$$

Furthermore, the index k now runs over spatial values only. In order to determine the propagators of the simple fields we shall start by determining the spectrum of the operator

$$\hat{H} = -\left(\partial^k + i A_{cl}^k \right) \left(\partial_k + i A_k^{cl} \right) + \frac{B}{4r^2}, \tag{4.13}$$

i.e. solving the eigenvalue problem

$$\hat{H}\Psi = E\Psi. \tag{4.14}$$

Apart from the last purely radial term, this problem is identical to Dirac's original quantum mechanical monopole problem [18] whose solution we now briefly review following [40–42], see also [43]. It is convenient to work in spherical coordinates where the Hamiltonian reads

$$\hat{H} = -\left(\nabla^2 + \frac{iB}{r^2(1+\cos\theta)} \partial_\phi - \frac{B^2}{4r^2} \frac{1-\cos\theta}{1+\cos\theta} - \frac{B^2}{4r^2} \right), \tag{4.15}$$

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[5]The gauge connection is $A = B(1 - \cos\theta)d\phi/2$, whose curvature is indeed the volume form of S^2: $F = Bd\phi \wedge d\theta/2$. The factor of $r\sin\theta$ arises from transformation to the local frame of the spherical coordinate system.

with ∇^2 the 3d Laplacian. Following [41] we define

$$L_\pm = L_x \pm iL_y = e^{\pm i\phi} \left[\pm \frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi} + \frac{B}{2}\frac{\sin\theta}{1+\cos\theta} \right], \tag{4.16}$$

$$L_z = -i\frac{\partial}{\partial\phi} + \frac{B}{2}. \tag{4.17}$$

Then we have

$$\hat{H} = p_r^2 + \frac{L^2(\theta,\phi)}{r^2}, \qquad \text{where} \qquad p_r = \frac{-i}{r}\frac{\partial}{\partial r}r, \tag{4.18}$$

and

$$L^2(\theta,\phi) = -\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{iB}{1+\cos\theta}\partial_\phi - \frac{B^2}{2}\frac{1}{1+\cos\theta} \right]. \tag{4.19}$$

Now, it holds that

$$\left[\vec{L}, H\right] = 0, \tag{4.20}$$

and the operators L_i fulfill the familiar SU(2) commutation relations

$$[L_i, L_j] = i\epsilon_{ijk}L_k. \tag{4.21}$$

The solution to the eigenvalue problem is found by assuming that the wave function factorizes as follows

$$\Psi = \rho(r)\,W(\theta,\phi), \tag{4.22}$$

which implies that (4.14) becomes equivalent to the following two equations

$$\frac{\partial^2\rho}{\partial r^2} + \frac{2}{r}\frac{\partial\rho}{\partial r} - \frac{\lambda}{r^2}\rho = -E\rho, \tag{4.23}$$

$$L^2(\theta,\phi)W(\theta,\phi) = \lambda W(\theta,\phi). \tag{4.24}$$

We can immediately conclude that the angular wave functions will constitute highest weight representations of SU(2). Introducing the further factorization

$$W(\theta,\phi) = U(\theta)e^{im\phi}, \quad m = 0,\pm 1, \pm 2, \ldots, \tag{4.25}$$

where m has to be integer for the wave function to be single valued, we find

$$L_z W(\theta,\phi) = \left(m + \frac{B}{2} \right) W(\theta,\phi) \equiv \overline{m}\,W(\theta,\phi), \tag{4.26}$$

and the angular wave equation takes the form

$$-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial U}{\partial\theta} + \left(\frac{m^2}{\sin^2\theta} + \frac{mB}{1+\cos\theta} + \frac{B^2}{2}\frac{1}{1+\cos\theta} \right) U(\theta) = \lambda U(\theta). \tag{4.27}$$

From SU(2) representation theory we know that the possible values of λ are

$$\lambda = l(l+1), \tag{4.28}$$

where l is a non-negative integer, when $B = 0$. In the presence of the monopole, we encounter the phenomenon of spectral flow, i.e. the fact that the energy levels re-organize themselves as B is varied. In figure 4 we illustrate the spectral flow between $B = 0$ and $B = 2$. For the case, we are interested in, i.e. $B = 1$, we see that the possible values of l have shifted from integer values at $B = 0$ to half-integer values at $B = 1$, cf. equation (4.26). Introducing the variable

$$x = \cos\theta, \tag{4.29}$$

equation (4.27) can be written as

$$\left[(1-x^2)\frac{\partial^2}{\partial x^2} - 2x\frac{\partial}{\partial x} - \frac{1}{2(1-x)}\left(\overline{m} - \frac{B}{2}\right)^2 - \frac{1}{2(1+x)}\left(\overline{m} + \frac{B}{2}\right)^2\right] U = -\lambda U. \tag{4.30}$$

Performing the following rescaling

$$\mathrm{U}(x) = (1-x)^{\alpha/2}(1+x)^{\beta/2}\,\widetilde{U}(x), \tag{4.31}$$

with

$$\alpha = \left|\overline{m} + \frac{B}{2}\right|, \quad \beta = \left|\overline{m} - \frac{B}{2}\right|, \tag{4.32}$$

the equation (4.30) turns into the following equation for $\widetilde{U}$

$$(1-x^2)\widetilde{U}'' + (\beta - \alpha - x(2+\alpha+\beta))\widetilde{U}' + \left(l(l+1) + \frac{1}{4}(\alpha+\beta)(2+\alpha+\beta)\right)\widetilde{U} = 0, \tag{4.33}$$

where the primes denote differentiation with respect to x. We recognize equation (4.33) as the defining equation for the Jacobi polynomials, i.e.

$$\widetilde{U}(x) = \mathcal{N}P_n^{(\alpha,\beta)}(x), \tag{4.34}$$

with $\mathcal{N}$ a normalization constant, provided

$$\left(l(l+1) + \frac{1}{4}(\alpha+\beta)(2+\alpha+\beta)\right) = n(n+\alpha+\beta+1), \quad n = 0, 1, 2, \ldots, \tag{4.35}$$

i.e.

$$l = n + \frac{1}{2}(\alpha+\beta) = n + \frac{1}{2}\left|\overline{m} + \frac{B}{2}\right| + \frac{1}{2}\left|\overline{m} - \frac{B}{2}\right|, \quad n = 0, 1, 2, \ldots. \tag{4.36}$$

More specifically the eigenfunctions of our angular problem read

$$W(\theta,\phi) = Y_{l\overline{m}}^{(q)}(\theta,\phi) \tag{4.37}$$

$$= C_{l\overline{m}}(1-\cos\theta)^{\alpha/2}(1+\cos\theta)^{\beta/2}P_{l-\alpha/2-\beta/2}^{(\alpha,\beta)}(\cos\theta)\,e^{i(\overline{m}-B/2)\phi}, \tag{4.38}$$

where $C_{l\overline{m}}$ is a normalization constant and where the superscript q denotes the monopole charge

$$q = \frac{B}{2}. \tag{4.39}$$

These are the so-called monopole spherical harmonics and for the case of interest in this paper, i.e. $B = 1$,

$$l = \frac{1}{2}, \frac{3}{2}, \ldots, \quad \overline{m} = -l, -l+1, \ldots, l-1, l, \tag{4.40}$$

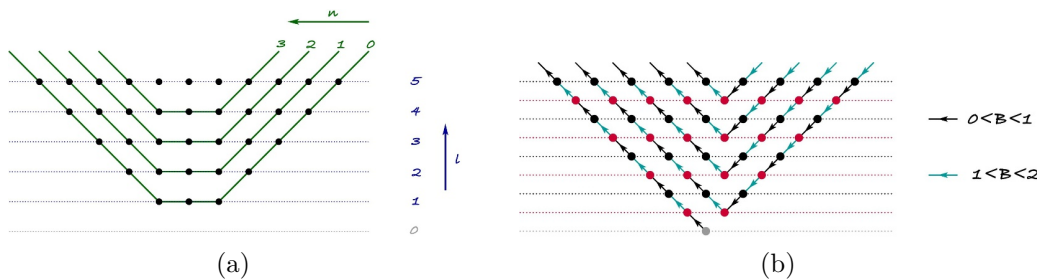exposing again the phenomenon of spectral flow.

Figure 4. (a) The spectrum, at $B = 2$. (b) The spectral flow from $B = 0$ to $B = 2$. The Dirac quantization condition corresponds to crossing (half-)integer-l lines when the spectrum recombines into the usual SU(2) multiplets. The zero angular momentum state disappears in the process.

The normalization constant can be determined explicitly and reads

$$C_{l\overline{m}} = \left(\frac{(2l+1)\left(-\frac{\alpha}{2}-\frac{\beta}{2}+l\right)!\,\Gamma\left(l+\frac{\alpha}{2}+\frac{\beta}{2}+1\right)}{2^{+\alpha+\beta+2}\,\pi\,\Gamma\left(l+\frac{\alpha}{2}-\frac{\beta}{2}+1\right)\Gamma\left(l-\frac{\alpha}{2}+\frac{\beta}{2}+1\right)} \right)^{1/2}. \tag{4.41}$$

Returning to the radial equation, we insert $\lambda = l(l+1)$ and rescale $\rho(r) = r^{-1/2}R(r)$ which leads to Bessel's equation

$$\frac{d^2 R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr} + \left(E - \frac{(l+1/2)^2}{r^2}\right)R(r) = 0. \tag{4.42}$$

Picking the solution which makes $\rho(r)$ regular at $r = 0$ gives

$$\rho(r) = (\sqrt{E}r)^{-1/2}J_\nu(\sqrt{E}r), \tag{4.43}$$

with

$$\nu = l + \frac{1}{2}. \tag{4.44}$$

Having completed the quantization of the monopole problem we can construct the propagators of the simple fields via spectral decomposition, cf. section 4.4

4.3 An $S = 1$ particle, a scalar and the monopole

Determining the propagators for the fields with flavour mixing amounts to determining the spectrum of a spin one particle interacting with a scalar in the monopole background. The problem involving the spin one particle alone was analyzed in [44, 45] (we will follow [44]). The starting point is the quadratic part of the Lagrangian density for the $\varnothing$ components of the fields with mixing, i.e. $\varphi_1 \equiv \varphi, A_1, A_2, A_3$, which reads

$$\begin{aligned}
\mathcal{L}_m = {}& \varphi_{j1}^\dagger \left(\partial_t^2 + (\partial^k + iA_{cl}^k)(\partial_k + iA_k^{cl}) - \frac{B}{4r^2} \right) \varphi_{j1} \\
& + \vec{A}_{j1}^\dagger \left(\partial_t^2 + (\nabla^k + iA_{cl}^k)(\nabla_k + iA_k^{cl}) - \frac{B}{4r^2} \right) \vec{A}_{j1} \\
& + i\varphi_{j1}^\dagger \left(\frac{B}{r^2}\,\hat{\mathbf{r}} \cdot \vec{A}_{j1} \right) - i\left(\vec{A}_{j1}^\dagger \cdot \frac{B}{r^2}\,\hat{\mathbf{r}} \right) \varphi_{j1} \\
& - i\vec{A}_{j1}^\dagger \cdot \left(\frac{B}{r^2}\,\hat{\mathbf{r}} \times \vec{A}_{j1} \right), j \neq 1,
\end{aligned} \tag{4.45}$$

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where we have underlined the fact that the A-field constitutes a vector field. As for the simple fields we will start by diagonalizing the time-independent part of the problem. From section 4.2 we already know how to diagonalize the term in the first line above making use of monopole spherical harmonics. In a similar manner we can diagonalize the second term making use of monopole vector spherical harmonics. The angular momentum operator components, L_x, L_y, L_z can be defined as in section 4.2 and the vector nature of the gauge field is taken into account by using a spin-one basis for the coordinate functions. Effectively, we are thus searching for eigenfunctions corresponding to the total angular momentum $J = L + S$ with the spin eigenvalue equal to one. We have already seen that in the monopole background with $B = 1$ the possible values of the angular momentum is $\ell = \frac{1}{2}, \frac{3}{2}, \ldots$ which means that we have the following possibilities for the total angular momentum, J

$$\text{For } \ell \geqslant \frac{3}{2} : J = \ell - 1, \ell, \ell + 1, \tag{4.46}$$

$$\text{For } \ell = \frac{1}{2} : J = \ell, \ell + 1. \tag{4.47}$$

Combining the monopole spherical harmonics with $S = 1$ representation matrices one can construct monopole vector spherical harmonics [44] which fulfill

$$(L + S)^2 \, \mathbf{Y}_{J\ell M}^{(q)}(\theta, \phi) = J(J+1) \mathbf{Y}_{J\ell M}^{(q)}(\theta, \phi), \tag{4.48}$$

$$(L_z + S_z) \, \mathbf{Y}_{J\ell M}^{(q)}(\theta, \phi) = M \, \mathbf{Y}_{J\ell M}^{(q)}(\theta, \phi), \tag{4.49}$$

$$L^2 \mathbf{Y}_{J\ell M}^{(q)}(\theta, \phi) = \ell(\ell + 1) \mathbf{Y}_{J\ell M}^{(q)}(\theta, \phi), \tag{4.50}$$

$$S^2 \mathbf{Y}_{J\ell M}^{(q)}(\theta, \phi) = 2 \, \mathbf{Y}_{J\ell M}^{(q)}(\theta, \phi). \tag{4.51}$$

These vector spherical harmonics have the following additional properties which will prove convenient in the following

$$\hat{\mathbf{r}} \cdot \mathbf{Y}_{J\ell M}^{(q)}(\theta, \phi) = \mathcal{C}_{J\ell}^{(q)} \, Y_{JM}^{(q)}(\theta, \phi), \tag{4.52}$$

$$\hat{\mathbf{r}} \times \mathbf{Y}_{J\ell M}^{(q)}(\theta, \phi) = i \sum_L \mathcal{A}_{J\ell L}^{(q)} \, \mathbf{Y}_{JLM}^{(q)}(\theta, \phi), \tag{4.53}$$

where the constants $\mathcal{A}_{J\ell L}^{(q)}$ and $\mathcal{C}_{J\ell}^{(q)}$ are known in closed form. We have listed these in appendix B. In addition, one has

$$\mathbf{r} \, Y_{JM}^{(q)} = \sum_\ell C_{J\ell}^{(q)} \, \mathbf{Y}_{J\ell M}^{(q)}(\theta, \phi). \tag{4.54}$$

We thus expand the scalar field in the basis of monopole spherical harmonics and the vector field in the basis of monopole vector spherical harmonics in the following way (leaving out the subscripts on the fields)

$$\varphi(r, \theta, \phi) = \sum_{J = \frac{1}{2}, \frac{3}{2}, \ldots} \sum_{M=-J}^{J} r^{-1} K^J(r) \, Y_{JM}^{(q)}(\theta, \phi), \tag{4.55}$$

$$\vec{A}(r, \theta, \phi) = \sum_{J = \frac{1}{2}, \frac{3}{2}, \ldots} \sum_{M=-J}^{J} \vec{A}_{JM}(r, \theta, \phi), \tag{4.56}$$

where

$$r\vec{A}_{JM}(r,\theta,\phi) = \tag{4.57}$$
$$F_-^J(r)\mathbf{Y}_{JJ-1M}^{(q)}(\theta,\phi) + F_0^J(r)\mathbf{Y}_{JJM}^{(q)}(\theta,\phi) + F_+^J(r)\mathbf{Y}_{JJ+1M}^{(q)}(\theta,\phi),$$

with the understanding that $F_-^{\frac{1}{2}}(r) \equiv 0$. Working in this basis the spectral problem that we need to solve reads

$$\hat{H}\begin{pmatrix}\varphi \\ \vec{A}\end{pmatrix} = \frac{1}{r^2}\begin{pmatrix} r^2 p_r^2 + \mathbf{L}^2 & -iB\,\hat{\mathbf{r}}^{\mathbf{T}} \\ iB\hat{\mathbf{r}} & r^2 p_r^2 + \mathbf{L}^2 - iB\,\hat{\mathbf{r}}\times \end{pmatrix}\begin{pmatrix}\varphi \\ \vec{A}\end{pmatrix} = E\begin{pmatrix}\varphi \\ \vec{A}\end{pmatrix}. \tag{4.58}$$

We will follow the strategy of [44] where the part of the problem involving only the gauge field was solved. Inserting the expansions of φ and $\vec{A}$ and making use of the relations (4.50), (4.52), and (4.53) one finds for each value of J a set of four coupled differential equations for the radial functions $F_-^J(r)$, $F_0^J(r)$, $F_+^J(r)$ and $K^J(r)$. The first of these equations reads

$$\left(z^2\frac{d^2}{dz^2} - z^2 - J(J+1)\right)K(z) \tag{4.59}$$
$$+ iB\left(\mathcal{C}_{JJ-1}^{(q)}F_-^J(z) + \mathcal{C}_{JJ}^{(q)}F_0^J(z) + \mathcal{C}_{JJ+1}^{(q)}F_+^J(z)\right) = 0,$$

where we have defined

$$z = \sqrt{E}\,r. \tag{4.60}$$

The other three equations are analogous. We now notice that the function $\sqrt{z}J_\nu(z)$ fulfils the equation

$$\left(z^2\frac{d^2}{dz^2} - z^2 - \left(\nu^2 - \frac{1}{4}\right)\right)\sqrt{z}J_\nu(z) = 0. \tag{4.61}$$

It thus makes sense to search for a solution to the set of four coupled differential equations where all the four unknown functions are proportional to $\sqrt{z}J_\nu(z)$ for some ν. Writing

$$K^J(z) = K^J\sqrt{z}J_\nu(z),$$

$$F_-^J(z) = F_-^J\sqrt{z}J_\nu(z), \quad F_0^J(z) = F_0^J\sqrt{z}J_\nu(z), \quad F_+^J(z) = F_+^J\sqrt{z}J_\nu(z),$$

where K^J, F_-^J, F_0^J and F_+^J are constants we get a set of four linear equations for these four unknown constants, which we can write as

$$\begin{bmatrix} a & -i\,\mathcal{C}_{J-1} & -i\,\mathcal{C}_J & -i\,\mathcal{C}_{J+1} \\ i\,\mathcal{C}_{J-1} & a + 2J + \mathcal{A}_{J-1J-1} & \mathcal{A}_{JJ-1} & 0 \\ i\mathcal{C}_J & \mathcal{A}_{J-1J} & a + \mathcal{A}_{JJ} & \mathcal{A}_{J+1J} \\ iC_{J+1} & 0 & \mathcal{A}_{JJ+1} & a - 2(J+1) + \mathcal{A}_{J+1J+1} \end{bmatrix}\begin{bmatrix} K \\ F_- \\ F_0 \\ F_+ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

where for readability we have suppressed the superscript q as well as the first indices on the $\mathcal{C}$'s and $\mathcal{A}$'s which obviously all take the value J, and likewise for the superscripts on the constants. Furthermore, we have introduced

$$a = \nu^2 - \left(J + \frac{1}{2}\right)^2, \tag{4.62}$$

and we have set $B = 1$. The possible values of ν for a given J are thus given by the values of a for which the determinant of the above matrix vanishes (i.e. by minus the eigenvalues of the matrix), which are

$$a = \{0, 0, -2J, 2(J+1)\}, \quad J \geqslant 3/2, \tag{4.63}$$

corresponding to

$$\nu = \left\{ J - \frac{1}{2}, J + \frac{1}{2}, J + \frac{1}{2}, J + \frac{3}{2} \right\} \equiv \{\nu_-^J, \nu_0^J, \nu_0^J, \nu_+^J\}, \quad J \geqslant 3/2. \tag{4.64}$$

It is interesting to note the difference between this result and the result of [44]. Of the three values of ν, which one gets in the situation where one considers only the gauge field, one value equals ν_0^J but the other two are irrational. The coupling to the scalar field, originating from the supersymmetry of $\mathcal{N} = 4$ SYM, clearly renders the problem analytically more tractable, manifesting the presence of integrability. We can choose a set of orthonormal eigenvectors of the above matrix as follows

$$V_-^J = \left(\frac{-i}{2\sqrt{J(2J+1)}}, \frac{\sqrt{(2J+1)(2J-1)}}{2J}, \frac{1}{2J}\sqrt{\frac{J+1}{2J+1}}, 0 \right), \tag{4.65}$$

$$V_0^J = \left(-\frac{i}{2}\sqrt{\frac{2J-1}{J}}, -\frac{1}{2J}, \frac{\sqrt{(2J-1)(J+1)}}{2J}, 0 \right),$$

$$\widehat{V}_0^J = \left(\frac{i}{2}\sqrt{\frac{2J+3}{J+1}}, 0, \frac{\sqrt{J(2J+3)}}{2J+2}, \frac{1}{2J+2} \right),$$

$$V_+^J = \left(\frac{-i}{2\sqrt{(J+1)(2J+1)}}, 0, \frac{-J}{2(1+J)\sqrt{J(1+2J)}}, \frac{\sqrt{(2J+1)(2J+3)}}{2(1+J)} \right).$$

For the special case of $J = 1/2$, where $F_-^{\frac{1}{2}}(r) \equiv 0$, we find

$$\nu = \{0, 1, 2\} \equiv \{\nu_-^{\frac{1}{2}}, \nu_0^{\frac{1}{2}}, \nu_+^{\frac{1}{2}}\}, \quad J = 1/2, \tag{4.66}$$

with the corresponding eigenvectors

$$V_-^{\frac{1}{2}} = \left(-\frac{i}{2}, \sqrt{\frac{3}{4}}, 0 \right), \quad V_0^{\frac{1}{2}} = \left(i\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, \frac{1}{3} \right),$$

$$V_+^{\frac{1}{2}} = \left(\frac{-i}{2\sqrt{3}}, -\frac{1}{6}, \frac{2\sqrt{2}}{3}, \right), \tag{4.67}$$

consistently with the above. These eigenvectors provide us with a basis for the space of solutions of the quantum mechanical problem (4.58) and we can determine the propagators of the fields with flavour mixing. In the following we will explicitly determine only the propagators for the scalars, but it is not difficult to reconstruct the vector propagator as well. The fermion propagators are naturally expanded in the spinor monopole harmonics [46].

4.4 The propagators for the scalars

The equation defining the propagators of the simple scalars, cf. eq. (4.11) reads (assuming Euclidean signature)

$$\left(-\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}L^2(\theta,\phi)\right)G(\mathbf{x},\mathbf{x}') =$$

$$\frac{g_{\text{YM}}^2}{2}\frac{\delta(t-t')\delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi')}{r^2\sin\theta}, \tag{4.68}$$

where $L^2(\theta,\phi)$ was defined in (4.19) and where $\mathbf{x} = (t,r,\theta,\phi)$. We now make the following ansatz for $G(\mathbf{x},\mathbf{x}')$

$$G(\mathbf{x},\mathbf{x}') = \frac{g_{\text{YM}}^2}{2}\sum_{l,\bar{m}}(r\,r')^{-1}G_l(r,r',t,t')\,Y_{l,\bar{m}}^{(q)}(\theta,\phi)Y_{l,\bar{m}}^{(q)*}(\theta',\phi'), \tag{4.69}$$

and find that $G_l(r,r',t,t')$ has to fulfill the following equation

$$\left(-\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2} + \frac{l(l+1)}{r^2}\right)G_l(r,r',t,t') = \delta(r-r')\delta(t-t'), \tag{4.70}$$

which we recognize as the propagator of a scalar in AdS$_2$ with mass parameter $m^2 = l(l+1)$. The appearance of an AdS$_2$ propagator was to be expected given the symmetries of the problem. We elaborate on this connection to AdS$_2$ in appendix C. A convenient way to write the scalar propagator with mass parameter m^2 for AdS$_{d+1}$ is exactly by means of its spectral decomposition [47]

$$G_\nu(r,r',\vec{x},\vec{y}) = (rr')^{d/2}\int\frac{d^dk}{(2\pi)^d}\int_0^\infty d\omega\frac{\omega}{\omega^2+\vec{k}^2}e^{i\vec{k}(\vec{x}-\vec{y})}J_\nu(\omega r)J_\nu(\omega r')$$

$$= (rr')^{d/2}\int\frac{d^dk}{(2\pi)^d}\,e^{i\vec{k}(\vec{x}-\vec{y})}I_\nu(|\vec{k}|r^<)K_\nu(|\vec{k}|r^>), \tag{4.71}$$

where I_ν and K_ν are modified Bessel functions and where

$$\nu = \sqrt{m^2 + \frac{1}{4}}. \tag{4.72}$$

For the scalar which mixes with the gauge field the relevant mode expansion is not in terms of orbital but in terms of total angular momentum, J. To get the propagator for this field we write a spectral decomposition like (4.71) for the propagator of the combined field $(\varphi_1, \vec{A})$ making use of the complete basis of eigenfunctions determined earlier and identify the desired propagator as the 11-component. This leads to an expression similar to (4.69) except that the single term G_l gets replaced with the sum

$$|(V_-^J)_1|^2 G_{\nu_-^J} + \left(|(V_0^J)_1|^2 + |(\widehat{V}_0^J)_1|^2\right)G_{\nu_0^J} + |(V_+^J)_1|^2 G_{\nu_+^J},$$

where $(V_-^J)_1$ is the first component of the eigenvector V_-^J etc., cf. eqs. (4.65)–(4.67), and where it is understood that $|(\widehat{V}_0^{\frac{1}{2}})_1| = 0$. We notice that the lowest mode which appears is $\nu_-^{1/2}$ which exactly saturates the BF-bound [48].

For the computation of one-loop corrections to one-point functions we will need to evaluate propagators with coinciding endpoints and we need a regularization procedure. For the AdS$_2$ part of the propagator we can use the method successfully employed in [38, 39], i.e. dimensional regularization. We regularize the propagator for coinciding points by going to polar coordinates in $d = 1 - 2\epsilon$ dimensions and obtain

$$G_\nu(r, r, t, t) = \frac{g_{\text{YM}}^2}{2} r \frac{2\pi^{1/2-\epsilon}}{\Gamma(1/2-\epsilon)} \int_0^\infty dk \frac{k^{-2\epsilon}}{(2\pi)^{1-2\epsilon}} I_\nu(kr) K_\nu(kr). \qquad (4.73)$$

Expanding to two leading orders in ϵ we get

$$G_\nu(r, r, t, t) = \frac{g_{\text{YM}}^2}{2} \frac{1}{4\pi} \left(\frac{1}{\epsilon} - \gamma + \log(4\pi) + 2\log(r) - 2\Psi(\nu + \frac{1}{2}) \right). \qquad (4.74)$$

Alternatively one can use propagators in AdS_2 and regularize by point-splitting. This gives the same result, see appendix C.

4.5 The correction to the one-point function

In order to compute the one-loop correction to the one-point function we start from the properly normalized operator (3.13), insert the complex scalar Z and expand around the classical field, ie.

$$Z = \Phi_1 + \varphi_1 + i\varphi_2, \qquad (4.75)$$

with Φ_1 given in eq. (2.2) and φ_1 and φ_2 the quantum fields. To one-loop order we only get contributions from configurations in the form of tadpoles where two quantum fields occupying neighboring sites get connected by a propagator, figure 3a. The fields propagating in the loop are from the $\mathring{a}$ and $\varnothing$ blocks in (4.7) and have color indices φ_a^{1j}, but the contribution of the 11 component is unaffected by the monopole and vanishes by chirality. The non-planar diagrams 3b are not only large-N suppressed, but identically vanish, because the fields propagating in the loop are φ_a^{11}.

In principle one could also get contributions from diagrams, denoted as "lollipop" diagrams in [38, 39], where a single quantum field gets connected by a propagator to a closed loop. However, the contributions from such diagrams are expected to vanish due to supersymmetry provided a supersymmetric regulator like dimensional regularization in combination with dimensional reduction is applied. This was demonstrated by explicit calculation for the supersymmetric D3-D5 probe brane set-up in [38, 39]. Accordingly, we shall therefore ignore these diagrams.

There are L tadpole configurations and $N - 1$ field components with a non-trivial coupling to Φ_1. This implies

$$\langle \mathcal{O}_L(\mathbf{x}) \rangle_T = \frac{1}{\sqrt{L}} \left(\frac{4\pi^2}{\lambda} \right)^{L/2} \left\{ \frac{1}{(2r)^L} + \frac{(N-1)L}{(2r)^{L-2}} \left[\langle \varphi_1(x)\varphi_1(x) \rangle - \langle \varphi_2(x)\varphi_2(x) \rangle \right] \right\}. \qquad (4.76)$$

Making use of the identity

$$\sum_M Y_{JM}^{(q)}(\theta, \phi) Y_{JM}^{(q)*}(\theta, \phi) = \frac{(2J+1)}{4\pi}, \qquad (4.77)$$

we can write the propagator for the simple field at coinciding points as (cf. eq. (4.69))

$$\langle \varphi_2(x)\varphi_2(x)\rangle = \frac{g_{\text{YM}}^2}{2}\frac{1}{r^2}\sum_{J=\frac{1}{2},\frac{3}{2},\ldots}\frac{2J+1}{4\pi}\,G_{\nu_0^J}(x,x). \tag{4.78}$$

For the field φ_1 which mixes with the gauge field we get a similar expression except that the single term $G_{\nu_0^J}(x,x)$ gets replaced with the sum

$$|(V_-^J)_1|^2 G_{\nu_-^J}(x,x) + \left(|(V_0^J)_1|^2 + |(\widehat{V}_0^J)_1|^2\right)G_{\nu_0^J}(x,x) + |(V_+^J)_1|^2 G_{\nu_+^J}(x,x).$$

Noting that the following relation is fulfilled

$$|(V_-^J)_1|^2 + |(V_0^J)_1|^2 + |(\widehat{V}_0^J)_1|^2 + |(V_+^J)_1|^2 = 1, \tag{4.79}$$

we see that all terms containing a pole in ϵ cancel out and we are left with the following expression for the one-loop correction to the one-point function

$$\langle \mathcal{O}_L(\mathbf{x})\rangle_T^{1-loop} = \frac{1}{\sqrt{L}}\left(\frac{4\pi^2}{\lambda}\right)^{L/2}\Bigg\{\frac{1}{(2r)^L}\frac{g_{YM}^2(N-1)L}{4\pi^2}\times$$
$$\sum_J (2J+1)\left[\left(1-|(V_0^J)_1|^2 - |(\widehat{V}_0^J)_1|^2\right)\Psi\left(\nu_0^J + \frac{1}{2}\right)\right.$$
$$\left.-|(V_-^J)_1|^2\,\Psi\left(\nu_- + \frac{1}{2}\right) - |(V_+^J)_1|^2\,\Psi\left(\nu_+ + \frac{1}{2}\right)\right]\Bigg\}. \tag{4.80}$$

Furthermore, upon using the identity

$$\Psi(z+1) = \Psi(z) + \frac{1}{z}, \tag{4.81}$$

as well as the exact expressions for the relevant components of the eigenvectors, the sum telescopes and we obtain:

$$\langle \mathcal{O}_L(\mathbf{x})\rangle_T^{1-loop} = \frac{1}{\sqrt{L}}\left(\frac{4\pi^2}{\lambda}\right)^{L/2}\frac{1}{(2r)^L}\left\{\frac{g_{YM}^2(N-1)L}{4\pi^2}\sum_{J=\frac{1}{2},\frac{3}{2},\ldots}\frac{1}{4}\left(\frac{1}{J^2}-\frac{1}{(J+1)^2}\right)\right\}$$
$$= \frac{1}{\sqrt{L}}\left(\frac{4\pi^2}{\lambda}\right)^{L/2}\frac{1}{(2r)^L}\left(\frac{g_{YM}^2(N-1)L}{4\pi^2}\right), \tag{4.82}$$

exactly as predicted by (3.23).

5 Unprotected operators and integrability

As we discussed in the introduction the straight 't Hooft line is expected to preserve integrability of $\mathcal{N}=4$ SYM. The evidence is based on classical string theory where the 't Hooft line is described by a D1-brane in the bulk of $AdS_5 \times S^5$ that features integrability-preserving boundary conditions [7]. Here we will study integrability from the weak-coupling perspective by probing the 't Hooft line with local operators. Single-trace operators can be

viewed as states in an integrable quantum spin chain, while a D-brane (the 't Hooft line in this case) maps to a boundary state whose overlaps with the Bethe eigenstates determine the one-point correlation functions [16, 17]. The boundary state may or may not be integrable. If it is, the one-point functions admit a compact determinant representation [16, 17] that potentially can be extended to all operators and all loop orders by a version of integrability bootstrap [13–15]. Our goal is to establish perturbative integrability of 't Hooft loops. We will limit ourselves to the leading perturbative order and to specific sectors where the spin chain takes a particularly simple form.

5.1 SO(6) sector

The SO(6) sector consists of generic scalar operators

$$\mathcal{O} = \Psi^{I_1\ldots I_L} \operatorname{tr} \Phi_{I_1} \ldots \Phi_{I_L}, \tag{5.1}$$

which at one loop only mix among themselves. The mixing matrix acting on the wavefunction $\Psi^{I_1\ldots I_L}$ coincides with the Hamiltonian of an integrable SO(6) spin chain [49]:

$$\Gamma = \frac{\lambda}{16\pi^2} \sum_{\ell=1}^{L} \left(2 - 2P_{\ell\ell+1} + K_{\ell\ell+1}\right), \tag{5.2}$$

where P and K are permutation and trace operators acting on the neighboring sites: $P_{JL}^{IK} = \delta_L^I \delta_J^K$, $K_{JL}^{IK} = \delta^{IK}\delta_{JL}$.

The model is solvable by Bethe ansatz with the standard Lie-algebraic Bethe equations:

$$\left(\frac{u_{aj} - \frac{iq_a}{2}}{u_{aj} + \frac{iq_a}{2}}\right)^L \prod_{bk} \frac{u_{aj} - u_{bk} + \frac{iM_{ab}}{2}}{u_{aj} - u_{bk} - \frac{iM_{ab}}{2}} = -1, \tag{5.3}$$

where M_{ab} is the Cartan matrix of SO(6) and q_a are the Dynkin labels of its vector representation:

$$M = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \qquad q = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \tag{5.4}$$

The one-loop anomalous dimension is given by

$$\Delta = L + \frac{\lambda}{8\pi^2} \sum_{aj} \frac{q_{aj}^2}{u_{aj}^2 + \frac{q_{aj}^2}{4}}, \tag{5.5}$$

for a state with K_a Bethe roots and quantum numbers (the Dynkin labels of the SO(6) representation):

$$\mu_a = Lq_a - \sum_b M_{ab} K_b. \tag{5.6}$$

Computing the correlator of (5.1) with the 't Hooft loop at the tree level is straightforward, each field is replaced by its classical expectation value (2.2). The spin-chain wavefunction gets projected on a fixed external state as a result:

$$\operatorname{Bst}_{I_1\ldots I_L} = n_{I_1} \ldots n_{I_L}. \tag{5.7}$$

The one-point function is the overlap with this state:

$$\langle \mathcal{O}(x) \rangle_T = \left(\frac{2\pi^2}{\lambda r^2} \right)^{\frac{L}{2}} L^{-\frac{1}{2}} \frac{\langle \mathrm{Bst} \,|\, \Psi \rangle}{\langle \Psi \,|\, \Psi \rangle^{\frac{1}{2}}} \,. \tag{5.8}$$

An extra λ-dependent prefactor arises because of the difference between the spin-chain norm and (2.7). Propagators in the two-point function add a factor of $\lambda/8\pi^2$ each and an overall cyclicity gives rise to a factor of L.

The boundary state (5.7) is known to be integrable [50]. Indeed, the boundary state picks up a single component of the wavefunction, say $\Psi^{1\cdots 1}$ for $n = (1, \mathbf{0})$. Such a projection can be shown to commute with all odd charges of the integrable hierarchy [50], which is how integrability is defined for boundary states [51, 52].

According to that definition non-zero overlaps are allowed only for parity-invariant configurations of Bethe roots:

$$\{u_{aj}\} = \{-u_{aj}\} \,. \tag{5.9}$$

Allowed configurations may either contain pairs of roots $(u_{aj}, -u_{aj})$ or in addition include a solitary root at zero.

For states satisfying the selection rule the overlap admits an elegant determinant representation [50]:

$$\langle \mathcal{O}(x) \rangle_T = \left(\frac{\pi}{\sqrt{\lambda}\, r} \right)^L \sqrt{ \frac{1}{L} \, \frac{\prod_j u_{2j}^2 \left(u_{2j}^2 + \frac{1}{4} \right)}{\prod_j u_{1j}^2 \left(u_{1j}^2 + \frac{1}{4} \right) \prod_j u_{3j}^2 \left(u_{3j}^2 + \frac{1}{4} \right)} \, \frac{\det G^+}{\det G^-} } \,, \tag{5.10}$$

where the products are over half of the roots (picking one root in each pair), and the Gaudin factors are $K/2 \times K/2$ determinants:

$$G^{\pm}_{aj,bk} = \left(\frac{L q_a}{u_{aj}^2 + \frac{1}{4}} - \sum_{cl} K^+_{aj,cl} \right) \delta_{ab} \delta_{jk} + K^{\pm}_{aj,bk},$$

$$K^{\pm}_{aj,bk} = \frac{M_{ab}}{(u_{aj} - u_{bk})^2 + \frac{M_{ab}^2}{4}} \pm \frac{M_{ab}}{(u_{aj} + u_{bk})^2 + \frac{M_{ab}^2}{4}} \,, \tag{5.11}$$

with indices labelling positive roots again. For configurations containing zero roots or singular roots at $\pm i/2$ the determinant formula needs to be slightly modified. The general expression that accommodates these special cases can be found in [53].

5.2 SL(2) sector

The SL(2) sector consists of operators

$$\mathcal{O} = \sum_{\{n_l\}} \frac{1}{n_1! \ldots n_L!} \, \Psi_{n_1 \ldots n_L} \, \mathrm{tr} \, D^{n_1} Z \ldots D^{n_L} Z, \qquad D \equiv D_0 + D_3. \tag{5.12}$$

The states at each site are conventionally labelled as

$$|n\rangle \;\longleftrightarrow\; \frac{1}{n!} D^n Z. \tag{5.13}$$

They form an infinite-dimensional unitary representation of $\mathfrak{sl}(2)$:

$$D \left| n \right\rangle = (n + 1) \left| n + 1 \right\rangle,$$
$$K \left| n \right\rangle = n \left| n - 1 \right\rangle,$$
$$S \left| n \right\rangle = \left(n + \frac{1}{2} \right) \left| n \right\rangle, \tag{5.14}$$

where $K = D^\dagger$ generates special conformal transformations and S is the spin.

Operator mixing in the $\mathfrak{sl}(2)$ sector is described by the spin-chain Hamiltonian

$$\Gamma = \frac{\lambda}{8\pi^2} \sum_{\ell=1}^{L} h_{\ell,\ell+1}, \tag{5.15}$$

with [54]

$$h \left| n, m \right\rangle = \sum_k \left[(H(n) + H(m)) \delta_{kn} - \frac{1 - \delta_{kn}}{|k - n|} \right] \left| k, n + m - k \right\rangle, \tag{5.16}$$

where $H(n)$ is the harmonic number:

$$H(n) = \sum_{s=1}^{n} \frac{1}{s}. \tag{5.17}$$

The spectral equations are

$$\left(\frac{u_j - \frac{i}{2}}{u_j + \frac{i}{2}} \right)^L \prod_k \frac{u_j - u_k - i}{u_j - u_k + i} = -1, \qquad E = \frac{\lambda}{8\pi^2} \sum_j \frac{1}{u_j^2 + \frac{1}{4}}. \tag{5.18}$$

To find the one-point function we just need to substitute the classical expectation value (2.2) for Z. The derivative D acts on Z as ∂_3, and we can use the identity

$$\partial_3^n \frac{1}{r} = \frac{(-1)^n n!}{r^{n+1}} P_n(\cos\theta) \tag{5.19}$$

to express the result of differentiation on each site. Here θ is the angle between the radius vector and the x_3 direction, and P_n are Legendre polynomials. The one-point function can then be written in the overlap form:

$$\langle \mathcal{O}(x) \rangle_T = \left(\frac{2\pi}{\sqrt{\lambda}} \right)^L \frac{L^{-\frac{1}{2}}}{r^{L+S}} \frac{\langle \mathrm{Bsl} | \Psi \rangle}{\langle \Psi | \Psi \rangle^{\frac{1}{2}}}, \tag{5.20}$$

where $S = n_1 + \ldots + n_L$ and

$$\langle \mathrm{Bsl} | = \langle B | \otimes \ldots \otimes \langle B |, \tag{5.21}$$

with

$$\langle B | = \sum_{n=0}^{\infty} (-1)^n P_n(\cos\theta) \langle n | = \langle 0 | \sum_{n=0}^{\infty} \frac{(-K)^n}{n!} P_n(\cos\theta). \tag{5.22}$$

We posit that this is an integrable boundary state.

In fact, at $\theta = 0$ this state is just a vacuum descendant. For $\theta = 0$ we have $P_n(1) = 1$ and

$$\langle B |\Big|_{\theta=0} = \langle 0 |\, \mathrm{e}^{-K}. \tag{5.23}$$

The additivity of the exponential then implies

$$\langle \mathrm{Bsl} |\Big|_{\theta=0} = \langle 0 |\, \mathrm{e}^{-K_{\mathrm{tot}}}, \tag{5.24}$$

where

$$K_{\mathrm{tot}} = \sum_{l=1}^{L} K_l. \tag{5.25}$$

And $K_{\mathrm{tot}} = D_{\mathrm{tot}}^{\dagger}$.

The special conformal generator annihilates any primary state:

$$K_{\mathrm{tot}} | \Psi_{\mathrm{primary}} \rangle = 0. \tag{5.26}$$

Hence, the one-point function of any conformal primary vanishes on the positive x_3 semi-axis:

$$\langle O_{\mathrm{primary}}(x) \rangle_T \Big|_{\theta=0} = 0, \tag{5.27}$$

the only exception being the spin-chain vacuum $\mathrm{tr}\, Z^L$.

The one-point function in general position is not protected by symmetries and in general does not vanish. We find that it is described by the following determinant formula:

$$\langle \mathcal{O}(x) \rangle_T = \left(\frac{2\pi}{\sqrt{\lambda}\, r} \right)^L \left(\frac{\sin\theta}{r} \right)^S \sqrt{ \frac{1}{L} \, \frac{Q(0)}{Q\left(\frac{i}{2}\right)} \, \frac{\det G^+}{\det G^-} } \,, \tag{5.28}$$

which applies to any state with paired roots:[6] $\{u_j, -u_j\}_{j=1\ldots\frac{S}{2}}$. The Q-functions are defined as

$$Q(x) = \prod_{j=1}^{\frac{S}{2}} \left(x^2 - u_j^2 \right), \tag{5.29}$$

and the Gaudin factors are given by

$$G_{jk}^{\pm} = \delta_{jk} \left(\frac{L}{u_j^2 + \frac{1}{4}} - \sum_s K_{js}^+ \right) + K_{jk}^{\pm},$$
$$K_{jk}^{\pm} = -\frac{2}{(u_j - u_k)^2 + 1} \mp \frac{2}{(u_j + u_k)^2 + 1} \,. \tag{5.30}$$

We were lead to this formula by taking inspiration from general results on integrable overlaps and thoroughly checked the formula on a few dozens of eigenstates. In addition, we checked that the overlap vanishes for unpaired configurations of Bethe roots. Explicit formulas of

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[6] A root at zero is also allowed and can be included by appropriately modifying the Gaudin factors [55]. For unpaired states the one-point function must vanish.

this type for the $\mathfrak{sl}(2)$ spin chain, but for different types of boundary states, have been presented in [56–58].

The most powerful approach to first-principles derivation of overlap formulas is based on the algebraic Bethe ansatz [14, 59–61] and is applicable to spin chains with an infinite-dimensional Hilbert space, such as the $\mathfrak{sl}(2)$ spin chain at hand. We were informed by Tamas Gombor that the boundary state (5.21), (5.22) falls into the general category of integrable boundary states considered in [60] and that the overlap formula (5.28) can be derived by the algebraic methods presented there.[7]

For illustration of the overlap representation of the one-point function, consider the two-magnon states [62]:

$$|\Psi\rangle = \left\{ \sum_{\ell < \ell'} \left[\mathrm{e}^{ip(\ell-\ell')} + \mathrm{e}^{ip(\ell'-\ell)+i\delta} \right] D(\ell)D(\ell') + \cos\frac{\delta}{2}\, \mathrm{e}^{\frac{i\delta}{2}} \sum_{\ell} D(\ell)^2 \right\} |0\rangle . \tag{5.31}$$

These are Bethe eigenstates with two Bethe roots $\{u, -u\}$, where

$$\mathrm{e}^{ip} = \frac{u + \frac{i}{2}}{u - \frac{i}{2}} = \mathrm{e}^{i\delta}. \tag{5.32}$$

The following sums appear in the overlap and the norm:

$$\sum_{\ell < \ell'} \left[\mathrm{e}^{ip(\ell-\ell')} + \mathrm{e}^{ip(\ell'-\ell)+i\delta} \right] = -L\, \mathrm{e}^{\frac{i\delta}{2}} \left(\cos\frac{\delta}{2} + \cot\frac{p}{2} \sin\frac{\delta}{2} \right),$$

$$\sum_{\ell < \ell'} \left| \mathrm{e}^{ip(\ell-\ell')} + \mathrm{e}^{ip(\ell'-\ell)+i\delta} \right|^2 = L\left(L - 1 - \cos\delta - \cot p\, \sin\delta \right), \tag{5.33}$$

where the Bethe equation $\mathrm{e}^{ipL+i\delta} = 1$ has been used to simplify the expressions. For the norm and the overlap we get:

$$\langle \Psi | \Psi \rangle = L(L+1),$$

$$\langle \mathrm{Bsl} | \Psi \rangle = -\frac{L}{r^2} \cos\frac{p}{2}\, \mathrm{e}^{\frac{i\delta}{2}} \sin^2\theta. \tag{5.34}$$

Here we also used (5.32). Dropping a phase we then get for the one-point function in (5.20):

$$\langle \mathcal{O}(x) \rangle_T = \left(\frac{2\pi}{\sqrt{\lambda}\, r} \right)^L \left(\frac{\sin\theta}{r} \right)^2 \frac{u}{\sqrt{(L+1)\left(u^2 + \frac{1}{4} \right)}}. \tag{5.35}$$

For the state with two roots the Gaudin factors are just numbers:

$$G^+ = \frac{L}{u^2 + \frac{1}{4}}, \qquad G^- = \frac{L+1}{u^2 + \frac{1}{4}}, \tag{5.36}$$

and the determinant representation (5.28) recovers the above result.

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[7]T. Gombor, private communication.

5.3 Gluon subsector

The field tensor $F_{\mu\nu}$ contains two representations of the Lorentz group. The irreducible components are the self-dual and anti-self-dual projections which can be singled out by contracting with the 't Hooft symbols [63]:

$$\eta^{\pm}_{i\mu\nu} = \varepsilon_{0i\mu\nu} \pm \delta_{0\mu}\delta_{i\nu} \mp \delta_{0\nu}\delta_{i\mu}. \tag{5.37}$$

The 't Hooft symbols satisfy:

$$\eta^{\pm}_{i\mu\nu} = \pm\frac{1}{2}\varepsilon_{\mu\nu\lambda\rho}\eta^{\pm}_{i\lambda\rho}, \tag{5.38}$$

and realize an embedding of $\mathfrak{su}(2)_{L/R}$ into $\mathfrak{so}(4) = \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$. The index i in $\eta^{+}_{i\mu\nu}$ is thus the vector index of $\mathfrak{su}(2)_L$ in its spin-1 representation. In a slight abuse of notation we use the same notation as for ordinary spatial indices, the latter corresponding to the diagonal subalgebra in $\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$. We hope this will not lead to confusion.

The (anti-)self-dual components of the field strength,

$$F^{\pm}_i = \frac{1}{2}\,\eta^{\pm}_{i\mu\nu}F^{\mu\nu}, \tag{5.39}$$

form a closed sector under dilatations, in the sense that operators

$$\mathcal{O} = \Psi^{i_1\ldots i_L}\,\mathrm{tr}\,F^{\pm}_{i_1}\ldots F^{\pm}_{i_L}, \tag{5.40}$$

mix only among themselves at one loop. Operators of this form appear naturally if the $\mathfrak{psu}(2,2|4)$ spin chain is formulated in the standard Cartan basis with a single fermion node on the Dynkin diagram, also known as the "Beast" grading [64].

The ground state (or rather the reference state) then is a gluon operator

$$\mathcal{O}_0 = \mathrm{tr}(F^{\pm}_z)^L, \tag{5.41}$$

where

$$F^{\pm}_z = F^{\pm}_1 + iF^{\pm}_2. \tag{5.42}$$

This operator is not protected and acquires a one-loop anomalous dimension $3\lambda L/8\pi^2$ [54, 64]. Excited states are constructed by replacing some of the F_z's by

$$F^{\pm}_x = F^{\pm}_3, \qquad F_{\bar{z}} = F^{\pm}_1 - iF^{\pm}_2. \tag{5.43}$$

All of them have lower dimension, the ground state (5.41) has the largest energy possible for a given length. For example, the length-2 singlets

$$\mathcal{T}^{\pm} = \mathrm{tr}[F^{\pm}_z F^{\pm}_{\bar{z}} + (F^{\pm}_x)^2] = \mathrm{tr}\,F^{\pm}_i F^{\pm}_i, \tag{5.44}$$

have anomalous dimension zero. These operators are in fact linear combinations of the Lagrangian density (its gluon part) and that of the topological charge:

$$\mathcal{T}^{\pm} = \frac{1}{2}\,\mathrm{tr}\,F_{\mu\nu}F^{\mu\nu} \pm \frac{1}{2}\,\mathrm{tr}\,F_{\mu\nu}\widetilde{F}^{\mu\nu}. \tag{5.45}$$

They are not renormalized because the beta-function is zero.

The one-loop mixing matrix of the gluon operators (5.40) is the Hamiltonian of an SO(3) spin chain [65, 66]:

$$\Gamma = \frac{\lambda}{16\pi^2} \sum_{l=1}^{L} \left(5 + P_{l,l+1} - 2K_{l,l+1}\right), \tag{5.46}$$

where P and K are permutation and trace operators acting on the SO(3) indices of $F_{i_l}^{\pm}$. The model is integrable [67–69] and its Bethe-ansatz solution [70–72] is given by:

$$\left(\frac{u_j - i}{u_j + i}\right)^L \prod_k \frac{u_j - u_k + i}{u_j - u_k - i} = -1, \qquad E = \frac{\lambda}{8\pi^2} \left(3L - \sum_j \frac{2}{u_j^2 + 1}\right). \tag{5.47}$$

The singlet state (5.44), for example, corresponds to $u = \left\{1/\sqrt{3}, -1/\sqrt{3}\right\}$.

The (anti-)self-dual components of the field strength are linear combinations of electric and magnetic fields $F_i^{\pm} = B_i \mp E_i$, and for the field of the monopole we have:

$$\left\langle F_i^{\pm} \right\rangle_T = \frac{B}{2} \frac{x_i}{r^3}. \tag{5.48}$$

To unambiguously define the one-point functions we need a proper normalization condition for tensor operators in (5.40). This is to some extent arbitrary, we take:

$$\left\langle \mathcal{O}^+(t, \mathbf{0}) \mathcal{O}^-(0, \mathbf{0}) \right\rangle = \frac{1}{t^{2\Delta}}. \tag{5.49}$$

Then, taking into account

$$\left\langle F_i^+(t, \mathbf{0}) F_j^-(0, \mathbf{0}) \right\rangle = \frac{\lambda \delta_{ij}}{\pi^2 t^4}, \tag{5.50}$$

and substituting the classical field (2.1) into the operator we find:

$$\langle \mathcal{O}(x) \rangle_T = \left(\frac{\pi}{2\sqrt{\lambda}\, r^3}\right)^L L^{-\frac{1}{2}} \frac{\langle \mathrm{Bgl} | \Psi \rangle}{\langle \Psi | \Psi \rangle^{\frac{1}{2}}}. \tag{5.51}$$

The boundary state $\langle \mathrm{Bgl}|$ is one-site factorizable:

$$\mathrm{Bgl}_{i_1 \ldots i_L} = x_{i_1} \ldots x_{i_L}. \tag{5.52}$$

It picks one particular component of the wave function where spins on all sites point in the same direction (that of the radius vector of the operator insertion).

We expect on general grounds that this state is integrable. As such it should have non-zero overlaps only with balanced Bethe eigenstates in which all roots are paired, and should have a determinant representation similar to (5.10) and (5.28). We have not proven integrability in any direct way, nor have we honestly derived the overlap formula, but we have extensively checked both. We computed numerically overlaps with all eigenstates up to length nine and checked that they vanish for unbalanced Bethe states. For overlaps with

paired states $u = \{u_j, -u_j\}_{j=1\ldots M/2}$ we found a concise determinant formula:[8]

$$\langle \mathcal{O}(x) \rangle_T = \left(\frac{\pi}{2\sqrt{2\lambda}} \right)^L \frac{z^{L-M}}{r^{3L-M}} \sqrt{\frac{1}{L} \frac{Q(0)}{Q\left(\frac{i}{2}\right)} \frac{\det G^+}{\det G^-}}, \qquad (5.53)$$

where $Q(x)$ is the Baxter polynomial:

$$Q(x) = \prod_{j=1}^{\frac{M}{2}} \left(x^2 - u_j^2 \right), \qquad (5.54)$$

and $G^\pm$ are the Gaudin factors:

$$G_{ij}^\pm = \left(\frac{2L}{u_j^2 + 1} - \sum_s K_{js}^+ \right) \delta_{jk} + K_{jk}^\pm, \qquad (5.55)$$

$$K_{jk}^\pm = \frac{2}{(u_j - u_k)^2 + 1} \pm \frac{2}{(u_j + u_k)^2 + 1}. \qquad (5.56)$$

The space-time dependence of the one-point function is very simple, simpler than required by explicit symmetries alone. Those require the spin-S overlap to be of the form $z^S \mathcal{P}$, where $\mathcal{P}$ is a homogeneous polynomial in $x \equiv x_3$ and $|z|$. For the highest-weight states the polynomial prefactor neatly combines into $(x^2 + |z|^2)^{(L-S)/2} = r^M$. This simplification, we believe, is also a consequence of integrability.

To exemplify the use of the overlap formula, we consider the two-magnon states of arbitrary length with Bethe roots at u and $-u$ constrained by the Bethe equations. The chiral components of the field strength form the canonical basis of the $\mathfrak{su}(2)$ spin-1 representation:

$$F_z^\pm \longleftrightarrow \sqrt{2} \, |1\rangle, \qquad F_x^\pm \longleftrightarrow |0\rangle, \qquad F_{\bar{z}}^\pm \longleftrightarrow \sqrt{2} \, |-1\rangle. \qquad (5.57)$$

The spin lowering operator acts as

$$L_- F_z^\pm = 2 F_x^\pm, \qquad L_- F_x^\pm = -F_{\bar{z}}^\pm. \qquad (5.58)$$

The two-magnon eigenstate of the Hamiltonian (5.46) can be generated by applying the lowering operators twice:

$$|\Psi\rangle = \frac{1}{2} \left\{ \sum_{\ell < \ell'} \left[e^{ip(\ell - \ell')} + e^{ip(\ell' - \ell) + i\delta} \right] L_-(\ell) L_-(\ell') + \cos\frac{\delta}{2} \, e^{\frac{i\delta}{2}} \sum_\ell L_-(\ell)^2 \right\} |0\rangle, \qquad (5.59)$$

quite similarly to (5.31), while the phase shift and the momentum of the magnons are now

$$e^{ip} = \frac{u + i}{u - i}, \qquad e^{i\delta} = \frac{u - \frac{i}{2}}{u + \frac{i}{2}}. \qquad (5.60)$$

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[8]The number of Bethe roots in a balanced state needs not to be even, there are exceptional states containing a triplet of singular roots at $i, -i, 0$ [73, 74] and an even number of paired regular roots. These states are also parity-invariant and have vanishing total momentum thus allowing for non-zero overlaps. The determinant formula is ill-defined on singular roots and has to be regularized, more details can be found in [75].

For two magnons the Gaudin factors are just numbers:

$$G^+ = \frac{2L}{u^2 + 1}, \qquad G^- = \frac{2L}{u^2 + 1} - \frac{1}{u^2 + \frac{1}{4}}, \tag{5.61}$$

which gives the following prediction for the one-point function:

$$\langle \mathcal{O}_{2-\text{magnon}}(x) \rangle_T = \left(\frac{\pi}{2\sqrt{2\lambda}} \right)^L \frac{z^{L-2}}{r^{3L-2}} \frac{u}{\sqrt{\left(u^2 + \frac{1}{4} \right) \left(L - \frac{1}{2} \frac{u^2+1}{u^2+\frac{1}{4}} \right)}}. \tag{5.62}$$

The the norm and the overlap can be computed with the help of (5.33). Taking into account the Bethe equations, we get:

$$\langle \Psi | \Psi \rangle = 2^L L \left(L - \frac{1}{2} \frac{u^2+1}{u^2+\frac{1}{4}} \right), \qquad \langle \text{Bgl} | \Psi \rangle = L r^2 z^{L-2} \frac{u}{\sqrt{u^2 + \frac{1}{4}}}. \tag{5.63}$$

Substitution of these expressions into (5.51) recovers (5.62).

6 Conclusions

Our results confirm that an infinite 't Hooft line in the $\mathcal{N} = 4$ super-Yang-Mills theory defines an integrable dCFT. Integrability opens an avenue for computing expectation values of local operators exactly without any approximations. As of now, we found exact expressions for protected operators by combining localization with S-duality. For unprotected operators our results are limited to the leading order in perturbation theory and to certain subsectors of the operator algebra, but they do reveal the integrability structure expected of the exact answer.

The diagram technique in the monopole field, here applied to protected one-point functions, can be used for non-protected operators as well, and also to more complicated correlation functions. For example, it should be relatively straightforward to construct loop corrections to the overlap formulas we derived.

It is not inconceivable that the full solution can be bootstrapped from integrability. Our results point to a relative simplicity of the underlying scattering theory. For comparison, scattering on the D3-D5 defect [13] (used to construct non-perturbative one-point functions in [13–15]) involves boundary resonances. The boundary states we found have no internal structure (require no auxiliary space), and we expect the boundary scattering to be free of bound states. The simplicity of the protected one-point functions (3.22) points in the same direction. Analogous one-point functions for the D3-D5 system are sums of k terms, each depending on x_a with $a = 1 \ldots k$ [76], plus wrapping corrections, where k happens to coincide with the number of bound states in the boundary scattering [13]. We only found one term that depends on x_1, in this sense the 't Hooft line should be similar to the D3-D5 dCFT at $k = 1$ which was studied in more detail in [58].

Finally, similar techniques should be applicable to monopole operators in the ABJM theory. In three dimensions monopole operators are point-like, but the one-point functions

similar to those we have studied may still be non-trivial because the monopole operators and those built from the fundamental fields are not mutually local. We expect integrability of AdS_4/CFT_3 to be part of that story.

Acknowledgments

A One-point function at strong coupling

When

$$\lambda \to \infty, \qquad L - \text{fixed}, \tag{A.1}$$

the correlator of the 't Hooft loop with a chiral primary is described by classical supergravity in $AdS_5 \times S^5$.

The localization prediction (3.19) in this limit becomes

$$\mathbb{C}_L \simeq 2\pi\sqrt{\frac{L}{\lambda}} \,. \tag{A.2}$$

Here we will reproduce this results from the explicit supergravity calculation.

A macroscopic object in the bulk perturbs the metric of $AdS_5 \times S^5$ and according to the standard AdS/CFT dictionary an expectation value of a local operator is the response to this perturbation as seen from the boundary. The computation of $\mathbb{C}_L$ thus involves a bulk-to-boundary propagator connecting the operator insertion to the vertex operator on the world-volume of the D1-brane that represents the 't Hooft line. We will use the method first applied to the fundamental string [32] and generalized to D-branes in [77]. A similar approach can be used to compute the correlator of the 't Hooft loop with the Wilson loop [78].

The dual of the circular 't Hooft loop is a D1-brane of the spherical shape:

$$x^\mu = \frac{R}{\cosh\tau} \left(\sin\sigma, \cos\sigma, 0, 0\right), \qquad z = \tanh\tau, \tag{A.3}$$

where (z, x^μ) are the standard Poincaré coordinates of $AdS_5 \times S^5$:

$$ds^2 = \frac{dx^2 + dz^2}{z^2} + d\theta^2 + \sin^2\theta \, d\Omega^2. \tag{A.4}$$

The calculation closely follows the footsteps of [32] where the circular Wilson loop was considered.

The supergravity dual of the chiral operator (3.13) is a scalar with $m^2 = L(L-4)$ and the canonically normalized propagator:[9]

$$\langle\phi_L(z,x)\phi_L(w,y)\rangle_{\text{SUGRA}} \stackrel{w \to 0}{=} \frac{L-1}{2\pi^2} \frac{z^L w^L}{[z^2 + (x-y)^2]^L} \,. \tag{A.5}$$

---

[9]We only need the bulk-to-boundary propagator and hence send one of the points, (w, y), to the boundary.

The response function is given by the standard AdS/CFT prescription:

$$\langle \mathcal{O}_L(y) \rangle = \sqrt{\frac{2\pi^2}{L-1}} \lim_{w \to 0} w^{-L} \langle \phi_L(w, y) \rangle_{\text{SUGRA}}, \tag{A.6}$$

assuming there is a source of ϕ_L in the bulk.

The source in our case is the D1-brane that couples to the 10D metric through the usual DBI action:

$$S_{\text{DBI}} = T_{D1} \int d^2\sigma \sqrt{\det_{ab} \partial_a X^M \partial_b X^N g_{MN}}. \tag{A.7}$$

And the field ϕ_L resides in the metric perturbation around the ambient $AdS_5 \times S^5$ background [79] (see also [80]):

$$\begin{aligned}
\delta g_{mn} &= 2Y_L \left[2\nabla_m \nabla_n - L(L-1) g_{mn} \right] \phi_L, \\
\delta g_{\alpha\beta} &= 2L(L+1) g_{\alpha\beta} Y_L \phi_L, \\
a_{mnpr} &= \varepsilon_{mnprs} (L+1) Y_L \nabla^s \phi_L, \\
a_{\alpha\beta\gamma\delta} &= -(L+1) \varepsilon_{\alpha\beta\gamma\delta\varepsilon} \nabla^\varepsilon Y_L \phi_L,
\end{aligned} \tag{A.8}$$

where Y_L is the spherical function on S^5. For the operator at hand:[10]

$$Y_L = \frac{\pi}{N\sqrt{8L(L-1)}} (\sin\theta)^L e^{iL\varphi}. \tag{A.9}$$

The D1-brane does not couple directly to the four-form and resides at one point on S^5. Therefore only the metric fluctuations in AdS_5 contribute to the one-point function. Expanding the D-brane action to the linear order we find:

$$\langle \mathcal{O}_L(y) \rangle_T = -\frac{T_{D1}}{2} \int d^2\sigma \sqrt{h} h^{ab} \partial_a X^M \partial_b X^N \delta g_{MN}, \tag{A.10}$$

where h_{ab} is the induced metric on the world-volume. Taking into account (A.8), we get:

$$\begin{aligned}
\langle \mathcal{O}_L(y) \rangle_T = &-\frac{\pi^2}{(L-1)\sqrt{\lambda L}} \int d^2\sigma \sqrt{h} h^{ab} \partial_a X^m \partial_b X^n \\
&\times \lim_{w \to 0} w^{-L} \langle [2\nabla_m \nabla_n - L(L-1) g_{mn}] \phi_L(X) \phi_L(w, y) \rangle_{\text{SUGRA}},
\end{aligned} \tag{A.11}$$

where we took into account various factors appearing in (3.12), (A.6) and (A.9), and set $\theta = \pi/2$, $\varphi = 0$ as appropriate for the 't Hooft line with the R-symmetry orientation $n_i = (1, \mathbf{0})$.

---

[10]The scalar fields Φ_i are in one-to-one correspondence with the Cartesian coordinates n_i on S^5. The general chiral primary operator $C^{i_1 \cdots i_L} \operatorname{tr} \Phi_{i_1} \ldots \Phi_{i_L}$, defined by a symmetric traceless tensor C_A, maps to the spherical function $Y_A = C_A^{i_1 \cdots i_L} n_{i_1} \ldots n_{i_L}$. For $\operatorname{tr} Z^L$ the spherical function is $(n_1 + i n_2)^L = (\sin\theta)^L e^{iL\varphi}$. There are two natural ways to normalize the spherical harmonics, one by the tensor squared of C_A, the other by integrating the square of Y_A over S^5. The L-dependent overall factor partly arises from this difference in normalization [79] and partly from mixing of the KK modes on S^5 [81], it also includes the square root of the 10D gravitational constant $\kappa_{10} = (2\pi)^5/8N^2$ appearing as an overall factor in the supergravity action but absent in the canonically normalized propagator that we use to compute the amplitude.

It remains to integrate the bulk-to-boundary propagator over the minimal surface (A.3). The answer anticipated from the conformal symmetry is rather contrived, eq. (2.8), but the whole dynamical information is contained in the overall constant, which can be computed by choosing any convenient y, for instance $y \to \infty$ [32], as in (2.9). Then,

$$\mathbb{C}_L = \frac{(L+1)\sqrt{L}}{2R^L\sqrt{\lambda}} \int d^2\sigma \sqrt{h}\, z^{L-2} \left[(\partial x)^2 - (\partial z)^2 \right]. \tag{A.12}$$

The solution (A.3) is in the conformal gauge, the induced metric is trivial, and we find:

$$\mathbb{C}_L = \frac{2\pi(L+1)\sqrt{L}}{\sqrt{\lambda}} \int_0^\infty d\tau \, \frac{\tanh^L \tau}{\cosh^2 \tau} = 2\pi \sqrt{\frac{L}{\lambda}}, \tag{A.13}$$

in agreement with (A.2).

There is another regime of interest:

$$\lambda \to \infty, \qquad L \to \infty, \qquad \frac{\lambda}{L^2} - \text{fixed}, \tag{A.14}$$

the BMN limit [82]. The localization prediction in this case is

$$\mathbb{C}_L \simeq \frac{2}{\sqrt{L}} \sinh \frac{\pi L}{\sqrt{\lambda}}. \tag{A.15}$$

The operator in the BMN limit is dual to a classical string rather than a supergravity mode. The form of the answer points to a one-loop origin, since sinh admits a product representation over $n^2 + \pi^2 L^2/\lambda$, which are exactly the BMN string modes [82]. It would be very interesting to make this argument more precise.

B   Clebsch Gordon coefficients

Below we list the Clebsch Gordon coefficients entering the relations (4.52) and (4.53) for the case of monopole charge, $q = 1/2$. Their explicit derivation as well as their counterparts for general value of the monopole charge can be found e.g. in [44]. For the $\mathcal{C}$'s one has

$$\mathcal{C}_{JJ-1} = \frac{1}{2}\sqrt{2 - \frac{1}{J}}, \quad \mathcal{C}_{JJ} = -\frac{1}{2\sqrt{J(J+1)}}, \quad \mathcal{C}_{JJ+1} = -\frac{1}{2}\sqrt{\frac{1}{J+1} + 2}, \tag{B.1}$$

and for the $\mathcal{A}$'s one finds

$$\mathcal{A}_{J-1J-1} = \frac{1}{2J}, \quad \mathcal{A}_{JJ} = -\frac{1}{2J(J+1)}, \quad \mathcal{A}_{J+1J+1} = -\frac{1}{2(J+1)}, \tag{B.2}$$

$$\mathcal{A}_{J-1J} = \mathcal{A}_{JJ-1} = \frac{\sqrt{(2J-1)(J+1)}}{2J}, \quad \mathcal{A}_{JJ+1} = \mathcal{A}_{J+1J} = \frac{\sqrt{J(2J+3)}}{2J+2}, \tag{B.3}$$

$$\mathcal{A}_{J-1J+1} = \mathcal{A}_{J+1J-1} = 0. \tag{B.4}$$

C Reduction to AdS_2

The connection of monopole quantization to AdS_2 can be made very explicit. The $\mathbb{R}^4$ in spherical coordinates is conformally equivalent to $AdS_2 \times S^2$:

$$ds^2 = r^2 \left(\frac{d\tau^2 + dr^2}{r^2} + d\theta^2 + \sin^2\theta \, d\varphi^2 \right). \tag{C.1}$$

The conformal factor drops out from the action because the scaling symmetry is preserved by the 't Hooft loop. Expansion in the monopole harmonics can be viewed as the Kaluza-Klein reduction on S^2:

$$\Phi(x) = \frac{1}{r} \sum_{lm} \chi_{lm}(\tau, r) Y_{lm}^{(q)}(\theta, \varphi), \tag{C.2}$$

resulting in an infinite tower of fields on AdS_2.

Take for example the simple scalar with the kinetic operator

$$-D^2 + \frac{B^2}{4r^2} = \frac{1}{r} \left(-\frac{\partial^2}{\partial \tau^2} - \frac{\partial}{\partial r^2} + \frac{L^2}{r^2} \right) r, \tag{C.3}$$

where L_i is the angular momentum operator in the monopole background. The action for a given KK mode is then

$$S = \frac{1}{2} \int d\tau dr \left[(\partial_\tau \chi)^2 + (\partial_r \chi)^2 + \frac{l(l+1)}{r^2} \chi^2 \right], \tag{C.4}$$

where we took into account the r^2 from the measure and used the explicit form of the L^2 eigenvalues. This is just the canonical action of the scalar field of mass $m^2 = l(l+1)$ in AdS_2. The momentum squared plays the role of the mass-squared operator in the reduced theory.

The scalar field in AdS_2 is characterized by the dimension of the dual operator in the effective CFT_1 on the 't Hooft line, which is related to the mass by the standard formula

$$\Delta(\Delta - 1) = m^2. \tag{C.5}$$

The paramater ν defined in (4.72) is related to Δ by

$$\nu = \Delta - \frac{1}{2}. \tag{C.6}$$

For the scalar with $m^2 = l(l+1)$, the dimension is $\Delta = l+1$ and $\nu = l+1/2$.

The propagator of the field χ is just the standard bulk-to-bulk propagator in AdS_2:

$$G_\Delta(\tau_1, r_1; \tau_2, r_2) = \frac{\Gamma(\Delta)\xi^\Delta}{2^{\Delta+1}\sqrt{\pi}\,\Gamma\left(\Delta + \frac{1}{2}\right)} \, {}_2F_1\left(\frac{\Delta}{2}, \frac{\Delta+1}{2}; \Delta + \frac{1}{2}; \xi^2 \right), \tag{C.7}$$

where

$$\xi = \frac{2r_1 r_2}{r_1^2 + r_2^2 + (\tau_1 - \tau_2)^2}. \tag{C.8}$$

The natural regularization in AdS_2 is covariant point-splitting. For $a^2 \equiv (\tau_1 - \tau_2)^2 + (r_1 - r_2)^2 \to 0$:

$$G_\Delta(\xi) \simeq -\frac{1}{2\pi} \left(\ln \frac{a}{2r} + \gamma + \Psi(\Delta) \right), \tag{C.9}$$

which coincides with (4.74) up to unimportant constant terms that encapsulate regularization ambiguity and should cancel in any properly renormalized (or finite) observable.

References

[1] G. 't Hooft, *On the Phase Transition Towards Permanent Quark Confinement*, [*Nucl. Phys. B* **138** (1978) 1](https://doi.org/10.1016/0550-3213(78)90153-0) [ɪɴSPIRE].

[2] A. Kapustin, *Wilson-'t Hooft operators in four-dimensional gauge theories and S-duality*, [*Phys. Rev. D* **74** (2006) 025005](https://doi.org/10.1103/PhysRevD.74.025005) [[hep-th/0501015](https://arxiv.org/abs/hep-th/0501015)] [ɪɴSPIRE].

[3] J. Gomis, T. Okuda and D. Trancanelli, *Quantum 't Hooft operators and S-duality in N = 4 super Yang-Mills*, [*Adv. Theor. Math. Phys.* **13** (2009) 1941](https://doi.org/10.4310/ATMP.2009.v13.n6.a9) [[arXiv:0904.4486](https://arxiv.org/abs/0904.4486)] [ɪɴSPIRE].

[4] J. Gomis and T. Okuda, *S-duality, 't Hooft operators and the operator product expansion*, [*JHEP* **09** (2009) 072](https://doi.org/10.1088/1126-6708/2009/09/072) [[arXiv:0906.3011](https://arxiv.org/abs/0906.3011)] [ɪɴSPIRE].

[5] J. Gomis, T. Okuda and V. Pestun, *Exact Results for 't Hooft Loops in Gauge Theories on S^4*, [*JHEP* **05** (2012) 141](https://doi.org/10.1007/JHEP05(2012)141) [[arXiv:1105.2568](https://arxiv.org/abs/1105.2568)] [ɪɴSPIRE].

[6] D.-E. Diaconescu, *D-branes, monopoles and Nahm equations*, [*Nucl. Phys. B* **503** (1997) 220](https://doi.org/10.1016/S0550-3213(97)00438-0) [[hep-th/9608163](https://arxiv.org/abs/hep-th/9608163)] [ɪɴSPIRE].

[7] A. Dekel and Y. Oz, *Integrability of Green-Schwarz Sigma Models with Boundaries*, [*JHEP* **08** (2011) 004](https://doi.org/10.1007/JHEP08(2011)004) [[arXiv:1106.3446](https://arxiv.org/abs/1106.3446)] [ɪɴSPIRE].

[8] P. Liendo and C. Meneghelli, *Bootstrap equations for $\mathcal{N} = 4$ SYM with defects*, [*JHEP* **01** (2017) 122](https://doi.org/10.1007/JHEP01(2017)122) [[arXiv:1608.05126](https://arxiv.org/abs/1608.05126)] [ɪɴSPIRE].

[9] C. Montonen and D.I. Olive, *Magnetic Monopoles as Gauge Particles?*, [*Phys. Lett. B* **72** (1977) 117](https://doi.org/10.1016/0370-2693(77)90076-4) [ɪɴSPIRE].

[10] E. Witten and D.I. Olive, *Supersymmetry Algebras That Include Topological Charges*, [*Phys. Lett. B* **78** (1978) 97](https://doi.org/10.1016/0370-2693(78)90357-X) [ɪɴSPIRE].

[11] H. Osborn, *Topological Charges for N = 4 Supersymmetric Gauge Theories and Monopoles of Spin 1*, [*Phys. Lett. B* **83** (1979) 321](https://doi.org/10.1016/0370-2693(79)91118-3) [ɪɴSPIRE].

[12] I. Buhl-Mortensen et al., *Asymptotic One-Point Functions in Gauge-String Duality with Defects*, [*Phys. Rev. Lett.* **119** (2017) 261604](https://doi.org/10.1103/PhysRevLett.119.261604) [[arXiv:1704.07386](https://arxiv.org/abs/1704.07386)] [ɪɴSPIRE].

[13] S. Komatsu and Y. Wang, *Non-perturbative defect one-point functions in planar $\mathcal{N} = 4$ super-Yang-Mills*, [*Nucl. Phys. B* **958** (2020) 115120](https://doi.org/10.1016/j.nuclphysb.2020.115120) [[arXiv:2004.09514](https://arxiv.org/abs/2004.09514)] [ɪɴSPIRE].

[14] T. Gombor and Z. Bajnok, *Boundary states, overlaps, nesting and bootstrapping AdS/dCFT*, [*JHEP* **10** (2020) 123](https://doi.org/10.1007/JHEP10(2020)123) [[arXiv:2004.11329](https://arxiv.org/abs/2004.11329)] [ɪɴSPIRE].

[15] T. Gombor and Z. Bajnok, *Boundary state bootstrap and asymptotic overlaps in AdS/dCFT*, [*JHEP* **03** (2021) 222](https://doi.org/10.1007/JHEP03(2021)222) [[arXiv:2006.16151](https://arxiv.org/abs/2006.16151)] [ɪɴSPIRE].

[16] M. de Leeuw, C. Kristjansen and K. Zarembo, *One-point Functions in Defect CFT and Integrability*, [*JHEP* **08** (2015) 098](https://doi.org/10.1007/JHEP08(2015)098) [[arXiv:1506.06958](https://arxiv.org/abs/1506.06958)] [ɪɴSPIRE].

[17] I. Buhl-Mortensen, M. de Leeuw, C. Kristjansen and K. Zarembo, *One-point Functions in AdS/dCFT from Matrix Product States*, [*JHEP* **02** (2016) 052](https://doi.org/10.1007/JHEP02(2016)052) [[arXiv:1512.02532](https://arxiv.org/abs/1512.02532)] [ɪɴSPIRE].

[18] P.A.M. Dirac, *Quantised singularities in the electromagnetic field,*, [*Proc. Roy. Soc. Lond. A* **133** (1931) 60](https://doi.org/10.1098/rspa.1931.0130) [ɪɴSPIRE].

[19] M. Bianchi, M.B. Green and S. Kovacs, *Instanton corrections to circular Wilson loops in N=4 supersymmetric Yang-Mills*, *JHEP* **04** (2002) 040 [`hep-th/0202003`] [InSPIRE].

[20] M.A. Shifman, *Wilson Loop in Vacuum Fields*, *Nucl. Phys. B* **173** (1980) 13 [InSPIRE].

[21] J.K. Erickson, G.W. Semenoff and K. Zarembo, *Wilson loops in $N = 4$ supersymmetric Yang-Mills theory*, *Nucl. Phys. B* **582** (2000) 155 [`hep-th/0003055`] [InSPIRE].

[22] N. Drukker and D.J. Gross, *An Exact prediction of $N = 4$ SUSYM theory for string theory*, *J. Math. Phys.* **42** (2001) 2896 [`hep-th/0010274`] [InSPIRE].

[23] G.W. Semenoff and K. Zarembo, *More exact predictions of SUSYM for string theory*, *Nucl. Phys. B* **616** (2001) 34 [`hep-th/0106015`] [InSPIRE].

[24] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, *Commun. Math. Phys.* **313** (2012) 71 [`arXiv:0712.2824`] [InSPIRE].

[25] S. Giombi and V. Pestun, *Correlators of local operators and 1/8 BPS Wilson loops on S^2 from 2d YM and matrix models*, *JHEP* **10** (2010) 033 [`arXiv:0906.1572`] [InSPIRE].

[26] K. Okuyama and G.W. Semenoff, *Wilson loops in $N = 4$ SYM and fermion droplets*, *JHEP* **06** (2006) 057 [`hep-th/0604209`] [InSPIRE].

[27] N. Drukker and B. Fiol, *All-genus calculation of Wilson loops using D-branes*, *JHEP* **02** (2005) 010 [`hep-th/0501109`] [InSPIRE].

[28] S.A. Hartnoll and S.P. Kumar, *Higher rank Wilson loops from a matrix model*, *JHEP* **08** (2006) 026 [`hep-th/0605027`] [InSPIRE].

[29] S. Kawamoto, T. Kuroki and A. Miwa, *Boundary condition for D-brane from Wilson loop, and gravitational interpretation of eigenvalue in matrix model in AdS/CFT correspondence*, *Phys. Rev. D* **79** (2009) 126010 [`arXiv:0812.4229`] [InSPIRE].

[30] M. Beccaria and A.A. Tseytlin, *On the structure of non-planar strong coupling corrections to correlators of BPS Wilson loops and chiral primary operators*, *JHEP* **01** (2021) 149 [`arXiv:2011.02885`] [InSPIRE].

[31] J. Polchinski, *String theory*, Cambridge University Press (1998) [`DOI:10.1017/CBO9780511816079`].

[32] D.E. Berenstein, R. Corrado, W. Fischler and J.M. Maldacena, *The Operator product expansion for Wilson loops and surfaces in the large N limit*, *Phys. Rev. D* **59** (1999) 105023 [`hep-th/9809188`] [InSPIRE].

[33] N. Drukker, D.J. Gross and H. Ooguri, *Wilson loops and minimal surfaces*, *Phys. Rev. D* **60** (1999) 125006 [`hep-th/9904191`] [InSPIRE].

[34] J. Ambjorn, R.A. Janik and C. Kristjansen, *Wrapping interactions and a new source of corrections to the spin-chain/string duality*, *Nucl. Phys. B* **736** (2006) 288 [`hep-th/0510171`] [InSPIRE].

[35] V. Borokhov, A. Kapustin and X.-K. Wu, *Topological disorder operators in three-dimensional conformal field theory*, *JHEP* **11** (2002) 049 [`hep-th/0206054`] [InSPIRE].

[36] V. Borokhov, A. Kapustin and X.-K. Wu, *Monopole operators and mirror symmetry in three-dimensions*, *JHEP* **12** (2002) 044 [`hep-th/0207074`] [InSPIRE].

[37] G. Grignani, L. Griguolo, N. Mori and D. Seminara, *Thermodynamics of theories with sixteen supercharges in non-trivial vacua*, *JHEP* **10** (2007) 068 [`arXiv:0707.0052`] [InSPIRE].

[38] I. Buhl-Mortensen et al., *One-loop one-point functions in gauge-gravity dualities with defects*, *Phys. Rev. Lett.* **117** (2016) 231603 [arXiv:1606.01886] [ɪɴSPIRE].

[39] I. Buhl-Mortensen et al., *A Quantum Check of AdS/dCFT*, *JHEP* **01** (2017) 098 [arXiv:1611.04603] [ɪɴSPIRE].

[40] I. Tamm, *Die verallgemeinerten Kugelfunktionen und die Wellenfunktionen eines Elektrons im Felde eines Magnetpoles*, *Z. Phys.* **71** (1931) 141 [ɪɴSPIRE].

[41] M. Fierz, *Zur Theorie magnetisch geladener Teilchen*, *Helv. Phys. Acta* **17** (1944) 27.

[42] T.T. Wu and C.N. Yang, *Dirac's Monopole Without Strings: Classical Lagrangian Theory*, *Phys. Rev. D* **14** (1976) 437 [ɪɴSPIRE].

[43] S.M. Roy and V. Singh, *Exact Solution of Schrodinger Equation in Aharonov-Bohm Plus Dirac Monopole Potential*, *Phys. Rev. Lett.* **51** (1983) 2069 [ɪɴSPIRE].

[44] H.A. Olsen, P. Osland and T.T. Wu, *On the Existence of Bound States for a Massive Spin 1 Particle and a Magnetic Monopole*, *Phys. Rev. D* **42** (1990) 665 [ɪɴSPIRE].

[45] E.J. Weinberg, *Monopole vector spherical harmonics*, *Phys. Rev. D* **49** (1994) 1086 [hep-th/9308054] [ɪɴSPIRE].

[46] Y. Kazama, C.N. Yang and A.S. Goldhaber, *Scattering of a Dirac Particle with Charge Ze by a Fixed Magnetic Monopole*, *Phys. Rev. D* **15** (1977) 2287 [ɪɴSPIRE].

[47] H. Liu and A.A. Tseytlin, *On four point functions in the CFT/AdS correspondence*, *Phys. Rev. D* **59** (1999) 086002 [hep-th/9807097] [ɪɴSPIRE].

[48] P. Breitenlohner and D.Z. Freedman, *Stability in Gauged Extended Supergravity*, *Annals Phys.* **144** (1982) 249 [ɪɴSPIRE].

[49] J.A. Minahan and K. Zarembo, *The Bethe ansatz for N = 4 superYang-Mills*, *JHEP* **03** (2003) 013 [hep-th/0212208] [ɪɴSPIRE].

[50] M. De Leeuw et al., *Spin Chain Overlaps and the Twisted Yangian*, *JHEP* **01** (2020) 176 [arXiv:1912.09338] [ɪɴSPIRE].

[51] S. Ghoshal and A.B. Zamolodchikov, *Boundary S matrix and boundary state in two-dimensional integrable quantum field theory*, *Int. J. Mod. Phys. A* **9** (1994) 3841 [*Erratum ibid.* **9** (1994) 4353] [hep-th/9306002] [ɪɴSPIRE].

[52] B. Pozsgay, L. Piroli and E. Vernier, *Integrable Matrix Product States from boundary integrability*, *SciPost Phys.* **6** (2019) 062 [arXiv:1812.11094] [ɪɴSPIRE].

[53] C. Kristjansen, D. Müller and K. Zarembo, *Duality relations for overlaps of integrable boundary states in AdS/dCFT*, *JHEP* **09** (2021) 004 [arXiv:2106.08116] [ɪɴSPIRE].

[54] N. Beisert, *The complete one loop dilatation operator of N = 4 superYang-Mills theory*, *Nucl. Phys. B* **676** (2004) 3 [hep-th/0307015] [ɪɴSPIRE].

[55] M. Brockmann, J.D. Nardis, B. Wouters and J.-S. Caux, *Néel-XXZ state overlaps: odd particle numbers and Lieb-Liniger scaling limit*, arXiv:1403.7469 [DOI:10.1088/1751-8113/47/34/345003].

[56] Y. Jiang, S. Komatsu and E. Vescovi, *Structure constants in $\mathcal{N} = 4$ SYM at finite coupling as worldsheet g-function*, *JHEP* **07** (2020) 037 [arXiv:1906.07733] [ɪɴSPIRE].

[57] Y. Jiang and B. Pozsgay, *On exact overlaps in integrable spin chains*, *JHEP* **06** (2020) 022 [arXiv:2002.12065] [ɪɴSPIRE].

[58] C. Kristjansen, D. Müller and K. Zarembo, *Integrable boundary states in D3-D5 dCFT: beyond scalars*, *JHEP* **08** (2020) 103 [arXiv:2005.01392] [INSPIRE].

[59] T. Gombor, *On the classification of rational K-matrices*, *J. Phys. A* **53** (2020) 135203 [arXiv:1904.03044] [INSPIRE].

[60] T. Gombor and B. Pozsgay, *On factorized overlaps: Algebraic Bethe Ansatz, twists, and Separation of Variables*, *Nucl. Phys. B* **967** (2021) 115390 [arXiv:2101.10354] [INSPIRE].

[61] T. Gombor, *On exact overlaps for $\mathfrak{gl}(N)$ symmetric spin chains*, *Nucl. Phys. B* **983** (2022) 115909 [arXiv:2110.07960] [INSPIRE].

[62] M. Staudacher, *The Factorized S-matrix of CFT/AdS*, *JHEP* **05** (2005) 054 [hep-th/0412188] [INSPIRE].

[63] G. 't Hooft, *Computation of the Quantum Effects Due to a Four-Dimensional Pseudoparticle*, *Phys. Rev. D* **14** (1976) 3432 [*Erratum ibid.* **18** (1978) 2199] [INSPIRE].

[64] N. Beisert and M. Staudacher, *The $N = 4$ SYM integrable super spin chain*, *Nucl. Phys. B* **670** (2003) 439 [hep-th/0307042] [INSPIRE].

[65] G. Ferretti, R. Heise and K. Zarembo, *New integrable structures in large-N QCD*, *Phys. Rev. D* **70** (2004) 074024 [hep-th/0404187] [INSPIRE].

[66] N. Beisert, G. Ferretti, R. Heise and K. Zarembo, *One-loop QCD spin chain and its spectrum*, *Nucl. Phys. B* **717** (2005) 137 [hep-th/0412029] [INSPIRE].

[67] A.B. Zamolodchikov and V.A. Fateev, *Model factorized S matrix and an integrable heisenberg chain with spin 1* (in Russian), *Sov. J. Nucl. Phys.* **32** (1980) 298 [INSPIRE].

[68] P.P. Kulish, N.Y. Reshetikhin and E.K. Sklyanin, *Yang-Baxter Equation and Representation Theory. I*, *Lett. Math. Phys.* **5** (1981) 393 [INSPIRE].

[69] N.Y. Reshetikhin, *Integrable Models of Quantum One-dimensional Magnets With $O(N)$ and $Sp(2k)$ Symmetry*, *Theor. Math. Phys.* **63** (1985) 555 [INSPIRE].

[70] L.A. Takhtajan, *The picture of low-lying excitations in the isotropic Heisenberg chain of arbitrary spins*, *Phys. Lett. A* **87** (1982) 479 [INSPIRE].

[71] H.M. Babujian, *Exact solution of the one-dimensional isotropic Heisenberg chain with arbitrary spin S*, *Phys. Lett. A* **90** (1982) 479 [INSPIRE].

[72] H.M. Babujian, *Exact solution of the isotropic Heisenberg chain with arbitrary spins: thermodynamics of the model*, *Nucl. Phys. B* **215** (1983) 317 [INSPIRE].

[73] W. Hao, R.I. Nepomechie and A.J. Sommese, *Singular solutions, repeated roots and completeness for higher-spin chains*, *J. Stat. Mech.* **1403** (2014) P03024 [arXiv:1312.2982] [INSPIRE].

[74] J. Hou, Y. Jiang and R.-D. Zhu, *Spin-s Rational Q-system*, arXiv:2303.07640 [INSPIRE].

[75] C. Kristjansen, D. Müller and K. Zarembo, *Overlaps and fermionic dualities for integrable super spin chains*, *JHEP* **03** (2021) 100 [arXiv:2011.12192] [INSPIRE].

[76] M. De Leeuw, C. Kristjansen and G. Linardopoulos, *Scalar one-point functions and matrix product states of AdS/dCFT*, *Phys. Lett. B* **781** (2018) 238 [arXiv:1802.01598] [INSPIRE].

[77] A. Bissi, C. Kristjansen, D. Young and K. Zoubos, *Holographic three-point functions of giant gravitons*, *JHEP* **06** (2011) 085 [arXiv:1103.4079] [INSPIRE].

[78] A. Gorsky, A. Monin and A.V. Zayakin, *Correlator of Wilson and t'Hooft Loops at Strong Coupling in $N = 4$ SYM Theory*, *Phys. Lett. B* **679** (2009) 529 [`arXiv:0904.3665`] [INSPIRE].

[79] S. Lee, S. Minwalla, M. Rangamani and N. Seiberg, *Three point functions of chiral operators in $D = 4$, $N = 4$ SYM at large N*, *Adv. Theor. Math. Phys.* **2** (1998) 697 [`hep-th/9806074`] [INSPIRE].

[80] G. Arutyunov and S. Frolov, *Scalar quartic couplings in type IIB supergravity on $AdS_5 \times S^5$*, *Nucl. Phys. B* **579** (2000) 117 [`hep-th/9912210`] [INSPIRE].

[81] H.J. Kim, L.J. Romans and P. van Nieuwenhuizen, *The Mass Spectrum of Chiral $N = 2$ $D = 10$ Supergravity on S^5*, *Phys. Rev. D* **32** (1985) 389 [INSPIRE].

[82] D.E. Berenstein, J.M. Maldacena and H.S. Nastase, *Strings in flat space and pp waves from $N = 4$ superYang-Mills*, *JHEP* **04** (2002) 013 [`hep-th/0202021`] [INSPIRE].