Published for SISSA by Springer

Received: May 5, 2023
Accepted: August 7, 2023
Published: August 24, 2023

Minimally modified Fritzsch texture for quark masses and CKM mixing

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Abstract: The Standard Model does not constrain the form of the Yukawa matrices and thus the origin of fermion mass hierarchies and mixing pattern remains puzzling. On the other hand, there are intriguing relations between fermion masses and mixing angles which may point towards specific textures of Yukawa matrices. One of the classic hypothesis is the zero texture proposed by Fritzsch which is, however, excluded by present precision tests since it predicts a too large value of $|V_{cb}|$ as well as a too small value of the ratio $|V_{ub}/V_{cb}|$. In this paper we discuss a minimal modification which still maintains the six zero entries as in the original Fritzsch ansatz. This modification consists in introducing an asymmetry between the 23 and 32 entries in the down-quark Yukawa matrix. We show that this flavour structure can naturally emerge in the context of models with inter-family $\mathrm{SU}(3)_H$ symmetry. We present a detailed analysis of this Fritzsch-like texture by testing its predictions and showing that it is perfectly compatible with the present precision data on quark masses and CKM mixing matrix.

Keywords: CKM Parameters, Flavour Symmetries, Theories of Flavour

ArXiv ePrint: 2305.00069

https://doi.org/10.1007/JHEP08(2023)162

Contents

1 Introduction

The replication of fermion families is one of the main puzzles of particle physics. Three fermion families are in identical representations of the Standard Model (SM) gauge symmetry $SU(3) \times SU(2) \times U(1)$. Left-handed quarks $q_{Li} = (u_L, d_L)_i$ and leptons $\ell_{Li} = (\nu_L, e_L)_i$ transform as weak doublets whereas right-handed components u_{Ri}, d_{Ri}, e_{Ri} are weak singlets, $i = 1, 2, 3$ being the family index. The fermion masses emerge after spontaneous breaking of the EW symmetry $SU(2) \times U(1)$ by the Higgs doublet ϕ, via the Yukawa couplings

$$Y_u^{ij} u_i^c q_j \phi + Y_d^{ij} d_i^c q_j \tilde{\phi} + Y_e^{ij} e_i^c \ell_j \tilde{\phi} + \text{h.c.} \tag{1.1}$$

where $Y_{e,u,d}$ are the Yukawa matrices, and $\tilde{\phi} = i\tau_2 \phi^*$. Here, instead of the right-handed fermion fields, we use their left-handed complex conjugates (antifields) as $u_L^c = C\overline{u_R}^T$ and omit in the following the subscript L for q, u^c, d^c etc. all being the left-handed Weyl spinors. With these notations, the description can be conveniently extended to a supersymmetric extensions of the SM and/or to a grand unified theory (GUT). The Yukawa couplings (1.1), after substituting the Higgs vacuum expectation value (VEV) $\langle \phi^0 \rangle = v_\mathrm{w} = 174\,\mathrm{GeV}$, originate the fermion mass matrices $\mathrm{M}_f = Y_f v_\mathrm{w}$, $f = u, d, e$. They can be brought to the diagonal form (the mass eigenstate basis) via the bi-unitary transformations:

$$V_f^c \mathrm{M}_f V_f = \mathrm{M}_f^{\mathrm{diag}} \tag{1.2}$$

so that the quark masses m_u, m_c, m_t and m_d, m_s, m_b are the eigenvalues of the mass matrices M_u and M_d. (In the following we discuss concretely the quark sector considering the presence of leptons implicitly.) The "right" matrices $V^c_{u,d}$ rotating the right-handed quarks have no physical meaning in the SM context, while the "left" ones $V_{u,d}$ give rise to the mixing in the quark charged currents coupled to weak $W^\pm$ bosons which is determined by the unitary Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix $V_{\rm CKM}$ [1, 2]:

$$V_{\rm CKM} = V_u^\dagger V_d = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \tag{1.3}$$

This matrix is unitary, and by rotating away the irrelevant phases, it can be conveniently parameterized in terms of four parameters, three mixing angles θ_{12}, θ_{23}, θ_{13} and one CP-violating phase δ [2]. In the so called standard parameterization [3] it is written as:

$$V_{\rm CKM} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \tag{1.4}$$

where $s_{ij} = \sin\theta_{ij}$, $c_{ij} = \cos\theta_{ij}$ and θ_{ij} can be chosen so that $s_{ij}, c_{ij} \geq 0$. As a measure of CP violation, the rephasing-invariant quantity $J \sum_{m,n} \epsilon_{ikm}\epsilon_{jln} = \mathrm{Im}[V_{ij}V_{kl}V^*_{il}V^*_{kj}]$ can be considered instead of the phase δ, the Jarlskog invariant [4], which in the standard parameterization reads:

$$J = \sin\delta\, s_{12}s_{23}s_{13}\, c_{12}c_{23}c_{13}^2 \tag{1.5}$$

The mass spectrum and the mixing angles of quarks present a strong inter-family hierarchy. Namely, by parameterizing masses and mixings between quarks with a small parameter $\epsilon \sim 1/20$, we have for down-type quarks $m_b : m_s : m_d = 1 : \epsilon : \epsilon^2$ and for up-type $m_t : m_c : m_u = 1 : \epsilon^2 : \epsilon^4$, with $V_{us} \sim \sqrt{\epsilon}$, $V_{cb} \sim \epsilon$, $V_{ub} \sim \epsilon^2$. The SM does not contain any theoretical input that could explain the inter-family hierarchy of fermion masses and the pattern of the CKM mixing angles. Besides, the same is true for its supersymmetric or grand unified extensions. In a sense, the SM is technically natural since it can tolerate any pattern of the Yukawa matrices Y_f, but it tells nothing about their structures which remain arbitrary. So the origin of the fermion mass hierarchy and their weak mixing pattern remains a mystery.

It is tempting to think that the fermion flavour structure is connected to some underlying theory which determines the pattern of the Yukawa matrices, and that relations between masses and mixing angles such as the well-known formula for the Cabibbo angle $V_{us} = \sqrt{m_d/m_s}$ are not accidental. In particular, relations between the fermion masses and mixing angles can be obtained by considering Yukawa matrix textures with reduced number of free parameters, with certain zero elements. This *zero-texture* approach was originally thought to calculate the Cabibbo angle in the two-family framework in refs. [5–7], in fact before the discovery of b and t quarks.

In the frame of six quarks, in refs. [8, 9] H. Fritzsch extended the zero texture for the mass matrices in the form:

$$\mathrm{Y}_{u,d} = \begin{pmatrix} 0 & A'_{u,d} & 0 \\ A_{u,d} & 0 & B'_{u,d} \\ 0 & B_{u,d} & C_{u,d} \end{pmatrix} \tag{1.6}$$

where all non-zero elements are generically complex, with the symmetricity condition $|A_{u,d}| = |A'_{u,d}|$, $|B_{u,d}| = |B'_{u,d}|$ motivated in the context of the left-right symmetric models.[1] Besides reproducing the formula for the Cabibbo angle, this texture exhibits at least two remarkable features.

- By a phase transformation of the quark fields, $F'_{u,d}\mathrm{M}_{u,d}F_{u,d} = \widetilde{\mathrm{M}}_{u,d}$, the matrices (1.6) can be brought to real symmetric matrices $\widetilde{\mathrm{M}}_{u,d}$, which then can be diagonalized by orthogonal transformations, $O^T_{u,d}\widetilde{\mathrm{M}}_{u,d}O_{u,d} = \mathrm{M}^{\mathrm{diag}}_{u,d}$. In this way, the three real parameters $|A_d|$, $|B_d|$, $|C_d|$ can be expressed in terms of the three eigenvalues of $\widetilde{\mathrm{M}}_d$, i.e. the down quark masses m_d, m_s, m_b, and so the three rotation angles in the orthogonal matrix O_d can be expressed in terms of the mass ratios m_d/m_s and m_s/m_b. Analogously, the three angles in O_u can be expressed in terms of the upper quarks mass ratios m_u/m_c and m_c/m_t. The CKM matrix (1.3) is obtained as $V_{\mathrm{CKM}} = O^T_u F^*_u F_d O_d$, where the diagonal matrix $F = F^*_u F_d$ can be parameterized by two phase parameters, $F = \mathrm{diag}(e^{i\alpha}, e^{i\beta}, 1)$. Then, the four physical elements of the CKM matrix, that is the three mixing angles $\theta_{12}, \theta_{23}, \theta_{13}$ and the CP-phase δ, can be expressed in terms of the known mass ratios, m_d/m_s, m_s/m_b, m_u/m_c and m_c/m_t, and of two unknown phases α and β.

- In view of the interfamily hierarchies, $m_d \ll m_s \ll m_b$ and $m_u \ll m_c \ll m_t$, the Fritzsch ansatz (1.6) demonstrates an interesting property coined as the *decoupling hypothesis* [11]. Since the CKM angles depend on the quark mass ratios, in the limit in which the masses of the first family vanish, $m_u, m_d \to 0$, the mixing of the latter with the heavier families should disappear, i.e. $\theta_{12}, \theta_{13} \to 0$. At the next step, in the limit of massless second family, $m_s, m_c \to 0$, also the 2-3 mixing should disappear, i.e. $\theta_{23} \to 0$.

However, in the original works [8, 9], the 'zeros' in these matrices were achieved at the price of introducing several Higgs bi-doublets differently transforming under some discrete flavor symmetry. This underlying theoretical construction looks rather obsolete. Namely, the need for several Higgs bi-doublets spoils the natural flavor conservation [12–14] and unavoidably leads to severe flavor-changing effects [15]. In a more natural way, without employing the left-right symmetry, the Fritzsch texture can be obtained in the context of models with $\mathrm{SU}(3)_H$ gauge symmetry between the three families [16, 17], as we shall describe in this work.

[1] It has been shown in ref. [10] that Yukawa matrices of the form in eq. (1.6) without the assumption of hermiticity can always be obtained by a weak-basis transformation starting from arbitrary Yukawa matrices, that is, eq. (1.6) is derived by a specific choice of weak basis, the nearest-neighbour interaction basis, and does not imply any relation for the Yukawa couplings.

Moreover, in light of present experimental and lattice results on quark masses and CKM elements, the symmetric Fritzsch texture for quarks must be excluded, since there is no parameter space in which these precise data can be reproduced [18, 19]. More concretely, the small enough value of $|V_{cb}|$ and large enough value of $|V_{ub}/V_{cb}|$ cannot be achieved for any values of the phase parameters. A possibility to obtain viable textures is to extend the original Fritzsch texture by replacing one of the zero entries with a non-zero one, e.g. by introducing a non-zero 13 element [20] or a non-zero 22 element, as recently analysed e.g. in refs. [21–26]. However, these modifications do not satisfy the *decoupling* feature and the introduction of new parameters reduces the predictivity.

On the other hand, instead of decreasing the number of zero entries, one can think to break the symmetricity condition. Namely, an asymmetry in the 23 blocks of the Yukawa matrices of the form in eq. (1.6) can be introduced, $|B_{u,d}| \neq |B'_{u,d}|$ [27–29]. It is worth noting that in this scenario the properties of the original texture are preserved. In fact, the decoupling feature does not require the equality of the moduli of non-diagonal elements in (1.6). We will give a detailed study of such minimally modified Fritzsch ansatz. We show that this texture can predict all the correct masses, mixing angles and CP-violating phase. Moreover, we find that a scenario presenting the minimal modification $|B_d| \neq |B'_d|$ in the down-quarks sector while maintaining the symmetricity $|B_u| = |B'_u|$ in the up-quarks sector can accommodate all experimental data.

The paper is organized as follows. In section 2 we describe how the Fritzsch texture can be obtained within the context of the inter-family gauge group $\mathrm{SU}(3)_H$, and how it can be minimally deformed in the 2-3 blocks in presence of a scalar field in adjoint (octet) representation of $\mathrm{SU}(3)_H$. In section 3 we analyse the relations between the parameters of the asymmetric Fritzsch texture and the quark mass ratios and mixing angles. In section 4 we confront the minimally modified Fritzsch matrices with the recent high precision determinations of quark masses and CKM matrix elements, and show that this flavour structure predicts all the masses, the mixing angles and the CP-violating phase in perfect agreement with the experimental results. In section 5 we summarize our results.

2 Fritzsch-like textures from horizontal symmetry $\mathrm{SU}(3)_H$

2.1 Fermion masses with horizontal symmetry: effective opearators

The key for understanding the replication of families, fermion mass hierarchy and mixing pattern may lie in symmetry principles. For example, one can assign to fermion species different charges of an abelian global flavor symmetry U(1) [30]. There are also models making use of an anomalous gauge symmetry $\mathrm{U}(1)_A$ to explain the fermion mass hierarchy while also tackling other naturalness issues [31–35]. Abelian flavour symmetries with extra Higgs doublets have been used to generate Yukawa matrices with vanishing entries [36–39]. However, it is difficult to obtain the highly predictive quark mass matrices with six texture zeros within this approach.

Nonetheless, one can point to a more complete picture by introducing the non-abelian horizontal gauge symmetry $\mathrm{SU}(3)_H$ between three families [16, 17, 40–45]. This symmetry should have a chiral character, with the left-handed and right-handed components of quarks (and leptons) transforming in different representations of the family symmetry, namely as $SU(3)_H$ triplets and anti-triplets respectively, so that the fermions cannot acquire masses without the breakdown of $SU(3)_H$ invariance. In our chiral notations this means that all left-handed fields must transform as triplets:

$$q_i,\, u_i^c, d_i^c \sim 3 \qquad (\ell_i,\, e_i^c \sim 3) \tag{2.1}$$

where $i = 1, 2, 3$ is the family $SU(3)_H$ index. Such an arrangement is compatible with the grand unified extensions of the SM. In particular, in the context of SU(5) GUT [46] each family is represented by the left-handed Weyl fields $\bar{F}_i = (d^c, \ell)_i$ and $T_i = (u^c, e^c, q)_i$ respectively in $\bar{5}$ and 10 representations of SU(5). Then, the fermions can be arranged in the following representations of $SU(5) \times SU(3)_H$ [17, 40, 41]:

$$\bar{F}_i = (d^c,\, \ell)_i \sim (\bar{5}, 3), \qquad T_i = (u^c,\, q,\, e^c)_i \sim (10, 3) \tag{2.2}$$

while in the context of $SO(10) \times SU(3)_H$ all these fermions, along with the "right-handed neutrinos" $\nu_L^c = C\overline{\nu_R}^T$ can be packed into the unique multiplet in the spinor representation of SO(10), $\Psi_i = (\bar{F}, T, \nu^c)_i \sim (16, 3)$.[2]

Due to the chiral character of the horizontal symmetry, the fermion masses cannot be induced without breaking $SU(3)_H$, which forbids the direct Yukawa couplings of fermions (2.1) with the Higgs doublets ϕ. As far as the fermion bilinears $u_i^c q_j$, $d_i^c q_j$ and $e^c \ell$ transform in representations $3 \times 3 = 6 + \bar{3}$, the fermion masses can be induced only via the higher order operators

$$\frac{\chi^{ij}}{M}\, u_i^c q_j \phi + \frac{\chi^{ij}}{M}\, d_i^c q_j \tilde{\phi} + \frac{\chi^{ij}}{M}\, e_i^c \ell_j \tilde{\phi} + \text{h.c.} \tag{2.3}$$

involving some horizontal scalars χ (coined as flavons) in symmetric (anti-sextets $\chi^{\{ij\}} \sim \bar{6}$) or antisymmetric (triplets $\chi^{[ij]} = \epsilon^{ijk}\chi_k \sim 3$) representations of $SU(3)_H$, where M is some effective scale (the coupling constants of different flavons are omitted). After inserting the flavon VEVs in the operators (2.3), the standard Yukawa couplings (1.1) are induced which will reflect the VEVs pattern. Extending the SM to SU(5) GUT, in the context of $SU(5) \times SU(3)_H$ theory [17, 40], the Yukawa couplings emerge from the decomposition of the SU(5)-invariant Yukawa couplings

$$G_u^{ij} T_i T_j H + G_d^{ij} \bar{F}_i T_j H^* + \text{h.c.} \tag{2.4}$$

where H is the scalar 5-plet which contains the SM Higgs doublet ϕ. G_u^{ij} and G_d^{ij} are effective Wilson coefficients of operators containing flavons, emerging from the structures $(\chi^{ij}/M) T_i T_j H$ and $(\chi^{ij}/M) \bar{F}_i T_j H^*$. Some χ-flavons can also be in adjoint representations of SU(5), or more generally these effective coefficients should involve a scalar 24-plet Σ of SU(5) in order to avoid the undesiderable relations between the down quark and lepton

[2] With this set of fermions, $SU(3)_H$ would have triangle anomalies. For their cancellation one can introduce additional chiral fermions transforming under $SU(3)_H$ [16, 17, 40, 41]. The easiest way to cancel the anomalies is to share the $SU(3)_H$ symmetry with mirror fermions [47] belonging to a parallel SM$'$ sector of particles identical to the SM sector of ordinary particles (for a review, see e.g. [48, 49]).

masses [17].[3] In the following, we mainly concentrate on the quark sector in the context of the SM, having in mind that in the context of grand unification analogous considerations can be extended to leptons.

Interestingly, operators (2.3) which are invariant under the local $\mathrm{SU}(3)_H$ symmetry by construction, in fact have a larger global symmetry $\mathrm{U}(3)_H$. Namely, they are invariant also under a global chiral $\mathrm{U}(1)_H$ symmetry, implying an overall phase transformation of fermions u^c_i, d^c_i, q_i and flavon scalars χ. Hence, all families can become massive only if $\mathrm{U}(3)_H$ symmetry is fully broken.

This feature allows to relate the fermion mass hierarchy and mixing pattern with the breaking pattern of $\mathrm{U}(3)_H$ symmetry, with a natural realization of the decoupling hypothesis. When $\mathrm{U}(3)_H$ breaks down to $\mathrm{U}(2)_H$, the third family of fermions become massive while the first two families remain massless, and mixing angles are zero. At the next step, when $\mathrm{U}(2)_H$ breaks down to $\mathrm{U}(1)_H$, the second family acquires masses and the CKM mixing angle θ_{23} can be non-zero, but the first family remains massless ($m_u, m_d = 0$) and unmixed with the heavier fermions ($\theta_{12}, \theta_{13} = 0$). Only at the last step, when $\mathrm{U}(1)_H$ is broken, also the first family can acquire masses and its mixing with heavier families can emerge. In this way, the inter-family mass hierarchy can be related to the hierarchy of flavon VEVs inducing the horizontal symmetry breaking $\mathrm{U}(3)_H \to \mathrm{U}(2)_H \to \mathrm{U}(1)_H \to$ *nothing*.

In the last step of this breaking chain, the chiral global $\mathrm{U}(1)_H$ symmetry can be associated with the Peccei-Quinn symmetry provided that $\mathrm{U}(1)_H$ is also respected by the Lagrangian of the flavon fields [17, 51]. This can be achieved by forbidding the trilinear terms between the χ-scalars by means of a discrete symmetry. Thus, in this framework, the Peccei-Quinn symmetry can be considered as an accidental symmetry emerging from the local symmetry $\mathrm{SU}(3)_H$. In this case the axion will have non-diagonal couplings between the fermions of different families, i.e. it will act as a familon [16, 51]. Phenomenological and cosmological implications of such flavor-changing axion were discussed in refs. [52–61].

Let us discuss now how Fritzsch zero textures can naturally emerge in this scenario with horizontal symmetry. As the simplest set of χ-flavons, we can choose two triplets χ_1, χ_2, and one anti-sextet χ_3, and arrange their VEVs in the following form [17]:

$$\langle \chi_3^{\{ij\}} \rangle = \mathrm{diag}(0,0,V_3) \qquad \langle \chi_{2i} \rangle = \begin{pmatrix} V_2 \\ 0 \\ 0 \end{pmatrix} \qquad \langle \chi_{1i} \rangle = \begin{pmatrix} 0 \\ 0 \\ V_1 \end{pmatrix} \tag{2.5}$$

[3]The minimal scenario, with matrices G_u and G_d being SU(5) singlets, would imply $Y_u = Y_u^T$ and $Y_e = Y_d^T$, the latter equality leading to incorrect relations between the down quark and charged lepton masses. However, this shortcoming can be avoided in a more general context, by considering $G_{u,d} = G_{u,d}(\Sigma/M)$ as functions of the scalar Σ in the adjoint representation (24-plet) which breaks SU(5) down to the SM gauge group $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ at the GUT scale $M_G \simeq 10^{16}$ GeV or so. This is equivalent to introducing higher order operators in powers of Σ/M, which can be obtained e.g. by integrating out some heavy vector-like fermions at the mass scale $M > M_G$. In this way, the expansions $G_{u,d}(\Sigma) = G^{(0)}_{u,d} + G^{(1)}_{u,d}(\Sigma/M) + G^{(2)}_{u,d}(\Sigma^2/M^2) + \ldots$ will in general contain terms in 1, 24 etc. representations of SU(5) which remove the above restrictive relations $Y_u = Y_u^T$ and $Y_e = Y_d^T$ and render the Yukawa matrices $Y_{u,d,e}$ in the low energy SM to be independent from each other (for a review, see e.g. [50]).

i.e. the VEV of χ_3 is given by a symmetric rang-1 matrix directed towards the 3rd axis in the SU(3)$_H$ space, the VEV of χ_1 is parallel to $\langle\chi_3\rangle$ and the VEV of χ_2 is orthogonal to it and without losing generality it can be oriented towards the 1st axis (for the analysis of the flavon potential allowing such a solution see ref. [40]). The total matrix of flavon VEVs has the form

$$\langle\chi^{ij}\rangle = \left\langle \chi_1^{[ij]} + \chi_2^{[ij]} + \chi_3^{\{ij\}} \right\rangle = \begin{pmatrix} 0 & V_1 & 0 \\ -V_1 & 0 & V_2 \\ 0 & -V_2 & V_3 \end{pmatrix} \tag{2.6}$$

Then, modulo different coupling constants of χ-flavons in the two operators in (2.3), the Yukawa matrices $Y_u, Y_d \propto \langle\chi\rangle/M$ will reflect the VEV pattern (2.6). Hence, the Yukawa matrices would acquire the 'symmetric' Fritzsch forms (1.6) with $A'_{u,d} = A_{u,d}$ and $B'_{u,d} = B_{u,d}$ (the $-$ signs can be eliminated by quark phase transformations). The hierarchies between the different Yukawa entries, corresponding to the inter-family mass hierarchies, can be related to a hierarchy $V_3 \gg V_2 \gg V_1$ in the horizontal symmetry breaking chain U(3)$_H$ $\to$ U(2)$_H$ $\to$ U(1)$_H$ $\to$ *nothing*. After this breaking, the theory reduces to the SM with one standard Higgs doublet ϕ, and so, in difference from the Fritzsch's original model [8, 9], in our construction the flavor will be naturally conserved in neutral currents [12–14].

2.2 UV completing and the role of heavy vector-like fermions

In the UV-complete pictures the operators (2.3) can be induced via renormalizable interactions after integrating out some extra heavy fields, scalars [40] or verctor-like fermions [16, 17, 51]. Hereafter we shall employ the second possibility. Namely, one can introduce the following set of left-handed fermions of up- and down-quark type in weak singlet representations

$$U_i,\ D_i \sim 3 \qquad U^{ci},\ D^{ci} \sim \bar{3} \tag{2.7}$$

These fermions are allowed to have SU(3)$_H$ invariant mass terms. More generically, their mass terms transform as $\bar{3} \times 3 = 1 + 8$ and they can emerge from the Yukawa couplings with the scalars in singlet and octet representations of SU(3)$_H$, $S \sim 1$ and $\Phi \sim 8$, namely

$$(g_D S\delta^i_j + f_D \Phi^i_j) D_i D^{cj} + \text{h.c.} \tag{2.8}$$

with analogous couplings for U_i, U^{ci}. In fact, one can introduce an adjoint scalar Φ of SU(3)$_H$ in analogy to the adjoint scalar of SU(5), the 24-plet Σ. We also assume that the cross-interaction terms of Φ with χ-flavons in the scalar potential align the Φ VEV towards the largest VEV V_3 in (2.6), i.e. proportionally to the λ_3 generator: $\langle\Phi\rangle = V\,\text{diag}(1, 1, -2)$. In this case, the heavy fermion mass matrices contributed by singlet and octet VEVs have the diagonal forms

$$\mathrm{M}_{U,D} = g_{U,D}\langle S\rangle + f_{U,D}\langle\Phi\rangle = M_{U,D}\,\text{diag}(X^{-1}_{U,D}, X^{-1}_{U,D}, 1) \tag{2.9}$$

where $M_{U,D} \sim M$ is an overall mass scale determined by the VEVs of S and Φ and generically $X_{U,D} \neq 1$ are complex numbers. Only in the absence of the octet contribution we have $X_{U,D} = 1$.

The following Yukawa couplings between the light quarks $q_i, u_i^c, d_i^c \sim 3$ (2.1) and heavy fermions (2.7) are allowed by the symmetry

$$\sum_{n=1}^{3} h_u^{(n)} \chi_n^{ij}\, u_i^c U_j + \sum_{n=1}^{3} h_d^{(n)} \chi_n^{ij}\, d_i^c D_j + y_u \phi\, U^{ci} q_i + y_d \tilde{\phi}\, D^{ci} q_i \tag{2.10}$$

with couplings $h_{u,d}$ and $y_{u,d}$. In this way, the matrices of Yukawa couplings $Y_{u,d}$ in eq. (1.1) are induced after integrating out the heavy fermions via universal seesaw mechanism [16, 17]. Namely, for the upper quarks this mechanism is illustrated by the first diagram in figure 1 while the analogous diagram involving D and D^c states will work for the down quarks. More generally, also heavy quarks in weak doublet representations $Q^i \sim \bar{3}$ and $Q_i^c \sim 3$ can be used for quark mass generation (see the second diagram in figure 1). However, this would not affect the final form of the Yukawa matrices (1.6) which we shall discuss in this work.[4] Analogously, the charged lepton Yukawa couplings can be induced by introducing the vector like lepton states, weak singlets $E_i \sim 3$, $E^{ci} \sim \bar{3}$ and weak doublets $L^i \sim \bar{3}$, $L_i^c \sim 3$. In particular, they will be at work in the case of $\mathrm{SU}(5) \times \mathrm{SU}(3)_H$ extension [17] where all these states fit into the set of fermions in vector-like representations $(\bar{5}, \bar{3}) = (D^{ci}, L^i)$, $(5, 3) = (D_i, L_i^c)$ and $(10, \bar{3}) = (U^{ci}, Q^i, E^{ci})$, $(\overline{10}, 3) = (U_i, Q_i^c, E_i)$. Interestingly, in the context of supersymmetry our mechanism can lead to interesting relations between the fermion Yukawa couplings and the soft SUSY breaking terms which allow to naturally realize the minimal flavor violation scenarios [47, 70–72].

After substituting the flavon VEVs into eq. (2.10), we obtain

$$Y_u = \chi_u \mathrm{M}_U^{-1} y_u, \qquad Y_d = \chi_d \mathrm{M}_D^{-1} y_d; \qquad\qquad \chi_{u,d}^{ij} = \sum_{n=1}^{3} h_{u,d}^{(n)} \langle \chi_n^{ij} \rangle \tag{2.11}$$

Therefore, the Yukawa couplings of quarks will have the forms

$$\begin{aligned} Y_d &= \frac{y_d}{M_D} \begin{pmatrix} 0 & X_D h_D^{(1)} V_1 & 0 \\ -X_D h_D^{(1)} V_1 & 0 & h_D^{(2)} V_2 \\ 0 & -X_D h_D^{(2)} V_2 & h_D^{(3)} V_3 \end{pmatrix}, \\ Y_u &= \frac{y_u}{M_U} \begin{pmatrix} 0 & X_U h_U^{(1)} V_1 & 0 \\ -X_U h_U^{(1)} V_1 & 0 & h_U^{(2)} V_2 \\ 0 & -X_U h_D^{(2)} V_2 & h_U^{(3)} V_3 \end{pmatrix} \end{aligned} \tag{2.12}$$

where the non-zero entries are generically complex. By the phase transformations $\widetilde{Y}_{u,d} = F'_{u,d} Y_{u,d} F_{u,d}$, where $F_{d,u} = \mathrm{diag}(e^{i\alpha_{d,u}}, e^{i\beta_{d,u}}, e^{i\gamma_{d,u}})$, the Yukawa matrices can be brought

[4] Mixing of the light quarks with vector-like quarks with mass of order TeV can be at the origin of the recently observed Cabibbo angle anomalies [62–68] (see ref. [69] for a review).

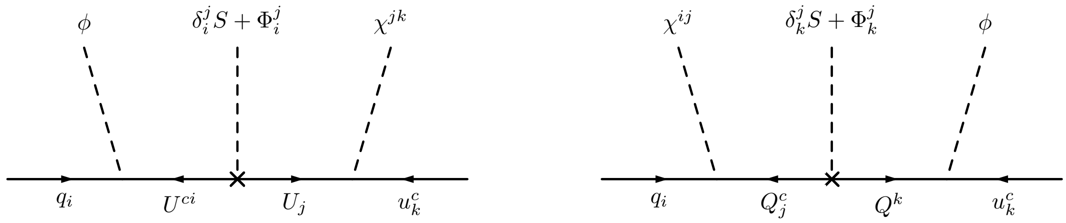

Figure 1. Seesaw diagrams inducing the Yukawa couplings of upper quarks via exchange of vector-like quarks U, U^c and Q, Q^c. Analogous diagrams with $U, U^c \to D, D^c$ and $\phi \to \tilde{\phi}$ will work for down quarks.

to the forms

$$\widetilde{Y}_d = \begin{pmatrix} 0 & A_d & 0 \\ A_d & 0 & x_d B_d \\ 0 & x_d^{-1} B_d & C_d \end{pmatrix}, \quad \widetilde{Y}_u = \begin{pmatrix} 0 & A_u & 0 \\ A_u & 0 & x_u B_u \\ 0 & x_u^{-1} B_u & C_u \end{pmatrix} \tag{2.13}$$

with all parameters being real and positive. In absence of the $\mathrm{SU}(3)_H$ octet contribution in the heavy fermion masses we would have $x_{u,d} = 1$ and thus we would effectively obtain the "symmetric" Fritzsch ansatz. However, this possibility is excluded since it predicts too large value of $|V_{cb}|$ and too small value of $|V_{ub}/V_{cb}|$.

2.3 A prototype model: $\mathbf{SU(5) \times SU(3)_{\mathit{H}}}$

Let us conclude this section by presenting a prototype model which gives $x_u = 1$ in the up quark Yukawa matrix in eq. (2.13), that is the symmetric Fritzsch texture, by symmetry-based arguments in the context of $\mathrm{SU}(5) \times \mathrm{SU}(3)_H$ theory. In this case the quark states should be "packed" together with leptons in multiplets $\bar{5}_i = (\bar{5}, 3)$ and $10_i = (10, 3)$, as in (2.2). For the 'generation' of fermion masses we introduce the extra vector-like fermions like $\mathcal{T}^i = (Q, U^c, E^c)^i$ and $\overline{\mathcal{T}}_i = (Q^c, U, E)_i$ respectively in representations $(10, \bar{3}) + (\overline{10}, 3)$, which are in principle sufficient for inducing the up quark, down quark and charged lepton masses. Additional contribution to the down quark and lepton masses can be obtained by involving also $\overline{\mathcal{F}}^i = (D^c, L)^i$ and $\mathcal{F}_i = (D, L^c)_i$ in $(\bar{5}, \bar{3}) + (5, 3)$ representations.

Let us involve the real scalars in the following representations: $\Sigma \sim (24, 1)$ and $\Phi \sim (1, 8)$ in the adjoint representations of $\mathrm{SU}(5)$ and $\mathrm{SU}(3)_H$ respectively, and two singlets $R, S \sim (1, 1)$. We also keep three complex flavon scalars $\chi_{1,2,3}$ with the VEVs forming the Fritzsch-like texture as in eq. (2.6) in the $\mathrm{SU}(3)_H$ space. In addition, flavons χ_1 and χ_3 remain in representations $(1, 3)$ and $(1, \bar{6})$ while χ_2 is taken in the mixed representation $(24, 3)$, for reasons that will become clear later. Needless to say, for inducing the quark and lepton masses we must also introduce the scalar 5-plet $H = (\phi, \tau) \sim (5, 1)$ of $\mathrm{SU}(5)$ which includes the SM Higgs doublet ϕ along with its color-triplet partner τ.

As for the fermion sector, it consists of the following multiplets:

$$\begin{aligned} &T \sim (10, 3), \qquad \overline{F}_{1,2} \sim (\bar{5}, 3), \qquad F \sim (5, \bar{3}), \\ &\mathcal{T} \sim (10, \bar{3}), \qquad \overline{\mathcal{T}} \sim (\overline{10}, 3); \qquad \overline{\mathcal{F}} \sim (\bar{5}, \bar{3}), \qquad \mathcal{F} \sim (5, 3) \end{aligned} \tag{2.14}$$

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Let us introduce now two discrete symmetries under which the following fields change sign:

$$\begin{aligned}
&Z_\Sigma: \quad \Sigma \to -\Sigma, \quad R \to -R, \quad \overline{\mathcal{T}} \to -\overline{\mathcal{T}}, \quad \mathcal{F} \to -\mathcal{F}, \quad \chi_{1,2,3} \to -\chi_{1,2,3}\\
&Z_\Phi: \quad \Phi \to -\Phi, \quad S \to -S, \quad F \to -F
\end{aligned} \tag{2.15}$$

while the rest of the fields remain invariant under these symmetries.

The Yukawa couplings invariant under SU(5) $\times$ SU(3)$_H$ and compatible with these discrete symmetries are the following:

$$\begin{aligned}
\mathcal{L}_{\rm Yuk} = & \, yT\mathcal{T}H + \tilde{y}T\overline{\mathcal{F}H} + y_1\overline{F}_1\mathcal{T}\overline{H} + y_2\overline{F}_2\mathcal{T}\overline{H}\\
& + (g\Sigma + g'R)\mathcal{T}\overline{\mathcal{T}} + (f\Sigma + f'R)\overline{\mathcal{F}}\mathcal{F}\\
& + (a_1S + a_1'\Phi)\overline{F}_1F + (a_2S + a_2'\Phi)\overline{F}_2F\\
& + h^{(n)}\chi_nT\overline{\mathcal{T}} + h_1^{(n)}\chi_n\overline{F}_1\mathcal{F} + h_2^{(n)}\chi_n\overline{F}_2\mathcal{F} + \text{h.c}
\end{aligned} \tag{2.16}$$

where $a, a', \ldots$ are couplings of order $O(1)$. The family indices as well as the indices of SU(5) are suppressed, and $\overline{H}$ stands for the complex conjugate of H. Notice that the terms in the first and second rows do not contain the scalars which break the family symmetry and they remain degenerate between the three families also after SU(3)$_H$ breaking. On the contrary, the third and fourth rows carry the information on the SU(3)$_H$ breaking pattern.

Let us remark that the discrete symmetry Z_Σ, namely $\chi_{1,2,3} \to -\chi_{1,2,3}$, forbids the trilinear couplings between the flavon fields. Then, at the level of renormalizable couplings, the Lagrangian of the theory acquires an accidental global symmetry under the flavon phase transformation (in fact, the Yukawa terms (2.16) automatically respect this symmetry, under proper transformation of the fermion phases). This chiral global symmetry U(1)$_H$ has color anomaly and it can be considered as the Peccei-Quinn symmetry which is spontaneously broken by the smallest of the flavon VEVs in eq. (2.5) (see also refs. [16, 17, 51]).

For a proper range of parameters in the Higgs potential, the scalar fields get VEVs, with $\langle\Sigma\rangle = V_\Sigma \cdot \text{diag}(1/3, 1/3, 1/3, -1, -1)$ breaking SU(5) down to SU(3) $\times$ SU(2) $\times$ U(1), $\langle\Phi\rangle = V_\Phi \cdot \text{diag}(1, 1, -2)$ breaking SU(3)$_H$ down to SU(2)$_H$ $\times$ U(1)$_H$, and $\langle R, S\rangle = V_{R,S}$. We assume that these VEVs are all of the order of the GUT scale V_Σ.[5] As for the flavon fields, they have the VEV pattern shown in eq. (2.5) in the SU(3)$_H$ space, while the VEV of χ_2 breaks also SU(5), $\langle\chi_2\rangle = (V_2, 0, 0) \otimes \text{diag}(1/3, 1/3, 1/3, -1, -1)$.

The masses of ordinary fermions, up quarks, down quarks and charged leptons of the three families, are induced by the Yukawa couplings in the first row of eq. (2.16) involving the SM Higgs doublet $\phi \subset H$, after integrating out the heavy fermion species. Notice that the discrete symmetries (2.15) prevent the vector-like fermion species in (2.14) to have arbitrarily large masses, much larger than the GUT scale V_Σ.

[5]We assume that the VEV $\langle H\rangle$ of order 100 GeV is due to fine-tuning, without discussing the questions of the grand-hierarchy and doublet-triplet splitting. The latter problems can be addressed in supersymmetric extension of the model, and most naturally in the context of supersymmetric SU(6) theory, with specific implications also for the fermion mass generation [74–76]. In SU(6) theory the fermions of one family are settled in representations $15 + \overline{6}_{1,2}$ which decomposition under SU(6) $\to$ SU(5) reads as $15 = 10 + 5$ and $\overline{6}_{1,2} = \overline{5}_{1,2}$ (plus two singlets, that can be considered as "right-handed neutrinos" for the neutrinos mass generation). In fact, our choice of multiplets T, F and $\overline{F}_{1,2}$ in first line of (2.14) is motivated by the possibility of embedding SU(5) in SU(6).

The couplings in the second line of eq. (2.16), after substituting the VEVs of Σ and R, induce the Dirac mass terms between the vector-like species $Q, U^c, E^c \subset \mathcal{T}$ and $Q^c, U, E \subset \overline{\mathcal{T}}$, and analogously between $D^c, L \subset \overline{\mathcal{F}}$ and $D, L^c \subset \mathcal{F}$. These masses are degenerate in $\mathrm{SU}(3)_H$ space, but they have SU(5) splitting induced by the non-zero contribution of $\langle\Sigma\rangle$ which depends on the U(1) hypercharge of the fermion species. Namely, we get

$$M_\Psi = M_{10}(1+\kappa r_{10} Y_\Psi), \quad \Psi = Q, U^c, E^c; \quad M_\Psi = M_5(1+\kappa r_5 Y_\Psi), \quad \Psi = D^c, L \tag{2.17}$$

where $M_{10} = g' V_R$ and $M_5 = f' V_R$ can be taken real without loss of generality. The Yukawa ratios $r_{10} = g/g'$ and $r_5 = f/f'$ are in general complex and $\kappa = V_\Sigma/V_R$ determines the size of SU(5) splittings between the fermion fragments, with Y_Ψ being the U(1) hypercharge of the given species (normalized as $Y_Q = 1/3$, $Y_{D^c} = 2/3$ etc.).

Analogously, the Yukawa terms of the third row in eq. (2.16) induce the Dirac masses between the 5-plets F^i and one combination $\overline{F}_i$ of the two identical species $\overline{F}_{1i}$ and $\overline{F}_{2i}$ ($i = 1,2,3$ being the family index), with mixing angles θ_i determined by the proportion between the mass values $M_{1i} = a_1 V_S + a_1' V_\Phi \mathcal{Y}_i$ and $M_{2i} = a_2 V_S + a_w' V_\Phi \mathcal{Y}_i$. The other (orthogonal) combination of $\bar{5}$-plets $\overline{F}'_i$ remains light and it represents in fact the multiplet which contains the light species $d_i^c, \ell_i \subset \overline{F}'_i$. The masses M_{1i} and M_{2i} have no splitting in the SU(5) space but they are split in the family space by the contribution of $\langle\Phi\rangle = V_\Phi \cdot \mathrm{diag}(1,1,-2)$, which breaks $\mathrm{SU}(3)_H$ down to $\mathrm{SU}(2)_H \times \mathrm{U}(1)_H$. Namely, the splitting is given by the values of $\mathrm{U}(1)_H$ hypercharges $\mathcal{Y}_{1,2} = 1$, $\mathcal{Y}_3 = -2$. Therefore, after this breaking, $\overline{F}'_i$ does not form anymore an $\mathrm{SU}(3)_H$ triplet but rather a $2+1$ representation of $\mathrm{SU}(2)_H \times \mathrm{U}(1)_H$. The mixing angle θ_i is different for $i = 1,2$ and $i = 3$, and thus the original states $\overline{F}_{1i}$ and $\overline{F}_{2i}$ entering in the Yukawa Lagrangian in eq. (2.16) contain the survived light states $\overline{F}'_i$ with different weights, respectively $\sin\theta_i$ and $\cos\theta_i$, $\theta_1 = \theta_2 \neq \theta_3$.

Let us consider first the up quark masses which are induced by integrating out the heavy species $\mathcal{T}, \overline{\mathcal{T}}$. Substituting the large VEVs in the relevant Yukawa couplings of (2.16), we obtain the full 9×9 mass matrix of all (light and heavy) up quark states

$$\begin{pmatrix} u^c & Q^c & U^c \end{pmatrix} \begin{pmatrix} 0 & yHI & \chi_{u^c}^T \\ \chi_q & M_Q I & 0 \\ yHI & 0 & M_{U^c} I \end{pmatrix} \begin{pmatrix} q \\ Q \\ U \end{pmatrix} \tag{2.18}$$

where the diagonal 3×3 blocks given by eq. (2.17) are degenerate in families (I denotes the 3×3 unit matrix). Namely, taken the hypercharges $Y_Q = 1/3$ and $Y_{U^c} = -4/3$, we get $M_Q = M_{10}(1 + \frac{1}{3}\kappa r_{10})$ and $M_{U^c} = M_{10}(1 - \frac{4}{3}\kappa r_{10})$.

The pattern of the flavor symmetry breaking is encoded in the off-diagonal blocks composed by the VEVs of the flavon fields:

$$\chi_\psi = \sum_n h^{(n)} \langle \chi_n \rangle = \begin{pmatrix} 0 & h^{(1)} V_1 & 0 \\ -h^{(1)} V_1 & 0 & h^{(2)} V_2 Y_\psi \\ 0 & -h^{(2)} Y_\psi V_2 & h^{(3)} V_3 \end{pmatrix} \tag{2.19}$$

where again we have $Y_q = 1/3$ in χ_q and $Y_{u^c} = -4/3$ in χ_{u^c}. Then, after rotating away the heavy states, for the up quark Yukawa matrix we obtain in the seesaw approximation:

$$Y_u = -y(M_Q^{-1}\chi_q + \chi_{u^c}^T M_{U^c}^{-1}) = \begin{pmatrix} 0 & A_u & 0 \\ -A_u & 0 & B_u \\ 0 & -B_u & C_u \end{pmatrix} \tag{2.20}$$

where we have

$$\begin{aligned} C_u &= \frac{(2-\kappa r_{10})y}{(1-\kappa r_{10}-\frac{4}{9}\kappa^2 r_{10}^2)}\cdot\frac{h^{(3)}V_3}{g'V_R}, \\ B_u &= \frac{5y}{3(1-\kappa r_{10}-\frac{4}{9}\kappa^2 r_{10}^2)}\cdot\frac{h^{(2)}V_2}{g'V_R}, \qquad & \frac{B_u}{C_u} &= \frac{5}{3(2-\kappa r_{10})}\cdot\frac{h^{(2)}V_2}{h^{(3)}V_3} \\ A_u &= \frac{-5\kappa r_{10} y}{3(1-\kappa r_{10}-\frac{4}{9}\kappa^2 r_{10}^2)}\cdot\frac{h^{(1)}V_1}{g'V_R}, \qquad & \frac{A_u}{B_u} &= -\kappa r_{10}\frac{h^{(1)}V_1}{h^{(2)}V_2} \end{aligned} \tag{2.21}$$

Hence, in this model the up-quarks Yukawa matrix has a Fritzsch texture without any deformation. By rotating the fermion phases, the matrix (2.20) can be transformed into the real symmetric form $\widetilde{Y}_u$ in (2.13) with $x_u = 1$. The matrix elements are related to the up-quark Yukawas $y_{u,c,t}$ approximately as $C_u \approx y_t$, $B_u \approx \sqrt{y_c y_t}$ and $A_u \approx \sqrt{y_u y_c}$.

Given that all Yukawa couplings in the model are assumed to be order 1, then the top Yukawa value $y_t \sim 1$ implies $V_3 \simeq V_R$. The Yukawa hierarchy $y_t : y_c : y_u \simeq 1 : 4\times 10^{-3} : 10^{-5}$ can be related to the milder hierarchy in $SU(3)_H$ symmetry breaking VEVs, $V_2/V_3 \sim B_u/C_u \approx \sqrt{y_c/y_t} \simeq 1/15$. As for the small eigenvalue y_u, in the absence of the Σ contribution in the masses of the heavy 10-plets $\mathcal{T}$, $\overline{\mathcal{T}}$, i.e. in the limit $\kappa = V_\Sigma/V_R \to 0$, we would get $A_u \to 0$ and thus $y_u \to 0$. This is because the antisymmetric flavon $\chi_1^{[ij]}$ cannot contribute since the combination $T_i T_j H$ is symmetric in family indexes i and j. However, the matrix $(g'\langle R\rangle + g\langle\Sigma\rangle)^{-1}$ in $SU(5)$ space contains a piece of 24-plet, and so the triplet flavon $\chi_1 \sim (1, 3)$ can effectively contribute anti-symmetrically in the entries $12-21$, in combination with the effective 45-plet contained in the tensor product ΣH, $45 \subset 24\times 5$. As for $\chi_2 \sim (24, 3)$, it also can contribute anti-symmetrically in the entries $23-32$ as an effective 45-plet.[6] The small value $A_u/B_u \approx \sqrt{y_u/y_t} \simeq 1/300$ can come from the ratio $\kappa = V_\Sigma/V_R \sim 0.1$. Namely, the correct relations between $y_{u,c,t}$ can be obtained by taking the VEV ratios of the same order, for example $V_1/V_2 \sim V_2/V_3 \sim V_\Sigma/V_R \sim 1/15$.

As regards the down quarks and charged leptons, as described above, after integrating out the heavy $F + \overline{F}$ states, the fragments d_i^c, ℓ_i are contained in the effective light combinations $\overline{F}'_i = \sin\theta_i \overline{F}_{1i} + \cos\theta_i \overline{F}_{2i}$ which do not form anymore an $SU(3)_H$ triplet since $\theta_{1,2} \neq \theta_3$. Then, the Yukawa terms $(y_1\overline{F}_{1i} + y_2\overline{F}_{2i})\mathcal{T}^i\overline{H}$ contained in eq. (2.16) can be reduced to $y'_i\overline{F}'_i\mathcal{T}^i\overline{H}$ where $y'_i = y_1\cos\theta_i + y_2\sin\theta_i$, and so we have $y'_1 = y'_2 \neq y'_3$. Therefore, by denoting $y'_3 = y'$ and $y'_{1,2} = X_D y'$, the above Yukawa terms can be written as $y'_i\overline{F}'_i X^i_j \mathcal{T}^j\overline{H}$, where $X = \mathrm{diag}(x, x, 1)$ is the diagonal matrix. Needless to say, these terms maintain $SU(5)$ degeneracy between the down quark d^c and lepton ℓ states.

[6] Interestingly, in the context of supersymmetric $SU(5)$ theory, such a structure can also lead to natural suppression of dangerous $D = 5$ operators since the quark couplings $q_i q_i \tau$ with the color-triplet in H are symmetric, and hence cannot be induced by antisymmetric flavons [73].

Hence, in terms of light states $d_i^c, \ell_i \subset \overline{F}'_i$ we obtain the full 9×9 mass matrices of all (light and heavy) down quark and charged lepton type species respectively as

$$\begin{pmatrix} d^c & Q^c & D^c \end{pmatrix} \begin{pmatrix} 0 & y'\overline{H}X & \chi_{d^c}^T \\ \chi_q & M_Q I & 0 \\ \tilde{y}\overline{H}I & 0 & M_{D^c}I \end{pmatrix} \begin{pmatrix} q \\ Q \\ D \end{pmatrix},$$
$$\begin{pmatrix} \ell & E & L \end{pmatrix} \begin{pmatrix} 0 & y'X\overline{H} & \chi_\ell^T \\ \chi_{e^c} & M_{E^c}I & 0 \\ \tilde{y}\overline{H}I & 0 & M_L I \end{pmatrix} \begin{pmatrix} e^c \\ E^c \\ L^c \end{pmatrix} \tag{2.22}$$

where the mass terms in the diagonal 3×3 entries M_{E^c}, M_{D^c} and M_L are given by eq. (2.17), modulo their U(1) hypercharges Y_{E^c}, etc. Notice also that the blocks $M_Q I$ and χ_q are the same as in the up-quark case (see eq. (2.22)). As for the off-diagonal leptonic entry χ_{e^c}, which is originated from the couplings $h^{(n)}\chi_n T\overline{\mathcal{T}}$ in eq. (2.16), it carries the Fritzsch-like texture given by eq. (2.19), with $Y_{e^c} = 2$.

The situation is different for the matrices χ_{d^c} and χ_ℓ. In fact, the couplings $(h_1^{(n)}\overline{F}_{1i} + h_2^{(n)}\overline{F}_{2i})\chi_n^{ij}\mathcal{F}^i$ in terms of the light states $\overline{F}'_i$ reduces to the terms $\overline{F}'_i(\tilde{h}_i^{(n)}\chi_n^{ij})\mathcal{F}^i$, where $\tilde{h}_i^{(n)} = h_1^{(n)}\cos\theta_i + h_2^{(n)}\sin\theta_i$. Hence, the matrix χ_{d^c} (and similarly for χ_ℓ) has a deformed Fritzsch-like form and it can be presented as $\tilde{\chi}_{d^c}\tilde{X}$ where $\tilde{\chi}_{d^c}$ has the Fritzsch texture like (2.19) and $\tilde{X} = \mathrm{diag}(\tilde{x}, \tilde{x}, 1)$ is the deformation matrix.

Thus, after integrating out the heavy states, for the Yukawa matrices of down quarks and charged leptons we obtain the modified Fritzsch textures

$$Y_d = -y'XM_Q^{-1}\chi_q - \tilde{y}\tilde{X}\tilde{\chi}_{d^c}^T M_{D^c}^{-1} = \begin{pmatrix} 0 & A_d & 0 \\ -A_d & 0 & x_d B_d \\ 0 & -x_d^{-1}B_d & C_d \end{pmatrix},$$
$$Y_e^T = -y'XM_{E^c}^{-1}\chi_{e^c} - \tilde{y}\tilde{X}\tilde{\chi}_\ell^T M_L^{-1} = \begin{pmatrix} 0 & A_e & 0 \\ -A_e & 0 & x_e B_e \\ 0 & -x_e^{-1}B_e & C_e \end{pmatrix} \tag{2.23}$$

where in general $x_{d,e} \neq 1$. The detailed analysis of the lepton sector is beyond the scope of this paper. Let us note, however, that the famous SU(5) degeneracy between the down quark and charged lepton masses is essentially violated between $23-32$ entries induced by the VEV of the flavon field $\chi_2 \sim (24, 3)$, $B_d \neq B_e$. As for other entries, we have the approximate relations $C_d \approx C_e$ and $A_d \approx A_e$ provided that $V_\Sigma < V_R$. This means that, e.g. for $\kappa = V_\Sigma/V_R \sim 0.1$ or so, there are interesting relations $y_b \approx y_\tau$ and $y_d y_s \approx y_e y_\mu$ fulfilled at about 10 % accuracy. For the hierarchy of the $\mathrm{SU}(3)_H$ breaking scales we expect $V_1/V_3 \sim A_e/C_e \approx \sqrt{y_e y_\mu}/y_\tau \simeq (1/15)^2$, and thus the relative smallness of the ratio $A_u/C_u \sim \kappa V_1/V_3$ is naturally explained by small value of $\kappa = V_\Sigma/V_R \sim 1/15$. In other words, In the limit $\kappa \to 0$ we would have $y_u = 0$ and $A_d = A_e$, $C_d = C_e$. Hence, the precision of the relations $y_b \approx y_\tau$ and $y_d y_s \approx y_e y_\mu$ is in fact due to the small value of y_u.

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3 Parameters of the asymmetric Fritzsch texture

The Yukawa matrices in the Fritzsch form (1.6) can be diagonalized by biunitary transformations parameterized as:

$$(F_d'O_d')^\dagger\, Y_d\,(F_dO_d) = \mathrm{diag}(y_d,y_s,y_b)\,, \qquad (F_u'O_u')^\dagger\, Y_u\,(F_uO_u) = \mathrm{diag}(y_u,y_c,y_t) \tag{3.1}$$

where $O_{d(u)}$, $O'_{d(u)}$ are orthogonal matrices and $F_{d(u)}$, $F'_{d(u)}$ are the phase transformations, so that the rephased matrices:

$$\widetilde{Y}_d = F_d'^\dagger Y_d F_d = \begin{pmatrix} 0 & A_d & 0 \\ A_d & 0 & B_d\,x_d \\ 0 & B_d/x_d & C_d \end{pmatrix}, \quad \widetilde{Y}_u = F_u'^\dagger Y_u F_u = \begin{pmatrix} 0 & A_u & 0 \\ A_u & 0 & B_u\,x_u \\ 0 & B_u/x_u & C_u \end{pmatrix} \tag{3.2}$$

present only real and positive entries. Then, the real matrices are diagonalized by the bi-orthogonal transformations

$$O_d'^T \widetilde{Y}_d O_d = \mathrm{diag}(y_d,y_s,y_b)\,, \qquad O_u'^T \widetilde{Y}_u O_u = \mathrm{diag}(y_u,y_c,y_t) \tag{3.3}$$

Therefore, for the CKM matrix of quark mixing we obtain

$$V_{\mathrm{CKM}} = O_u^T F O_d = O_u^T \begin{pmatrix} e^{i(\tilde\beta+\tilde\delta)} & 0 & 0 \\ 0 & e^{i\tilde\beta} & 0 \\ 0 & 0 & 1 \end{pmatrix} O_d \tag{3.4}$$

where the matrix $F = F_u^* F_d$ without loss of generality can be parameterized by the two phases $\tilde\beta$ and $\tilde\delta$ while the orthogonal matrices $O_{u,d}$ can be parametrized as

$$O_d = O_{d23}O_{d13}O_{d12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23}^d & s_{23}^d \\ 0 & -s_{23}^d & c_{23}^d \end{pmatrix} \begin{pmatrix} c_{13}^d & 0 & s_{13}^d \\ 0 & 1 & 0 \\ -s_{13}^d & 0 & c_{13}^d \end{pmatrix} \begin{pmatrix} c_{12}^d & s_{12}^d & 0 \\ -s_{12}^d & c_{12}^d & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.5}$$

with $c_{ij}^d = \cos\theta_{ij}^d$ and $s_{ij}^d = \sin\theta_{ij}^d$, and analogously for up-quarks, with $O_u = O_{u23}O_{u13}O_{u12}$. The rotations of right-handed states $O'_{u,d}$, can be parameterized in the same way, with sines $s_{ij}^{d(u)\prime}$ and cosines $c_{ij}^{d(u)\prime}$.

Hence, $\widetilde{Y}_d$ contains four parameters, A_d, B_d, C_d and x_d, which determine the three Yukawa eigenvalues $y_{d,s,b}$ and the three rotation angles in O_d. Analogously, the four parameters in $\widetilde{Y}_u$ determine the Yukawa eigenvalues $y_{u,c,t}$ and the three angles in O_u. Therefore, we have 10 real parameters $A_{u/d}, B_{u/d}, C_{u/d}, x_{u/d}$ and two phases $\tilde\beta, \tilde\delta$ which have to match 10 observables, the 6 Yukawa eigenvalues and 4 independent parameters of the CKM matrix (1.4).

The Yukawa eigenvalues and rotation matrices O and O' can be found by considering the "symmetric" squares respectively of the Yukawa matrices $\widetilde{Y}_f^T\widetilde{Y}_f$ and $\widetilde{Y}_f\widetilde{Y}_f^T$, $f = u,d$. In doing so, we obtain the following relations

$$\begin{aligned} C^2 + (x^2 + x^{-2})B^2 + 2A^2 &= Y_3^2 + Y_2^2 + Y_1^2 \\ B^4 + 2C^2A^2 + (x^2+x^{-2})B^2A^2 + A^4 &= Y_3^2Y_2^2 + Y_3^2Y_1^2 + Y_2^2Y_1^2 \\ A^2C &= Y_1Y_2Y_3 \end{aligned} \tag{3.6}$$

where we omit the indices $f = u, d$ and imply $Y_{1,2,3} = y_{u,c,t}$ for the Yukawa eigenvalues of upper quarks and $Y_{1,2,3} = y_{d,s,b}$ for down quarks.

It is useful to expand the parameters having in mind the approximate hierarchy $y_t : y_c : y_u \sim 1 : \epsilon_u : \epsilon_u^2$ and $y_b : y_s : y_d \sim 1 : \epsilon_d : \epsilon_d^2$, where it can also be noted that phenomenologically the rough relation $\epsilon_u \sim \epsilon_d^2$ applies. In leading order approximation (up to corrections of order $\epsilon \sim Y_2/Y_3 \sim Y_1/Y_2$) we have (see also ref. [28])

$$C \approx Y_3, \quad B \approx \sqrt{Y_2 Y_3}, \quad A \approx \sqrt{Y_1 Y_2} \tag{3.7}$$

so that $C_f : B_f : A_f \sim 1 : \epsilon_f^{1/2} : \epsilon_f^{3/2}$. Since these ratios in fact reflect the hierarchy in the horizontal symmetry breaking (2.5), $C : B : A \sim V_3 : V_2 : V_1$, this means that the inter-family mass hierarchy can actually be induced by a milder hierarchy between the VEVs. The matrix entries A_f, B_f and C_f depend on the deformation x_f only at higher orders in ϵ_f:

$$\begin{aligned}
&\frac{A}{Y_3} = \sqrt{\frac{Y_1 Y_2}{Y_3^2}\frac{1}{c_{23}^{f\prime} c_{23}^f}} + \mathcal{O}(\epsilon^{7/2})\,, \qquad \frac{B}{Y_3} = \sqrt{\frac{Y_2}{Y_3}\left[1 - \frac{1}{2}\frac{Y_1}{Y_2}\left(\frac{c_{23}^f}{c_{23}^{f\prime}} + \frac{c_{23}^{f\prime}}{c_{23}^f}\right)\right]} + \mathcal{O}(\epsilon^{7/2})\,,\\
&\frac{C}{Y_3} = \sqrt{1 - \frac{B^2}{Y_3^2}\left(x^2 + \frac{1}{x^2}\right) + \frac{Y_2^2}{Y_3^2} - 2\frac{A^2}{Y_3^2}} + \mathcal{O}(\epsilon^4)\,.
\end{aligned} \tag{3.8}$$

On the other hand, considering again the hierarchy $Y_3 : Y_2 : Y_1 \sim 1 : \epsilon : \epsilon^2$, the rotation angles in (3.5) appear to be small, so that $c_{ij}^{d,u} \approx 1$, and in the leading approximation we obtain

$$s_{23}^f \approx \frac{x^{-1} B}{C} \approx \frac{1}{x}\sqrt{\frac{Y_2}{Y_3}}, \quad s_{12}^f \approx \frac{AC}{B^2} \approx \sqrt{\frac{Y_1}{Y_2}}, \quad \frac{s_{13}^f}{s_{23}^f} \approx \frac{x^2 A}{C} \approx x^2 s_{12}^f \frac{Y_2}{Y_3} \tag{3.9}$$

so that $s_{23}^f, s_{12}^f \sim \epsilon_f^{1/2}$ and $s_{13}^f \sim \epsilon_f^2$, $f = u, d$. More precisely, we have

$$\begin{aligned}
\tan(2\theta_{23}^f) &= \frac{2}{x}\sqrt{\frac{Y_2 - Y_1}{Y_3}}\,\frac{\sqrt{1 - (x^{-2} + x^2)\frac{Y_2 - Y_1}{Y_3} + \frac{Y_2^2}{Y_3^2}}}{1 - \frac{2}{x^2}\frac{Y_2 - Y_1}{Y_3} + \frac{Y_2^2}{Y_3^2}} + \mathcal{O}(\epsilon^{7/2})\\
\tan(2\theta_{12}^f) &= -2\sqrt{\frac{Y_1\, c_{23}^{f\prime}}{Y_2\, c_{23}^f}}\,\frac{1}{1 + \frac{1}{2}\frac{Y_1}{Y_3}(3x^2 - x^{-2})} + \mathcal{O}(\epsilon^{7/2})\\
\tan(2\theta_{13}^f) &= \frac{2 A s_{23}^{f\prime}}{Y_3} + \mathcal{O}(\epsilon^4) = 2x\frac{Y_2}{Y_3}\sqrt{\frac{Y_1}{Y_3}}\left[1 - \frac{1}{2}\frac{Y_1}{Y_2} - \frac{1}{4}\left(x^2 - \frac{1}{x^2}\right)\frac{Y_2}{Y_3}\right] + \mathcal{O}(\epsilon^4)
\end{aligned} \tag{3.10}$$

which, up to relative corrections of order $O(\epsilon^2)$, correspond to:

$$\begin{aligned}
s_{23}^f &\approx \frac{1}{x}\sqrt{\frac{Y_2}{Y_3}}\left(1 - \frac{1}{2}x^2\frac{Y_2}{Y_3} - \frac{1}{2}\frac{Y_1}{Y_2}\right) + \mathcal{O}(\epsilon^{5/2}),\\
s_{12}^f &\approx -\sqrt{\frac{Y_1\, c_{23}^{f\prime}}{Y_2\, c_{23}^f}}\left[1 - \frac{3}{2}\frac{Y_1\, c_{23}^{f\prime}}{Y_2\, c_{23}^f}\right] + \mathcal{O}(\epsilon^{5/2}),\\
s_{13}^f &\approx \frac{A s_{23}^{f\prime}}{Y_3} + \mathcal{O}(\epsilon^4) = x\frac{Y_2}{Y_3}\sqrt{\frac{Y_1}{Y_3}}\left[1 - \frac{1}{2}\frac{Y_1}{Y_2} - \frac{1}{4}\left(x^2 - \frac{1}{x^2}\right)\frac{Y_2}{Y_3}\right] + \mathcal{O}(\epsilon^4)
\end{aligned} \tag{3.11}$$

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The expressions for $\theta_{23}^{f\prime}$, $\theta_{12}^{f\prime}$ and $\theta_{13}^{f\prime}$ are the same with the replacement $x \to 1/x$, $s_{23}^{f\prime} \to s_{23}^{f}$, $c_{23}^{f\prime} \to c_{23}^{f}$.

These equations show that the yukawa matrix elements $A_{u/d}, B_{u/d}, C_{u/d}, x_{u/d}$ and the rotation angles in the matrices $O_{u,d}$ can be computed respectively in terms of the Yukawa ratios y_s/y_b, y_d/y_s and y_u/y_c, y_c/y_t and the 'deformation' parameters x_d and x_u.

Then, up to relative corrections $O(\epsilon_d^2)$, we have for the CKM matrix elements

$$
\begin{aligned}
|V_{us}| &= \left| s_{12}^d - s_{12}^u\, c_{12}^d c_{23}^d\, e^{-i\tilde{\delta}} \right| + \mathcal{O}(\epsilon^{5/2})\,, \\
|V_{cb}| &= \left| s_{23}^d - s_{23}^u\, c_{23}^d\, e^{-i\tilde{\beta}} \right| + \mathcal{O}(\epsilon^{5/2})\,, \\
|V_{ub}| &= \left| s_{13}^d\, e^{i\tilde{\delta}} - s_{12}^u\, \left(s_{23}^d\, c_{23}^u - s_{23}^u\, c_{23}^d\, e^{-i\tilde{\beta}} \right) \right| + \mathcal{O}(\epsilon^4)\,.
\end{aligned} \tag{3.12}
$$

It can be noticed that for fixed values of the asymmetries x_d, x_u, V_{us} depends on the phase $\tilde{\delta}$ while V_{cb} only on the phase $\tilde{\beta}$. It is also worth noting that for $x_d = 1$, the contribution of s_{13}^d in $|V_{ub}|$ is negligible and the Fritzsch texture implies the prediction $|V_{ub}/V_{cb}| \approx \sqrt{y_u/y_c}$. Similar considerations can be inferred for the other off-diagonal elements

$$
\begin{aligned}
|V_{cd}| &= \left| s_{12}^d\, c_{23}^d - s_{12}^u\, c_{12}^d\, e^{i\tilde{\delta}} + s_{12}^d s_{23}^d s_{23}^u\, e^{-i\tilde{\beta}} \right| + \mathcal{O}(\epsilon^{5/2})\,, \\
|V_{ts}| &= \left| \left(s_{23}^d - s_{23}^u\, c_{23}^d\, e^{i\tilde{\beta}} \right) c_{12}^d \right| + \mathcal{O}(\epsilon^{5/2})\,, \\
|V_{td}| &= \left| s_{13}^d\, e^{i\tilde{\delta}} - s_{12}^d\, \left(s_{23}^d - s_{23}^u\, c_{23}^d\, e^{i\tilde{\beta}} \right) \right| + \mathcal{O}(\epsilon^3)\,,
\end{aligned} \tag{3.13}
$$

with the prediction $|V_{td}/V_{ts}| \approx \sqrt{y_d/y_s}$ for $x_d = 1$. As regards the complex part of $V_{\rm CKM}$, we can consider the rephasing-invariant quantity $J = -\text{Im}(V_{us}^* V_{cb}^* V_{ub} V_{cs})$, the Jarlskog invariant. In our scenario we have

$$
\begin{aligned}
J = &- \sin\tilde{\delta}\, s_{12}^u s_{12}^d\, \left[(s_{23}^d)^2\, c_{23}^d c_{12}^d - 2\cos\tilde{\beta}\, s_{23}^d s_{23}^u + (s_{23}^u)^2 \right] \\
&+ \left(\sin\tilde{\delta} s_{12}^u s_{23}^d + \sin\tilde{\beta} s_{12}^d s_{23}^u \right) s_{13}^d + \mathcal{O}(\epsilon_d^4)
\end{aligned} \tag{3.14}
$$

4 Analysis of the Yukawa parameters and CKM mixing

4.1 Observables

The input values in our analysis will be the ratios of the Yukawa eigenvalues and the CKM matrix elements. More specifically, since we do not need to make assumptions on the energy scale at which the Yukawa matrices assume the Fritzsch form, we want to reproduce Yukawas ratios and CKM elements at different energy scales. Yukawa matrices evolve according to the renormalization group equations, as a function of the energy scale. For energy scales $\mu \lesssim m_t$, the running is basically determined by the strong coupling $\alpha_s(\mu)$ and QCD renormalization factors cancel in quark-mass ratios. We can derive the ratios of Yukawa couplings through the ratios of running quark masses at $\mu = m_t$. The latter ratios can be deduced from the data collected in table 1. For up quarks we also need the ratio m_t/m_b, from renormalization group equations (see for example ref. [80]) we obtain

m_s/m_{ud}	27.31(10)	PDG [3]*
m_u/m_d	0.477(19)	PDG [3]*
m_c/m_s	11.768(34)	FLAG $N_f = 2+1+1$ [77]
m_b/m_s	53.94(12)	Bazavov et al. 2018 [78]
m_b/m_c	4.579(9)	PDG [3]
Q	22.9(4)	PDG [3]*
	22.1(7)	phenomenological [79]
M_t	172.69 ± 0.30 GeV	PDG [3]
$m_b(m_b)$	4.203(11) GeV	FLAG 2021 $N_f = 2+1+1$ [77]
M_Z	91.1876(21) GeV	PDG [3]
$\alpha_s(M_Z)$	0.1185(16)	PDG [3]
$\alpha(M_Z)^{-1}$	127.951(9)	PDG [3]
$\sin^2\theta_W(M_Z)$	0.2299(43)	PDG [3]

*Value adopted by Particle Data Group (PDG), averaging $N_f = 2+1+1$ and $N_f = 2+1+1$ flavours lattice results [77].

Table 1. Determinations of quark mass ratios used in this work. In the first line, $m_{ud} = (m_u+m_d)/2$.

$m_t/m_b = 59.46 \pm 0.55$. Then, at $\mu = m_t$ we have

$$\frac{m_d}{m_s} = \frac{1}{20.17 \pm 0.27}\,, \qquad \frac{m_u}{m_c} = \frac{1}{498 \pm 21}$$
$$\frac{m_s}{m_b} = \frac{1}{53.94 \pm 0.12}\,, \qquad \frac{m_c}{m_t} = \frac{1}{272.3 \pm 2.6}\,, \tag{4.1}$$

where we extracted the value m_s/m_d through the relation $m_s/m_d = m_s/m_{ud}\,(m_u/m_d+1)/2$ (blue band in figure 2). The main source of uncertainty in the mass ratios belongs to the ratio $r_{ud} = m_u/m_d$, which affects the ratios $m_d/m_s(r_{ud})$, $m_u/m_c(r_{ud})$.[7]

For energy scales μ larger than m_t, the set of coupled differential equations for the running of the Yukawa and gauge couplings should be considered (see refs. [81–83]). Namely, the renormalization group evolution of y_t at one loop reads:

$$\mu\frac{dy_t}{d\mu} \approx y_t\frac{1}{16\pi^2}\left(\frac{9}{2}y_t^2 - \frac{17}{20}g_1^2 - \frac{9}{4}g_2^2 - 8g_s^2\right) \tag{4.2}$$

where g_s, g_2, and g_1 are normalized as in SU(5), so that the electroweak gauge coupling constants are $g_2^2/(4\pi) = g^2/(4\pi) = \alpha/\sin^2\theta_W$ and $g_1^2/(4\pi) = \frac{5}{3}g'^2/(4\pi) = \frac{5}{3}\alpha/\sin^2\theta_W$. Since the light generations evolve in the same way with gauge couplings and trace of Yukawa matrices, the ratios y_d/y_s, y_u/y_c remain invariant. The third generation instead receives additional Yukawa contributions. Consequently the ratios with the heaviest generations

[7]In the following we are going to neglect the other small errors contributing in eq. (4.1) and consider only this larger uncertainty.

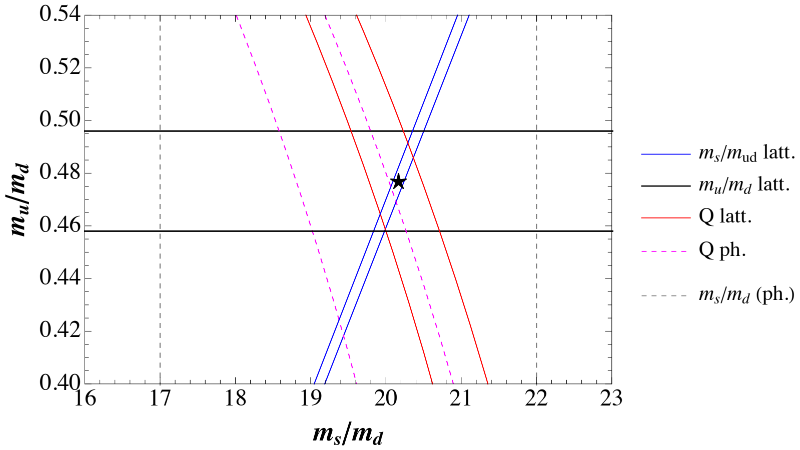

Figure 2. Light quarks mass ratios (see also table 1). Black continuous lines show the average of three and four flavours lattice determinations of the ratio m_u/m_d; blue lines represent the average of three and four flavours lattice determinations of the ratio m_s/m_{ud}, $m_{ud} = (m_u + m_d)/2$; red lines are obtained from the relation $Q^2 = (m_s^2 - m_{ud}^2)/(m_d^2 - m_u^2)$, using lattice determinations of quark mass ratios. We also indicate the phenomenological determinations $Q = 22.1(7)$ [79] (dashed magenta) and $m_s/m_d = 17$–22 [3] (dashed grey). The black star represents the central value $(m_u/m_d, m_s/m_d) = (0.477, 20.17)$.

evolve as:

$$\mu \frac{d}{d\mu}\frac{y_c}{y_t} \approx -\frac{y_c}{y_t}\frac{1}{16\pi^2}\frac{3}{2}y_t^2\,, \qquad \mu \frac{d}{d\mu}\frac{y_s}{y_b} \approx \frac{y_s}{y_b}\frac{1}{16\pi^2}\frac{3}{2}y_t^2 \tag{4.3}$$

As regards the CKM matrix, the mixing angles involving the third generation change according to renormalization group equations [84, 85]:

$$\mu\frac{dV_{cb}}{d\mu} \approx V_{cb}\frac{1}{16\pi^2}\frac{3}{2}y_t^2\,, \qquad \mu\frac{dV_{ub}}{d\mu} \approx V_{ub}\frac{1}{16\pi^2}\frac{3}{2}y_t^2 \tag{4.4}$$

and similarly for V_{td} and V_{ts}, while the mixings between the first two families (V_{us}, V_{ud}, V_{cd}, V_{cs}) remain unchanged. As regards the CP violating Jarlskog invariant, the scaling of J at leading order is the same as $|V_{cb}|^2$, $|V_{td}|^2$, etc., see eq. (4.4).

In order for the Yukawa matrices (3.2) to be a viable texture, we must verify that we can obtain the correct determinations of the quark masses and of moduli and phases of the elements of the CKM matrix $V_{\rm CKM}$. Since $V_{\rm CKM}$ is unitary, there are only 4 independent observables. In the standard parameterization (1.4) these quantities correspond to:

$$\begin{aligned} s_{13}^2 &= |V_{ub}|^2\,, \quad s_{12}^2 = \frac{|V_{us}|^2}{1-|V_{ub}|^2}\,, \quad s_{23}^2 = \frac{|V_{cb}|^2}{1-|V_{ub}|^2} \\ J &= -\mathrm{Im}(V_{ub}V_{cs}V_{us}^*V_{cb}^*) \end{aligned} \tag{4.5}$$

where we indicated the invariant J instead of the phase δ. The results of the latest global fit for the low energy observables are [3] (see also table 3):

$$s_{13} = 0.00369(11)\,, \quad s_{12} = 0.22500(67)\,, \quad s_{23} = 0.04182^{+0.00085}_{-0.00074} \tag{4.6}$$

Quantity	Value	Quantity	Value
$\lvert V_{ud}\rvert$	0.97373(31)	$\lvert V_{cs}\rvert$	0.975(6)
$\lvert V_{us}\rvert$	0.2243(8)	$\lvert V_{cd}\rvert$	0.221(4)
$\lvert V_{ub}\rvert$	0.00382(20)	$\lvert V_{td}\rvert$	0.0086(2)
$\lvert V_{cb}\rvert$	0.0408(14)	$\lvert V_{ts}\rvert$	0.0415(9)
$\lvert V_{td}/V_{ts}\rvert$	0.207(3)	$\lvert V_{ub}/V_{cb}\rvert$	0.084(7)

Table 2. Magnitudes and phases of CKM elements as quoted by Particle Data Group [3].

Parameter	Global fit value [3]
$\sin\theta_{12}$	0.22500 ± 0.00067
$\sin\theta_{23}$	$0.04182^{+0.00085}_{-0.00074}$
$\sin\theta_{13}$	0.00369 ± 0.00011
J	$(3.08^{+0.15}_{-0.13}) \times 10^{-5}$
δ	1.144 ± 0.027

Table 3. Result of global fit for CKM parameters, including constraints implied by the unitarity of the three generation CKM matrix, as reported by Particle Data Group [3].

and concerning CP violation:

$$J = (3.08^{+0.15}_{-0.13}) \times 10^{-5} \tag{4.7}$$

or $\delta = 1.144(27)$. These values produce the CKM matrix [3]

$$|V_{\rm CKM}| = \begin{pmatrix} 0.97435 \pm 0.00016 & 0.22500 \pm 0.00067 & 0.00369 \pm 0.00011 \\ 0.22486 \pm 0.00067 & 0.97349 \pm 0.00016 & 0.04182^{+0.00085}_{-0.00074} \\ 0.00857^{+0.00020}_{-0.00018} & 0.04110^{+0.00083}_{-0.00072} & 0.999118^{+0.000031}_{-0.000036} \end{pmatrix} \tag{4.8}$$

which may be compared to the determinations collected in table 2.

In our numerical analysis we will consider three benchmark scales at which the Yukawa matrices are assumed in the Fritzsch form in eq. (3.2): 10^3 GeV, 10^6 GeV and 10^{16} GeV. In order to test the viability of the model against different hypothesis for the scale of new physics, in our numerical analysis we will consider three benchmark scales at which we assume that the Yukawa matrices acquire the Fritzsch form in eq. (3.2): 10^3 GeV, 10^6 GeV and 10^{16} GeV. This choice of benchmark scales is also related to scenarios with interesting phenomenological or theoretical implications.

Namely, in a scenario with gauge horizontal symmetry $SU(3)_H$, the corresponding gauge bosons mediate flavour-changing processes such as $K \to \overline{K}$ mixing, $K \to \pi\mu e$, $K \to \mu e$, ... which set a limit on the 'horizontal' symmetry breaking scale $V_2 \gtrsim 10^6$ GeV or so. In fact, these processes are contributed by the gauge bosons of $SU(2)_H \subset SU(3)_H$ which act between the first two families and whose masses are related to the intermediate scale V_2 (the largest scale V_3 of $SU(3)_H$ breaking can be estimated from the VEV ratio of order

$V_2/V_3 \sim 0.1$, which is needed for the explanation of the quark mass hierarchy and mixing pattern). Then, if $V_2 \lesssim 10^6$ GeV these flavor changing processes would be detectable. For larger values of $\mathrm{SU}(3)_H$ symmetry breaking scales these phenomena would be strongly suppressed.

The choice of scale $\mu \sim 10^3$ GeV is motivated by some present anomalies which could be related to the presence of the vector-like fermion species at the TeV scale. Vector-like fermions mixing with SM families are present in the model. If the masses of these new fermions are large, then mixings with the SM fields are small and the effect on the CKM matrix (or equivalently on the PMNS matrix in the lepton sector) are negligible. However, there are hints of tensions in different determinations of CKM elements, which can be a sign of the presence of such vector-like species with masses at the TeV scale and large mixings with SM light quarks [62–68]. In this scenario, violation of CKM unitarity would be expected, as well as several new phenomena (e.g. flavour changing neutral currents processes) which may be detected. Moreover, mixing with the top quark can generate a positive contribution to m_W [68, 86–89], as would be needed taking into account the recent result by the CDF Collaboration [90], which exhibits a large discrepancy from the SM expectation. However, since the spontaneous breaking scale of the gauge $\mathrm{SU}(3)_H$ symmetry should be larger then $V_3 > 10^6$ GeV, the choice of few TeV scale for the vector-like quarks would require that the flavon fields should have extremely small Yukawa constants.

Finally, the choice of the GUT scale $\mu \sim 10^{16}$ GeV is natural in the context of the $\mathrm{SU}(5) \times \mathrm{SU}(3)_H$ picture, and in particular in the context of the prototype model presented at the end of section 2.

4.2 Analysis and results

In this section we are going to verify that asymmetric Fritzsch textures can predict quark masses together with moduli and phases of the mixing elements, given present precision of experimental data and recent results from lattice computations. We will test the validity of this flavour pattern considering its formation at different energy scales. As already noted, we have 10 real parameters $A_{d,u}$, $B_{d,u}$, $C_{d,u}$, $x_{d,u}$, $\tilde{\beta}$, $\tilde{\delta}$ which have to match 10 observables, the 6 Yukawa eigenvalues and the 4 independent parameters of the CKM matrix.

4.2.1 Symmetric Fritzsch texture: why it does not work

The canonical Fritzsch texture with $x_{u,d} = 1$, employing 8 parameters to determine 10 observables, would be the most predictive structure. However, it is in contradiction with the experimental data, as can be seen in refs. [18, 19]. Before proceeding with the modified texture, we illustrate the reasons behind the failure of the Fritzsch matrices, taking into account present precision data and using our formalism, in order to demonstrate how these predictions can be adjusted by a deformation of the texture.

In this scenario, the 6 yukawa eigenvalues and the element V_{us} (or equivalently the Cabibbo angle s_{12}) can be reproduced by the symmetric Yukawa matrices by using 7 parameters, the 6 yukawa elements and the phase $\tilde{\delta}$. In fact, as it is apparent from eq. (3.12), besides the ratios of Yukawas, the value of V_{us} is selected only by the phase $\tilde{\delta}$ while it has almost no dependence on the phase $\tilde{\beta}$. On the contrary, the value of V_{cb} is

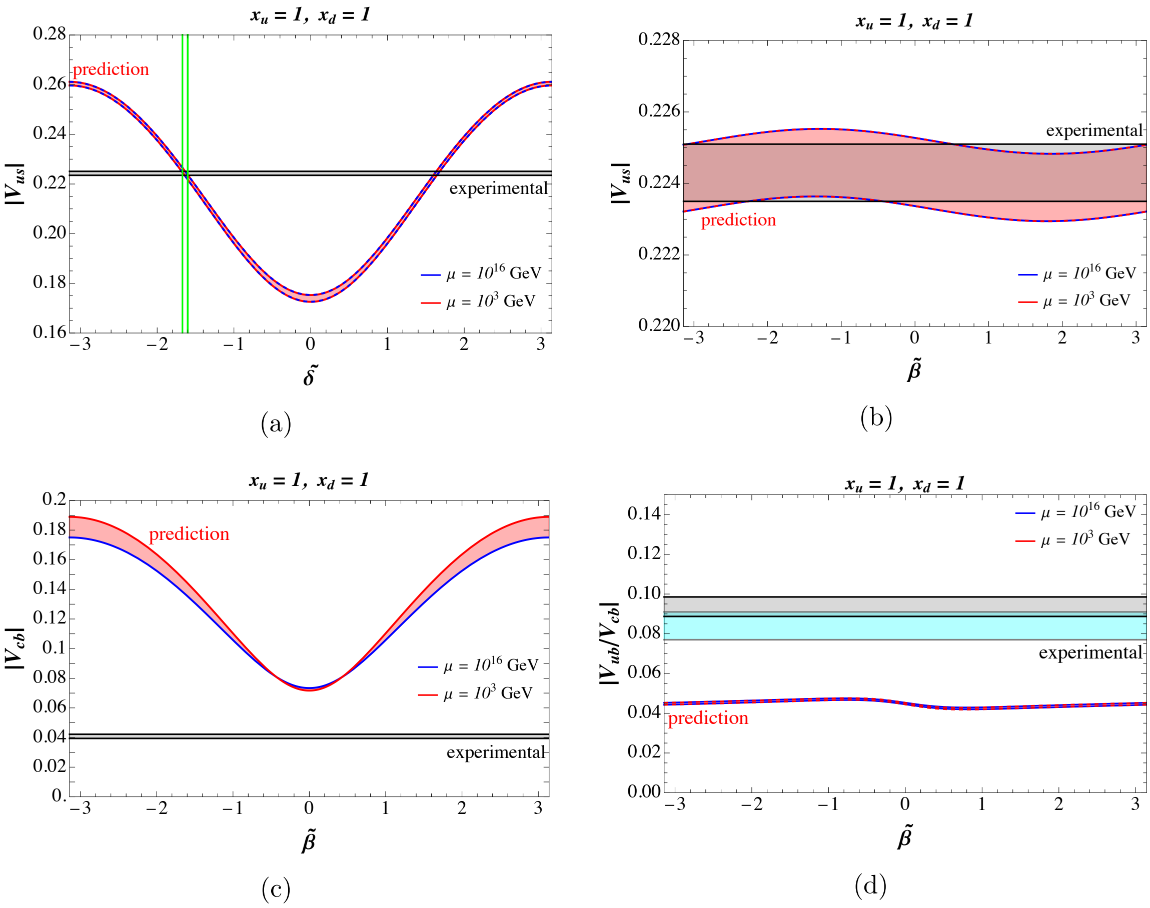

Figure 3. Symmetric Fritzsch textures for quark Yukawa matrices do not predict the correct value of V_{cb} and of the ratio V_{ub}/V_{cb}.

determined by the phase $\tilde{\beta}$, independently of the phase $\tilde{\delta}$. Along with that, in the symmetric case, V_{ub} only presents a mild dependence on $\tilde{\delta}$, given the smallness of s_{13}^d. Therefore, the symmetric Fritzsch texture also implies a clean prediction of the ratio $|V_{ub}/V_{cb}| \approx \sqrt{m_u/m_c}$, basically independent of $\tilde{\delta}$ but also of $\tilde{\beta}$. However, the experimental determinations of $|V_{cb}|$ and of the ratio $|V_{ub}/V_{cb}|$ cannot be accommodated by any of the values of the phase $\tilde{\beta}$, regardless of the energy scale at which the Yukawa matrices assume the Fritzsch texture. In a similar way, the predicted values of $|V_{td}|$, $|V_{ts}|$ and $|V_{td}|/|V_{ts}|$ appear to be too large.

We illustrate this result in figure 3. The value of V_{us} determines the phase $\tilde{\delta}$ as shown in figure 3(a), 3(b). In figures 3(c) and 3(d) we show the value of V_{cb} and V_{ub}/V_{cb} in this case, given V_{us} within 1σ of the experimental constraint. We indicate the predictions (red bands) assuming that the Yukawa matrices present the symmetric Fritzsch texture at a scale between 10^3 GeV (blue lines) and 10^{16} GeV (red lines), confronted with the experimental determinations (grey bands) at 1σ confidence level. In figure 3(d), in addition to the experimental determination $|V_{ub}/V_{cb}| = 0.094(5)$ (grey) obtained from the separate determinations of V_{ub} and V_{cb} (see table 2), we also indicate the independent measurement of the ratio $|V_{ub}/V_{cb}| = 0.084(7)$ (cyan). The width of the prediction in this case is given by

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the uncertainty on the ratio m_u/m_d and on V_{us}. It is clear that the expectations implied by this scenario largely disagree with the experimental requirements.

We are therefore going to consider the minimally modified Fritzsch texture with asymmetry parameters x_d and x_u. We will pay a special attention to the 9 parameters case with $x_u = 1$, $x_d \neq 1$ which can be motivated in the context of SU(5) grand unification and demonstrate that such a predictive ansatz can perfectly work.

4.2.2 Asymmetric Fritzsch texture: how it works

Let us first provide an example in order to convey a comparison with the expectations of the standard Fritzsch texture displayed in figure 3. For this purpose, we fix $x_d = 3.3$ (keeping $x_u = 1$). As it is clear from eq. (3.12), again the expected value of V_{us} manifests almost no dependence from the phase $\tilde{\beta}$ whereas it is determined by the phase $\tilde{\delta}$. Conversely, V_{cb} remains independent of the phase $\tilde{\delta}$. However, as effect of the presence of the asymmetry, the rotation angle s^d_{23} decreases while s^d_{13} increases. This modification causes the prediction of $|V_{cb}|$ to shift towards lower values. As a result, an interval of values of the phase $\tilde{\beta}$ intercepts the experimental determination. Furthermore, the asymmetry originates a dependence of $|V_{ub}|$ on $\tilde{\delta}$ and of the ratio $|V_{ub}/V_{cb}|$ on $\tilde{\beta}$. Similarly, the predictions of V_{td} and V_{ts} adjust to lower values.

Remarkably, the values of $\tilde{\delta}$ and $\tilde{\beta}$ selected by $|V_{us}|$ and $|V_{cb}|$ provide a prediction of all other observables which results within 1σ of their experimental constraints for any scale of new physics. In fact, a χ^2 fit using the determinations of $|V_{us}|$, $|V_{ub}|$, $|V_{cb}|$ and J (again we have 8 parameters against 6 eigenvalues plus 4 CKM observables) returns in the minimum $\chi^2_{\rm min} \approx 0$. In figure 4 we illustrate the new predictions (red bands) for Fritzsch-like texture at a scale between 10^3 GeV (blue lines) and 10^{16} GeV (red lines), confronted with the experimental determinations (grey bands) at 1σ confidence level. We also indicate the 1σ interval of the phases $\tilde{\delta}$ and $\tilde{\beta}$ (green and yellow bands respectively). In displaying the plots, the other variables are allowed to move inside the 1σ confidence region. The 1σ and 2σ confidence intervals of $\tilde{\delta}$ and $\tilde{\beta}$ are displayed in figure 5. In figure 6 we report the expectations on V_{td} and V_{ts} produced by these parameters. It is evident that all these observables can be obtained within 1σ of the experimental constraints independently on the scale of new physics. We will now describe a more detailed analysis, allowing different values of the asymmetries in order to find the parameter space for which all observables are in agreement with the experimental constraints.

4.2.3 Global numerical analysis

In the general scenario in which both up and down Yukawa matrices can present the asymmetry, $x_d, x_u \neq 0$, we have 10 conditions for 10 parameters. Therefore, we can wonder if an exact solution exists. We proceed as follows. We first evaluate the matrix entries $A_{d,u}$, $B_{d,u}$, $C_{d,u}$ in eq. (3.2) in terms of the Yukawa ratios and of the parameters x_d, x_u, $\tilde{\beta}$, $\tilde{\delta}$. Hence, we fit the CKM mixing elements. In particular, we want to find the values of the asymmetry parameters x_d, x_u and the phases $\tilde{\beta}, \tilde{\delta}$ for which, if they exist, the mixing matrix in eq. (3.4) can reproduce the 4 independent quantities describing the unitary matrix $V_{\rm CKM}$ as determined by present global data reported in eqs. (4.6), (4.7). Next, we want

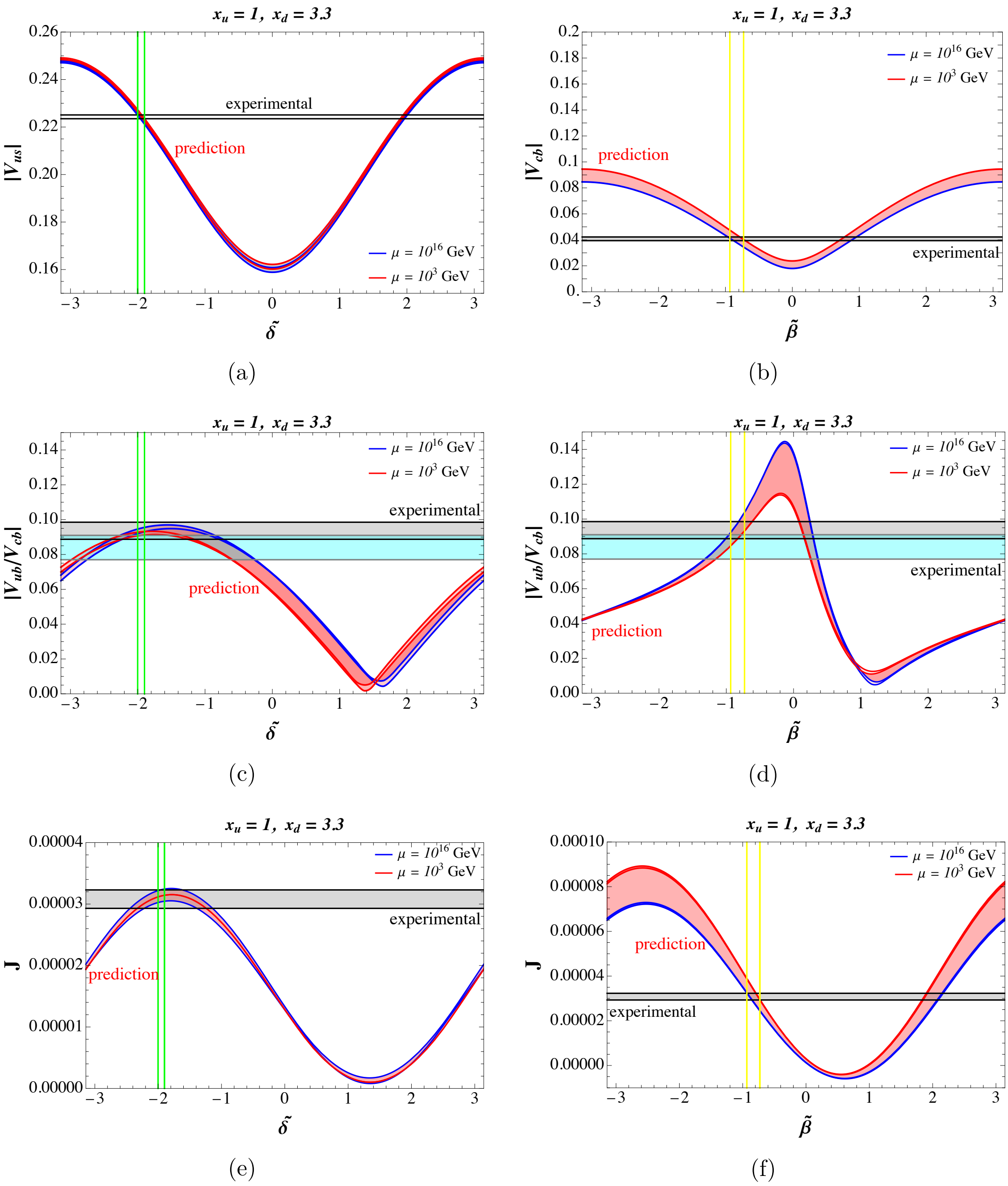
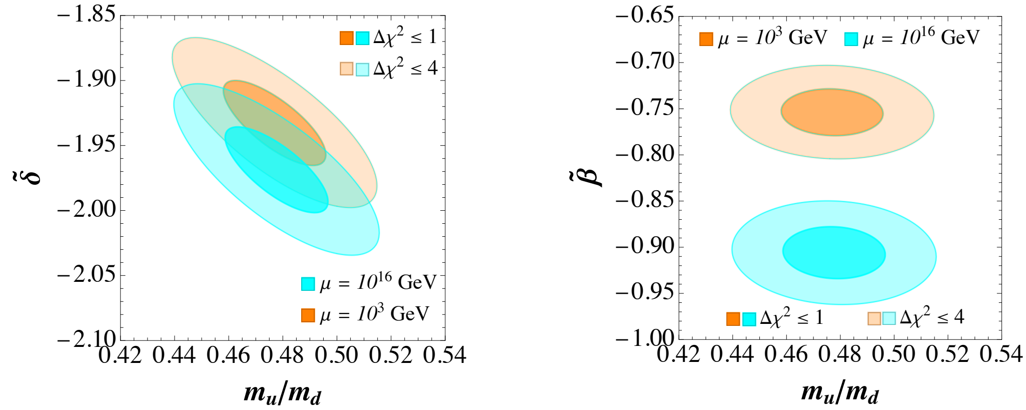
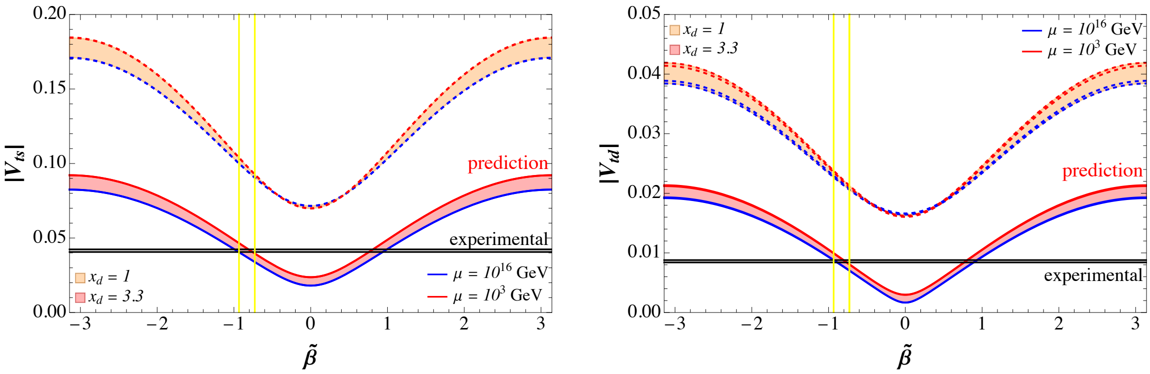


Figure 4. Predictions of the asymmetric Fritzsch-like textures (see eq. (3.2)) with $x_d = 3.3$, $x_u = 1$, confronted with experimental data.



(a) (b)

Figure 5. 1σ and 2σ confidence intervals of the parameters.



(a) (b)

Figure 6. Predictions of V_{ts} and V_{td} from asymmetric Fritzsch textures, with paramters determined as in figure 4. We also show the wrong predictions of the symmetric Fritzsch texture (orange region) at a scale between 10^3 GeV (blue dashed lines) and 10^{16} GeV (red dashed lines).

to investigate the SU(5) motivated scenario with symmetric texture for up-type quarks, by imposing $x_u = 1$. Hence, we perform a χ^2 fit of the four CKM observables with three parameters, the asymmetry parameter x_d and the phases $\tilde{\beta}, \tilde{\delta}$.

In the following, we will use the central values of Yukawas ratios. The ratios m_u/m_d and m_s/m_d are not well-known, as shown in figure 2. For this reason, in principle $r_{ud} = m_u/m_d$ could be left as a parameter, so that we would obtain the functions $\frac{y_d}{y_s}(r_{ud})$, $\frac{y_u}{y_c}(r_{ud})$, $V_{\text{CKM}}(r_{ud}, x_d, x_u, \tilde{\beta}, \tilde{\delta})$ which can vary with this ratio (as we did in the previous example). However, since the central value of the determination m_u/m_d turns out to be also a good point in our fit, we also impose $m_u/m_d = 0.477$, in the central value. In order to test the viability of the model against different hypothesis for the scale of new physics, we illustrate the analysis assuming that the Yukawa matrices acquire the Fritzsch form in eq. (3.2) at the benchmark scales of 10^3 GeV, 10^6 GeV and 10^{16} GeV.

$\boldsymbol{\mu = 10^3}$ **GeV.** At 1 TeV, we have (recalling that the ratios y_d/y_s, y_u/y_c in eq. (4.1) remain renormalization invariant)

$$\frac{y_s}{y_b} = \frac{1}{53.21}\,, \qquad \frac{y_c}{y_t} = \frac{1}{276.0}\,, \qquad \frac{y_d}{y_b} = \frac{1}{1073}\,, \qquad \frac{y_u}{y_t} = \frac{1}{1.37 \times 10^5}\,, \tag{4.9}$$

corresponding to $y_c/y_t(\mu = 1\,\text{TeV}) \approx (1 - 0.014)\, m_c/m_t$, $y_s/y_b(\mu = 1\,\text{TeV}) \approx 1.014\, m_s/m_b$, For the mixing with the third generation we find $V_{cb}(1\,\text{TeV}) = 1.014\, V_{cb}$, $V_{ub}(1\,\text{TeV}) = 1.014\, V_{ub}$, and the same for V_{td} and V_{ts}. After imposing theYukawas ratios (4.9), we can write the Yukawa matrices in terms of the four parameters $x_d, x_u, \tilde{\beta}, \tilde{\delta}$. Hence, we get a system of four equations (we have to match three angles and J) to be solved with the four parameters $x_d, x_u, \tilde{\beta}, \tilde{\delta}$. This system turns out to have a solution:

$$x_d = 3.15\,, \quad x_u = 0.97\,, \quad \tilde{\beta} = -0.75\,, \quad \tilde{\delta} = -1.91\,. \tag{4.10}$$

This means that the Fritzsch-like pattern in eq. (3.2) can be considered as a good flavour structure which gives the right predictions of masses and mixings of quarks. A second solution can be found with $(x_d, x_u, \tilde{\beta}, \tilde{\delta}) = (5.28, 1.35, -1.25, 3.09)$, which requires larger asymmetries and we are not going to consider here.

Given the result in eq. (4.10), we are interested to analyse the scenario in which the up-quark Yukawa matrix assumes the original symmetric Fritzsch texture, that is $x_u = 1$. Having one less parameter, we perform a χ^2 fit of the three CKM angles and the CP-violating quantity J with the parameters x_d, $\tilde{\delta}$, $\tilde{\beta}$. We obtain $\chi^2_{\text{min}} = 0.25$ in the minimum, with best fit values in:

$$x_d = 3.16 \pm 0.17\,, \quad \tilde{\beta} = -0.78 \pm 0.02\,, \quad \tilde{\delta} = -1.92 \pm 0.04 \tag{4.11}$$

where we also indicated the 1σ interval of the parameters $(\chi^2_{\text{min}} + 1)$. We conclude that the canonical symmetric Fritzsch texture for up quarks is a good predictive ansatz in models in which an asymmetry is generated in the mixing between the second and third generation in the down sector.

$\boldsymbol{\mu = 10^6}$ **GeV.** Assuming 10^3 TeV as the scale of new physics, we have (with the ratios y_d/y_s, y_u/y_c in eq. (4.1))

$$\frac{y_s}{y_b} = \frac{1}{51.31}\,, \qquad \frac{y_c}{y_t} = \frac{1}{286.2}\,, \qquad \frac{y_d}{y_b} = \frac{1}{1034}\,, \qquad \frac{y_u}{y_t} = \frac{1}{1.42 \times 10^5}\,, \tag{4.12}$$

corresponding to $y_c/y_t(10^3\,\text{TeV}) \approx 0.95\, m_c/m_t$ and $y_s/y_b(10^3\,\text{TeV}) \approx 1.05\, m_s/m_b$. For the mixing elements we find $V_{cb}(10^3\,\text{TeV}) = 1.051\, V_{cb}$, $V_{ub}(10^3\,\text{TeV}) = 1.051\, V_{ub}$ and the same for V_{td} and V_{ts}. By imposing the values of the Yukawas and the four CKM parameters we find the solution:

$$x_d = 3.09\,, \quad x_u = 0.92\,, \quad \tilde{\beta} = -0.75\,, \quad \tilde{\delta} = -1.91\,, \tag{4.13}$$

and a second solution in $(x_d, x_u, \tilde{\beta}, \tilde{\delta}) = (5.18, 1.28, -1.25, 3.09)$.

Again we can fix $x_u = 1$ so that the up-quark Yukawa matrix assumes the original symmetric Fritzsch texture. After imposing the Yukawa eigenvalues, we perform the fit of

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the three CKM angles and J with the parameters x_d, $\tilde{\delta}$, $\tilde{\beta}$. We obtain $\chi^2_{\rm min} = 0.03$ in the minimum and the 1σ intervals of parameters ($\chi^2_{\rm min} + 1$):

$$x_d = 3.14 \pm 0.16\,, \quad \tilde{\beta} = -0.84 \pm 0.02\,, \quad \tilde{\delta} = -1.92 \pm 0.04 \tag{4.14}$$

$\mu = 10^{16}$ GeV. At 10^{16} GeV, we have (with the ratios y_d/y_s, y_u/y_c in eq. (4.1))

$$\frac{y_s}{y_b} = \frac{1}{48.40}\,, \qquad \frac{y_c}{y_t} = \frac{1}{303.4}\,, \qquad \frac{y_d}{y_b} = \frac{1}{976}\,, \qquad \frac{y_u}{y_t} = \frac{1}{1.51 \times 10^5}\,, \tag{4.15}$$

meaning $y_c/y_t(10^{16}\,\text{GeV}) \approx (1-0.1)\, m_c/m_t$, $y_s/y_b(10^{16}\,\text{GeV}) \approx 1.1\, m_s/m_b$. For the mixing elements we have $V_{cb}(10^{16}\,\text{GeV}) = 1.114\, V_{cb}$, $V_{ub}(10^{16}\,\text{GeV}) = 1.114\, V_{ub}$. We find the solution

$$x_d = 3.00\,, \quad x_u = 0.85\,, \quad \tilde{\beta} = -0.75\,, \quad \tilde{\delta} = -1.91\,, \tag{4.16}$$

and a second solution in $(x_d, x_u, \tilde{\beta}, \tilde{\delta}) = (5.03, 1.17, -1.25, 3.09)$, with larger asymmetries.

By imposing the condition $x_u = 1$, we perform the χ^2 fit of the three CKM angle and invariant J. We receive the minimum $\chi^2_{\rm min} = 0.75$ in the best fit values of the parameters (and relative 1σ interval, $\chi^2_{\rm min} + 1$)

$$x_d = 3.12 \pm 0.16\,, \quad \tilde{\beta} = -0.93 \pm 0.02\,, \quad \tilde{\delta} = -1.94 \pm 0.04 \tag{4.17}$$

Summary. We conclude that a symmetric Fritzsch texture for up quarks ($x_u \approx 1$) and a minimally modified Fritzsch texture as in eq. (3.2) for down quarks with $x_d \approx 3$ are good flavour structures, which can predict the right masses of quarks as well as CKM mixings and phase. In particular, $3 \lesssim x_d \lesssim 3.3$ is a good interval for any energy scale at which the Fritzsch-like textures exist. In figure 7 we present the results of the analysis in the x_d-$\tilde{\beta}$, $\tilde{\delta}$-$\tilde{\beta}$ and x_d-$\tilde{\beta}$ planes in the scenario with $x_u = 1$, marginalizing over the other variable, assuming Fritzsch-like texture at 10^3 GeV, 10^6 GeV, 10^{16} GeV (left, centre and right respectively). We show the 1σ, 2σ, 3σ (blue, red and green regions) confidence intervals ($\chi^2_{\rm min} + 1$, $\chi^2_{\rm min} + 4$, $\chi^2_{\rm min} + 9$). The best fit values give for the three benchmark scales respectively the magnitudes of CKM elements:

$$\begin{pmatrix} 0.97435 & 0.22500 & 0.00369 \\ 0.22486 & 0.97347 & 0.0418 \\ 0.00865 & 0.0410 & 0.99910 \end{pmatrix}, \begin{pmatrix} 0.97435 & 0.22500 & 0.00367 \\ 0.22485 & 0.97341 & 0.0416 \\ 0.00880 & 0.0409 & 0.99903 \end{pmatrix}, \begin{pmatrix} 0.97435 & 0.22500 & 0.00365 \\ 0.22481 & 0.97330 & 0.0415 \\ 0.00901 & 0.0407 & 0.99892 \end{pmatrix} \tag{4.18}$$

in perfect agreement with the observables in eq. (4.8).

By adopting different choices of experimental results, e.g. some of the determinations in table 2 instead of the global fit, the result of the numerical analysis would be very similar. In fact, we find that this flavour pattern is able to reproduce all CKM observables within 1σ.

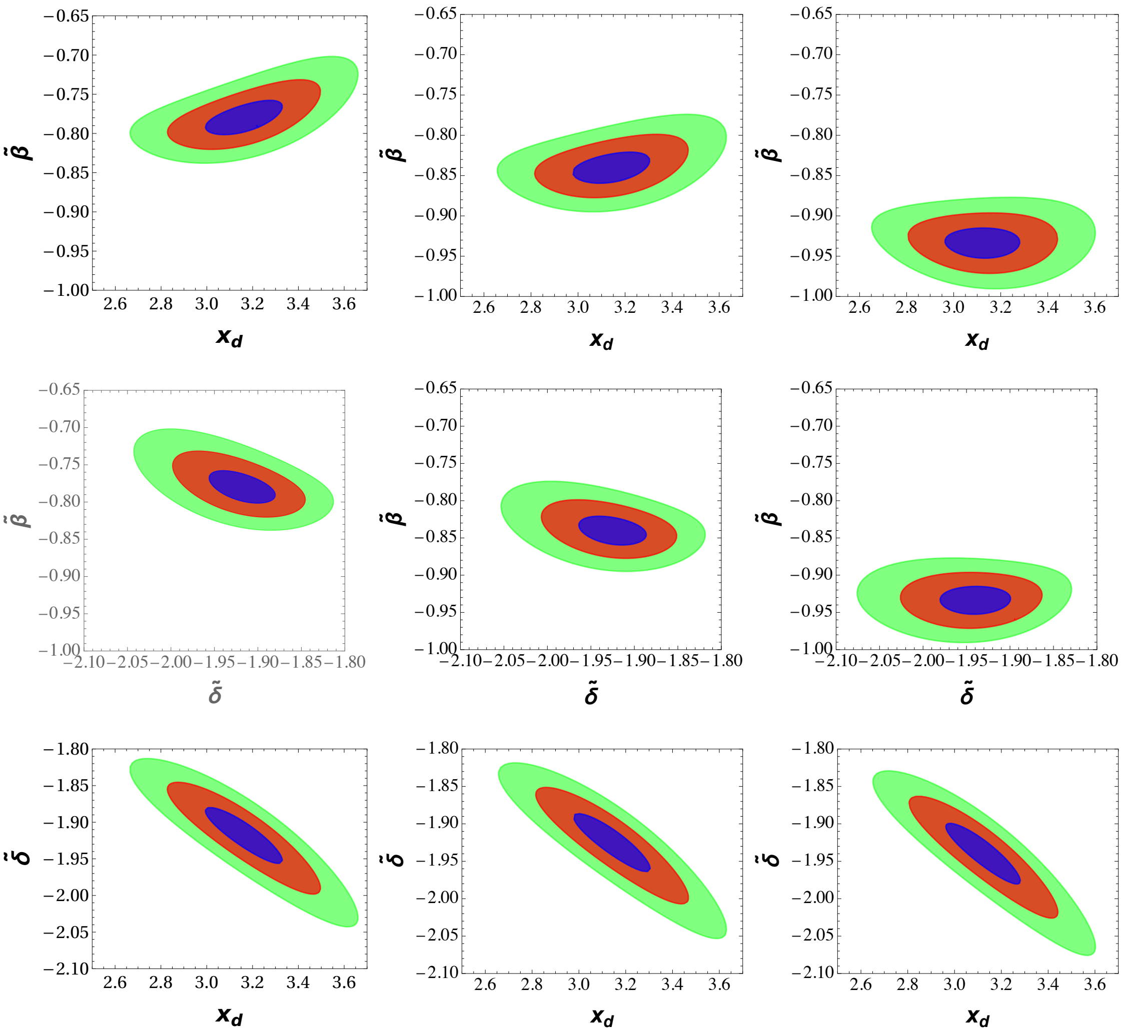

Figure 7. Parameter space in the scenario with $x_u = 1$ (symmetric Fritzsch texture for up-type quarks, asymmetric 23 entries for down-type quarks, see eq. (3.2)) in the x_d-$\tilde{\beta}$, $\tilde{\delta}$-$\tilde{\beta}$ and x_d-$\tilde{\beta}$ planes, marginalizing over the other variable. 1σ, 2σ and 3σ preferred regions of the parameters are indicated ($\chi^2_{\text{min}} + 1$, $\chi^2_{\text{min}} + 4$, $\chi^2_{\text{min}} + 9$), assuming Yukawa matrices of Fritzsch-like form at 10^3 GeV (left), 10^6 TeV (centre), 10^{16} TeV (right).

5 Conclusion

The hierarchy between fermion masses, their mixing pattern as well as the replication of families itself remain a mystery in the context of the Standard Model or grand unified theories. It is intriguing to think that clues for an explanation may be found in the existing relations between mass ratios and mixing angles, as the formula for the Cabibbo angle $V_{us} = \sqrt{m_d/m_s}$, which may be regarded as not accidental but rather connected to an underlying flavour theory. These relations can be predicted by mass matrices with reduced number of parameters. Moreover, present experimental data and lattice computations have reached enough precision to scrutinize some of the hypothesis on the Yukawa textures.

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In this work, we concentrated on the predictive Fritzsch texture for quark masses, with 6 zero entries (three in the up-quarks Yukawa matrix and three in the down-type one). This flavour structure contains 8 parameters which should match 10 observables: six quarks masses, three mixing angles and one CP-violating phase. However, the original symmetric ansatz is excluded by present data. Nevertheless, an asymmetry in the mixing between the second and third generation can be introduced. We analyzed this asymmetric version of the Fritzsch texture, with the same vanishing elements, considering its possible origin and confronting its predictions with recent precise experimental and lattice results on quark masses and mixings.

In particular, we showed that the canonical symmetric Fritzsch form for up-type quarks can be combined with the asymmetric texture for down quarks. In this way, with 9 parameters to match 10 observables, all values of mass ratios and CKM matrix observables can be reproduced within $\sim 1\sigma$, independently on the energy scale at which the Fritzsch structure is generated.

We showed how this flavour pattern, the symmetric texture for up quarks and the asymmetric one for down quarks, can naturally arise from models with $\mathrm{SU}(3)_H$ gauge family symmetry in the context of Standard Model or grand unified theories.

Acknowledgments

We would like to thank Paolo Panci, Robert Ziegler and Antonio Rodríguez Sánchez for useful discussions. The work of Z.B. was supported in part by Ministero dell'Istruzione, Università e della Ricerca (MIUR) under the program PRIN 2017, Grant 2017X7X85K "The dark universe: synergic multimessenger approach". We would like to remember Harald Fritzsch who recently passed away for his pioneering contribution towards the understanding of flavour structures.

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