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The attractor flow for AdS_5 black holes in $\mathcal{N} = 2$ gauged supergravity

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Abstract: We study the flow equations for BPS black holes in $\mathcal{N} = 2$ five-dimensional gauged supergravity coupled to any number of vector multiplets via FI couplings. We develop the Noether-Wald procedure in this context and exhibit the conserved charges as explicit integrals of motion, in the sense that they can be computed at any radius on the rotating spacetime. The boundary conditions needed to solve the first order differential equations are discussed in great detail. We extremize the entropy function that controls the near horizon geometry and give explicit formulae for all geometric variables at their supersymmetric extrema. We have also considered a complexification of the near-horizon variables that elucidates some features of the theory from the near-horizon perspective.

Keywords: AdS-CFT Correspondence, Black Holes, Supergravity Models

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Contents

1 Introduction

The radial dependence of physical fields in a black hole background relates the field configuration far from the black hole to the region near the black hole horizon. It is important in holography because the radial evolution is identified with the renormalization group flow in the dual quantum field theory, so it determines the low energy QFT observables in terms of UV data. For supersymmetric black holes in asymptotically flat spacetimes, these ideas are realized beautifully by the so-called attractor flow. Unfortunately, the analogous construction for BPS black holes in asymptotically AdS spacetimes, where holography is more precise, is less developed. The goal of this article is to give an explicit and detailed account of the attractor flow for BPS black holes in AdS_5.

All extremal black holes, whether supersymmetric or not, enjoy an attractor *mechanism*, in that the end point of the radial flow is a horizon region with enhanced $\mathrm{SO}(2,1)$ symmetry. The field configuration in this AdS_2 region is determined by the entropy function formalism, an extremization principle that was studied in many contexts [1–9], including its applications to subleading corrections of the black hole entropy [10–12]. The attractor *flow* refers to the entire evolution from the asymptotic space to the event horizon. The long throat characterizing the final approach to the horizon gives a geometric reason to intuit that only a restricted set of endpoints are possible. In this case the flow is *attracted* to specific points in configuration space.

The study of supersymmetric attractor flows was initiated in $\mathcal{N}=2$ ungauged supergravity [13, 14]. In this context an attractor mechanism was realized. The horizon values of scalars in $\mathcal{N}=2$ vector multiplets is independent of their freely chosen asymptotic values. The attractor flow for asymptotically flat black holes was later generalized to different amounts of supersymmetry in various dimensions [15–24]. In this work we inquire about the analogous radial flow in gauged supergravity, the setting for asymptotically AdS black holes. This research direction is very well motivated by holography, but it is technically more involved and much less developed. Moreover, the reference to an *attractor* is a misnomer in gauged supergravity: in general no *initial* data is introduced at the asymptotic AdS boundary, so nothing is lost when the horizon is reached. However, the attractor terminology is so ingrained by now that we keep it, even though it can be misleading. Indeed, versions of attractor flows and attractor mechanisms in gauged supergravity previously appeared in [25–30].

In the canonical set-up [13, 14], ungauged $\mathcal{N}=2$ supergravity coupled to vector multiplets, scalar fields are the only variables needed to characterize the attractor flow. Classical black hole solutions involve vector fields as well but, in stationary black hole backgrounds, their radial dependence is entirely determined by the conserved electric charges, augmented by magnetic charges in $D=4$. The scalars can take any values in the asymptotically flat space as they are moduli that parametrize the vacuum. However, for given charges, their radial flow is governed by an effective black hole potential that guides them to an attractor value that depends only on the charges.

BPS black holes in AdS are fundamentally different because the scalar fields in gauged supergravity are subject to a potential that depends on the couplings of the theory, so generally the scalars are not moduli that parametrize vacua. When scalars do not depend on free parameters at infinity, there is no scope for an attractor mechanism that imposes horizon values independent of such parameters. There is an exception when FI-couplings are fine-tuned to create flat directions in the scalar potential. In this case the scalar fields do obey an attractor mechanism.

This work focuses on black holes with nonzero electrical charge and rotation in 5D asymptotically AdS spacetimes [31–35] whose solution has been well studied in various contexts. Several of the technical challenges we encounter in our setup arise because all known BPS black holes in AdS_5 have non-vanishing angular momentum. In our implementation of the attractor flow, the angular momentum J is a conserved quantity like any other: it can be computed as a flux integral over any topological sphere surrounding the black hole. The attractor flow from asymptotically AdS to the horizon corresponds to decreasing radial coordinate. Because of the rotation, the spatial sphere at any given radius is squashed and co-rotating.

There has been much previous work on the attractor mechanism in gauged supergravity, including [25–30, 36–49] and references therein. As a guide, we highlight some of the features we focus on:

- We construct the entire attractor flow of gauged supergravity, ie. the radial solution that interpolates between the AdS_5 vacuum far from the black hole and the horizon vacuum of AdS_2-type. We do this directly in 5D, without the dimensional reduction to 4D pursued by some researchers [26, 30].

- We construct *conserved charges* in terms of flux integrals. They are not specifically defined asymptotically far from the black hole, nor in the horizon region. Rather, they can be evaluated on any member of a nested family of surfaces that interpolate between the horizon and a homologous asymptotic surface. This is essential for the attractor flow.

  Technically, we construct these conserved charges using the covariant Noether-Wald procedure. This is similar to what is done in [50], although their focus is on the near-horizon region. One can also take the approach of a generalized Komar integral [51–56], where a specific gauge must be chosen to deal with diverging asymptotics in AdS spacetimes.

- Our work incorporates *general special geometry*, including symplectic invariance. As such, we incorporate general cubic prepotentials. We provide a reader friendly guide to the widely studied STU and minimal SUGRA models that correspond to special cases. For a recent study on the attractor mechanism in the STU model, see [57].

- We carry out the *entropy extremization* procedure in complete detail, reconstructing all details of the horizon structure. This elucidates the relation between real and complex fields.

It may be useful to also note some aspects of the attractor mechanism for gauged supergravity that are interesting but *not* developed in this article:

- We focus on electric black holes in *only five* spacetime dimensions.
- We study *supersymmetric* attractor flows but other extremal flows are interesting as well.
- We specialize to black holes with *equal angular momenta*, so our ansatz for the geometry preserves an S^2 throughout the flow. More general AdS$_5$ black holes with unequal angular momenta, studied in [33, 34], depend on a complex structure on $S^1 \times S^3$ that evolves radially and is defined only in Euclidean signature [58, 59].
- We study AdS$_5$ supergravity with FI-gauging, so only U(1) gauge groups appear. This class of theories is technically simpler, because it is entirely specified by a linear superpotential, and there is no need for the moment map. Moreover, in this theory all scalar fields are neutral [60].
- We do not connect to standard aspects of the AdS$_5$/CFT$_4$ correspondence, such as holographic renormalization, time-dependent sources, and so on. This direction was recently studied in [57].

We hope to expand on some of these omissions in future work.

This article is organized as follows. In section 2, we introduce the $\mathcal{N} = 2$ gauged supergravity action, the 5D form of the black hole solution, and its dimensional reduction to both 2D and 1D. section 3 is devoted to an analysis of Noether-Wald surface charges, and the subtleties of gauge invariant conserved charges for actions with Chern-Simons terms. section 4 is the longest and most detailed. We derive the first order differential equations imposed by preserved supersymmetry, in the context of our ansatz. We study the boundary conditions needed to solve the equations perturbatively, from both the horizon and the asymptotic AdS point of view. With both these perspectives, we recover the known black hole solutions by establishing truncation of the pertubative expansion. In section 5 we develop the entropy extremization formalism and compute all near horizon aspects of the black hole, including its entropy. We also construct a complexification of the near-horizon variables that elucidates some aspects of the solution. We conclude in section 6 with a discussion of open problems concerning the attractor flow in gauged supergravity and related topics. A series of appendices are devoted to technical details, conventions and notations regarding differential forms in appendix A, real special geometry in B, and the supersymmetry conditions in appendix C.

2 The Effective 2D Lagrangian

In this section we introduce the action of $\mathcal{N} = 2$ 5D supergravity with coupling to n_V vector multiplets and gauging by Fayet-Iliopoulos couplings, as well as its dimensional reduction to a 2D theory. This also serves to define conventions and notation. For additional details on real special geometry and supersymmetry we refer to appendices B and C, respectively.

2.1 The 5D theory

We study five dimensional $\mathcal{N} = 2$ gauged supergravity with bosonic action

$$S = \frac{1}{16\pi G_5} \int_{\mathcal{M}} \mathcal{L}_5 + \frac{1}{8\pi G_5} \int_{\partial\mathcal{M}} d^4x \sqrt{|h|}\, \mathrm{Tr}\, K\,, \tag{2.1}$$

where the 5D Lagrangian density is given by

$$\mathcal{L}_5 = (-\mathcal{R}_5 - 2V) \star_5 1 - G_{IJ} F_5^I \wedge \star_5 F_5^J + G_{IJ} dX^I \wedge \star_5 dX^J - \frac{1}{6} c_{IJK} F_5^I \wedge F_5^J \wedge A_5^K\,. \tag{2.2}$$

We have included the subscript 5 to emphasize that we are in five dimensions and the five dimensional Hodge dual is given by $\star_5$. The Gibbons-Hawking-York boundary term must be included to have a well-defined variation of the action (2.1) and is given by the trace of the second fundamental form K which is integrated over the induced metric h on the boundary. Other conventions and notations regarding differential forms and the Hodge dual are in appendix A.

The field content includes the field strengths $F_5^I = dA_5^I$ where $I = 1, \ldots, n$ and the scalars X^I, correspond to $n-1$ physical scalars, constrained via the following relation

$$\frac{1}{6} c_{IJK} X^I X^J X^K = 1. \tag{2.3}$$

The scalar potential is given by

$$V = -c^{IJK} \xi_I \xi_J X_K = -\xi_I \xi_J \left(X^I X^J - \frac{1}{2} G^{IJ} \right)\,, \tag{2.4}$$

where ξ_I are the real Fayet-Iliopoulos parameters. The scalars with lowered index

$$X_I = 2 G_{IJ} X^J\,, \tag{2.5}$$

obey the analogous constraint

$$\frac{1}{6} c^{IJK} X_I X_J X_K = 1\,, \tag{2.6}$$

when closure relation (B.3) is satisfied. For further details on definitions, conventions and identities, we refer the reader to appendix B.

Alternatively, the scalar potential can be expressed as

$$V = -\left(\frac{2}{3} W^2 - \frac{1}{2} G^{IJ} D_I W D_J W \right)\,, \tag{2.7}$$

where the superpotential W is

$$W = \xi_I X^I\,, \tag{2.8}$$

and the Kähler covariant derivative D_I takes the constraint (2.6) into acount. Using this form of the potential V, the condition for a supersymmetric minimum becomes

$$D_I W = \xi_I - \frac{1}{3} X_I (\xi \cdot X) \underset{\min}{=} 0. \tag{2.9}$$

According to this equation, the asymptotic values of the scalars $X_{I,\infty}$ must be parallel to ξ_I, in the sense of real special geometry vectors, and the constraint (2.6) determines the proportionality constant between the two:

$$X_{I,\infty} = \left(\frac{1}{6}c^{JKL}\xi_J\xi_K\xi_L\right)^{-1/3}\xi_I. \tag{2.10}$$

The value of the potential V at the minimum must be related to the AdS$_5$ length scale ℓ and the cosmological constant in the usual manner

$$V_\infty = -c^{IJK}\xi_I\xi_J X_{K,\infty} \equiv -6\ell^{-2}. \tag{2.11}$$

This gives the constraint

$$\frac{1}{6}c^{IJK}\xi_I\xi_J\xi_K = \ell^{-3}\,, \tag{2.12}$$

on the FI-parameters ξ_I and the simple relation for the asymptotic values of the scalars

$$X_{I,\infty} = \ell\xi_I. \tag{2.13}$$

Thus the n_V+1 independent FI-parameters ξ_I determine the asymptotic values X^I_∞ of the n_V scalars, as well as the AdS$_5$ scale ℓ.

For contrast, recall that in ungauged supergravity, the scalar fields are moduli as they experience no potential. Then their asymptotic values $X_{I,\infty}$ far from the black hole are set arbitrarily by boundary conditions, which is related to the fact that the spacetime is asymptotically flat. The fact that the value of the scalars X_I at the *horizon* is independent of the asymptotic values $X_{I,\infty}$ is the attractor mechanism for BPS black holes in ungauged supergravity.

As we have seen, the present context is very different in that the asymptotic values of the scalars are set by the theory through the FI-parameters ξ_I, rather than by boundary conditions. This is a generic feature of gauged supergravity, theories with asymptotically AdS vacuum. It precludes an attractor mechanism that is analogous to the one in asymptotically flat space. We will discuss this point in more depth in section 4 when we study the linear flow equations derived from supersymmetry.

The equations of motion $\mathcal{E}_\Phi$, where Φ is any field in the theory corresponding to the Lagrangian density (2.2), are the Einstein equation

$$\begin{aligned}\mathcal{E}_g = R_{AB} - \tfrac{1}{2}g_{AB}R + G_{IJ}\left(F^I_{5,AC}F^{J,C}_{5,B} - \tfrac{1}{4}g_{AB}F^I_{5,CD}F^{J,CD}_5\right)\\ - G_{IJ}\left(\nabla_A X^I\nabla_B X^J - \tfrac{1}{2}g_{AB}\nabla_C X^I\nabla^C X^J\right) - g_{AB}V = 0\,,\end{aligned} \tag{2.14}$$

and the matter equations for the Maxwell field A^I_5 and the constrained scalars X^I

$$\mathcal{E}_A = d\left(G_{IJ}\star F^J_5\right) + \tfrac{1}{4}c_{IJK}F^J_5\wedge F^K_5 = 0, \tag{2.15}$$

$$\begin{aligned}\mathcal{E}_{X^I} = -d\star dX_I + \tfrac{1}{3}X_IX^Jd\star dX_J + 2c^{JKL}\xi_K\xi_L\left(\tfrac{2}{3}X_IX_J - c_{IJM}X^M\right)\star 1 + \Big(X_JX^Lc_{IKL}\\ -\tfrac{1}{2}c_{IJK} - \tfrac{2}{3}X_IX_JX_K + \tfrac{1}{6}X_Ic_{JKN}X^N\Big)\left(F^J_5\wedge\star F^K_5 - dX^J\wedge\star dX^K\right) = 0.\end{aligned} \tag{2.16}$$

2.2 The effective 2D theory

We do not study all solutions to the 5D theory (2.1), just stationary black holes. Then it is sufficient to consider a reduction to 2D — and eventually to 1D. We impose the metric ansatz[1]

$$ds_5^2 = ds_2^2 - e^{-U_1} d\Omega_2^2 - e^{-U_2}(\sigma_3 + a^0)^2\,, \tag{2.17}$$

with ds_2^2 a general 2D metric and the 1-form ansatz for the gauge potential

$$A_5^I = a^I + b^I(\sigma_3 + a^0). \tag{2.18}$$

In our conventions, the left invariant 1-forms

$$\begin{aligned} \sigma_1 &= \sin\phi\, d\theta - \cos\phi \sin\theta\, d\psi\,, \\ \sigma_2 &= \cos\phi\, d\theta + \sin\phi\sin\theta\, d\psi\,, \\ \sigma_3 &= d\phi + \cos\theta\, d\psi\,, \end{aligned} \tag{2.19}$$

parametrize SU(2) with

$$0 \le \theta \le \pi\,, \qquad 0 \le \phi \le 4\pi\,, \qquad 0 \le \psi \le 2\pi\,. \tag{2.20}$$

The ansatz (2.17) suggests the vielbein

$$\begin{aligned} &e^0 = e^0_\mu dx^\mu\,, \qquad && e^1 = e^1_\mu dx^\mu\,, \qquad && e^2 = e^{-\frac{1}{2}U_1}\sigma_1\,, \\ &e^3 = e^{-\frac{1}{2}U_1}\sigma_2\,, && e^4 = e^{-\frac{1}{2}U_2}(\sigma_3 + a^0). \end{aligned} \tag{2.21}$$

We use Greek indices to denote the curved coordinates t and R in 2D. For extremal near-horizon geometries, the 2D coordinates describe the AdS_2 throat of the solution. The dimensional reduction via (2.17) and (2.18) of the 5D Lagrangian (2.2) introduces the scalar fields U_1, U_2 and b^I, along with the 1-forms a^0, a^I. All these fields depend only on the 2D coordinates.

The effective 2D Lagrangian density that follows from (2.2) is given by

$$\begin{aligned} \mathcal{L}_2 = \frac{\pi}{G_5} e^{-U_1 - \frac{1}{2}U_2} \Big\{ &\left(-\mathcal{R}_2 + 2e^{U_1} - \tfrac{1}{2}e^{2U_1 - U_2}\right) \star 1 - \tfrac{1}{2} dU_1 \wedge \star d\left(U_1 + 2U_2\right) \\ &- \tfrac{1}{2} e^{-U_2} da^0 \wedge \star da^0 - 2V - G_{IJ}\Big((da^I + b^I da^0) \wedge \star (da^J + b^J da^0) + e^{2U_1} b^I b^J \star 1 \\ &+ e^{U_2} db^I \wedge \star db^J - dX^I \wedge \star dX^J \Big) + \tfrac{1}{3} e^{U_1 + \frac{1}{2}U_2} c_{IJK} \left(\tfrac{3}{2} b^I b^J da^K + b^I b^J b^K da^0 \right) \Big\} \\ &+ \frac{\pi}{G_5} d\left((e^{-U_1 - \frac{1}{2}U_2} \star d(2U_1 + U_2)) - \tfrac{1}{6} b^I b^J a^K \right)\,. \end{aligned} \tag{2.22}$$

We denote the Ricci scalar of the reduced 2D metric $\mathcal{R}_2$ and the Hodge dual is now in 2D. The overall exponential factor $e^{-U_1 - \frac{1}{2}U_2}$ comes from the 5D metric on a deformed S^3. The first line in (2.22) is due to the reduction of the 5D Ricci scalar, which introduces

[1] In our conventions the metric has a mostly negative signature.

additional kinetic and potential terms associated to the scalars U_1 and U_2, as well as for the 1-form a^0 in the beginning of the second line. The terms preceded by G_{IJ} are the reduction of the Maxwell field which yield kinetic terms for the 1-forms a^0, a^I and the scalars b^I, and the reduction of the kinetic term of X^I. The remainder of the third line of (2.22) is the Chern-Simons term. Finally, in the last line, there is a total derivative that is inconsequential for the equations of motion but is required in order that $\mathcal{L}_2$ (2.22) is the dimensional reduction of the 5D Lagrangian (2.2). The latter does not include the Gibbons-Hawking-York boundary term, the extrinsic curvature that appears separately in (2.1).

Boundary terms present an important subtlety that we will return to repeatedly in our study. The Chern-Simons term in the 5D Lagrangian (2.2) is not manifestly gauge invariant, but it transforms to a total derivative under a gauge variation. Gauge invariance could be restored by introducing a total derivative in the action. Such a term does not change the equations of motion but the resulting theory is not covariant in 5D, so there is a tension between important principles. The bulk part of the 2D Lagrangian (2.22) is not only covariant, it is also manifestly gauge invariant: a^I appears only as the field strength da^I. Manifest gauge invariance also applies to a^0 which encodes 5D rotational invariance. These are benefits of reducing to 2D.

From the dimensionally reduced Lagrangian density (2.22), we can derive the equations of motion for the fields U_1, U_2, a^0, a^I and b^I. The solutions to these 2D equations of motion are solutions of the 5D theory. The field equations for the 2D scalar fields are given by

$$\begin{aligned}\mathcal{E}_{U_1} = {}& d(e^{-U_1-\frac{1}{2}U_2}\star(dU_1+dU_2)) + e^{-U_1-\frac{1}{2}U_2}\Big\{\Big(\mathcal{R}_2+2V-\tfrac{1}{2}e^{2U_1-U_2}\Big)\star 1\\ &+\tfrac{1}{2}dU_1\wedge\star(dU_1+2dU_2)+\tfrac{1}{2}e^{-U_2}da^0\wedge\star da^0+G_{IJ}\Big((b^I da^0+da^I)\wedge\star(b^J da^0+da^J)\\ &-e^{2U_1}b^Ib^J\star 1+e^{U_2}db^I\wedge\star db^J-dX^I\wedge\star dX^J\Big)\Big\}=0\,,\end{aligned} \tag{2.23}$$

$$\begin{aligned}\mathcal{E}_{U_2} = {}& d(e^{-U_1-\frac{1}{2}U_2}\star dU_1) + \tfrac{1}{2}e^{-U_1-\frac{1}{2}U_2}\Big\{\Big(\mathcal{R}_2+2V-2e^{U_1}+\tfrac{3}{2}e^{2U_1-U_2}\Big)\star 1\\ &+\tfrac{1}{2}dU_1\wedge\star(dU_1+2dU_2)+\tfrac{3}{2}e^{-U_2}da^0\wedge\star da^0+G_{IJ}\Big((b^I da^0+da^I)\wedge\star(b^J da^0+da^J)\\ &+e^{2U_1}b^Ib^J\star 1-e^{U_2}db^I\wedge\star db^J-dX^I\wedge\star dX^J\Big)\Big\}=0\,,\end{aligned} \tag{2.24}$$

$$\begin{aligned}\mathcal{E}_{b^I} = {}& 2d(e^{-U_1+\frac{1}{2}U_2}G_{IJ}\star db^J) - 2e^{-U_1-\frac{1}{2}U_2}G_{IJ}da^0\wedge\star(b^J da^0+da^J) - 2G_{IJ}e^{U_1-\frac{1}{2}U_2}b^J\star 1\\ &+c_{IJK}b^J da^K + c_{IJK}b^Jb^K da^0 = 0\,,\end{aligned} \tag{2.25}$$

and the 1-forms satisfy

$$\begin{aligned}\mathcal{E}_{a^0} &= -d(e^{-U_1-\frac{3}{2}U_2}\star da^0) - 2d(G_{IJ}b^I e^{-U_1-\frac{1}{2}U_2}\star(b^J da^0+da^J)) + \tfrac{1}{3}c_{IJK}d(b^Ib^Jb^K) = 0\,,\\ \mathcal{E}_{a^I} &= -2d\left(e^{-U_1-\frac{1}{2}U_2}G_{IJ}\star\left(b^J da^0+da^J\right)\right)+\tfrac{1}{2}c_{IJK}d\left(b^Jb^K\right)=0.\end{aligned} \tag{2.26}$$

2.3 An effective 1D theory

We conclude the section by reducing the 2D reduced Lagrangian (2.22) to a one-dimensional radial effective theory where all of the functions that appear in the effective Lagrangian (2.22) are set to be exclusively radial functions, with respect to the radial coordinate R. In this additional reduction we pick a diagonal gauge for the 2d line element of (2.17):

$$ds_2^2 = e^{2\rho}dt^2 - e^{2\sigma}dR^2. \tag{2.27}$$

The operators d and $\star$ acting on the fields in the Lagrangian (2.22) simplify with this ansatz. For example, the 2D Ricci scalar becomes

$$\mathcal{R}_2 = 2e^{-\rho-\sigma}\partial_R(e^{-\sigma}\partial_R e^{\rho})\,. \tag{2.28}$$

Second derivatives are awkward so it is advantageous to rewrite this term as

$$e^{\rho+\sigma-U_1-\frac{1}{2}U_2}\mathcal{R}_2 = 2\partial_R\left(e^{\rho-\sigma-U_1-\frac{1}{2}U_2}\partial_R\rho\right) + e^{\rho+\sigma-U_1-\frac{1}{2}U_2}(e^{-2\sigma}\partial_R\rho)\partial_R\left(2U_1+U_2\right)\,. \tag{2.29}$$

The first term is a total derivative, an additional boundary term. To examine the total boundary contribution, we consider a constant radial slice at infinity. As we are now reducing to 1D, the boundary terms must be evaluated at the bounds for the time coordinate. This is trivial since there is no explicit time dependence. After dimensional reduction, the Gibbons-Hawking-York term in (2.1) corresponds to the total derivative

$$\mathcal{L}_{\text{GHY}} = \frac{2\pi}{G_5}\partial_R\left(e^{\rho-\sigma-U_1-\frac{1}{2}U_2}\partial_R(\rho-U_1-\tfrac{1}{2}U_2)\right)\,. \tag{2.30}$$

The total derivative term in (2.29), the Gibbons-Hawking-York term (2.30), and the boundary terms in the last line of (2.22) after dimensional reduction to 1D, precisely cancel. This leaves only the contribution arising from the Chern-Simons term

$$\mathcal{L}_{\text{bdry}} = -\frac{1}{6}\frac{\pi}{G_5}d\left(c_{IJK}b^Ib^Ja_t^K\right)\,. \tag{2.31}$$

This remaining boundary term in (2.31) is crucial as it will affect the conserved charges we seek to compute. We will comment on this in depth in the subsequent section 3. In summary, the 1D Lagrangian density takes the form

$$\begin{aligned}\mathcal{L}_1 = \frac{\pi}{G_5}e^{\rho+\sigma-U_1-\frac{1}{2}U_2}\Big[&-e^{-2\sigma}(\partial_R\rho)\partial_R\left(2U_1+U_2\right) - \tfrac{1}{2}e^{-2\sigma}(\partial_R U_1)(\partial_R U_1+2\partial_R U_2)\\ &-G_{IJ}e^{-2\sigma}\left(\partial_R X^I\partial_R X^J - e^{U_2}\partial_R b^I\partial_R b^J\right) + \tfrac{1}{2}e^{-U_2-2\rho-2\sigma}(\partial_R a_t^0)^2\\ &+G_{IJ}e^{-2\rho-2\sigma}(\partial_R a_t^I + b^I\partial_R a_t^0)(\partial_R a_t^J + b^J\partial_R a_t^0) - 2V + 2e^{U_1} - \tfrac{1}{2}e^{2U_1-U_2}\\ &-G_{IJ}e^{2U_1}b^Ib^J\Big] + \frac{\pi}{G_5}\frac{1}{3}c_{IJK}\left[-\frac{3}{2}b^Ib^J\partial_R a_t^K - b^Ib^Jb^K\partial_R a_t^0\right]\,.\end{aligned} \tag{2.32}$$

Having established the effective Lagrangian in 2D (2.22) and 1D (2.32), we proceed in the next subsection with construction of the Noether-Wald surface charges in our theory.

3 Noether-Wald surface charges

In this section, we review the Noether-Wald procedure for computing the conserved charge due to a general symmetry [61, 62]. We specifically consider an isometry generated by a Killing vector and a gauge symmetry in the presence of Chern-Simon terms. In each case, we express the conserved charge as a flux integral that is the same for any surface surrounding the black hole.

3.1 The Noether-Wald surface charge: general formulae

We consider a theory in D dimensions described by a Lagrangian $\mathcal{L}$ that is presented as a D-form. The Lagrangian depends on fields Φ_i that include both the metric $g_{\mu\nu}$ and matter fields, as well as the derivatives of these fields.

A symmetry ζ is such that the variation of $\mathcal{L}$ with respect to ζ is a closed form (locally), i.e. d acting on a $D-1$ form $\mathcal{J}_\zeta$:

$$\mathcal{L} \underset{\zeta}{\rightarrow} \mathcal{L} + \delta\mathcal{L} = \mathcal{L} + d\mathcal{J}_\zeta. \tag{3.1}$$

The variation of the Lagrangian due to *any* change in the fields is given by[2]

$$\begin{aligned}\delta\mathcal{L} &= \delta\Phi_i \frac{\partial\mathcal{L}}{\partial\Phi_i} + (\partial_\mu\delta\Phi_i)\frac{\delta\mathcal{L}}{\delta\partial_\mu\Phi_i} \\ &= \delta\Phi_i\left[\frac{\partial\mathcal{L}}{\partial\Phi_i} - \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\Phi_i}\right)\right] + \partial_\mu\left(\delta\Phi_i\frac{\delta\mathcal{L}}{\delta\partial_\mu\Phi_i}\right).\end{aligned} \tag{3.2}$$

The usual variational principle determines the equations of motion $\mathcal{E}_\Phi$ as the vanishing of the expression in the square bracket. The remaining term, by definition, is the total derivative of the presymplectic potential

$$\Theta^\mu \equiv \delta\Phi_i \frac{\delta\mathcal{L}}{\delta\partial_\mu\Phi_i}. \tag{3.3}$$

In our informal notation, the left hand side of this equation is indistinguishable from a vector. However, the Lagrangian is a D-form and the δ-type "derivative" removes an entire 1-form. Therefore, the presymplectic potential Θ becomes a $D-1$ form, with indices obtained by contracting the volume form with the vector that is normal to the boundary. A more precise version of (3.2) reads

$$\delta\mathcal{L} = \delta\Phi_i\left[\frac{\partial\mathcal{L}}{\partial\Phi_i} - \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\Phi_i}\right)\right] + d\Theta[\Phi_i, \delta\Phi_i]. \tag{3.4}$$

Comparing this formula for a general variation with its analogue (3.1) for a symmetry establishes $d\mathcal{J}_\zeta = d\Theta$ and so the $D-1$ form

$$J_\zeta = \mathcal{J}_\zeta - \Theta[\Phi_i, \delta\Phi_i] \tag{3.5}$$

[2] In practice, when we solve for the Einstein equations, we will consider a variation of the metric and will not directly use (3.2).

is closed when the equations of motion $\mathcal{E}_\Phi$ are imposed. This identifies the familiar conserved Noether current associated to the symmetry ζ. The corresponding Noether charge is

$$Q_{\zeta,\text{Noether}} = \int_\Sigma J_\zeta \,, \tag{3.6}$$

where Σ is a Cauchy surface on the background manifold. Conservation amounts to this charge being the same on all Cauchy surfaces. Conceptually, the total charge is the same at all times. That is the point of conservation in a truly dynamical setting, but it is not terribly interesting in a stationary black hole spacetime which is, by definition, independent of time.

For black holes it is important that, given the closed $(D-1)$ form J_ζ, there exists a $(D-2)$-form Q_ζ such that

$$J_\zeta \cong dQ_\zeta. \tag{3.7}$$

The Q_ζ is the Noether-Wald *surface* charge. It amounts to a conserved *flux* in the sense of Gauss' law: integration of the flux over any surface enclosing the source gives the same result.

The surface charge Q_ζ is more subtle than the conserved charge integrated over an entire Cauchy surface. The semi-equality $\cong$ reminds us that generally the closed form J_ζ is only d of something locally so, in general, the charge Q_ζ, is only defined up to d of some $D-3$ form. Therefore, it does not necessarily satisfy Gauss' law.

One way around this is to evaluate the surface charge at infinity. For example, the presence of a Chern-Simons term can be interpreted physically as a charge density that obstructs flux conservation but this contribution is subleading at infinity and will not contribute to $Q_{\zeta,\text{Noether}}$.

Alternatively, following [52, 53, 55, 56, 63], we can modify our definition of the surface charge by adding a $D-2$ form to Q_ζ. This new surface charge satisfies a Gauss law and can be integrated at any given surface Σ.

A third approach [4], is the one taken in this paper. It is to compute the surface charges in a dimensionally reduced 2D theory.

Moreover, we integrate by parts such that in the process of dimensional reduction to 2D, we ensure gauge invariance. We will carry this procedure out in section 3.4. Therefore, in this case, Q_ζ will satisfy a Gauss law.

The procedure for computing the conserved charges is extremely general. In the following, we make the abstract procedure explicit for two particular symmetries: isometries generated by a spacetime Killing vector ξ and gauge symmetries λ in the presence of Chern-Simons terms.

3.2 Killing vector fields

A Killing vector ξ generates a spacetime isometry. It transforms the Lagrangian as

$$\delta_\xi \mathcal{L} = L_\xi \mathcal{L}. \tag{3.8}$$

Here L_ξ is the Lie derivative along the Killing vector ξ.

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The Lie derivative acting on a general form ω is given by Cartan's magic formula

$$L_\xi \omega = d(i_\xi \omega) + i_\xi d\omega. \tag{3.9}$$

Since $\mathcal{L}$ is a D-form it must be closed $d\mathcal{L} = 0$ and then the Lie derivative becomes

$$\delta_\xi \mathcal{L} = L_\xi \mathcal{L} = d(i_\xi \mathcal{L}) + i_\xi(d\mathcal{L}) = d(\xi \cdot \mathcal{L})\,, \tag{3.10}$$

where $\cdot$ denotes the contraction of ξ with the first index of $\mathcal{L}$. Comparing (3.10) with (3.1), we identify

$$\mathcal{J}_\xi = \xi \cdot \mathcal{L}\,, \tag{3.11}$$

up to a closed form that is unimportant in our application. Thus, for a Killing vector ξ, the Noether current (3.5) becomes

$$J_\xi = \xi \cdot \mathcal{L} - \Theta[\Phi, \mathcal{L}_\xi \Phi]. \tag{3.12}$$

The computations show that this current $(D-1)$ form is closed on-shell. In other words, it is conserved when the equations of motion are satisfied.

3.3 Incorporating gauge invariance

We now consider a *gauge invariant* Lagrangian and compute the conserved current as defined in (3.5) for the conserved charges of the theory, whether derived from spacetime isometries or gauge invariance.

The relevant gauge invariant Lagrangian is the one defined in (2.1) *without* the Chern-Simons term. In other words, we consider the Lagrangian density

$$\begin{aligned}
\mathcal{L}_{5,\text{pot}} &= -\frac{1}{16\pi G_5}\sqrt{g_5}\left(\mathcal{R}_5 + 2V\right)\,, \\
\mathcal{L}_{5,\text{kin}} &= \frac{1}{16\pi G_5}\sqrt{g_5}\left(-\frac{1}{2}G_{IJ}F^I_{5,AB}F^{J,AB}_5 + G_{IJ}\nabla^A X^I \nabla_A X^J\right).
\end{aligned} \tag{3.13}$$

We use early capital Latin indices $A, B, \ldots$ to denote 5D coordinates. The Lagrangian $\mathcal{L}_{5,\text{kin}} + \mathcal{L}_{5,\text{pot}}$ is manifestly gauge invariant

$$\delta_\alpha(\mathcal{L}_{5,\text{pot}} + \mathcal{L}_{5,\text{kin}}) = 0\,, \tag{3.14}$$

As detailed in the previous subsection, there is a conserved charge for any Killing vector that generates a spacetime isometry. According to (3.10), the Lagrangian $\mathcal{L}_{5,\text{kin}} + \mathcal{L}_{5,\text{pot}}$ transforms as

$$\delta_\xi(\mathcal{L}_{5,\text{pot}} + \mathcal{L}_{5,\text{kin}}) = \nabla_A\left(\xi^A(\mathcal{L}_{5,\text{pot}} + \mathcal{L}_{5,\text{kin}})\right). \tag{3.15}$$

The presymplectic potential (3.3) for $\mathcal{L}_{5,\text{pot}}$ is

$$\begin{aligned}
\Theta^{A,5}_{\xi,\text{pot}} &= \frac{1}{16\pi G_5}\sqrt{g_5}\left(\nabla_B\nabla^A\xi^B + \nabla_B\nabla^B\xi^A - 2\nabla^A\nabla_B\xi^B\right) \\
&= \frac{1}{16\pi G_5}\sqrt{g_5}\left(\nabla_B\left(\nabla^B\xi^A - \nabla^A\xi^B\right) + 2R^{AB}\xi_B\right)\,,
\end{aligned} \tag{3.16}$$

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where in the second line, we have used the commutator relation for two covariant derivatives. In addition, the presymplectic potential for the kinetic terms $\mathcal{L}_{5,\mathrm{kin}}$ is

$$\Theta^{A,5}_{\alpha,\xi,\mathrm{kin}} = \frac{1}{16\pi G_5}\sqrt{g_5}G_{IJ}\left(2F_5^{I,AB}\left(\xi^C F^J_{5,CB} + \nabla_B\left(\xi^C A^J_{5,C} + \alpha^J_5\right)\right)\right)\,, \tag{3.17}$$

where we have used the variation

$$\delta A^I_{A,5} = \delta_\xi A^I_{5,A} + \delta_\alpha A^I_{5,A} = \xi^B F^I_{5,BA} + \nabla_A\left(\xi^B A^I_{5,B}\right) + \nabla_A \alpha^I\,. \tag{3.18}$$

Inserting the variations (3.14) and (3.15) and the presympletic potentials given in (3.16) and (3.17) into the current density (3.5), we find

$$\begin{aligned} J^A_{\alpha,\xi} = \frac{1}{16\pi G_5}\sqrt{g_5}\Big[&-\nabla_B\left(\nabla^B\xi^A - \nabla^A\xi^B\right) - 2\nabla_B\left(G_{IJ}F^{I,AB}(\xi^C A^J_C + \alpha^J)\right) \\ &- 2\xi_B\mathcal{E}^B_g - 2\mathcal{E}^A_{J,A_5}(\xi^C A^J_C + \alpha^J)\Big]\,, \end{aligned} \tag{3.19}$$

where the second line is proportional to the equations of motion $\mathcal{E}^B_g$ and $\mathcal{E}^A_{J,A_5}$ and vanish on-shell giving

$$J^A_{\alpha,\xi} = -\frac{1}{16\pi G_5}\sqrt{g_5}\nabla_B\Big[\left(\nabla^B\xi^A - \nabla^A\xi^B\right) + 2\left(G_{IJ}F^{I,AB}(\xi^C A^J_C + \alpha^J)\right)\Big]. \tag{3.20}$$

The Noether-Wald surface charges of the theory can now be read off from the current (3.20). To find the conserved charges, we integrate over a surface Σ enclosing the source and we find

$$\begin{aligned} Q_\alpha &= -\frac{1}{8\pi G_5}\int_\Sigma d\Sigma_{AB}\sqrt{g_5}G_{IJ}F^{I,AB}\alpha^J\,, \\ Q_\xi &= -\frac{1}{16\pi G_5}\int_\Sigma d\Sigma_{AB}\sqrt{g_5}\left[\left(\nabla^B\xi^A - \nabla^A\xi^B\right) + 2G_{IJ}F^{I,AB}\xi^C A^J_C\right]\,. \end{aligned} \tag{3.21}$$

3.4 Chern-Simons terms

The charge Q_ξ that corresponds to angular momentum depends explicitly on the gauge field A^J whereas the electric charges Q_α depend on the field strength. When Chern-Simons terms are taken into account, Q_α also depends on the gauge field A^J. This gauge dependence renders the value of the charges ambiguous.

To address the situation, we dimensionally reduce the theory (2.1) to 2D, as was done in subsection 2.2 and express the resulting action as a covariant theory in 2D [4]. As part of the process, we must ensure that the field strength does not have a nonzero flux through the squashed sphere. This can be achieved by adding total derivatives before the dimensional reduction to remove the derivatives acting on the gauge potentials and gauge fields that act nontrivally through the squashed sphere.

We now show how this can be done. Let us consider the Lagrangian (3.13) along with the five-dimensional Chern-Simons term of the form

$$\mathcal{L}_{5,\mathrm{CS}} = -\frac{1}{16\pi G_5}\frac{1}{6}c_{IJK}F^I_5\wedge F^J_5\wedge A^K_5\,. \tag{3.22}$$

We are interested in transforming (3.22) by the inclusion of total derivatives such that the potential term associated to the electric charge is manifestly gauge invariant. Note this procedure is not covariant in 5D and therefore we explicitly break covariance along the way. However, because of the dimensional reduction, the 2D Lagrangian still remains covariant.

We consider the ansatz in (2.17) such that the potential and gauge fields are of the form

$$\begin{aligned} A^I_5 &= A^I_{5,A} dx^A = A^I_{5,\mu} dx^\mu + A^I_{5,a} dx^a\,, \\ F^I_5 &= \frac{1}{2} F^I_{5,AB}\, dx^A \wedge dx^B = \frac{1}{2} F^I_{5,\mu\nu} dx^\mu \wedge dx^\nu + F^I_{5,\mu a} dx^\mu \wedge dx^a + \frac{1}{2} F^I_{5,ab} dx^a \wedge dx^b\,, \end{aligned} \tag{3.23}$$

where lowercase Latin indices denote the indices on the compact space and as before, the Greek indices correspond to the 2D space. Expanding out the Chern-Simons term in component form using (3.23), there are two types of terms, having the following structure of indices: $F^I_{\mu\nu} F^J_{bc} A^K_a$ and $F^I_{\mu a} F^J_{bc} A^K_\nu$. Only for the second expression we must transfer the derivative such that in the process of dimensional reduction, we find it to be gauge invariant in the 2D theory. This means the integration by parts of this term takes the form

$$c_{IJK} \epsilon^{\mu abc\nu} F^I_{\mu a} F^J_{bc} A^K_\nu = 2 c_{IJK} \epsilon^{\mu abc\nu} \left(\partial_\mu (A^K_\nu A^I_a F^J_{bc}) - (\partial_\mu A^K_\nu)(A^I_a F^J_{bc}) \right)\,, \tag{3.24}$$

and the presymplectic potential is found to be

$$\begin{aligned} \Theta^{A,5}_{\alpha,\xi,\mathrm{CS}} = &\frac{1}{16\pi G_5} \sqrt{g_5} \left[\frac{1}{6} c_{IJK} \left(\epsilon^{ABCDE} F^I_{BC} A^J_D \left(\xi^F F^K_{FE} + \nabla_E (\xi^F A^K_F) + \nabla_E \alpha^K \right) \right) \right] \\ &- \frac{1}{8\pi G_5} \sqrt{g_5} \left[c_{IJK} \epsilon^{Aabc\nu} A^I_a F^J_{bc} \nabla_\nu \alpha^K \right]\,, \end{aligned} \tag{3.25}$$

where the last term is the contribution of (3.24) coming from adding a total derivative. To investigate the current and the Noether-Wald surface charges, we proceed to dimensionally reduce over the squashed S^3 where covariance over the 2D spacetime is still maintained. The 5D rotational isometries in φ and ψ take on a different role in the 2D perspective. Moreover, we find that they become 2D gauge transformations of a^0 and a^I coming from the dimensionally reduced potential A^I (2.18).

3.5 The 2D conserved charges

The 2D Lagrangian (2.22) inherits some symmetries from the 5D theory (2.2), including gauge symmetry associated with the 5D gauge potential A^I and rotational isometries associated to the Killing vectors ∂_ϕ and ∂_ψ. In the 2D theory, all symmetries become gauge symmetries and have associated charges. We denote the 2D charge originally coming from the 5D rotational isometries J and the 2D charges originally coming from the 5D gauge transformations Q_I. These 2D gauge transformations are associated to a^0 and a^I as they come from the dimensionally reduced potential A^I (2.18). Therefore, we consider the following symmetries

$$\delta_\lambda a^0 = d\lambda, \qquad \delta_\chi a^I = d\chi^I\,, \tag{3.26}$$

with total corresponding conserved current

$$J_{\lambda,\chi} = J_\lambda + J_\chi = \sum_{i=\lambda,\chi} (\mathcal{J}_i - \Theta_i)\,, \tag{3.27}$$

where J_λ and J_χ are the currents corresponding to λ and χ, respectively, and the second equality is given by (3.5). The effective 2D Lagrangian (2.22) is manifestly gauge invariant and the variations with respect to each symmetry (3.26) yield

$$\delta_\lambda \mathcal{L}_2 = d\mathcal{J}_\lambda = 0\,, \quad \delta_\chi \mathcal{L}_2 = d\mathcal{J}_\chi = -\frac{\pi}{G_5}\frac{1}{6} c_{IJK}\, d\left(b^I b^J d\chi^K\right). \tag{3.28}$$

The presymplectic potentials given in (3.4) become

$$\Theta_\lambda = -\frac{\pi}{G_5} e^{-U_1-\frac{1}{2}U_2}\left[e^{-U_2} \star da^0 + 2G_{IJ} b^I \star \left(b^J da^0 + da^J\right)\right] d\lambda + \frac{\pi}{G_5}\frac{1}{3} c_{IJK} b^I b^J b^K d\lambda\,, \tag{3.29}$$

$$\Theta_\chi = -\frac{\pi}{G_5} e^{-U_1-\frac{1}{2}U_2}\left[2G_{IJ} d\chi^I \wedge \star\left(b^J da^0 + da^J\right)\right] + \frac{\pi}{G_5}\frac{1}{3} c_{IJK} b^I b^J d\chi^K. \tag{3.30}$$

We used the symmetries (3.26) and included the additional total derivative term (3.24). Using the equations of motion (2.26), the on-shell current (3.27) can be recast in the form of (3.7):

$$\begin{aligned} J_{\lambda,\chi} \cong \frac{\pi}{G_5} d\Bigg[&\lambda\left(e^{-U_1-\frac{1}{2}U_2}\left[e^{-U_2} \star da^0 + 2G_{IJ} b^I \star \left(b^J da^0 + da^J\right)\right] - \frac{1}{3} c_{IJK} b^I b^J b^K\right) \\ &+\chi^I\left(2e^{-U_1-\frac{1}{2}U_2} G_{IJ} \star \left(b^J da^0 + da^J\right) - \frac{1}{2} c_{IJK} b^J b^K\right)\Bigg]. \end{aligned} \tag{3.31}$$

The conserved charges J and Q_I can be directly read off from (3.31) and we find

$$\begin{aligned} J &= \frac{\pi}{G_5}\left[e^{-U_1-\frac{1}{2}U_2}\left[e^{-U_2} \star da^0 + 2G_{IJ} b^I \star \left(b^J da^0 + da^J\right)\right] - \frac{1}{3} c_{IJK} b^I b^J b^K\right], \\ Q_I &= \frac{\pi}{G_5}\left[2e^{-U_1-\frac{1}{2}U_2} G_{IJ} \star \left(b^J da^0 + da^J\right) - \frac{1}{2} c_{IJK} b^J b^K\right]. \end{aligned} \tag{3.32}$$

From now on, we use the rescaled charges

$$\widetilde{J} \equiv \frac{4G_5}{\pi} J\,, \qquad \widetilde{Q} \equiv \frac{4G_5}{\pi} Q_I. \tag{3.33}$$

The charges and the current are indeed conserved including the charge associated to a^I since we demanded gauge invariance at the level of the Lagrangian in (2.22). This added a total derivative that shifted the charge but did not affect the equations of motion. Moreover, the charges computed in 2D are proportional to those computed in 5D. In the 1D reduction (2.32), the charges take the following form

$$\begin{aligned} \widetilde{J} &= 4\left(e^{-U_1-\frac{3}{2}U_2-\rho-\sigma}\left(\partial_R a_t^0 + 2G_{IJ} e^{U_2} b^I\left(\partial_R a_t^J + b^J \partial_R a_t^0\right)\right) - \frac{1}{3} c_{IJK} b^I b^J b^K\right), \\ \widetilde{Q}_I &= 4\left(2e^{-U_1-\frac{1}{2}U_2-\rho-\sigma} G_{IJ}\left(\partial_R a_t^J + b^J \partial_R a_t^0\right) - \frac{1}{2} c_{IJK} b^J b^K\right). \end{aligned} \tag{3.34}$$

These formulae are essential for the radial flow in the black hole background. A *very* rough reading is that each of the conserved charges $\widetilde{J}$ and $\widetilde{Q}_I$ are radial derivatives of their conjugate potentials a_t^0, a_t^I, as in elementary electrodynamics. With this naïve starting point, the overall factors depending on U_1, U_2, ρ and σ serve to take on the non-flat spacetime into account and G_{IJ} incorporates special geometry as required by symmetry. All remaining terms depend on b^I and take rotation into account in a manner that combines kinematics (rotation "looks" like a force) and electrodynamics (electric and magnetic fields mix in a moving frame). These effects defy simple physical interpretations.

From our point of view, the formulae (3.34) for the charges $\widetilde{J}$ and $\widetilde{Q}_I$ are complicated functions of various fields, each of which are themselves nontrivial functions of the radial coordinate. Our construction shows that symmetry guarantees that these *combinations* must be independent of the radial position, within the framework of our *ansatz*.

In the following section we study the conditions that *supersymmetric* AdS_5 black holes must satisfy. The vanishing of the supersymmetry variations of the theory for a subset of the supersymmetries always imposes first-order radial differential equations on the joint geometry/matter configuration. We refer to these first order equations as *flow equations*. They are very constraining but, as usual for first order equations imposed by supersymmetry, they are not sufficient to determine the solution. The raison d'être of this entire section is that the additional data needed, sometimes referred to as the integrability conditions, is furnished by the conserved charges.

4 The flow equations

In this section we derive the first order flow equations for AdS_5 black holes. They follow from preservation of supersymmetry, complemented by conservation of the charges. We study the flow equations using two perturbative expansions: one starting at the near-horizon and one starting at the asymptotic boundary. Enforcing the conservation of charges at both the horizon and at the asymptotic boundary allows us to make contact between the two expansions.

4.1 Supersymmetry conditions

We study bosonic backgrounds that preserve some supersymmetry [32, 64, 65]. Thus there exists a supersymmetric spinor ϵ^α for which the gravitino and the dilatino variations vanish. This condition amounts to

$$0 = \left[G_{IJ}\left(\frac{1}{2}\gamma^{AB}F^J{}_{AB} - \gamma^A\nabla_A X^J\right)\epsilon^\alpha - \xi_I\epsilon^{\alpha\beta}\epsilon^\beta\right]\partial_i X^I\,, \tag{4.1}$$

$$0 = \left[(\partial_A - \frac{1}{4}\omega_A^{BC}\gamma_{BC}) + \frac{1}{24}(\gamma_A{}^{BC} - 4\delta_A{}^B\gamma^C)X_I F^I_{BC}\right]\epsilon^\alpha + \frac{1}{6}\xi_I(3A^I_A - X^I\gamma_A)\epsilon^{\alpha\beta}\epsilon^\beta\,, \tag{4.2}$$

where ϵ^α $(\alpha = 1, 2)$ are symplectic Majorana spinors. In the following, we recast these variations as radial flow equations.

For the analysis of supersymmetry, it is convenient to split the 5D spacetime geometry into $(1+4)$ dimensions as

$$ds_5^2 = f^2(dt + w\sigma_3)^2 - f^{-1}ds_4^2\,, \tag{4.3}$$

$$ds_4^2 = g_m^{-1}dR^2 + \frac{1}{4}R^2(\sigma_1^2 + \sigma_2^2 + g_m\sigma_3^2)\,. \tag{4.4}$$

This form highlights the 4D base space ds_4^2 which is automatically Kähler. This can be shown by picking the flat vielbein

$$e^1 = g_m^{-1/2}dR\,, \quad e^2 = \frac{1}{2}R\sigma_1\,, \quad e^3 = \frac{1}{2}R\sigma_2\,, \quad e^4 = \frac{1}{2}Rg_m^{1/2}\sigma_3\,, \tag{4.5}$$

which gives the manifestly closed Kähler 2-form

$$J^{(1)} = \epsilon(e^1 \wedge e^4 - e^2 \wedge e^3). \tag{4.6}$$

The symbol $\epsilon = \pm 1$ denotes the orientation of the base manifold. It should not be confused with the supersymmetry parameter ϵ^α.

The $(1+4)$ split of the 5D gauge potential A^I defined in (2.18) can be expressed as

$$A^I = fY^I(dt + w\sigma_3) + u^I\sigma_3. \tag{4.7}$$

In the rest of the paper, as well as in appendix C, we use lowercase Latin letters for the four spatial indices.

4.2 Dictionary between the $(1+4)$ and the $(2+3)$ splits

We can relate the $(1+4)$ split introduced in the previous subsection to simplify the supersymmetric variations (4.1) and (4.2), to the $(2+3)$ split (2.17) and (2.18) that was used earlier to perform the reduction from 5D to 2D. The 5D geometry (2.17) with the diagonal gauge (2.27) for the 2D line element is

$$ds_5^2 = e^{2\rho}dt^2 - e^{2\sigma}dR^2 - e^{-U_1}(\sigma_1^2 + \sigma_2^2) - e^{-U_2}(\sigma_3 + a^0)^2. \tag{4.8}$$

By identifying the metric components of (4.3) and (4.8), we find the dictionary of variables in the $(2+3)$ split of the 5D line element ds_5^2, expressed in terms of the variables in the $(1+4)$ split

$$\begin{aligned} e^{-U_1} &= \frac{1}{4}R^2 f^{-1}\,, \qquad e^{-U_2} = \frac{1}{4}R^2 g_m f^{-1} - f^2w^2\,, \qquad b^I = fY^I w + u^I\,, \\ e^{2\sigma} &= f^{-1}g_m^{-1}\,, \qquad a_t^0 = \frac{-f^2 w}{\frac{1}{4}R^2 g_m f^{-1} - f^2 w^2}\,, \qquad e^{2\rho} = f^2 - \frac{f^4w^2}{(\frac{1}{4}R^2 g_m f^{-1} - f^2w^2)^2}. \end{aligned} \tag{4.9}$$

In this section we primarily use the $(1+4)$ variables fX^I, u^I, w and g_m, along with the conserved charges Q_I and J.

As noted in the previous subsection, the 4D base of the $(1+4)$ split (4.3) is automatically Kähler. In the variables of the $(2+3)$ split in (4.9), the Kähler condition amounts to the relation

$$e^{\sigma+\rho-U_2/2} = \frac{1}{2}R\,, \tag{4.10}$$

between ρ, σ, and U_2. This is explained further in appendix C.1.

4.3 The attractor flow equations

The preserved supersymmetries are defined by the projections on the spinors ϵ^α

$$\gamma^0\epsilon^\alpha = \epsilon^\alpha\,, \tag{4.11}$$

$$\frac{1}{4}J^{(1)}_{mn}\gamma^{mn}\epsilon^\alpha = -\epsilon^{\alpha\beta}\epsilon^\beta. \tag{4.12}$$

The $J^{(1)}_{mn}$ are components of the Kähler form $J^{(1)}$ (4.6), and the spatial gamma matrices γ^m satisfy the usual Clifford algebra. The details on the simplification of the equations (4.1) and (4.2) are presented in appendix C. The result is the following differential conditions on the variables fX^I, u^I, w and g_m

$$0 = G_{IJ}\left(\partial_R(fY^I) - \partial_R(fX^I)\right)\partial_i X^J\,, \tag{4.13}$$

$$0 = \left(\partial_{R^2} + \frac{1}{R^2}\right)u^I - \frac{1}{2}\epsilon f^{-1}c^{IJK}X_J\xi_K\,, \tag{4.14}$$

$$0 = \left(\partial_{R^2} - \frac{1}{R^2}\right)w + \frac{1}{2}f^{-1}X_I\left(\partial_{R^2} - \frac{1}{R^2}\right)u^I\,, \tag{4.15}$$

$$0 = -\epsilon R^2(\partial_{R^2}g_m) + 2\epsilon(1-g_m) + 2\xi_I u^I. \tag{4.16}$$

The variation (4.13) allows for the electric potential fY^I in (4.7) to be identified with the scalar field fX^I, and thus

$$A^I = fX^I(dt + w\sigma_3) + u^I\sigma_3. \tag{4.17}$$

This sets the variables a^I, a^0 and b^I in the decomposition (2.18) to

$$a^I + b^I a^0 = fX^I dt\,,\ b^I = fX^I w + u^I\,, \tag{4.18}$$

with $a^0 = a^0_t dt$ given in (4.9).

In the limit of ungauged supergravity, fX^I is a harmonic function. Then, the identification of $X^I = Y^I$ means that the corresponding electric potential is also a harmonic function. We may expect the same functional dependence in the case of gauged supergravity.[3]

In the context of a black hole, solutions to the supersymmetry conditions (4.13)–(4.16) are specified in part by the conserved charges of the theory. The charges in $\widetilde{J}$ and $\widetilde{Q}_I$ in (3.34) are expressed in terms of the $(2+3)$ variables $U_1, U_2, b^I, a^0_t, a^I_t$ but we can recast them in terms of $(1+4)$ variables f, u^I, w, g_m using the dictionary for the geometry (4.9) and the potential (4.18). We can also remove most of the derivatives in the equations (3.34) for the charges $\widetilde{J}$ and $\widetilde{Q}_I$ using the radial equations for the variables u^I, w and g_m (4.14)–(4.16). Our final expressions of the charges, which we use for the remainder of the section are given by

$$\begin{aligned}\widetilde{J} &= \widetilde{Q}_I u^I + \frac{2}{3}c_{IJK}u^I u^J u^K - R^2 g_m\left(f^{-1}X\cdot u + 2w - \frac{1}{2}\epsilon R^2 f^{-3}\xi\cdot fX\right)\\ &\quad + 2R^2 w(1+\epsilon\xi\cdot u)\,,\\ \widetilde{Q}_I &= -2c_{IJK}u^J u^K - 2\epsilon w R^2\xi_I - g_m R^4\partial_{R^2}(f^{-1}X_I).\end{aligned} \tag{4.19}$$

[3]There are n_V+1 potentials Y^I and n_V scalars X^I so there is freedom to adjust a single integration constant that we do not exploit. It is unclear to us if this freedom is physically significant.

The second of these equations is a first order differential equation for $f^{-1}X_I$. Together with the three radial differential equations (4.14)–(4.16) for the variables u^I, g_m and w, we find the four equations

$$\left(\partial_{R^2} + \frac{1}{R^2}\right) u^I = \frac{1}{2}\epsilon c^{IJK}(f^{-1}X_J)\xi_K \,, \tag{4.20a}$$

$$\left(\partial_{R^2} + \frac{2}{R^2}\right) g_m = \frac{2}{R^2} + \frac{2}{R^2}\epsilon\xi_I u^I, \tag{4.20b}$$

$$\left(\partial_{R^2} - \frac{1}{R^2}\right) w = -\frac{1}{2}f^{-1}X_I\left(\partial_{R^2} - \frac{1}{R^2}\right) u^I \,, \tag{4.20c}$$

$$R^4\partial_{R^2}(f^{-1}X_I) = -\frac{1}{g_m}\left(\widetilde{Q}_I + 2c_{IJK}u^J u^K + 2\epsilon w R^2\xi_I\right). \tag{4.20d}$$

We refer to the set of first order differential equations (4.20a)–(4.20d) as the attractor flow equations for black hole solutions to the theory (2.1).

4.4 Solution of the attractor flow equations

The attractor flow equations (4.20a)–(4.20d) are first order differential equations. In this subsection we discuss the boundary conditions needed to specify their solutions completely. This turns out to be surprisingly subtle. We then solve the equations using perturbative expansions.

4.4.1 Boundary conditions

The attractor flow equations (4.20a)–(4.20d) are first order differential equations with ξ_I and $\tilde{Q}_I$ as given parameters. As such the superficial expectation is that the specification of all the unknown functions $u^I, g_m, w, f^{-1}X_I$ at any coordinate R^2 yields the corresponding derivatives at that position. Further iterations should then be sufficient to reconstruct the entire radial dependence, at least in principle. We seek to implement this strategy starting from either asymptotically AdS_5, or from the horizon. We consider each in turn.

For a solution to be asymptotically AdS_5, the metric ansatz (4.3) requires the leading order behavior $f \to R^0$, $g_m \to R^2$, and $w \to R^2$ as $R \to \infty$. With these boundary conditions for f, g_m, and w, (4.20a) and (4.20d) yield $u^I \to c^{IJK}\xi_J\xi_K R^2$ and $X_I \to \xi_I \cdot R^0$ for the matter fields as $R \to \infty$.

Alternatively, we can impose boundary conditions at the horizon of the black hole. There, the near-horizon geometry has a manifest AdS_2 factor of the form

$$ds_2^2 = R^4 dt^2 - \frac{dR^2}{R^2}\,, \tag{4.21}$$

and so $g_m \to R^0$, $f \to R^2$, and $w \to R^{-2}$. With these leading asymptotics, the attractor flow equations (4.20a) and (4.20d) determine $u^I \to R^0$ and $X_I \to R^0$ near the horizon.

Staying with our superficial expectation, we would start from either asymptotically AdS_5 or from the AdS_2 horizon. Mathematically, it could be a concern that the differential equations are coupled and non-linear, because then the expansions might fail to converge. This situation is most likely incompatible with a black hole solution that is

	$R \to \infty$	$R \to 0$
g_m	R^2	R^0
f	R^0	R^2
w	R^2	R^{-2}
u^I	R^2	R^0
X_I	R^0	R^0

Table 1. Asymptotics of the various functions.

regular throughout the entire flow from asymptotically AdS_5 to the AdS_2 horizon, or *vice versa*. Nonlinearity does not appear to pose a conceptual challenge.

In order to study the unknown functions u^I, g_m, w and $f^{-1}X_I$ around a regular point, we multiply by an appropriate factor of R^2. Near the horizon we consider R^2u^I, R^2g_m, R^2w, and $R^2f^{-1}X_I$. At infinity, we expand the functions $R^{-2}u^I, R^{-2}g_m, R^{-2}w$ and $R^{-2}f^{-1}X_I$. After such rescalings the left hand sides of each of the attractor flow equations (4.20a)–(4.20d) will take the form

$$\left(\partial_{R^2} + \frac{\alpha+\beta}{R^2}\right) P = R^{-2\beta}\left(\partial_{R^2} + \frac{\alpha}{R^2}\right)(R^{2\beta}P), \tag{4.22}$$

for some field P and some integers α and β that can either be positive or negative. The challenge we will encounter repeatedly is that, when $P \sim R^{-2(\alpha+\beta)}$, this expression vanishes. We refer to this situation as a *zero-mode* of the perturbative expansion. What it means is that an attractor flow equation does not reveal a derivative, contrary to expectation. Instead, it yields a constraint between the unknown functions on the right hand side of the equation in question. This constraint will be nonlinear and, in general, difficult to implement. In other words, the initial value problem, at both the horizon and at asymptotic infinity, turns out to be unexpectedly complicated.

In the following subsections we develop this general theme explicitly, first starting from the horizon, and then from asymptotic infinity. We subsequently merge the two perturbative expansions to seek a global understanding.

4.4.2 Perturbative solution starting from the horizon

To satisfy the regularity conditions at the horizon, we take $\beta = 1$ in (4.22) and rewrite the attractor flow equations (4.20a)–(4.20d) as

$$\partial_{R^2}(R^2u^I) = \frac{1}{2}\epsilon c^{IJK}(R^2f^{-1}X_J)\xi_K\,, \tag{4.23a}$$

$$\left(\partial_{R^2} + \frac{1}{R^2}\right)(R^2g_m) = 2 + \frac{2}{R^2}\epsilon\xi_I(R^2u^I)\,, \tag{4.23b}$$

$$\left(\partial_{R^2} - \frac{2}{R^2}\right)(R^2w) = -\frac{1}{2}R^{-2}(R^2f^{-1}X_I)\left(\partial_{R^2} - \frac{2}{R^2}\right)(R^2u^I)\,, \tag{4.23c}$$

$$\left(\partial_{R^2} - \frac{1}{R^2}\right)(R^2f^{-1}X_I) = -R^{-2}g_m^{-1}\left(\widetilde{Q}_I + 2R^{-4}c_{IJK}(R^2u^J)(R^2u^K) + 2\epsilon(R^2w)\xi_I\right). \tag{4.23d}$$

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We then expand the unknown functions near the horizon. Since the radial dependence is of the form R^{2n} where n is some integer, the expansion can be written as

$$R^2 u^I = \sum_{n=1}^{\infty} u^I_{(n)} R^{2n}\,, \tag{4.24a}$$

$$R^2 g_m = \sum_{n=1}^{\infty} g_{m,(n)} R^{2n}\,, \tag{4.24b}$$

$$R^2 w = \sum_{n=0}^{\infty} w_{(n)} R^{2n}\,, \tag{4.24c}$$

$$R^2 f^{-1} X_I = \sum_{n=0}^{\infty} x_{I,(n)} R^{2n}. \tag{4.24d}$$

With the asymptotic structure of table 1, the expansions for $R^2 u^I$ and $R^2 g_m$ do not start with a constant term. Moreover, with the horizon expansions above, the differential operators on the left hand sides of (4.23c) and (4.23d) are such that the coefficients $w_{(2)}$ and $x_{(1)}$ drop out. These coefficients are the zero-modes that make the initial value problem more complicated. There are no analogous zero-modes for u^I and g_m.

To study the structure of the attractor equations (4.23a)–(4.23d), we temporarily treat the unknown scalar field $f^{-1}X_I$ as a given function of the radial coordinate R. Then the linear flow equation (4.23a), which is sourced by $f^{-1}X_I$, yields all the coefficients $u^I_{(n)}$ in terms of $x_{I,(n)}$. At this point we know both of the functions $f^{-1}X_I$ and u^I and then the attractor flow equation (4.23b) similarly yields the series coefficients $g_{m,(n)}$ in terms of $x_{I,(n)}$. Given all of $f^{-1}X_I$, u^I, and g_m, it would seem straightforward to exploit (4.23c) and find all the coefficients $w_{(n)}$ in terms of $x_{I,(n)}$. This mostly works, but the zero-mode $w_{(2)}$ can *not* be determined this way. That is the obstacle where, as advertized, the derivatives are such that an expansion coefficient simply drops out.

The final flow equation (4.23d), due to the conserved charge Q_I (4.19), is crucial for the complete story. Assuming for a moment that the zero mode $w_{(2)}$ is given as an initial condition, along with the entire function $f^{-1}X_I$, this equation determines the expansion parameters $x_{I,(n)}$ in terms of the $x_{I,(n)}$ themselves, and so the entire system would appear to be solved. However, this last equation also has a zero mode $x_{I,(1)}$ which cannot be determined by the iterative procedure. In short, a careful analysis must be considered and accordingly, this is what we proceed to do now: we solve the expansion coefficients order by order, following the procedure we have outlined.

First, inserting the expansions (4.24a) and (4.24d) into the flow equations (4.23a), we find

$$\sum_{n=1}^{\infty} n u^I_{(n)} R^{2n-2} = \frac{\epsilon}{2} \sum_{n=0}^{\infty} c^{IJK} \xi_K x_{J,(n)} R^{2n}. \tag{4.25}$$

Comparing each order in R^2 leads to a relation between the coefficients $u^I_{(n)}$ and $x_{I,(n)}$:

$$u^I_{(n)} = \frac{\epsilon}{2n} c^{IJK} \xi_K x_{J,(n-1)}\,, \quad n \geq 1. \tag{4.26}$$

We insert this result in the flow equation (4.23b) for the expansion of g_m (4.24b) and find

$$\sum_{n=1}^{\infty}(n+1)g_{m,(n)}R^{2n-2} = 2 + \sum_{n=1}^{\infty}\frac{1}{n}c^{IJK}\xi_I\xi_J x_{K,(n-1)}R^{2n-2}. \tag{4.27}$$

Thus all expansion coeficients of g_m can be expressed in terms of the $x_{I,(n)}$:

$$g_{m,(n)} = \frac{1}{n+1}\left(2\delta_{n,1} + \frac{1}{n}c^{IJK}\xi_I\xi_J x_{K,(n-1)}\right), \quad n \geq 1\,. \tag{4.28}$$

The steps we have taken so far leads us to express the functions u^I and g_m solely in terms of $f^{-1}X_I$. This was expected from the general discussion.

We then start with (4.23c) and use the expansions (4.24a)–(4.24d) to obtain

$$\sum_{n=0}^{\infty}(n-2)w_{(n)}R^{2n-2} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(n-1-k)x_{I,(k)}u^I_{(n+1-k)}\right)R^{2n-2}. \tag{4.29}$$

Comparing powers of R^2 we find

$$(n-2)w_{(n)} = -\frac{\epsilon}{4}\sum_{k=0}^{n}\frac{n-1-k}{n+1-k}c^{IJK}\xi_I x_{J,(k)}x_{K,(n-k)}. \tag{4.30}$$

We see explicitly that the differential equation (4.23c) fails to express the zero mode $w_{(2)}$ in the expansion (4.24c) in terms of other data. However, the right hand side of (4.30) still reveals important information at $n=2$ since it imposes a constraint on the $x_{I,(n)}$ expansion coefficients

$$0 = \frac{\epsilon}{3}c^{IJK}x_{I,(0)}x_{J,(2)}\xi_K. \tag{4.31}$$

Thus determination of $w_{(2)}$ is replaced by a constraint on the functions $f^{-1}X_I$ which we have considered given so far.

To make further progress it remains to study the constants of motion due to the conservation (4.19) of electric charge and angular momentum. The electric charge (4.23d) yields

$$\begin{aligned}-\sum_{n=1}^{\infty}\sum_{k=1}^{n}&(n-1-k)g_{m,(k)}x_{I,(n-k)}R^{2n-2}\\ &= \tilde{Q}_I + 2c_{IJK}\sum_{n=1}^{\infty}\sum_{k=1}^{n}u^J_{(k)}u^K_{(n+1-k)}R^{2n-2} + 2\epsilon\xi_I\sum_{n=0}^{\infty}w_{(n)}R^{2n}.\end{aligned} \tag{4.32}$$

Comparing each power of R^2 we find the sequence of relations

$$-\sum_{k=1}^{n+1}(n-k)g_{m,(k)}x_{I,(n-k+1)} = \tilde{Q}_I\delta_{n,0} + 2c_{IJK}\sum_{k=1}^{n+1}u^J_{(k)}u^K_{(n+2-k)} + 2\epsilon\xi_I w_{(n)}\,, \tag{4.33}$$

where it is understood in this relation that the coefficients $g_{m,(k)}$, $u^I_{(k)}$, and $w_{(k\neq 2)}$ depend on the $x_{I,(k)}$ through (4.26), (4.28), and (4.30). Thus the relations (4.33) constrain the given functions $f^{-1}X_I$ significantly. Unfortunately, these constraints are nonlinear and difficult to solve.

For the constant order in the R^2 expansion, we take $n = 0$ in (4.33) and find the electric charge

$$\begin{aligned}\widetilde{Q}_I = x_{I,(0)}\left(1+\tfrac{1}{2}c^{JKL}x_{J,(0)}\xi_K\xi_L\right) - \tfrac{1}{2}c_{IJK}c^{JML}c^{KNP}\xi_M\xi_N x_{L,(0)}x_{P,(0)} \\ + \tfrac{1}{4}\xi_I c^{LMN}x_{L,(0)}x_{M,(0)}\xi_N.\end{aligned} \tag{4.34}$$

The charges $\widetilde{Q}_I$ depend only on the scalar fields at the horizon $x_{I,(0)}$ and the FI-parameters ξ_I. In fact, if positivity conditions are imposed on the $x_{I,(0)}$, the $x_{I,(0)}$ in (4.34) can be inverted in terms of the $\widetilde{Q}_I$, allowing to replace the pair of inputs $(\xi_I, x_{I,(0)})$ by the pair $(\xi_I, \widetilde{Q}_I)$. This fits nicely with the understanding of the physical inputs and charges that go in defining the radial flow at every radial hypersurface.

To rewrite (4.34) in a more canonical form, we simplify the second term involving a triple product of c_{IJK} by contracting (B.3) with $\xi_M\xi_Q x_{L,(0)}x_{P,(0)}$. This gives

$$\widetilde{Q}_I = x_{I,(0)} - \frac{1}{2}\xi_I\left(\frac{1}{2}c^{JKL}x_{J,(0)}x_{K,(0)}\xi_L\right) + \frac{1}{2}c_{IJK}\left(\frac{1}{2}c^{JNO}\xi_N\xi_O\right)c^{KLM}x_{L,(0)}x_{M,(0)}. \tag{4.35}$$

This expression makes contact with the form of the charge given in [32].[4] The improvement in our work is that we introduce the charge independently of the radial coordinate so it can be computed at any hypersurface we choose, which — in this case — is the black hole horizon.

Before analyzing the consequences of electric charge conservation (4.33) for $n \geq 1$, we consider the analogous equations due to conservation of the black hole angular momentum J (4.19). As the first line of (4.19) involves the scalar field X^I, we must recast it in terms of the scalar field with a lowered index, as our expansion (4.24d) dictates. Utilizing (B.6), we have

$$f^{-3}fX^I = \frac{1}{2}c^{IJK}(f^{-1}X_J)(f^{-1}X_K). \tag{4.37}$$

Introducing the near-horizon expansions (4.24a)–(4.24d) we find

$$\begin{aligned}\widetilde{J}\delta_{n,0} = \widetilde{Q}_I u^I_{(n+1)} &+ \frac{2}{3}c_{IJK}\sum_{k=1}^{n+1}\sum_{\ell=1}^{k}u^I_{(\ell)}u^J_{(k+1-\ell)}u^K_{(n+2-k)} - \sum_{k=0}^{n}\sum_{\ell=0}^{n-k}g_{m,(k+1)}x_{I,(\ell)}u^I_{(n+1-\ell-k)} \\ &- 2\sum_{k=0}^{n}g_{m,(k+1)}w_{(n-k)} + \frac{\epsilon}{4}c^{IJK}\xi_I\sum_{k=0}^{n}\sum_{\ell=0}^{n-k}g_{m,(k+1)}x_{J,(\ell)}x_{K,(n-k-\ell)} \\ &+ 2w_{(n)} + 2\epsilon\xi_I\sum_{k=0}^{n}w_{(k)}u^I_{(n-k+1)}.\end{aligned} \tag{4.38}$$

[4]The equation agrees with Q_I given in (3.53) of [32] with the following map between notations:

$$q_I = \frac{1}{3}x_{I,(0)},\ \bar{X}_I = \frac{1}{3}\ell\xi_I\,,\ \bar{X}^I = \frac{1}{2}\ell^2c^{IJK}\xi_J\xi_K\,,\ Q_{\text{there}} = \frac{\pi}{4G}\widetilde{Q}_{\text{here}}. \tag{4.36}$$

As before, it is understood that $g_{m,(k)}$, $u^I_{(k)}$, and $w_{(k)}$ depends on the $x_{I,(k)}$ according to (4.26), (4.28) and (4.30), and here we also need the explicit form of $\widetilde{Q}_I$ (4.35). Thus angular momentum conservation gives another infinite set of relations between the $x_{I,(k)}$. Unfortunately, they are even more nonlinear than their analogues for conservation of electric charge.

For $n=0$ (4.38) gives the angular momentum expressed in terms of $x_{I(0)}$ and ξ_I

$$\begin{aligned}\widetilde{J} = &\frac{\epsilon}{4}c^{IJK}x_{I,(0)}x_{J,(0)}\xi_K - \frac{\epsilon}{4}c^{IJK}c^{LMN}\xi_I\xi_N\xi_K x_{J,(0)}x_{L,(0)}x_{M,(0)} \\ &+\epsilon c_{IJK}c^{ILM}c^{JNO}c^{KPQ}\left(\tfrac{1}{8}x_{L,(0)}x_{P,(0)}x_{Q,(0)}\xi_N\xi_O\xi_M + \tfrac{1}{12}x_{N,(0)}x_{P,(0)}\xi_M\xi_O\xi_Q\right),\end{aligned} \tag{4.39}$$

where we have used the value of $\widetilde{Q}_I$ given in (4.35). To make contact with the form of the angular momentum in [32], we rewrite the formula as[5]

$$\widetilde{J} = \frac{\epsilon}{4}c^{IJK}x_{I,(0)}x_{J,(0)}\xi_K + \frac{1}{36}\left(c^{IJK}\xi_I\xi_J\xi_K\right)\left(c^{LMN}x_{L,(0)}x_{M,(0)}x_{N,(0)}\right). \tag{4.41}$$

Again, we are able to express the final result for the conserved charge entirely in terms of near horizon data. Moreover, since $\widetilde{Q}_I$ and $\widetilde{J}$ depend on the same integration constants $x_{I,(0)}$, the charges are indeed not independent of each other.

We now turn to the $n=1$ component of (4.33), i.e., electric charge conservation at order R^2 away from the horizon. It amounts to

$$g_{m,(2)}x_{I,(0)} = 4c_{IJK}u^J_{(1)}u^K_{(2)} + 2\epsilon\xi_I w_{(1)}. \tag{4.42}$$

The absence of $x_{I,(1)}$ in this equation is due to the zero-mode in (4.24d). However, a constraint on the $x_{I,(1)}$ will follow, in analogy with the zero-mode $w_{(2)}$ giving the condition (4.31). The values of $g_{m,(2)}$, $u^I_{(1)}$, $u^I_{(2)}$, and $w_{(1)}$ from (4.26), (4.28) and (4.30) give the vector relation

$$\left[\frac{1}{6}c^{JKL}\xi_K\xi_L x_{I,(0)} - \frac{1}{2}c_{IKM}c^{JLM}c^{KNP}\xi_L\xi_N x_{P,(0)} + \frac{1}{2}c^{JKL}\xi_K x_{L,(0)}\xi_I\right]x_{J,(1)} = 0. \tag{4.43}$$

We see that $x_{I,(1)}$ is constrained even though it is a zero-mode of the differential operator. Using the cubic condition on the c_{IJK} (B.4), we can show that the matrix in square brackets has the null vector

$$x_{I,(1)} = \ell\xi_I. \tag{4.44}$$

It is unique, at least for generic structure constants c_{IJK} and generic charges, which are parametrized by $x_{I,(0)}$. The constraint (4.43) does not determine the overall normalization. However, the scale of the radial coordinate R^2 is arbitrary from the near horizon point of view, so the choice (4.44) involves no loss of generality.

[5]This agrees with the angular momentum reported in (3.50) of [32] with the following map between conventions:

$$q_I = \frac{1}{3}x_{I,(0)}\,, \quad \bar{X}_I = \frac{1}{3}\ell\xi_I\,, \quad J_{\text{there}} = \frac{\pi}{4G}\widetilde{J}_{\text{here}}. \tag{4.40}$$

The $n = 2$ component of (4.33) gives another vector-valued relation

$$T_I^J x_{J,(2)} = 2\epsilon \left(w_{(2)} + \frac{\epsilon}{2\ell} \right) \xi_I \,, \tag{4.45}$$

where we have simplified using (4.44) and

$$T_I^J = -\left(1 + \frac{1}{2} c^{KLM} \xi_K \xi_L x_{M,(0)} \right) \delta_I^J + \frac{1}{12} c^{JKL} \xi_K \xi_L x_{I,(0)} - \frac{1}{3} c_{IKL} c^{KMJ} c^{LNP} \xi_M \xi_N x_{P,(0)}. \tag{4.46}$$

The matrix T_I^J is invertible so, given the inputs $\xi_I, x_{I,(0)}, w_{(2)}$, the coefficient $x_{J,(2)}$ is completely determined by (4.45). However, the value of $x_{J,(2)}$ computed this way fails to satisfy the previously established constraint (4.31). This apparent contradiction can be avoided only if

$$x_{I,(2)} = 0. \tag{4.47}$$

Because the left hand side of (4.45) vanishes, the right side requires

$$w_{(2)} = -\frac{\epsilon}{2\ell}. \tag{4.48}$$

At this point, we can finally consider generic components of the electric charge conservation (4.33), i.e. the infinite set of equations $n \geq 3$. The coefficients $w_{(n\geq 3)}$ can be eliminated using (4.30) for $w_{(n)}$. The $u^I_{(n)}$ and $g_{m,(n)}$ are similarly traded for $x_{I,(n)}$, this time using (4.26) and (4.28). For all $n \geq 3$ this gives

$$\begin{aligned}
&-\sum_{k=0}^{n} \frac{n-1-k}{k+2} \left(2\delta_{0,k} + \frac{1}{k+1} c^{JKL} \xi_J \xi_K x_{L,(k)} \right) x_{I,(n-k)} \\
&= \frac{1}{2} c_{IJK} c^{JLM} c^{KNP} \xi_L \xi_N \sum_{k=0}^{n} \frac{1}{(k+1)(n-k+1)} x_{M,(k)} x_{P,(n-k)} \\
&\quad - \frac{1}{2(n-2)} \xi_I \sum_{k=0}^{n} \frac{n-1-k}{n+1-k} c^{JKL} \xi_J x_{K,(k)} x_{L,(n-k)}.
\end{aligned} \tag{4.49}$$

This messy expression can be reorganized as a recurrence relation giving $x_{I,(n)}$ in terms of the preceding $x_{I,(0\leq k\leq n-1)}$:

$$\begin{aligned}
&\Bigg[-\frac{n-1}{2} \left(2 + c^{KLM} \xi_K \xi_L x_{M,(0)} \right) \delta_I^J + \frac{1}{(n+1)(n+2)} c^{JKL} \xi_K \xi_L x_{I,(0)} \\
&- \frac{1}{n+1} c_{IKL} c^{KMJ} c^{LPQ} \xi_M \xi_P x_{Q,(0)} - \frac{1}{(n-2)(n+1)} c^{JKL} \xi_K x_{L,(0)} \xi_I \Bigg] x_{J,(n)} \\
&= \sum_{k=1}^{n-1} \Bigg[\frac{(n-1-k)}{(k+1)(k+2)} c^{JKL} \xi_J \xi_K x_{L,(k)} x_{I,(n-k)} + \frac{1}{2} c_{IJK} c^{JLM} c^{KNP} \xi_L \xi_N x_{M,(k)} x_{P,(n-k)} \\
&\quad - \frac{1}{2(n-2)} \xi_I \frac{n-1-k}{n+1-k} c^{JKL} \xi_J x_{K,(k)} x_{L,(n-k)} \Bigg].
\end{aligned} \tag{4.50}$$

The left hand side can be inverted, at least for some specific models of c_{IJK}, such as the STU model ($c_{IJK} = |\epsilon_{IJK}|$ for I, J, K running from 1 to 3). In such cases the recurrence relation (4.50) determines all higher-order $x_{I,(n\geq 3)}$ in terms of the coefficients $x_{I,(0)}$, $x_{I,(1)}$ and $x_{I,(2)}$ as well as the FI-parameters ξ_I. In fact, the constraints (4.44) and (4.47) from low n will be sufficient to show that the series *truncates* at $n=2$. This is discussed in subsection 4.4.4.

At this point we have exhausted the information that comes from the conservation of electric charge $\widetilde{Q}_I$ charge in (4.33). We did not yet study the $\widetilde{J}$ conservation relations (4.38). As noted already, the constant order $n=0$ determines the angular momentum from a near horizon perspective. We have worked out the first few orders $n\geq 1$ and found either redundant relations, involving already known coefficients such as $x_{I,(0)}$, ξ_I and $x_{I,(1)}$, or relations that tie together higher order $x_{I,(k\geq 2)}$ with lower-order ones. We do not foresee any further constraints due to the $\widetilde{J}$ relations.

In summary, starting from the near horizon region, we have exploited supersymmetry and found the entire black hole solution. The fields u^I, g_m, w and $f^{-1}X_I$ are reported in (4.26), (4.28), (4.30) and (4.45). Additionally, we computed the electric charges $\widetilde{Q}_I$ and the angular momenta in terms of the horizon values of the scalars $x_{I,(0)}$, and the subleading coefficients $x_{I,(1)}$ which, according to (4.44), coincide with the FI-parameters ξ_I.

4.4.3 Perturbative solution starting from asymptotic AdS

We now adapt the approach from the previous subsection and expand the unknown functions u^I, g_m, w and $f^{-1}X_I$ at large R, near the asymptotic AdS$_5$ boundary.

Given the asymptotic behaviors listed in table 1, regularity requires taking $\beta=-1$ in (4.22). We then recast the flow equations (4.20a)–(4.20d) as

$$\left(\partial_{R^2}+\frac{2}{R^2}\right)(R^{-2}u^I)=\frac{1}{2}\epsilon c^{IJK}(R^{-2}f^{-1}X_J)\xi_K\,, \tag{4.51a}$$

$$\left(\partial_{R^2}+\frac{3}{R^2}\right)(R^{-2}g_m)=\frac{2}{R^4}+\frac{2}{R^2}\epsilon\xi_I(R^{-2}u^I). \tag{4.51b}$$

$$\partial_{R^2}(R^{-2}w)=-\frac{1}{2}R^2(R^{-2}f^{-1}X_I)\partial_{R^2}(R^{-2}u^I)\,, \tag{4.51c}$$

$$\begin{aligned}\left(\partial_{R^2}+\frac{1}{R^2}\right)(R^{-2}f^{-1}X_I)=-\frac{R^{-8}}{(R^{-2}g_m)}\Big(&\widetilde{Q}_I+2c_{IJK}R^4(R^{-2}u^J)(R^{-2}u^K)\\ &+2\epsilon R^4(R^{-2}w)\xi_I\Big)\,.\end{aligned} \tag{4.51d}$$

We define the perturbative expansions at infinity as

$$R^{-2}u^I=\sum_{n=0}^{\infty}\bar{u}^I_{(n)}R^{-2n}\,, \tag{4.52a}$$

$$R^{-2}g_m=\sum_{n=0}^{\infty}\bar{g}_{m,(n)}R^{-2n}, \tag{4.52b}$$

$$R^{-2}w=\sum_{n=0}^{\infty}\bar{w}_{(n)}R^{-2n}\,, \tag{4.52c}$$

$$R^{-2}f^{-1}X_I=\sum_{n=1}^{\infty}\bar{x}_{I,(n)}R^{-2n}. \tag{4.52d}$$

The bar distinguishes the expansion coefficients at the asymptotically AdS_5 boundary from their analogues at the horizon.

As before, we initially specify the entire series $\bar{x}_{I,(n)}$. Additionally, examination of (4.51a)–(4.51d) shows that $\bar{u}^I_{(2)}$, $\bar{g}_{m,(3)}$, $\bar{w}_{(0)}$, and $\bar{x}_{I,(1)}$ do not appear on the left hand sides of the equations. These are the zero modes that we also regard as inputs, at least provisionally. Among the zero-modes, we can determine $\bar{x}_{I,(1)}$ from the outset because they give the asymptotic values of the scalars

$$\bar{x}_{I,(1)} = \ell\xi_I \,, \tag{4.53}$$

as we found in (2.13), by extremizing the potential of gauged supergravity.

We now proceed to solve for the expansion coefficients of each variable, order by order. Starting with the u^I flow equation (4.20a), and using the expansions (4.52a) and (4.52d), we find

$$\sum_{n=0}^{\infty}(2-n)\bar{u}^I_{(n)}R^{-2n-2} = \frac{1}{2}\epsilon c^{IJK}\xi_J\sum_{n=1}^{\infty}\bar{x}_{K,(n+1)}R^{-2n-2}. \tag{4.54}$$

Comparing inverse powers of R^2, we find $\bar{u}^I_{(n)}$ for $n \neq 2$:

$$(2-n)\bar{u}^I_{(n)} = \frac{1}{2}\epsilon c^{IJK}\xi_J\bar{x}_{K,(n+1)}\,, \quad n \geq 0\,. \tag{4.55}$$

The zero mode $\bar{u}^I_{(2)}$ drops out of the equation. Instead, we find a vectorial constraint on $x_{I,(3)}$

$$c^{IJK}\xi_J\bar{x}_{K,(3)} = 0. \tag{4.56}$$

It has the obvious solution

$$\bar{x}_{I,(3)} = 0\,, \tag{4.57}$$

for all values of I. This solution is unique if the matrix $c^{IJK}\xi_J$ is non-singular. In (B.19), we show that it is indeed invertible

$$(c^{IJK}\xi_J)^{-1} = \frac{1}{2}\ell^3\left(c_{IJK}c^{JLM}\xi_L\xi_M - \xi_I\xi_K\right). \tag{4.58}$$

Next, we consider the g_m flow equation (4.20b). The expansions (4.52a) and (4.52b) give

$$\sum_{n=0}^{\infty}(3-n)\bar{g}_{m,(n)}R^{-2n-2} = 2R^{-4} + 2\epsilon\xi_I\sum_{n=0}^{\infty}\bar{u}^I_{(n)}R^{-2n-2}\,. \tag{4.59}$$

The expansion coefficients $\bar{g}_{m,(n)}$ — with the exception of the zero mode $g_{m,(3)}$ — can be expressed in terms of $x_{I,(n)}$ and the zero mode $u^I_{(2)}$ as

$$(3-n)\bar{g}_{m,(n)} = \begin{cases} 2\delta_{1,n} + \frac{1}{2-n}c^{IJK}\xi_I\xi_J\bar{x}_{K,(n+1)} & n \neq 2, \\ 2\epsilon\xi_I\bar{u}^I_{(2)} & n = 2\,. \end{cases} \tag{4.60}$$

In compensation for not determining $\bar{g}_{m,(3)}$, we find the constraint $\xi_I \bar{u}^I_{(3)} = 0$. Rewriting this constraint using (4.55) gives a projection on $x_{I,(4)}$

$$c^{IJK}\xi_I\xi_J\bar{x}_{K,(4)} = 0. \tag{4.61}$$

This constraint is a real special geometry scalar, unlike the vector-valued condition (4.56). It will nevertheless prove useful when simplifying results at large R.

We now turn to the nonlinear flow equation for w (4.20c). After using the expansions (4.52a), (4.52c) and (4.52d), we find

$$\sum_{n=1}^{\infty}(n-1)\bar{w}_{(n-1)}R^{-2n} = -\frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=1}^{n}(n-k)\bar{x}_{I,(k)}\bar{u}^I_{(n-k)}R^{-2n}\,, \tag{4.62}$$

and so

$$(n-1)\bar{w}_{(n-1)} = -\frac{1}{2}\sum_{k=1}^{n}(n-k)\bar{x}_{I,(k)}\bar{u}^I_{(n-k)}\,, \qquad n\geq 1\,. \tag{4.63}$$

For $n=1$, the left hand side vanishes, so the zero-mode $\bar{w}_{(0)}$ is undetermined. The right hand side also vanishes for $n=1$ so in this case the equation with a zero-mode offers no additional information. We omit the $n=1$ case and rewrite (4.63) to

$$\bar{w}_{(n)} = -\frac{1}{2n}\sum_{k=1}^{n}(n+1-k)\bar{x}_{I,(k)}\bar{u}^I_{(n+1-k)}\,, \qquad n\geq 1. \tag{4.64}$$

We have refrained from eliminating $\bar{u}^I_{(k)}$ in favor of $\bar{x}_{I,(n)}$ via (4.55) because, generally, the equation involves the zero mode $\bar{u}^I_{(2)}$ which cannot be removed this way.

The final flow equation (4.51d) was derived by combining supersymmetry with conservation of electric charge. Using the expansions (4.52a)–(4.52d), we find

$$\begin{aligned}
&R^{-4}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\bar{g}_{m,(k)}(n-k)\bar{x}_{I,(n+1-k)}R^{-2n}\\
&= R^{-4}\widetilde{Q}_I\delta_{n,2} + 2c_{IJK}R^{-4}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\bar{u}^J_{(k)}\bar{u}^K_{(n-k)}R^{-2n} + 2\epsilon\xi_I R^{-4}\sum_{n=0}^{\infty}\bar{w}_{(n)}R^{-2n}.
\end{aligned} \tag{4.65}$$

For all $n\geq 0$, this gives

$$\sum_{k=0}^{n}\bar{g}_{m,(k)}(n-k)\bar{x}_{I,(n+1-k)} = \widetilde{Q}_I\delta_{n,2} + 2c_{IJK}\sum_{k=0}^{n}\bar{u}^J_{(k)}\bar{u}^K_{(n-k)} + 2\epsilon\xi_I\bar{w}_{(n)}. \tag{4.66}$$

Again, we have chosen to maintain (4.66) as implicit functions of $\bar{x}_{I,(n)}$ due to the presence of the zero modes.

It is worth examining the first few orders of (4.66) in detail. The $n=0$ component of (4.66) gives

$$\xi_I\bar{w}_{(0)} = -\epsilon c_{IJK}\bar{u}^J_{(0)}\bar{u}^K_{(0)} = -\frac{\epsilon}{16}c_{IJK}c^{JLM}c^{KNP}\bar{x}_{L,(1)}\xi_M\bar{x}_{N,(1)}\xi_P = -\frac{\epsilon}{2\ell}\xi_I\,, \tag{4.67}$$

where we have used (4.53). Thus, it provides the value of the zero mode $w_{(0)}$

$$\bar{w}_{(0)} = -\frac{\epsilon}{2\ell}\,. \tag{4.68}$$

At the next order, the $n = 1$ component of (4.66) is redundant as it just confirms the value for $\bar{w}_{(1)}$ already obtained from (4.64).

The $n = 2$ component of (4.66) is particularly important, because it relates the electric charge to the expansion parameters at infinity

$$\begin{aligned}\widetilde{Q}_I &= \bar{x}_{I,(2)} - \frac{1}{2}\xi_I\left(\frac{1}{2}c^{JKL}\bar{x}_{J,(2)}\bar{x}_{K,(2)}\xi_L\right) + \frac{1}{2}c_{IJK}\left(\frac{1}{2}c^{JNO}\xi_N\xi_O\right)c^{KLM}\bar{x}_{L,(2)}\bar{x}_{M,(2)}\\ &\quad + \epsilon\ell\left(\xi_I\xi_J - c_{IJK}c^{KLM}\xi_L\xi_M\right)\bar{u}^J_{(2)}\,,\end{aligned} \tag{4.69}$$

where we imposed (4.53) and (4.57) and recast the charge in a form similar to (4.35). Since the electric charge is conserved, the expression (4.69) for $\widetilde{Q}_I$, written in terms of the expansion parameters at infinity, must be equal to its analogue (4.35) obtained from expansion near the horizon.

We have established that $\widetilde{Q}_I$ at infinity has been determined with the only inputs necessary being the coefficients $\bar{x}_{I,(2)}$, ξ_I and $\bar{u}^I_{(2)}$. We now move on to the components of (4.66) for $n \geq 3$, to establish the recursion relation for the coefficients at infinity.

For $n = 3$, (4.66) simplifies after eliminating the g_m, u^I and w coefficients using (4.55), (4.60) and (4.64) to

$$4\ell^{-2}\bar{x}_{I,(4)} + 2\epsilon\left(\bar{x}_{I,(2)}\xi_J + \frac{1}{3}\xi_I\bar{x}_{J,(2)} - c_{IJK}c^{KLM}\xi_L\bar{x}_{m,(2)}\right)\bar{u}^J_{(2)} = 0. \tag{4.70}$$

This relation indicates that $\bar{x}_{I,(4)}$ is described only with the help of $\bar{x}_{I,(2)}$, ξ_I and $\bar{u}^I_{(2)}$.

Furthermore, we note that for $n \geq 4$, the simplification of (4.66) yields a generalization of (4.70)

$$\begin{aligned}&\frac{(n-1)^2}{n-2}\ell^{-2}\bar{x}_{I,(n+1)}\\ &= -(n-1)\left(1 + \tfrac{1}{2}(c\cdot\xi\xi\bar{x}_{(2)})\right)\bar{x}_{I,(n)} - 2\epsilon(n-2)\xi_J\bar{u}^J_{(2)}\bar{x}_{I,(n-1)} - (n-3)\bar{g}_{m,(3)}\bar{x}_{I,(n-2)}\\ &\quad - \sum_{k=4}^{n}\frac{n-k}{(k-2)(k-3)}(c\cdot\xi\xi\bar{x}_{(k+1)})\bar{x}_{I,(n+1-k)} + 4c_{IJK}\bar{u}^J_{(2)}\bar{u}^K_{(n-2)}\\ &\quad + 2c_{IJK}c^{JLM}c^{KNP}\xi_L\xi_N\sum_{k=1}^{n-1}{}' \frac{\bar{x}_{M,(k+1)}\bar{x}_{P,(n-k+1)}}{4(k-2)(n-k-2)} + \xi_I\sum_{k=2}^{n}{}' \frac{n+1-k}{n-1-k}\frac{(c\cdot\xi\bar{x}_{(k)}\bar{x}_{(n-k+2)})}{2n}\,,\end{aligned} \tag{4.71}$$

where the apostrophes on the summation symbols indicate that we exclude the terms in the sum with vanishing denominators. We have imposed the value of $\bar{x}_{I,(1)}$ as given in (4.53) and products of the form $c \cdot xyz$ indicate special geometry contractions under c^{JKL} of the form $c^{JKL}x_Jy_Kz_L$. The expression (4.71) has been expanded to distinguish contributions coming from $\bar{x}_{I,(n+1)}$, given by the left hand side of (4.71), and $\bar{x}_{I,(2\leq k\leq n)}$, given by the right hand side of the equality in (4.71). It becomes clear that a given $\bar{x}_{I,(n+1)}$ depends only on the expansion parameters $\bar{x}_{I,(1)}, \bar{x}_{I,(2)}, \ldots, \bar{x}_{I,(n)}$. By recursion, i.e. applying (4.71) repeatedly, we can now determine all $\bar{x}_{I,(4\leq k\leq n)}$ in terms of $\bar{x}_{I,(2)}$, ξ_I, $\bar{u}^I_{(2)}$ and $\bar{g}_{m,(3)}$.

Lastly, we analyse the conservation of angular momentum $\widetilde{J}$ by expanding the first equation in (4.19) at infinity, making sure to rescale the functions u^I, g_m, w and $f^{-1}X_I$ appropriately

$$\begin{aligned}\widetilde{J} = R^2 Q_I(R^{-2}u^I) + \tfrac{2}{3}R^6 c_{IJK}(R^{-2}u^I)(R^{-2}u^J)(R^{-2}u^K) - R^8(R^{-2}g_m)\Big(2R^{-2}(R^{-2}w)\\ +(R^{-2}f^{-1}X)\cdot(R^{-2}u) - \tfrac{1}{2}\epsilon R^{-2}f^{-3}\xi\cdot fX\Big) + 2R^4(R^{-2}w)(1+\epsilon R^2\xi\cdot(R^{-2}u)).\end{aligned} \tag{4.72}$$

This can be expanded like (4.38) in terms of the expansions at infinity (4.52a)–(4.52b), leading to a relation for the component of R^{-2n}. At constant order ($n=0$), we obtain

$$\begin{aligned}\widetilde{J} = \frac{\epsilon}{4}c^{IJK}\bar{x}_{I,(2)}\bar{x}_{J,(2)}\xi_K + \frac{1}{36}\left(c^{IJK}\xi_I\xi_J\xi_K\right)\left(c^{LMN}\bar{x}_{L,(2)}\bar{x}_{M,(2)}\bar{x}_{N,(2)}\right) + \frac{\epsilon}{\ell}\bar{g}_{m,(3)}\\ + \ell\left(\frac{1}{2}\xi_I c^{JKL}\xi_J\xi_K\bar{x}_{L,(2)} + \xi_I + \frac{5}{3}\ell^{-3}\bar{x}_{I,(2)}\right)\bar{u}^I_{(2)}\,,\end{aligned} \tag{4.73}$$

where we have imposed (4.53) and (4.57). This expression at most depends on the inputs $\bar{x}_{I,(2)}$, ξ_I, $\bar{g}_{m,(3)}$ and $\bar{u}^I_{(2)}$. All higher order powers of the $\widetilde{J}$ relation at infinity in (4.72) are redundant because they yield relations between the coefficients that have already been established.

In summary, we have studied the first order equations (4.51a)–(4.51d) due to supersymmetry and conservation of electric charge, by expanding perturbatively near infinity. Given the asymptotic values of the scalars fields (4.53), as well as the conserved charges $\widetilde{Q}_I$, $\widetilde{J}$ defined by fall-off conditions at infinity, the simplest outcome would have been for supersymmetry to determine the entire black hole geometry. Our finding is much more complicated: all physical fields can be expressed as a perturbative series with expansion parameters that depend not only on ξ_I and $\bar{x}_{I,(2)}$ but also the zero-modes $\bar{u}^I_{(2)}$ and $\bar{g}_{m,(3)}$.

4.4.4 Summary and discussion of perturbative solutions

The study in the subsection so far focused on technical details. This was needed because the interplay between supersymmetry, boundary conditions, and conserved charges proved to be rather intricate. We now conclude the subsection with a summary of the final results and discussion of their interpretation.

The black hole solution is parametrized primarily by the matter fields: scalar fields $f^{-1}X_I$, with the prefactor f such that the combination $f^{-1}X_I$ is unconstrained by real special geometry, and the magnetic potentials u^I. Because of supersymmetry, the electric potentials fY^I can be identified with the scalar fields fX^I, see (C.47). Given the matter fields, $f^{-1}X_I$ and u^I, as well as supersymmetry, the geometry is specified by a Kähler base that depends on the function g_m, and a fibre encoding rotation through the potential w. All unknown functions $f^{-1}X_I$, u_I, g_m and w can depend only on a single radial coordinate R^2 and they must satisfy specified first order differential equations (4.20a)–(4.20d).

Supersymmetry is never sufficient to specify an entire solution, because it is first order, and there is always an integrability condition that is of second order. Taking into account the Noether-Wald procedure, we find a second order constraint that satisfies a Gauss' law that was subject to detailed discussion in section 3. With this augmentation, the first order differential equations form a complete system. Angular momentum, with its conservation law also discussed in section 3, yields nothing new, except for a formula giving the angular momentum in terms of the same parameters that define the electric charge in the near-horizon expansion. In the case of the asymptotic infinity expansion, the electric charge depends on one of the zero modes $\bar{u}^I_{(2)}$ in addition to $\bar{x}_{I,(2)}$, ξ_I whereas the angular momentum depends on both the zero modes $\bar{g}_{m,(3)}$ and $\bar{u}^I_{(2)}$ as well as $\bar{x}_{I,(2)}$ and ξ_I.

Consistent boundary conditions for the differential equations can be specified at any radius, in principle. They must depend, at the very least, on the FI-coupling constants ξ_I and the electric charges Q_I. We find that, when *starting from the black hole horizon*, this data is sufficient. Because of supersymmetry, these parameters specify the entire near horizon geometry, including the squashing of the horizon due to angular momentum. This explains why the electric charge and the angular momentum are only described with the use of the leading $x_{I,(0)}$ term in $f^{-1}X_I$, but any further subleading information about any of the fields u^I, g_m or w requires subleading contributions away from the horizon, with *derivative* information of the $f^{-1}X_I$ expansion (4.24d) supplied by the $x_{I,(1)}$ coefficients.

The linchpin for establishing this claim about the near horizon expansion is the scalar field. In the series expansion for $R^2 f^{-1}X_I$, we have the constant at the horizon $x_{I,(0)}$ and then at $\mathcal{O}(R^2)$, we have $x_{I,(1)}$. For the third expansion coefficent, we find $x_{I,(2)} = 0$ (4.47). With this starting point, the recursion relation (4.50) shows that all $x_{I,(k\geq 2)}$ actually *vanish*. The fact that the scalar field $f^{-1}X_I$ truncates after the first two terms is the near horizon version of the fact that $f^{-1}X_I$ is a harmonic function, as is familiar from ungauged supergratity.

When analyzing the supersymmetry conditions we provisionally considered $f^{-1}X_I$ an input that all other variables were expressed in terms of. The truncation $x_{I,(k\geq 2)} = 0$ has the immediate effect of truncating $u^I_{(n\geq 3)} = g_{m,(n\geq 3)} = w_{(n\geq 3)} = 0$. The expansions at the horizon (4.24a)–(4.24d) simplify and we find

$$R^2 u^I = \frac{\epsilon}{2} c^{IJK} \xi_J x_{K,(0)} R^2 + \frac{\epsilon\ell}{4} c^{IJK} \xi_J \xi_K R^4 \,, \tag{4.74a}$$

$$R^2 g_m = \left(1 + \frac{1}{2} c^{IJK} \xi_I \xi_J x_{K,(0)}\right) R^2 + \frac{1}{\ell^2} R^4 \,, \tag{4.74b}$$

$$R^2 w = -\frac{\epsilon}{8} c^{IJK} \xi_I x_{J,(0)} x_{K,(0)} - \frac{\epsilon\ell}{4} c^{IJK} \xi_I \xi_J x_{K,(0)} R^2 - \frac{\epsilon}{2\ell} R^4 \,, \tag{4.74c}$$

$$R^2 f^{-1} X_I = x_{I,(0)} + \ell \xi_I R^2 . \tag{4.74d}$$

These expressions exactly match the well-known Gutowski-Reall solution [32], with the appropriate identifications of notation.[6] The electric charge $\widetilde{Q}_I$ (4.35) and the angular momentum $\widetilde{J}$ (4.39) computed in the near horizon expansion similarly agree with the familiar results.

[6] u^I, g_m, w and $f^{-1}X_I$ match U^I, g, w and $f^{-1}X_I$ respectively in [32] via:

$$q_I = \frac{1}{3} \bar{x}_{I,(2)} \,, \;\; \bar{X}_I = \frac{1}{3} \ell \xi_I . \tag{4.75}$$

The analogous analysis starting from asymptotic AdS$_5$ turned out to be less straightforward. Recalling that coefficients starting from infinity are denoted by barred expansion coefficients, we find that, when shooting in (going from infinity towards the horizon), we must not only specify ξ_I and $\bar{x}_{I,(2)}$, but also $\bar{u}_{I,(2)}$ and $\bar{g}_{m,(3)}$. The harmonic function we established at the horizon reproduces the Gutowski-Reall solution and has features in common with their very familiar analogues in ungauged supergravity. At infinity, it corresponds to

$$x_{I,(0)} = \bar{x}_{I,(2)}\,, \quad \bar{u}^I_{(2)} = \bar{g}_{m,(3)} = 0. \tag{4.76}$$

With these special values, the recursion relation (4.71) simplifies greatly

$$\begin{aligned}
&\frac{(n-1)^2}{n-2}\ell^{-2}\bar{x}_{I,(n+1)} \\
&= -(n-1)\left(1+\tfrac{1}{2}(c\cdot\xi\xi\bar{x}_{(2)})\right)\bar{x}_{I,(n)} - \sum_{k=4}^{n}\frac{n-k}{(k-2)(k-3)}(c\cdot\xi\xi\bar{x}_{(k+1)})\bar{x}_{I,(n+1-k)} \\
&\quad + 2c_{IJK}c^{JLM}c^{KNP}\xi_L\xi_N{\sum_{k=1}^{n-1}}'\frac{\bar{x}_{M,(k+1)}\bar{x}_{P,(n-k+1)}}{4(k-2)(n-k-2)} + \xi_I{\sum_{k=2}^{n}}'\frac{n+1-k}{n-1-k}\frac{(c\cdot\xi\bar{x}_{(k)}\bar{x}_{(n-k+2)})}{2n}.
\end{aligned} \tag{4.77}$$

Since we already know $\bar{x}_{I,(3)} = 0$ from (4.57), and vanishing $\bar{u}^I_{(2)}$ leads to vanishing $\bar{x}_{I,(4)}$ as well via (4.61), it is not difficult to show that the expansion coefficients $\bar{x}_{I,(k\geq 3)}$ all vanish. Thus the perturbative series for $\bar{x}_{I,(n)}$ truncates after two terms, as expected for a harmonic function. The identification (4.76) identifies the subleading coefficient in the harmonic function at infinity with the leading one at the horizon, and *vice versa*.

While (4.76) are the default, it is interesting that asymptotic boundary conditions with nonzero $\bar{u}_{I,(2)}$, $\bar{g}_{m,(3)}$ are consistent with supersymmetry. It has been argued that there may be missing solutions in certain supergravity theories that may not satisfy the canonical nonlinear charge constraint, see for example [60, 66–68]. Since the value of the conserved charges do not take the canonical form, one way wonder if those parameters are somehow related to these missing solutions.

From this point of view, the possibility of $\bar{u}_{I,(2)}$, $\bar{g}_{m,(3)}$ perturbing asymptotic AdS$_5$ might be desirable. In the following, we discuss this possibility.

First, recall that the electric charge $\widetilde{Q}_I$ and the angular momentum $\widetilde{J}$ are conserved charges, which means that they are the same whether evaluated at infinity or the horizon. Identifying (4.35) with (4.69) we find

$$\begin{aligned}
&x_{I,(0)} - \frac{1}{2}\xi_I\left(\frac{1}{2}c^{JKL}x_{J,(0)}x_{K,(0)}\xi_L\right) + \frac{1}{2}c_{IJK}\left(\frac{1}{2}c^{JNO}\xi_N\xi_O\right)c^{KLM}x_{L,(0)}x_{M,(0)} \\
&= \bar{x}_{I,(2)} - \frac{1}{2}\xi_I\left(\frac{1}{2}c^{JKL}\bar{x}_{J,(2)}\bar{x}_{K,(2)}\xi_L\right) + \frac{1}{2}c_{IJK}\left(\frac{1}{2}c^{JNO}\xi_N\xi_O\right)c^{KLM}\bar{x}_{L,(2)}\bar{x}_{M,(2)} \\
&\quad + \epsilon\ell\left(\xi_I\xi_J - c_{IJK}c^{KLM}\xi_L\xi_M\right)\bar{u}^J_{(2)}\,,
\end{aligned} \tag{4.78}$$

from matching $\widetilde{Q}_I$, and similarly (4.41) with (4.73) give

$$\begin{aligned}&\frac{\epsilon}{4}c^{IJK}x_{I,(0)}x_{J,(0)}\xi_K+\frac{1}{36}\left(c^{IJK}\xi_I\xi_J\xi_K\right)\left(c^{LMN}x_{L,(0)}x_{M,(0)}x_{N,(0)}\right)\\ =&\frac{\epsilon}{4}c^{IJK}\bar{x}_{I,(2)}\bar{x}_{J,(2)}\xi_K+\frac{1}{36}\left(c^{IJK}\xi_I\xi_J\xi_K\right)\left(c^{LMN}\bar{x}_{L,(2)}\bar{x}_{M,(2)}\bar{x}_{N,(2)}\right)\\ &+\frac{\epsilon}{\ell}\bar{g}_{m,(3)}+\ell\left(\frac{1}{2}\xi_I c^{JKL}\xi_J\xi_K\bar{x}_{L,(2)}+\xi_I+\frac{5}{3}\ell^{-3}\bar{x}_{I,(2)}\right)\bar{u}^I_{(2)}\,,\end{aligned}\tag{4.79}$$

from matching $\widetilde{J}$. These conservation laws are consistent with a UV solution specified by $x_{I,(0)}$ (and the FI-couplings ξ_I) that flows to an IR configuration with $\bar{x}_{I,(2)}$ that may not even remotely agree with (4.76). This consideration suggests that supersymmetry and charge conservation do little to constrain the IR limit of the flow.

However, there is a different source of intuition. If the perturbative series of $f^{-1}X_I$ from infinity did *not* truncate after exactly two terms, the third term would diverge at the horizon $R^2\to 0$, rather than approaching a constant. Other fields excited at the same order would similarly suggest a singularity. It could happen that, taking into account successive powers R^{-2k} to all orders, there would be a finite limit $R^2\to 0$, after all, but determining by explicit computation whether this possibility is realized for any $\bar{u}_{I,(2)}$, $\bar{g}_{m,(3)}$ is technically challenging.

From a different perspective, since the conserved charges from the near-horizon expansion do satisfy the typical charge constraint, the possibly new black hole solutions that do not seem to satisfy the typical charge constraint at infinity, would not flow to the expected near-horizon extremal AdS_2 geometry, which implies that these solutions may not be black holes after all.

Moreover, a change in the electric potential $fY^I\to fY^I+\beta^I$ with β^I constant is trivial as it does not change the electromagnetic field strength. However, with the vielbein we have picked, such a shift must be accompanied by $u^I\to u^I-w\beta^I$. Because w includes a term $w\sim R^{-2}$ at large R, such a gauge transformation has the ability to remove $\bar{u}^I_{(2)}$. This mechanism shows the $\bar{u}^I_{(2)}$ are allowed, in principle, but also that they are not physical deformations. Indeed, these coefficients diverge at the horizon, so they correspond to a singular gauge which is ill-advised.

5 Entropy Extremization

In this section, we consider the near-horizon limit of the Legendre transform of the radial Lagrangian (2.32), leading to a near-horizon entropy function. Extremizing this entropy function with respect to the near-horizon variables leads to an expression for the entropy in terms of the aforementioned charges.

5.1 Near-horizon setup

First, we consider the near-horizon of the line element ds_2^2 (2.27), where we recall that $e^{2\rho}$ and $e^{2\sigma}$ can be expressed in terms of the variables f, g_m, and w as in (4.9). At the horizon

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$R \to 0$, these variables have known near-horizon behaviors according to table 1. Thus, the near-horizon limit of (2.27) becomes

$$ds^2_{2,\text{nh}} = v\left(\frac{R^4}{\ell_2^2}dt^2 - \frac{dR^2}{R^2}\right), \tag{5.1}$$

with v and ℓ_2 defined based on the $R \to 0$ behavior of $e^{2\rho}$ and $e^{2\sigma}$:

$$e^{2\sigma}\Big|_{\text{nh}} \equiv \frac{v}{R^2}, \qquad e^{2\rho}\Big|_{\text{nh}} \equiv \frac{v}{\ell_2^2}R^4. \tag{5.2}$$

Furthermore, $v^{\frac{1}{2}}$ and $\ell_2^{\frac{1}{3}}$ are near-horizon length scales defining the 2D (t, R) part of the line element (4.8)

$$ds^2_{5,\text{nh}} = v\left(\frac{R^4}{\ell_2^2}dt^2 - \frac{dR^2}{R^2}\right) - e^{-U_1}(\sigma_1^2 + \sigma_2^2) - e^{-U_2}(\sigma_3 + a^0)^2. \tag{5.3}$$

The role of the variable ℓ_2 is elucidated by noting the near-horizon limit of the Kähler condition (4.10):

$$\ell_2 = 2ve^{-\frac{1}{2}U_2}. \tag{5.4}$$

This relation will be used to eliminate ℓ_2 in the rest of the near-horizon analysis.

Having reviewed the near-horizon 2D line element, and in anticipation of applying the entropy function formalism [1–9] to the Lagrangian density in (2.32), we use the following coordinate transformation

$$\begin{aligned} dt &\to \frac{1}{2}\ell_2 dt\,, \\ dR &\to \frac{1}{2R}dR\,, \end{aligned} \tag{5.5}$$

to bring the coordinates (t, R) in (5.1) to the canonical AdS_2 form

$$ds^2_{2,\text{nh}} = \frac{v}{4}\left(R^2dt^2 - \frac{dR^2}{R^2}\right), \tag{5.6}$$

where now it is clear that $v^{\frac{1}{2}}$ is related to the AdS_2 length scale.

The coordinate transformation (5.5) will have the effect of rescaling the Lagrangian 2-form $\mathcal{L}_2 = \mathcal{L}_1 dt \wedge dR$ (2.22) by a factor

$$\mathcal{L}_2 \to \frac{\ell_2}{4R}\mathcal{L}_2. \tag{5.7}$$

With the prescription of defining the entropy function through omitting the $dt \wedge dR$ volume form from the dimensionally-reduced action, we anticipate dividing the density $\mathcal{L}_1$ by a factor of $\frac{4R}{\ell_2}$.

Combining the near-horizon behaviors of $e^{2\rho}$ and $e^{2\rho}$ studied above with the dictionary definitions (4.9), e^{-U_1}, e^{-U_2} and b^I can be shown to be constants to leading order in the near-horizon limit based on the leading-order behaviors of f, g_m and w consistent with the small R asymptotics in table 1.

Concerning the matter fields, the electric fields a^I and a^0 in (2.18) become in the near-horizon limit

$$a^I\Big|_{\rm nh} \equiv \frac{e^I R^2}{2v} e^{\frac{1}{2}U_2} dt, \quad a^0\Big|_{\rm nh} \equiv -\frac{e^0 R^2}{2v} e^{\frac{1}{2}U_2} dt. \tag{5.8}$$

The total 1D Lagrangian density (2.32) then becomes

$$\begin{aligned}\mathcal{L}_{1,\rm nh} = & \frac{\pi}{2G_5} e^{-U_1-\frac{1}{2}U_2} \frac{4R}{\ell_2} v \Bigg[\frac{1}{v^2} e^{-U_2}(e^0)^2 - \frac{4}{v} + e^{U_1} - \frac{1}{4} e^{2U_1-U_2} - \frac{1}{2} G_{IJ} e^{2U_1} b^I b^J \\ & + \frac{2}{v^2} G_{IJ}(e^I - b^I e^0)(e^J - b^J e^0) - V \Bigg] - \frac{\pi}{2G_5} \frac{4R}{\ell_2} \frac{1}{2} c_{IJK} b^I b^J \left(e^K - \frac{2}{3} e^0 b^K \right).\end{aligned} \tag{5.9}$$

Apart from an overall prefactor in the integration measure, every other appearance of ℓ_2 has been re-expressed in terms of v and U_2 by using (5.4). The $\frac{4R}{\ell_2}$ factor has been factored out of the volume element, and we follow the prescription made earlier to exactly cancel it out with the $\frac{\ell_2}{4R}$ factor from (5.7) in order to obtain the Lagrangian density suitable for the entropy function. We also note the presence of the Chern-Simons boundary terms in (5.9) that are crucial for calculating the near-horizon charges.

We now obtain a near-horizon Lagrangian (5.9) that is a function of the variables v, U_1, U_2, e^I, e^0, and b^I. We will next derive the charges Q_I and J from $\mathcal{L}_{1,\rm nh}$, with the goal of Legendre transforming the Lagrangian into an entropy function $\mathcal{S}$ that can be ultimately extremized towards a function purely of the charges $\mathcal{S} = \mathcal{S}(Q_I, J)$.

We note from the earlier Noether procedure (3.32) that Q_I and J were obtained in terms of radially dependent variables. In terms of the near-horizon limit of the electric fields (5.8), this becomes

$$Q_I = \frac{\pi}{G_5} \left[e^{-U_1-\frac{1}{2}U_2} \frac{4}{v} G_{IJ}(e^J - b^J e^0) - \frac{1}{2} c_{IJK} b^J b^K \right], \tag{5.10}$$

$$J = \frac{\pi}{G_5} \left[-\frac{2}{v} e^{-U_1-\frac{3}{2}U_2} e^0 + \frac{4}{v} e^{-U_1-\frac{1}{2}U_2} G_{IJ} b^I (e^J - b^J e^0) - \frac{1}{3} c_{IJK} b^I b^J b^K \right]. \tag{5.11}$$

This introduces the charges Q_I and J as conjugates to the electric fields e^I and e^0, allowing for the inversion

$$e^I - b^I e^0 = \frac{v}{16} e^{U_1+\frac{1}{2}U_2} G^{IJ} \left(\widetilde{Q}_J + 2c_{JKL} b^K b^L \right), \tag{5.12}$$

$$e^0 = -\frac{v}{8} e^{U_1+\frac{3}{2}U_2} \left(\widetilde{J} - \widetilde{Q}_I b^I - \frac{2}{3} c_{IJK} b^I b^J b^K \right), \tag{5.13}$$

where now the rescaled $\widetilde{Q}_I$ and $\widetilde{J}$ (3.33) have been used. The near-horizon entropy function can now be defined as a Legendre transform of the Lagrangian density (5.9) with fixed charges

$$\mathcal{S} = 2\pi \left(e^I \frac{\partial \mathcal{L}_{1,\rm nh}}{\partial e^I} + e^0 \frac{\partial \mathcal{L}_{1,\rm nh}}{\partial e^0} - \mathcal{L}_{1,\rm nh} \right), \tag{5.14}$$

which, after eliminating the electric fields through (5.12) and (5.13), yields

$$\begin{aligned}\mathcal{S} = & \frac{4\pi^2}{G_5} e^{-U_1-\frac{1}{2}U_2} \Bigg[1 + \frac{v}{4} \Bigg(\frac{1}{4} e^{2U_1-U_2} - e^{U_1} + \frac{1}{64} e^{2U_1+2U_2} \left(\widetilde{J} - \widetilde{Q}_I b^I - \frac{2}{3} c_{IJK} b^I b^J b^K \right)^2 \\ & + V + \frac{1}{128} e^{2U_1+U_2} G^{IJ} \left(\widetilde{Q}_I + 2c_{IKL} b^K b^L \right) \left(\widetilde{Q}_J + 2c_{JMN} b^M b^N \right) + \frac{1}{2} e^{2U_1} G_{IJ} b^I b^J \Bigg) \Bigg].\end{aligned} \tag{5.15}$$

This entropy function depends on the physical variables v, U_1, U_2, b^I, X^I describing the near horizon geometry and matter fields, with the conserved charges $\widetilde{Q}_I$, $\widetilde{J}$ appearing as fixed parameters. At its extremum, it yields the physical variables and the black hole entropy as a function of the charges.

5.2 Extremization of the Entropy Function

It is exceedingly simple to extremize with respect to v which appears only as a Lagrange multiplier in front of the large round bracket that comprises nearly all of (5.15). This leaves the extremized value of $\mathcal{S}$:

$$\mathcal{S} = \frac{4\pi^2}{G_5} e^{-U_1 - \frac{1}{2}U_2}. \tag{5.16}$$

This is exactly the black hole entropy computed via the area law for a horizon defined by the volume 3-form $e^{-U_1-\frac{1}{2}U_2}\sigma_1 \wedge \sigma_2 \wedge \sigma_3$ with the angular ranges specified in (2.20). However, the explicit dependence of U_1 and U_2 on the charges remains to be determined. For this we must extremize with respect to the remaining variables

$$\partial_{U_1}\mathcal{S} = \partial_{U_2}\mathcal{S} = \partial_{b^I}\mathcal{S} = D_I\mathcal{S} = 0. \tag{5.17}$$

Here D_I is the Kähler-covariantized derivative with respect to the scalars X^I. It is defined such that $X^I D_I = 0$, which is the correct way to vary the scalars while also implementing the constraint (B.1). The conditions (5.17) give

$$0 = V - e^{U_1} + \frac{1}{4}e^{2U_1-U_2} + \frac{1}{64}e^{2U_1+2U_2}\mathcal{M}^2 + G^{IJ}\frac{1}{128}e^{2U_1+U_2}\mathcal{K}_I\mathcal{K}_J + \frac{1}{2}G_{IJ}e^{2U_1}b^Ib^J\,, \tag{5.18}$$

$$0 = -4 - vV + \frac{v}{4}e^{2U_1-U_2} + \frac{v}{64}e^{2U_1+2U_2}\mathcal{M}^2 + G^{IJ}\frac{v}{128}e^{2U_1+U_2}\mathcal{K}_I\mathcal{K}_J + \frac{v}{2}G_{IJ}e^{2U_1}b^Ib^J\,, \tag{5.19}$$

$$\begin{aligned}0 = &-2 - \frac{v}{2}V + \frac{v}{2}e^{U_1} - \frac{3v}{8}e^{2U_1-U_2} + \frac{3v}{128}e^{2U_1+2U_2}\mathcal{M}^2 + G^{IJ}\frac{v}{256}e^{2U_1+U_2}\mathcal{K}_I\mathcal{K}_J \\ &- \frac{v}{4}G_{IJ}e^{2U_1}b^Ib^J\,,\end{aligned} \tag{5.20}$$

$$0 = vD_IV + \frac{v}{128}e^{2U_1+U_2}(D_IG^{JK})\mathcal{K}_J\mathcal{K}_K + \frac{v}{2}e^{2U_1}(D_IG_{JK})b^Jb^K\,, \tag{5.21}$$

$$0 = \frac{1}{32}e^{U_2}\left(-e^{U_2}\mathcal{K}_I\mathcal{M} + 2G^{JK}c_{IJN}b^N\mathcal{K}_K\right) + G_{IJ}b^J\,, \tag{5.22}$$

where $\mathcal{M}$ and $\mathcal{K}_I$ are shorthand for

$$\mathcal{M} \equiv \widetilde{J} - \widetilde{Q}_Ib^I - \frac{2}{3}c_{IJK}b^Ib^Jb^K\,,\ \ \mathcal{K}_I \equiv \widetilde{Q}_I + 2c_{IJK}b^Jb^K. \tag{5.23}$$

Additionally, D_I acts on the scalars X^J following[7]

$$D_IX^J = \delta_I^J - \frac{1}{3}X_IX^J. \tag{5.25}$$

[7]This can be generalized to other quantities via the product rule on D_I, for instance,

$$D_IX_J = \frac{1}{2}c_{JKL}D_I(X^KX^L) = c_{IJK}X^K - \frac{2}{3}X_IX_J. \tag{5.24}$$

Ideally we seek the most general extremum that solves (5.18)–(5.22) that is consistent with our ansatz (5.3). However, the extremization equations are highly nonlinear in the variables of interest (v, U_1, U_2, X^I, b^I). Therefore, in the following we specialize and find all supersymmetric solutions.

Near-horizon supersymmetric conditions. It is straightforward to take the near-horizon limit of the supersymmetry conditions (4.14)–(4.16), along with the identification $X^I = Y^I$ (C.47). After inverting e^I and e^0 in terms of $\widetilde{Q}_I$ and $\widetilde{J}$, following (5.12) and (5.13)), we obtain the following near-horizon supersymmetric relations

$$0 = \widetilde{Q}_I + 2c_{IJK}b^J b^K - 4e^{-U_2}X_I \,, \tag{5.26}$$

$$0 = \widetilde{J} - \widetilde{Q}_I b^I - \frac{2}{3}c_{IJK}b^I b^J b^K - 4e^{-U_1-U_2}(\xi \cdot X) \,, \tag{5.27}$$

$$0 = b^I - e^{-U_1}\left(X^I(\xi \cdot X) - G^{IJ}\xi_J\right) \,, \tag{5.28}$$

$$0 = e^{U_1} - \frac{4}{v} - 2V. \tag{5.29}$$

We also need the near-horizon version of the Kähler condition (4.10). We can trade the variables ρ and σ describing the 2D geometry for f and g_m following (4.9) and eliminate a_t^0 that results in favor of the charges through (5.13). These steps lead to

$$\frac{4}{v} - (\xi \cdot X)^2 = e^{2U_1-U_2}. \tag{5.30}$$

We have verified that when the five supersymmetric relations (5.26)–(5.30) are satisfied, then the five $\mathcal{S}$ extremization equations (5.18)–(5.22) are satisfied as well. The details of this computation are not instructive so we omit them. The reverse logic would be that *all* extremal solutions within the scope of our ansatz are supersymmetric. This we have not shown, and it is indeed not true, i.e. there are no known nonextremal supersymmetric Lorentzian black holes. Thus the specialization to supersymmetric solutions addresses a genuine subset of the extremal black holes.

In the remainder of this subsection we solve the supersymmetry relations (5.26)–(5.30) explicitly and find all variables as functions of the charges Q_I and J.

5.2.1 Solving for the entropy and the charge constraint

The supersymmetry conditions (5.26)–(5.30) are all algebraic, but they are far from trivial. Straightforward contractions, followed by taking simple linear combinations, give scalar identities

$$\begin{aligned} X \cdot b &= e^{-U_1}\xi \cdot X \,, \\ \widetilde{Q} \cdot b &= \frac{3}{2}\widetilde{J} - 8e^{-U_1-U_2}\xi \cdot X \,, \\ \xi \cdot b &= e^{U_1-U_2} - 1 \,, \end{aligned} \tag{5.31}$$

which will prove useful later. Our strategy will be to exploit identities like these to find simple combinations of variables that can be expressed entirely in terms of the charges

Q_I, J and the couplings ξ_I. Combinations of those will in turn give explicit formulae for physical variables.

In this spirit, we expand $\widetilde{Q}_J\widetilde{Q}_K$ using the square of (5.26), and then simplify the terms that are products of b's using (5.27). We obtain

$$\frac{1}{2}c^{IJK}\widetilde{Q}_J\widetilde{Q}_K = 32e^{-2U_1-U_2}c^{IJK}\xi_J\xi_K + 16e^{-U_1-U_2}X^I - 2\widetilde{J}b^I. \tag{5.32}$$

Contracting (5.32) with ξ_I, the first term on the right becomes proportional to $e^{-2U_1-U_2}$, which is related to the black hole entropy through $\mathcal{S}$ (5.16). There will also be a term $(\xi \cdot X)$ that we can eliminate with the help of

$$\widetilde{J} = 8e^{-2U_1}(\xi \cdot X) + \frac{32}{3}e^{-3U_1}c^{IJK}\xi_I\xi_J\xi_K\,, \tag{5.33}$$

which is a simplification of (5.27) with $\widetilde{Q}_I$ and b^I eliminated using (5.26) and (5.28), respectively. These steps give

$$\frac{1}{6}c^{IJK}\xi_I\xi_J\xi_K\left(\frac{\mathcal{S}}{2\pi}\right)^2 = \frac{1}{2}c^{IJK}\xi_I Q_J Q_K - \frac{\pi}{4G_5}2J\,, \tag{5.34}$$

which amounts to an explicit formula for the black hole entropy as function of the conserved charges

$$\mathcal{S} = 2\pi\sqrt{\frac{1}{2}c^{IJK}\ell^3\xi_I Q_J Q_K - N^2 J}. \tag{5.35}$$

This is in full agreement with the entropy of supersymmetric extremal AdS_5 black holes [69–71]. We have expressed the entropy using the untilded charges Q_I and J (3.33) and traded G_5 for N^2 using $\frac{\pi\ell^3}{4G_5} = \frac{1}{2}N^2$ and applied $\frac{1}{6}c^{IJK}\xi_I\xi_J\xi_K = \ell^{-3}$ from (2.12) to explicitly show all the dimensionful quantities.

Continuing with the strategy of evaluating natural combinations of the conserved charges, we evaluate the cubic invariant of the charges $c^{IJK}\widetilde{Q}_I\widetilde{Q}_J\widetilde{Q}_K$ by taking the cube of $\widetilde{Q}_I$ from (5.26), with the resulting contractions of X_I and b^I such as $c_{IJK}X^Ib^Jb^K$ and X_Ib^I simplified using the b^I relation (5.28) as well as (5.29) and (5.30) for the terms quadratic in ξ. This yields

$$c^{IJK}\widetilde{Q}_I\widetilde{Q}_J\widetilde{Q}_K = 6e^{-U_1-2U_2} + \frac{1}{8}\widetilde{J}c_{IJK}b^Ib^Jb^K. \tag{5.36}$$

Alternatively, we can arrive at the cubic product of electric charges by contracting (5.32) with Q_I from (5.26), giving

$$\begin{aligned}&\tfrac{1}{64}c^{IJK}\widetilde{Q}_I\widetilde{Q}_J\widetilde{Q}_K\\ &= 4e^{-U_1-2U_2} + 2e^{-2U_1-U_2} + e^{-2U_1-U_2}c^{IJK}\widetilde{Q}_I\xi_J\xi_K - \tfrac{1}{4}e^{-U_1-U_2}\widetilde{J}(\xi\cdot X) + \tfrac{1}{8}\widetilde{J}c_{IJK}b^Ib^Jb^K.\end{aligned} \tag{5.37}$$

Comparing (5.36) and (5.37), we find the identity

$$\frac{1}{2}c^{IJK}\widetilde{Q}_I\xi_J\xi_K + 1 = e^{U_1}\left(e^{-U_2} + \frac{1}{8}\widetilde{J}\;\xi\cdot X\right). \tag{5.38}$$

It is useful because it gives access to a useful combination of U_1, U_2, and $\xi \cdot X$. Indeed, we can simplify the cube of the electric charge in the form (5.36) using the b identity (5.28), and then (5.33) to eliminate $(\xi \cdot X)$, to give

$$\frac{1}{6}c^{IJK}\widetilde{Q}_I\widetilde{Q}_J\widetilde{Q}_K + \widetilde{J}^2 = 64 e^{-U_1-U_2}\left(e^{-U_2} + \frac{1}{8}\widetilde{J}\,\xi \cdot X\right). \tag{5.39}$$

The right-hand side of this equation differs from that of (5.38) only by a factor proportional $e^{-2U_1-U_2}$ which is the square of the geometric measure on the black hole horizon. As such, it is related to the black hole entropy $\mathcal{S}$ both through the area law (5.16) and as a function of charges (5.35). Collecting these relations, and reintroducing Q_I and J (3.33) in order to align with the conventional units for this result, we find

$$\left(\frac{1}{6}c^{IJK}Q_IQ_JQ_K + \frac{\pi}{4G_5}J^2\right) = \ell^3\left(\frac{1}{2}c^{IJK}\xi_I\xi_JQ_K + \frac{\pi}{4G_5}\right)\left(\frac{1}{2}c^{IJK}\xi_IQ_JQ_K - \frac{\pi}{2G_5}J\right). \tag{5.40}$$

This is the prototypical 5D nonlinear charge constraint [69–71]. However, the charge constraint (5.40) does not make progress towards solving the supersymmetry equations nor determining the near horizon solution in terms of conserved charges. Rather, it is a relation between the conserved charges that, if taken at face value, all supersymmetric black holes dual to $\mathcal{N} = 4$ SYM must satisfy. This is extremely important and the continuing questions regarding this constraint is one of the motivations for the detailed study reported in this article.

Even at this point of our discussion where we are deep into solving certain nonlinear equations, it is worth noting that the black hole entropy (5.34) and the charge constraint (5.40) can be combined into one complex-valued equation

$$\frac{1}{6}c^{IJK}\left(Q_I + i\frac{\mathcal{S}}{2\pi}\xi_I\right)\left(Q_J + i\frac{\mathcal{S}}{2\pi}\xi_J\right)\left(Q_K + i\frac{\mathcal{S}}{2\pi}\xi_K\right) + \frac{\pi}{4G_5}\left(-J + i\frac{\mathcal{S}}{2\pi}\right)^2 = 0. \tag{5.41}$$

The real part gives the constraint (5.40) and the imaginary part gives the formula for the entropy (5.34). Complexified equations are natural in problems involving supersymmetry. Also, (5.41) appears as the condition for a complex saddle point that provides an accounting in N = 4 SYM for the entropy of black hole preserving 1/16 of the maximal supersymmetry [69–72].

5.2.2 Near-horizon variables as function of conserved charges

Having discussed the black hole entropy and the constraint on charges, we move on to expressing all other aspects of the near-horizon geometry and the matter content in terms of the fixed charges Q_I and J.

For the following computations, we make use of the expressions (5.26) and (5.27) for J and Q_I simplified using the relation for b^I in (5.28). We find

$$Q_I = \frac{\pi}{G_5}e^{-2U_1}\left[X_I e^{2U_1-U_2} + X_I(\xi\cdot X)^2 - 2\xi_I(\xi\cdot X) - 4G_{IJ}c^{JKL}\xi_K\xi_L\right], \tag{5.42}$$

$$J = \frac{16\pi}{G_5}e^{-3U_1}\left[\frac{1}{8}e^{U_1}\xi\cdot X + \ell^{-3}\right]. \tag{5.43}$$

Multiplying both sides of (5.38) by J, using the relation (5.43), we can solve for $e^{U_1}\,\xi\cdot X$ in terms of the charges

$$\frac{1}{8}\xi\cdot X e^{U_1}=\frac{J\left(\frac{1}{2}c^{IJK}Q_I\xi_J\xi_K+\frac{\pi}{4G_5}\right)-\left(\frac{\mathcal{S}}{2\pi}\right)^2\ell^{-3}}{J^2+\left(\frac{\mathcal{S}}{2\pi}\right)^2}. \tag{5.44}$$

This result ties a geometrical quantity — as it appears in the near-horizon Kähler relation (5.30) — to the charges. It also has an additional immediate value as (5.43) relates it to e^{-3U_1}, giving

$$e^{-3U_1}=\frac{G_5}{16\pi}\frac{J^2+\left(\frac{\mathcal{S}}{2\pi}\right)^2}{\frac{1}{2}c^{IJK}Q_I\xi_J\xi_K+\frac{\pi}{4G_5}+J\ell^{-3}}. \tag{5.45}$$

This expression sets the scale of the non-deformed S^3 which has line element $e^{-U_1}(\sigma_1^2+\sigma_2^2+\sigma_3^2)$.

Due to the rotation of the black hole, the horizon geometry (4.8) is deformed away from S^3. We can quantify the deformation by computing $e^{U_1-U_2}$ via e^{3U_1} from (5.45) and $e^{2U_1+U_2}$ from (5.16):

$$e^{U_1-U_2}=\frac{\frac{4G_5}{\pi}\left(\frac{1}{2}c^{IJK}Q_I\xi_J\xi_K+\frac{\pi}{4G_5}+J\ell^{-3}\right)\left(\frac{\mathcal{S}}{2\pi}\right)^2}{J^2+\left(\frac{\mathcal{S}}{2\pi}\right)^2}. \tag{5.46}$$

The only scalar near-horizon parameter that was not yet computed is the AdS$_2$-volume v. Due to the alternate near-horizon Kähler condition (5.30), the combination ve^{U_1} can be expressed in terms of $\mathcal{S}$, $(\xi\cdot X)e^{U_1}$ and e^{-3U_1}. These three quantities were given as functions of the conserved charges in (5.16), (5.44) and (5.45). After simplifications, we find

$$\frac{v}{4}e^{U_1}=\frac{\pi\ell^3}{4G_5}\frac{\ell^3\left(\frac{1}{2}c^{IJK}Q_I\xi_J\xi_K+\frac{\pi}{4G_5}\right)+J}{\ell^6\left(\frac{1}{2}c^{IJK}Q_I\xi_J\xi_K+\frac{\pi}{4G_5}\right)^2+\left(\frac{\mathcal{S}}{2\pi}\right)^2}. \tag{5.47}$$

This completes the explicit extremization of the entropy function for the scalar variables which at this point have all been expressed in terms of conserved charges Q_I, J and FI-couplings ξ_I.

We must similarly determine the vectors b^I and X^I at the extremum which may be determined, in principle, by the input vectors ξ_I and $\widetilde{Q}_I$. However, the position of the vector indices I do not match so the full real special geometry enters. We exploit only the vectorial symmetry, and then, X^I and b^I must be linear combinations of *three* vectors: $c^{IJK}\widetilde{Q}_IQ_J$, $c^{IJK}\widetilde{Q}_I\xi_J$, and $c^{IJK}\xi_I\xi_J$. One linear relation of this kind was given in (5.32). To find another, we contract (5.32) with $\widetilde{Q}_I$ and simplify using (5.37). This gives

$$\widetilde{Q}\cdot X=8e^{-U_2}+4e^{-U_1}. \tag{5.48}$$

Combining this with (5.31) and (5.44), we have all four inner products of b^I, X^I, Q_I and ξ_I. We already determined the scalar combinations $c^{IJK}\xi_I\xi_J\widetilde{Q}_K$ and $c^{IJK}\xi_I\widetilde{Q}_J\widetilde{Q}_K$ from (5.34) and (5.38), so we can establish the vectorial equation

$$\frac{1}{2}c^{IJK}\widetilde{Q}_J\xi_K = b^I + \frac{1}{8}\widetilde{J}e^{U_1}X^I. \tag{5.49}$$

Inversion of (5.32) and (5.49) give

$$b^I = \frac{4G_5}{\pi}\cdot\frac{1}{2}\frac{c^{IJK}\xi_J Q_K\left(\frac{\mathcal{S}}{2\pi}\right)^2 - \left(\frac{1}{2}c^{IJK}Q_JQ_K - \frac{1}{2}c^{IJK}\xi_J\xi_K\left(\frac{\mathcal{S}}{2\pi}\right)^2\right)J}{J^2+\left(\frac{\mathcal{S}}{2\pi}\right)^2}, \tag{5.50}$$

and

$$X^I = 4e^{-U_1}\frac{Jc^{IJK}\xi_J Q_K + \left(\frac{1}{2}c^{IJK}Q_JQ_K - \frac{1}{2}c^{IJK}\xi_J\xi_K\left(\frac{\mathcal{S}}{2\pi}\right)^2\right)}{J^2+\left(\frac{\mathcal{S}}{2\pi}\right)^2}, \tag{5.51}$$

where, once we impose the value of e^{-U_1} given in (5.45), X^I is a function of the entropy (5.35) and the charges of the black hole.

In summary, we have found that the near-horizon limit of the supersymmetric equations implies that the near-horizon fields and variables of the geometry/matter ansatz are given by the charges Q_I and J, through the relations (5.45)–(5.51), which themselves parametrize a special extremum of the near-horizon entropy function (5.15), for general FI coupling ξ_I and c_{IJK}.

5.3 Complexification of the near-horizon variables

Each of the main results derived in the previous subsection are complicated formulae. However, they resemble one another and, in particular, it stands out that several expressions, such as (5.50) and (5.51), share a common denominator. Indeed, there is an elegant way to pair them into complexified near-horizon variables

$$Z^I = b^I - ie^{-\frac{1}{2}U_2}X^I = \frac{G_5}{\pi}\frac{c^{IJK}(Q_J + i\frac{\mathcal{S}}{2\pi}\xi_J)(Q_K + i\frac{\mathcal{S}}{2\pi}\xi_K)}{-J + i\frac{\mathcal{S}}{2\pi}}. \tag{5.52}$$

To the extent X^I can be interpreted is an electric field it is indeed natural that its partner is a magnetic field b^I. In addition to the real part b^I being given by (5.50), we recognize in the imaginary part the combination of the factor $e^{-U_1-\frac{1}{2}U_2}$ in the entropy $\mathcal{S}$ and $X^Ie^{U_1}$ given respectively by (5.35) and (5.51).

Some discussions of the AdS$_5$ black hole geometry invoke from the outset principles that are inherently complex, such as the Euclidean path integral or special geometry in four dimensions. This can give conceptual challenges so, in our discussion of entropy extremization, complex variables such as (5.52) are introduced [26, 57] only for their apparent convenience. To make precise connections with the literature, we now to reintroduce the electric fields e^I and e^0 conjugate to the conserved charges $\tilde{Q}_I$ and $\tilde{J}$. For e^0 defined in (5.13), simplification using (5.27), gives an expression for e^0 that depends on ve^{U_1} in (5.47), e^{-3U_1} in (5.45), $\mathcal{S}$ in (5.16), and $(\xi\cdot X)e^{U_1}$ in (5.44). Collecting formulae, we then find

$$e^0 = -\frac{4\pi}{\mathcal{S}}\frac{\pi\ell^3}{4G_5}\frac{J\ell^3\left(\frac{1}{2}c^{IJK}Q_I\xi_J\xi_K+\frac{\pi}{4G_5}\right)-\left(\frac{\mathcal{S}}{2\pi}\right)^2}{\ell^6\left(\frac{1}{2}c^{IJK}Q_I\xi_J\xi_K+\frac{\pi}{4G_5}\right)^2+\left(\frac{\mathcal{S}}{2\pi}\right)^2}. \tag{5.53}$$

This expression for e^0 combines nicely with (5.47) and gives the complex potential

$$\frac{1}{2}e^0+i\frac{v}{4}e^{U_1} = \frac{\pi\ell^3}{4G_5}\left(\frac{2\pi}{\mathcal{S}}\right)\frac{-J+i\frac{\mathcal{S}}{2\pi}}{\ell^3\left(\frac{1}{2}c^{IJK}Q_I\xi_J\xi_K+\frac{\pi}{4G_5}\right)+i\left(\frac{\mathcal{S}}{2\pi}\right)}. \tag{5.54}$$

Given e^0 in (5.53) as well as (5.50) and (5.51), the electrical potentials dual to the vectorial charges become (5.12):

$$e^I = \frac{2\pi}{\mathcal{S}}\frac{\ell^6\left(\frac{1}{2}c^{IJK}Q_JQ_K-\frac{1}{2}c^{IJK}\xi_J\xi_K\left(\frac{\mathcal{S}}{2\pi}\right)^2\right)\left(\frac{1}{2}c^{IJK}Q_I\xi_J\xi_K+\frac{\pi}{4G_5}\right)+\ell^3c^{IJK}Q_J\xi_K\left(\frac{\mathcal{S}}{2\pi}\right)^2}{\ell^6\left(\frac{1}{2}c^{IJK}Q_I\xi_J\xi_K+\frac{\pi}{4G_5}\right)^2+\left(\frac{\mathcal{S}}{2\pi}\right)^2}. \tag{5.55}$$

As preparation for the complexified version, we combine (5.50) and (5.51) as

$$\begin{aligned}&\frac{v}{2}\left(b^Ie^{U_1}+X^I\xi\cdot X\right)\\&=\frac{\ell^3c^{IJK}Q_J\xi_K\left(\frac{1}{2}c^{LMN}Q_L\xi_M\xi_N+\frac{\pi}{4G_5}\right)-\ell^3\left(\frac{1}{2}c^{IJK}Q_JQ_K-\frac{1}{2}c^{IJK}\xi_J\xi_K\left(\frac{\mathcal{S}}{2\pi}\right)^2\right)}{\ell^6\left(\frac{1}{2}c^{IJK}Q_I\xi_J\xi_K+\frac{\pi}{4G_5}\right)^2+\left(\frac{\mathcal{S}}{2\pi}\right)^2},\end{aligned} \tag{5.56}$$

where we have imposed $\frac{v}{4}e^{U_1}$ in (5.47), e^{-U_1} in (5.45), and $(\xi\cdot X)e^{U_1}$ in (5.44). We then find the complex special geometry vector

$$e^I+i\frac{v}{2}\left(b^Ie^{U_1}+X^I(\xi\cdot X)\right) = \frac{2\pi}{\mathcal{S}}\frac{\frac{1}{2}c^{IJK}(Q_J+i\frac{\mathcal{S}}{2\pi}\xi_J)(Q_K+i\frac{\mathcal{S}}{2\pi}\xi_K)}{\ell^3(\frac{1}{2}c^{LMN}Q_L\xi_M\xi_N+\frac{\pi}{4G_5})+i\frac{\mathcal{S}}{2\pi}}. \tag{5.57}$$

The complex potentials (5.54)–(5.57) appear commonly in the literature, albeit with the normalization

$$\frac{\omega}{\pi} = \frac{\pi\ell^3}{4G_5}\left(\frac{2\pi}{\mathcal{S}}\right)\frac{-J+i\frac{\mathcal{S}}{2\pi}}{\ell^3\left(\frac{1}{2}c^{IJK}Q_I\xi_J\xi_K+\frac{\pi}{4G_5}\right)+i\left(\frac{\mathcal{S}}{2\pi}\right)} = \frac{1}{2}e^0+i\frac{v}{4}e^{U_1}, \tag{5.58}$$

and

$$\frac{\Delta^I}{\pi} = \frac{2\pi}{\mathcal{S}}\frac{\frac{1}{2}c^{IJK}\ell^3(Q_J+i\frac{\mathcal{S}}{2\pi}\xi_J)(Q_K+i\frac{\mathcal{S}}{2\pi}\xi_K)}{\ell^3\left(\frac{1}{2}c^{IJK}Q_I\xi_J\xi_K+\frac{\pi}{4G_5}\right)+i\left(\frac{\mathcal{S}}{2\pi}\right)} = e^I+i\frac{v}{2}(b^Ie^{U_1}+X^I\xi X). \tag{5.59}$$

The real and imaginary parts of the complexified potentials ω and Δ^I are related to one another through

$$\xi_Ie^I+e^0=0. \tag{5.60}$$

and from the identities (5.28) and (5.30), we find

$$2\omega + \xi_I \Delta^I = 2\pi i. \tag{5.61}$$

This is our version of the well-known complex constraint that is imposed on the chemical potentials conjugate to J and Q_I in analyses involving complex saddle points from the outset. An important example is the Hosseini-Hristov-Zaffaroni (HHZ) extremization principle for 5D rotating BPS black holes [26, 69–71]. The complexified potentials ω and Δ^I can be exploited to simplify the Lagrangian density (5.9). The linchpin is the identity

$$\frac{\frac{1}{6}c_{IJK}\Delta^I\Delta^J\Delta^K}{\omega^2} = \left(\frac{2\pi^2}{\mathcal{S}}\right)\frac{(-J+\frac{i\mathcal{S}}{2\pi})^2}{\ell^3\left(\frac{1}{2}c^{IJK}Q_I\xi_J\xi_K + \frac{\pi}{4G_5}\right) + i\left(\frac{\mathcal{S}}{2\pi}\right)}. \tag{5.62}$$

It is established using the cube of (5.59), along with (B.4) to simplify the products of c_{IJK}, as well as the square of (5.58) and the complexified charge relation (5.41). The same combination of terms appears when evaluating instead

$$\Delta^I Q_I - 2\omega J = -\left(\frac{N^2}{2}\right)\left(\frac{2\pi^2}{\mathcal{S}}\right)\frac{(-J+\frac{i\mathcal{S}}{2\pi})^2}{\ell^3\left(\frac{1}{2}c^{IJK}Q_I\xi_J\xi_K + \frac{\pi}{4G_5}\right) + i\left(\frac{\mathcal{S}}{2\pi}\right)} + \mathcal{S}\,, \tag{5.63}$$

with the use of the definitions of ω and Δ^I in (5.58) and (5.59) respectively, as well as the complex relations (5.41) and (5.61), and exchanging G_5 for N via $\frac{\pi\ell^3}{4G_5} = \frac{N^2}{2}$. This allows us to rewrite the black hole entropy $\mathcal{S}$ as

$$\mathcal{S} = \Delta^I Q_I - 2\omega J + \frac{N^2}{2}\frac{\frac{1}{6}c_{IJK}\Delta^I\Delta^J\Delta^K}{\omega^2}. \tag{5.64}$$

Referring back to the real-valued entropy functional $\mathcal{S}$ as the Legendre transform of the on-shell Lagrangian $\mathcal{L}_1$ (5.14), and noting that (e^I, e^0) constitute the real parts of (Δ^I, ω), we obtain the greatly simplified expression

$$2\pi\mathcal{L}_{1,\text{nh}} = -\frac{N^2}{2}\text{Re}\left(\frac{\frac{1}{6}c_{IJK}\Delta^I\Delta^J\Delta^K}{\omega^2}\right). \tag{5.65}$$

We have thus been able to reproduce the standard HHZ entropy function result [26], although while remaining entirely in 5D (no reduction to 4D), with the help of the entropy function formalism. The derivation of (5.64) also makes the Legendre transformation between the entropy $\mathcal{S}$ and the complexified entropy function manifest.

6 Discussion

We have analysed the first order attractor flow equations derived from the vanishing of the supersymmetric variations in $D = 5$ $\mathcal{N} = 2$ gauged supergravity with FI-couplings to $\mathcal{N} = 2$ vector multiplets. We focus on solutions with electric charges Q_I and one independent angular momentum J. In order to analyze the flow equations and find the conserved charges, we first assume a perturbative expansion at either the near-horizon geometry or the asymptotic boundary. As usual, the supersymmetry conditions are not sufficient to guarantee a solution to the equation of motion, but we find that the conserved Noether-Wald surface charges fill this gap. This leads to a self-contained set of first order differential equations.

To integrate these differential equations we need boundary conditions, or more generally integration constants. In the present setting, this turns out to be somewhat complicated. Generically, first order differential equations, even coupled ones, just need values at one point to compute the derivative and then, by iteration, the complete solution follows.[8] We find that, whether starting from the black hole horizon or the asymptotic AdS_5, solving the first order equations is subtle. Supersymmetry conditions exhibit zero-modes which fail to provide a derivative, as a first order differential equation is expected to do. On the positive side, in these situations supersymmetry give relations between the first few coefficients near a boundary.

After exploiting conserved charges extensively, the initial value problem simplifies. Indeed, at the horizon, all fields must satisfy the entropy extremization principle, discussed in detail in section 5. The relative simplicity of *shooting out* from the horizon can be construed as black hole attractor behavior. The situation starting from asymptotic AdS is much more involved, as detailed in subsection 4.4.

We are far from the first to investigate the attractor flow for rotating AdS_5 black holes. Some notable works are [26, 57]. In our procedure, we have remained in five dimensions, without dimensionally reducing to four dimensions, where the metric no longer contains a fibration. Our approach is complementary, in that the role of rotation is highlighted. Additionally, we have allowed for backgrounds that go beyond the omnipresent STU model. Finally, we have also considered a complexification of the near-horizon variables that elucidates some features of the theory from the near-horizon perspective. This includes the well-known complex constraint on the chemical potentials.

Many open problems persist after our analysis of AdS_5 rotating black holes. For example, we derived the first order attractor flow equations from the supersymmetric variations of the $\mathcal{N} = 2$ gauged theory, but it would be instructive to also derive them from the Lagrangian. After a suitable Legendre transform, the dimensionally reduced Lagrangian can be written as a sum of squares, up to a total derivative. In minimizing the Lagrangian, each square gives a condition that is equivalent to the vanishing of the supersymmetric variations. It would be interesting to extract the flow equations from this method as it can also be more directly related to the entropy extremization once the near-horizon limit is taken. We also expect this now radial entropy function to greatly simplify once the fields and variables in it are suitably complexified, such as was done at the near-horizon level. This would allow for an understanding of the underlying complex structure of the rotating AdS_5 black hole spacetime without the customary reduction to 4D.

[8] Locality is among the major caveats. In principle, first order differential equation give derivatives, and then the derivatives of the derivatives, and so on for the whole series. Generally, it is not easy to prove convergence for a series obtained this way, but this obstacle, and other mathematical fine points, do not appear significant at our level of analysis.

Higher derivative corrections in the context of AdS_5 black holes have been studied by [63, 73–75] and references therein, and it would be interesting to understand the role of higher derivative corrections in the attractor flow. This is also interesting from the entropy extremization point of view and allows us to probe higher derivative corrections to the entropy from the near-horizon, which can be checked via holography. Finally, a similar analysis can then be completed in other dimensions, including the rotating AdS black holes in six and seven dimensions [76, 77]. The product of the scalar fields with one of the parameters of the metric yields a harmonic function and we would expect that one can solve the flow equations using a similar approach via a perturbative expansion. We hope to comment on these ideas in the near future.

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A Conventions and notations

In this appendix, we summarize the conventions and notations used in the various expressions involving differential geometry as well as real special geometry.
We introduce components as

$$\xi = \xi^\mu \frac{\partial}{\partial x^\mu}\,, \quad \omega = \frac{1}{r!}\omega_{\mu_1\ldots\mu_r}\,\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_r}. \tag{A.1}$$

In this notation the interior product i_ξ of ω with respect to ξ is

$$\begin{aligned} i_\xi\omega &= \frac{1}{(r-1)!}\xi^\nu\omega_{\nu\mu_2\ldots\mu_r}\,\mathrm{d}x^{\mu_2}\wedge\ldots\wedge\mathrm{d}x^{\mu_r} \\ &= \frac{1}{r!}\sum_{s=1}^{r}\xi^{\mu_s}\omega_{\mu_1\ldots\mu_s\ldots\mu_r}(-1)^{s-1}\,\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\widehat{\mathrm{d}x^{\mu_s}}\wedge\ldots\wedge\mathrm{d}x^{\mu_r}. \end{aligned} \tag{A.2}$$

The wide hat indicates that dx^{μ_s} is removed.

The Hodge dual is defined by

$$\star_r\left(\,\mathrm{d}x^{\mu_1}\wedge\mathrm{d}x^{\mu_2}\wedge\ldots\wedge\mathrm{d}x^{\mu_r}\right) = \frac{\sqrt{|g|}}{(m-r)!}\varepsilon^{\mu_1\mu_2\ldots\mu_r}{}_{v_{r+1}\ldots v_m}\,\mathrm{d}x^{v_{r+1}}\wedge\ldots\wedge\mathrm{d}x^{v_m}\,, \tag{A.3}$$

where the subscript r denotes the dimension of the spacetime and the totally antisymmetric tensor is

$$\varepsilon_{\mu_1\mu_2\ldots\mu_m} = \begin{cases} +1 & \text{if } (\mu_1\mu_2\ldots\mu_m) \text{ is an even permutation of } (12\ldots m) \\ -1 & \text{if } (\mu_1\mu_2\ldots\mu_m) \text{ is an odd permutation of } (12\ldots m) \\ 0 & \text{otherwise.} \end{cases} \tag{A.4}$$

The indices on the totally antisymmetric symbol $\varepsilon_{\mu_1\mu_2\ldots\mu_m}$ can be raised by the metric through

$$\varepsilon^{\mu_1\mu_2\ldots\mu_m} = g^{\mu_1 v_1} g^{\mu_2 v_2} \ldots g^{\mu_m v_m} \varepsilon_{v_1 v_2 \ldots v_m} = g^{-1} \varepsilon_{\mu_1\mu_2\ldots\mu_m}. \tag{A.5}$$

The Hodge dual of the identity 1 gives the invariant volume element

$$\star_r 1 = \frac{\sqrt{|g|}}{m!} \varepsilon_{\mu_1\mu_2\ldots\mu_m} \, \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_m} = \sqrt{|g|} \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m. \tag{A.6}$$

We define the r-forms U and V as

$$U = \frac{1}{r!} U_{\mu_1\ldots\mu_r} \, \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_r}, \qquad V = \frac{1}{r!} V_{\mu_1\ldots\mu_r} \, \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_r}\,, \tag{A.7}$$

such that

$$U \wedge \star_r V = V \wedge \star_r U = \frac{1}{r!} U_{\mu_1\ldots\mu_r} V^{\mu_1\ldots\mu_r} \sqrt{|g|} \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m. \tag{A.8}$$

B Real special geometry

In this appendix we summarize the conventions and formulae needed for manipulations in real special geometry. We study $\mathcal{N} = 2$ theories with n_V vector multiplets and $n_H = 0$ hyper-multiplets. The starting point is a collection of real 5D scalar fields X^I with $I = 0, 1, \ldots, n_V$. They are subject to the constraint

$$\frac{1}{6} c_{IJK} X^I X^J X^K = 1\,, \tag{B.1}$$

where the structure constants c_{IJK} are real numbers, completely symmetric in I, J, and K, that satisfy the closure relation

$$c_{IJK} c_{J'(LM} c_{PQ)K'} \delta^{JJ'} \delta^{KK'} = \frac{4}{3} \delta_{I(L} c_{MPQ)}. \tag{B.2}$$

The index I takes $n_V + 1$ distinct values but, because of the constraint (B.1), there are n_V independent scalar fields, one for each $\mathcal{N} = 2$ vector multiplet in 5D. Round brackets $(\cdots)$ indicate symmetrization of indices with weight one so, for example, $c_{IJK} = c_{(IJK)}$.

Using the Euclidean metric to define c^{IJK} with upper indices, meaning $c^{IJK} = \delta^{II'} \delta^{JJ'} \delta^{KK'} c_{I'J'K'}$, the closure relation (B.2) can be rewritten as

$$c_{IJK} c^{J(LM} c^{PQ)K} = \frac{4}{3} \delta_I^{(L} c^{MPQ)}. \tag{B.3}$$

We also note the following identities involving symmetrizations

$$c_{IJK}c^{J(LM}c^{PQ)K} = \frac{1}{3}c_{IJK}\left(c^{JLM}c^{PQK} + c^{JLP}c^{MQK} + c^{JPM}c^{LQK}\right), \tag{B.4}$$

$$\delta_I^{(L}c^{MPQ)} = \frac{1}{4}\left(\delta_I^L c^{MPQ} + \delta_I^M c^{LPQ} + \delta_I^P c^{LMQ} + \delta_I^Q c^{LMP}\right). \tag{B.5}$$

Given the scalars X^I and c_{IJK} as inputs, we define the scalar X_I (with lower index) and the metric on field space G_{IJ} as

$$\begin{aligned} X_I &= \frac{1}{2}c_{IJK}X^J X^K, \\ G_{IJ} &= \frac{1}{2}\left(X_I X_J - c_{IJK}X^K\right). \end{aligned} \tag{B.6}$$

In manipulations we often use the formulae

$$\begin{aligned} G_{IJ}X^J &= \frac{1}{2}X_I, \\ X_I X^I &= 3. \end{aligned} \tag{B.7}$$

The closure relation (B.3) then requires that the inverse matrix G^{IJ} satisfies

$$c^{IJK}X_K = X^I X^J - \frac{1}{2}G^{IJ}. \tag{B.8}$$

It follows that, just as G_{IJ} lowers indices on X^J indices (up to a factor of $\frac{1}{2}$), the inverse G^{IJ} raises indices on X_J

$$G^{IJ}X_J = 2(X^I X^J - c^{IJK}X_K)X_J = 2X^I. \tag{B.9}$$

We also note the identity

$$(c^{IJK}X_K)(c_{ILM}X^M) = \left(X^I X^J - \tfrac{1}{2}G^{IJ}\right)(X_I X_L - 2G_{IL}) = \delta_L^J + X^J X_L. \tag{B.10}$$

In the literature, it is common to summarize real special geometry through the cubic polynomial

$$\mathcal{V} = \frac{1}{6}c_{IJK}X^I X^J X^K. \tag{B.11}$$

The constraint (B.1) is simply $\mathcal{V} = 1$. Differentiating *first* and *then* imposing the constraint $\mathcal{V} = 1$, we find

$$\mathcal{V}_I \equiv \frac{\partial\mathcal{V}}{\partial X^I} = \frac{1}{2}c_{IJK}X^J X^K = X_I, \tag{B.12}$$

$$\mathcal{V}_{IJ} \equiv \frac{\partial^2\mathcal{V}}{\partial X^I \partial X^J} = c_{IJK}X^K, \tag{B.13}$$

$$G_{IJ} = -\frac{1}{2}\frac{\partial^2 \ln\mathcal{V}}{\partial X^I \partial X^J} = \frac{1}{2}(\mathcal{V}_I\mathcal{V}_J - \mathcal{V}_{IJ}) = \frac{1}{2}\left(X_I X_J - c_{IJK}X^K\right). \tag{B.14}$$

The inverse $\mathcal{V}^{IJ}$ of $\mathcal{V}_{IJ}$ (meaning it satisfies $\mathcal{V}^{IJ}\mathcal{V}_{JK} = \delta_K^I$) is given by

$$\mathcal{V}^{IJ} = \frac{1}{2}(X^I X^J - G^{IJ}). \tag{B.15}$$

The STU-model is an important example. In this special case $n_V = 2$ and we shift the labels so $I = 1, 2, 3$ (rather than $I = 0, 1, 2$). The only nonvanishing c_{IJK} are $c_{123} = 1$ and all its permutations. In our normalizations, the STU model has

$$X^1 X^2 X^3 = 1 \,, \quad X_I^{-1} = X_I \,, \quad G_{IJ} = \frac{1}{2} X_I^2 \delta_{IJ}.$$

In these formulae there is no sum over $I = 1, 2, 3$. We add a special note about adapting the formalism of real special geometry, this time adapted to ξ_I given by the constraint

$$\frac{1}{6} c^{IJK} \xi_I \xi_J \xi_K = \ell^{-3}. \tag{B.16}$$

Following similar steps in terms of defining a raised version of the ξ_I, imposing consistency with the raised c^{IJK} through the condition (B.4), we can define the following

$$\begin{aligned} \xi^I &= \frac{1}{2} c^{IJK} \xi_J \xi_K \,, \\ \xi_I &= \frac{1}{2} \ell^3 c_{IJK} \xi^J \xi^K. \end{aligned} \tag{B.17}$$

We then go on defining a version of the G_{IJ} and G^{IJ} for the ξ_I

$$\begin{aligned} \tilde{G}^{IJ} &= 2 \left(\ell^3 \xi^I \xi^J - c^{IJK} \xi_K \right) \,, \\ \tilde{G}_{IJ} &= \frac{1}{2} \ell^3 \left(\xi_I \xi_J - c_{IJK} \xi^K \right) \,, \end{aligned} \tag{B.18}$$

which leads to the crucial inversion identity on the ξ_I:

$$\frac{1}{2} \ell^3 (c_{IKM} c^{MNP} \xi_N \xi_P - \xi_I \xi_K) \left(c^{IJL} \xi_L \right) = \delta_K^J. \tag{B.19}$$

A final comment: in this article, we take 5D supergravity as the starting point. For an introduction to the geometric interpretation of the 5D fields and the formulae they satisfy in terms of Calabi-Yau compactification of 11D supergravity, we refer to [17].

C Supersymmetry conditions

In this appendix we establish the conditions that our *ansatz* (4.3) must satisfy in order to preserve supersymmetry.

C.1 The Kähler condition on the base geometry

We want to establish the conditions on the variables in the 4D base geometry in (4.3) that ensure that it is Kähler. For a given vielbein basis e^a on the base $ds_4^2 = \eta_{ab} e^a e^b$, such as (4.5), the Kähler condition is

$$d(e^1 \wedge e^4 - e^2 \wedge e^3) = 0. \tag{C.1}$$

In the $(1 + 4)$ split (4.3), the base space (4.4) is automatically Kähler as

$$\left(g_m^{-1/2} \right) \left(\tfrac{1}{2} R g_m^{1/2} dR \wedge \sigma_3 \right) - \tfrac{1}{4} R^2 \sigma_1 \wedge \sigma_2 = d \left(\tfrac{1}{4} R^2 \sigma_3 \right) \,, \tag{C.2}$$

which is automatically closed. We look instead to the $(2+3)$ split in (4.8) to obtain a nontrivial Kähler condition. For that, we rewrite (4.8) in the form $ds_5^2 = f^2(dt+\omega)^2 - f^{-1}ds_4^2$, and find the warp factor

$$f = (e^{2\rho} - e^{-U_2}(a_t^0)^2)^{1/2}\,, \tag{C.3}$$

the 1-form

$$\omega = -f^{-2}e^{-U_2}a_t^0\;\sigma_3\,, \tag{C.4}$$

and the 4D base geometry

$$ds_4^2 = fe^{2\sigma}dR^2 + \frac{1}{4}R^2(\sigma_1^2+\sigma_2^2) + f^{-1}e^{2\rho-U_2}\sigma_3^2. \tag{C.5}$$

To find the condition for which (C.5) is Kähler, we introduce the basis 1-forms

$$e^1 = f^{1/2}e^{\sigma}dR\,, \tag{C.6}$$

$$e^2 = \frac{1}{2}R\sigma_1\,, \tag{C.7}$$

$$e^3 = \frac{1}{2}R\sigma_2\,, \tag{C.8}$$

$$e^4 = f^{-1/2}e^{\rho-U_2/2}\sigma_3. \tag{C.9}$$

The Kähler 2-form $J = e^1\wedge e^4 - e^2\wedge e^3$ becomes

$$J = e^{\sigma+\rho-U_2/2}dR\wedge\sigma_3 - \frac{1}{4}R^2\sigma_1\wedge\sigma_2. \tag{C.10}$$

The Kähler condition demands that J is closed

$$dJ = e^{\sigma+\rho-U_2/2}dR\wedge\sigma_1\wedge\sigma_2 - \frac{1}{2}RdR\wedge\sigma_1\wedge\sigma_2 = 0. \tag{C.11}$$

We therefore find

$$e^{\sigma+\rho-U_2/2} = \frac{1}{2}R. \tag{C.12}$$

This condition must be satisfied so that the general ansatz (4.8) can support supersymmetry. The Kähler condition allow us to rewrite the base geometry (C.5) as

$$ds_4^2 = fe^{2\sigma}dR^2 + \frac{1}{4}R^2(\sigma_1^2+\sigma_2^2+f^{-1}e^{-2\sigma}\sigma_3^2). \tag{C.13}$$

This form of the base geometry depends on a single function $fe^{2\sigma}$.

C.2 Kähler potential

The Kähler condition (C.12) relates the 1-forms e^1 and e^4 in (4.5). If we define a radial coordinate r such that

$$\partial_r R = f^{-\frac{1}{2}}e^{-\sigma}\,, \tag{C.14}$$

the tetrad simplifies so

$$e^1 = dr\,, \tag{C.15}$$

$$e^4 = \partial_r\left(\tfrac{1}{4}R^2\right)\sigma_3. \tag{C.16}$$

with e^2 and e^3 unchanged. In these coordinates, the unique spin connections solving Cartan's equations $de^a + \omega^a{}_b e^b = 0$ are

$$ {}^4\omega^2{}_1 = {}^4\omega^4{}_3 = \frac{\partial_r R}{R} e^2 \,, \tag{C.17} $$

$$ {}^4\omega^3{}_1 = {}^4\omega^2{}_4 = \frac{\partial_r R}{R} e^3 \,, \tag{C.18} $$

$$ {}^4\omega^4{}_1 = \left(\frac{\partial_r R}{R} + \frac{\partial_r^2 R}{\partial_r R} \right) e^4 \,, \tag{C.19} $$

$$ {}^4\omega^2{}_3 = \left(\frac{\partial_r R}{R} - \frac{2}{R \partial_r R} \right) e^4 \,, \tag{C.20} $$

where the ^{4} superscript distinguishes these 4D spin connections from the 5D spin connections that will appear in later computations. The resulting curvature 2-forms $R^a{}_b = d\omega^a{}_b + \omega^a{}_c \omega^c{}_b$ on the 4D base become

$$ R^2{}_1 = R^4{}_3 = \frac{\partial_r^2 R}{R} (e^1 e^2 + e^3 e^4) \,, \tag{C.21} $$

$$ R^3{}_1 = R^2{}_4 = \frac{\partial_r^2 R}{R} (e^1 e^3 - e^2 e^4) \,, \tag{C.22} $$

$$ R^4{}_1 = \left(\frac{\partial_r^3 R}{\partial_r R} + 3 \frac{\partial_r^2 R}{R} \right) e^1 e^4 + 2 \frac{\partial_r^2 R}{R} e^3 e^2 \,, \tag{C.23} $$

$$ R^2{}_3 = 2 \frac{\partial_r^2 R}{R} e^1 e^4 + \frac{4}{R^2} ((\partial_r R)^2 - 1) e^3 e^2 . \tag{C.24} $$

The components of the Riemann curvature are read off from $R^a{}_b = \frac{1}{2} \mathrm{Riem}^a{}_{bcd} e^c e^d$. For a complex manifold they are collected succinctly in the Kähler curvature 2-form with components $\mathcal{R}_{ab} = \frac{1}{2} \mathrm{Riem}_{abcd} J^{cd}$. In the context of our *ansatz* (4.4), we have

$$ \begin{aligned} \mathcal{R}_{14} &= \epsilon (\mathrm{Riem}_{1423} - \mathrm{Riem}_{1414}) = \epsilon \left(\frac{\partial_r^3 R}{\partial_r R} + 5 \frac{\partial_r^2 R}{R} \right) \,, \\ \mathcal{R}_{23} &= \epsilon (\mathrm{Riem}_{2323} - \mathrm{Riem}_{1423}) = -\epsilon \left(2 \frac{\partial_r^2 R}{R} + \frac{4}{R^2} ((\partial_r R)^2 - 1) \right) \,, \end{aligned} \tag{C.25} $$

and so the Kähler curvature 2-form becomes

$$ \begin{aligned} \mathcal{R} &= \frac{1}{2} \epsilon \left(R \partial_r^3 R + 5 \partial_r R \partial_r^2 R \right) dr \sigma_3 - \frac{1}{2} \epsilon \left(R \partial_r^2 R + 2((\partial_r R)^2 - 1 \right) \sigma_1 \sigma_2 \\ &= \epsilon d \left((\frac{1}{2} R \partial_r^2 R + (\partial_r R)^2 - 1) \sigma_3 \right) . \end{aligned} \tag{C.26} $$

It is manifestly of the form $\mathcal{R} = dP$ where $P = p\sigma_3$ with

$$ p = \epsilon \left(\frac{1}{2} R \partial_r^2 R + (\partial_r R)^2 - 1 \right) = \epsilon \left(\frac{1}{4} R \partial_R (\frac{1}{f} e^{-2\sigma}) + \frac{1}{f} e^{-2\sigma} - 1 \right) . \tag{C.27} $$

The second equation follows by repeated use of (C.14). Since $\mathcal{R}$ is the exterior derivative of something, it is clearly closed. Thus the base manifold is Kähler.

The final expression (C.27) depends on the single scalar function $fe^{2\sigma}$ that determines the base geometry (C.13). It encapsulates everything about the curvature of the 4D base.

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C.3 Supersymmetry conditions

The $\mathcal{N} = 2$ supergravity theory we consider is, in particular, invariant under the fermionic transformations of the gaugino and the gravitino

$$\delta\lambda = \left[G_{IJ}\left(\frac{1}{2}\gamma^{\mu\nu}F^J_{\mu\nu} - \gamma^\mu\nabla_\mu X^J\right)\epsilon^\alpha - \xi_I\epsilon^{\alpha\beta}\epsilon^\beta\right]\partial_i X^I, \tag{C.28}$$

$$\delta\psi^\alpha_\mu = \left[(\partial_\mu - \frac{1}{4}\omega^{\nu\rho}_\mu\gamma_{\nu\rho}) + \frac{1}{24}(\gamma_\mu{}^{\nu\rho} - 4\delta_\mu{}^\nu\gamma^\rho)X_I F^I_{\nu\rho}\right]\epsilon^\alpha + \frac{1}{6}\xi_I(3A^I_\mu - X^I\gamma_\mu)\epsilon^{\alpha\beta}\epsilon^\beta, \tag{C.29}$$

where ϵ^α $(\alpha = 1, 2)$ are symplectic Majorana spinors. For bosonic solutions to the theory that respect at least some supersymmetry these variations vanish for the spinors ϵ^α that generate the preserved supersymmetry. Supersymmetric black holes in AdS_5 with finite horizon area preserve the supersymmetry generated by the spinors ϵ^α that satisfy the projections

$$\gamma^0\epsilon^\alpha = \epsilon^\alpha\,, \tag{C.30}$$

$$\frac{1}{4}J^{(1)}_{mn}\gamma^{mn}\epsilon^\alpha = -\epsilon^{\alpha\beta}\epsilon^\beta. \tag{C.31}$$

Each of these equations impose two projections on the spinor ϵ^α. All these projections commute, so the resulting black holes preserve $2^{-4} = 1/16$ of the maximal supersymmetry.

We seek to work out the conditions that set the supersymmetric variations (C.28) and (C.29) to zero, satisfying the projections (C.30) and (C.31) imposed on purely bosonic solutions. We use the matter ansatz and geometry in (2.18) and (4.3), respectively.

The gamma matrices are defined with respect to a flat 5D space and satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\,, \tag{C.32}$$

with the flat 5D space defined in (4.3) via the following veilbein

$$E^0 = f(dt + w\sigma_3)\,, \tag{C.33}$$

$$E^i = f^{-\frac{1}{2}}e^i\,, \tag{C.34}$$

where e^i with spatial indices refers to the 4D veilbein introduced in (4.5). Furthermore, the gamma matrices γ^μ following the projection (4.12) satisfy

$$-\frac{\epsilon}{2}(\gamma^{23} - \gamma^{14}) = \epsilon\epsilon^{\alpha\beta}\epsilon^\beta\,, \tag{C.35}$$

where $\gamma^{\mu\nu}$ is the antisymmetrized product for $a \neq b$, which means that after squaring (C.35) we obtain

$$\gamma^{1234}\epsilon^\alpha = \epsilon^\alpha\,, \tag{C.36}$$

and thus

$$\gamma^{14}\epsilon^\alpha = -\gamma^{23}\epsilon^\alpha. \tag{C.37}$$

This becomes relevant for evaluating inner products of components of 2-forms and γ^{ab} as well as their decomposition into self-dual and anti-self-dual terms.

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C.3.1 The gaugino equation

Recall that the gaugino equation is given by (C.28), where the 5D 2-form $F^I = dA^I$ can be computed from (4.7)

$$F^I = \partial_R(fY^I)e^{-\sigma}f^{-1}E^1\wedge E^0+4f\left(fY^I\partial_{R^2}w+\partial_{R^2}u^I\right)E^1\wedge E^4-\frac{4f}{R^2}\left(fY^Iw+u^I\right)E^2\wedge E^3. \tag{C.38}$$

The spatial F^I_{mn} components can be rearranged into self-dual and anti-self-dual terms

$$\begin{aligned}F^I &= \partial_R(fY^I)e^{-\sigma}f^{-1}E^1\wedge E^0\\ &\quad+2f\left(fY^I\left(\partial_{R^2}-\frac{1}{R^2}\right)w+\left(\partial_{R^2}-\frac{1}{R^2}\right)u^I\right)(E^1\wedge E^4+E^2\wedge E^3)\\ &\quad+2f\left(fY^I\left(\partial_{R^2}+\frac{1}{R^2}\right)w+\left(\partial_{R^2}+\frac{1}{R^2}\right)u^I\right)(E^1\wedge E^4-E^2\wedge E^3).\end{aligned} \tag{C.39}$$

Since $(\gamma^{14}+\gamma^{23})\epsilon^\alpha=0$ per (C.37), only the anti-self-dual components of $F^{\mu\nu}$ via $F^J_{\mu\nu}\gamma^{\mu\nu}$ contributes to the gaugino variation. We thus simplify $G_{IJ}\frac{1}{2}\gamma^{\mu\nu}F^J_{\mu\nu}$ to find

$$\begin{aligned}&G_{IJ}\frac{1}{2}\gamma^{\mu\nu}F^J_{\mu\nu}\epsilon^\alpha\\ &=G_{IJ}\left[\partial_R(fY^I)e^{-\sigma}f^{-1}\gamma^{10}+2f\left(fY^I\left(\partial_{R^2}+\tfrac{1}{R^2}\right)w+\left(\partial_{R^2}+\tfrac{1}{R^2}\right)u^I\right)(\gamma^{14}-\gamma^{23})\right]\epsilon^\alpha.\end{aligned} \tag{C.40}$$

We then move on to the second term of (C.28), noting that X^I is only a function of R

$$G_{IJ}(-\gamma^\mu\nabla_\mu X^J)\epsilon^\alpha = G_{IJ}\left(-\gamma^1e^{-\sigma}\partial_RX^J\right)\epsilon^\alpha. \tag{C.41}$$

Lastly, the third term of (C.28) becomes

$$-\xi_I\epsilon^{\alpha\beta}\epsilon^\beta = +\epsilon\xi_I\gamma^{23}\epsilon^\alpha. \tag{C.42}$$

Combining all three contributions, we obtain the following equations

$$G_{IJ}\left[\partial_R(fY^I)-f\partial_RX^I\right]\partial_iX^I=0\,, \tag{C.43}$$

$$\left[4G_{IJ}f\left(\left(\partial_{R^2}+\frac{1}{R^2}\right)u^J+fY^J\left(\partial_{R^2}+\frac{1}{R^2}\right)w\right)+\epsilon\xi_I\right]\partial_iX^I=0. \tag{C.44}$$

Since $X_I\partial_iX^I=\frac{1}{2}\partial_i(X_IX^I)=0$, the $f\partial_RX^J$ term can be rewritten as $\partial_R(fX^J)$ and thus we obtain

$$\boxed{G_{IJ}\left[\partial_R(fY^J)-\partial_R(fX^J)\right]\partial_iX^I=0\,,} \tag{C.45}$$

This can be reexpressed by defining a vector $\delta^I=fY^I-fX^I$, to imply that $\partial_R\delta^I$ is orthogonal to ∂_iX^I, and thus proportional to X^I:

$$\partial_R\delta^I = kX^I\,, \tag{C.46}$$

for some constant k. We will focus on the special solution where δ^I vanishes, meaning

$$X^I = Y^I. \tag{C.47}$$

Using this relation, we now move on to (C.44), the second gaugino variation result. It is a projection of the vector quantity in the square brackets, along the direction of $\partial_i X^I$. The immediate consequence of it is that this quantity is proportional to X_I. Rearranging terms, we obtain the ambiguous result

$$\left(\partial_{R^2} + \frac{1}{R^2}\right) u^I = \frac{1}{2}\epsilon f^{-1} c^{IJK} X_J \xi_K + \frac{1}{2} f^{-1} \lambda X^I \,, \tag{C.48}$$

with λ a scalar coefficient that arises from the ambiguity in defining the quantity in square brackets in (C.44) as orthogonal to $\partial_i X^I$. Determining this quantity requires resorting to further supersymmetry relations, which leads us to the vanishing of the gravitino variation (C.29).

C.3.2 The gravitino equation

In order to simplify the vanishing of the gravitino equation (C.29), we need to establish the components of the 5D spin connection that appears in the term $-\frac{1}{4}\, \omega_\mu^{\nu\rho}\gamma_{\nu\rho}\epsilon^\alpha$. Based on the vielbein (C.33) and (C.34), we have

$$\begin{aligned}
&{\omega^0}_1 = f^{-1}e^{-\sigma}\partial_R f E^0 + 2f^2 \partial_{R^2} w E^4 \,, && {\omega^0}_2 = -\frac{2f^2 w}{R^2} E^3 \,,\\
&{\omega^0}_3 = \frac{2f^2 w}{R^2} E^2 \,, && {\omega^0}_4 = -2f^2\partial_{R^2} w E^1 \,,\\
&{\omega^2}_1 = {}^4\omega^2_1 - \frac{1}{2} f^{-1} e^{-\sigma}\partial_R f E^2 \,, && {\omega^3}_1 = {}^4{\omega^3}_1 - \frac{1}{2} f^{-1} e^{-\sigma}\partial_R f E^3 \,,\\
&{\omega^4}_1 = {}^4{\omega^4}_1 - \frac{1}{2} f^{-1} e^{-\sigma}\partial_R f E^4 - 2f^2\partial_{R^2} w E^0 \,, && {\omega^2}_3 = {}^4{\omega^2}_3 - \frac{2f^2 w}{R^2} E^0 \,,\\
&{\omega^3}_4 = {}^4{\omega^3}_4 \,, && {\omega^4}_2 = {}^4{\omega^4}_2 \,,
\end{aligned} \tag{C.49}$$

where ${}^4\omega^m_n$ represents the 4D spin connections (C.17)–(C.20), and e^m are the 4D tetrad 1-forms, which are related to the E^μ (C.33) and (C.34) via $e^m = f^{1/2}E^m$. We now proceed to evaluate the components of the gravitino variation (C.29), starting with $\mu = 0$:

$$\left(\partial_0 - \frac{1}{4}\omega_0^{\nu\rho}\gamma_{\nu\rho}\right)\epsilon^\alpha = \left(\partial_0 + \gamma^{23} f^2\left(\partial_{R^2} + \frac{1}{R^2}\right) w - \frac{1}{2} f^{-1} e^{-\sigma}\partial_R f \gamma^1\right)\epsilon^\alpha \,, \tag{C.50}$$

$$\begin{aligned}
\frac{1}{24}(\gamma_0^{\nu\rho} - 4\delta_0^\nu\gamma^\rho) X_I F^I_{\nu\rho}\epsilon^\alpha = \Bigg(&-\gamma^{23} f^2 \left[\left(\partial_{R^2} + \frac{1}{R^2}\right) w + \frac{1}{3} f^{-1} X_I \left(\partial_{R^2} + \frac{1}{R^2}\right) u^I\right]\\
&+\frac{1}{2} f^{-1} e^{-\sigma}\partial_R f \gamma^1\Bigg)\epsilon^\alpha \,,
\end{aligned} \tag{C.51}$$

$$\frac{1}{6}\xi_I (3A_0^I - X^I\gamma_0)\epsilon^{\alpha\beta}\epsilon^\beta = \epsilon\frac{1}{3}\xi_I X^I \gamma^{23}\epsilon^\alpha \,, \tag{C.52}$$

where A_0^I stands for the component of the A^I 1-form along the E^0 flat veilbein, which amounts to $A_0^I = f^{-1}A_t^I = f^{-1}(fX^I) = X^I$. Thus, adding up the three contributions

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in (C.50), (C.51) and (C.52), we note that the terms proportional to γ^1 cancel out identically. What is left is terms proportional to the identity and to γ^{23}, which when made to vanish separately, lead to two results

$$\partial_0 \epsilon = 0\,, \tag{C.53}$$

$$f X_I \left(\partial_{R^2} + \frac{1}{R^2}\right) u^I - \epsilon \xi_I X^I = 0. \tag{C.54}$$

This equation is another expression involving a projection of the quantity $\left(\partial_{R^2} + \frac{1}{R^2}\right) u^I$. Rather than a redundant relation, it can in fact be used to further constrain the ambiguity in u^I that arose from the projection in the gaugino variation (C.44). In fact, (C.48) and (C.54) imply that

$$f X_I \left(\frac{1}{2}\epsilon f^{-1} c^{IJK} X_J \xi_K + \frac{1}{2} f^{-1} \lambda X^I\right) - \epsilon \xi_I X^I = 0\,, \tag{C.55}$$

which immediately means that $\lambda = 0$. The final result is given by

$$\boxed{\left(\partial_{R^2} + \frac{1}{R^2}\right) u^I = \frac{1}{2}\epsilon f^{-1} c^{IJK} X_J \xi_K.} \tag{C.56}$$

We now move on to the spatial components of (C.29). For $\mu = 1$:

$$\left(\partial_1 - \frac{1}{4}\omega_1^{\nu\rho}\gamma_{\nu\rho}\right)\epsilon^\alpha = \left(\partial_1 + \gamma^4 f^2 \partial_{R^2} w\right)\epsilon^\alpha\,, \tag{C.57}$$

$$\begin{aligned}\frac{1}{24}(\gamma_1^{\nu\rho} - 4\delta_1^\nu \gamma^\rho) X_I F^I_{\nu\rho}\epsilon^\alpha = \Bigg(&-\gamma^4 f^2 \left[\left(2\partial_{R^2} - \frac{1}{R^2}\right) w + \frac{1}{3} f^{-1} X_I \left(2\partial_{R^2} - \frac{1}{R^2}\right) u^I\right] \\ &-\frac{1}{2} f^{-1} e^{-\sigma} \partial_R f \gamma^1\Bigg)\epsilon^\alpha\end{aligned}\,, \tag{C.58}$$

$$\frac{1}{6}\xi_I (3A_1^I - X^I \gamma_1)\epsilon^{\alpha\beta}\epsilon^\beta = \frac{1}{6}\xi_I X^I \gamma^4 \epsilon^\alpha = \frac{1}{6} f X_I \left(\partial_{R^2} + \frac{1}{R^2}\right) u^I \gamma^4 \epsilon^\alpha. \tag{C.59}$$

Again, adding the contributions (C.57), (C.58) and (C.59), and separating out the terms proportional to the identity, γ^1 and γ^4, we obtain

$$\boxed{\left(\partial_{R^2} - \frac{1}{R^2}\right) w + \frac{1}{2} f^{-1} X_I \left(\partial_{R^2} - \frac{1}{R^2}\right) u^I = 0\,,} \tag{C.60}$$

as well as the spatial dependence of the spinor ϵ: $\partial_R \epsilon = \frac{1}{2} f^{-1}(\partial_R f)\epsilon$, which leads to $\epsilon = \epsilon_0 f^{1/2}$ for some constant ϵ_0. The $\mu = 2$ and $\mu = 3$ components of (C.29) yield the same condition (C.60), which leaves us with $\mu = 4$ that introduces an additional term due

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to the appearance of the 4D spin connection terms:

$$\left(\partial_4 - \frac{1}{4}\omega_4^{\nu\rho}\gamma_{\nu\rho}\right)\epsilon^\alpha = \left(\partial_4 - \gamma^1 f^2 \partial_{R^2} w - \frac{1}{4} f^{-1} e^{-\sigma}\partial_R f \gamma^{14} + \frac{1}{4}\,{}^4\omega_{mn}\gamma^{mn}\right)\epsilon^\alpha\,, \tag{C.61}$$

$$\begin{aligned}\frac{1}{24}(\gamma_4^{\nu\rho} - 4\delta_4^\nu\gamma^\rho)X_I F^I_{\nu\rho}\epsilon^\alpha = &\left(\gamma^1 f^2\left[(2\partial_{R^2} - \frac{1}{R^2})w + \frac{1}{3}f^{-1}X_I(2\partial_{R^2} - \frac{1}{R^2})u^I\right]\right.\\ &\left. + \frac{1}{4}f^{-1}e^{-\sigma}\partial_R f\gamma^{14}\right)\epsilon^\alpha\,,\end{aligned} \tag{C.62}$$

$$\frac{1}{6}\xi_I(3A_4^I - X^I\gamma_4)\epsilon^{\alpha\beta}\epsilon^\beta = \left(\frac{1}{2}\epsilon\xi_I u^I\gamma^{23} - \frac{1}{6}fX_I(\partial_{R^2} + \frac{1}{R^2})u^I\gamma^1\right)\epsilon^\alpha. \tag{C.63}$$

Combining these terms leads to the condition (C.60) as well as the 4D relation

$$\left(\frac{1}{4}\,{}^4\omega_{mn}\gamma^{mn} + \frac{1}{2}\xi_I u^I\gamma^{23}\right)\epsilon^\alpha = 0. \tag{C.64}$$

Using the 4D spin connections (C.17)–(C.20), we find that ${}^4\omega_{mn}\gamma^{mn}\epsilon^\alpha = -2p\gamma^{23}\epsilon^\alpha$ with p from (C.27). We relate $fe^{2\sigma}$ to g_m based on (4.9), and find that

$$\boxed{p = \epsilon\left(\frac{1}{2}R^2(\partial_{R^2}g_m) + g_m - 1\right) = \xi_I u^I.} \tag{C.65}$$

We can now gather the four main supersymmetry relations that were derived:

$$0 = G_{IJ}\left(\partial_R(fY^I) - \partial_R(fX^I)\right)\partial_i X^J\,, \tag{C.66}$$

$$0 = \left(\partial_{R^2} + \frac{1}{R^2}\right)u^I - \frac{1}{2}\epsilon f^{-1}c^{IJK}X_J\xi_K\,, \tag{C.67}$$

$$0 = \left(\partial_{R^2} - \frac{1}{R^2}\right)w + \frac{1}{2}f^{-1}X_I\left(\partial_{R^2} - \frac{1}{R^2}\right)u^I\,, \tag{C.68}$$

$$0 = -\epsilon R^2(\partial_{R^2}g_m) + 2\epsilon(1 - g_m) + 2\xi_I u^I. \tag{C.69}$$

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