

Study of gapped phases of 4d gauge theories using temporal gauging of the \mathbb{Z}_N 1-form symmetry

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ABSTRACT: To study gapped phases of 4d gauge theories, we introduce the temporal gauging of \mathbb{Z}_N 1-form symmetry in 4d quantum field theories (QFTs), thereby defining effective 3d QFTs with $\tilde{\mathbb{Z}}_N \times \mathbb{Z}_N$ 1-form symmetry. In this way, spatial fundamental Wilson and 't Hooft loops are simultaneously genuine line operators. Assuming a mass gap and Lorentz invariant vacuum of the 4d QFT, the $\tilde{\mathbb{Z}}_N \times \mathbb{Z}_N$ symmetry must be spontaneously broken to an order- N subgroup H , and we can classify the 4d gapped phases by specifying H . This establishes the 1-to-1 correspondence between the two classification schemes for gapped phases of 4d gauge theories: one is the conventional Wilson-'t Hooft classification, and the other is the modern classification using the spontaneous breaking of 4d 1-form symmetry enriched with symmetry-protected topological states.

KEYWORDS: Confinement, Topological States of Matter, Wilson, 't Hooft and Polyakov loops

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1 Introduction

Color confinement is one of the most remarkable phenomena in 4d non-Abelian gauge theories, and we are continuously developing various techniques to understand its physical mechanism. Importantly, we are interested in the vacuum structure in the space of gauge theories as well as its properties for a specific theory. This motivation naturally leads us to classify the possible vacua as states of quantum phases of matter.

The classification problem of possible gapped phases of 4d $SU(N)$ gauge theories (with adjoint matter) has a long history, and one of the key ideas is to study the behavior of the interparticle potential for probe particles. We can introduce the test quark as the Wilson loop operator, and the electric charge of the test quark is characterized by the center elements of the gauge group, $\mathbb{Z}_N \subset SU(N)$. Then, Wilson proposed that confinement and Higgs phases are discriminated by studying whether the Wilson loop shows the area law or the perimeter law [1]. Interestingly, we can also consider magnetic particles as well as electric ones. The magnetic charges belong to $\tilde{\mathbb{Z}}_N = \pi_1(SU(N)/\mathbb{Z}_N)$, whose elements specify possible Dirac strings, and we can describe their worldlines using 't Hooft loops. The above observation leads to the Wilson-'t Hooft classification, which says that the gapped phases are classified according to the set of deconfined dyonic lines in $\tilde{\mathbb{Z}}_N \times \mathbb{Z}_N$ [2–4].

In the modern perspective of generalized global symmetry in quantum field theories (QFTs), this Wilson-'t Hooft classification is a bit mysterious. When we consider a 4d gauge theory with (generalized) locality, we need to specify the global structure of the

gauge group, such as $SU(N)$ vs. $SU(N)/\mathbb{Z}_N$. Once the global structure is specified, we cannot have both Wilson and 't Hooft loops as genuine line operators [5, 6]. Only N of N^2 dyonic lines are genuine line operators and they have to be mutually local. The other lines are non-genuine and live on the boundaries of topological surface operators, which explains the Wilson-'t Hooft commutation relation kinematically. These mutually-local dyonic lines specify an order- N group $G(\subset \tilde{\mathbb{Z}}_N \times \mathbb{Z}_N)$, and the theory has a G 1-form symmetry [7]. It should be noted that so far we have *not* discussed the dynamics of gauge theories at all in this paragraph; everything is just about the definition of genuine line operators even though the order- N subgroup of $\tilde{\mathbb{Z}}_N \times \mathbb{Z}_N$ appears similarly as in the Wilson-'t Hooft classification of gapped phases.

In this paper, let us always choose the global structure of the gauge group to be $SU(N)$. Then the 1-form symmetry is denoted by $\mathbb{Z}_N^{[1]}$, which measures the \mathbb{Z}_N electric charge of the Wilson loop. Since the 't Hooft lines are not genuine line operators, we do not have the magnetic counterpart of the 1-form symmetry that measures $\tilde{\mathbb{Z}}_N$. This situation would naturally raise the question of why we need the whole set of dyonic lines to characterize gapped phases in the Wilson-'t Hooft classification. Here, we wish to answer this question and make a clear connection between the Wilson-'t Hooft classification and the classification via the 1-form symmetry.

To achieve this goal, we introduce “temporal gauging” of the 1-form symmetry and apply this technique to produce 3d QFTs with $\tilde{\mathbb{Z}}_N \times \mathbb{Z}_N$ 1-form symmetry out of 4d QFTs with \mathbb{Z}_N 1-form symmetry. We study the partition function of these 3d QFTs with $\tilde{\mathbb{Z}}_N^{[1]} \times \mathbb{Z}_N^{[1]}$ in the presence of the background gauge fields, which we call the 't Hooft partition function as it was first introduced by 't Hooft in ref. [3]. Let us emphasize that the temporal gauging is reversible, so the 't Hooft partition function carries the same amount of information as the 4d partition function. The 't Hooft partition function turns out to be strongly constrained by the 4d Lorentz invariance of the original theory, and this is exactly the setup that justifies the Wilson-'t Hooft classification. We show in section 2 that the classification of the 4d gapped phases according to the spontaneous breaking of the 4d 1-form symmetry enriched with symmetry-protected topological (SPT) states is in 1-to-1 correspondence with the Wilson-'t Hooft classification via the temporal gauging operation:

$$\begin{aligned} & (\mathbb{Z}_N^{[1]})_{4d} \xrightarrow{\text{SSB}} (\mathbb{Z}_n^{[1]})_{4d} \text{ enriched with the } \mathbb{Z}_n^{[1]} \text{ level-}k \text{ SPT state} \\ \xleftrightarrow{1:1} & \tilde{\mathbb{Z}}_N \times \mathbb{Z}_N \xrightarrow{\text{SSB}} H = \{x(n, 0) + y(k, -N/n) \in \tilde{\mathbb{Z}}_N \times \mathbb{Z}_N\}. \end{aligned} \tag{1.1}$$

The left-hand-side describes the characterization of the gapped phases in 4d QFT language and the right-hand-side describes it after performing the temporal gauging, and these two are shown to be completely equivalent.

In section 3, we discuss the situation where the 4d 1-form symmetry has a mixed 't Hooft anomaly. We shall see that the anomaly relation in 4d is translated into the higher-group structure of 3d QFTs after the temporal gauging. We can reproduce the anomaly matching constraint by combining the higher-group structure with the symmetry breaking, $\tilde{\mathbb{Z}}_N^{[1]} \times \mathbb{Z}_N^{[1]} \xrightarrow{\text{SSB}} H^{[1]}$, while the higher-group structure itself is not sufficient to reach this conclusion. We then introduce the S and T operations to study the connection

between different gapped phases in section 4. These operations generate an $SL(2, \mathbb{Z})$ action on the space of 4d QFTs with $\mathbb{Z}_N^{[1]}$ symmetry, and these operations give automorphisms on $\tilde{\mathbb{Z}}_N \times \mathbb{Z}_N$ that relate different order- N subgroups $H_1 \simeq H_2$. We apply it to the $\mathcal{N} = 1^*$ supersymmetric Yang-Mills theory and study its rich vacuum structure from this viewpoint.

2 Temporal gauging of 1-form symmetry and 't Hooft partition function

Throughout this paper, we will analyze the gapped phases of 4d QFTs with \mathbb{Z}_N 1-form symmetry, which is denoted by $\mathbb{Z}_N^{[1]}$. For this purpose, we introduce the background gauge field B_{4d} for the 1-form symmetry and study properties of the partition function

$$\mathcal{Z}[B_{4d}]. \tag{2.1}$$

This partition function is defined on any general 4-dimensional Riemannian manifold M_4 . In the following, we especially pay attention to the case

$$M_4 = M_3 \times S^1. \tag{2.2}$$

We refer to this S^1 as the temporal direction, and denote its coordinate by $\tau \sim \tau + L$. As we are still interested in the four-dimensional dynamics, we basically assume that the size L of S^1 is sufficiently large and the phase is smoothly connected to the ground states. In the following, we choose a spin structure for $M_3 \times S^1$.

By regarding the size of M_3 to be much larger than that of S^1 , we can pretend that we are dealing with 3d QFTs. Then, the 4d $\mathbb{Z}_N^{[1]}$ symmetry splits into [7]

$$(\mathbb{Z}_N^{[1]})_{4d} \implies (\mathbb{Z}_N^{[1]})_{3d} \times (\mathbb{Z}_N^{[0]})_{3d}. \tag{2.3}$$

Let B_m and A denote the background gauge fields for $(\mathbb{Z}_N^{[1]})_{3d}$ and $(\mathbb{Z}_N^{[0]})_{3d}$, respectively. Then they can be related to the 4d background gauge fields B_{4d} as [8–11]

$$B_{4d} = B_m + A \wedge \frac{d\tau}{L}. \tag{2.4}$$

Here, B_m does not have the temporal component, and we sometimes call it the magnetic flux. The temporal-spatial component is expressed by the 1-form gauge field A .

We define the temporal gauging by the path integral in terms of A :¹

$$\mathcal{Z}_{\text{tH}}[\tilde{B}_e, B_m] = \int \mathcal{D}a \exp\left(\frac{2\pi i}{N} \int_{M_3} \tilde{B}_e \cup a\right) \mathcal{Z}[B_m + a \wedge (d\tau/L)]. \tag{2.5}$$

Here, \tilde{B}_e is the background gauge field for the \mathbb{Z}_N 1-form symmetry, $\tilde{\mathbb{Z}}_N^{[1]}$, dual to the original $\mathbb{Z}_N^{[0]}$ symmetry. Regarding \mathcal{Z}_{tH} as the partition function of the 3d QFT defined on M_3 , it enjoys the $\tilde{\mathbb{Z}}_N^{[1]} \times \mathbb{Z}_N^{[1]}$ symmetry and (\tilde{B}_e, B_m) is the corresponding background

¹We follow the convention that the background fields are denoted with upper case and the dynamical ones are with lower case. When we promote the background gauge fields to the dynamical ones, we change their letters to the corresponding lower case letters, accordingly.

2-form gauge field. As this partition function (2.5) was first introduced by 't Hooft in ref. [3] for the case of $M_4 = T^4$, we shall refer to it as the 't Hooft partition function.

As this theory enjoys the $\tilde{\mathbb{Z}}_N \times \mathbb{Z}_N$ 1-form symmetry, we must have the corresponding line operators. Since the $\mathbb{Z}_N^{[1]}$ symmetry is the 1-form symmetry in the original 4d theory, let us refer to the corresponding operator as the Wilson loop, $W_{(0,1)}(C)$. It is then natural to refer to the charged object of $\tilde{\mathbb{Z}}_N^{[1]}$ as the 't Hooft loop, $W_{(1,0)}(C)$. In general, the dyonic loop operator (with the magnetic charge $m \in \mathbb{Z}_N$ and the electric charge $e \in \mathbb{Z}_N$) is denoted as

$$W_{(m,e)}(C) e^{i \int_D (m \tilde{B}_e + e B_m)} \tag{2.6}$$

with some $\partial D = C$ in the presence of the background gauge fields.²

Here, we need to emphasize that the 't Hooft partition function (2.5) is introduced to understand the possible phases of 4d gauge theories, while \tilde{B}_e and B_m there are 3d \mathbb{Z}_N 2-form gauge fields. Even though $\mathcal{Z}[B_{4d}]$ is covariant under Lorentz transformations, the temporal-gauging procedure does not respect it, and thus \mathcal{Z}_{tH} may seem at first sight to be less useful compared with the original one $\mathcal{Z}[B_{4d}]$ for studying the 4d dynamics. Let us point out, however, that the temporal gauging is a reversible operation, and thus $\mathcal{Z}[B_{4d}]$ and $\mathcal{Z}_{\text{tH}}[\tilde{B}_e, B_m]$ should carry the same amount of information. Moreover, it turns out in the following that $\mathcal{Z}_{\text{tH}}[\tilde{B}_e, B_m]$ provides a convenient tool for the classification of gapped phases, and the physical meaning of each phase also becomes quite transparent.

2.1 Positivity of the 't Hooft partition function $\mathcal{Z}_{\text{tH}}[\tilde{B}_e, B_m]$

An important property of the 't Hooft partition function is its semi-positivity, and one can easily show it using reflection positivity.³ Regard S^1 as the temporal direction. We pick the antipodal points $\{\tau = 0, L/2\} \subset S^1$ and choose $M_3 \times \{0, L/2\} \subset M_4$ as the reflection plane for the Osterwalder-Schrader reflection.

Using the 1-form gauge invariance, or the topological nature of the codim-2 defects, we may set the specific alignment of the discrete gauge fields. For the spatial part B_m , we require that B_m does not depend on τ at all so that B_m is invariant under the Osterwalder-Schrader reflection. To discuss the temporal gauge field $a \wedge (d\tau/L)$, we note the following trivial identity,

$$\int \mathcal{D}a F[a] = \int \mathcal{D}a_1 \mathcal{D}a_2 F[a_1 - a_2], \tag{2.7}$$

which holds for any functionals $F[a]$. Using this identity, we find

$$\mathcal{Z}_{\text{tH}}[\tilde{B}_e, B_m] = \int \mathcal{D}a_1 \mathcal{D}a_2 e^{\frac{2\pi i}{N} \int_{M_3} \tilde{B}_e \cup (a_1 - a_2)} \mathcal{Z} \left[B_m + a_1 \wedge \frac{d\tau}{L} - a_2 \wedge \frac{d\tau}{L} \right]. \tag{2.8}$$

Thus, the defects for the temporal directions can be doubled, and these two defects can be put on arbitrary locations due to their topological nature. By a suitable choice, they can be related by the Osterwalder-Schrader reflection. Then, the reflection positivity ensures that

$$\mathcal{Z}_{\text{tH}}[\tilde{B}_e, B_m] \geq 0 \tag{2.9}$$

for any $\tilde{B}_e, B_m \in H^2(M_3; \mathbb{Z}_N)$.

²We note that these are genuine line operators in the effective 3d QFT on M_3 as the surface dependence appears only when we turn on background 2-form gauge fields.

³We assume that the 4d theory is unitary and thus its path integral satisfies reflection positivity.

We note that this positivity is achieved by the temporal gauging procedure. Indeed, the ordinary partition function, $\mathcal{Z}[B_{4d}]$, can take complex values in general in the presence of the background gauge fields, and it often provides us important information on the quantum phases of matter. In general, temporal components of the gauge field flip their sign under the Osterwalder-Schrader reflection, so this complex phase is consistent with reflection positivity. In the case of the 't Hooft partition function, the positivity argument works as we sum up all the possible gauge fields having temporal components (note that \tilde{B}_e only has purely spatial components), and we find (2.9).

The physical meaning of the positivity (2.9) becomes more transparent if we consider it in the operator formalism [3]. Let \mathcal{H} be the Hilbert space when we quantize the theory on M_3 , and let $\hat{H}[B_m]$ be the Hamiltonian operator that contains the magnetic flux B_m . We can further define the projection operator onto the electric flux sector $\hat{P}[\tilde{B}_e]$, which satisfies $\hat{P}[\tilde{B}_e]^\dagger = \hat{P}[\tilde{B}_e]$, $\hat{P}[\tilde{B}_e]^2 = \hat{P}[\tilde{B}_e]$ and $\sum_{\tilde{B}_e} \hat{P}[\tilde{B}_e] = \mathbf{1}_{\mathcal{H}}$. Then the 't Hooft partition function can be written as

$$\mathcal{Z}_{\text{tH}}[\tilde{B}_e, B_m] = \text{Tr}_{\mathcal{H}} \left[\hat{P}[\tilde{B}_e] e^{-L\hat{H}[B_m]} \right], \tag{2.10}$$

and then its positivity is quite manifest.

2.2 $\mathcal{Z}_{\text{tH}}[\tilde{B}_e, B_m]$ for gapped phases and constraints from Lorentz invariance

In ref. [3], 't Hooft found the “duality equation” for $\mathcal{Z}_{\text{tH}}[\tilde{B}_e, B_m]$ by considering a discrete rotation of the torus $M_4 = T^4$. By assuming that the system is gapped, the duality equation implies that $\tilde{\mathbb{Z}}_N^{[1]} \times \mathbb{Z}_N^{[1]}$ should be spontaneously broken to an order- N subgroup, i.e.,

$$\tilde{\mathbb{Z}}_N^{[1]} \times \mathbb{Z}_N^{[1]} \xrightarrow{\text{SSB}} H^{[1]}, \tag{2.11}$$

where the unbroken symmetry H has order N , $|H| = N$, and satisfies⁴

$$\forall (x_1, y_1), (x_2, y_2) \in H, \quad \langle (x_1, y_1), (x_2, y_2) \rangle \equiv x_1 y_2 - x_2 y_1 = 0 \pmod{N}. \tag{2.12}$$

This condition is referred to as the mutual locality condition.

In order to see how such a constraint on gapped phases arises, let us consider what would happen if $\tilde{\mathbb{Z}}_N^{[1]} \times \mathbb{Z}_N^{[1]}$ were not broken at all. By gauging \tilde{B}_e , we can undo the temporal gauging procedure and this gives the delta-functional constraint on A for $(\mathbb{Z}_N^{[0]})_{3d}$. This implies that $(\mathbb{Z}_N^{[0]})_{3d}$ is spontaneously broken, while $(\mathbb{Z}_N^{[1]})_{3d}$ is unbroken by assumption. Since both of these symmetries arise from $(\mathbb{Z}_N^{[1]})_{4d}$, this option obviously violates the 4d Lorentz invariance. Similarly, if we assume $\tilde{\mathbb{Z}}_N^{[1]} \times \mathbb{Z}_N^{[1]}$ were completely broken, we find that $(\mathbb{Z}_N^{[0]})_{3d}$ is unbroken while $(\mathbb{Z}_N^{[1]})_{3d}$ is broken, and again Lorentz invariance is violated. These quick observations already tell us that Lorentz invariance puts severe constraints

⁴We note that when H is an order- N subgroup of $\tilde{\mathbb{Z}}_N \times \mathbb{Z}_N$, then the mutual locality condition (2.12) is automatically satisfied. To see this, let us assume (to derive a contradiction) that the mutual locality is violated, so that there are $(x_1, y_1), (x_2, y_2) \in H$ such that $M \equiv x_1 y_2 - x_2 y_1 \neq 0 \pmod{N}$. Without loss of generality, this $M (< N)$ can be taken to be a positive divisor of N . Then, the subgroup $\{a(x_1, y_1) + b(x_2, y_2)\} \subset H$ has order $N^2/M > N$, and this contradicts with $|H| = N$.

and requires the correct amount of symmetry breaking for $\widetilde{\mathbb{Z}}_N^{[1]} \times \mathbb{Z}_N^{[1]}$, and we can actually find that it is broken down to an exactly order- N subgroup (with mutual locality) when assuming a mass gap (and also one technical assumption).⁵

We shall give a review of the original argument by 't Hooft in appendix A to be self-contained. Here, instead, let us perform explicit calculations of the 4d partition function $\mathcal{Z}[B_{4d}]$ and $\mathcal{Z}_{\text{tH}}[\widetilde{B}_e, B_m]$ for gapped phases. We here assume that the 4d \mathbb{Z}_N 1-form symmetry is spontaneously broken to a subgroup,

$$(\mathbb{Z}_N^{[1]})_{4d} \xrightarrow{\text{SSB}} (\mathbb{Z}_n^{[1]})_{4d}, \tag{2.13}$$

where n is a positive divisor of N , and the vacuum state further acquires a nontrivial SPT phase for the unbroken $(\mathbb{Z}_n^{[1]})_{4d}$ symmetry. The low-energy theory becomes $\mathbb{Z}_{N/n}$ topological field theory, and the partition function can be modeled as⁶

$$\begin{aligned} \mathcal{Z}[B_{4d}] &= \frac{|H^0(M_4; \mathbb{Z}_{N/n})|}{|H^1(M_4; \mathbb{Z}_{N/n})|} \sum_{b \in H^2(M_4; \mathbb{Z}_{N/n})} \exp\left(\frac{2\pi i}{N/n} \int b \cup B_{4d}\right) \\ &\times \exp\left(\frac{2\pi i k}{n} \int \frac{1}{2} P_2\left(\frac{B_{4d}}{N/n}\right)\right), \end{aligned} \tag{2.14}$$

where $P_2(B) = B \cup B + B \cup_1 dB$ is the Pontryagin square.⁷ The b field refers to the discrete $\mathbb{Z}_{N/n}$ 2-form gauge field (*not* a \mathbb{Z}_N gauge field) for the topological field theory, and its path integral gives the delta-functional constraint on B_{4d} so that

$$\int_{\Sigma} B_{4d} \in \frac{N}{n} \mathbb{Z} \tag{2.15}$$

for any closed 2-cycle Σ . As $\int B_{4d}$ is well-defined mod N , we can regard $B_{4d}/(N/n)$ as the \mathbb{Z}_n 2-form gauge field, and the second line on the right-hand-side of (2.14) describes the level- k SPT action for this unbroken $\mathbb{Z}_n^{[1]}$ symmetry with $k \sim k + n$ (given a spin structure).

Let us compute the 't Hooft partition function for (2.14), which is given by

$$\begin{aligned} \mathcal{Z}_{\text{tH}}[\widetilde{B}_e, B_m] &= \frac{1}{|H^0(M_3; \mathbb{Z}_N)|} \sum_{a \in H^1(M_3; \mathbb{Z}_N)} \exp\left(\frac{2\pi i}{N} \int \widetilde{B}_e \cup a\right) \\ &\times \frac{|H^0(M_3 \times S^1; \mathbb{Z}_{N/n})|}{|H^1(M_3 \times S^1; \mathbb{Z}_{N/n})|} \sum_{b_m \in H^2(M_3; \mathbb{Z}_{N/n})} \sum_{a' \in H^1(M_3; \mathbb{Z}_{N/n})} \\ &\times \exp\left(\frac{2\pi i}{N/n} \int (b_m \cup a + B_m \cup a') + \frac{2\pi i}{n} \frac{k}{(N/n)^2} \int B_m \cup a\right). \end{aligned} \tag{2.16}$$

⁵As this consequence is very similar to that of the anomaly-matching constraint, one might wonder if this can be understood from the mixed anomaly for $\widetilde{\mathbb{Z}}_N \times \mathbb{Z}_N$ 1-form symmetry. However, this is not the case, and we emphasize that the 4d Lorentz invariance of the original theory plays a pivotal role here.

⁶We may consider more general 2-group gauge theories as possible models (see, e.g., refs. [12–17]). Here, let us restrict our attention to these simplest possibilities.

⁷As long as working on torsion-free 4-manifolds, we can always take an integral lift of the discrete gauge field B and $P_2(B)$ can be simply thought of as $B \cup B$ by identifying B with one of the integral lifts. We shall use this property throughout this paper to simplify computations.

We note that $|H^1(M_3 \times S^1; \mathbb{Z}_{N/n})| = (N/n)^{\beta_1(M_3)+1}$, so $\frac{|H^0(M_3 \times S^1; \mathbb{Z}_{N/n})|}{|H^1(M_3 \times S^1; \mathbb{Z}_{N/n})|} = \frac{1}{(N/n)^{\beta_1(M_3)}}$. Here, $\beta_i(M) = \text{rank } H^i(M; \mathbb{Z})$ is the i -th Betti number. The summation over a' gives the delta-functional constraint on B_m , so we separate it from other path integrals:

$$\begin{aligned} \mathcal{Z}_{\text{tH}}[\tilde{B}_e, B_m] &= \frac{1}{(N/n)^{\beta_1(M_3)}} \sum_{a' \in H^1(M_3; \mathbb{Z}_{N/n})} \exp\left(\frac{2\pi i}{N/n} \int a' \cup B_m\right) \\ &\quad \times \frac{1}{N} \sum_{a \in H^1(M_3; \mathbb{Z}_N)} \sum_{b_m \in H^2(M_3; \mathbb{Z}_{N/n})} \exp\left(\frac{2\pi i}{N} \int a \cup \left(n b_m + \tilde{B}_e + k \frac{n}{N} B_m\right)\right) \\ &= \frac{1}{(N/n)^{\beta_1(M_3)}} (N/n)^{\beta_1(M_3)} \delta_N[nB_m] \times \frac{1}{N} N^{\beta_1(M_3)} \delta_N\left[\frac{N}{n} \left(\tilde{B}_e + k \frac{n}{N} B_m\right)\right] \\ &= N^{\beta_1(M_3)-1} \delta_N\left[nB_m, \frac{N}{n} \tilde{B}_e + k B_m\right]. \end{aligned} \tag{2.17}$$

Here, $\delta_N[B]$ is the delta functional that gives 1 when $\int_\Sigma B \in N\mathbb{Z}$ for every closed cycle Σ and gives 0 otherwise. This shows that the deconfined lines are generated by

$$W_{(0,n)}(C) e^{i \int_D n B_m}, \quad W_{(N/n,k)}(C) e^{i \int_D ((N/n)\tilde{B}_e + k B_m)}, \tag{2.18}$$

with $\partial D = C$, and thus the unbroken subgroup H is given by

$$H = \left\{ x(n, 0) + y\left(k, -\frac{N}{n}\right) \right\} \subset \tilde{\mathbb{Z}}_N \times \mathbb{Z}_N, \tag{2.19}$$

and we can readily confirm that the mutual locality condition (2.12) is satisfied. We can also check that $|H| = N$ and moreover that every order N subgroup of $\tilde{\mathbb{Z}}_N \times \mathbb{Z}_N$ appears in this way.⁸

In the above discussion, we start from the 4d partition function (2.14) and derive (2.17) by the temporal gauging, but we can reverse the logic to reproduce (2.14) by performing the path integral of \tilde{B}_e of the 't Hooft partition function (2.17), which achieves the equivalence mentioned in (1.1), and let us recapitulate it here:

$$\begin{aligned} (\mathbb{Z}_N^{[1]})_{4d} &\xrightarrow{\text{SSB}} (\mathbb{Z}_n^{[1]})_{4d} \text{ enriched with the } \mathbb{Z}_n^{[1]} \text{ level-}k \text{ SPT state} \\ \xleftrightarrow{1:1} \tilde{\mathbb{Z}}_N \times \mathbb{Z}_N &\xrightarrow{\text{SSB}} H = \{x(n, 0) + y(k, -N/n) \in \tilde{\mathbb{Z}}_N \times \mathbb{Z}_N\}. \end{aligned} \tag{2.20}$$

Therefore, the order- N subgroup H of $\tilde{\mathbb{Z}}_N \times \mathbb{Z}_N$ correctly characterizes the gapped phases of 4d QFTs with $\mathbb{Z}_N^{[1]}$ symmetry: vacuum states with different H are distinguished as quantum phases.

⁸To see that every order N subgroup of $\mathbb{Z}_N \times \mathbb{Z}_N$ appears, note that any such subgroup K arises from an index N sublattice L of $\mathbb{Z} \times \mathbb{Z}$ containing $N\mathbb{Z} \times N\mathbb{Z}$ such that $K = L/(N\mathbb{Z} \times N\mathbb{Z})$. Then, since $(N, 0)$ and $(0, N)$ are linearly independent vectors in L , a theorem on lattices implies that we can find a basis u, v of L of the form $u \equiv \frac{1}{q}(N, 0), v \equiv \frac{k}{N}(N, 0) + \frac{1}{n}(0, N)$, with q, n positive divisors of N and k an integer. The condition that L have index N in $\mathbb{Z} \times \mathbb{Z}$ then implies that $N^2/qn = \langle u, v \rangle = N$, i.e., that $N/q = n$. Hence, K is precisely of the form (2.19).

2.3 Example: lattice $SU(N)$ Yang-Mills theory at strong coupling

It would be useful to compute $\mathcal{Z}[B_{4d}]$ and $\mathcal{Z}_{\text{tH}}[\tilde{B}_e, B_m]$ in some microscopically solvable model for concrete understanding of their behaviors. Here, let us consider the strong-coupling expansion of the lattice $SU(N)$ gauge theory with the Wilson action.

The Wilson action with the \mathbb{Z}_N two-form gauge field is given by

$$S_W[U_\ell, B_p] = -\frac{1}{2g^2} \sum_p \text{tr} \left(e^{-\frac{2\pi i}{N} B_p U_p} + e^{\frac{2\pi i}{N} B_p U_p^\dagger} \right), \quad (2.21)$$

where U_ℓ denotes the $SU(N)$ -valued link variable, $U_p = \mathcal{P} \prod_{\ell \in p} U_\ell$ is the path-ordered products along the plaquette p , and B_p denotes the \mathbb{Z}_N -valued plaquette variable, which is identified with B_{4d} . The partition function is given by

$$\mathcal{Z}[B_{4d}] = \int \mathcal{D}U_\ell \exp(-S_W[U_\ell, B_p]). \quad (2.22)$$

We expand this partition function in terms of $1/g^2$ in the strong-coupling expansion by using formulas of Haar integration, such as $\int dU (U)_{i_1 i_2} (U^\dagger)_{j_1 j_2} = \frac{1}{N} \delta_{i_1 j_2} \delta_{i_2 j_1}$. Let us then expand the path-integral weight up to the $O(1/g^2)$ term for each plaquette,

$$\exp(-S_W) \simeq \prod_p \left(1 + \frac{1}{2g^2} \text{tr} \left(e^{-\frac{2\pi i}{N} B_p U_p} + e^{\frac{2\pi i}{N} B_p U_p^\dagger} \right) \right). \quad (2.23)$$

Then the partition function can be represented as a sum over closed surfaces,

$$\mathcal{Z}[B_{4d}] \simeq \sum_{\Sigma: \text{closed surface}} N^{\chi(\Sigma)} \left(\frac{1}{2Ng^2} \right)^{\text{Area}(\Sigma)} e^{-\frac{2\pi i}{N} \int_\Sigma B_{4d}}. \quad (2.24)$$

We can think of this expression as the sum over the worldsheets of confining strings with the string tension $\sigma = \ln(2Ng^2)$ in lattice units. When Σ is a contractible closed surface, we have $\int_\Sigma B_{4d} = 0 \pmod{N}$. Thus, the nontrivial B_{4d} dependence appears only if the confining-string worldsheet wraps around nontrivial 2-cycles, and such processes are exponentially suppressed:

$$\mathcal{Z}[B_{4d}] - \mathcal{Z}[0] \simeq O(e^{-\sigma L^2}) \xrightarrow{L \rightarrow \infty} 0, \quad (2.25)$$

where L is the length of T^4 . This shows that, in the infinite-volume limit, we can regard $\mathcal{Z}[B_{4d}] \rightarrow 1$, which corresponds to $n = N$ and $k = 0$ in (2.14).

Now, let us perform the temporal gauging of (2.24) to find $\mathcal{Z}_{\text{tH}}[\tilde{B}_e, B_m]$. As we have found that the B_{4d} dependence of $\mathcal{Z}[B_{4d}]$ is exponentially small, its Fourier transform localizes to $\tilde{B}_e = 0$ and we get $\mathcal{Z}_{\text{tH}}[\tilde{B}_e, B_m] \propto \delta_N[\tilde{B}_e]$. More precisely, for $\tilde{B}_e = 0$,

$$\begin{aligned} \mathcal{Z}_{\text{tH}}[0, B_m] &= \int \mathcal{D}a \mathcal{Z}[B_m + a \wedge d\tau/L] \\ &\simeq \mathcal{Z}[0] + O(e^{-\sigma L^2}) \xrightarrow{L \rightarrow \infty} \mathcal{Z}[0]. \end{aligned} \quad (2.26)$$

On the other hand, if we take $\int_{(T^2)_{12}} \tilde{B}_e = 1$ as an example of $\tilde{B}_e \neq 0$, then $\exp(i \int \tilde{B}_e \cup a) = \exp(i \int_{S^1} a_3 dx^3)$. To cancel this phase in the summation of a , the confining-string

worldsheet should wrap once around the 3-4 cycle, and we get

$$\begin{aligned} \mathcal{Z}_{\text{tH}}[\tilde{B}_e(\neq 0), B_m] &= \int \mathcal{D}a e^{\frac{2\pi i}{N} \int a_3 dx^3} \mathcal{Z}[B_m + a \wedge d\tau/L] \\ &\simeq O(e^{-\sigma L^2}) \xrightarrow{L \rightarrow \infty} 0. \end{aligned} \quad (2.27)$$

We actually find $\mathcal{Z}_{\text{tH}}[\tilde{B}_e, B_m] = \mathcal{Z}[0] \delta_N[\tilde{B}_e]$ neglecting the exponentially small contributions as $L \rightarrow \infty$, and the unbroken order- N subgroup is $H = \{0\} \times \mathbb{Z}_N \subset \tilde{\mathbb{Z}}_N \times \mathbb{Z}_N$.

3 Anomaly matching and the higher-group structure

In general, global symmetry in QFTs may have an 't Hooft anomaly, which is an obstruction to the promotion of the global symmetry to local gauge redundancy. The 't Hooft anomaly is invariant under any local and symmetric deformations of QFTs, and thus the low-energy effective theory is strongly constrained as it must reproduce the anomaly computed in ultraviolet. In this section, we shall discuss the structure of the 't Hooft partition function $\mathcal{Z}_{\text{tH}}[\tilde{B}_e, B_m]$ when the original 4d $\mathbb{Z}_N^{[1]}$ symmetry has a mixed 't Hooft anomaly.

Pure Yang-Mills theory. As an example, let us consider the generalized anomaly, or global inconsistency, of pure $SU(N)$ Yang-Mills theory. The 4d Yang-Mills partition function $\mathcal{Z}_\theta^{\text{YM}}$ has an 't Hooft anomaly involving the θ periodicity [8, 18–20] and we can detect it by introducing the background \mathbb{Z}_N two-form gauge field B_{4d} :

$$\mathcal{Z}_{\theta+2\pi}^{\text{YM}}[B_{4d}] = \exp\left(\frac{2\pi i}{N} \int \frac{1}{2} P_2(B_{4d})\right) \mathcal{Z}_\theta^{\text{YM}}[B_{4d}]. \quad (3.1)$$

To satisfy the anomaly matching condition in the confined phase, the level crossing of the ground state is mandatory, as the two confined states at θ and $\theta + 2\pi$ are distinct as 4d SPT states with $\mathbb{Z}_N^{[1]}$ symmetry.

Let us interpret this result using the 't Hooft partition function:

$$\mathcal{Z}_{\text{tH},\theta}^{\text{YM}}[\tilde{B}_e, B_m] = \int \mathcal{D}a \exp\left(\frac{2\pi i}{N} \int \tilde{B}_e \cup a\right) \mathcal{Z}_\theta^{\text{YM}}[B_m + a \wedge d\tau/L]. \quad (3.2)$$

By performing the temporal gauging on both sides of (3.1), we find that

$$\begin{aligned} \mathcal{Z}_{\text{tH},\theta+2\pi}^{\text{YM}}[\tilde{B}_e, B_m] &= \int \mathcal{D}a e^{\frac{2\pi i}{N} \int \tilde{B}_e a} \mathcal{Z}_{\theta+2\pi}^{\text{YM}}[B_m + a \wedge d\tau/L] \\ &= \int \mathcal{D}a e^{\frac{2\pi i}{N} \int \tilde{B}_e a} e^{\frac{2\pi i}{N} \int B_m a} \mathcal{Z}_{\theta+2\pi}^{\text{YM}}[B_m + a \wedge d\tau/L] \\ &= \mathcal{Z}_{\text{tH},\theta}^{\text{YM}}[\tilde{B}_e + B_m, B_m]. \end{aligned} \quad (3.3)$$

This is nothing but the Witten effect [21], which claims that the purely magnetic line at $\theta + 2\pi$ is equivalent to the dyonic line at θ . Equivalently, the (-1) -form transformation, $\theta \rightarrow \theta + 2\pi$, induces the nontrivial action on the 1-form symmetries, $\tilde{B}_e \rightarrow \tilde{B}_e + B_m$, which is an example of the higher-group structure [22–27].

We note that the Witten effect, or the higher-group structure, itself does not give non-trivial constraints, in contrast to the 't Hooft anomaly. The trivial state, $\mathcal{Z}_{\text{tH},\theta}^{\text{YM}}[\tilde{B}_e, B_m] =$

1, is consistent with the transformation, $\tilde{B}_e \rightarrow \tilde{B}_e + B_m$, and this state is indeed realized by the high-temperature Yang-Mills theory at any value of θ . The anomaly matching constraint is reproduced by considering the 4d Lorentz invariance. As discussed in section 2, the gapped state with 4d Lorentz invariance must have the symmetry breaking $\tilde{\mathbb{Z}}_N^{[1]} \times \mathbb{Z}_N^{[1]} \xrightarrow{\text{SSB}} H^{[1]}$, and thus we can set the 't Hooft partition function for a given θ to be

$$\mathcal{Z}_{\text{tH},\theta}^{\text{YM}}[\tilde{B}_e, B_m] = \delta_N[nB_m]\delta_N[(N/n)\tilde{B}_e + kB_m]. \tag{3.4}$$

Dialing the θ parameter, $\theta \rightarrow \theta + 2\pi$, we should obtain

$$\mathcal{Z}_{\text{tH},\theta+2\pi}^{\text{YM}}[\tilde{B}_e, B_m] = \delta_N[nB_m]\delta_N[(N/n)\tilde{B}_e + (k + (N/n))B_m]. \tag{3.5}$$

As $k \sim k + n$, we should encounter a phase transition as a function of θ if N/n is not a multiple of n . This is always the case for ordinary confinement phases, $n = N$, while the totally Higgs phase, $n = 1$, does not need the phase transition in θ . This reproduces the consequence of the 4d 't Hooft anomaly (3.1).

$\mathcal{N} = 1$ supersymmetric Yang-Mills theory. The higher group structure may be more evident in the $\mathcal{N} = 1$ super Yang-Mills (SYM) case, where the shift of the θ angle is related to the discrete chiral symmetry $(\mathbb{Z}_{2N})_\chi$. Introducing the discrete chiral gauge field A_χ , we find the 't Hooft anomaly,

$$\mathcal{Z}^{\text{SYM}}[A_\chi + d\lambda_\chi, B_{4d}] = \exp\left(\frac{2\pi i}{N} \int \lambda_\chi \cup \frac{1}{2}P_2(B_{4d})\right) \mathcal{Z}^{\text{SYM}}[A_\chi, B_{4d}]. \tag{3.6}$$

By performing a similar computation as in (3.3), this relation is translated as

$$\mathcal{Z}_{\text{tH}}^{\text{SYM}}[A_\chi + d\lambda_\chi, \tilde{B}_e, B_m] = \mathcal{Z}_{\text{tH}}^{\text{SYM}}[A_\chi, \tilde{B}_e + \lambda_\chi \cup B_m, B_m]. \tag{3.7}$$

The 0-form chiral symmetry causes the Witten effect and induces a nontrivial action on the 1-form symmetry, $\tilde{B}_e \rightarrow \tilde{B}_e + \lambda_\chi \cup B_m$.

Again, the higher-group symmetry itself does not require the degeneracy of ground states, but it gives a nontrivial consequence when we further impose the 4d Lorentz invariance. If we assume that the system is in a confined phase (i.e. $n = N$), then the 't Hooft argument shows that the partition function of a given vacuum should be described by

$$\delta_N[\tilde{B}_e + kB_m], \tag{3.8}$$

with some $k \sim k + N$. Then, the higher-group structure discussed above indicates that the discrete chiral transformation interchanges the vacuum with label k to the vacuum with label $k + 1$:

$$\delta_N[\tilde{B}_e] \xrightarrow{\text{chiral}} \delta_N[\tilde{B}_e + B_m] \xrightarrow{\text{chiral}} \dots \xrightarrow{\text{chiral}} \delta_N[\tilde{B}_e + (N-1)B_m] \xrightarrow{\text{chiral}} \delta_N[\tilde{B}_e] \tag{3.9}$$

The 't Hooft partition function of $\mathcal{N} = 1$ SYM theory is then given by

$$\mathcal{Z}_{\text{tH}}^{\text{SYM}}[\tilde{B}_e, B_m] = \sum_{k=1}^N \delta_N[\tilde{B}_e + kB_m]. \tag{3.10}$$

These N vacua are understood as the chiral broken vacua, $(\mathbb{Z}_{2N})_X \xrightarrow{\text{SSB}} \mathbb{Z}_2$, and the label k specifies the phase of the gluino condensate, $\langle \lambda^2 \rangle = \Lambda^3 e^{2\pi i k/N}$. This is exactly the vacuum structure expected from the anomaly matching condition obtained before the temporal gauging, and the same information is found via the higher-group structure combined with the constraint from 4d Lorentz invariance.

4 S and T operations on 4d QFTs with the \mathbb{Z}_N 1-form symmetry

Let us introduce the formal operations, S and T, that act on 4d QFTs with the \mathbb{Z}_N 1-form symmetry:

$$S: \mathcal{Z}[B_{4d}] \mapsto S\mathcal{Z}[B_{4d}] \equiv \int \mathcal{D}b_{4d} \mathcal{Z}[b_{4d}] \exp\left(\frac{2\pi i}{N} \int B_{4d} \cup b_{4d}\right), \quad (4.1)$$

$$T: \mathcal{Z}[B_{4d}] \mapsto T\mathcal{Z}[B_{4d}] \equiv \mathcal{Z}[B_{4d}] \exp\left(\frac{2\pi i}{N} \int \frac{1}{2} P_2(B_{4d})\right). \quad (4.2)$$

The S operation dynamically gauges the $\mathbb{Z}_N^{[1]}$ in 4d spacetime, and thus the original background gauge field is promoted to the dynamical field b_{4d} . The gauged theory acquires the dual $\mathbb{Z}_N^{[1]}$ symmetry, and we introduce the background gauge field B_{4d} that couples to it. The T operation just shifts the local counterterm for the background gauge field B_{4d} .

Here, we would like to emphasize that these S and T operations do not necessarily imply the duality/symmetry of a given 4d QFT. We can always apply these operations as long as the 4d QFTs have a \mathbb{Z}_N 1-form symmetry, and generically these operations generate different QFTs. One may say that S and T are morphisms in the category of 4d QFTs with the $\mathbb{Z}_N^{[1]}$ symmetry (see refs. [28, 29] for the case of 3d U(1) symmetry). When the generated QFT is accidentally the same as the original one, these operations may be regarded as self-duality operations. The $SU(N)$ Yang-Mills theory with adjoint matter always has the self T duality associated with $\theta \rightarrow \theta + 2\pi$ as we see in (3.1). Examples with the full $SL(2, \mathbb{Z})$ self-duality are the Cardy-Rabinovici model [30–33] and $\mathcal{N} = 4$ SYM theory [34–37]. In section 4.2, we shall discuss the $\mathcal{N} = 1^*$ SYM theory in detail.

4.1 S and T operations on 't Hooft partition functions

Let us study how these operations act on the 't Hooft partition function (2.5). We define the S and T operations on $\mathcal{Z}_{\text{tH}}[\tilde{B}_e, B_m]$ by the temporal gauging of the S- and T-transformed partition functions, respectively:

$$S\mathcal{Z}_{\text{tH}}[\tilde{B}_e, B_m] \equiv \int \mathcal{D}a \exp\left(\frac{2\pi i}{N} \int_{M_3} \tilde{B}_e \cup a\right) S\mathcal{Z}\left[B_m + a \wedge \frac{d\tau}{L}\right], \quad (4.3)$$

$$T\mathcal{Z}_{\text{tH}}[\tilde{B}_e, B_m] \equiv \int \mathcal{D}a \exp\left(\frac{2\pi i}{N} \int_{M_3} \tilde{B}_e \cup a\right) T\mathcal{Z}\left[B_m + a \wedge \frac{d\tau}{L}\right]. \quad (4.4)$$

We can compute these path integrals explicitly to express the right-hand-side using \mathcal{Z}_{tH} . The S transformation is given by

$$\begin{aligned} \text{S}\mathcal{Z}_{\text{tH}}[\tilde{B}_e, B_m] &= \int \mathcal{D}a e^{\frac{2\pi i}{N} \int_{M_3} \tilde{B}_e a} \int \mathcal{D}b_m \mathcal{D}a' e^{\frac{2\pi i}{N} \int_{M_3} (B_m a' + b_m a)} \mathcal{Z}\left[b_m + a' \wedge \frac{d\tau}{L}\right] \\ &= \int \mathcal{D}b_m \mathcal{D}a' N^{b_1(M_3)-1} \delta_N[\tilde{B}_e + b_m] e^{\frac{2\pi i}{N} \int_{M_3} B_m a'} \mathcal{Z}\left[b_m + a' \wedge \frac{d\tau}{L}\right] \\ &= \mathcal{Z}_{\text{tH}}[B_m, -\tilde{B}_e]. \end{aligned} \tag{4.5}$$

Here, we decompose the dynamical 4d \mathbb{Z}_N gauge field as $b = b_m + a' \wedge \frac{d\tau}{L}$. The T transformation is given by

$$\begin{aligned} \text{T}\mathcal{Z}_{\text{tH}}[\tilde{B}_e, B_m] &= \int \mathcal{D}a e^{\frac{2\pi i}{N} \int_{M_3} \tilde{B}_e a} e^{\frac{2\pi i}{N} \int_{M_3} B_m a} \mathcal{Z}\left[B_m + a \wedge \frac{d\tau}{L}\right] \\ &= \mathcal{Z}_{\text{tH}}[\tilde{B}_e + B_m, B_m]. \end{aligned} \tag{4.6}$$

Let us choose the generators $S, T \in \text{SL}(2, \mathbb{Z})$ to be

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \tag{4.7}$$

so that $\text{SL}(2, \mathbb{Z}) = \langle S, T \mid S^2 = (ST^{-1})^3, S^4 = 1 \rangle$ and $C \equiv S^2 = (ST^{-1})^3$ corresponds to charge conjugation. If we write $\vec{B} = (\tilde{B}_e, B_m)^t$ as a column vector, then the transformations (4.5) and (4.6) can be expressed as

$$\text{S}\mathcal{Z}_{\text{tH}}[\vec{B}] = \mathcal{Z}_{\text{tH}}[S^{-1}\vec{B}], \quad \text{T}\mathcal{Z}_{\text{tH}}[\vec{B}] = \mathcal{Z}_{\text{tH}}[T^{-1}\vec{B}]. \tag{4.8}$$

Thus, S and T transformations generate the $\text{SL}(2, \mathbb{Z})$ action on the space of 4d QFTs with the $\mathbb{Z}_N^{[1]}$ symmetry. The explicit form of the action can be easily identified when we use the 't Hooft partition function $\mathcal{Z}_{\text{tH}}[\tilde{B}_e, B_m]$, as we just need to perform an $\text{SL}(2, \mathbb{Z})$ transformation on the background field $\vec{B} = (\tilde{B}_e, B_m)^t$. For example, we find

$$(S^{p_1} T^{q_1} S^{p_2} T^{q_2} \dots) \mathcal{Z}_{\text{tH}}[\vec{B}] = \mathcal{Z}_{\text{tH}}[(S^{p_1} T^{q_2} S^{p_2} T^{q_2} \dots)^{-1} \vec{B}]. \tag{4.9}$$

As we can express

$$(S^{p_1} T^{q_2} S^{p_2} T^{q_2} \dots) = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \text{SL}(2, \mathbb{Z}), \tag{4.10}$$

with some $p, q, r, s \in \mathbb{Z}$ with $ps - qr = 1$, the above relation becomes

$$(S^{p_1} T^{q_1} S^{p_2} T^{q_2} \dots) \mathcal{Z}_{\text{tH}}[\tilde{B}_e, B_m] = \mathcal{Z}_{\text{tH}}[s\tilde{B}_e - qB_m, -r\tilde{B}_e + pB_m]. \tag{4.11}$$

Let us now consider two different gapped phases specified by order- N subgroups H_1 and H_2 , and assume that there is an isomorphism $H_1 \xrightarrow{\sim} H_2$ induced by an automorphism of $\tilde{\mathbb{Z}}_N \times \mathbb{Z}_N$. Then, there should be an $\text{SL}(2, \mathbb{Z})$ operation that relates H_1 and H_2 , and thus these different phases are connected in the web of S, T operations.

4.2 Non-Abelian gapped phases in the $\mathcal{N} = 1^*$ SYM theory

$\mathcal{N} = 1^*$ SYM theory is defined as the mass deformation of the 4d $\mathcal{N} = 4$ SYM theory. It is one of the most interesting theoretical playgrounds for 4d theories as its vacuum structure is very rich [35–37]. Our goal below is to express the 't Hooft partition functions for the massive vacua of $\mathcal{N} = 1^*$ theory.

First, we provide a quick overview of $\mathcal{N} = 1^*$ theory with $SU(N)$ gauge group starting with its $\mathcal{N} = 4$ origin. In the $\mathcal{N} = 1$ notation, the field content of $\mathcal{N} = 4$ SYM theory is a vector multiplet V and three chiral multiplets Φ_i ($i = 1, 2, 3$) all in the adjoint representation of $SU(N)$. The action of the theory consists of gauge invariant kinetic terms of these fields plus a unique superpotential,

$$W = \frac{1}{g^2} \text{tr}(\Phi_1[\Phi_2, \Phi_3]). \quad (4.12)$$

The $\mathcal{N} = 1^*$ theory is obtained from $\mathcal{N} = 4$ by adding a mass term:

$$\Delta W = \frac{m}{2g^2} \left(\text{tr} \Phi_1^2 + \text{tr} \Phi_2^2 + \text{tr} \Phi_3^2 \right). \quad (4.13)$$

In the $m \rightarrow \infty$ limit with arbitrarily small coupling constant at the cut-off m , $i\tau(m) \rightarrow \infty$, with $\Lambda^3 = m^3 e^{i2\pi\tau(m)/N}$ fixed, the theory reduces to pure $\mathcal{N} = 1$ theory, which is believed to be a confining gauge theory. At finite m , the theory has an extremely rich classical and quantum vacuum structure. Since we take the three masses non-zero, the moduli space of the $\mathcal{N} = 4$ theory is completely lifted and the theory has isolated vacua. The classical vacua are determined by the solutions of the F -term equations, $\partial W/\partial \Phi_i = 0$, given by

$$[\Phi_i, \Phi_j] = -m\varepsilon_{ijk}\Phi_k. \quad (4.14)$$

Therefore, the supersymmetric classical vacua can be expressed by three $N \times N$ matrices which obey the standard commutation relations for the $\mathfrak{su}(2)$ algebra. Up to gauge transformations, the classical vacua are described as

$$\Phi'_i \equiv \frac{1}{im} \Phi_i = \underbrace{J_i^{(d_1)} \oplus \dots \oplus J_i^{(d_1)}}_{k_{d_1}} \oplus \underbrace{J_i^{(d_2)} \oplus \dots \oplus J_i^{(d_2)}}_{k_{d_2}} \oplus \dots, \quad (4.15)$$

where $J_i^{(d)}$ are the generators of d -dimensional irreducible representation of $\mathfrak{su}(2)$, k_d denotes its multiplicity, and $N = d_1 k_{d_1} + d_2 k_{d_2} + \dots$. Ignoring discrete factors momentarily, the gauge structure at these vacua is reduced to $[\otimes_d U(k_d)]/U(1)$ at the classical level. If the classical vacuum contains different $\mathfrak{su}(2)$ representations, i.e. $N \neq dk_d$, then there will be unbroken $U(1)$'s in the infrared and it becomes a gapless Coulomb vacuum.

Our primary interest in this paper is the 't Hooft partition functions of massive vacua, and they arise when the Higgs expectation values take the following form:

$$\Phi'_i = \mathbf{1}_{N/d} \otimes J_i^{(d)}, \quad (i = 1, 2, 3), \quad (4.16)$$

where d is a divisor of N . As a result, the $SU(N)$ gauge group is Higgsed as

$$SU(N) \xrightarrow{\text{Higgs}} \frac{SU(N/d) \times \mathbb{Z}_N}{\mathbb{Z}_{N/d}}, \quad (4.17)$$

where $SU(N/d)$ is the remaining continuous gauge group acting on the $\mathbf{1}_{N/d}$ component, \mathbb{Z}_N in the numerator is the center subgroup of $SU(N)$, and they need to be divided by the common center $\mathbb{Z}_{N/d}$. As the classically massless contents describe the $\mathcal{N} = 1$ $SU(N/d)$ SYM theory with emergent $\mathbb{Z}_{2(N/d)}$ discrete chiral symmetry, they are expected to be gapped by quantum fluctuations with N/d distinct vacua. In addition to these confinement dynamics, the remnant discrete $\mathbb{Z}_N/\mathbb{Z}_{N/d} \simeq \mathbb{Z}_d$ gauge field describes the topological field theory for deconfined Wilson lines. As we have this dynamics for each divisor of N , the total number of the massive vacua is given by the divisor function,

$$\sigma(N) = \sum_{d|N} d = \sum_{d|N} \left(\frac{N}{d}\right). \tag{4.18}$$

As pointed out in ref. [35], every gapped state in the Wilson-'t Hooft classification can be realized as one of these vacua in the $\mathcal{N} = 1^*$ SYM theory, and we will find their 't Hooft partition functions and study how these vacua behave under S and T transformations.⁹

Depending on whether the prime factorization of N is square free, the $SL(2, \mathbb{Z})$ orbit of the massive vacua shows different features. In the following, let us work on concrete examples, $N = 6$ and $N = 4$, as demonstrations.

Massive vacua of $\mathcal{N} = 1^*$ $SU(6)$ SYM theory (the case with N square-free).

Let us consider $SU(6)$ gauge theory as a pedagogical example. Positive divisors of $N = 6$ are $d = 1, 2, 3$, and 6 , and we get the following table for the Higgs expectation values to have massive vacua:

Higgs vev	$SU(6) \xrightarrow{\text{Higgs}}$	$(\mathbb{Z}_6^{[1]})_{4d} \xrightarrow{\text{SSB}}$	$\mathcal{Z}_{\text{tH}}[\tilde{B}_e, B_m]$
$J_i^{(d=6)}$	\mathbb{Z}_6	1	$\delta_6[B_m]$
$\mathbf{1}_2 \otimes J_i^{(d=3)}$	$SU(2) \times \mathbb{Z}_3$	$(\mathbb{Z}_2^{[1]})_{4d}$	$\delta_6[2B_m, 3\tilde{B}_e + kB_m]$
$\mathbf{1}_3 \otimes J_i^{(d=2)}$	$SU(3) \times \mathbb{Z}_2$	$(\mathbb{Z}_3^{[1]})_{4d}$	$\delta_6[3B_m, 2\tilde{B}_e + kB_m]$
$\mathbf{1}_6 \otimes J_i^{(d=1)} = 0$	$SU(6)$	$(\mathbb{Z}_6^{[1]})_{4d}$	$\delta_6[\tilde{B}_e + kB_m]$

(4.19)

Let us discuss each of these vacuum structures to obtain the 't Hooft partition functions and their relation under $SL(2, \mathbb{Z})$.

Let us first discuss the Higgs phase, where $\Phi'_i = J_i^{(6)}$ and the gauge group is Higgsed to the center \mathbb{Z}_6 . We note that this classical vacuum cannot be consistent with the nonzero magnetic flux $B_{4d} \neq 0$, and a vortex must be created to satisfy the boundary condition. Thus, the partition function with nonzero flux becomes exponentially small,

$$\mathcal{Z}[B_{4d}] = \delta_6[B_{4d}], \tag{4.20}$$

and the corresponding 't Hooft partition function is given by

$$\mathcal{Z}_{\text{tH}}[\tilde{B}_e, B_m] = \delta_6[B_m]. \tag{4.21}$$

⁹A recent paper [38] also studies the global properties of the $\mathcal{N} = 1^*$ gapped vacua using the analogous S and T operations.

Indeed, in this Higgs phase, the electrically charged particles associated with the triplet of adjoint scalars condense, and we naturally expect the confinement of magnetic charges.

In the totally confined phase, the Higgs expectation values are zero, $\Phi'_i = \mathbf{1} \otimes J_i^{(1)} = 0$, and the $\mathbb{Z}_6^{[1]}$ symmetry is unbroken. At the classical level, gapless gluons and gluinos are associated with the $SU(6)$ gauge group with $\mathcal{N} = 1$ supersymmetry, and quantum fluctuations should generate confinement. In the $\mathcal{N} = 1^*$ theory, the counterpart of discrete chiral symmetry of $\mathcal{N} = 1$ SYM does not exist as the potentially anomalous $U(1)$ symmetry is broken at the classical level by the superpotential (4.12), but the low-energy effective theory for the confinement phase acquires an emergent \mathbb{Z}_{2N} axial symmetry. As computed in (3.9), the 't Hooft partition functions for these gapped chiral-broken vacua are given by

$$\mathcal{Z}_{\text{tH}}[\tilde{B}_e, B_m] = \delta_6[\tilde{B}_e + kB_m], \tag{4.22}$$

where $k = 1, \dots, N$ specifies the phase of the gluino condensate and they are cyclically permuted by the T operation. Another important remark is that the $k = 0$ confining state is dual to the Higgs state by the S operation as expected from electromagnetic duality.

The analyses of the remaining two cases are similar, so let us focus on the case of $\Phi'_i = \mathbf{1}_2 \otimes J_i^{(3)}$, where the gauge group becomes

$$SU(6) \xrightarrow{\text{Higgs}} SU(2) \times \mathbb{Z}_3. \tag{4.23}$$

The Higgs expectation value $\Phi'_i = \mathbf{1}_2 \otimes J_i^{(3)}$ is compatible with the magnetic fluxes $\int_{\Sigma} \frac{2\pi}{6} B_{4d} \in \pi\mathbb{Z}$, so the partition function should contain the factor $\delta_6[2B_{4d}]$. The possible 't Hooft partition functions for gapped phases are then given by

$$\delta_6[2B_m, 3\tilde{B}_e + kB_m], \tag{4.24}$$

with $k \sim k + 2$. We note that these two states are related by the T operation since

$$T: \delta_6[2B_m, 3\tilde{B}_e] \longleftrightarrow \delta_6[2B_m, 3\tilde{B}_e + 3B_m] = \delta_6[2B_m, 3\tilde{B}_e + B_m]. \tag{4.25}$$

Let us discuss why this is the case. To proceed, we need to study the confinement dynamics of the effective $\mathcal{N} = 1$ $SU(2)$ SYM theory, and we should note that the θ angle for the $SU(2)$ gauge group becomes $\theta_{\text{eff}} = 3\theta$ in the Higgsing (4.23). The T operation, $\theta \rightarrow \theta + 2\pi$, acts as $\theta_{\text{eff}} \rightarrow \theta_{\text{eff}} + 6\pi = \theta_{\text{eff}} + 2\pi \pmod{4\pi}$ and the chiral broken vacua for the $SU(2)$ theory are actually related by the T transformation.

The complete list of the 't Hooft partition functions for gapped phases are listed in figure 1, and we also show how the $SL(2, \mathbb{Z})$ operations relate those phases.¹⁰ All the gapped states are exchanged by S and T operations. This is a general fact when N is a square-free integer, and we demonstrate it here for $N = 6$ as an example. In this figure, we

¹⁰This figure has the identical structure with the S, T -duality orbit for $\mathcal{N} = 4$ SYM of the gauge Lie algebra $\mathfrak{g} = \mathfrak{su}(6)$ in ref. [5], and some readers may be confused about the difference between them. The key difference is that ref. [5] discusses the global structure of the gauge group itself, which is about the kinematics not the dynamics, while we here discuss the vacuum structure of the $SU(6)$ gauge theory. In particular, in ref. [5], one constructs genuine line operators in $(SU(N)/\mathbb{Z}_n)_k$ theory, while in our case, we are depicting the dyonic charges that are screened in the $SU(N)$ theory.

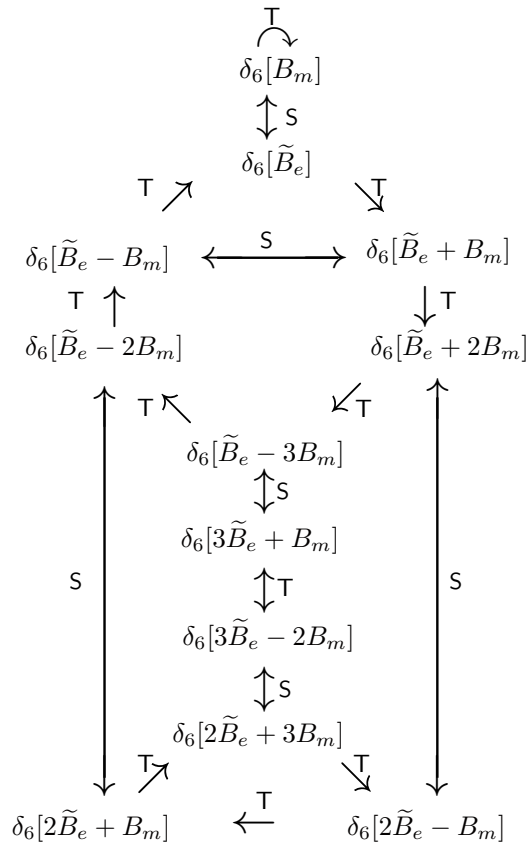


Figure 1. 't Hooft partition functions for massive vacua of $SU(6) \mathcal{N} = 1^*$ SYM theory, and their $SL(2, \mathbb{Z})$ transformations. There are $\sigma(6) = 1 + 2 + 3 + 6 = 12$ supersymmetric gapped vacua and these phases are in 1:1 correspondence with the symmetry breaking pattern $(\mathbb{Z}_N^{[1]})_{4d} \rightarrow (\mathbb{Z}_n^{[1]})_{4d}$ enriched with the $\mathbb{Z}_n^{[1]}$ level- k SPT state. Since $N = 6$ is a square-free integer, all massive phases are connected by S and T transformations.

use the fact that the 't Hooft partition function for this case can be written using a single delta functional. For example, (4.24) contains two different delta functions, but it can be equivalently rewritten as

$$\delta_6[2B_m, 3\tilde{B}_e + kB_m] = \delta_6[3\tilde{B}_e + kB_m] \quad (4.26)$$

for $k = 1, 2$ (Since $k \sim k + 2$, we can always choose this way). This is related to the fact that all the order N subgroups of $\tilde{\mathbb{Z}}_N \times \mathbb{Z}_N$ is isomorphic to \mathbb{Z}_N when N is square free.¹¹

Massive vacua of $\mathcal{N} = 1^*$ $SU(4)$ SYM theory (the case with N not square-free). When N contains squares in its prime factorization, the $SL(2, \mathbb{Z})$ structure of massive vacua

¹¹Let us prove this fact using the 't Hooft partition function (in a physically intuitive way). We start from the 't Hooft partition function, $\delta_N[nB_m, \frac{N}{n}\tilde{B}_e + kB_m]$, and we assume that $\gcd(n, N/n) = 1$, which is always the case if N is square free. The T transformation shifts $k \rightarrow k + (N/n)$, and this surveys all possible $k \sim k + n$ due to $\gcd(n, N/n) = 1$. In particular, we can reach the vacuum, $\delta_N[nB_m, \frac{N}{n}\tilde{B}_e + B_m] = \delta_N[\frac{N}{n}\tilde{B}_e + B_m]$, and its S transformation gives one of the totally confined phase, $\delta_N[\tilde{B}_e - \frac{N}{n}B_m]$.

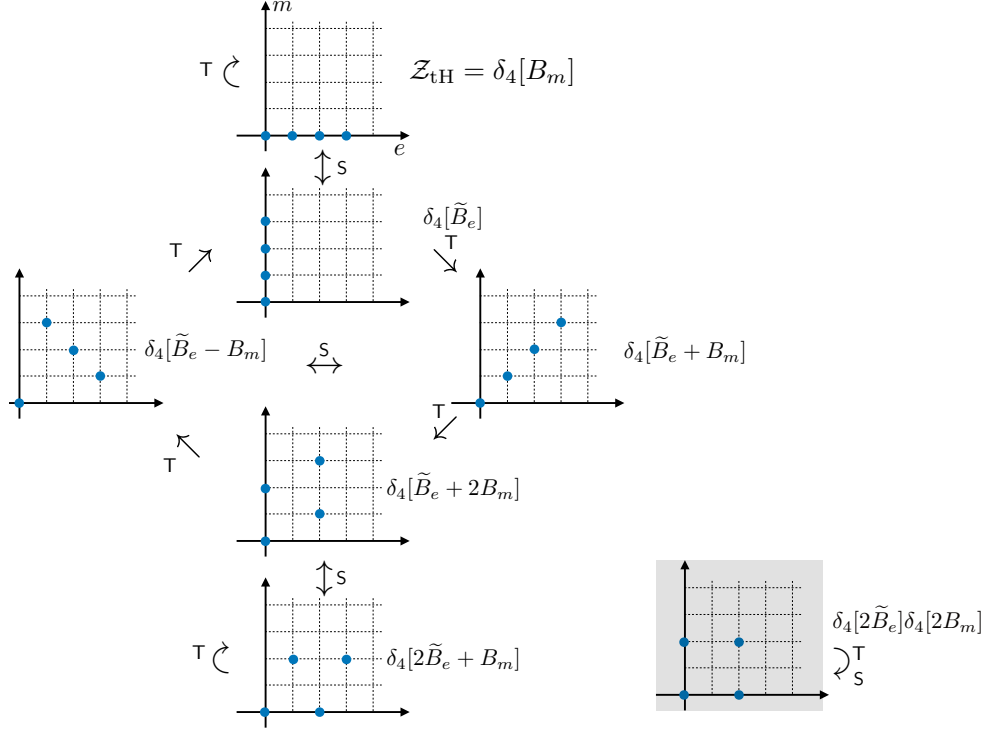


Figure 2. Phases and 't Hooft partition functions for the $\mathcal{N} = 1^*$ SYM theory with $SU(4)$ gauge group. The marked points on the lattice indicate the electromagnetic charges of deconfined dyonic lines in each phase. As $N = 4$ is not square free, the web of S and T transformations is disconnected.

has disconnected components. Let us discuss $N = 4$ as the simplest example. The massive vacua are listed in the following table:

Higgs vev	$SU(4) \xrightarrow{\text{Higgs}}$	$(\mathbb{Z}_4^{[1]})_{4d} \xrightarrow{\text{SSB}}$	$\mathcal{Z}_{\text{tH}}[\tilde{B}_e, B_m]$
$J_i^{(d=4)}$	\mathbb{Z}_4	1	$\delta_4[B_m]$
$\mathbf{1}_2 \otimes J_i^{(d=2)}$	$[SU(2) \times \mathbb{Z}_4]/\mathbb{Z}_2$	$(\mathbb{Z}_2^{[1]})_{4d}$	$\delta_4[2B_m, 2\tilde{B}_e + kB_m]$
$\mathbf{1}_4 \otimes J_i^{(d=1)} = 0$	$SU(4)$	$(\mathbb{Z}_4^{[1]})_{4d}$	$\delta_4[\tilde{B}_e + kB_m]$

(4.27)

The analysis of the Higgs and totally confining phases is completely the same as we have done for $N = 6$, so let us here focus on the case $\Phi'_i = \mathbf{1}_2 \otimes J_i^{(2)}$.

The Higgs expectation value $\Phi'_i = \mathbf{1}_2 \otimes J_i^{(2)}$ causes the Higgsing of the gauge group,

$$SU(4) \xrightarrow{\text{Higgs}} \frac{SU(2) \times \mathbb{Z}_4}{\mathbb{Z}_2}, \tag{4.28}$$

and it breaks 1-form symmetry as $(\mathbb{Z}_4^{[1]})_{4d} \rightarrow (\mathbb{Z}_2^{[1]})_{4d}$. The $SU(2)$ gauge group is confined in the infrared and it exhibits two vacua due to the spontaneous breaking of emergent chiral symmetry. These two phases are distinguished by the SPT actions of the unbroken $(\mathbb{Z}_2^{[1]})_{4d}$ symmetry, and thus their 4d partition functions are given by

$$\mathcal{Z}[B_{4d}] = \delta_4[2B_{4d}] \exp\left(\frac{2\pi ik}{2} \int \frac{1}{2} P_2(B_{4d}/2)\right), \tag{4.29}$$

with $k \sim k + 2$. The corresponding 't Hooft partition functions are given by

$$\mathcal{Z}_{\text{tH}}[\tilde{B}_e, B_m] = \delta_4[2B_m, 2\tilde{B}_e + kB_m]. \tag{4.30}$$

Importantly, these chiral broken vacua are *not* related by the \mathbb{T} transformation, $\tilde{B}_e \rightarrow \tilde{B}_e + B_m$, and each chiral-broken vacuum is invariant under \mathbb{T} . To understand their \mathbb{T} invariance from the microscopic viewpoint, we should notice that the effective θ angle of the $\text{SU}(2)$ theory is given by $\theta_{\text{eff}} = 2\theta$ using the θ angle of the $\text{SU}(4)$ theory. Under the 2π shift of θ , the effective θ angle is shifted as $\theta_{\text{eff}} \rightarrow \theta_{\text{eff}} + 4\pi$, and each chiral-broken vacua of $\mathcal{N} = 1$ $\text{SU}(2)$ SYM theory is invariant under this operation.

In figure 2, we give the complete list of the 't Hooft partition functions for gapped vacua and their relations under the \mathbb{S} and \mathbb{T} operations. We also show the set of deconfined lines of each gapped vacua. Unlike the $\text{SU}(6)$ case, there are disconnected components in the duality web of \mathbb{S} and \mathbb{T} operations for $\text{SU}(4)$ theory, and in particular, $\mathcal{Z}_{\text{tH}}[\tilde{B}_e, B_m] = \delta_4[2B_m, 2\tilde{B}_e]$ is invariant under both \mathbb{S} and \mathbb{T} operations. The presence of disconnected components is a general feature for cases with N not square free, and $\text{SU}(4)$ is an illustrative example.

5 Summary and discussion

In this paper, we introduce the temporal gauging for 4d QFTs with $\mathbb{Z}_N^{[1]}$ symmetry and define the 't Hooft partition function. This operation does not respect the 4d Lorentz invariance, but it introduces the spatial $\tilde{\mathbb{Z}}_N^{[1]} \times \mathbb{Z}_N^{[1]}$ symmetry, and thus the spatial Wilson and 't Hooft lines become genuine line operators (while all the temporal line operators are no longer genuine). This allows us to justify the classification of gapped phases using the Wilson-'t Hooft criterion, and we establish its 1-to-1 correspondence with the spontaneous breakdown of 4d $\mathbb{Z}_N^{[1]}$ symmetry enriched with the SPT phase of the unbroken 1-form symmetry.

In other words, our argument justifies the use of dyonic line operators to classify the SPT phases for the vacua of 4d gauge theories. This may be reminiscent of the use of the string order parameter [39] to characterize the Haldane gap or AKLT state [40, 41]. The Kennedy-Tasaki (KT) transformation [42–44] maps this nonlocal string order parameter to a local correlation function, and the SPT nature of the AKLT state can be understood as the spontaneous breaking of hidden (or dual) $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry (see ref. [45] for its field-theoretic description from a modern viewpoint). In this context, we may say that the temporal gauging of 4d $\mathbb{Z}_N^{[1]}$ symmetry gives a suitable KT transformation for 4d gauge theories to classify their gapped phases solely by spontaneous symmetry breaking. Honestly, it is quite astonishing that 't Hooft had already introduced all the essential ingredients in refs. [2–4] before any of these developments.

In this paper, we compute the 't Hooft partition function for the simplest 4d topological states and make the connection with the Wilson-'t Hooft criterion. As the general 4d topological states are known to be described by the 2-group gauge theories [15–17], it would be an interesting future study to compute the 't Hooft partition functions of these topological states, which would give us better understanding of 4d gapped phases.

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A Duality equation on T^4 and classification of 4d gapped phases

In the main text, we have taken the point of view that the mutual locality condition (2.12) is nothing but the statement that the spontaneous breakdown of 1-form symmetry be consistent with 4d Lorentz invariance. Here, we review 't Hooft's original derivation of this condition [3], which is quite insightful and also interesting for its elementary character.

We take the Euclidean spacetime manifold to be a flat four-torus with $L_{\mu=1,\dots,4}$ the circumferences of the various circles in $T^4 = (S^1)^4$. With this topology, the background gauge fields A, B_m, \tilde{B}_e are associated with triplets of mod N integers $\vec{n} = (n_1, n_2, n_3) \equiv (n_{14}, n_{24}, n_{34}), \vec{m} = (m_1, m_2, m_3) \equiv (n_{23}, n_{31}, n_{12}), \vec{e} = (e_1, e_2, e_3) \equiv (e_{23}, e_{31}, e_{12})$ via

$$B_{4d} = B_m + A \wedge \frac{d\tau}{L_4} = \sum_{i<j} n_{ij} \frac{dx_i \wedge dx_j}{L_i L_j} + \sum_i n_{i4} \frac{dx_i \wedge d\tau}{L_i L_4}, \quad (\text{A.1})$$

$$\tilde{B}_e = \sum_{i<j} e_{ij} \frac{dx_i \wedge dx_j}{L_i L_j}. \quad (\text{A.2})$$

Accordingly, we shall write the ordinary and 't Hooft partition functions as

$$\mathcal{Z}[B_{4d}] \equiv \mathcal{Z}[\vec{n}, \vec{m}], \quad \mathcal{Z}_{\text{tH}}[\tilde{B}_e, B_m] \equiv \mathcal{Z}_{\text{tH}}[\vec{e}, \vec{m}], \quad (\text{A.3})$$

and the relation between them as¹²

$$\mathcal{Z}_{\text{tH}}[\vec{e}, \vec{m}] = \frac{1}{N^3} \sum_{\vec{n}} \exp\left(\frac{2\pi i}{N} \vec{e} \cdot \vec{n}\right) \mathcal{Z}[\vec{n}, \vec{m}]. \quad (\text{A.4})$$

4d Lorentz invariance provides a important constraint on the 't Hooft partition function. In particular, let us consider the Euclidean Lorentz transformation

$$\Lambda = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad (\text{A.5})$$

¹²When we use the definition (2.5), the canonical normalization is given by $\frac{1}{|H^0(M_3; \mathbb{Z}_N)|} = \frac{1}{N}$ instead of $\frac{1}{N^3}$, and the inverse operation $\int \mathcal{D}\tilde{B}_e$ has the normalization factor $\frac{|H^0(M_3; \mathbb{Z}_N)|}{|H^1(M_3; \mathbb{Z}_N)|} = \frac{1}{N^{\beta_1(M_3)-1}} = \frac{1}{N^2}$. Here, we follow the original normalization by 't Hooft that assigns $1/N^3$ for the 't Hooft partition function. The following argument basically works in both normalizations, but we would like here to note that the normalization of (2.17) becomes 1 instead of $N^{\beta_1(M_3)-1}$ in this convention.

then the magnetic flux is transformed as $(n'_{ij}) = \Lambda^t(n_{ij})\Lambda$. This has the effect of interchanging the pairs (n_1, n_2) and (m_1, m_2) as we have

$$\begin{pmatrix} -m'_2 & n'_1 \\ m'_1 & n'_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -m_2 & n_1 \\ m_1 & n_2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (\text{A.6})$$

while keeping n_3, m_3 fixed. Using the notation $\tilde{q} \equiv (q_1, q_2)$ for a three-vector $\vec{q} = (q_1, q_2, q_3)$, the covariance of the ordinary partition function under the above Lorentz transformation reads (we also have $L_1 \leftrightarrow L_2, L_3 \leftrightarrow L_4$, but this is suppressed in our notation)

$$\mathcal{Z}[\tilde{n}, n_3; \tilde{m}, m_3] = \mathcal{Z}[\tilde{m}, n_3; \tilde{n}, m_3]. \quad (\text{A.7})$$

After Fourier transformation, we then obtain

$$\mathcal{Z}_{\text{tH}}[\tilde{e}, e_3; \tilde{m}, m_3] = \frac{1}{N^2} \sum_{\tilde{e}', \tilde{m}'} \exp\left(\frac{2\pi i}{N}(\tilde{e} \cdot \tilde{m}' - \tilde{m} \cdot \tilde{e}')\right) \mathcal{Z}_{\text{tH}}[\tilde{e}', e_3; \tilde{m}', m_3], \quad (\text{A.8})$$

which is what 't Hooft calls the ‘‘duality relation.’’

To obtain the constraints on the gapped vacua, 't Hooft makes a technical assumption: if the vacuum is gapped, then the ratio $\mathcal{Z}_{\text{tH}}[\tilde{e}, \tilde{m}]/\mathcal{Z}_{\text{tH}}[\tilde{0}, \tilde{0}]$ should approach either 0 or 1 as $L_{\mu=1,\dots,4} \rightarrow \infty$. Although this seems to be a nontrivial assumption, we have checked in (2.17) that it is actually valid for the $\mathbb{Z}_{N/n}$ topological states with a \mathbb{Z}_n SPT phase. It would be nice if we could prove/disprove it for general 4d topological states with $\mathbb{Z}_N^{[1]}$.

Under the above assumption, a gapped phase is then characterized by the set of all fluxes that are ‘light,’ i.e., the set of fluxes (\tilde{e}, \tilde{m}) with $\mathcal{Z}_{\text{tH}}[\tilde{e}, \tilde{m}]/\mathcal{Z}_{\text{tH}}[\tilde{0}, \tilde{0}] \rightarrow 1$. The possible sets of light fluxes are strongly constrained by the duality relation (A.8). As shown in ref. [3], any two light fluxes $(\tilde{e}, \tilde{m}), (\tilde{e}', \tilde{m}')$ with $e_3 = e_3', m_3 = m_3'$ must satisfy

$$\tilde{e} \cdot \tilde{m}' - \tilde{m} \cdot \tilde{e}' = 0 \pmod{N}, \quad (\text{A.9})$$

and there are either exactly N^2 or 0 light fluxes out of the possible N^4 fluxes with given e_3, m_3 .

Proof. Let us fix the normalization $\mathcal{Z}_{\text{tH}}[\tilde{0}, \tilde{0}] = 1$. Suppose $(\tilde{e}, e_3; \tilde{m}, m_3)$ is a light flux. Let us first establish that $(\tilde{0}, e_3; \tilde{0}, m_3)$ is also light. To see this, we note that

$$\begin{aligned} 1 = \mathcal{Z}_{\text{tH}}[\tilde{e}, e_3; \tilde{m}, m_3] &= \frac{1}{N^2} \sum_{\tilde{e}', \tilde{m}'} \exp\left(\frac{2\pi i}{N}(\tilde{e} \cdot \tilde{m}' - \tilde{m} \cdot \tilde{e}')\right) \mathcal{Z}_{\text{tH}}[\tilde{e}', e_3; \tilde{m}', m_3] \\ &\leq \frac{1}{N^2} \sum_{\tilde{e}', \tilde{m}'} \mathcal{Z}_{\text{tH}}[\tilde{e}', e_3; \tilde{m}', m_3] = \mathcal{Z}_{\text{tH}}[\tilde{0}, e_3; \tilde{0}, m_3] \leq 1, \end{aligned} \quad (\text{A.10})$$

where the second step uses the duality relation, the third step uses the positivity of \mathcal{Z}_{tH} , and the fourth step uses the duality relation again. So indeed $\mathcal{Z}_{\text{tH}}[\tilde{0}, e_3; \tilde{0}, m_3] = 1$.

Replacing the final inequality with an equality in (A.10), we find

$$\frac{1}{N^2} \sum_{\tilde{e}', \tilde{m}'} \mathcal{Z}_{\text{tH}}[\tilde{e}', e_3; \tilde{m}', m_3] = \mathcal{Z}_{\text{tH}}[\tilde{0}, e_3; \tilde{0}, m_3] = 1. \quad (\text{A.11})$$

Clearly, this can hold only if precisely N^2 of the N^4 terms on the left-hand side are equal to 1, with all the remaining terms equal to 0. In other words, we have established that precisely N^2 among the N^4 fluxes $(\tilde{e}', e_3; \tilde{m}', m_3)$ are light.

Let us now look at the duality relation again,

$$1 = \mathcal{Z}_{\text{tH}}[\tilde{e}, e_3; \tilde{m}, m_3] = \frac{1}{N^2} \sum_{\tilde{e}', \tilde{m}'} \exp\left(\frac{2\pi i}{N}(\tilde{e} \cdot \tilde{m}' - \tilde{m} \cdot \tilde{e}')\right) \mathcal{Z}_{\text{tH}}[\tilde{e}', e_3; \tilde{m}', m_3].$$

As exactly N^2 of the $\mathcal{Z}_{\text{tH}}[\tilde{e}', e_3; \tilde{m}', m_3]$ equal 1 while all the others vanish, this equality can be true iff $\exp\left(\frac{2\pi i}{N}(\tilde{e} \cdot \tilde{m}' - \tilde{m} \cdot \tilde{e}')\right) = 1$ for each light flux $(\tilde{e}', e_3; \tilde{m}', m_3)$. \square

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