

# Probing phase structure of black holes with Lyapunov exponents

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Xiaobo Guo,<sup>b</sup> Yuhang Lu,<sup>c</sup> Benrong Mu<sup>a</sup> and Peng Wang<sup>c</sup>

<sup>a</sup>Physics Teaching and Research Section, College of Medical Technology,  
Chengdu University of Traditional Chinese Medicine, Chengdu, 611137, P.R. China

<sup>b</sup>Mechanical and Electrical Engineering School, Chizhou University,  
Chizhou, Anhui, 247000, P.R. China

<sup>c</sup>Center for Theoretical Physics, College of Physics, Sichuan University,  
Chengdu, 610064, P.R. China

E-mail: [guoxiaobo@czu.edu.cn](mailto:guoxiaobo@czu.edu.cn), [luyuhang668@stu.scu.edu.cn](mailto:luyuhang668@stu.scu.edu.cn),  
[benrongmu@cdutcm.edu.cn](mailto:benrongmu@cdutcm.edu.cn), [pengw@scu.edu.cn](mailto:pengw@scu.edu.cn)

**ABSTRACT:** We conjecture that there exists a relationship between Lyapunov exponents and black hole phase transitions. To support our conjecture, Lyapunov exponents of the motion of particles and ring strings are calculated for Reissner-Nordström-AdS black holes. When a phase transition occurs, the Lyapunov exponents become multivalued, and branches of the Lyapunov exponents coincide with black hole phases. Moreover, the discontinuous change in the Lyapunov exponents can be treated as an order parameter, and has a critical exponent of  $1/2$  near the critical point. Our findings reveal that Lyapunov exponents can be an efficient tool to study phase structure of black holes.

**KEYWORDS:** Black Holes, Spacetime Singularities

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## 1 Introduction

Black hole thermodynamics lies in the interdisciplinary area of general relativity, quantum mechanics, information theory and statistical physics, and can provide profound insights into the nature of gravity. The area theorem, which asserts that the total horizon area of black holes is a non-decreasing function of time [1], suggests that black holes may be endowed with thermodynamic properties. Inspired by the resemblance between the area theorem and the second law of thermodynamics, Bekenstein postulated that black hole entropy can be described by the horizon area [2, 3]. The analogy between usual thermodynamics and black hole thermodynamics was further enhanced by the discovery of Hawking radiation, assigning black holes a temperature [4, 5].

Later on, Hawking and Page discovered that there exists a phase transition between Schwarzschild-AdS black holes and a thermal space [6]. With the advent of the AdS/CFT correspondence [7, 8], thermodynamics and phase structure of various AdS black holes have been widely studied [9–17]. Specifically, Reissner-Nordström-AdS (RN-AdS) black holes exhibit a van der Waals-like phase transition, which consists of a first-order phase transition terminating at a second-order critical point, in a canonical ensemble [12, 13], and a Hawking-Page-like phase transition in a grand canonical ensemble [15]. In the extended phase space with the cosmological constant being treated as a thermodynamic pressure [18–20], phase behavior and  $P$ - $V$  criticality have been explored for AdS black holes, which discovered a broad range of new phenomena [21–42]. In the extended phase space, the analogy between RN-AdS black holes and the van der Waals fluid becomes more complete, in that the coexistence lines in the  $P$ - $T$  diagram are both finite and terminate at critical points, and the  $P$ - $V$  criticality matches with one another [20].

Since the nature of black hole thermodynamics has not yet been fully understood, it is of great interest to explore phase structure of black holes from various perspectives.

For example, the Ruppeiner geometry can be exploited to probe the microstructure of black holes [43–49]. Motivated by the Ruppeiner geometry, RN-AdS black holes have been proposed to be built of some unknown micromolecules [50, 51]. More interestingly, there have been attempts to associate phase transitions of black holes with some observational signatures, such as quasinormal modes [52–56], circular orbit radius of a test particle [57–59] and black hole shadow radius [60, 61]. It showed that phase structure of black holes can be revealed by behavior of the aforementioned physical quantities, and the discontinuity in the physical quantities across phase transitions behaves similarly to an order parameter.

Lyapunov exponents characterize the rate of separation of adjacent trajectories, and positive/negative Lyapunov exponents correspond to divergent/convergent trajectories [62]. Lyapunov exponents can be used to study chaotic dynamics in general relativity, which is a nonlinear dynamical theory. The chaotic motion of particles in black hole spacetime has been extensively studied, such as static axisymmetric spacetimes [63, 64], rotating charged black hole spacetimes [65, 66], multi-black hole spacetimes [67], bumpy spacetimes [68], weakly magnetized Schwarzschild black holes [69], black holes with discs or rings [70], Schwarzschild-Melvin black holes [71], accelerating black holes [72], spacetimes with a quadrupole mass moment [73] and black holes with quantum gravity corrections [74, 75]. Particularly, the motion of a particle near the black hole horizon was studied in [76, 77], which found that the Lyapunov exponent obeys an universal upper bound proposed in the framework of gauge/gravity duality [78]. Nevertheless, counterexamples that violate the upper bound have been reported [79, 80]. Moreover, partly motivated by gauge/gravity duality, chaotic dynamics of a ring string has been studied in Schwarzschild-AdS and charged AdS black holes [81–87]. Intriguingly, Lyapunov exponents of unstable null geodesics have been revealed to be closely related to the imaginary part of a class of quasinormal modes of perturbations in black hole spacetime [88, 89].

In this paper, we aim to explore the relationship between phase structure of RN-AdS black holes and Lyapunov exponents of particles and ring strings moving in the black holes. The rest of this paper is organized as follows. Thermodynamics and phase transitions of RN-AdS black holes are briefly reviewed in section 2. Focusing on particles, we examine the relationship between Lyapunov exponents of unstable circular geodesics and phase structure of RN-AdS black holes in section 3. The case of a ring string is discussed in section 4. We summarize our results with a brief discussion in section 5. For simplicity, we set  $G = \hbar = k_B = c = 1$  in this paper.

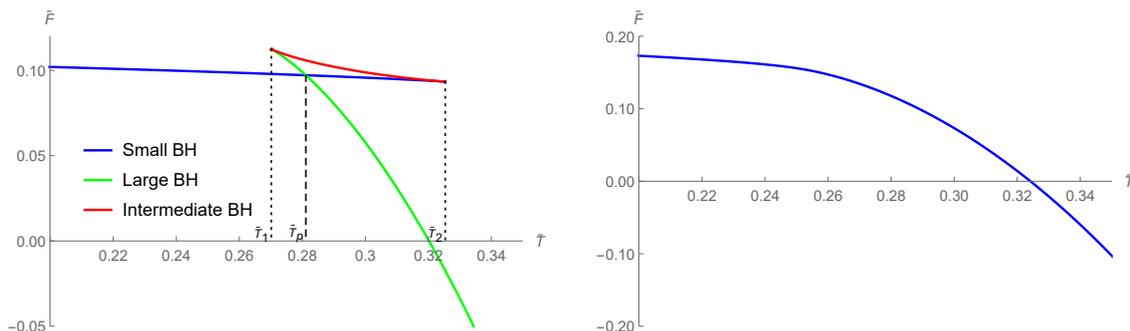
## 2 Phase structure of RN-AdS black holes

In this section, we review thermodynamic properties and phase structure of RN-AdS black holes. The 4-dimensional static charged RN-AdS black hole solution is described by

$$ds^2 = -f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \quad (2.1)$$

where the metric function  $f(r)$  is

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{l^2}, \quad (2.2)$$



**Figure 1.** Free energy  $\tilde{F}$  as a function of the temperature  $\tilde{T}$  for RN-AdS black holes. *Left:*  $\tilde{Q} = 0.11 < \tilde{Q}_c$ . There coexist three black hole solutions when  $\tilde{T}_1 < \tilde{T} < \tilde{T}_2$ , and a first-order phase transition between Small BH and Large BH occurs at  $\tilde{T}_p$ . *Right:*  $\tilde{Q} = 0.20 > \tilde{Q}_c$ . Only one black hole solution exists, and hence no phase transition occurs.

and  $l$  is the AdS radius. Here, the parameters  $M$  and  $Q$  can be interpreted as the black hole mass and charge, respectively. The RN-AdS black hole has an event horizon at  $r = r_+$ , and the horizon radius  $r_+$  satisfies  $f(r_+) = 0$ . In terms of  $r_+$ , the Hawking temperature  $T$  and the mass  $M$  are given by [47]

$$T = \frac{1}{4\pi r_+} \left( 1 - \frac{Q^2}{r_+^2} + \frac{3r_+^2}{l^2} \right), \quad M = \frac{r_+}{2} \left( 1 + \frac{Q^2}{r_+^2} + \frac{r_+^2}{l^2} \right), \quad (2.3)$$

respectively. Moreover, the RN-AdS black hole obeys the first law of thermodynamics,

$$dM = TdS + \Phi dQ, \quad (2.4)$$

where  $S = \pi r_+^2$  and  $\Phi = Q/r_+$  are the entropy and the potential of the black hole, respectively. By computing the Euclidean action in the semiclassical approximation, we obtain the free energy

$$F = M - TS = \frac{1}{4} \left( \frac{3Q^2}{r_+} + r_+ - \frac{r_+^3}{l^2} \right). \quad (2.5)$$

By dimensional analysis, we find that the physical quantities scale as powers of  $l$ ,

$$\tilde{Q} = Q/l, \quad \tilde{r}_+ = r_+/l, \quad \tilde{T} = Tl, \quad \tilde{F} = F/l, \quad \tilde{M} = M/l, \quad \tilde{r} = r/l, \quad (2.6)$$

where the tildes denote dimensionless quantities.

Using eq. (2.3), we can express  $\tilde{r}_+$  as a function of  $\tilde{T}$ . If  $\tilde{r}_+(\tilde{T})$  is multivalued, there is more than one black hole solution for fixed values of  $\tilde{Q}$  and  $\tilde{T}$ , corresponding to multiple phases in a canonical ensemble. The critical point is an inflection point determined by

$$\frac{\partial \tilde{T}}{\partial \tilde{r}_+} = 0, \quad \frac{\partial^2 \tilde{T}}{\partial \tilde{r}_+^2} = 0, \quad (2.7)$$

which gives the corresponding quantities evaluated at the critical point,

$$\tilde{r}_{+c} = \frac{1}{\sqrt{6}}, \quad \tilde{Q}_c = \frac{1}{6}, \quad \tilde{T}_c = \frac{1}{\pi} \sqrt{\frac{2}{3}}, \quad \Phi_c = \frac{1}{\sqrt{6}}. \quad (2.8)$$

To study phase transitions, we express  $\tilde{F}$  with respect to  $\tilde{T}$  by plugging  $\tilde{r}_+(\tilde{T})$  into eq. (2.5) and plot  $\tilde{F}$  against  $\tilde{T}$  for various  $\tilde{Q}$  in figure 1. The left panel shows that, when  $\tilde{Q} < \tilde{Q}_c$ , there are three black solutions, dubbed as Small BH, Intermediate BH and Large BH. The three black hole solutions coexist for some range of  $\tilde{T}$ , and a first-order phase transition occurs at  $\tilde{T}_p$ . When  $\tilde{Q} > \tilde{Q}_c$ , there is only one black hole solution and no phase transition, which is shown in the right panel.

### 3 Phase transitions and Lyapunov exponents of particles

In this section, we investigate the relationship between Lyapunov exponents of massless and massive particles and phase transitions of RN-AdS black holes. In particular, we focus on unstable circular geodesics on the equatorial hyperplane with  $\theta = \pi/2$ , which are described by the Lagrangian

$$2\mathcal{L} = -f(r)\dot{t}^2 + \frac{1}{f(r)}\dot{r}^2 + r^2\dot{\varphi}^2. \tag{3.1}$$

Here dots and primes denote derivatives with respect to the proper time and the areal radius  $r$ , respectively. Then the radial motion can be expressed as

$$\dot{r}^2 + V_{\text{eff}}(r) = E^2, \tag{3.2}$$

where the constant  $E$  can be treated as the energy and the energy per unit mass for massless and massive particles, respectively. Here, we introduce the effective potential,

$$V_{\text{eff}}(r) = f(r) \left[ \frac{L^2}{r^2} + \delta_1 \right], \tag{3.3}$$

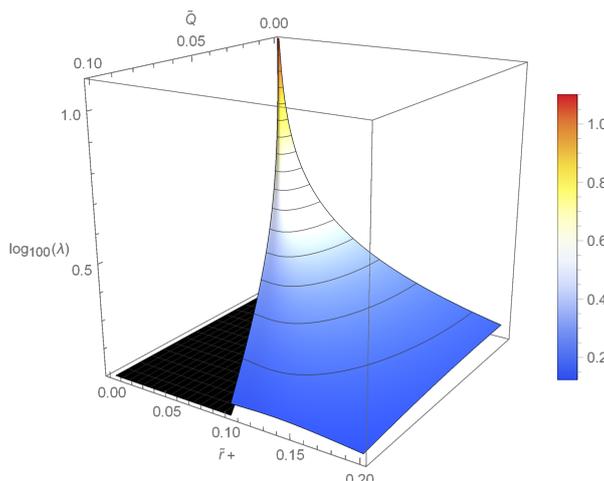
where  $L$  is identified as the angular momentum of the particles, and  $\delta_1 = 1$  and  $0$  correspond to massless and massive particles, respectively. The radius of an unstable circular geodesic is determined by

$$V'_{\text{eff}}(r) = 0, \quad V''_{\text{eff}}(r) < 0. \tag{3.4}$$

In this paper, we focus on unstable circular geodesics and explore the relationship between their Lyapunov exponents and phase transitions. When unstable circular motions are perturbed, the perturbation will increase exponentially, which indicates that motions near unstable circular orbits are highly sensitivity to initial conditions. Therefore, unstable circular motions are closely related to chaotic motions in black holes, and studying unstable circular motions can provide insight into properties of chaos. For example, Maldacena, Shenker and Stanford conjectured that there is a universal upper bound on Lyapunov exponents  $\lambda$  of chaos,

$$\lambda \leq \frac{2\pi T}{\hbar}, \tag{3.5}$$

where  $T$  is the temperature of the system [78]. In [90], it was found that the bound (3.5) can be violated for unstable circular motions of charged particles near a charged black hole. In addition, unstable circular null geodesics constitute photon spheres outside the event horizon, which play an important role in black hole observations.



**Figure 2.** Three-dimensional plot of  $\log_{100} \lambda$  as a function of  $\tilde{Q}$  and  $\tilde{r}_+$  for unstable null circular geodesics. Black hole solutions do not exist in the black region.

### 3.1 Massless particles

For massless particles, there is always (except for  $L = 0$ ) an unstable circular geodesic outside the event horizon at

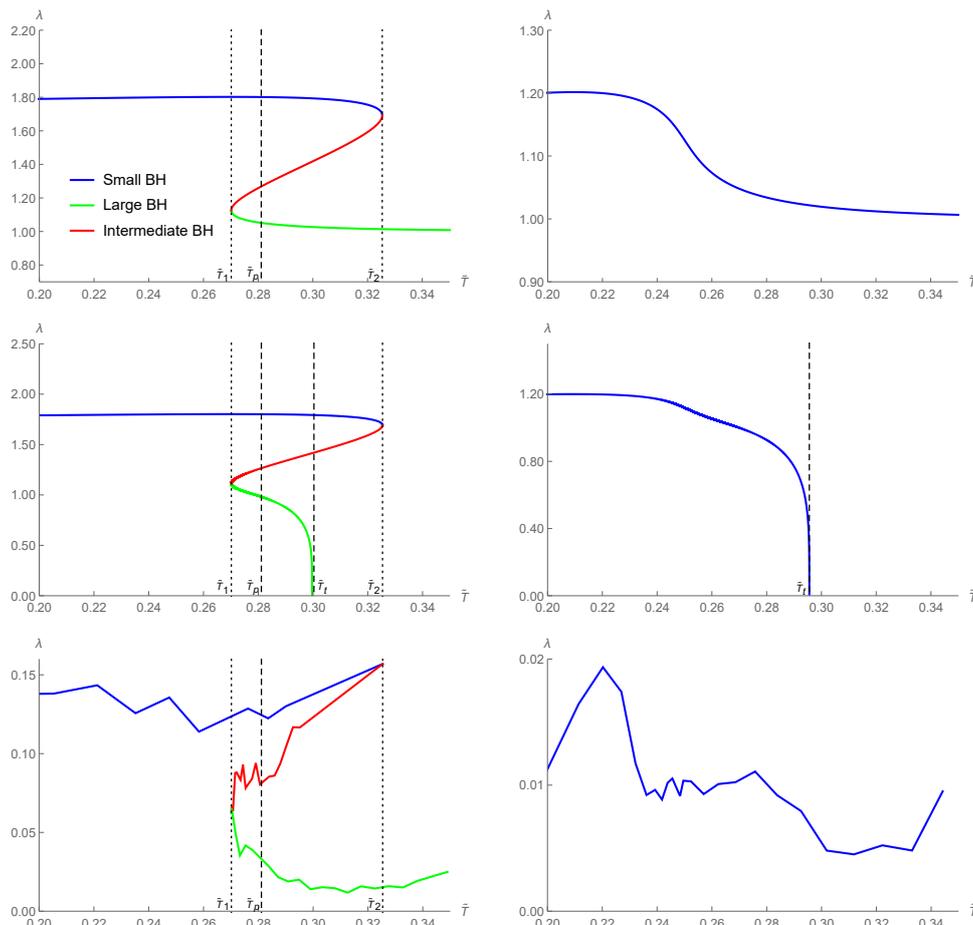
$$r_o = \frac{l}{2} \left[ \frac{3}{2} \tilde{r}_+ \left( \frac{\tilde{Q}^2}{\tilde{r}_+^2} + \tilde{r}_+^2 + 1 \right) + \sqrt{\frac{9}{4} \tilde{r}_+^2 \left( \frac{\tilde{Q}^2}{\tilde{r}_+^2} + \tilde{r}_+^2 + 1 \right)^2 - 8 \tilde{Q}^2} \right], \quad (3.6)$$

which is independent of  $L$ . Furthermore, the Lyapunov exponent of the unstable null circular geodesic is given by [88]

$$\lambda = \sqrt{-\frac{r_o^2 f(r_o)}{L^2} V_{\text{eff}}''(r_o)}, \quad (3.7)$$

which depends only on  $\tilde{Q}$  and  $\tilde{r}_+$ . To be self-contained, the derivation of eq. (3.7) is provided in the appendix. We present the 3D plot of  $\log_{100} \lambda$  as a function of  $\tilde{Q}$  and  $\tilde{r}_+$  in figure 2, where black hole solutions with the event horizon at  $\tilde{r}_+$  do not exist in the black region. It shows that  $\lambda$  diverges when  $\tilde{r}_+$  decrease to zero. In addition,  $\lambda$  approaches 1 as  $\tilde{Q}$  or  $\tilde{r}_+$  increases to infinity. In fact, eqs. (3.6) and 3.7 indicate that, when  $\tilde{r}_+$  or  $\tilde{Q}$  approaches infinity,  $\tilde{r}_o$  goes to infinity, and hence  $\lambda$  goes to 1.

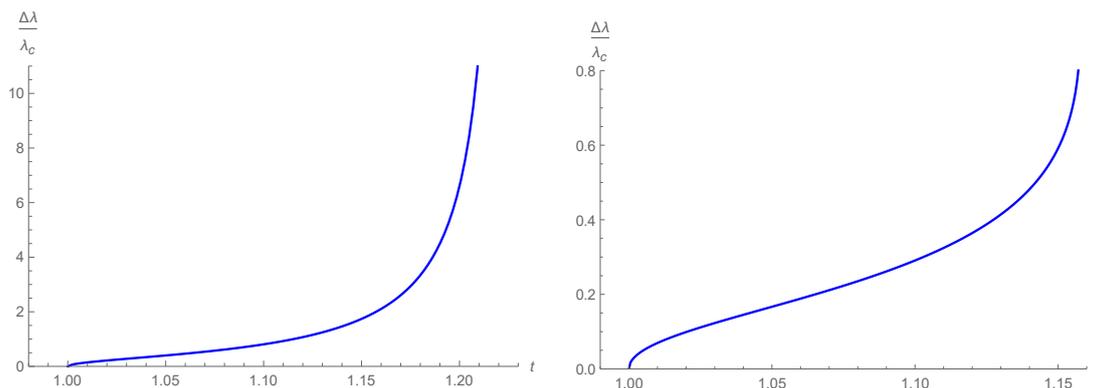
Plugging  $\tilde{r}_+(\tilde{T})$  into eq. (3.7), one can express the Lyapunov exponent  $\lambda$  in terms of  $\tilde{T}$ . In figure 3,  $\lambda$  is plotted against  $\tilde{T}$  for various values of  $\tilde{Q}$  in the top row. For the case of  $\tilde{Q} = 0.11 < \tilde{Q}_c$  in the left column, three black hole solutions (namely Small BH, Intermediate BH and Large BH) coexist for  $\tilde{T}_1 < \tilde{T} < \tilde{T}_2$ . Moreover, the first-order phase transition between Small BH and Large BH occurs at  $\tilde{T}_p$ . When  $\tilde{T}_1 < \tilde{T} < \tilde{T}_2$ ,  $\lambda$  possesses three branches, which exactly match Small BH, Intermediate BH and Large BH, respectively. For Small BH,  $\lambda$  first slightly rises to a maximum and then declines as  $\tilde{T}$  increases toward  $\tilde{T}_2$ . On the other hand,  $\lambda$  of Intermediate BH and Large BH increases and



**Figure 3.** Lyapunov exponents  $\lambda$  of particles and strings as a function of the temperature  $\tilde{T}$  for  $\tilde{Q} = 0.11 < \tilde{Q}_c$  (Left Column) and  $\tilde{Q} = 0.20 > \tilde{Q}_c$  (Right Column). Top Row: Massless particles on the unstable null circular geodesic; Middle Row: Massive particles with  $L = 20l$  on the unstable time-like circular geodesic; Bottom row: Motion of ring strings with the initial conditions  $\theta_0 = 0$ ,  $E = 1000$ ,  $\frac{d}{d\tilde{r}}(r \cos \theta) = 0$  and  $r_0 = 5.92$  (Left) and  $7.30$  (Right). Three black hole solutions, i.e., Small BH, Intermediate BH and Large BH, coexist for  $\tilde{T}_1 < \tilde{T} < \tilde{T}_2$ . The phase transition between Small BH and Large BH occurs at  $\tilde{T} = \tilde{T}_p$ , and  $\lambda$  of massive particles vanishes at  $\tilde{T} = \tilde{T}_t$ . When  $\tilde{Q} < \tilde{Q}_c$ ,  $\lambda$  is a multivalued function of  $\tilde{T}$  with three branches, which coincide with the three black hole solutions, respectively. When  $\tilde{Q} > \tilde{Q}_c$ ,  $\lambda$  is single-valued, demonstrating that there is only one black hole solution. These observations imply that  $\lambda$  as a function of  $\tilde{T}$  can reflect the phase structure of RN-AdS black holes.

decreases, respectively, with increasing  $\tilde{T}$  from  $\tilde{T}_1$ . As expected,  $\lambda$  approaches 1 as  $\tilde{T}$  goes to infinity. The right column displays the case of  $\tilde{Q} = 0.20 > \tilde{Q}_c$ , where there is only black hole solution and no phase transition. When  $\tilde{T}$  increases toward infinity,  $\lambda$  first increases to a maximum, then decreases and finally approaches 1. These observations suggest that the Lyapunov exponent  $\lambda$  can be used to probe phase structure of black holes.

Interestingly, the phase transition can be characterized by the discontinuous change in the Lyapunov exponent,  $\Delta\lambda = \lambda_S - \lambda_L$ , where  $\lambda_S$  and  $\lambda_L$  are the Lyapunov exponents of



**Figure 4.** Rescaled discontinuity in the Lyapunov exponent  $\Delta\lambda/\lambda_c$  during the phase transition as a function of the rescaled phase transition temperature  $t \equiv \tilde{T}_p/\tilde{T}_c$  near the critical temperature  $t = 1$ . *Left:* Massless particles on the unstable null circular geodesic; *Right:* Massive particles with  $L = 20l$  on the unstable time-like circular geodesic. The parameter  $\Delta\lambda$  is nonzero at the first-order phase transition and vanishes at the critical point, which indicates that  $\Delta\lambda$  plays a role of an order parameter.

Small BH and Large BH evaluated at  $\tilde{T} = \tilde{T}_p$ , respectively. Note that, for the second-order phase transition at the critical point, one has  $\lambda_S = \lambda_L = \lambda_c$ , and hence  $\Delta\lambda = 0$ . We display  $\Delta\lambda/\lambda_c$  as a function of  $t \equiv \tilde{T}_p/\tilde{T}_c$ , for which  $t = 1$  at the critical point, in the left panel of figure 4. It shows that, when RN-AdS black holes undergo the first-order phase transition from Small BH to Large BH, the Lyapunov exponent  $\lambda$  jumps from  $\lambda_S$  to  $\lambda_L$  with a nonzero  $\Delta\lambda$ . Consequently,  $\Delta\lambda$  can be treated as an order parameter. To investigate the critical behavior of  $\Delta\lambda$ , we expand  $\Delta\lambda$  in terms of  $t$  near the critical point and obtain

$$\frac{\Delta\lambda}{\lambda_c} \sim 0.700\sqrt{t-1}, \tag{3.8}$$

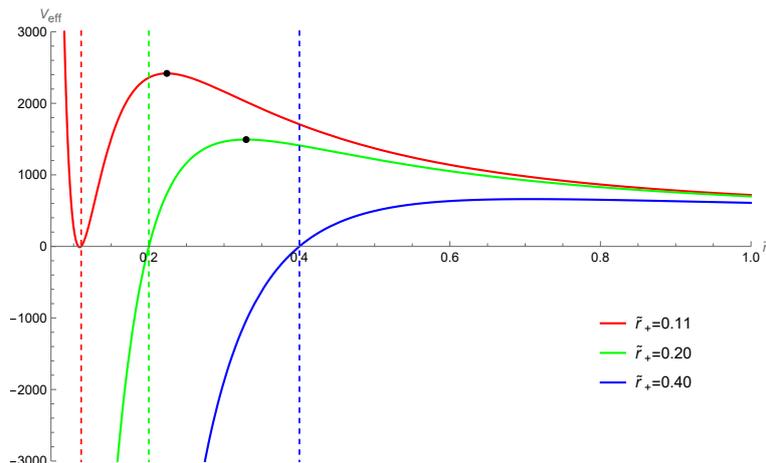
which gives that the critical exponent of  $\Delta\lambda$  is  $1/2$ . Our result shows that the critical exponent of  $\Delta\lambda$  is identical to that of the order parameter in the van der Waals fluid predicted by the mean field theory. It is worth emphasizing that the critical exponent of the circular orbit radius was also found to be  $1/2$  for charged AdS black holes [57].

### 3.2 Massive particles

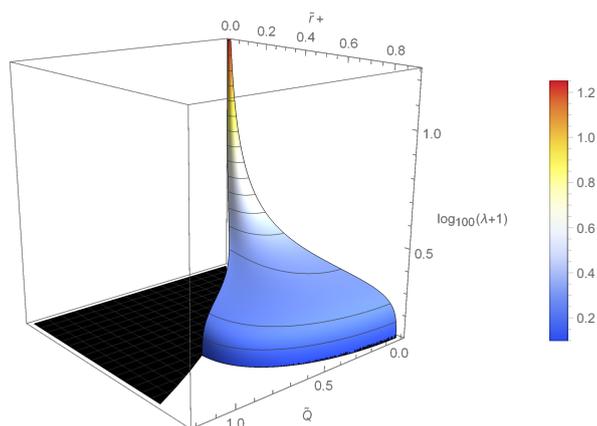
For massive particles, both stable and unstable circular geodesics can exist in RN-AdS black holes. Since unstable orbits are related to the conjectured universal upper bound on Lyapunov exponents [76, 79], we here focus on unstable time-like circular geodesics. Specifically, we consider the Lyapunov exponent of unstable circular geodesics for massive particles with a given angular momentum. In figure 5, we plot the effective potential energy of massive particles with  $L = 20l$  in RN-AdS black holes with  $\tilde{Q} = 0.11$  for various  $\tilde{r}_+$ . It shows that, if  $\tilde{r}_+$  is too large, unstable time-like circular geodesics cease to exist.

The requirement  $V'_{\text{eff}}(r) = 0$  for a circular orbit at  $r = r_o$  yields

$$L^2 = \frac{r_o^3 f'(r_o)}{2f(r_o) - r_o f'(r_o)}, \tag{3.9}$$



**Figure 5.** Effective potential of massive particles with the angular momentum  $L = 20l$  in the RN-AdS black holes with  $\tilde{Q} = 0.11$  for  $\tilde{r}_+ = 0.11, 0.20$  and  $0.40$ . The vertical dashed lines denote the radius of the event horizon. The black dots represent the maxima of the effective potentials, corresponding to unstable time-like circular geodesics. When  $\tilde{r}_+ = 0.40$ , the effective potential has no maximum.

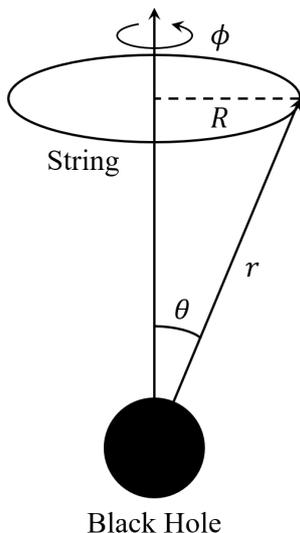


**Figure 6.** Three-dimensional plot of  $\log_{100}(\lambda + 1)$  as a function of  $\tilde{Q}$  and  $\tilde{r}_+$  for massive particles with  $L = 20l$ . There is no black hole solution in the black region, and black holes in the white region have no unstable time-like circular orbits.

which can be used to express  $r_o$  in terms of  $L$ . The Lyapunov exponent of the time-like circular orbit is given by [88]

$$\lambda = \frac{1}{2} \sqrt{[r_o f'(r_o) - 2f(r_o)] V''_{\text{eff}}(r_o)}, \tag{3.10}$$

which can be rewritten as a function of  $L, \tilde{Q}$  and  $\tilde{r}_+$  by using eq. (3.9). In the appendix, we also present the derivation of eq. (3.10). The 3D plot of  $\log_{100}(\lambda + 1)$  as a function of  $\tilde{Q}$  and  $\tilde{r}_+$  is displayed for  $L = 20l$  in figure 6, where black holes in the white region possess no unstable time-like circular geodesics, and black hole solutions do not exist in the black region. Note that the Lyapunov exponent  $\lambda$  vanishes on the boundary of the white region.



**Figure 7.** An oscillating ring string moves along the axis of a black hole.

Plugging  $\tilde{r}_+(\tilde{T})$  into eq. (3.10) leads to  $\lambda$  as a function of  $\tilde{T}$ , which is plotted in the middle row of figure 3 with various  $\tilde{Q}$  and  $L = 20l$ . Unlike the massless case, there exists a terminate temperature  $\tilde{T}_t$ , at which the unstable time-like circular orbit disappears, and  $\lambda$  becomes zero. When  $\tilde{Q} = 0.11 < \tilde{Q}_c$ ,  $\lambda$  is multivalued for  $\tilde{T}_1 < \tilde{T} < \tilde{T}_t$ , for which three black hole solutions coexist. When  $\tilde{Q} = 0.20 > \tilde{Q}_c$ ,  $\lambda$  monotonically decreases and becomes zero at the terminate temperature  $\tilde{T}_t$ . In addition, the discontinuous change in the Lyapunov exponent  $\Delta\lambda$  is plotted against the temperature  $\tilde{T}$  for massive particles with  $L = 20l$  in the right panel of figure 4, which indicates that  $\Delta\lambda$  can serve as an order parameter. Near the critical temperature, we find

$$\frac{\Delta\lambda}{\lambda_c} \sim 0.731\sqrt{t-1}, \tag{3.11}$$

which confirms that the critical exponent of  $\Delta\lambda$  is also 1/2.

#### 4 Phase transitions and Lyapunov exponents of strings

In contrast to motion of particles in RN-AdS black holes, equations of motion governing strings are non-integrable, showing chaotic behavior of strings in RN-AdS black holes [91]. Therefore, numerical computations are often required to obtain Lyapunov exponents of motion of strings. Following [83], we consider a ring string coaxially moving in the RN-AdS black hole spacetime, which is illustrated in figure 7. The equations of motion for a string is determined by the Polyakov action,

$$S_p(\gamma, X) = \frac{-1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{-\gamma} \gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}, \tag{4.1}$$

where  $X^\mu$  are the target space coordinates, the indices  $\{\alpha, \beta\} = 1$  and 2 correspond to the  $(\tau, \sigma)$  coordinates on the worldsheet of the string, respectively,  $\gamma^{\alpha\beta}$  is the worldsheet

metric, and  $G_{\mu\nu}$  is the target space metric. It should be emphasized that the configuration described in figure 7 is axially symmetric, which greatly facilitates our numerical study. Indeed, the variables  $\tau$  and  $\sigma$  are separable in this configuration, and hence we only need to solve ordinary differential equations. On the other hand, although other string configurations, e.g., open strings and ring strings moving on the equatorial plane, are also worth considering. They are generally not axially symmetric, which usually leads to solving coupled partial differential equations.

The ring string configuration considered in our paper is described by the following ansatz for the coordinates of the target space

$$t = t(\tau), \quad r = r(\tau), \quad \theta = \theta(\tau), \quad \phi = n\sigma, \tag{4.2}$$

where  $n$  is the winding number of the string along the  $\phi$  direction, and  $\tau$  is the proper time. For the above ansatz with the conformal gauge  $\gamma^{\alpha\beta} = \eta^{\alpha\beta}$ , the Lagrangian for the ring string in RN-AdS black holes becomes

$$\mathcal{L} = \frac{-1}{2\pi\alpha'} \left[ f(r) \dot{t}^2 - \frac{\dot{r}^2}{f(r)} - r^2 \dot{\theta}^2 + r^2 n^2 \sin^2 \theta \right], \tag{4.3}$$

where dots and primes denote derivatives with respect to  $r$  and  $\tau$ , respectively. Using the Legendre transformation, one obtains the Hamiltonian

$$\mathcal{H} = \frac{\pi\alpha'}{2} \left[ f(r) P_r^2 + \frac{P_\theta^2}{r^2} - \frac{P_\theta^2}{f^2(r)} \right] + \frac{n^2 r^2 \sin^2 \theta}{2\pi\alpha'}, \tag{4.4}$$

where  $P_t$ ,  $P_r$  and  $P_\theta$  are the canonical momenta, and the Hamiltonian satisfies the constraint  $\mathcal{H} = 0$ . The canonical equations of motion are then given by

$$\begin{aligned} \dot{t} &= \frac{-\pi\alpha'}{f(r)} P_t, \\ \dot{P}_t &= 0, \\ \dot{r} &= \pi\alpha' f(r) P_r, \\ \dot{P}_r &= \pi\alpha' \left[ -\frac{f'(r) P_r^2}{2} - \frac{f'(r) P_t^2}{f^2(r)} + \frac{P_\theta^2}{r^3} \right] - \frac{n^2 r \sin^2 \theta}{\pi\alpha'}, \\ \dot{\theta} &= \frac{\pi\alpha' P_\theta}{r^2}, \\ \dot{P}_\theta &= -\frac{n^2 r^2 \sin \theta \cos \theta}{\pi\alpha'}, \end{aligned} \tag{4.5}$$

which gives that  $P_t = E$  is the conserved energy. As shown in [83], there are three different scenarios depending on the initial conditions of the string and the black hole parameters:

- (1) The string oscillates back and forth around the black hole.
- (2) The string oscillates a finite number of times around the black hole before being captured by it.

- (3) The string oscillates a finite number of times around the black hole before escaping to infinity.

To calculate the Lyapunov exponent of the motion of the string, we evolve two adjacent trajectories with an initial distance  $d_0$  in the phase space spanned by  $r$ ,  $\theta$ ,  $P_r$  and  $P_\theta$ . When the distance between the trajectories  $d_{t_i}$  exceeds the upper threshold at  $t = t_i$ , one initializes a rescaling of one trajectory back to having the initial distance  $d_0$ . In [92], it showed that the maximum Lyapunov exponent  $\lambda$  can be well approximated by the average of the time-local Lyapunov exponent

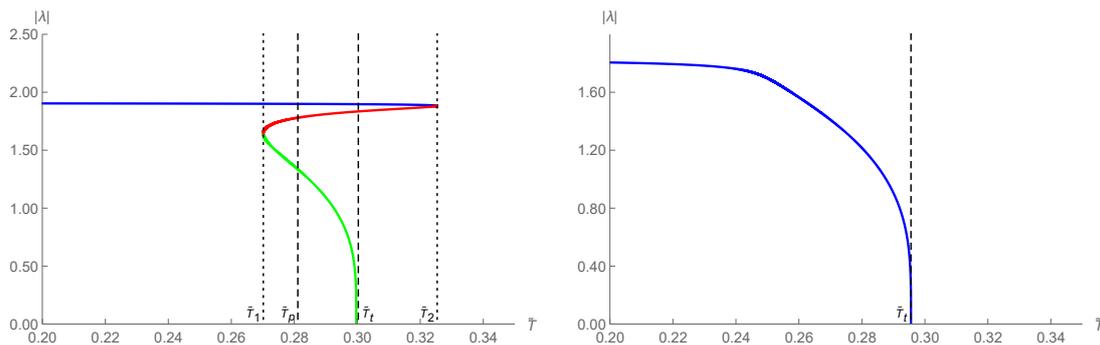
$$k_n = \frac{1}{t_n - t_0} \sum_{i=1}^n \ln \left( \frac{d_{t_i}}{d_0} \right). \tag{4.6}$$

Indeed, the authors of [92] found that the limit  $\lim_{n \rightarrow \infty} k_n = k$  exists, and one can identify  $k$  with  $\lambda$  except for an error which tends to zero with  $d_0$ . With a large total time of evolution, we intend to numerically calculate  $k_n$ , which provides an accurate enough estimate of  $\lambda$ . Note that the maximum Lyapunov exponent is of particularly interesting since a strictly positive maximum Lyapunov exponent can be considered as an indication of deterministic chaos. Here, we adopt to Verner’s “most efficient” Runge-Kutta 9(8) method [93], which can achieve high accuracy solving (tolerances like  $< 10^{-12}$ ).

In the bottom row of figure 3, we plot the Lyapunov exponent  $\lambda$  against the temperature  $\tilde{T}$  for ring strings of the scenarios (1) and (3) in RN-AdS black holes with  $\tilde{Q} = 0.11$  and 0.20. Note that due to the finite  $n$  used in our numerical study, the Lyapunov exponent  $\lambda$  displayed in the bottom row of figure 3 is not smooth. The bumpy features result from the numerical method used to estimate  $\lambda$  with  $k_n$ , and carry no physical information. The findings in [92] guarantee that the fluctuations vanish when  $n$  goes to infinity. Similar to the case of particles, the Lyapunov exponent  $\lambda$  of strings is multivalued when multiple black hole phases coexist. On the other hand, the Lyapunov exponent  $\lambda$  is single-valued if there exists only one phase. Our results suggest that it is quite universal to explore phase structure of black holes with Lyapunov exponents. Finally, one may wonder whether the corresponding critical exponent of  $\Delta\lambda$  can be also obtained for strings. In principle, this can be achieved by calculating  $\Delta\lambda$  near the critical temperature and interpolating  $\Delta\lambda$  as a function of the temperature. To compute the critical exponent accurately enough, one needs to reduce fluctuations of  $\lambda$  by evolving the system for a considerably long time and calculate  $\Delta\lambda$  for a large enough number of  $\tilde{Q}$  just slightly below the critical charge  $\tilde{Q}_c$ . Nevertheless, such a procedure is rather computationally expensive, and hence we only consider the relationship between the Lyapunov exponents and the temperature for strings.

## 5 Conclusions

In this paper, we calculated Lyapunov exponents of massless particles, massive particles and ring strings in RN-AdS black holes, and found that the behavior of the Lyapunov exponents can be employed to explore phase structure of black holes. In particular, when the black hole charge is less than the critical charge, the Lyapunov exponents as a function



**Figure 8.** The modulus of the Lyapunov exponent  $|\lambda|$  of stable time-like circular orbits as a function of the temperature  $\tilde{T}$  for massive particles with  $L = 20l$  in RN-AdS black holes with  $\tilde{Q} = 0.11 < \tilde{Q}_c$  (Left) and  $\tilde{Q} = 0.20 > \tilde{Q}_c$  (Right). The behavior of Lyapunov exponents in the stable case is strikingly similar to that in the unstable case.

of the temperature demonstrate three branches, which correspond to three coexisting black hole phases. When the charge is greater than the critical charge, the Lyapunov exponents are singled-valued functions of the temperature, which coincides with one black hole phase. At the first-order phase transition, the discontinuity in the Lyapunov exponent  $\Delta\lambda$  can act as an order parameter to characterize the black hole phase transition. Remarkably,  $\Delta\lambda$  was shown to have a critical exponent of  $1/2$  at the critical point.

As discussed above, there exist time-like stable circular orbits for massive particles besides unstable circular orbits. Can Lyapunov exponents of stable circular orbits also reflect phase structure of black holes? In figure 8, we plot the modulus of the Lyapunov exponent  $|\lambda|$  of time-like stable circular orbits as a function of  $\tilde{T}$  with  $L = 20l$  for  $\tilde{Q} = 0.11$  and  $0.20$  in the left and right panels, respectively. Note that time-like stable circular orbits have pure imaginary Lyapunov exponents. Interestingly, figure 8 displays that the behavior of Lyapunov exponent in both stable and unstable cases show striking resemblance, which suggests that Lyapunov exponents of stable circular orbits can also be used to explore black hole phase transitions.

Our results support the conjectured relationship between Lyapunov exponents and phase transition for RN-AdS black holes, which could open a new window to study thermodynamics of black holes. It will be of great interest if our analysis can be generalized to more general black hole spacetimes beyond RN-AdS black holes. More importantly, it is highly desirable to investigate the relationship between Lyapunov exponents and black hole phase transitions in the extended phase space, in which the cosmological constant is identified as a thermodynamic pressure.

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## A Lyapunov exponents of circular orbits

From the Lagrangian (3.1), one obtains the particle's canonical momenta

$$p_t = -f(r)\dot{t}, \quad p_r = \frac{\dot{r}}{f(r)}, \quad p_\varphi = r^2\dot{\varphi}, \quad (\text{A.1})$$

which gives the Hamiltonian

$$\mathcal{H} = -\frac{p_t^2}{2f(r)} + \frac{f(r)p_r^2}{2} + \frac{p_\varphi^2}{2r^2}. \quad (\text{A.2})$$

Since  $t$  and  $\varphi$  are cyclic coordinates, the corresponding momenta  $p_t$  and  $p_\varphi$  are constant, i.e.,  $p_t = -E$  and  $p_\varphi = L$ . As a result, the Hamiltonian can be reduced to

$$\mathcal{H} = \frac{V_{\text{eff}}(r) - E^2}{2f(r)} + \frac{f(r)p_r^2}{2} - \frac{\delta_1}{2}, \quad (\text{A.3})$$

where eq. (3.3) is used. The equations of motion for  $r$  and  $p_r$  are then given by

$$\begin{aligned} \dot{r} &= f(r)p_r, \\ \dot{p}_r &= -\frac{V'_{\text{eff}}(r)}{2f(r)} - \frac{f'(r)p_r^2}{2} + \frac{V_{\text{eff}}(r) - E^2}{2f^2(r)}f'(r). \end{aligned} \quad (\text{A.4})$$

Linearizing the equations of motion around a circular orbit at  $r = r_o$  leads to

$$\begin{aligned} \delta\dot{r} &= \delta p_r, \\ \delta\dot{p}_r &= -\frac{V''_{\text{eff}}(r_o)\delta r}{2f(r_o)}, \end{aligned} \quad (\text{A.5})$$

where we use  $E^2 - V_{\text{eff}}(r_o) = V'_{\text{eff}}(r_o) = 0$ . In terms of the coordinate time  $t$ , the above equations become

$$\begin{pmatrix} d(\delta r)/dt \\ d(\delta p_r)/dt \end{pmatrix} = K \begin{pmatrix} \delta r \\ \delta p_r \end{pmatrix}, \quad (\text{A.6})$$

where the linear stability matrix  $K$  is

$$K = \begin{pmatrix} 0 & f(r_o)t^{-1} \\ -V''_{\text{eff}}(r_o)f^{-1}(r_o)t^{-1}/2 & 0 \end{pmatrix}. \quad (\text{A.7})$$

The principal Lyapunov exponent  $\lambda$  of the circular orbit is determined by the eigenvalues of the matrix  $K$ , i.e.,

$$\lambda = \sqrt{-\frac{V''_{\text{eff}}(r_o)}{2t^2}}, \quad (\text{A.8})$$

where the  $+$  sign is chosen for  $\lambda$  [88].

If the particle is massless, the conditions  $p_t = -f(r_o)\dot{t}$  and  $E^2 - V_{\text{eff}}(r_o) = 0$  yield

$$\dot{t} = \frac{L}{f^{1/2}(r_o)r_o}. \quad (\text{A.9})$$

So the Lyapunov exponent of the circular null geodesics can be written as

$$\lambda = \sqrt{-\frac{r^2 f(r_o)}{L^2} V_{\text{eff}}''(r_o)}, \quad (\text{A.10})$$

which is eq. (3.7). Note that  $V_{\text{eff}}''(r_o)/L^2$  is independent of the angular momentum  $L$ . For the massive particle, the conditions  $p_t = -f(r_o)\dot{t}$ ,  $E^2 - V_{\text{eff}}(r_o) = 0$  and  $V_{\text{eff}}'(r_o) = 0$  lead to

$$\dot{t} = \sqrt{\frac{2}{2f(r_o) - r_o f'(r_o)}}. \quad (\text{A.11})$$

Consequently, the Lyapunov exponent of the circular time-like geodesic can be expressed as

$$\lambda = \frac{1}{2} \sqrt{[r_o f'(r_o) - 2f(r_o)] V_{\text{eff}}''(r_o)}, \quad (\text{A.12})$$

which is eq. (3.10).

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