

Doubled space and extended supersymmetry

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ABSTRACT: The doubled formulation of the worldsheet provides a description of string theory in which T-duality is promoted to a manifest symmetry. Here we extend this approach to $\mathcal{N} = (2, 2)$ superspace providing a doubled formulation for bi-Hermitian/generalised Kähler target spaces. The theory is described by a single function, a doubled-generalised Kähler potential, supplemented with a manifestly $\mathcal{N} = (2, 2)$ constraint. Several examples serve to illustrate this construction, including a discussion of the $\mathcal{N} = (2, 2)$ description of T-folds.

KEYWORDS: Extended Supersymmetry, String Duality, Superspaces, Superstrings and Heterotic Strings

ARXIV EPRINT: [2205.06017](https://arxiv.org/abs/2205.06017)

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1 Introduction

The interplay between the physics of the string worldsheet and the geometry of target space is a rich and fascinating subject. In this paper, we focus on the supersymmetric worldsheet non-linear σ -model. While any target manifold, \mathcal{M} , allows for $\mathcal{N} = (1, 1)$ supersymmetry on the worldsheet, requiring greater amounts of supersymmetry enforces stringent conditions on the metric, G , and the Kalb-Ramond two-form B defined on \mathcal{M} .

Our interest is in $\mathcal{N} = (2, 2)$ supersymmetry, as originally investigated long ago [1]. Requiring $\mathcal{N} = (2, 2)$ worldsheet supersymmetry gives particular conditions on the geometrical data in G and B . When the two-form B is absent, the criteria is that the target space be Kähler. With the inclusion of B , the metric is required to be Hermitian with respect to two (appropriately covariantly constant) complex structures and is accordingly said to be *bi-Hermitian*.

A seminal result of Gualtieri [2] is that bi-Hermitian geometry can be elegantly reinterpreted as a *Generalised Kähler Geometry* (GKG) structure on the generalised tangent bundle $E \simeq T\mathcal{M} \oplus T^*\mathcal{M}$. With $\dim \mathcal{M} = D$, the natural pairing between vectors and covectors endows E with an $O(D, D)$ invariant bilinear pairing, usually denoted by η . On E , in general and not just for a GKG, the geometric data, G and B , are combined together to form the components of a *generalised metric* denoted \mathcal{H} or with index raised $\mathcal{E} = \eta^{-1}\mathcal{H}$. This generalised metric itself is an $O(D, D)$ element, and hence obeys $\mathcal{E}^2 = 1$. In the specialisation to GKG, this data also is encoded in a pair $\mathcal{J}_1, \mathcal{J}_2$ of commuting *generalised complex* structures on the generalised tangent bundle. The generalised metric is identified with the product of the two generalised complex structures, via $\mathcal{E} = -\mathcal{J}_1\mathcal{J}_2$.

The natural appearance of $O(D, D)$ means that the extended bundle $T\mathcal{M} \oplus T^*\mathcal{M}$ is relevant more generally in string theory. It is a natural space on which T-duality can act linearly. When \mathcal{M} possesses d commuting isometries, such that $T^d \hookrightarrow \mathcal{M} \rightarrow \mathcal{B}$ is a torus fibration over a base manifold \mathcal{B} , a subgroup $O(d, d; \mathbb{Z})$ of this $O(D, D)$ becomes the exact T-duality symmetry.¹

T-duality symmetries are a powerful organisational tool shedding light on the full structure of string theory and its low energy effective descriptions. Generalised geometry, and the closely related double field theory, have been used to reframe supergravity in a way in which the $O(d, d)$ symmetry becomes a manifest property [3–6]. This is closely related to ideologically similar pursuits in the context of the string *worldsheet* theory [7–11]. In these worldsheet duality symmetric approaches, the idea is to consider a target space \mathbb{M} that is itself *doubled*, i.e. forms a bundle $T^d \times T^d \hookrightarrow \mathbb{M} \rightarrow \mathcal{B}$ in which the torus fibres of which have been doubled (the base space remains the same).

This goes a step further than generalised geometry as it is the manifold itself (rather than its tangent bundle) that is enlarged. Heuristically, the democratic treatment of momentum and winding modes interchanged by T-duality is accommodated by the inclusion of d additional coordinated directions in this doubled torus target. An appealing feature of the doubled worldsheet theory is that it may allow for a description of strings in T-folds, a class of non-geometric backgrounds in which locally geometric patches are glued with T-duality transformations.

The purpose of this note is to understand how to combine the requirements of extended supersymmetry with the T-duality symmetric worldsheet. Given how well adapted concepts of generalised geometry are to both extended supersymmetry and T-duality, it is of some surprise that the combination of them is not straightforward. There are, in fact, several technical reasons why this is challenging.

¹In supergravity these will both be $O(d, d; \mathbb{R})$ and $O(D, D; \mathbb{R})$, but in the full string theory the true T-duality is $O(d, d; \mathbb{Z})$. In this work, where the field is left unspecified in a group it should be assumed to be over \mathbb{R} .

Firstly, in duality symmetric approaches one typically introduces an extra T-dual boson, \tilde{x} , for each boson, x , corresponding to a toroidal isometry direction. In extended supersymmetry however, the real field x is part of a multiplet whose bosonic content contains a second real scalar, y , together forming a complex superfield $z = y + ix$. The y coordinate need not (and generally will not) correspond to some adapted coordinate of an isometry. We will see that in order to implement the doubled torus fibration specified by fields $\mathbb{X} = (x, \tilde{x})$ we are also compelled to double the base coordinates $\mathbb{Y} = (y, \tilde{y})$ as well. Having implemented the doubling one should also have a procedure to constrain the theory so as to not introduce any additional physical degrees of freedom. On the $\mathbb{X} = (x, \tilde{x})$ this amounts to imposing a worldsheet chirality constraint, $d\mathbb{X} = \star\mathcal{E}(d\mathbb{X})$, that encodes the canonical transformation between dual models. On the \mathbb{Y} we will require a *topological* constraint $d\mathbb{Y} = \mathcal{E}(d\mathbb{Y})$. In fact, the entire theory can be understood as being defined by a single function — a doubled-generalised-Kähler potential $\mathbb{V}(\mathbb{Y})$ — in terms of which the topological constraint on the \mathbb{Y} can be integrated to reveal an *algebraic* relation that simply undoes the over-doubling so introduced.

Secondly, implementing the chirality constraint on the \mathbb{X} at the Lagrangian level is challenging; doing so in the spirit of Floreanini-Jackiw and Tseytlin requires breaking manifest worldsheet Lorentz invariance which is evidently undesirable from the point of view of the supersymmetry algebra. The PST approach to chiral bosons does allow one to retain worldsheet Lorentz covariance, at the expense of introducing additional fields, but the extension of this to $\mathcal{N} = (2, 2)$ is very involved [11, 12]. In this work we will somewhat side step this point and instead, as with Hull’s approach to the doubled world sheet [9, 10], view the chirality constraint as an additional ingredient to be imposed off shell by hand.

The third challenge is there exists a variety of $\mathcal{N} = (2, 2)$ multiplets (chiral, twisted chiral and semi-chiral) and T-duality can change the type of multiplets required to furnish a particular theory. To simplify the situation we will restrict our attention first to so-called BiLP geometries which are described without semi-chiral multiplets. Rather appealingly, within our approach once we have appropriately doubled the BiLP geometry we find that we can access T-duality frames in which the conventional description requires semi-chiral multiplets.

There exist previous studies in the literature making contact between the string worldsheet and generalised complex geometry. The generalised complex structures of generalised Kähler geometry have been shown to appear naturally in the string worldsheet theory in first order or Hamiltonian approaches [13–15]. Our approach is rather fully covariant on the worldsheet (though see [16] for an outline of a ‘covariant Hamiltonian’ alternative).

A doubled superspace formulation exists for certain backgrounds with extended $\mathcal{N} = (4, 4)$ supersymmetry. This links to the T-folds mentioned earlier. One of the most studied examples of a T-fold background is the exotic 5_2^2 brane obtained by T-dualising the NS5 brane on two traverse directions (see [17] for a detailed discussion). This geometry admits extended $\mathcal{N} = (4, 4)$ worldsheet supersymmetry and can be studied via an $\mathcal{N} = (2, 2)$ description (with a generalised Kähler potential) such as in [18]. It can alternatively be described via an $\mathcal{N} = (4, 4)$ gauged linear sigma model (GLSM). In [19] a doubled $\mathcal{N} = (4, 4)$ GLSM was written down capable of describing the duality between the NS5 and

Kaluza-Klein monopole, and providing a natural setting with which to discuss worldsheet instanton corrections to the naive Buscher T-duality rules. Interestingly, here it was already noted that supersymmetry required the doubling of both the fibre and base directions. This was later extended to include the 5_2^2 obtained via a further T-duality in [20, 21]. The generalised complex structures of this duality chain were investigated very recently [22, 23].

The paper is structured as follows:

In section 2 we begin with a telegraphic bi-lingual (conventional/generalised) precis of the geometry associated to extended supersymmetry. Here we pay special attention to how generalised geometric considerations lead to a very elegant derivation of the transformations of complex structures under T-duality.

In section 3, we proceed with the presentation of the $\mathcal{N} = (2, 2)$ doubled worldsheet theory and demonstrate that upon reducing to $\mathcal{N} = (1, 1)$ superspace it reproduces the existing doubled string Lagrangian and constraints. We elaborate on how $O(d, d)$ acts in this doubled model, showing in particular that the transformation of complex structures in the doubled space matches that of the usual complex structures of bi-Hermitian geometry.

The final section, section 4, is dedicated to examples displaying how the construction of the doubling works in practice, with some particular applications illustrated. In subsection 4.1, we showcase a simple toroidal model. In subsection 4.2, we look at $SU(2) \times U(1)$. In subsection 4.3, we discuss the $\mathcal{N} = (2, 2)$ realisation of T-folds, and present codimension-1 and codimension-2 examples. Lastly, in subsection 4.4, we discuss a non-geometric T-duality transformation of a Kähler geometry leading to a semi-chiral geometry which we describe without introducing semi-chirals.

2 The geometry of extended supersymmetry

2.1 Bi-Hermitian geometry and the $\mathcal{N} = (2, 2)$ worldsheet

Consider a non-linear sigma-model defined on a target $\{\mathcal{M}, G, B\}$ represented by the Lagrangian

$$\mathcal{L} = (G_{IJ} + B_{IJ})\partial_+ x^I \partial_- x^J. \tag{2.1}$$

Regardless of the choice of metric G and two-form B this theory can be extended to one with $\mathcal{N} = (1, 1)$ supersymmetry by moving to superspace; replacing x^I with the superfield $X^I(x, \psi, F)$ (ψ a fermion and F an auxiliary boson), and upgrading the derivatives ∂_{\pm} to super-covariant derivatives D_{\pm} and integrating:

$$\mathcal{L} = \int d^2\theta (G_{IJ} + B_{IJ}) D_+ X^I D_- X^J. \tag{2.2}$$

One can then postulate the existence of an additional supersymmetry that mixes these superfields

$$\delta X^I = \epsilon_+ (J_+)^I{}_J D_+ X^J + \epsilon_- (J_-)^I{}_J D_- X^J. \tag{2.3}$$

The analysis of [1] shows for this transformation to be consistent that J_{\pm} are complex structures obeying, for all vectors $U, V \in \Gamma(T\mathcal{M})$,

$$J_+^2 = J_-^2 = -1, \quad [U, V] + J_{\pm}[J_{\pm}U, V] + J_{\pm}[U, J_{\pm}V] - [J_{\pm}U, J_{\pm}V] = 0. \tag{2.4}$$

To leave the Lagrangian invariant under the additional supersymmetry we further require that the metric G is Hermitian with respect to both complex structures

$$G(J_{\pm}U, J_{\pm}V) = G(U, V), \tag{2.5}$$

and that the complex structures define two-forms $\omega_{\pm}(U, V) = -G(U, J_{\pm}V)$ which satisfy,

$$d\omega_{\pm}(U, V, W) = \mp H(J_{\pm}U, J_{\pm}V, J_{\pm}W), \tag{2.6}$$

for any three vectors U, V, W . The last condition is equivalent to the requirement that the complex structures are covariantly constant with respect to the Levi-Civita connection \pm torsion $H = dB$. A target space possessing these properties is often called a bi-Hermitian geometry.

To manifest this second supersymmetry off-shell requires the introduction of $\mathcal{N} = (2, 2)$ superspace (see appendix for details). There are three pertinent superfields and which should be used depends crucially on the properties of J_{\pm} : chiral fields $z^{\alpha}, \bar{z}^{\alpha}$ parametrise $\ker(J_{+} - J_{-})$; twisted chiral fields w^{μ}, \bar{w}^{μ} parametrise $\ker(J_{+} + J_{-})$; and semi-chiral fields [24] (l, r, \bar{l}, \bar{r}) parametrise the remaining directions $\text{im}([J_{+}, J_{-}]G^{-1})$. Should the target geometry be of the type that only chiral and twisted chiral superfields are required, i.e. $[J_{+}, J_{-}] = 0$, then it is said to be BiLP (bi-Hermitian local product). This BiLP class, which we will largely focus on in the present work, clearly encompasses Kähler geometry as a special case where $J_{+} = J_{-}$.

In $\mathcal{N} = (2, 2)$ superspace, a single real function known as the generalised Kähler potential, $V(z, w, \bar{z}, \bar{w})$ for a BiLP, specifies the Lagrangian

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} V(z, w, \bar{z}, \bar{w}). \tag{2.7}$$

Passing back to $\mathcal{N} = (1, 1)$ superspace (discarding total derivatives) one obtains²

$$\begin{aligned} \mathcal{L} = & 2V_{\alpha\bar{\beta}}(D_{+}z^{\alpha}D_{-}\bar{z}^{\beta} + D_{+}\bar{z}^{\beta}D_{-}z^{\alpha}) - 2V_{\mu\bar{\nu}}(D_{+}w^{\mu}D_{-}\bar{w}^{\nu} + D_{+}\bar{w}^{\nu}D_{-}w^{\mu}) \\ & - 2V_{\alpha\bar{\mu}}(D_{+}z^{\mu}D_{-}\bar{w}^{\mu} - D_{+}\bar{w}^{\mu}D_{-}z^{\mu}) - 2V_{\bar{\alpha}\mu}(D_{+}\bar{z}^{\mu}D_{-}w^{\mu} - D_{+}w^{\mu}D_{-}\bar{z}^{\mu}), \end{aligned} \tag{2.8}$$

which shows that the metric and B-field are extracted as derivatives of this potential according to³

$$G_{\alpha\bar{\beta}} = 2V_{\alpha\bar{\beta}}, \quad G_{\mu\bar{\nu}} = -2V_{\mu\bar{\nu}}, \quad B_{\alpha\bar{\mu}} = -2V_{\alpha\bar{\mu}}, \quad B_{\bar{\alpha}\mu} = -2V_{\bar{\alpha}\mu}. \tag{2.9}$$

If in addition semi-chiral multiplets are required similar, albeit significantly more involved, expressions are found for the geometric data in terms of derivatives of the generalised Kähler potential.

²We use the notation $V_{\alpha} \equiv \frac{\partial V}{\partial z^{\alpha}}$, $V_{\bar{\alpha}} \equiv \frac{\partial V}{\partial \bar{z}^{\alpha}}$, and so on.

³The perhaps unseemly factors of 2 are to match with our later conventions involving real coordinates.

2.2 Generalized Kähler geometry

We now briefly review how the bi-Hermitian geometry invoked by $\mathcal{N} = (2, 2)$ supersymmetry can be reformulated in terms of generalized Kähler geometry on the generalized tangent space $E \simeq T\mathcal{M} \oplus T^*\mathcal{M}$ [2, 25]. Given two sections of this bundle $\mathbb{U} = (U, \alpha)$ and $\mathbb{V} = (V, \beta)$ we can construct the $O(D, D)$ invariant pairing

$$\eta(\mathbb{U}, \mathbb{V}) = \iota_U \beta + \iota_V \alpha \quad (2.10)$$

and the H twisted Courant bracket

$$[\mathbb{U}, \mathbb{V}]_H := [U, V] + \iota_U d\beta - \iota_V d\alpha + \frac{1}{2} d(\iota_U \beta - \iota_V \alpha) + \iota_V \iota_U H. \quad (2.11)$$

Given a metric and two-form on \mathcal{M} we can also endow E with a generalised metric \mathcal{H} , a representative of the coset $O(D, D)/O(D) \times O(D)$ that provides an additional symmetric pairing. We shall often work in a basis of E for which the inner products are given by

$$\eta = \begin{pmatrix} 0 & 1_D \\ 1_D & 0 \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} G - BG^{-1}B & -BG^{-1} \\ G^{-1}B & G^{-1} \end{pmatrix}. \quad (2.12)$$

In places it is more natural to view the generalised metric as an endomorphism of E defined by $\mathcal{E} = \eta^{-1} \cdot \mathcal{H}$. As \mathcal{H} is an $O(D, D)$ element it follows that $\mathcal{E}^2 = \text{id}$ and hence defines projectors

$$\mathcal{P}_{\pm} = \frac{1}{2} (\text{id} \pm \mathcal{E}). \quad (2.13)$$

In general \mathcal{E} can be decomposed as

$$\mathcal{E} = \begin{pmatrix} 1_D & 0 \\ -B & 1_D \end{pmatrix} \begin{pmatrix} 0 & G^{-1} \\ G & 0 \end{pmatrix} \begin{pmatrix} 1_D & 0 \\ B & 1_D \end{pmatrix}. \quad (2.14)$$

A generalised Kähler structure on E consists of a pair, \mathcal{J}_1 and \mathcal{J}_2 , of generalised complex structures such that $\mathcal{J}_1^2 = \mathcal{J}_2^2 = -\text{id}$ and $[\mathcal{J}_1, \mathcal{J}_2] = 0$ which are Courant-integrable,

$$[\mathbb{U}, \mathbb{V}]_H + \mathcal{J}_{1,2}[\mathcal{J}_{1,2}\mathbb{U}, \mathbb{V}]_H + \mathcal{J}_{1,2}[\mathbb{U}, \mathcal{J}_{1,2}\mathbb{V}]_H - [\mathcal{J}_{1,2}\mathbb{U}, \mathcal{J}_{1,2}\mathbb{V}]_H = 0 \quad \forall \mathbb{U}, \mathbb{V} \in \Gamma(E). \quad (2.15)$$

Furthermore, a generalized metric can now be obtained from the product of two generalized complex structures

$$\mathcal{E} = -\mathcal{J}_1 \mathcal{J}_2. \quad (2.16)$$

A remarkable result is that the conditions of bi-Hermitian geometry of eqs. (2.4)–(2.6) can be equated to the definition of a generalised complex structure. This is achieved explicitly through the Gualtieri map⁴

$$\mathcal{J}_{1,2} = \frac{1}{2} \begin{pmatrix} 1_D & 0 \\ -B & 1_D \end{pmatrix} \begin{pmatrix} J_+ \pm J_- & \omega_+^{-1} \mp \omega_-^{-1} \\ -(\omega_+ \mp \omega_-) & -(J_+^t \pm J_-^t) \end{pmatrix} \begin{pmatrix} 1_D & 0 \\ B & 1_D \end{pmatrix}. \quad (2.17)$$

⁴In which \mathcal{J}_1 corresponds to the lower choice in \pm .

2.3 Transformation of complex structures under T-duality

A virtue of the generalised Kähler perspective is that the action of $O(D, D; \mathbb{R})$, i.e. the transformations \mathcal{O} that preserve $\eta = \mathcal{O}^t \eta \mathcal{O}$, have a linear action on the tensors:

$$\mathbb{U} \rightarrow \mathcal{O}^{-1} \mathbb{U}, \quad \mathcal{J}_{1,2} \rightarrow \mathcal{O}^{-1} \mathcal{J}_{1,2} \mathcal{O}, \quad \mathcal{E} \rightarrow \mathcal{O}^{-1} \mathcal{E} \mathcal{O}, \quad \mathcal{H} \rightarrow \mathcal{O}^t \mathcal{H} \mathcal{O}. \quad (2.18)$$

In this way the complicated T-duality transformations — which involve a subgroup of the full $O(D, D; \mathbb{R})$ acting only in isometry directions — of geometric quantities, G, B, J_{\pm} , can be made transparent. Here we illustrate this by rederiving the transformation rules under T-duality of the spacetime complex structure originally worked out by Hassan [26].

Now let's take the spacetime metric $G = (G_{IJ})$ and choose for this a vielbein $e = (e^A_I)$ and flat metric $h = (h_{AB})$ such that $G = e^t h e$. In terms of this we can now construct a natural basis $\{\mathcal{V}_{+A}, \mathcal{V}_{-A}\}$, $A \in \{1, \dots, D\}$, for the generalized tangent bundle $E \simeq TM \oplus TM^*$,

$$\mathcal{V}_{\pm} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} \begin{pmatrix} \pm e^{-1} \\ e^t h \end{pmatrix}, \quad (2.19)$$

and its dual $\{\bar{\mathcal{V}}_+^A, \bar{\mathcal{V}}_-^A\}$,

$$\bar{\mathcal{V}}_{\pm} = \frac{1}{\sqrt{2}} \left(e, \pm h^{-1} e^{-t} \right) \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}. \quad (2.20)$$

This basis is orthonormal in the sense that:

$$\bar{\mathcal{V}}_+^A \mathcal{V}_{+B} = \delta_B^A, \quad \bar{\mathcal{V}}_-^A \mathcal{V}_{-B} = -\delta_B^A, \quad \bar{\mathcal{V}}_+^A \mathcal{V}_{-B} = 0, \quad \bar{\mathcal{V}}_-^A \mathcal{V}_{+B} = 0, \quad (2.21)$$

and enjoys the completeness relation:

$$\sum_A \mathcal{V}_{+A} \otimes \bar{\mathcal{V}}_+^A - \mathcal{V}_{-A} \otimes \bar{\mathcal{V}}_-^A = 1_{2D}. \quad (2.22)$$

Furthermore we can reconstruct the previously defined projectors via $\sum_A \mathcal{V}_{\pm A} \otimes \bar{\mathcal{V}}_{\pm}^A = \mathcal{P}_{\pm}$.

As these vielbein inherit the natural action of $O(D, D; \mathbb{R})$ i.e. $\mathcal{V}_{\pm} \rightarrow \tilde{\mathcal{V}}_{\pm} = \mathcal{O}^{-1} \mathcal{V}_{\pm}$, $\bar{\mathcal{V}}_{\pm} \rightarrow \tilde{\bar{\mathcal{V}}}_{\pm} = \bar{\mathcal{V}}_{\pm} \mathcal{O}$, we can use their transformation to deduce the T-duality rules on e to find a T-dual vielbein \tilde{e} . In fact, depending on if we consider \mathcal{V}_- or \mathcal{V}_+ we find two different results for the T-dual vielbein \tilde{e}_{\pm} . This is neither a contradiction nor a surprise; as $\tilde{e}_+^t h \tilde{e}_+ = \tilde{e}_-^t h \tilde{e}_-$ we have that \tilde{e}_{\pm} are related by a local Lorentz transformation reflecting the fact that on the worldsheet left and right movers transform differently under T-duality. To make this precise we will need to use some facts about the structure of transformations $\mathcal{O} \in O(D, D; \mathbb{R})$. Representing the matrix \mathcal{O} in a block diagonal form,

$$\mathcal{O} = \begin{pmatrix} \mathbb{A} & \mathbb{C} \\ \mathbb{B} & \mathbb{D} \end{pmatrix}, \quad (2.23)$$

one verifies that $O(D, D)$ invariance implies

$$\mathbb{B}^t \mathbb{A} + \mathbb{A}^t \mathbb{B} = 0 = \mathbb{D}^t \mathbb{C} + \mathbb{C}^t \mathbb{D}, \quad \mathbb{B}^t \mathbb{C} + \mathbb{A}^t \mathbb{D} = 1_D, \quad (2.24)$$

or equivalently,

$$\mathbb{A}\mathbb{C}^t + \mathbb{C}\mathbb{A}^t = 0 = \mathbb{B}\mathbb{D}^t + \mathbb{D}\mathbb{B}^t, \quad \mathbb{B}\mathbb{C}^t + \mathbb{D}\mathbb{A}^t = 1_D. \quad (2.25)$$

Then, starting from $\mathcal{V}_+ \rightarrow \mathcal{O}^{-1}\mathcal{V}_+$ one gets,

$$e \rightarrow \tilde{e}_+ = e \left(\mathbb{C}^t E^t + \mathbb{D}^t \right)^{-1} \equiv e \mathbb{O}_+^{-1}, \quad (2.26)$$

and for $E = G + B$ one finds,

$$E \rightarrow \tilde{E} = (E\mathbb{C} + \mathbb{D})^{-1}(E\mathbb{A} + \mathbb{B}). \quad (2.27)$$

On the other hand when starting from \mathcal{V}_- one gets,

$$e \rightarrow \tilde{e}_- = e \left(-\mathbb{C}^t E + \mathbb{D}^t \right)^{-1} \equiv e \mathbb{O}_-^{-1}, \quad (2.28)$$

and for E one finds,

$$E \rightarrow \tilde{E} = (\mathbb{A}^t E - \mathbb{B}^t)(-\mathbb{C}^t E + \mathbb{D}^t)^{-1}. \quad (2.29)$$

Using the identities eqs. (2.24) and (2.25) one easily shows that the transformation rules in eqs. (2.27) and (2.29) are the same. One additionally has that $\Lambda = e\mathbb{O}_+^{-1}\mathbb{O}_-e^{-1}$ is the Lorentz transformation relating \tilde{e}_+ and \tilde{e}_- . The utility of this is revealed by noting that we can construct $O(D, D)$ invariant quantities by contracting generalised tensors with the above generalised vielbeins and in particular we have

$$\bar{\mathcal{V}}_{\pm}^A \mathcal{J}_1 \mathcal{V}_{\pm B} = \pm (e J_{\pm} e^{-1})^A_B, \quad \bar{\mathcal{V}}_{\pm}^A \mathcal{J}_2 \mathcal{V}_{\pm B} = (e J_{\pm} e^{-1})^A_B, \quad \bar{\mathcal{V}}_{\pm}^A \mathcal{J}_i \mathcal{V}_{\mp B} = 0. \quad (2.30)$$

As these are inert under $O(D, D)$ transformations, using equations (2.26) and (2.28), we immediately read off the transformation rules for the complex structures J_{\pm} :

$$J_{\pm} \rightarrow \tilde{J}_{\pm} = \mathbb{O}_{\pm} J_{\pm} \mathbb{O}_{\pm}^{-1}. \quad (2.31)$$

$O(d, d)$ versus $O(D, D)$. To describe genuine T-duality transformations in this language, we assume that $T^d \hookrightarrow M \rightarrow B$ and the vector fields that generate the torus action should be isometries of G and $H = dB$. In this case a subgroup $O(d, d; \mathbb{Z})$ of this $O(D, D)$ relates equivalent string theory backgrounds. For our purpose we will be slightly more restrictive by assuming the two-form potential B is also invariant⁵ and so too the J_{\pm} . We choose adapted coordinates such that all geometric data, i.e. G , B and J_{\pm} , are independent of the adapted coordinates. We denote these adapted coordinates by x^i , $i = 1, \dots, d$, and the remaining non-isometric coordinates by y^a , $a = 1, \dots, D - d$. We then focus on the subgroup $O(d, d; \mathbb{R}) \subseteq O(D, D; \mathbb{R})$ with $d \leq D$. The explicit embedding of $\mathcal{O} \in O(d, d; \mathbb{R})$ in $O(D, D; \mathbb{R})$ is given by eq. (2.23) where,

$$\begin{aligned} \mathbb{A} &= \begin{pmatrix} \mathcal{A} & 0 \\ 0 & 1_{(D-d)} \end{pmatrix}, & \mathbb{D} &= \begin{pmatrix} \mathcal{D} & 0 \\ 0 & 1_{(D-d)} \end{pmatrix}, \\ \mathbb{B} &= \begin{pmatrix} \mathcal{B} & 0 \\ 0 & 0_{(D-d)} \end{pmatrix}, & \mathbb{C} &= \begin{pmatrix} \mathcal{C} & 0 \\ 0 & 0_{(D-d)} \end{pmatrix}, \end{aligned} \quad (2.32)$$

such that we can define an element \mathcal{O} of $O(d, d; \mathbb{R})$ given by

$$\mathcal{O} \equiv \begin{pmatrix} \mathcal{A} & \mathcal{C} \\ \mathcal{B} & \mathcal{D} \end{pmatrix}. \quad (2.33)$$

⁵In any case, even to speak of this potential means that we are working in patchwise fashion.

Example: Buscher transformations. The particular case of a Buscher transformation in a particular direction labelled by i corresponds to:

$$\mathcal{A} = \mathcal{D} = 1_d - \varrho_i, \quad \mathcal{B} = \mathcal{C} = \rho_i, \quad (2.34)$$

where $(\varrho_i)_{jk} = \delta_{ij}\delta_{ik}$, $i, j, k \in \{1, \dots, d\}$. The Buscher transformation in all d isometry directions corresponds is given by the product of all such individual Buscher transformations, and corresponds to:

$$\mathcal{A} = \mathcal{D} = 0, \quad \mathcal{B} = \mathcal{C} = 1_d. \quad (2.35)$$

In this latter case, writing $E = G + B$ as

$$E = \begin{pmatrix} E_{xx} & E_{xy} \\ E_{yx} & E_{yy} \end{pmatrix}, \quad (2.36)$$

(where we let x and y schematically denote the isometry and non-isometry indices) we get from eq. (2.27),

$$E \rightarrow \tilde{E} = \begin{pmatrix} E_{xx}^{-1} & E_{xx}^{-1} E_{xy} \\ -E_{yx} E_{xx}^{-1} & E_{yy} - E_{yx} E_{xx}^{-1} E_{xy} \end{pmatrix}, \quad (2.37)$$

which are exactly the Buscher rules [27, 28]. The complex structures then transform as in eq. (2.31) where \mathbb{O}_{\pm} are given by,

$$\mathbb{O}_+ = \begin{pmatrix} E_{xx}^t & E_{yx}^t \\ 0 & 1 \end{pmatrix}, \quad \mathbb{O}_- = \begin{pmatrix} -E_{xx} & -E_{xy} \\ 0 & 1 \end{pmatrix}. \quad (2.38)$$

If J_+ and J_- commute, then this property is preserved by Buscher transformations.

Example: bivector transformation of a Kähler geometry. We present a simple example where a T-duality transformation renders commuting complex structures non-commuting. Let us consider a $D = 2d$ -dimensional Kähler geometry, with d isometries. The complex structures are equal, $J_+ = J_-$. Coordinates can be chosen such that metric takes the form $G_{xx} = G_{yy} \equiv g$, $G_{xy} = 0$. Hence we can write

$$J_+ = J_- = \begin{pmatrix} 0 & 1_d \\ -1_d & 0 \end{pmatrix}, \quad G = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}. \quad (2.39)$$

There is no B -field.

We consider an $O(d, d)$ transformation generated by

$$\mathcal{A} = \mathcal{D} = 1_d, \quad \mathcal{B} = 0 \quad \mathcal{C} = \beta, \quad (2.40)$$

where $\beta^T = -\beta$ has the interpretation of a bivector. The transformation matrices for the complex structures are

$$\mathbb{O}_+ = \begin{pmatrix} 1_d - \beta g & 0 \\ 0 & 1_d \end{pmatrix}, \quad \mathbb{O}_- = \begin{pmatrix} 1_d + \beta g & 0 \\ 0 & 1_d \end{pmatrix}, \quad (2.41)$$

leading to

$$J'_+ = \begin{pmatrix} 0 & 1_d - \beta g \\ -(1_d - \beta g)^{-1} & 0 \end{pmatrix}, \quad J'_- = \begin{pmatrix} 0 & 1_d + \beta g \\ -(1_d + \beta g)^{-1} & 0 \end{pmatrix}. \quad (2.42)$$

These do not commute for any non-zero β . Hence the resulting geometry, with metric and B -field given by

$$G' + B' = \begin{pmatrix} g' + b' & 0 \\ 0 & g \end{pmatrix}, \quad g' + b' = (g^{-1} + \beta)^{-1}, \quad (2.43)$$

will (ordinarily) be described by semi-chiral superfields in an $\mathcal{N} = (2, 2)$ description.

We can use coordinate transformations of the non-isometric coordinates y (which are unaffected by the T-duality) to simplify one or other of the two complex structures (2.42). For instance, if we define new coordinates

$$y' = y'(y), \quad \text{such that} \quad \frac{\partial y'}{\partial y} = 1 - \beta g, \quad (2.44)$$

then we obtain

$$J'_+ = \begin{pmatrix} 0 & 1_d \\ -1_d & 0 \end{pmatrix}, \quad J'_- = \begin{pmatrix} 0 & (1_d + \beta g)(1_d - \beta g)^{-1} \\ -(1_d - \beta g)(1_d + \beta g)^{-1} & 0 \end{pmatrix}. \quad (2.45)$$

This corresponds to diagonalising J'_+ in a complex basis. It also brings the metric and B -field into a more symmetric form with

$$G' + B' = \begin{pmatrix} g' + b' & 0 \\ 0 & g' \end{pmatrix}, \quad g' + b' = (g^{-1} + \beta)^{-1}, \quad (2.46)$$

in which one notices in particular that the base-base component of $G' + B'$ has been transformed. In later examples, a democracy between the fibre and base will be restored by adding also b' as a two-form potential into this base-base component — however such a change will arise as the incorporation of a total derivative on the worldsheet.

3 Doubled worldsheet for extended supersymmetry

3.1 Review of $\mathcal{N} = (1, 1)$ sigma model

In this section we review how one can rewrite a $\mathcal{N} = (1, 1)$ sigma model in an $O(d, d)$ -covariant language. The $\mathcal{N} = (1, 1)$ non-linear σ -model Lagrangian is given by (2.2) We may introduce a split of the coordinates $X^I = (y^a, x^i)$, with $i = 1, \dots, d$, and decompose the metric and two-form as

$$G_{ab} = g_{ab} + A_a^i A_b^j g_{ij}, \quad G_{ai} = g_{ij} A_a^j, \quad G_{ij} = g_{ij}, \quad (3.1)$$

$$B_{ab} = b_{ab} + A_{[a}^i A_{b]i} + b_{ij} A_a^i A_b^j, \quad B_{ai} = -A_{ai} - b_{ij} A_a^j, \quad B_{ij} = b_{ij}. \quad (3.2)$$

In practice, we will consider manifolds with isometries such that the x^i are adapted coordinates for the isometries, along which we can T-dualise, and refer to the y^a as ‘spectators’.

The Lagrangian (2.2) can be written in terms of this split as

$$\mathcal{L} = (g_{ab} + b_{ab} - A_{[a}{}^i A_{b]i}) D_+ y^a D_- y^b + A_{ai} (\nabla_+ x^i D_- y^a - D_+ y^a \nabla_- x^i) + (g_{ij} + b_{ij}) \nabla_+ x^i \nabla_- x^j, \quad (3.3)$$

where $\nabla_{\pm} x^i \equiv D_{\pm} x^i + A_a{}^i D_{\pm} y^a$. We can rewrite (3.3) in terms of $O(d, d)$ symmetric variables using the doubled formalism of Hull [9, 10]. Firstly, we introduce dual coordinates \tilde{x} and define

$$\mathbb{X}^M = \begin{pmatrix} x^i \\ \tilde{x}_i \end{pmatrix}. \quad (3.4)$$

The geometric data contained in g_{ij} and b_{ij} can be encoded instead in an $O(d, d)$ generalised metric⁶

$$\mathcal{H}_{MN} = \begin{pmatrix} g_{ij} - b_{ik} g^{kl} b_{lj} & -b_{ik} g^{kj} \\ g^{ik} b_{kj} & g^{ij} \end{pmatrix}, \quad (3.5)$$

which obeys $\eta^{-1} \mathcal{H} \eta^{-1} \mathcal{H} = 1$ with η the $O(d, d)$ invariant. We further introduce $\mathbb{A}_a{}^M = (A_a{}^i A_{ai})$ which serves as a connection for the doubled space with coordinates \mathbb{X}^M , if we view it as a fibration over a base with coordinates y^a . Defining $\nabla_{\pm} \mathbb{X}^M \equiv D_{\pm} \mathbb{X}^M + \mathbb{A}_a{}^M D_{\pm} y^a$, the doubled worldsheet Lagrangian is:

$$\mathcal{L}_{\text{Hull}} = (g_{ab} + b_{ab}) D_+ y^a D_- y^b + \frac{1}{2} \mathcal{H}_{MN} \nabla_+ \mathbb{X}^M \nabla_- \mathbb{X}^N + \frac{1}{2} \mathbb{A}_a{}^N \eta_{NM} (\nabla_+ \mathbb{X}^M D_- y^a - D_+ y^a \nabla_- \mathbb{X}^M) - \frac{1}{2} \Omega_{MN} D_+ \mathbb{X}^M D_- \mathbb{X}^N. \quad (3.6)$$

The very final term is a total derivative, with

$$\Omega = \begin{pmatrix} 0 & 1_d \\ -1_d & 0 \end{pmatrix}. \quad (3.7)$$

This can be interpreted as a symplectic form in the doubled space.

To avoid introducing new degrees of freedom, the Lagrangian eq. (3.6) is supplemented by the constraints

$$\mathcal{P}_- \nabla_+ \mathbb{X} = \mathcal{P}_+ \nabla_- \mathbb{X} = 0, \quad (3.8)$$

where we recall the projectors

$$\mathcal{P}_{\pm} = \frac{1}{2} (1 \pm \eta^{-1} \mathcal{H}). \quad (3.9)$$

These constraints can be viewed as a (twisted) self-duality condition obeyed by the scalars \mathbb{X}^M on the worldsheet whose solution determines half of the coordinates $\{x, \tilde{x}\}$ in terms of the other half.

This elimination can be done explicitly at the level of the action by gauging the shift symmetry in the dual coordinates. This involves introducing an auxiliary worldvolume

⁶For want of symbols we reuse $\eta, \mathcal{H}, \mathcal{P}$ as doubled quantities but now for $O(d, d)$ rather than $O(D, D)$; hopefully context serves to disambiguate.

gauge field which we write as $C_{\pm}^M = (0, \tilde{C}_{\pm i})$. The gauged action has the following formally $O(d, d; \mathbb{R})$ covariant form:

$$\begin{aligned} \mathcal{L}_{\text{Hull gauged}} = & E_{ab} D_+ y^a D_- y^b + \frac{1}{2} \mathcal{H}_{MN} (\nabla_+ \mathbb{X}^M + C_+^M) (\nabla_- \mathbb{X}^N + C_-^N) \\ & + \frac{1}{2} \mathbb{A}_a{}^N \eta_{NM} ((\nabla_+ \mathbb{X}^M + C_+^M) D_- y^a - D_+ y^a (\nabla_- \mathbb{X}^M + C_-^M)) \\ & + \frac{1}{2} \eta_{MN} (C_+^M D_- \mathbb{X}^N - D_+ \mathbb{X}^M C_-^N) - \frac{1}{2} \Omega_{MN} D_+ \mathbb{X}^M D_- \mathbb{X}^N. \end{aligned} \quad (3.10)$$

The equations of motion of $\tilde{C}_{\pm i}$ are algebraic and equivalent to the constraints. They allow the \tilde{x} coordinates to be completely eliminated from (3.10), giving the original action (3.3).

3.2 The doubled $\mathcal{N} = (2, 2)$ model

In this section we demonstrate how to formulate the doubled sigma model directly in $\mathcal{N} = (2, 2)$ superspace.

We focus first on the simpler case of BiLP geometries, i.e. generalized Kähler manifolds with commuting complex structures. This allows for a description solely in terms of chiral and twisted chiral fields. We summarise our $\mathcal{N} = (2, 2)$ superspace conventions in appendix A. We denote chiral fields z^α , $\alpha = 1, \dots, n_c$ and twisted chiral fields by w^μ , $\mu = 1, \dots, n_t$. We let $d = n_c + n_t$. These fields are complex: we introduce their real and imaginary parts by writing

$$z^\alpha = \frac{1}{2}(y^\alpha + i x^\alpha), \quad w^\mu = \frac{1}{2}(y^\mu + i x^\mu). \quad (3.11)$$

The $N = (2, 2)$ Lagrangian describing a BiLP geometry is encoded in a single function, the generalized Kähler potential, $V(z, w, \bar{z}, \bar{w})$, with the action

$$S = \int d^2\sigma d^2\theta d^2\hat{\theta} V(z, w, \bar{z}, \bar{w}). \quad (3.12)$$

We now restrict our attention to geometries admitting isometries, with the assumption that the imaginary parts of the chiral and twisted chiral superfields correspond to adapted coordinates for these geometries, i.e. the geometry is independent of the coordinates (x^μ, x^α) . We can accordingly take the generalized Kähler potential to be independent of the isometry directions, thus $V = V(z + \bar{z}, w + \bar{w})$.

Given a generalized Kähler potential of this form, we can define T-dual potentials by Legendre transforming with respect to any of the combinations $z + \bar{z}$ or $w + \bar{w}$. Our construction requires the partial T-dual potentials, $V^{(c)}(z + \bar{z}, \tilde{z} + \bar{\tilde{z}})$ and $V^{(t)}(\tilde{w} + \bar{\tilde{w}}, w + \bar{w})$, obtained as Legendre transformations of V with respect to all the twisted chiral and chiral fields respectively

$$\begin{aligned} V^{(c)}(z + \bar{z}, \tilde{z} + \bar{\tilde{z}}) &= V(z + \bar{z}, p) + p^\mu (\tilde{z}_\mu + \bar{\tilde{z}}_\mu), \\ V^{(t)}(\tilde{w} + \bar{\tilde{w}}, w + \bar{w}) &= V(q, w + \bar{w}) - q^\alpha (\tilde{w}_\alpha + \bar{\tilde{w}}_\alpha). \end{aligned} \quad (3.13)$$

This Legendre transformation introduces n_t (dual) chiral fields \tilde{z}_μ and n_c (dual) twisted chiral fields \tilde{w}_α . We again write the real and imaginary parts as

$$\tilde{w}_\alpha = \frac{1}{2}(\tilde{y}_\alpha + i \tilde{x}_\alpha), \quad \tilde{z}_\mu = \frac{1}{2}(\tilde{y}_\mu + i \tilde{x}_\mu). \quad (3.14)$$

We then define the *doubled-generalised Kähler potential* \mathbb{V} as the sum of the two partial duals:

$$\mathbb{V}(z, w, \tilde{w}, \tilde{z}) = \frac{1}{2}(V^{(c)}(z + \bar{z}, \tilde{z} + \bar{\tilde{z}}) + V^{(t)}(\tilde{w} + \bar{\tilde{w}}, w + \bar{w})). \quad (3.15)$$

We can then define the action for our doubled $\mathcal{N} = (2, 2)$ sigma model:

$$\int d^2\sigma d^2\theta d^2\hat{\theta} \mathbb{V}(z, w, \tilde{w}, \tilde{z}). \quad (3.16)$$

This action is supplemented with the following constraints, which result from the Legendre transformations:

$$\begin{aligned} V_\alpha(z + \bar{z}, w + \bar{w}) &= (\tilde{w} + \bar{\tilde{w}})_\alpha, \\ V_\mu(z + \bar{z}, w + \bar{w}) &= -(\tilde{z} + \bar{\tilde{z}})_\mu, \end{aligned} \quad (3.17)$$

or concisely in terms of the real parts of superfields as⁷

$$V_\alpha(y) = \tilde{y}_\alpha, \quad V_\mu(y) = -\tilde{y}_\mu. \quad (3.18)$$

3.3 Reduction from $\mathcal{N} = (2, 2)$ doubled model to $\mathcal{N} = (1, 1)$ doubled model

Here we verify that the doubled $\mathcal{N} = (2, 2)$ model is sensible in that when it is reduced to $\mathcal{N} = (1, 1)$ language by performing the integration over the second additional superspace coordinate, $\int d^2\hat{\theta} \equiv \hat{D}_+ \hat{D}_-$, we recover the $\mathcal{N} = (1, 1)$ doubled sigma model introduced above.

Let us first consider the usual $\mathcal{N} = (1, 1)$ description of the BiLP geometry introduced in (2.7) but now imposing isometries in the x^α, x^μ directions (the imaginary parts of chiral and twisted chiral multiplets respectively). The $\mathcal{N} = (1, 1)$ Lagrangian that one finds can be written as $\mathcal{L} = \mathcal{L}_x + \mathcal{L}_y$ with

$$\begin{aligned} \mathcal{L}_x &= g_{\alpha\beta} D_+ x^\alpha D_- x^\beta + g_{\mu\nu} D_+ x^\mu D_- x^\nu + b_{\alpha\nu} D_+ x^\alpha D_- x^\nu + b_{\mu\beta} D_+ x^\mu D_- x^\beta, \\ \mathcal{L}_y &= g_{\alpha\beta} D_+ y^\alpha D_- y^\beta + g_{\mu\nu} D_+ y^\mu D_- y^\nu + b_{\alpha\nu} D_+ y^\alpha D_- y^\nu + b_{\mu\beta} D_+ y^\mu D_- y^\beta, \end{aligned} \quad (3.19)$$

where the metric and B -field are derived as second derivatives (implicitly understood to be with respect to the coordinates y) of the potential:

$$g_{\alpha\beta} = +V_{\alpha\beta}, \quad g_{\mu\nu} = -V_{\mu\nu}, \quad b_{\alpha\nu} = -V_{\alpha\nu}, \quad b_{\mu\beta} = +V_{\mu\beta}. \quad (3.20)$$

Observe that the fibre and spectator terms have identical metric and B -field components. The B -field terms in \mathcal{L}_y are in fact a total derivative. Discarding these terms, the doubled $\mathcal{N} = (1, 1)$ sigma model (3.6) derived from (3.19) has the Lagrangian

$$\mathcal{L} = \frac{1}{2} \mathcal{H}_{MN} D_+ \mathbb{X}^M D_- \mathbb{X}^N + g_{\alpha\beta} D_+ y^\alpha D_- y^\beta + g_{\mu\nu} D_+ y^\mu D_- y^\nu. \quad (3.21)$$

The $O(d, d)$ generalised metric appearing here has the functional form of (3.5) with the specific d -dimensional metric and B -field:

$$g_{ij} \equiv \begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & g_{\mu\nu} \end{pmatrix}, \quad b_{ij} \equiv \begin{pmatrix} 0 & b_{\alpha\nu} \\ b_{\mu\beta} & 0 \end{pmatrix}. \quad (3.22)$$

⁷Henceforth, we use the notation $V_\mu \equiv \partial_\mu V$, $V_{\mu\alpha} \equiv \partial_\mu \partial_\alpha V$, etc., where now we consider the derivatives with respect to y_α and y_μ .

The doubled coordinates are $\mathbb{X}^M = (x^i, \tilde{x}_i)$ with $x^i = (x^\alpha, x^\mu)$, $\tilde{x}_i = (\tilde{x}_\alpha, \tilde{x}_\mu)$, obeying the constraint (3.8) which is equivalent to

$$D_\pm \tilde{x}_i = \pm(g \mp b)_{ij} D_\pm x^j. \tag{3.23}$$

Now we wish to show how the $\mathcal{N} = (2, 2)$ doubled model arising from the doubled generalised Kähler potential eq. (3.15) reproduces the doubled model in $\mathcal{N} = (1, 1)$ super-space. We again have to perform the integration $\int d^2\hat{\theta} \equiv \hat{D}_+ \hat{D}_-$, and therefore we need the double derivatives of the doubled potential (3.15). We can easily obtain these derivatives, and relate them to derivatives of the original potential V , using properties of the Legendre transformation defining $V^{(c)}$ and $V^{(t)}$ in (3.13). For instance, by taking a total derivative of the defining relation

$$V^{(c)}(y^\alpha, \tilde{y}_\mu) = V(y^\alpha, y^\mu) + y^\mu \tilde{y}_\mu, \tag{3.24}$$

we have the conditions

$$V_\mu = -\tilde{y}_\mu, \quad V_\alpha = V_\alpha^{(c)}, \quad V^{(c)\mu} = y^\mu. \tag{3.25}$$

A subsequent total derivative of each these conditions produces

$$\begin{aligned} V_{\mu\nu} dy^\nu + V_{\mu\alpha} dy^\alpha &= -d\tilde{y}_\mu, \\ V_{\alpha\nu} dy^\nu + V_{\alpha\beta} dy^\beta &= V_\alpha^{(c)\nu} d\tilde{y}_\nu + V_{\alpha\beta}^{(c)} dy^\beta, \\ V^{(c)\mu\nu} d\tilde{y}_\nu + V^{(c)\mu}{}_\alpha dy^\alpha &= dy^\mu. \end{aligned} \tag{3.26}$$

Solving simultaneously returns the result that

$$\begin{aligned} V_{\alpha\beta}^{(c)} &= V_{\alpha\beta} - V_{\alpha\mu} (V^{-1})^{\mu\nu} V_{\nu\beta}, \quad V_\alpha^{(c)\nu} = V^{(c)\nu}{}_\alpha = -V_{\alpha\mu} (V^{-1})^{\mu\nu}, \\ V^{(c)\mu\nu} &= -(V^{-1})^{\mu\nu}, \end{aligned} \tag{3.27}$$

in which $(V^{-1})^{\mu\nu}$ means the matrix inverse of $V_{\mu\nu}$. The same manipulations for the dualisation to twisted chirals gives

$$\begin{aligned} V_{\mu\nu}^{(t)} &= V_{\mu\nu} - V_{\mu\alpha} (V^{-1})^{\alpha\beta} V_{\beta\nu}, \quad V_\mu^{(t)\beta} = V^{(t)\beta}{}_\mu = V_{\mu\alpha} (V^{-1})^{\alpha\beta}, \\ V^{(t)\alpha\beta} &= -(V^{-1})^{\alpha\beta}. \end{aligned} \tag{3.28}$$

Invoking eqs. (3.27) and (3.28) we immediately observe that these derivatives correspond to the components of the generalised metric:

$$\mathcal{H}_{MN} = \begin{pmatrix} V_{\alpha\beta}^{(c)} & 0 & 0 & V_\alpha^{(c)\mu} \\ 0 & -V_{\mu\nu}^{(t)} & -V_\mu^{(t)\alpha} & 0 \\ 0 & -V^{(t)\alpha}{}_\nu & -V^{(t)\alpha\beta} & 0 \\ V^{(c)\mu}{}_\alpha & 0 & 0 & V^{(c)\mu\nu} \end{pmatrix}. \tag{3.29}$$

Notice that this is not simply the Hessian of the doubled potential $\mathbb{V}(\mathbb{Y})$ with respect to the doubled coordinates $\mathbb{Y}^M = (y^\alpha, y^\mu, \tilde{y}_\alpha, \tilde{y}_\mu)$ but instead contains some additional signs. These

extra signs arise, when passing from $\mathcal{N} = (2, 2)$ to $\mathcal{N} = (1, 1)$, from the complex structures to which the chiral and twisted chiral fields are adapted.

Consequently, the doubled model in $\mathcal{N} = (1, 1)$ superspace is given by

$$\mathcal{L} = \frac{1}{2} D_+ \mathbb{X}^M \mathcal{H}_{MN} D_- \mathbb{X}^N + \frac{1}{2} D_+ \mathbb{Y}^M \mathcal{H}_{MN} D_- \mathbb{Y}^N, \quad (3.30)$$

where $\mathbb{X}^M = (x^\alpha \ x^\mu \ \tilde{x}_\alpha \ \tilde{x}_\mu)$ and $\mathbb{Y}^M = (y^\alpha \ y^\mu \ \tilde{y}_\alpha \ \tilde{y}_\mu)$ and \mathcal{H}_{MN} as given in equation (3.5) in terms of the metric and B -field (3.22). A priori the “spectator” part of the Lagrangian looks quite different from (3.6) as it is also, formally, doubled. However, we need to also consider the constraints (3.17) or (3.18) which in $\mathcal{N} = (1, 1)$ superspace derivatives of (3.18) imply

$$\begin{aligned} D_\pm \tilde{y}_\alpha &= V_{\alpha\beta} D_\pm y^\beta + V_{\alpha\mu} D_\pm y^\mu, \\ D_\pm \tilde{y}_\mu &= -V_{\mu\nu} D_\pm y^\nu - V_{\mu\alpha} D_\pm y^\alpha, \end{aligned} \quad (3.31)$$

which can be rewritten using $y^i \equiv (y^\alpha \ y^\mu)$ and $\tilde{y}_i \equiv (\tilde{y}_\alpha \ \tilde{y}_\mu)$ as

$$D_\pm \tilde{y}_i = (g - b)_{ij} D_\pm y^j. \quad (3.32)$$

Note this is *not* the standard T-duality constraint; left and right movers (i.e. plus and minus derivatives in (3.32)) are related in the same fashion whereas in T-duality the transformation of left and right movers differ by a target space local Lorentz rotation (as per discussion in section 2.3).

Next for the isometry directions we act on (3.18) using \hat{D}_\pm . Using the $\mathcal{N} = (2, 2)$ superspace constraints for chiral and twisted chiral multiplets, see eq. (A.22), this gives

$$\begin{aligned} D_\pm \tilde{x}_\alpha &= \pm V_{\alpha\beta}(y) D_\pm x^\beta + V_{\alpha\nu}(y) D_\pm x^\nu, \\ D_\pm \tilde{x}_\mu &= \mp V_{\mu\nu}(y) D_\pm x^\nu - V_{\mu\beta}(y) D_\pm x^\beta, \end{aligned} \quad (3.33)$$

which can be rewritten using $x^i \equiv (x^\alpha \ x^\mu)$ and $\tilde{x}_i \equiv (\tilde{x}_\alpha \ \tilde{x}_\mu)$ as

$$D_\pm \tilde{x}_i = \pm (g_{ij} \mp b_{ij}) D_\pm x^j, \quad (3.34)$$

i.e. exactly recovering the anticipated chirality constraints (3.8) of the doubled $\mathcal{N} = (1, 1)$ string in the form (3.23).

We now return to the Lagrangian (3.30). Although it would be inconsistent to use the constraints (3.33) in the action, we are allowed insert (3.31) or (3.32) in order to eliminate the \tilde{y} coordinates in terms of the y coordinates. This is because these are related by the (local) field redefinition (3.18) whereas the relationship between x and \tilde{x} has the interpretation of a chirality constraint. Once we eliminate the coordinates \tilde{y} from (3.30) we obtain exactly the $\mathcal{N} = (1, 1)$ doubled Lagrangian (3.6).

Furthermore, we can also directly obtain both the x and y equations of motion by taking further derivatives of the constraint in the form (3.33). For instance, the equation of motion of x^α follows from

$$\begin{aligned} 0 &= D_- D_+ \tilde{x}_\alpha + D_+ D_- \tilde{x}_\alpha \\ &= D_- (V_{\alpha\beta}(y) D_+ x^\beta + V_{\alpha\nu}(y) D_+ x^\nu) + D_+ (-V_{\alpha\beta}(y) D_- x^\beta + V_{\alpha\nu}(y) D_- x^\nu), \end{aligned} \quad (3.35)$$

while acting again with \hat{D}_\pm we have for example

$$\begin{aligned} \hat{D}_- D_+ \tilde{x}_\alpha &= D_+ D_- \tilde{y}_\alpha = \hat{D}_- (V_{\alpha\beta}(y) D_+ x^\beta + V_{\alpha\nu}(y) D_+ x^\nu) \\ &= -V_{\alpha\beta\gamma} D_- x^\gamma D_+ x^\beta + V_{\alpha\beta\mu} D_- x^\mu D_+ x^\beta - V_{\alpha\beta} D_+ D_- y^\beta \\ &\quad - V_{\alpha\nu\gamma} D_- x^\gamma D_+ x^\nu + V_{\alpha\mu\nu} D_- x^\mu D_+ x^\nu + V_{\alpha\mu} D_+ D_- y^\mu, \end{aligned} \quad (3.36)$$

and after substituting in for \tilde{y}_α using (3.18) this can be checked to reproduce the y^α equation of motion.

A final comment concerns the topological term of the $\mathcal{N} = (1, 1)$ doubled string [9, 10]. Whilst this term breaks $O(d, d)$ symmetry, its inclusion is important in gauging and to obtain correct answers at the quantum level. Here we point out that such a term arises in $\mathcal{N} = (2, 2)$ double model from the following quantity:

$$\varpi(y, \tilde{y}) = \frac{1}{2} (y^\alpha \tilde{y}_\alpha - y^\mu \tilde{y}_\mu). \quad (3.37)$$

One has that $\hat{D}_+ \hat{D}_- \varpi$ gives (without dropping total derivatives)

$$\mathcal{L}_\varpi = \frac{1}{2} \Omega_{MN} D_+ \mathbb{X}^M D_- \mathbb{X}^N + \frac{1}{2} \Omega_{MN} \mathbb{Y}^M D_- D_+ \mathbb{Y}^N, \quad (3.38)$$

where the constant antisymmetric matrix Ω_{MN} has non-zero components $\Omega^\alpha_\beta = +\delta^\alpha_\beta = -\Omega_\beta^\alpha$, $\Omega^\mu_\nu = +\delta^\mu_\nu = -\Omega_\nu^\mu$. Each term in (3.38) is a total derivative. The doubled potential \mathbb{V} is such that $V = \mathbb{V} + \varpi$. Hence adding or subtracting the total derivative ϖ in $\mathcal{N} = (2, 2)$ superspace, and using the constraint, breaks the duality symmetry.

In the $\mathcal{N} = (1, 1)$ doubled approach, making different choices of Ω_{MN} corresponds to picking different choices of duality frame. Here we can implement this by making different choices of ϖ . For instance, flipping the sign allows us to pick out the totally T-dual potential

$$\tilde{V} \equiv \mathbb{V} - \varpi = V - y^\alpha \tilde{y}_\alpha + y^\mu \tilde{y}_\mu. \quad (3.39)$$

An alternative choice such as $\varpi^{(c)} = \frac{1}{2} (y^\alpha \tilde{y}_\alpha + y^\mu \tilde{y}_\mu)$ gives back $V^{(c)} = \mathbb{V} + \varpi^{(c)}$. These different choices of ϖ (with different choices of sign) lead to different Ω_{MN} related by Buscher T-duality.

3.4 $O(d, d)$ and the $\mathcal{N} = (2, 2)$ doubled model

3.4.1 The constraint as a Lagrangian

Critical to the $\mathcal{N} = (2, 2)$ doubled model was the constraint of eq. (3.18) that encodes not only the duality relations between coordinates x and \tilde{x} interchanged by T-duality on the fibre but also an un-doubling of the formally doubled base coordinates y and \tilde{y} . To get a better understanding of this, and how T-duality acts on the doubled model, we first formulate the constraint (3.18) in a more democratic form. From the definitions of $V^{(c)}$ and $V^{(t)}$ in (3.13), it follows that

$$d\mathbb{V} = \frac{1}{2} (\tilde{y}_\alpha dy^\alpha - y^\alpha d\tilde{y}_\alpha - \tilde{y}_\mu dy^\mu + y^\mu d\tilde{y}_\mu). \quad (3.40)$$

In terms of $\mathbb{Y}^M = (y^\alpha \ y^\mu \ \tilde{y}_\alpha \ \tilde{y}_\mu)$ we can recast this expression as

$$\mathbb{V}_M \equiv \frac{\partial \mathbb{V}}{\partial \mathbb{Y}^M} = \Theta_{MN} \mathbb{Y}^N, \quad (3.41)$$

with

$$\Theta_{MN} = \frac{1}{2} \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}, \quad \sigma \equiv \begin{pmatrix} 1_{n_c} & 0 \\ 0 & -1_{n_t} \end{pmatrix}. \quad (3.42)$$

The object Θ_{MN} is suggestive of a symplectic structure. Indeed, we could regard (3.40) as being the statement that $d\mathbb{V} = \vartheta$ with $\vartheta \equiv \Theta_{MN} \mathbb{Y}^N d\mathbb{Y}^M$ a potential for a symplectic form $\omega = -d\vartheta$ given by

$$\omega = \Theta_{MN} d\mathbb{Y}^M \wedge d\mathbb{Y}^N = d\tilde{y}_\alpha \wedge dy^\alpha - d\tilde{y}_\mu \wedge dy^\mu. \quad (3.43)$$

The constraint (3.40) implies that we have $\omega = 0$ on the constraint surface. Hence, the doubled model is defined on a Lagrangian submanifold of the ‘doubled’ base space with coordinates \mathbb{Y}^M and symplectic form ω .

3.4.2 Doubled and over-doubled complex structures

Although we have been working in coordinates manifestly adapted to both isometries and complex structures, it is helpful to note that the over-doubled space itself inherits complex structures:

$$\hat{D}_\pm \begin{pmatrix} \mathbb{X} \\ \mathbb{Y} \end{pmatrix} = \mathbb{J}_\pm D_\pm \begin{pmatrix} \mathbb{X} \\ \mathbb{Y} \end{pmatrix}, \quad (3.44)$$

with \mathbb{J}_\pm such that

$$\mathbb{J}_+ = \begin{pmatrix} 0 & j_+ \\ -j_+ & 0 \end{pmatrix}, \quad \mathbb{J}_- = \begin{pmatrix} 0 & j_- \\ -j_- & 0 \end{pmatrix} \quad (3.45)$$

where for n_c chiral, n_t twisted chiral, n_c dual twisted chiral and n_t dual chiral fields, with $n_c + n_t = d$, we have that j_+ and j_- are $2d$ matrices given by:

$$j_+ = 1_{2d}, \quad j_- = \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix}, \quad (3.46)$$

with σ as in (3.42). A simple calculation then shows that the $\mathcal{N} = (1, 1)$ Lagrangian that follows on carrying out the integration over the extra $\mathcal{N} = (2, 2)$ fermionic coordinates has the form

$$\mathcal{L} = \mathbb{V}_{KL} j_+^K j_-^L D_+ \mathbb{X}^M D_- \mathbb{X}^N + \mathbb{V}_{MK} j_-^K j_+^L D_+ \mathbb{Y}^M D_- \mathbb{Y}^N, \quad (3.47)$$

where the derivatives of $\mathbb{V}(\mathbb{Y})$ are always with respect to \mathbb{Y}^M . Comparing with (3.30) this implies that the $O(d, d)$ generalised metric which appears in both terms is given by

$$\mathcal{H}_{MN} = 2\mathbb{V}_{KL} j_+^K j_-^L = 2\mathbb{V}_{MK} j_-^K j_+^L. \quad (3.48)$$

Given that $j_+^M j_-^N = \delta_N^M$, these two expressions are indeed equal.

We can further make an identification

$$\Theta_{MN} = \frac{1}{2} j_-^K j_+^L \eta_{LK}, \quad (3.49)$$

using η the $O(d, d)$ structure. We then note that taking an exterior derivative of (3.41) implies

$$\mathbb{V}_{MN}d\mathbb{Y}^N = \Theta_{MN}d\mathbb{Y}^N \Rightarrow d\mathbb{Y}^M = \eta^{MN}\mathcal{H}_{NK}d\mathbb{Y}^K \quad (3.50)$$

which reproduces the constraint (3.32) (and to reiterate for emphasis, the absence of a Hodge star in this relation is in contradistinction to that of the T-duality relation obeyed by the \mathbb{X}).

3.4.3 Transformation rules

Let us now look at the T-duality transformation properties of our model. From the $\mathcal{N} = (1, 1)$ case, we know that $\mathcal{O} \in O(d, d)$ acts on \mathbb{X} as,

$$\mathbb{X} \rightarrow \mathbb{X}' = \mathcal{O}^{-1}\mathbb{X}. \quad (3.51)$$

We require that these transformations respect the $\mathcal{N} = (2, 2)$ supersymmetry transformations, which implies we should also have an action of \mathcal{O} on \mathbb{Y} ,

$$\mathbb{Y} \rightarrow \mathbb{Y}' = \mathcal{O}^{-1}\mathbb{Y}. \quad (3.52)$$

This is accompanied by a transformation of the complex structures \mathbb{J}_{\pm} of (3.45):

$$\mathbb{J}_{\pm} \rightarrow \mathbb{J}'_{\pm} = \begin{pmatrix} \mathcal{O}^{-1} & 0 \\ 0 & \mathcal{O}^{-1} \end{pmatrix} \mathbb{J}_{\pm} \begin{pmatrix} \mathcal{O} & 0 \\ 0 & \mathcal{O} \end{pmatrix}. \quad (3.53)$$

In order to reproduce the behavior of the doubled action in $\mathcal{N} = (1, 1)$ superspace under $O(d, d)$ transformation we then have to require that the doubled potential transforms as a scalar,

$$\mathbb{V}(\mathbb{Y}) \rightarrow \mathbb{V}'(\mathbb{Y}') = \mathbb{V}(\mathbb{Y}). \quad (3.54)$$

We must also require that the constraint transformations covariantly, in particular we need

$$\Theta' = \mathcal{O}^T\Theta\mathcal{O}. \quad (3.55)$$

The T-duality invariance of the doubled model means that although we initially defined the doubled potential starting from a BiLP geometry, it can describe more complicated cases. For example, as we saw at the end of section 2.3, T-duality transformations can take us from a BiLP geometry with commuting complex structures to a geometry with non-commuting complex structures. The latter would normally have to be described with semi-chiral superfields. However, we can alternatively describe the geometry using appropriate combinations of our original doubled chiral and twisted chiral superfields, and thereby avoid introducing semi-chirals. We will demonstrate this explicitly in an example in section 4.4.

3.4.4 Transformation of complex structures in detail

In order to verify that the above transformation rules are sensible, we should consider more carefully the implications of the transformation (3.53) of the complex structures of (3.45).

These are initially given in adapted coordinates, and their geometric nature is unclear. Despite this, we can show that their transformation (3.53) under $O(d, d)$, together with the transformation of the constraint, allow us to recover the correct transformation of the usual complex structures J_{\pm} as derived in section 2.3.

Firstly, we note that as we start with \mathbb{J}_+ of the form (3.45) with $j_+ = 1_{2d}$, the $O(d, d)$ transformation (3.53) always preserves $\mathbb{J}'_+ = \mathbb{J}_+$. However, generically \mathbb{J}_- will transform non-trivially and lead to a very non-standard complex structure which in particular mixes physical and dual superfields. The correct prescription to recover sensible complex structures acting solely on the physical superfields is to apply the constraint to eliminate the dual ones. Then, after applying $O(d, d)$ transformation and the constraint, one obtains new complex structures J'_{\pm} , with the special feature that J'_+ is always in ‘diagonal’ (w.r.t. a complex basis) form. This corresponds to doing a standard $O(d, d)$ transformation as in section 2.3 followed by a coordinate transformation of the non-isometric directions in order to bring J'_+ into the desired form.

We now prove this. Let’s start with the complex structures corresponding to our $2d$ -dimensional BiLP geometry with n_c chiral superfields and n_t twisted chiral superfields. In adapted coordinates for the d isometry directions we can write (as in appendix A.3)

$$J_+ = \begin{pmatrix} 0 & 1_d \\ -1_d & 0 \end{pmatrix}, \quad J_- = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}. \quad (3.56)$$

The metric and B -field are meanwhile given by

$$G + B = \begin{pmatrix} g + b & 0 \\ 0 & g + b \end{pmatrix}, \quad g = \begin{pmatrix} g_{cc} & 0 \\ 0 & g_{tt} \end{pmatrix}, \quad b = \begin{pmatrix} 0 & b_{ct} \\ -b_{ct} & 0 \end{pmatrix}, \quad (3.57)$$

where we schematically exhibit the non-zero metric and B -field components in chiral and twisted chiral directions.

The T-duality transformation rule (2.31) implies that under an $\mathcal{O} \in O(d, d) \subset O(D, D)$ transformation with

$$\mathcal{O} = \begin{pmatrix} \mathcal{A} & \mathcal{C} \\ \mathcal{B} & \mathcal{D} \end{pmatrix}, \quad (3.58)$$

we obtain new complex structures

$$J'_+ = \begin{pmatrix} 0 & O_+ \\ -O_+^{-1} & 0 \end{pmatrix}, \quad J'_- = \begin{pmatrix} 0 & O_- \sigma \\ -\sigma O_-^{-1} & 0 \end{pmatrix}, \quad (3.59)$$

where

$$O_+ = \mathcal{C}^t(g - b) + \mathcal{D}^t, \quad O_- = -\mathcal{C}^t(g + b) + \mathcal{D}^t. \quad (3.60)$$

We ‘diagonalise’ J'_+ by using a coordinate transformation $y \rightarrow y'(y)$ such that $\partial y'/\partial y = O_+$. This leads to:

$$J'_+ = \begin{pmatrix} 0 & 1_d \\ -1_d & 0 \end{pmatrix}, \quad J'_- = \begin{pmatrix} 0 & O_- \sigma O_+^{-1} \\ -O_+ \sigma O_-^{-1} & 0 \end{pmatrix}. \quad (3.61)$$

We then have

$$\begin{aligned} \hat{D}_- x' &= O_- \sigma O_+^{-1} D_- y' = (-\mathcal{C}^t(g + b) + \mathcal{D}^t) \sigma (\mathcal{C}^t(g - b) + \mathcal{D}^t)^{-1} D_- y', \\ \hat{D}_- y' &= -O_+ \sigma O_-^{-1} D_- x' = -(\mathcal{C}^t(g - b) + \mathcal{D}^t) \sigma (-\mathcal{C}^t(g + b) + \mathcal{D}^t)^{-1} D_- x'. \end{aligned} \quad (3.62)$$

We wish to reproduce these relationships from our $\mathcal{N} = (2, 2)$ doubled model. According to (3.53), we obtain new complex structures \mathbb{J}'_{\pm} with $\mathbb{J}'_+ = \mathbb{J}_+$ and

$$\mathbb{J}'_- = \begin{pmatrix} 0 & j'_- \\ -j'_- & 0 \end{pmatrix}, \quad j'_- = \begin{pmatrix} \mathcal{D}^t \sigma \mathcal{A} - \mathcal{C}^t \sigma \mathcal{B} & \mathcal{D}^t \sigma \mathcal{C} - \mathcal{C}^t \sigma \mathcal{D} \\ \mathcal{B}^t \sigma \mathcal{A} - \mathcal{A}^t \sigma \mathcal{B} & \mathcal{B}^t \sigma \mathcal{C} - \mathcal{A}^t \sigma \mathcal{D} \end{pmatrix}. \quad (3.63)$$

Hence we have

$$\begin{aligned} \hat{D}_- x' &= (\mathcal{D}^t \sigma \mathcal{A} - \mathcal{C}^t \sigma \mathcal{B}) D_- y' + (\mathcal{D}^t \sigma \mathcal{C} - \mathcal{C}^t \sigma \mathcal{D}) D_- \tilde{y}' \\ \hat{D}_- \tilde{y}' &= -(\mathcal{D}^t \sigma \mathcal{A} - \mathcal{C}^t \sigma \mathcal{B}) D_- x' - (\mathcal{D}^t \sigma \mathcal{C} - \mathcal{C}^t \sigma \mathcal{D}) D_- \tilde{x}', \end{aligned} \quad (3.64)$$

We eliminate the dual coordinates from (3.64) using the transformed constraints:

$$\begin{aligned} D_- \tilde{y}' &= (g' - b') D_- y' = (\mathcal{A}^t (g - b) + \mathcal{B}^t) O_+^{-1} D_- y', \\ D_- \tilde{x}' &= -(g' + b') D_- y' = -(\mathcal{A}^t (g + b) - \mathcal{B}^t) O_-^{-1} D_- y', \end{aligned} \quad (3.65)$$

using (2.27) and (2.29) to express the transformation of the geometry. Starting with $\hat{D}_- x'$, we can thus write:

$$\begin{aligned} \hat{D}_- x' &= \left((\mathcal{D}^t \sigma \mathcal{A} - \mathcal{C}^t \sigma \mathcal{B}) O_+ + (\mathcal{D}^t \sigma \mathcal{C} - \mathcal{C}^t \sigma \mathcal{D}) (\mathcal{A}^t (g - b) + \mathcal{B}^t) \right) O_+^{-1} D_- y' \\ &= \left((\mathcal{D}^t \sigma (\mathcal{A} \mathcal{C}^t + \mathcal{C} \mathcal{A}^t) - \mathcal{C}^t \sigma (\mathcal{B} \mathcal{C}^t + \mathcal{D} \mathcal{A}^t)) (g - b) \right. \\ &\quad \left. + \mathcal{D}^t \sigma (\mathcal{A} \mathcal{D}^t + \mathcal{C} \mathcal{B}^t) - \mathcal{C}^t \sigma (\mathcal{B} \mathcal{D}^t + \mathcal{D} \mathcal{B}^t) \right) O_+^{-1} D_- y' \\ &= (-\mathcal{C}^t \sigma (g - b) \sigma + \mathcal{D}^t \sigma) O_+^{-1} D_- y', \end{aligned} \quad (3.66)$$

where we used the conditions resulting from the fact that the blocks $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ give an $O(d, d)$ transformation. Finally, as $\sigma(g - b)\sigma = g + b$ we recover exactly the correct relationship (3.62). The calculation of $\hat{D}_- y'$ proceeds similarly.

4 Examples

In this section we illustrate the construction of the doubled potential in a number of examples. In each case the recipe is as follows:

- We start with the generalised Kähler potential $V(z + \bar{z}, w + \bar{w})$ describing a BiLP geometry with isometries.
- We construct the totally chiral and twisted chiral potentials, $V^{(c)}(z + \bar{z}, \tilde{z} + \bar{\tilde{z}})$ and $V^{(t)}(\tilde{w} + \bar{\tilde{w}}, w + \bar{w})$, defined by Legendre transformation in (3.13).
- We hence write down the doubled potential \mathbb{V} given by (3.15) and the constraints given by (3.17) or (3.18).

4.1 Torus with constant B-field

A first simple example is a torus with constant B -field. This can be realised via the following generalised Kähler potential:

$$V(z, w) = \frac{1}{2}(z + \bar{z})^2 - \frac{1}{2}(w + \bar{w})^2 + \lambda(z + \bar{z})(w + \bar{w}). \quad (4.1)$$

Defining $y_1 = z + \bar{z}$ be the real part of the chiral superfield and $y_2 = w + \bar{w}$ that of the twisted chiral superfield we can rewrite the potential as

$$V(y_1, y_2) = \frac{1}{2}y_1^2 - \frac{1}{2}y_2^2 + \lambda y_1 y_2. \quad (4.2)$$

The corresponding geometry is simply:

$$ds^2 = dx_1^2 + dy_1^2 + dx_2^2 + dy_2^2, \quad B = -\lambda dy_1 \wedge dy_2 - \lambda dx_1 \wedge dx_2. \quad (4.3)$$

We now give the doubled $\mathcal{N} = (2, 2)$ description. The dual potentials (3.13) used in the construction of the doubled potential are:

$$V^{(c)}(y_1, \tilde{y}_2) = \frac{1}{2}(1 + \lambda^2)y_1^2 + \frac{1}{2}\tilde{y}_2^2 + \lambda y_1 \tilde{y}_2, \quad (4.4)$$

$$V^{(t)}(\tilde{y}_1, y_2) = -\frac{1}{2}\tilde{y}_1^2 - \frac{1}{2}(1 + \lambda^2)y_2^2 + \lambda \tilde{y}_1 y_2. \quad (4.5)$$

The doubled potential (3.15) is thus:

$$2\mathbb{V} = \frac{1}{2}(1 + \lambda^2)y_1^2 + \frac{1}{2}\tilde{y}_2^2 - \frac{1}{2}\tilde{y}_1^2 - \frac{1}{2}(1 + \lambda^2)y_2^2 + \lambda(y_1 \tilde{y}_2 + \tilde{y}_1 y_2), \quad (4.6)$$

and the constraints (3.18) are:

$$\begin{pmatrix} y_1 \\ \tilde{y}_1 \\ y_2 \\ \tilde{y}_2 \end{pmatrix} = \begin{pmatrix} -\tilde{y}_1 + \lambda y_2 \\ (1 + \lambda^2)y_1 + \lambda \tilde{y}_2 \\ \tilde{y}_2 + \lambda y_1 \\ -(1 + \lambda^2)y_2 + \lambda \tilde{y}_1 \end{pmatrix}. \quad (4.7)$$

4.2 $SU(2) \times U(1)$

Another example is provided by the WZW model on $SU(2) \times U(1)$, for details of which see [29]. In its BiLP description, the generalized Kähler potential [30] reads,

$$V = -\frac{1}{2}(w + \bar{w})^2 + \int^{z+\bar{z}-w-\bar{w}} dr \ln(1 + e^r), \quad (4.8)$$

which in our conventions describes the geometry

$$ds^2 = \frac{1}{1 + e^{y_1 - y_2}} \left(e^{y_1 - y_2} (dy_1^2 + dx_1^2) + dy_2^2 + dx_2^2 \right), \quad (4.9)$$

$$B = \frac{e^{y_1 - y_2}}{1 + e^{y_1 - y_2}} (dy_1 \wedge dy_2 + dx_1 \wedge dx_2).$$

The Legendre dual potentials are:

$$\begin{aligned}
 V^{(c)}(z + \bar{z}, \tilde{z} + \bar{\tilde{z}}) &= \frac{1}{2} (z + \bar{z})^2 + \int^{z+\bar{z}-p} dr \ln(1 + e^{-r}) + p(\tilde{z} + \bar{\tilde{z}} - 1(z + \bar{z})) \\
 &= \frac{1}{2} (\tilde{z} + \bar{\tilde{z}})^2 + \int^{\tilde{z}+\bar{\tilde{z}}+z+\bar{z}} dr \ln(1 - e^{-r}), \\
 V^{(t)}(\tilde{w} + \bar{\tilde{w}}, w + \bar{w}) &= -\frac{1}{2} (w + \bar{w})^2 + \int^{q-w-\bar{w}} dr \ln(1 + e^r) - q(\tilde{w} + \bar{\tilde{w}}) \\
 &= -\frac{1}{2} (\tilde{w} + \bar{\tilde{w}} + (w + \bar{w}))^2 - \int^{\tilde{w}+\bar{\tilde{w}}} dr \ln(1 - e^{-r}). \tag{4.10}
 \end{aligned}$$

Hence we get for the doubled-generalised Kähler potential,

$$\begin{aligned}
 \mathbb{V}(z + \bar{z}, w + \bar{w}, \tilde{w} + \bar{\tilde{w}}, \tilde{z} + \bar{\tilde{z}}) &= \frac{1}{2} \left(\frac{1}{2} (\tilde{z} + \bar{\tilde{z}})^2 - \frac{1}{2} (\tilde{w} + \bar{\tilde{w}} + w + \bar{w})^2 \right. \\
 &\quad \left. + \int_{\tilde{w}+\bar{\tilde{w}}}^{\tilde{z}+\bar{\tilde{z}}-z-\bar{z}} dr \ln(1 - e^{-r}) \right). \tag{4.11}
 \end{aligned}$$

The constraints are

$$\begin{pmatrix} z + \bar{z} \\ w + \bar{w} \\ \tilde{w} + \bar{\tilde{w}} \\ \tilde{z} + \bar{\tilde{z}} \end{pmatrix} = \begin{pmatrix} w + \bar{w} + \ln(e^{\tilde{w}+\bar{\tilde{w}}} - 1) \\ \ln(e^{\tilde{z}+\bar{\tilde{z}}} - e^{z+\bar{z}}) \\ -\ln(1 - e^{z+\bar{z}-\tilde{z}-\bar{\tilde{z}}}) \\ w + \bar{w} + \tilde{w} + \bar{\tilde{w}} \end{pmatrix}. \tag{4.12}$$

4.3 T-fold examples

We now consider some more novel examples of $\mathcal{N} = (2, 2)$ non-geometries. The general idea is the following. Recall that a generalised Kähler potential $V(z, w)$ is defined only up to generalised Kähler transformations:

$$V(z, w) \rightarrow V(z, w) + F(z, w) + \bar{F}(\bar{z}, \bar{w}) + G(\bar{z}, w) + \bar{G}(z, \bar{w}), \tag{4.13}$$

for arbitrary functions F and G . The examples we consider below will have this feature when we carry out some transformation of z and w under which the geometry is globally identified. We will refer to this as a monodromy of the generalised Kähler potential. After T-dualising, this will induce a non-trivial monodromy of the dual generalised Kähler potential, which in terms of the dual superfields \tilde{z}, \tilde{w} , will not be interpretable as a generalised Kähler transformation.

More exactly, we will consider potentials with monodromies of the form

$$V(z + \bar{z}, w + \bar{w}) \rightarrow V(z + \bar{z}, w + \bar{w}) + \alpha_1(z + \bar{z}) + \alpha_2(w + \bar{w}) + \gamma(z + \bar{w})(w + \bar{w}), \tag{4.14}$$

with some constants $\alpha_1, \alpha_2, \gamma, \delta$. The parameter γ corresponds to a constant shift (large gauge transformation) of the B -field. More precisely, we will phrase this monodromy as arising from some transformation acting on the real parts $y_1 \equiv z + \bar{z}$, $y_2 = w + \bar{w}$. We will write this transformation as $(y_1, y_2) \rightarrow (y_1^*, y_2^*) = (y_1 + n_1, y_2 + n_2)$. This can describe global shifts by constants n_1, n_2 , under which we make a periodic identification, or in the non-compact case rotations by 2π (with $n_1 = n_2 = 0$). We assume the potential behaves as

$$V(y_1^*, y_2^*) = V(y_1, y_2) + \alpha_1 y_1 + \alpha_2 y_2 + \gamma y_1 y_2 + \delta. \tag{4.15}$$

We want to consider the effect of this monodromy on the dual coordinates and the dual potentials. The Legendre transformations are:

$$V^{(c)}(y_1, \tilde{y}_2) = V(y_1, y_2) + y_2 \tilde{y}_2 \quad V^{(t)}(\tilde{y}_1, y_2) = V(y_1, y_2) - y_1 \tilde{y}_1. \quad (4.16)$$

It follows that the corresponding shifts of the dual coordinates $\tilde{y}_1 = V_1$ and $\tilde{y}_2 = -V_2$ are:

$$\tilde{y}_1 \rightarrow \tilde{y}_1^* = \tilde{y}_1 + \alpha_1 + \gamma y_2, \quad \tilde{y}_2 \rightarrow \tilde{y}_2^* = \tilde{y}_2 - \alpha_2 - \gamma y_1. \quad (4.17)$$

By carefully considering the definition of the Legendre transformed potentials, we can show (see appendix B) that:

$$V^{(c)}(\tilde{y}_1^*, \tilde{y}_2^*) = V^{(c)}(y_1, \tilde{y}_2) + \alpha_1 y_1 + \delta + n_2(\tilde{y}_2 - \alpha_2 - \gamma y_1), \quad (4.18)$$

$$V^{(t)}(\tilde{y}_1^*, y_2^*) = V^{(t)}(\tilde{y}_1, y_2) + \alpha_2 y_2 + \delta - n_1(\tilde{y}_1 + \alpha_1 + \gamma y_2). \quad (4.19)$$

In both (4.18) and (4.19), the shift of the generalised Kähler potential is still a generalised Kähler transformation. However, if we consider the doubly dual potential, $\tilde{V}(\tilde{y}_1, \tilde{y}_2)$, we have

$$\begin{aligned} \tilde{V}(\tilde{y}_1^*, \tilde{y}_2^*) &= \tilde{V}(\tilde{y}_1, \tilde{y}_2) - \gamma y_1(\tilde{y}_1, \tilde{y}_2) y_2(\tilde{y}_1, \tilde{y}_2) + \delta \\ &\quad - n_1(\tilde{y}_1 + \alpha_1 + \gamma y_2(\tilde{y}_1, \tilde{y}_2)) + n_2(\tilde{y}_2 - \alpha_2 - \gamma y_1(\tilde{y}_1, \tilde{y}_2)). \end{aligned} \quad (4.20)$$

When expressed in terms of the dual superfields, this is not necessarily of the form of a generalised Kähler transformation. The corresponding metric and B -field configuration will then not describe a geometric space but rather a T-fold. We further see that while \tilde{V} behaves less than optimally, both $V^{(c)}$ and $V^{(t)}$ transform nicely: and it is these that our doubled-generalised potential is constructed from. We now describe two examples illustrating these points.

4.3.1 Torus fibration with constant H-flux and codimension-1 T-fold

We start with the example of a torus with constant H -flux. It is well-known that T-duality transformations of such backgrounds can lead outside the scope of conventional geometry, producing non-geometric spaces known as T-folds. We consider again a setup with one chiral superfield (with real part y_1) and a twisted chiral superfield (with real part y_2) and the following potential:

$$V(y_1, y_2) = \frac{1}{2} \left((c + m y_2) y_1^2 - c y_2^2 - \frac{1}{3} m y_2^3 \right), \quad (4.21)$$

where c and m are constants. This gives metric and B -field

$$ds^2 = (c + m y_2)(dx_1^2 + dy_1^2 + dx_2^2 + dy_2^2), \quad B = -m y_1 dy_1 \wedge dy_2 - m y_1 dx_1 \wedge dx_2, \quad (4.22)$$

describing the transverse geometry to a thrice-smearred NS5 brane, which is of codimension 1. If we take y_1, x_1, x_2 to be periodic (with period 2π) then this is a fibration of a three-torus over an interval parametrised by y_2 . The three-torus has constant H -flux, $H = dB = -m dy_1 \wedge dx_1 \wedge dx_2$. The gauge choice for the B -field implies that for $y_1 \rightarrow y_1 + 2\pi$

we patch by a large gauge transformation, which corresponds to a generalised Kähler transformation of the potential:

$$V(y_1 + 2\pi, y_2) = V(y_1, y_2) + 2\pi(c + my_2)(y_1 + \pi), \quad (4.23)$$

or in complex terms

$$V(z + \bar{z} + 2\pi, w + \bar{w}) = V(z + \bar{z}, w + \bar{w}) + 2\pi(c\pi + c(z + \bar{z}) + m\pi(w + \bar{w}) + m(zw + \bar{z}\bar{w} + z\bar{w} + \bar{z}w)). \quad (4.24)$$

T-dualising the geometry (4.22) on the x_1 and x_2 directions leads to

$$\begin{aligned} \widetilde{ds}^2 &= f(y_2)(dy_1^2 + dy_2^2) + \frac{f(y_2)}{f(y_2)^2 + (my_1)^2}(d\tilde{x}_1^2 + d\tilde{x}_2^2), \\ \tilde{B} &= -my_1 dy_1 \wedge dy_2 + \frac{my_1}{f(y_2)^2 + (my_1)^2} d\tilde{x}_1 \wedge d\tilde{x}_2. \end{aligned} \quad (4.25)$$

For $y_1 \rightarrow y_1 + 2\pi$ this is not patched by any combination of diffeomorphisms or gauge transformations: instead the geometry is only well-defined up to a non-trivial T-duality transformation. This is associated to a so-called ‘exotic’ 5_2^2 brane which is non-geometric in nature (see [17] for a detailed discussion). Here we have the codimension-1 version of this exotic brane, which has been recently studied in [31] for example.

Now we consider how this T-fold behaviour manifests in the generalised Kähler potentials. The Legendre dual coordinates are defined by

$$\tilde{y}_1 = V_1 = f(y_2)y_1, \quad \tilde{y}_2 = -V_2 = -\frac{1}{2}my_1^2 + \frac{1}{2m}(f(y_2)^2 - c^2), \quad (4.26)$$

where $f(y_2) \equiv c + my_2$. To solve for y_1 and y_2 , we write $y_1 = \tilde{y}_1/f$ where $f = f(y_2(\tilde{y}_1, \tilde{y}_2))$ is determined via (4.26). Explicitly:

$$f(\tilde{y}_1, \tilde{y}_2)^2 = m \left(\tilde{y}_2 + \frac{c^2}{2m} \pm \sqrt{\left(\tilde{y}_2 + \frac{c^2}{2m}\right)^2 + \tilde{y}_1^2} \right). \quad (4.27)$$

The fully T-dual potential can then be written down as

$$\begin{aligned} \tilde{V}(\tilde{y}_1, \tilde{y}_2) &= -\frac{1}{2} \frac{\tilde{y}_1^2}{f(\tilde{y}_1, \tilde{y}_2)} + \frac{f(\tilde{y}_1, \tilde{y}_2) - c}{m} \tilde{y}_2 \\ &\quad - \frac{1}{2m^2} \left(c(f(\tilde{y}_1, \tilde{y}_2) - c)^2 - \frac{1}{3}(f(\tilde{y}_1, \tilde{y}_2) - c)^3 \right). \end{aligned} \quad (4.28)$$

From (4.26) we see that under $y_1 \rightarrow y_1 + 2\pi$ we have

$$\tilde{y}_1 \rightarrow \tilde{y}_1 + 2\pi f, \quad \tilde{y}_2 \rightarrow \tilde{y}_2 - 2m\pi(f^{-1}\tilde{y}_1 + \pi). \quad (4.29)$$

As the function f is originally independent of y_1 , it is invariant under these shifts (this can be double checked at the level of the quadratic equation solved by f^2 obtained from (4.26) on substituting $y_1 = \tilde{y}_1/f$ into the second equation). It follows that the monodromy of the dual potential is

$$\tilde{V} \rightarrow \tilde{V} - 4\pi\tilde{y}_1 + 2\pi cf^{-1}\tilde{y}_1 - 4\pi^2 f + 2\pi^2 c, \quad (4.30)$$

which matches what one gets from the general expression (4.20). To emphasise the unpleasant nature of this transformation, we write it more explicitly and in terms of complex coordinates as:

$$\begin{aligned} \tilde{V} \rightarrow \tilde{V} - 4\pi(\tilde{w} + \bar{\tilde{w}}) + \frac{2\pi c(\tilde{w} + \bar{\tilde{w}})}{\sqrt{m \left[\tilde{z} + \bar{\tilde{z}} + \frac{c^2}{2m} \pm \sqrt{(\tilde{z} + \bar{\tilde{z}})^2 + (\tilde{w} + \bar{\tilde{w}})^2} \right]}} \\ - 4\pi^2 \sqrt{m \left[\tilde{z} + \bar{\tilde{z}} + \frac{c^2}{2m} \pm \sqrt{(\tilde{z} + \bar{\tilde{z}})^2 + (\tilde{w} + \bar{\tilde{w}})^2} \right]} + 2\pi^2 c \end{aligned} \quad (4.31)$$

In terms of the superfields \tilde{z} and \tilde{w} , this is not a generalised Kähler transformation. This reflects the non-geometric nature of the background. We could rewrite in terms of the original superfields that

$$\begin{aligned} \tilde{V}(\tilde{z}, \tilde{w}) \rightarrow \tilde{V}(\tilde{z}, \tilde{w}) - 4\pi(c + m(w + \bar{w}))(z + \bar{z}) \\ + 2\pi c(w + \bar{w}) - 4\pi^2(c + m(w + \bar{w})) + 2\pi^2 c \end{aligned} \quad (4.32)$$

which would be a generalised Kähler transformation if we had access to the dual description involving z and w . However, strictly speaking this requires going a doubled formalism.

Accordingly, we now construct the doubled potential $\mathbb{V} = \frac{1}{2}(V^{(c)} + V^{(t)})$ using the two partial duals:

$$\begin{aligned} V^{(c)} &= \frac{1}{2} \left(f(y_1, \tilde{y}_2) y_1^2 - \frac{1}{m^2} \left(c(f(y_1, \tilde{y}_2) - c)^2 + \frac{1}{3}(f(y_1, \tilde{y}_2) - c)^3 \right) \right) \\ &\quad + \frac{1}{m} \tilde{y}_2 (f(y_1, \tilde{y}_2) - c), \\ V^{(t)} &= -\frac{1}{2} \left(\frac{\tilde{y}_1^2}{c + m y_2} + c y_2^2 + \frac{1}{3} m y_2^3 \right), \end{aligned} \quad (4.33)$$

where

$$f(y_1, \tilde{y}_2)^2 = m^2 y_1^2 + c^2 + 2m \tilde{y}_2^2. \quad (4.34)$$

The monodromy transformation arises from the following shifts:

$$y_1 \rightarrow y_1 + 2\pi, \quad \tilde{y}_1 \rightarrow \tilde{y}_1 + 2\pi f(y_2), \quad \tilde{y}_2 \rightarrow \tilde{y}_2 - 2m\pi(y_1 + \pi), \quad (4.35)$$

in terms of which

$$\begin{aligned} V^{(c)} &\rightarrow V^{(c)} + 2\pi c(y_1 + \pi), \\ V^{(t)} &\rightarrow V^{(t)} - 2\pi(\tilde{y}_1 + \pi f(y_2)). \end{aligned} \quad (4.36)$$

It follows that \mathbb{V} shifts by a linear function of the coordinates which is a generalised Kähler transformation of the doubled-generalised potential. Once separating \tilde{V} from \mathbb{V} These transformation turn into (4.31) upon expressing dual coordinates in terms of the physical ones.

4.3.2 Codimension-2 T-fold

Next, we start with the codimension-2 NS5 solution for which an $\mathcal{N} = (2, 2)$ description is also available [18, 32].⁸ Let $u = y_1 + iy_2$ and let $f(u)$ be an arbitrary holomorphic function. Then let $V = \text{Im}(f(u))$. The metric and B -field have components

$$g_{11} = g_{22} = \text{Im}f'' , \quad B_{12} = -\text{Re}f'' , \quad (4.37)$$

where f'' denotes the second derivative of f . The double T-dual of this model will describe a T-fold if there is an initial monodromy giving a constant shift of the original B -field. To realise this, we view y_1 and y_2 as coordinates parametrising the \mathbb{R}^2 transverse space of a codimension-2 brane. We then consider functions which are not single-valued under rotations about the origin. In particular, if $f(e^{2\pi i}u) = f(u) + \frac{\gamma}{2}u^2$ then under such a rotation $V \rightarrow V + \gamma y_1 y_2$ and $B_{12} \rightarrow B_{12} - \gamma$. A particular choice could be

$$f(u) = -\frac{i\gamma}{4\pi}u^2 \left(\ln u - \frac{3}{2} \right) \Rightarrow f''(u) = -\frac{i\gamma}{2\pi} \ln u . \quad (4.38)$$

Writing $y_1 + iy_2 = re^{i\theta}$, the geometry is thus

$$ds^2 = -\frac{\gamma}{2\pi} \ln r \left(dr^2 + r^2 d\theta^2 + dx_1^2 + dx_2^2 \right) , \quad B = -\frac{\gamma}{2\pi} \theta r dr \wedge d\theta - \frac{\gamma}{2\pi} \theta dx_1 \wedge dx_2 . \quad (4.39)$$

This can be viewed as describing the geometry near a codimension-two NS5 brane. The dual T-fold geometry is:

$$\begin{aligned} \tilde{ds}^2 &= -\frac{\gamma}{2\pi} \ln r \left(dr^2 + r^2 d\theta^2 \right) + \frac{-\frac{\gamma}{2\pi} \ln r}{\left(\frac{\gamma}{2\pi} \ln r\right)^2 + \left(\frac{\gamma}{2\pi} \theta\right)^2} (d\tilde{x}_1^2 + d\tilde{x}_2^2) \\ \tilde{B} &= -\frac{\gamma}{2\pi} \theta r dr \wedge d\theta + \frac{\frac{\gamma}{2\pi} \theta}{\left(\frac{\gamma}{2\pi} \ln r\right)^2 + \left(\frac{\gamma}{2\pi} \theta\right)^2} d\tilde{x}_1 \wedge d\tilde{x}_2 \end{aligned} \quad (4.40)$$

which transforms by a non-geometric T-duality for $\theta \rightarrow \theta + 2\pi$.

Rather than explicitly determine the dual potentials in this case, we simply note that the general discussion at the start of this subsection allows us to infer that they will inherit monodromies under

$$\tilde{y}_1 \rightarrow \tilde{y}_1^* = \tilde{y}_1 + \gamma y_2 , \quad \tilde{y}_2 \rightarrow \tilde{y}_2^* = \tilde{y}_2 - \gamma y_1 , \quad (4.41)$$

in terms of which $V^{(c)}$ and $V^{(t)}$ are in fact invariant

$$V^{(c)}(y_1^*, y_2^*) = V^{(c)}(y_1, y_2) , \quad V^{(t)}(\tilde{y}_1^*, \tilde{y}_2^*) = V^{(t)}(\tilde{y}_1, \tilde{y}_2) \quad (4.42)$$

but

$$\tilde{V}(\tilde{y}_1^*, \tilde{y}_2^*) = \tilde{V}(\tilde{y}_1, \tilde{y}_2) - \gamma y_1(\tilde{y}_1, \tilde{y}_2) y_2(\tilde{y}_1, \tilde{y}_2) . \quad (4.43)$$

Accordingly while \tilde{V} in this case behaves poorly under the global shift leading to the T-duality identification, the doubled-generalised Kähler potential built from $V^{(c)}$ and $V^{(t)}$

⁸Doubled formulations of the codimension-2 NS5 to S^2 duality in terms of $\mathcal{N} = (4, 4)$ gauged linear sigma models in superspace were studied in [19–21].

will be well-behaved: it is invariant. Note in this case that the transformation of the doubled coordinates \mathbb{Y}^M leading to the monodromy is exactly that of a B -field shift in the original choice of polarisation with y_1 and y_2 as physical coordinates, and hence a bivector shift in the dual choice with \tilde{y}_1 and \tilde{y}_2 as physical coordinates.⁹

4.4 Semi-chiral geometry without semi-chirals

We now consider an example where we can describe a geometry that ordinarily requires semi-chiral superfields, without introducing semi-chiral superfields. The starting point is the $D = 2d$ -dimensional Kähler geometry of (2.39). In this case, the generalised Kähler potential is a genuine Kähler potential, $V = V(y^\alpha)$, which we suppose to depend only on the real parts of d chiral superfields. The doubled potential is then

$$\mathbb{V}(\mathbb{Y}) = \frac{1}{2}(V(y) + \tilde{V}(\tilde{y})), \tag{4.44}$$

where $\tilde{V}(\tilde{y}_a)$ is the Legendre dual of $V(y)$, and depends only on the real parts of d dual twisted chiral superfields. We recall that the constraints (3.18) can be written democratically as

$$\mathbb{Y}^M = \Theta^{MN} \partial_N \mathbb{V}, \quad \Theta^{MN} = 2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{4.45}$$

implying here

$$y^\alpha = -\tilde{V}^\alpha, \quad \tilde{y}_\alpha = V_\alpha. \tag{4.46}$$

The components of the original Kähler metric are $g_{\alpha\beta} = V_{\alpha\beta}$ and we have $\tilde{V}^{\alpha\beta} = -(V^{-1})^{\alpha\beta}$.

Now we consider an $O(d, d)$ transformation according to the prescription of section 3.4, with $\mathbb{X}'^M = \mathcal{O}^{-1M}{}_N \mathbb{X}^N$, $\mathbb{Y}'^M = \mathcal{O}^{-1M}{}_N \mathbb{Y}^N$, and the bivector transformation given by

$$\mathcal{O}^{-1M}{}_N = \begin{pmatrix} 1 & -\beta \\ 0 & 1 \end{pmatrix}, \quad \beta \equiv \lambda \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.47}$$

Explicitly, this is (where possible we drop the indices in expressions below):

$$x' = x - \beta \tilde{x}, \quad \tilde{x}' = \tilde{x}, \tag{4.48}$$

$$y' = y - \beta \tilde{y}, \quad \tilde{y}' = \tilde{y}, \tag{4.49}$$

with the doubled potential transforming as a scalar, hence

$$\mathbb{V}'(\mathbb{Y}') = \frac{1}{2}(V(y' + \beta \tilde{y}') + \tilde{V}(\tilde{y}')). \tag{4.50}$$

The transformed constraint can be written as

$$\mathbb{Y}'^M = \Theta'^{MN} \partial'_N \mathbb{V}', \tag{4.51}$$

⁹This is a bivector shift acting on a geometry described by one chiral and one twisted chiral superfield, which preserves the fact that the complex structures commute. This is unlike the case of the bivector shift acting on a purely chiral or purely twisted chiral (i.e. Kähler) geometry, which appears in the next example, albeit not as a T-fold. It would be interesting to find explicit examples of T-folds which map from BiLP to non-BiLP geometries.

with

$$\Theta'^{MN} = (\mathcal{O}^{-1}\Theta\mathcal{O}^{-T})^{MN} = 2 \begin{pmatrix} -2\beta & -1 \\ 1 & 0 \end{pmatrix} \quad (4.52)$$

hence

$$y'^\alpha = -\tilde{V}^\alpha - \beta^{\alpha\beta}V_\beta, \quad \tilde{y}'_\alpha = V_\alpha \quad (4.53)$$

Using (4.49) this can be checked to be equivalent to the original constraints (4.46).

The initial set of superfields obey an equation of the form (3.44) with over-doubled complex structures \mathbb{J}_\pm given by

$$\mathbb{J}_+ = \begin{pmatrix} 0 & 1_{2d} \\ -1_{2d} & 0 \end{pmatrix}, \quad \mathbb{J}_- = \begin{pmatrix} 0 & 0 & 1_d & 0 \\ 0 & 0 & 0 & -1_d \\ -1_d & 0 & 0 & 0 \\ 0 & 1_d & 0 & 0 \end{pmatrix}. \quad (4.54)$$

After the $O(d, d)$ transformation we obtain:

$$\mathbb{J}'_+ = \begin{pmatrix} 0 & 1_{2d} \\ -1_{2d} & 0 \end{pmatrix}, \quad \mathbb{J}'_- = \begin{pmatrix} 0 & 0 & 1_d & 2\beta \\ 0 & 0 & 0 & -1_d \\ -1_d & -2\beta & 0 & 0 \\ 0 & 1_d & 0 & 0 \end{pmatrix}. \quad (4.55)$$

This means that while $(\tilde{x}'_\alpha, \tilde{y}'_\alpha)$ on their own obey the defining conditions of twisted chiral superfields, once we look also at (x'^α, y'^α) we have

$$\hat{D}_+x' = +D_+y', \quad \hat{D}_+y' = -D_+x', \quad (4.56)$$

$$\hat{D}_-x' = +D_-y' + 2\beta D_- \tilde{x}', \quad \hat{D}_-y' = -D_-x' - 2\beta D_- \tilde{x}'. \quad (4.57)$$

In particular there is a mixing between the physical and dual superfields!

To make sense of this, we follow the general discussion in section 3.4.4 and take into account the constraints obeyed by the derivatives of the superfields. Taking derivatives of (4.53) leads as expected to

$$D_\pm \tilde{y}' = (g' - b')D_\pm y', \quad D_\pm \tilde{x}' = \pm(g' \mp b')D_\pm x', \quad (4.58)$$

with the correct transformations of the metric and B -field:

$$g' \pm b' = (g^{-1} \pm \beta)^{-1}. \quad (4.59)$$

This agrees with (2.46) on noting that the B -field components restricted to non-isometric directions are a total derivative.

Using these constraints in the non-standard relationships (4.57) to eliminate \tilde{x}_α and \tilde{y}_α , we obtain (after some algebraic manipulation):

$$\hat{D}_-x' = (1 + \beta g)(1 - \beta g)^{-1}D_- \tilde{x}', \quad \hat{D}_-y' = -(1 - \beta g)(1 + \beta g)^{-1}D_-x'. \quad (4.60)$$

From (4.56) and (4.60) we can read off the spacetime complex structures J'_\pm for the coordinates (x', y') . They match those obtained in (2.45) by using a bivector $O(d, d)$

transformation as well as a coordinate transformation of the non-isometric directions. This shows that the somewhat peculiar $O(d, d)$ transformations of our doubled formulation correctly reproduce the usual T-duality, once the right coordinates have been identified to match with.

This is strongly reminiscent of the work in [33] where a potential containing semi-chiral fields was obtained as a (local) quotient of a space described solely in terms of chiral and twisted chiral superfields. This approach allows a model containing n semi-chiral multiplets to be viewed as the quotient space of a model where the semi-chiral subspace was doubled and described in terms of n chiral and n twisted chiral superfields.

Acknowledgments

The authors acknowledge the support of the FWO-Vlaanderen through the project G006119N, as well as that of the Vrije Universiteit Brussel through the Strategic Research Program “High-Energy Physics”. CB is supported by an FWO-Vlaanderen Postdoctoral Fellowship. DCT is supported by The Royal Society through a University Research Fellowship *Generalised Dualities in String Theory and Holography* URF 150185 and in part by STFC grant ST/P00055X/1. We would like to thank Alex Arvanitakis, Sibylle Driezen, Martin Roček, David Svoboda and Rikard von Unge for discussions. CB, AS and DCT would like to thank the Simons Center for Geometry and Physics for providing a stimulating environment during the conference “Generalized Geometry and T-dualities” where some early findings of this project were presented [34].

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A Conventions

A.1 Worldsheet superspace conventions

In this section we lay down two dimensional superspace conventions which are being used throughout the text. We follow the conventions used in [29] where more details are provided.

The worldsheet coordinates are denoted by τ, σ , using which we can define light-cone coordinates as

$$\sigma^{\pm} = \tau + \sigma, \quad \sigma^{\mp} = \tau - \sigma. \tag{A.1}$$

The $\mathcal{N} = (1, 1)$ fermionic coordinates are denoted by θ^+ and θ^- and the corresponding super derivatives are

$$D_+ = \partial_{\theta^+} - \frac{i}{2}\theta^+\partial_{\sigma^+}, \quad D_- = \partial_{\theta^-} - \frac{i}{2}\theta^-\partial_{\sigma^-} \tag{A.2}$$

which satisfy

$$D_+^2 = -\frac{i}{2}\partial_{\sigma^+}, \quad D_-^2 = -\frac{i}{2}\partial_{\sigma^-}, \quad \{D_+, D_-\} = 0. \tag{A.3}$$

The $\mathcal{N} = (1, 1)$ integration measure is explicitly given by,

$$\int d^2\sigma d^2\theta = \int d\tau d\sigma D_+ D_- | \tag{A.4}$$

Passing from $\mathcal{N} = (1, 1)$ to $\mathcal{N} = (2, 2)$ superspace requires the introduction of two more real fermionic coordinates $\hat{\theta}^+$ and $\hat{\theta}^-$ where the corresponding fermionic derivatives satisfy,

$$\hat{D}_+^2 = -\frac{i}{2} \partial_+, \quad \hat{D}_-^2 = -\frac{i}{2} \partial_-, \quad (\text{A.5})$$

whereas all the anticommutators with the $(1, 1)$ superderivative vanish. The $\mathcal{N} = (2, 2)$ integration measure is extended to,

$$\int d^2\sigma d^2\theta d^2\hat{\theta} = \int d\tau d\sigma D_+ D_- \hat{D}_+ \hat{D}_- | \quad (\text{A.6})$$

In order to effectively work with the constraint $(2, 2)$ superfields we introduce a complex basis as

$$\mathbb{D}_\pm \equiv \hat{D}_\pm + i D_\pm \quad \bar{\mathbb{D}}_\pm \equiv \hat{D}_\pm - i D_\pm \quad (\text{A.7})$$

satisfying

$$\{\mathbb{D}_+, \bar{\mathbb{D}}_+\} = -2i \partial_+, \quad \{\mathbb{D}_-, \bar{\mathbb{D}}_-\} = -2i \partial_-. \quad (\text{A.8})$$

A.2 $\mathcal{N} = (2, 2)$ superfields

In extended superspace formulation fields have to be supplemented with constraints, here we review some of the most commonly used superfields in $N = (2, 2)$ superspace occurring in the text.

1. Chiral field is a complex superfield z satisfying the following constraints

$$\bar{\mathbb{D}}_\pm z = \mathbb{D}_\pm \bar{z} = 0. \quad (\text{A.9})$$

In terms of $N = (1, 1)$ superfields z_0 and \bar{z}_0 , this becomes,

$$z = z_0 + i \hat{\theta}^+ D_+ z_0 + i \hat{\theta}^- D_- z_0 + \hat{\theta}^+ \hat{\theta}^- D_+ D_- z_0, \quad (\text{A.10})$$

$$\bar{z} = \bar{z}_0 - i \hat{\theta}^+ D_+ \bar{z}_0 - i \hat{\theta}^- D_- \bar{z}_0 + \hat{\theta}^+ \hat{\theta}^- D_+ D_- \bar{z}_0. \quad (\text{A.11})$$

2. Twisted chiral field is a complex superfield w (see e.g. [1]) satisfying

$$\bar{\mathbb{D}}_+ w = \mathbb{D}_- w = \mathbb{D}_+ \bar{w} = \bar{\mathbb{D}}_- \bar{w} = 0. \quad (\text{A.12})$$

In terms of $N = (1, 1)$ superfields w_0 and \bar{w}_0 , this becomes,

$$w = w_0 + i \hat{\theta}^+ D_+ w_0 - i \hat{\theta}^- D_- w_0 - \hat{\theta}^+ \hat{\theta}^- D_+ D_- w_0, \quad (\text{A.13})$$

$$\bar{w} = \bar{w}_0 - i \hat{\theta}^+ D_+ \bar{w}_0 + i \hat{\theta}^- D_- \bar{w}_0 - \hat{\theta}^+ \hat{\theta}^- D_+ D_- \bar{w}_0. \quad (\text{A.14})$$

3. Semi-chiral multiplet l, \bar{l}, r and \bar{r} [24]:

$$\bar{\mathbb{D}}_+ l = \mathbb{D}_+ \bar{l} = \bar{\mathbb{D}}_- r = \mathbb{D}_- \bar{r} = 0. \quad (\text{A.15})$$

In terms of $N = (1, 1)$ superfields r_0 and \bar{r}_0 , ψ_- and $\bar{\psi}_-$, l_0 and \bar{l}_0 , ϕ_+ and $\bar{\phi}_+$, this becomes,

$$l = l_0 + i \hat{\theta}^+ D_+ l_0 + i \hat{\theta}^- \psi_- + \hat{\theta}^+ \hat{\theta}^- D_+ \psi_-, \quad (\text{A.16})$$

$$\bar{l} = \bar{l}_0 - i \hat{\theta}^+ D_+ \bar{l}_0 + i \hat{\theta}^- \bar{\psi}_- - \hat{\theta}^+ \hat{\theta}^- D_+ \bar{\psi}_-, \quad (\text{A.17})$$

$$r = r_0 + i \hat{\theta}^+ \phi_+ + i \hat{\theta}^- D_- r_0 - \hat{\theta}^+ \hat{\theta}^- D_- \phi_+, \quad (\text{A.18})$$

$$\bar{r} = \bar{r}_0 + i \hat{\theta}^+ \bar{\phi}_+ - i \hat{\theta}^- \bar{D}_- r_0 + \hat{\theta}^+ \hat{\theta}^- D_- \bar{\phi}_+. \quad (\text{A.19})$$

A.3 Complex structures in adapted coordinates

The passage from $(1, 1)$ to $(2, 2)$ superspace requires a choice of a target space complex structure. However, due to the passage to adapted coordinates this dependence is usually hidden. Here we write out the dependence of the $(2, 2)$ constraints on the complex structure as well as the constraints on the dual fields.

In order to do this we rewrite the above formulas in real coordinates $X = (x^\alpha, x^\mu, y^\alpha, y^\mu)$. The $N = (2, 2)$ constraint in which the complex structures appear explicitly then reads

$$\hat{D}_\pm X^I = (J_\pm)^I{}_J D_\pm X, \quad (\text{A.20})$$

and following our superfield conventions (summarised in appendix A) we have in these real coordinates that

$$J_+ = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}, \quad J_- = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}, \quad \sigma \equiv \begin{pmatrix} I_{n_c} & 0 \\ 0 & -I_{n_t} \end{pmatrix}. \quad (\text{A.21})$$

Explicitly, this means

$$\hat{D}_\pm x^\alpha = +D_\pm y^\alpha, \quad \hat{D}_\pm y^\alpha = -D_\pm x^\alpha, \quad \hat{D}_\pm x^\mu = \pm D_\pm y^\mu, \quad \hat{D}_\pm y^\mu = \mp D_\pm x^\mu. \quad (\text{A.22})$$

For dual superfields

$$\tilde{w}_\alpha = \frac{1}{2}(\tilde{y}_\alpha + i\tilde{x}_\alpha), \quad \tilde{z}_\mu = \frac{1}{2}(\tilde{y}_\mu + i\tilde{x}_\mu), \quad (\text{A.23})$$

we have

$$\tilde{J}_+ = -J_+^t = J_+, \quad \tilde{J}_- = +J_-^t = -J_-, \quad (\text{A.24})$$

which appear in the constraints $\hat{D}_\pm \tilde{X} = \tilde{J}_\pm D_\pm \tilde{X}$ with $\tilde{X} = (\tilde{x}_\alpha, \tilde{x}_\mu, \tilde{y}_\alpha, \tilde{y}_\mu)$. Explicitly,

$$\hat{D}_\pm \tilde{x}_\alpha = \pm D_\pm \tilde{y}_\alpha, \quad \hat{D}_\pm \tilde{y}_\alpha = \mp D_\pm \tilde{x}_\alpha, \quad \hat{D}_\pm \tilde{x}_\mu = +D_\pm \tilde{y}_\mu, \quad \hat{D}_\pm \tilde{y}_\mu = -D_\pm \tilde{x}_\mu. \quad (\text{A.25})$$

In these adapted coordinates it is easy to extract expressions for the action in $\mathcal{N} = (1, 1)$ superspace since

$$\begin{aligned} \mathcal{L} &= \hat{D}_+ \hat{D}_- V(X) | \\ &= \partial_{IJ} V (J_+ D_+ X)^I (J_- D_- X)^J + \partial_J V (J_- J_+)^J{}_K D_+ D_- X^K \\ &= (J_+ D_+ X)^I V_{IJ} (J_- D_- X)^J - D_+ X^I V_{IJ} (J_- J_+)^J{}_K D_- X^K. \end{aligned} \quad (\text{A.26})$$

For the case where we have isometries such that $V = V(y^\alpha, y^\mu)$ then the Hessian matrix becomes

$$\text{Hess} = V_{IJ} = \begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix}, \quad M = \begin{pmatrix} V_{\alpha\beta} & V_{\alpha\mu} \\ V_{\mu\alpha} & V_{\mu\nu} \end{pmatrix}. \quad (\text{A.27})$$

Then we find

$$E = G + B = J_+^T \text{Hess} J_- - \text{Hess} J_+ J_- = \begin{pmatrix} M\sigma & 0 \\ 0 & M\sigma \end{pmatrix}. \quad (\text{A.28})$$

B Calculation of monodromies of Legendre transformed potentials

This appendix serves to verify the formula (4.18) for the monodromy of the Legendre transformed potential $V^{(c)}$. The calculations for $V^{(t)}$ and \tilde{V} are similar. The monodromy is inherited from the transformations $y_1^* = y_1 + n_1$ and $y_2^* = y_2 + n_2$ of the original coordinates, in terms of which $V(y_1^*, y_2^*)$ is given by (4.15). Given the definitions of the dual coordinates we have $\tilde{y}_1^* = \tilde{y}_1 + \alpha_1 + \gamma y_2$, $\tilde{y}_2^* = \tilde{y}_2 - \alpha_2 - \gamma y_1$. We write $V^{(c)}$ as

$$V^{(c)}(y_1, \tilde{y}_2) = V(y_1, \hat{y}_2) + \hat{y}_2 \tilde{y}_2 \tag{B.1}$$

where $\hat{y}_2 = \hat{y}_2(y_1, \tilde{y}_2)$ by definition is derived from the equation $\tilde{y}_2 = -V_2$. Then, $\hat{y}_2^* = \hat{y}_2(y_1 + n_1, \tilde{y}_2 - \alpha_2 - \gamma y_1)$ solves

$$\tilde{y}_2 - \alpha_2 - \gamma y_1 = -V_2(y_1 + n_1, y_2 - n_2 + n_2) = -V_2(y_1, y_2 - n_2), \tag{B.2}$$

hence $\hat{y}_2^* = \hat{y}_2 + n_2$. It follows that

$$\begin{aligned} V^{(c)}(y_1^*, \tilde{y}_2^*) &= V(y_1^*, \hat{y}_2^*) + \hat{y}_2^* \tilde{y}_2^* \\ &= V(y_1, \hat{y}_2) + \alpha_1 y_1 + \alpha_2 \hat{y}_2 + \gamma y_1 \hat{y}_2 + \delta + (\hat{y}_2 + n_2)(\tilde{y}_2 - \alpha_2 - \gamma y_1). \end{aligned} \tag{B.3}$$

Multiplying out and cancelling leads to (4.18).

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