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On Type IIA AdS_3 solutions and massive GK geometries

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Abstract: We give necessary and sufficient conditions for warped AdS_3 (and $Mink_3$) solutions of Type II supergravities to preserve $\mathcal{N} = (2,0)$ supersymmetry, in terms of geometric conditions on their internal space M_7. Such solutions possess a canonical ten-dimensional Killing vector that can be either time-like or null. In this work we classify the null case in massive Type IIA supergravity which necessitates that M_7 decomposes as a circle fibration over a six-dimensional base with orthogonal SU(2)-structure containing a complex four-manifold. We narrow our focus to solutions for which M_7 becomes $\mathbb{T}^2$ fibred over a foliation of a Kähler manifold over an interval. We find a class of solutions which are the massive Type IIA version of GK geometries and present an extremal problem which computes the central charge of the solution using just topology. Finally, we present geometric conditions for AdS_3 solutions to preserve arbitrary extended chiral supersymmetry.

Keywords: AdS-CFT Correspondence, Flux Compactifications

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Contents

1 Introduction

Two-dimensional conformal field theories (CFTs) hold a special place in the landscape of CFTs. They feature prominently in string theory, describing the world-sheet dynamics of strings and have been studied extensively in the literature. As they admit an infinite dimensional conformal algebra they are heavily constrained and in certain cases completely solvable. When 2d CFTs preserve (at least) $\mathcal{N} = (2, 0)$ supersymmetry, c-extremization [1, 2] computes the central charge and R-charges (thus also conformal dimensions of certain operators) of the strongly coupled IR fixed point using only UV data. The key observation is that 2d $\mathcal{N} = (2, 0)$ SCFTs have a U(1) R-symmetry. Though the R-symmetry may mix with flavour symmetries along the RG flow, the exact R-symmetry in the IR extremizes the central charge viewed as a functional of possible R-symmetry choices, at the IR fixed point. Thus, from knowing just UV data, and with some mild assumptions in tow, one can obtain information about the IR fixed point. This is a direct 2d analogue of a-maximisation for 4d SCFTs [3].

As one may expect given AdS/CFT, in gravity, there are geometric extremal problems dual to the field-theoretic ones. The geometric dual of a-maximization was derived in [4, 5] for $AdS_5\times SE_5$ geometries and the geometric dual of c-extremization was found in [6] for so called "GK" geometries [7–9]. The geometric extremal problem was later extended in [10] to the F-theoretic extension of GK geometries of [11]. Further advances in the geometric dual of c-extremization for the GK geometry class have been made in [12–17].

However, whereas a-maximization and c-extremization work for generic field theories obeying certain mild assumptions, the geometric extremal problems are only defined for certain classes of solutions and there are holographic SCFTs whose duals are not contained within these classes. It is natural to conjecture that there is an extremal problem for any AdS_3 solution in supergravity with at least $\mathcal{N}=(2,0)$ supersymmetry which is the geometric dual of c-extremization for the putative dual field theory.[1] It is therefore an interesting problem to extend these geometric extremal problems to cover the full complement of holographic SCFTs. One of the key results needed for progress on these geometric extremal problems was a thorough understanding of the underlying geometry of the system. For [6] the underlying geometries are GK geometries which were first studied in [7] and arise from classifying AdS_3 solutions of Type IIB supergravity with 5-form flux and an SU(3)-structure. Therefore, extending the classification of all AdS solutions preserving fixed amounts of supersymmetry is a necessary requirement for making progress in extending these geometric extremal problems to further classes of geometries.[2] There has been a lot of interest in classifying AdS_3 solutions with various amounts of supersymmetry [7, 20–59][3] yet the analysis is still incomplete and there are interesting AdS_3 geometries still to be classified and constructed; this paper tightens the noose on classifying all AdS_3 solutions in Type II supergravity.

Another reason for interest in AdS_3 solutions stems from their presence in the near-horizon limit of black strings. The power of the near-horizon geometry is that many of the interesting observables of the black string may be obtained from the near-horizon rather than the asymptotic geometry. The entropy, angular momentum, electric and magnetic charges of a black string can all be obtained this way. However, the electrostatic potential and angular velocity require UV knowledge which is washed out in the near-horizon limit. As such if one restricts to understanding any of the first class of observables one needs only the near-horizon limit of the black string and not the full interpolating flow. In addition, the 2d CFT dual to the AdS_3 near-horizon gives a microscopic description for the Bekenstein-Hawking entropy of the black strings.[4] The first example of computing the microstates of a black string using the near-horizon geometry was performed in the mid 90's by Strominger and Vafa [65]. This was later extended to the MSW string [66] in

[1] Similar comments of course apply for any AdS solution where there is a dual field-theoretic extremal problem.

[2] A complementary approach to these geometric duals studies the extremal problem from gauged supergravity, see [18] for a-maximization and [19] for c-extremization. From this perspective it is also interesting to classify supersymmetric AdS_3 solutions and then to obtain consistent truncations which uplift on these geometries.

[3] See [60–64] for examples of further studies.

[4] There can be some subtleties regarding the contribution of "hair" to the entropy, which the 2d SCFT does not see, however this is a subleading contribution to the entropy and can therefore be neglected in most examples.

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M-theory and in F-theory in [67]. More recently there have been many advancements in studying the near-horizon of black strings. There has been recent interest in black strings with non-constant curvature horizons: in [68–70] black strings with spindle horizons were investigated whilst in [57, 71] black strings with disc horizons were studied. In a tangential direction, there have been further advancements in studying black strings in F-theory probing various four-dimensional asymptotically flat spaces [47, 55, 72–74] using their AdS_3 near-horizon geometries.

AdS_3 vacua are also interesting when considering whether AdS vacua can be parametrically scale separated. One can construct minimally supersymmetric AdS_3 vacua in Type II supergravity of the form $\mathrm{AdS}_3\times\mathrm{M}_7$ with M_7 a G_2 manifold, see for example [46, 54, 75]. Since the CFT dual is two-dimensional and benefits from the infinite dimensional conformal symmetry, one may hope that this extra control allows one to answer this question conclusively. With suitable projection conditions imposed on the $\mathcal{N}=(2,0)$ solutions discussed in this work one is able to obtain solutions preserving only $\mathcal{N}=(1,0)$ supersymmetry which may be candidates for solutions with scale separation. It would be interesting to address this point in the future.

The layout of the paper is as follows:

In section 2 we present necessary and sufficient conditions for a warped $\mathrm{AdS}_3\times\mathrm{M}_7$ solution of Type II supergravity to preserve $\mathcal{N}=(2,0)$ supersymmetry in terms of geometric conditions on the internal space M_7. These conditions depend on the inverse AdS radius m: for $m\neq 0$ we find that M_7 necessarily decomposes as a U(1) fibration over a six-dimensional base M_6, with the U(1) realising the R-symmetry of the superconformal group $\mathrm{OSp}(2|2)$ as expected. When $m=0$ conditions for warped $\mathcal{N}=2$ three-dimensional Minkowski (Mink_3) vacua are recovered; while these still generically contain a U(1) isometry, this is no longer an R-symmetry and for restricted classes need to be present. For AdS_3 solutions specifically ($m\neq 0$) we find that solutions fall into two classes depending on whether a canonical ten-dimensional Killing vector is time-like or null. This section is supplemented by the technical appendices A and B.

In section 3, and for the rest of the paper, we narrow are focus to solution in massive Type IIA supergravity supporting a null Killing vector. For such solutions M_6 necessarily supports an orthogonal SU(2)-structure, decomposing in terms of a complex vector and a four-manifold which in this case is complex. We introduce local coordinates for the complex vector and perform an analysis in terms of the SU(2)-structure torsion classes, reviewed in appendix C.

In section 4, in order to find solutions, we impose the existence of an additional Killing vector. This allows us to further refine the necessary and sufficient conditions for a supersymmetric solution. Inserting an ansatz for the SU(2)-structure manifold of the form of the product of two warped Riemann surfaces we find three classes of solutions. Buoyed by finding explicit solutions, in section 5 we study a more general ansatz for the SU(2)-structure manifold consisting of a warped Kähler manifold. We find a class of geometries which are the massive extension of GK geometries in five dimensions. These solutions are determined by the same master equation as GK geometries but contain D8-branes. In section 5.2 we show that one can define an extremal problem for determining the central charge of

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the solution using just the topology. This is the first example of an extremal problem for solutions of massive Type IIA supergravity.

Finally, in section 6 we use the $\mathcal{N}=(2,0)$ supersymmetry conditions in appendix B to derive necessary and sufficient conditions for AdS_3 solutions of Type II supergravity to preserve arbitrary extended chiral supersymmetry. We find that an $\mathcal{N}=(n,0)$ solution for $n\geq 2$ necessarily comes equipped with an anti-symmetric matrix of Killing vectors that should span the R-symmetry of whatever superconformal algebra a solution realises (there are many options [82]). The various $d=2$ superconformal algebras can be classified in terms of their R-symmetry and an associated representation [83].[5] We make a conjecture that precisely relates the R-symmetry and representation to the anti-symmetric matrix of Killing vectors.

2 Supersymmetry equations

We consider a bosonic background of Type II supergravity that preserves the symmetries of three-dimensional anti-de Sitter spacetime AdS_3. The ten-dimensional metric takes the form of a warped product of a metric on AdS_3 and a Riemanian metric on a seven-dimensional manifold M_7:

$$ds^2 = e^{2A} ds^2(\mathrm{AdS}_3) + ds^2(\mathrm{M}_7)\,, \tag{2.1}$$

where the warp factor e^{2A} is a function on M_7 and we assume that $\mathrm{Ricci}(\mathrm{AdS}_3) = -2m^2 g(\mathrm{AdS}_3)$. The NS-NS three-form, $H^{(10\mathrm{d})}$ and the R-R fluxes F take the form

$$H^{(10\mathrm{d})} = e^{3A} h_0 \mathrm{vol}(\mathrm{AdS}_3) + H\,, \quad F = f_\pm + e^{3A}\mathrm{vol}(\mathrm{AdS}_3)\wedge \star_7 \lambda(f_\pm)\,. \tag{2.2}$$

Here, H has support on M_7, and the Bianchi identity for $H^{(10\mathrm{d})}$ enforces $e^{3A}h_0$ to be a constant. F is a polyform, the sum of the p-form R-R field-strengths with p even for Type IIA supergravity and p odd for Type IIB. The forms $f_\pm$ have support on M_7, with the upper sign corresponding to Type IIA and the lower to Type IIB. In particular, $f_+ = \sum_{p=0}^3 f_{2p}$ and $f_- = \sum_{p=0}^3 f_{2p+1}$. The operator λ acts on the p-component of $f_\pm$ as

$$\lambda(f_\pm|_p) = (-1)^{\lfloor \frac{p}{2}\rfloor} f_\pm \Big|_p\,, \tag{2.3}$$

and $\star_7$ is the Hodge operator involving the metric on M_7. Finally, the dilaton Φ is a function on M_7.

The conditions stemming from requiring that this background preserves $\mathcal{N}=(2,0)$ supersymmetry are derived in appendix B. They involve two doublets of Spin(7) Majorana spinors, χ_1^I and χ_2^I, $I=1,2$ under the R-symmetry $\mathfrak{so}(2)\simeq\mathfrak{u}(1)$ of the supersymmetry algebra. Without loss of generality (see appendix) they are taken to satisfy

$$\chi_1^{I\dagger}\chi_1^J + \chi_2^{I\dagger}\chi_2^J = 2e^A\delta^{IJ}\,, \qquad \chi_1^{I\dagger}\chi_1^J - \chi_2^{I\dagger}\chi_2^J = ce^{-A}\delta^{IJ}\,, \tag{2.4}$$

for c a constant and δ^{IJ} the Kronecker delta.

[5] Strictly speaking this applies to the simple Lie superalgebras, but as these are in one-to-one correspondence with the algebras embeddable into supergravities with an AdS_3 factor, this subtelty is unimportant.

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The supersymmetry constraints can then be expressed in terms of two 1-forms and a set of bi-spinors. The one 1-forms are given by

$$\xi = -i\left(\chi_1^1\gamma_a\chi_1^2 \mp \chi_2^1\gamma_a\chi_2^2\right)\mathrm{e}^a\,,\quad \tilde{\xi} = -i\left(\chi_1^1\gamma_a\chi_1^2 \pm \chi_2^1\gamma_a\chi_2^2\right)\mathrm{e}^a\,,\qquad \langle\xi,\tilde{\xi}\rangle = 2c\,, \tag{2.5}$$

where γ^a, $a = 1,2,\ldots,7$ are the generators of Cliff(7) and e^a gives an orthonormal frame on $T^*\mathrm{M}_7$. Note that the 1-form ξ cannot be set to zero globally when $m \neq 0$ without reducing supersymmetry to $\mathcal{N} = (1,0)$. The set of bi-spinors is

$$\Psi^{IJ} \equiv \chi_1^I \otimes \chi_2^{J\dagger} = \frac{e^A}{2}\left(\delta^{IJ}\Psi^{(0)} + \sigma_1^{IJ}\Psi^{(1)} + i\sigma_2^{IJ}\Psi^{(2)} + \sigma_3^{IJ}\Psi^{(3)}\right), \tag{2.6}$$

where we have expanded in terms of δ^{IJ} and the Pauli matrices $(\sigma_1,\sigma_2,\sigma_3)$. They can be further decomposed into even, denoted by a plus subscript, and odd, denoted by a minus subscript, real parts as

$$\Psi^{(0,1,2,3)} = \Psi_+^{(0,1,2,3)} + i\Psi_-^{(0,1,2,3)}\,. \tag{2.7}$$

The supersymmetry constraints are

$$d\tilde{\xi} = \iota_\xi H \qquad e^{3A}h_0 = -mc\,, \tag{2.8a}$$

$$d_H\left(e^{2A-\Phi}\left(\Psi_\mp^{(1)} + i\Psi_\mp^{(3)}\right)\right) = 0\,, \tag{2.8b}$$

$$d_H\left(e^{3A-\Phi}\left(\Psi_\pm^{(1)} + i\Psi_\pm^{(3)}\right)\right) \mp 2me^{2A-\Phi}\left(\Psi_\mp^{(1)} + i\Psi_\mp^{(3)}\right) = 0\,, \tag{2.8c}$$

$$d_H\left(e^{3A-\Phi}\Psi_\pm^{(0)}\right) \mp 2me^{2A-\Phi}\Psi_\mp^{(0)} = \frac{1}{4}e^{3A}\star_7\lambda f_\pm\,, \tag{2.8d}$$

$$d_H\left(e^{A-\Phi}\Psi_\pm^{(2)}\right) = \frac{1}{8}\left(\tilde{\xi}\wedge + \iota_\xi\right)f_\pm\,, \tag{2.8e}$$

$$e^A\left(\Psi_\mp^{(0)}, f_\pm\right) = \mp\left(m + \frac{1}{2}e^{-A}ch_0\right)e^{-\Phi}\mathrm{vol}\left(\mathrm{M}_7\right),. \tag{2.8f}$$

Note there are further conditions implied by this, for instance acting on (2.8e) with $\tilde{\xi}\wedge + \iota_\xi$ leads to

$$d_H\left(e^{2A-\Phi}\Psi_\mp^{(0)}\right) = \mp\frac{1}{8}cf_\pm\,, \tag{2.9}$$

from which it follows that in Type IIA $cf_0 = 0$ and that in general pure R-R sources are only possible when $c = 0$. The vector dual to ξ generates an isometry and in fact is a symmetry of the whole background which corresponds to the R-symmetry. Under this symmetry the bi-spinors transform as

$$\mathcal{L}_\xi\Psi^{(0)} = \mathcal{L}_\xi\Psi^{(2)} = 0, \qquad \mathcal{L}_\xi\left(\Psi^{(1)} + i\Psi^{(3)}\right) = -4im\left(\Psi^{(1)} + i\Psi^{(3)}\right). \tag{2.10}$$

The background also possesses a symmetry generated by the vector dual of a ten-dimensional 1-form

$$K = \frac{1}{32}\left(2e^{2A}k - \xi\right), \tag{2.11}$$

where k is the 1-form dual to a time-like Killing vector on AdS$_3$, see appendix A. Two classes of backgrounds can be dissociated depending on whether K is time-like or null. The latter case occurs for $||\xi|| = 2e^A$.

All the conditions presented so far hold for $m = 0$, for which AdS$_3$ becomes Mink$_3$. Generically $\xi^a\partial_a$ is now an isometry with respect to which the spinors and bi-linears are singlets. Clearly there exist $\mathcal{N} = 2$ Mink$_3$ solutions for which no such isometry exists, the flat space D2-brane with no rotational invariance in its co-dimensions for instance. However for Mink$_3$ it is now possible to fix $\xi = 0$ so there is no isometry, this also implies $c = 0$ (the converse only implies $\langle\xi, \tilde{\xi}\rangle = 0$). This gives a concrete physical interpretation for $c \neq 0$ in this case, i.e.

$$\mathcal{N} = 2 \text{ Mink}_3: \; c \neq 0 \qquad \Rightarrow \qquad \text{U(1) flavour isometry } \xi^a\partial_a\,. \tag{2.12}$$

It is extremely common that classes of solutions with necessary flavour isometries can be mapped to more general classes for which this isometry is not necessary after T-duality. Thus we expect the most general classes of $\mathcal{N} = 2$ Mink$_3$ vacua (modulo duality) to be constrained such that $\xi = c = 0$. An exploration of such Minkowski vacua is beyond the scope of this work, but would be interesting to pursue.

In the present work we will focus on AdS$_3$ solutions in Type IIA and take $c = 0$, as we are primarily interested in backgrounds with non-zero Romans mass.

In general, the four Majorana spinors $(\chi_1^1, \chi_1^2, \chi_2^1, \chi_2^2)$, can be decomposed in terms of a single unit-norm Majorana spinor χ, and three real unit-norm 1-forms (V_1, V_2, V_3) whose interior products we parameterise as

$$\langle V_1, V_2\rangle = c_3\,, \qquad \langle V_2, V_3\rangle = c_1\,, \qquad \langle V_3, V_1\rangle = c_2\,, \tag{2.13}$$

for real functions (c_1, c_2, c_3). In order to solve

$$\chi_1^{I\dagger}\chi_1^J = \chi_2^{I\dagger}\chi_2^J = e^A\delta^{IJ}\,, \tag{2.14}$$

we take, without loss of generality, the following parameterisation

$$\chi_1^1 = e^{\frac{A}{2}}\chi\,, \qquad \chi_1^2 = -ie^{\frac{A}{2}}V_1\chi\,, \qquad \chi_2^1 = e^{\frac{A}{2}}(a - ibV_2)\chi\,, \qquad \chi_2^2 = -ie^{\frac{A}{2}}V_3(a - ibV_2)\chi\,, \tag{2.15}$$

where a, b are real functions constrained as

$$a^2 + b^2 = 1\,. \tag{2.16}$$

The Majorana spinor χ defines a G_2-structure characterised by a 3-form Φ_3 such that

$$\chi\otimes\chi^\dagger = \Psi_+^{(G_2)} + i\Psi_-^{(G_2)} = \frac{1}{8}\left(1 - i\Phi_3 - \star_7\Phi_3 + i\text{vol}_7\right)\,, \qquad \Phi_3\wedge\star_7\Phi_3 = 7\text{vol}_7\,. \tag{2.17}$$

In what follows we will work with

$$\xi_\pm \equiv V_1 \pm V_3\,, \qquad V \equiv V_2\,, \tag{2.18}$$

in terms of which

$$\xi = -e^A\xi_-\,, \qquad \tilde{\xi} = -e^A\xi_+\,. \tag{2.19}$$

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We will also define an auxiliary SU(3)-structure via

$$\Phi_3 = \frac{\xi_-}{|\xi_-|} \wedge J_2 - \text{Im}\Omega_3 \,, \qquad \star_7 \Phi_3 = \frac{1}{2} J_2 \wedge J_2 - \frac{\xi_-}{|\xi_-|} \wedge \text{Re}\Omega_3 \,. \tag{2.20}$$

A branching of possible solutions now appears depending on how (V_1, V_2) are aligned. Generically both $\xi_\pm$ are non trivial and can be used to define components of the vielbein — the exception is when $V_1 = \pm V_3$, which sets one of $\xi_\pm$ to zero; these cases need to be analysed separately. For AdS_3 we must have $\xi_- \neq 0$, but there is no barrier to fixing $\xi_+ = 0$, which one can check is equivalent to imposing that the ten-dimensional Killing vector $K^M \partial_M$ is null. The rest of this paper will be focused on classifying such AdS_3 solutions and finding new explicit examples.

3 The null case

In this work we will study the case of K being null which is equivalent to $\xi_+ = 0$. From this point we also take

$$m \neq 0 \,. \tag{3.1}$$

We are left with (ξ_-, V) and we introduce v such that

$$V = \cos\theta v + \frac{1}{2} \sin\theta \xi_- \,, \qquad \iota_v \xi_- = 0, \qquad ||v|| = 1 \,. \tag{3.2}$$

We can then further decompose the auxiliary SU(3)-structure as

$$J_2 = j_2 + u \wedge v \,, \qquad \Omega_3 = (u + iv) \wedge \omega_2 \,, \tag{3.3}$$

with u a unit-norm 1-form orthogonal to (ξ_-, v), and (j_2, ω_2) defining an SU(2)-structure.

In order to parameterise the $d = 7$ bi-spinors in as simple a fashion as possible we find it convenient to decompose the functions of the spinor, and redefine the SU(2)-structure forms as

$$a + ib\sin\theta = \cos\beta e^{i\psi} \,, \qquad b\cos\theta = \sin\beta \,, \qquad \omega_2 \to e^{-i\psi}\omega_2 \,, \quad z = u + iv \,, \tag{3.4}$$

and introduce

$$\Psi_+^{(\text{SU}(2))} = \frac{e^{i\psi}}{8}\Big[\cos\beta e^{-ij_2} - \sin\beta\omega_2\Big] \wedge e^{\frac{1}{2} z\wedge\bar{z}} \,, \quad \Psi_-^{(\text{SU}(2))} = \frac{1}{8} z \wedge \Big[\sin\beta e^{-ij_2} + \cos\beta\omega_2\Big] \,. \tag{3.5}$$

In terms of these we have

$$\begin{aligned}
\Psi_+^{(0)} &= \xi_- \wedge \text{Re}\Psi_-^{(\text{SU}(2))} \,, & \Psi_-^{(0)} &= 2\text{Im}\Psi_-^{(\text{SU}(2))} \,, \\
\Psi_+^{(1)} &= 2\text{Im}\Psi_+^{(\text{SU}(2))} \,, & \Psi_-^{(1)} &= -\xi_- \wedge \text{Re}\Psi_+^{(\text{SU}(2))} \,, \\
\Psi_+^{(2)} &= -\xi_- \wedge \text{Im}\Psi_-^{(\text{SU}(2))} \,, & \Psi_-^{(2)} &= 2\text{Re}\Psi_-^{(\text{SU}(2))} \,, \\
\Psi_+^{(3)} &= 2\text{Re}\Psi_+^{(\text{SU}(2))} \,, & \Psi_-^{(3)} &= \xi_- \wedge \text{Im}\Psi_+^{(\text{SU}(2))} \,.
\end{aligned} \tag{3.6}$$

One can readily check that the supersymmetry equations impose

$$\beta = 0\,, \tag{3.7}$$

without loss of generality. In this case, the first non-trivial component of (2.8c) is the 3-form, which imposes

$$d(e^{3A-\Phi}e^{i\psi}\omega_2) + ie^{2A-\Phi}me^{i\psi}\xi_- \wedge \omega_2 = 0\,. \tag{3.8}$$

Taking the general ansatz

$$\frac{1}{2}\xi_- = e^C(d\psi + \mathcal{A})\,, \tag{3.9}$$

then fixes

$$e^C = -\frac{e^A}{2m}\,, \qquad d\left(e^{3A-\Phi}\omega_2\right) = i\mathcal{A} \wedge \left(e^{3A-\Phi}\omega_2\right)\,. \tag{3.10}$$

The 5-form component of (2.8c) then yields

$$(H + id(u \wedge v)) \wedge \omega_2 = 0\,. \tag{3.11}$$

Equation (2.8b) contains the following constraints:

$$d\left(e^{2A-\Phi}v\right) = 0\,, \tag{3.12}$$

$$d\left(e^{2A-\Phi}u \wedge j_2\right) + e^{2A-\Phi}H \wedge v = 0\,, \tag{3.13}$$

$$(dj_2 \wedge v - H \wedge u) \wedge j_2 = 0\,. \tag{3.14}$$

We may decompose the flux f_+ as

$$f_+ = g_+ + \frac{1}{2}\xi_- \wedge g_-\,, \tag{3.15}$$

and given what has been derived thus far, (2.8e) imposes

$$g_- = -\frac{e^{A-\Phi}}{2m}(v - u \wedge j_2) \wedge \mathcal{F}\,, \tag{3.16}$$

where $\mathcal{F} = d\mathcal{A}$. Now since $\Psi_-^{(0)}$ is orthogonal to ξ_- we immediately get from (2.8f) that

$$\left(\mathcal{F} - 2m^2e^{-2A}j_2\right) \wedge j_2 \wedge v \wedge u = 0\,. \tag{3.17}$$

Combining this with $\mathcal{F} \wedge \omega_2 = 0$ implies

$$\mathcal{F} = 2m^2e^{-2A}\left(j_2 + \mathcal{F}^{(1,1)}\right)\,, \tag{3.18}$$

where $\mathcal{F}^{(1,1)}$ is a primitive $(1,1)$-form.

To proceed it is helpful to consider the torsion classes of an SU(2)-structure in six dimensions, see appendix C for the general form of these. We shall compute the torsion classes on

$$ds^2(\hat{\mathrm{M}}_6) = e^{3A-\Phi}ds^2(\mathrm{M}_6)\,, \tag{3.19}$$

where we add hats to the various forms to indicate this. The relevant classes are

$$d\hat{u} = s_1\text{Re}\hat{\omega}_2+s_2\text{Im}\hat{\omega}_2+s_3\hat{j}_2+s_4\hat{u}\wedge\hat{v}+T_1+\hat{u}\wedge W_1+\hat{v}\wedge W_2\,, \qquad d\hat{j}_2 = W_3\wedge\hat{j}_2+\hat{u}\wedge T_2+\hat{v}\wedge T_3\,, \tag{3.20}$$

with s_i real functions, W_i real 1-forms and T_i real primitive (1,1)-forms. We will also introduce some holomorphic 1-forms along the way, V_i. Expanding the NS-NS field-strength as

$$H = H_3 + \hat{u}\wedge H_2^1 + \hat{v}\wedge H_2^2 + \hat{u}\wedge\hat{v}\wedge H_1\,, \tag{3.21}$$

and the exterior derivative as[6]

$$d = \tilde{d}_4 + \hat{u}\wedge\iota_{\hat{u}}d + \hat{v}\wedge\iota_{\hat{v}}d\,, \tag{3.22}$$

then plugging all this into the derived constraints we find they reduce to

$$\begin{aligned}
&d(e^{\frac{1}{2}(A-\Phi)}\hat{v})=0\,,\quad d\hat{\omega}_2=i\mathcal{A}\wedge\hat{\omega}_2\,,\quad e^{3A-\Phi}H_1=\text{Re}V_1\,,\quad \text{Im}V_1=-W_1-\frac{1}{2}\tilde{d}_4(7A-3\Phi)\,,\\
&\qquad s_3=0\,,\qquad W_3=W_1+\frac{1}{2}\tilde{d}_4(5A-\Phi)\,,\qquad e^{3A-\Phi}H_3=W_2\wedge\hat{j}_2\,,\\
&e^{3A-\Phi}H_2^1=T_3-\frac{1}{2}(\iota_{\hat{v}}d(5A-\Phi)+2s_4)\hat{j}_2\,,\\
&e^{3A-\Phi}H_2^2=H^{(1,1)}+(s_2\text{Re}\hat{\omega}_2-s_1\text{Im}\hat{\omega}_2)+\iota_{\hat{u}}d(3A-\Phi)\hat{j}_2\,.
\end{aligned} \tag{3.23}$$

For the remaining flux component g_+, from (2.8d) we have

$$\begin{aligned}
&\star_7\lambda(g_+)\\
&=e^{C-\Phi}D\psi\wedge\Big[v\wedge(W_1\wedge j_2-W_2)+\frac{1}{2}v\wedge\tilde{d}_4(3A+\Phi)\wedge j_2-u\wedge(W_1+W_2\wedge j_2)+\frac{1}{2}u\wedge\tilde{d}_4(5A-\Phi)\\
&\quad -e^{-\frac{1}{2}(3A-\Phi)}T_1-e^{\frac{1}{2}(3A-\Phi)}(s_1\text{Re}\omega_2+s_2\text{Im}\omega_2)+e^{\frac{1}{2}(3A-\Phi)}u\wedge v\wedge\Big(\frac{1}{2}(\iota_{\hat{v}}d(5A-\Phi)-2s_4)\\
&\quad -2\iota_{\hat{u}}Aj_2-e^{\Phi-3A}H^{(1,1)}-e^{\Phi-3A}T_2-(s_2\text{Re}\omega_2-s_1\text{Im}\omega_2)-\frac{1}{4}(\iota_{\hat{v}}(3A+\Phi)+2s_4)j_2\wedge j_2\Big)\Big]\,,
\end{aligned} \tag{3.24}$$

where the lack of hats is intentional, we need to take the Hodge dual of this after all. To do so we define

$$\text{Re}V_2 = W_1\,,\qquad \text{Re}V_3 = W_2\,,\qquad \text{Re}V_4 = \tilde{d}_4 A\,,\qquad \text{Re}V_5 = \tilde{d}_4\Phi\,, \tag{3.25}$$

[6]The tilde on d_4 indicates a potentially twisted derivative as u may be fibered over M_4, v can only be trivially fibred.

and we find

$$f_0 = g_0 = -\frac{1}{2}e^{\frac{3}{2}(A-\Phi)}\left(\iota_{\hat{v}}d(3A+\Phi)+2s_4\right), \qquad g_6 = 0,$$

$$g_2 = e^{\frac{3}{2}(A-\Phi)}\Big[2\iota_{\hat{u}}dAj_2 + s_2\text{Re}\omega_2 - s_1\text{Im}\omega_2 - e^{\Phi-3A}T_2 - e^{\Phi-3A}H^{(1,1)}$$
$$+e^{-\frac{1}{2}(3A-\Phi)}\text{Im}V_3\wedge v + e^{-\frac{1}{2}(3A-\Phi)}\text{Im}\left(V_2+\frac{3}{2}V_4+\frac{1}{2}V_5\right)\wedge u\Big],$$

$$g_4 = e^{\frac{3}{2}(A-\Phi)}\Big[\frac{1}{4}\left(\iota_{\hat{v}}d(5A-\Phi)-2s_4\right)j_2\wedge j_2 - u\wedge v\wedge\left(s_1\text{Re}\omega_2 + s_2\text{Im}\omega_2 - e^{-3A+\Phi}T_1\right)$$
$$+e^{-\frac{1}{2}(3A-\Phi)}\text{Im}V_3\wedge j_2\wedge u - e^{-\frac{1}{2}(3A-\Phi)}\text{Im}\left(V_2-\frac{5}{2}V_4+\frac{1}{2}V_5\right)\wedge j_2\wedge v\Big]. \tag{3.26}$$

3.1 Introducing coordinates

Above we have presented the general conditions for a solution to preserve supersymmetry. In this section we will further reduce the system of equations by introducing coordinates for the system. Firstly, since $\hat{v}$ is conformally closed we can introduce a coordinate via

$$\hat{v} = e^{\frac{1}{2}(\Phi-A)}dy\,. \tag{3.27}$$

We may introduce an additional coordinate for $\hat{u}$ via

$$\hat{u} = e^U(d\varphi + \sigma + \tau dy) \equiv e^U D\varphi\,, \tag{3.28}$$

where σ has legs only along the SU(2)-structure manifold. It is natural to assume that both σ and τ are independent of φ, and moreover it is natural to assume ∂_φ is a Killing direction. For the time being we will not assume this, but instead reduce to this more restrictive class in section 4 and construct explicit solutions there.

With these coordinates we have,

$$i_{\hat{u}}d = e^{-U}\partial_\varphi\,, \qquad i_{\hat{v}}d = e^{\frac{1}{2}(A-\Phi)}\left(\partial_y - \tau\partial_\varphi\right) \equiv e^{\frac{1}{2}(A-\Phi)}\tilde{\partial}_y\,, \tag{3.29}$$

and the exterior derivative takes the form

$$d = dy\wedge\tilde{\partial}_y + D\varphi\wedge\partial_\varphi + \tilde{d}_4\,, \tag{3.30}$$

where

$$\tilde{d}_4 = d_4 - \sigma\partial_\varphi\,, \qquad \tilde{\partial}_y = \partial_y - \tau\partial_\varphi\,. \tag{3.31}$$

Note that the twisted exterior derivative $\tilde{d}_4$ satisfies

$$\tilde{d}_4^2 = -\left(\tilde{d}_4\sigma\right)\wedge\partial_\varphi\,, \tag{3.32}$$

and is therefore generically not nilpotent. Note, that for this to define a genuine exterior derivative, as opposed to twisted, we require $s_1 = s_2 = T_1 = 0$ or for ∂_φ to be a Killing vector.

Using the local coordinates we may further decompose the torsion conditions. For $\hat{u}$ we find

$$\begin{aligned} \tilde{d}_4\sigma &= e^{-U}\left(s_1\text{Re}\hat{\omega}_2 + s_2\text{Im}\hat{\omega}_2 + T_1\right), && (3.33)\\ \partial_\varphi\sigma - \tilde{d}_4 U &= W_1\,, && (3.34)\\ \partial_\varphi\tau - \tilde{\partial}_y U &= s_4 e^{\frac{1}{2}(\Phi-A)}\,, && (3.35)\\ e^{U-\frac{1}{2}(\Phi-A)}\left(\tilde{\partial}_y\sigma - \tilde{d}_4\tau\right) &= W_2\,. && (3.36) \end{aligned}$$

A similar decomposition for $\hat{j}_2$ gives

$$d\hat{j}_2 = \frac{1}{2}\Big(2\partial_\varphi\sigma - \tilde{d}_4(2U-5A+\Phi)\Big)\wedge\hat{j}_2 + e^U D\varphi\wedge T_2 + e^{\frac{1}{2}(\Phi-A)}dy\wedge T_3\,, \tag{3.37}$$

which in components reads

$$\begin{aligned} \tilde{d}_4\hat{j}_2 &= \frac{1}{2}\Big(2\partial_\varphi\sigma - \tilde{d}_4(2U-5A+\Phi)\Big)\wedge\hat{j}_2\,, && (3.38)\\ \partial_\varphi\hat{j}_2 &= e^U T_2\,, && (3.39)\\ \tilde{\partial}_y\hat{j}_2 &= e^{\frac{1}{2}(\Phi-A)}T_3\,. && (3.40) \end{aligned}$$

Observe that if φ defines a symmetry, then $\hat{j}_2$ is conformally closed.

Next, consider the decomposition of the connection one-form $\mathcal{A}$,

$$\mathcal{A} = P + \mathcal{A}_y dy + \mathcal{A}_\varphi D\varphi\,. \tag{3.41}$$

With this decomposition we may decompose the torsion conditions for $\hat{\omega}_2$, which gives

$$\begin{aligned} \tilde{d}_4\hat{\omega}_2 &= iP\wedge\hat{\omega}_2\,, && (3.42)\\ \tilde{\partial}_y\hat{\omega} &= i\mathcal{A}_y\hat{\omega}_2\,, && (3.43)\\ \partial_\varphi\hat{\omega} &= i\mathcal{A}_\varphi\hat{\omega}_2\,, && (3.44) \end{aligned}$$

where the first of these implies that the four-manifold supporting the SU(2)-structure forms is complex, with an associated complex structure I. Finally the torsion condition (3.17) leads to

$$\left(\tilde{d}_4 P - 2m^2 e^{\Phi-5A}\hat{j}_2\right)\wedge\hat{j}_2 = 0\,. \tag{3.45}$$

Having rewritten the torsion conditions in terms of the local coordinates we can proceed with the decomposition of the fluxes in coordinate form. The NS-NS 3-form field-strength is

$$\begin{aligned} H = {}& e^{U+\frac{1}{2}(\Phi-5A)}\left(\tilde{\partial}_y\sigma-\tilde{d}_4\tau\right)\wedge\hat{j}_2 + e^{U+\Phi-3A}D\varphi\wedge\left(T_3-\frac{1}{2}e^{\frac{1}{2}(A-\Phi)}\Big(2\partial_\varphi\tau-\tilde{\partial}_y(2U+\Phi-5A)\Big)\hat{j}_2\right)\\ &+e^{\frac{1}{2}(3\Phi-7A)}dy\wedge\left(H^{(1,1)}+s_2\text{Re}\hat{\omega}_2-s_1\text{Im}\hat{\omega}_2+e^{-U}\partial_\varphi(3A-\Phi)\hat{j}_2\right)\\ &+e^{U+\frac{1}{2}(3\Phi-7A)}D\varphi\wedge dy\wedge\Big(-I\cdot(\partial_\varphi\sigma)-\frac{1}{2}\tilde{d}_4^c(2U+3\Phi-7A)\Big)\,, \end{aligned} \tag{3.46}$$

with

$$\tilde{d}_4^c \equiv -I \cdot (\tilde{d}_4)\,. \tag{3.47}$$

We may rewrite the condition for the Romans mass as

$$\frac{1}{2}\tilde{\partial}_y(2U - \Phi - 3A) - e^{2(\Phi-A)} f_0 = \partial_\varphi \tau\,, \tag{3.48}$$

by using (3.35). The 2-form g_2 takes the form

$$\begin{aligned} g_2 = {} & e^{-\frac{1}{2}(\Phi+3A)}\Big[s_2 \mathrm{Re}\hat{\omega}_2 - s_1 \mathrm{Im}\hat{\omega}_2 - T_2 - H^{(1,1)}\Big] - e^{U-\frac{1}{2}(\Phi+3A)} I \cdot \Big(\tilde{\partial}_y \sigma - \tilde{d}_4 \tau\Big) \wedge dy \\ & + e^{U-\frac{1}{2}(\Phi+3A)} I \cdot \Big(\tilde{d}_4 (U - \tfrac{1}{2}(3A+\Phi)) - \partial_\varphi \sigma\Big) \wedge D\varphi + 2 e^{U-\frac{1}{2}(\Phi+3A)} \partial_\varphi A\, \hat{j}_2\,, \end{aligned} \tag{3.49}$$

whilst the 4-form g_4 is

$$\begin{aligned} g_4 = {} & e^{-4A}\Big(\tilde{\partial}_y (U + \tfrac{1}{2}(5A - \Phi)) - \partial_\varphi \tau\Big) \frac{1}{2} \hat{j}_2 \wedge \hat{j}_2 - e^{U+\Phi-5A} D\varphi \wedge dy \wedge \Big(s_1 \mathrm{Re}\hat{\omega}_2 + s_2 \mathrm{Im}\hat{\omega}_2 - T_1\Big) \\ & - e^{2U-4A} I \cdot \Big(\tilde{\partial}_y \sigma - \tilde{d}_4 \tau\Big) \wedge \hat{j}_2 \wedge D\varphi + e^{\Phi-5A} I \cdot \Big(\partial_\varphi \sigma - \tilde{d}_4 (U + \tfrac{1}{2}(5A - \Phi))\Big) \wedge \hat{j}_2 \wedge dy\,. \end{aligned} \tag{3.50}$$

Finally the Bianchi identity reads

$$dg_+ - H \wedge g_+ = -\frac{1}{4m^2} dy \wedge d\mathcal{A} \wedge d\mathcal{A}\,. \tag{3.51}$$

4 Solutions with additional symmetries

In the above we have presented the general decomposition of the torsion conditions using a set of coordinates. The resultant conditions are difficult to solve and therefore to make further progress we will impose some assumptions which make the problem more tractable.

4.1 Imposing an additional Killing vector

As we emphasised earlier one natural assumption to make is to impose that ∂_φ is a Killing vector. We will therefore assume that all the scalars and 2-form $\hat{j}_2$ are independent of φ. This assumption lets us drop $\tilde{d}_4$ for d_4 since everything (but $\hat{\omega}_2$) is independent of φ. It follows that $\hat{j}_2$ is conformally Kähler

$$d_4\Big(e^{U+\frac{1}{2}(\Phi-5A)} \hat{j}_2\Big) = 0\,, \tag{4.1}$$

and we therefore redefine our Kähler form to be

$$J = e^{\frac{1}{2}(2U-5A+\Phi)} \hat{j}_2\,. \tag{4.2}$$

This implies that the internal metric takes the form

$$ds^2_7 = \frac{e^{2A}}{4m^2}(d\psi + \mathcal{A})^2 + e^{\Phi-3A}\Big[e^{2U} D\varphi^2 + e^{\Phi-A} dy^2 + e^{\frac{1}{2}(5A-\Phi)-U} ds^2(\mathrm{M}_4)\Big]\,, \tag{4.3}$$

with the metric on M_4 Kähler at fixed y coordinate.

The torsion conditions for the new Kähler 2-form read

$$d_4 J = 0\,, \tag{4.4}$$

$$\partial_\varphi J = e^{\frac{1}{2}(4U-5A+\Phi)} T_2\,, \tag{4.5}$$

$$\partial_y J = \frac{1}{2}\partial_y(2U-5A+\Phi)J + e^{U+\Phi-3A}T_3\,, \tag{4.6}$$

however since we assume φ is an isometry we must set $T_2 = 0$ in the following.

Performing the same rescaling for the holomorphic volume form,

$$\Omega = e^{\frac{1}{2}(2U-5A+\Phi)}\hat{\omega}_2\,, \tag{4.7}$$

we find the torsion conditions

$$\begin{aligned} d_4\Omega &= \Big(i(P+\mathcal{A}_\varphi\sigma) + \frac{1}{2}d_4(2U-5A+\Phi)\Big)\wedge\Omega\,,\\ \partial_y\Omega &= \Big(i(\mathcal{A}_y+\mathcal{A}_\varphi\tau) + \frac{1}{2}\partial_y(2U-5A+\Phi)\Big)\Omega\,,\\ \partial_\varphi\Omega &= i\mathcal{A}_\varphi\Omega\,. \end{aligned} \tag{4.8}$$

It follows that $\mathcal{A}_\varphi$ should be a constant and therefore we may solve the final constraint simply by introducing a phase for Ω. Moreover, integrability implies

$$\partial_y P - d_4\mathcal{A}_y + \mathcal{A}_\varphi(\partial_y\sigma - d_4\tau) = 0\,, \tag{4.9}$$

$$d_4P + \mathcal{A}_\varphi d_4\sigma \in H^{(1,1)}(M_4)\,. \tag{4.10}$$

For non-trivial $\mathcal{A}_\varphi$ we may solve for P in terms of σ up to the addition of a term whose derivative is a $(1,1)$-form. In addition we have that

$$d_4\sigma = e^{-U}\left(s_1 e^{-\frac{1}{2}(2U-5A+\Phi)}\mathrm{Re}\Omega + s_2 e^{-\frac{1}{2}(2U-5A+\Phi)}\mathrm{Im}\Omega + T_1\right)\,. \tag{4.11}$$

It is useful to redefine s_1 and s_2 here to absorb the exponential factors but since the classes of solutions we consider later do not have such a term switched on we will refrain from doing so here. The supersymmetry condition (3.45) becomes

$$\left(d_4P - 2m^2 e^{-U-\frac{1}{2}(5A-\Phi)}J\right)\wedge J = 0\,, \tag{4.12}$$

which we may rewrite as the scalar equation

$$R_4 = 2\Box_4\Big(U - \tfrac{1}{2}(5A-\Phi)\Big) + 8m^2 e^{-U-\frac{1}{2}(5A-\Phi)}\,. \tag{4.13}$$

Note that if the first term on the right-hand side vanishes this is reminiscent of the condition for GK geometries [7–9] after a little redefinition.[7] As we will see later one of the classes of solution we obtain are the T-dual of the GK geometries in Type IIB with a torus and

[7] In [76] similar types of condition for the Ricci scalar appears with a Laplacian like term. The geometries of [76] generalise GK geometries by including rotation, though this is not the origin of such a term here.

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3-form flux in massless Type IIA. In fact we are able to generalise these solutions further by turning on a non-trivial Romans mass.

The NS-NS 3-form after this simplification becomes

$$
\begin{aligned}
H=&\Big(\partial_y\sigma-d_4\tau\Big)\wedge J+D\varphi\wedge\Big(e^{U+\Phi-3A}T_3+\frac{1}{2}\partial_y(2U+\Phi-5A)J\Big)\\
&+e^{-U-A+\Phi}dy\wedge\Big(e^{U-\frac{1}{2}(5A-\Phi)}H^{(1,1)}+s_2\mathrm{Re}\Omega-s_1\mathrm{Im}\Omega\Big)-D\varphi\wedge dy\wedge d_4^c\Big(e^{U+\frac{1}{2}(3\Phi-7A)}\Big)\,.
\end{aligned}
\tag{4.14}
$$

The condition for the Romans mass simplifies to

$$
\partial_y(2U-\Phi-3A)=2e^{2(\Phi-A)}f_0\,, \tag{4.15}
$$

whilst the R-R 2-form becomes

$$
\begin{aligned}
g_2=&e^{-U-\Phi+A}\left(s_2\mathrm{Re}\Omega-s_1\mathrm{Im}\Omega-e^{U-\frac{1}{2}(5A-\Phi)}H^{(1,1)}\right)-e^{U-\frac{1}{2}(\Phi+3A)}I\cdot(\partial_y\sigma-d_4\tau)\wedge dy\\
&-d_4^c\left(e^{U-\frac{1}{2}(\Phi+3A)}\right)\wedge D\varphi\,,
\end{aligned}
\tag{4.16}
$$

and the 4-form is

$$
\begin{aligned}
g_4=&e^{-2U+A-\Phi}\partial_y\left(U+\tfrac{1}{2}(5A-\Phi)\right)\frac{1}{2}J\wedge J-D\varphi\wedge dy\wedge\left(e^{\frac{1}{2}(\Phi-5A)}(s_1\mathrm{Re}\Omega+s_2\mathrm{Im}\Omega)-e^{U+\Phi-5A}T_1\right)\\
&-e^{U-\frac{1}{2}(\Phi+3A)}I\cdot(\partial_y\sigma-d_4\tau)\wedge J\wedge D\varphi-d_4^c\left(e^{-U-\frac{1}{2}(5A-\Phi)}\right)\wedge J\wedge dy\,.
\end{aligned}
\tag{4.17}
$$

We may construct the magnetic fluxes using (3.15) and find

$$
\begin{aligned}
f_2&=g_2\,,\\
f_4&=g_4+\frac{1}{4m^2}(d\psi+\mathcal{A})\wedge dy\wedge d\mathcal{A}\,, &(4.18)\\
f_6&=-\frac{1}{4m^2}(d\psi+\mathcal{A})\wedge D\varphi\wedge J\wedge d\mathcal{A}\,. &(4.19)
\end{aligned}
$$

4.2 Product of Riemann surface ansatz

Having introduced coordinates and made the assumption of an additional Killing vector we are in a position where we can introduce an ansatz for the four-dimensional base. The simplest choice is that we may decompose the base as the direct product of two Riemann surfaces. We take

$$
ds^2(M_4)=e^{2f_1(y)}ds^2(\Sigma_1)+e^{2f_2(y)}ds^2(\Sigma_2)\,, \tag{4.20}
$$

with $f_i(y)$ arbitrary functions of y and we take the metric on Σ_i to be the constant curvature one given by

$$
ds^2(\Sigma_i)=\frac{1}{1-\kappa_i x_i^2}dx_i^2+\left(1-\kappa_i x_i^2\right)dz_i^2\,. \tag{4.21}
$$

The Ricci scalar for the Riemann surface is $R=2\kappa_i$ and we take the structure forms to be

$$
J_i=dx_i\wedge dz_i\,,\qquad \Omega_i=\frac{1}{\sqrt{1-\kappa_i x_i^2}}\left(dx_i+i\left(1-\kappa_i x_i^2\right)dz_i\right)\,, \tag{4.22}
$$

which satisfy

$$d_4 J_i = 0\,, \quad d_4\Omega_i = i\kappa_i x_i dz_i \wedge \Omega_i\,. \tag{4.23}$$

In terms of the structure forms of the Riemann surfaces the SU(2)-structure forms are

$$J = e^{2f_1(y)}J_1 + e^{2f_2(y)}J_2\,, \quad \Omega = e^{f_1(y)+f_2(y)}\Omega_1 \wedge \Omega_2\,. \tag{4.24}$$

For the given Kähler form we can construct a single primitive $(1,1)$-form which preserves the symmetries, namely

$$\nu_2 = e^{2f_1(y)}J_1 - e^{2f_2(y)}J_2\,. \tag{4.25}$$

Note that $d_4\nu_2 = 0$, indeed without breaking the symmetries of the Riemann surfaces no other choice is possible.

Let us define

$$T_3 = t_3'(y)e^{-\mathrm{U}(y)-\Phi(y)+3A(y)}\nu_2\,, \tag{4.26}$$

then (4.6) implies

$$\begin{aligned} 4f_1(y) &= 2\mathrm{U}(y) - 5A(y) + \Phi(y) + 2t_3(y) + c_1\,, \\ 4f_2(y) &= 2\mathrm{U}(y) - 5A(y) + \Phi(y) - 2t_3(y) + c_2\,. \end{aligned} \tag{4.27}$$

We solve for $t_3(y)$ and $\mathrm{U}(y)$, giving

$$e^{2\mathrm{U}(y)} = C_1^2 e^{2f_1(y)+2f_2(y)+5A(y)-\Phi(y)}\,, \tag{4.28}$$

$$t_3(y) = C_2 + f_1(y) + f_2(y)\,. \tag{4.29}$$

Next take the primitive 2-form T_1 to be

$$T_1 = e^{\mathrm{U}(y)}t_1(y)\nu_2\,, \tag{4.30}$$

the integrability condition for (4.11) implies $s_1 = s_2 = 0$ unless $\kappa_1 = \kappa_2 = 0$. Let

$$A_i = x_i dz_i\,, \tag{4.31}$$

then σ takes the form

$$\sigma = \sum_{i=1}^{2} u_i(y)A_i\,. \tag{4.32}$$

Plugging this into (4.11) gives

$$d_4\sigma = \sum_{i=1}^{2} u_i(y)J_i = t_1(y)\nu_2\,, \tag{4.33}$$

and therefore

$$u_1(y) = t_1(y)e^{2f_1(y)}\,, \quad u_2(y) = -t_1(y)e^{2f_2(y)}\,. \tag{4.34}$$

It follows that a non-trivial T_1 leads to $D\varphi$ being non-trivially fibered over the base. Next consider the conditions on the holomorphic volume form (4.8). From the first we find

$$P + \mathcal{A}_\varphi\sigma = \kappa_1 A_1 + \kappa_2 A_2\,, \tag{4.35}$$

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whilst the second implies

$$\mathcal{A}_y = -\mathcal{A}_\varphi \tau \,. \tag{4.36}$$

We must require that τ is a function of y only, and does not have any Riemann surface dependence. It then follows that a coordinate transformation can be performed which sets $\tau = 0$ and therefore without loss of generality we may take $\tau = 0$ and therefore also $\mathcal{A}_y = 0$. Note that $\mathcal{A}_\varphi$ is a constant which we can pick by rescaling the holomorphic volume form by a φ dependent phase, we can therefore set it to vanish without loss of generality. It follows that the 1-form P is

$$P = \kappa_1 A_1 + \kappa_2 A_2 \,. \tag{4.37}$$

From the expression for the NS-NS flux in (4.14) the Bianchi identity imposes

$$\partial_y\Big(e^{2f_1(y)} f_1'(y)\Big) = \partial_y\Big(e^{2f_2(y)} f_2'(y)\Big) = \partial_y^2\Big(e^{2(f_1(y)+f_2(y))} t_1(y)\Big) = 0 \,, \tag{4.38}$$

note that it is independent of the primitive two-form $H^{(1,1)}$ which is necessarily closed on the four-dimensional base in order to preserve the symmetries of the Riemann surfaces. We may solve the first two by

$$e^{2f_i(y)} = a_i y + b_i \,, \tag{4.39}$$

which, upon substituting into the third gives

$$t_1(y) = \frac{\alpha y + \beta}{(a_1 y + b_1)(a_2 y + b_2)} \,. \tag{4.40}$$

So far we have solved for $t_1(y), t_3(y), f_i(y), \mathrm{U}(y)$ and it remains to determine $A(y)$ and $\Phi(y)$. From (4.13) we find

$$e^{-\Phi} = \frac{4m^2 e^{f_1(y)+f_2(y)-5A(y)}}{C_1\left(\kappa_1 e^{2f_2(y)} + \kappa_2 e^{2f_1(y)}\right)} \,. \tag{4.41}$$

The condition from the Romans mass reads

$$\partial_y\Big(2\mathrm{U}(y) - \Phi(y) - 3A(y)\Big) = 2e^{2\left(\Phi(y)-A(y)\right)} f_0 \,, \tag{4.42}$$

which we may solve for A giving

$$e^{8A(y)} = \frac{48m^2(a_1y+b_1)^2(a_2y+b_2)^2}{C_1^2 f_0(\kappa_1(a_2y+b_2)+\kappa_2(a_1y+b_1))^2(3b_1y(2b_2+a_2y)+a_1y^2(3b_2+2a_2y)-12\delta)} \,, \tag{4.43}$$

in the massive case and

$$e^{4A(y)} = \frac{\hat{\delta}(b_1+a_1y)(b_2+a_2y)}{\kappa_1(b_2+a_2y)+\kappa_2(b_1+a_1y)} \,, \tag{4.44}$$

for the massless case.[8]

[8] Note that if one redefines the constant $\delta \to -4m^4(f_0 C_1^2 \hat{\delta}^2)^{-1}$ in the massive case one can take the massless limit and land upon the massless solution we have presented.

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We have now solved for all the functions appearing in the solution, but for the primitive two-form $H^{(1,1)}$, using the Bianchi identity for the NS-NS flux, the Romans mass Bianchi identity and the Ricci scalar supersymmetry equation. In solving these conditions we have introduced eight integration constants, $(C_1, a_1, a_2, b_1, b_2, \alpha, \beta, \delta)$. We will see that the two remaining Bianchi identities will restrict these integration constants further. Recall that the primitive form $H^{(1,1)}$ was not constrained by the Bianchi identity for H, a convenient choice to make is

$$H^{(1,1)} = e^{-\frac{1}{2}(\Phi(y)-7A(y))} h^{(1,1)}(y) \nu_2 \,, \tag{4.45}$$

with ν_2 as defined in (4.25). This is the most general form we can pick without breaking additional symmetries of the Riemann surfaces.

The Bianchi identity for g_2 then imposes

$$\begin{aligned} 0 &= a_1 f_0 = a_2 f_0 = \alpha f_0 \,, \\ h^{(1,1)}(y) &= \frac{q_1}{(a_1 y + b_1)\sqrt{6 b_1 b_2 y + 3(a_2 b_1 + a_1 b_2) y^2 + 2 a_1 a_2 y^3 - 12\delta}} \,, \\ q_2 (a_1 y + b_1) &= q_1 (a_2 y + b_2) \,, \end{aligned} \tag{4.46}$$

where q_2 is some constant of proportionality. It is clear to see from the above conditions that there are different branches of solutions to consider. The Bianchi identity for g_4 leads to further branching conditions and it is therefore convenient to first solve the g_2 Bianchi identity before attempting to solve the g_4 one. We will first consider the massive case which turns out to have a unique family of solutions, before studying the massless case.

4.2.1 Massive class

We see from above that we must set

$$a_1 = a_2 = \alpha = 0 \,, \tag{4.47}$$

which implies that $h^{(1,1)}(y)$ is equal to

$$h^{(1,1)}(y) = \frac{q}{\sqrt{2 b_1 b_2 y - 4\delta}} \,. \tag{4.48}$$

The g_2 Bianchi identity is completely solved with these restrictions and we may move onto the g_4 Bianchi identity. One finds that this Bianchi identity is solved if we set $q = 0$ and thus the primitive two-form $H^{(1,1)} = 0$ and

$$\beta = \frac{\mathcal{A}_\varphi (b_2 \kappa_1 - b_1 \kappa_2) \pm \sqrt{\mathcal{A}_\varphi^2 (b_2 \kappa_1 + b_1 \kappa_2)^2 + 16 b_1^2 b_2^2 C_1^2 m^2}}{2(\mathcal{A}_\varphi^2 + 4 b_1 b_2 C_1^2 m^2)} \,. \tag{4.49}$$

It is useful to make the redefinitions

$$L f_0 = \hat{f}_0 \,, \quad y = \frac{2\delta}{b_1 b_2} + \frac{L h_8(\hat{y})^2}{2 \hat{f}_0} \,, \quad h_8(\hat{y}) = \hat{f}_0 \hat{y} + \hat{c} \,, \tag{4.50}$$

so that

$$f_0 = \partial_{\hat{y}} h_8(\hat{y}) \,, \tag{4.51}$$

and for simplicity to set $\mathcal{A}_\varphi = 0$ by multiplying the holomorphic volume form by a suitable φ-dependent phase. Let us also define

$$\begin{aligned} &L = m^{-1}\,, \quad b_i = L^2 \hat{b}_i\,, \quad C_1 = L^{-2}\hat{C}_1\,, \quad \varphi = L\hat{\varphi}\,, \\ &e^B = \frac{\hat{C}_1(\hat{b}_2\kappa_1 + \hat{b}_1\kappa_2)}{4\sqrt{\hat{b}_1\hat{b}_2}}\,, \quad \hat{\beta} = \frac{\sqrt{\kappa_1\kappa_2}}{2}\,. \end{aligned} \tag{4.52}$$

The final metric is

$$\begin{aligned} ds^2 = L^2 \frac{e^{-B/2}}{\sqrt{h_8(\hat{y})}} \Bigg[& ds^2\,(\mathrm{AdS}_3)^{\mathrm{unit}} + \frac{1}{4}\,(d\psi + \kappa_1 A_1 + \kappa_2 A_2)^2 \\ & + \hat{C}_1^2 \hat{b}_1 \hat{b}_2 \left(d\hat{\varphi} + \frac{\hat{\beta}}{\hat{C}_1 \hat{b}_1 \hat{b}_2} \left(\hat{b}_1 A_1 - \hat{b}_2 A_2 \right) \right)^2 \\ & + e^B \left(h_8\,(\hat{y})\, d\hat{y}^2 + \frac{1}{\hat{C}_1 \sqrt{\hat{b}_1\hat{b}_2}} \left(\hat{b}_1 ds^2\,(\Sigma_1) + \hat{b}_2 ds^2\,(\Sigma_2) \right) \right) \Bigg]\,, \end{aligned} \tag{4.53}$$

with dilaton and magnetic fluxes

$$\begin{aligned} &e^{-4\Phi} = e^B h_8(\hat{y})^5\,, \quad H = 0\,, \quad f_2 = 0\,, \\ &L^{-3} f_4 = \hat{\beta}\hat{C}_1 h_8(\hat{y}) D\hat{\varphi} \wedge d\hat{y} \wedge \nu_2 + \frac{h_8(\hat{y})}{4} D\psi \wedge d\hat{y} \wedge (\kappa_1 J_1 + \kappa_2 J_2)\,, \\ &L^{-5} f_6 = -\frac{\kappa_1 \hat{b}_2 + \kappa_2 \hat{b}_1}{4} D\psi \wedge J_1 \wedge J_2 \wedge D\hat{\varphi}\,. \end{aligned} \tag{4.54}$$

Note that for the solution to be well-defined we require

$$\hat{b}_1 > 0\,, \quad \hat{b}_2 > 0\,, \quad C_1 > 0\,, \quad \kappa_1\kappa_2 \geq 0\,. \tag{4.55}$$

There are therefore two choices one can make for the Riemann surface. Either one of the Riemann surfaces is a torus and the either is a torus or two-sphere or both are round two-spheres. We will see later in section 5.1 that this is in fact contained within a more general class of solution.

To bound the line interval parametrised by $\hat{y}$ we take the Romans mass to have jumps at positions $\hat{y}_i$, without loss of generality we can take $\hat{y}_0 = 0$ and take $h_8(0) = 0$. This signifies the presence of an O8-plane which caps off the space. By allowing the Romans mass to have jumps at the $\hat{y}_i$ whilst keeping the function $h_8(\hat{y})$ continuous we may obtain a second root at $\hat{y}_{p+1}$ which bounds the space between a second O8-plane. The solution is thus compact and well-defined. Since we find that this solution is a specialisation of a more general solution we discuss later we will not present the quantisation of flux here and instead refer the reader to the later section 5.2.

4.2.2 Massless case

Having considered the massive class of solution let us consider the massless solutions. We saw that in the massive case the solution was essentially unique, it turns out that this is not the case here and there is further branching. The g_4 Bianchi implies

$$h^{(1,1)}(y) = q_1 e^{-2f_1(y)} = q_2 e^{-2f_2(y)}\,, \tag{4.56}$$

which has two solutions, either $h^{(1,1)} = 0$ or $f_1(y)$ and $f_2(y)$ are proportional.

Case 1: $\boldsymbol{h^{(1,1)} = 0}$. Let us first consider the case where $h^{(1,1)} = 0$. It follows that we must set

$$\alpha = \beta = a_1 = \kappa_2 = 0 \tag{4.57}$$

with the other parameters free. For the metric to have the correct signature we require that $\kappa_1 = 1$ and therefore we have a round S^2. In fact this combines with the R-symmetry direction to form a round S^3. Performing the rescalings

$$\begin{aligned} &m \to L^{-1}\,, \quad b_i \to \hat{b}_i\,, \quad a_2 \to \frac{\hat{a}_2}{2L}\,, \quad C_1 \to L^{-2}\hat{C}_1\,, \quad \delta \to \hat{\delta}\,, \\ &y \to 2L\hat{y}\,, \quad \varphi \to L^3\frac{\hat{\varphi}}{2}\,, \end{aligned} \tag{4.58}$$

the final solution takes the form

$$ds^2 = L^2\sqrt{\hat{b}_1\hat{\delta}}\left[ds^2\left(\mathrm{AdS}_3\right)^{\mathrm{unit}} + \frac{1}{4}ds^2\left(S^3\right) + \frac{\left(\hat{a}_2\hat{y}+\hat{b}_2\right)}{4\hat{b}_1}\left(\hat{b}_1^2\hat{C}_1^2 d\hat{\varphi}^2 + \hat{b}_1\hat{C}_1^2\hat{\delta}dy^2 + ds^2\left(\mathbb{T}^2\right)\right)\right], \tag{4.59}$$

with dilaton and magnetic fluxes

$$\begin{aligned} e^{4\Phi} &= \frac{\hat{b}_1^3\hat{C}_1^4\hat{\delta}^5\left(\hat{a}_2\hat{y}+\hat{b}_2\right)^2}{256}\,, & L^{-2}H &= \frac{a_2}{4}d\hat{\varphi}\wedge\mathrm{vol}\left(\mathbb{T}^2\right), \quad f_2 = 0\,, \\ L^{-3}f_4 &= \frac{1}{2}\mathrm{vol}(S^3)\wedge d\hat{y}\,, & L^{-5}f_6 &= \frac{\hat{a}_2\hat{y}+\hat{b}_2}{8}\mathrm{vol}(S^3)\wedge d\varphi\wedge\mathrm{vol}(\mathbb{T}^2)\,. \end{aligned} \tag{4.60}$$

This solution is in fact a special limit of a later solution and therefore we will not analyse it further. One can of course also quotient the S^3 with a subgroup of $\mathrm{SU}(2)_R$ and preserve $\mathcal{N} = (2,0)$ supersymmetry.

Case 2: $\boldsymbol{h^{(1,1)} \neq 0}$. The second and final, class of massless solution allows for a non-trivial primitive two-form $H^{(1,1)}$. The final solution (after our favourite rescalings to make the metric coordinates and parameters dimensionless) is

$$\begin{aligned} ds^2 = L^2\sqrt{\frac{\hat{b}_1\hat{b}_2\hat{\delta}}{\hat{b}_2\kappa_1+\hat{b}_1\kappa_2}}&\left[ds^2\left(\mathrm{AdS}_3\right)^{\mathrm{unit}} + \frac{1}{4}D\psi^2 + \hat{b}_1\hat{b}_2\hat{C}_1^2 D\varphi^2 + \frac{\hat{C}_1\hat{\delta}\left(\hat{b}_2\kappa_1+\hat{b}_1\kappa_2\right)}{16}dy^2\right. \\ &\left. + \frac{\hat{b}_2\kappa_1+\hat{b}_1\kappa_2}{4\hat{b}_2}d\left(\Sigma_1\right)^2 + \frac{\hat{b}_2\kappa_1+\hat{b}_1\kappa_2}{4\hat{b}_1}d\left(\Sigma_2\right)^2\right], \end{aligned} \tag{4.61}$$

$$D\varphi \equiv d\varphi + \frac{\hat{\beta}}{\hat{b}_1\hat{b}_2}\left(\hat{b}_1A_1 - \hat{b}_2A_2\right), \quad D\psi \equiv d\psi + \kappa_1A_1 + \kappa_2A_2\,,$$

with dilaton and magnetic fluxes

$$\begin{aligned}
&e^{4\Phi}=\frac{\hat{b}_1^3\hat{b}_2^3\hat{C}_1^4\hat{\delta}^5}{256\left(\hat{b}_2\kappa_1+\hat{b}_1\kappa_2\right)}\,,\quad L^{-2}H=\frac{\delta^2\hat{q}_1\hat{b}_2\hat{C}_1^2}{16}dy\wedge\left(\hat{b}_1J_1-\hat{b}_2J_2\right)\,,\quad L^{-1}f_2=\frac{\hat{q}_1}{\hat{b}_1}\left(\hat{b}_2J_2-\hat{b}_1J_1\right)\,,\\
&L^{-3}f_4=-\hat{C}_1^2\beta dy\wedge D\varphi\wedge\left(\hat{b}_1J_1-\hat{b}_2J_2\right)+\frac{1}{4}D\psi\wedge dy\wedge(\kappa_1J_1+\kappa_2J_2)\,,\\
&L^{-5}f_6=-\frac{\hat{b}_2\kappa_1+\hat{b}_1\kappa_2}{4}D\psi\wedge D\varphi\wedge J_1\wedge J_2\,,
\end{aligned}\tag{4.62}$$

with

$$\hat{\beta}=\frac{\sqrt{4\kappa_1\kappa_2-\hat{q}_1^2\hat{C}_1^2\hat{b}_2^2\hat{\delta}^2}}{4\hat{C}_1}\,.\tag{4.63}$$

Note that setting $\hat{\beta}=\hat{q}_1=0$ leads to $\kappa_1\kappa_2=0$ and thus a solution in the previous section. Regularity imposes the inequalities

$$\hat{b}_1\hat{b}_2>0\,,\quad 4\kappa_1\kappa_2-\hat{q}_1^2\hat{C}_1^2\hat{b}_2^2\hat{\delta}^2>0\,,\quad \hat{\delta}\left(\hat{b}_2\kappa_1+\hat{b}_1\kappa_2\right)>0\,,\quad \left(\hat{b}_2\kappa_1+\hat{b}_1\kappa_2\right)\hat{b}_1>0\,.\tag{4.64}$$

Clearly we require $\kappa_1\kappa_2>0$ unless $\hat{q}_1=\hat{\beta}=0$ in which case we have $\kappa_1\kappa_2=0$ as discussed in the previous section. Since $\hat{b}_1\hat{b}_2>0$ it follows that we must set $\kappa_1=\kappa_2=1$, otherwise we violate the last bound. The solution therefore contains two round two-spheres and turns out to be T-dual to a solution in the literature, namely the solution in section 3.1 of [77].

5 General Kähler base

We will now assume that the conformally Kähler base is a non-trivial four-manifold, that is we take the y-dependence to come from an overall warp factor, so that

$$ds^2(M_4(\vec{x},y))=e^{2f(y)}ds^2(\mathcal{M}_4(\vec{x}))\,,\tag{5.1}$$

with $\mathcal{M}_4$ Kähler. We may then decompose the SU(2)-structure forms as

$$J=e^{2f(y)}J_K\,,\qquad \Omega=e^{2f(y)}\Omega_K\,,\tag{5.2}$$

where J_K and Ω_K are the SU(2)-structure forms on $\mathcal{M}_4(\vec{x})$ and are independent of y. They satisfy

$$d_4J_K=0\,,\qquad d_4\Omega_K=iP_K\wedge\Omega_K\,,\tag{5.3}$$

with P_4 the Ricci-form potential of the Kähler metric. We will allow the base to admit a closed primitive (1,1)-form which we denote by ν_2. Locally we may write it as

$$\nu_2=d\Sigma\,,\tag{5.4}$$

for some 1-form Σ defined on $\mathcal{M}_4(\vec{x})$. Since the 2-form is primitive it implies that $\star_4\nu_2=-\nu_2$ and consequently it is a harmonic 2-form. We will allow for all scalars to depend on both y and the Kähler coordinates in the following. When a function does not depend on both sets we will explicitly give the coordinate dependence, but otherwise omit the arguments unless necessary.

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Our assumptions on the base and the torsion conditions for J implies that we must set the primitive forms T_2 and T_3 to vanish. By construction equation (4.4) is satisfied whilst (4.6) imposes

$$e^U = g(\vec{x}) e^{2f(y)+\frac{1}{2}(5A-\Phi)}\,, \tag{5.5}$$

with $g(\vec{x})$ an arbitrary non-zero function on the base.

From the torsion conditions for the holomorphic volume form we find,

$$P = P_K - \mathcal{A}_\varphi \sigma + d_4^c \log g(\vec{x})\,, \tag{5.6}$$

$$\mathcal{A}_y = -\mathcal{A}_\varphi \tau(\vec{x}, y)\,. \tag{5.7}$$

We find that $\mathcal{A}_\varphi$ is constant and can therefore be removed by multiplying the holomorphic volume form by a φ-dependent phase. We may solve (4.11) by taking

$$T_1 = e^U t_1(\vec{x}, y)\nu_2\,. \tag{5.8}$$

Integrability implies

$$d_4 t_1(\vec{x}, y) = 0\,, \tag{5.9}$$

and therefore

$$\sigma = t_1(y)\Sigma\,. \tag{5.10}$$

From (4.13) we find

$$R_K + 2\Box_K \log g(\vec{x}) = 8m^2 g(\vec{x})^{-1} e^{\Phi-5A}\,, \tag{5.11}$$

where we used (5.5). Note that the left-hand side is independent of y and therefore we have that $\Phi - 5A$ must be independent of y too.

The Romans mass condition is equivalent to

$$e^{2(\Phi-A)} f_0 = 2f'(y) + \partial_y(A - \Phi)\,. \tag{5.12}$$

For the massless theory it is easy to see that this has solution

$$e^{\Phi-A} = c(\vec{x}) e^{2f(y)}\,, \tag{5.13}$$

with $c(\vec{x})$ a non-zero, but possibly constant, integration function. In the massive case for $f'(y) = 0$ the general solution is

$$e^{2(A-\Phi)} = 2f_0 y + b_1(\vec{x})\,, \tag{5.14}$$

but more generally we can only solve this condition once we have fixed $f(y)$.

Let us proceed with the conditions from the Bianchi identities. We must fix the second primitive two-form $H^{(1,1)}$, which we take to be

$$H^{(1,1)} = e^{2f(y)+\frac{1}{2}(A+3\Phi)} h^{(1,1)}(\vec{x}, y)\nu_2\,, \tag{5.15}$$

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then the Bianchi identity for the R-R 2-form implies the four conditions

$$d_4^c\Big(g(\vec{x})\partial_y e^{2f(y)+A-\Phi}\Big)=f_0 d_4^c\Big(g(\vec{x})e^{2f(y)+\Phi-A}\Big)\,, \tag{5.16}$$

$$d_4 d_4^c\Big(g(\vec{x})e^{A-\Phi}\Big)+2f_0 f'(y)J_K=0\,, \tag{5.17}$$

$$\Big(d_4\Big(e^{\Phi-A}h^{(1,1)}(\vec{x},y)\Big)-t_1(y)d_4^c\big(g(\vec{x})e^{A-\Phi}\big)\Big)\wedge\nu_2=-f_0 t_1'(y)\Sigma\wedge J_K\,, \tag{5.18}$$

$$t_1'(y)\Big(d_4\big(g(\vec{x})e^{A-\Phi}I\cdot\Sigma\big)+d_4^c\big(g(\vec{x})e^{A-\Phi}\big)\wedge\Sigma\Big)=-e^{\Phi-A}\Big(\partial_y h^{(1,1)}(\vec{x},y)+4f'(y)h^{(1,1)}(\vec{x},y)\Big)\nu_2\,. \tag{5.19}$$

The first condition is satisfied immediately after using the condition for the Romans mass in (5.12). The non-primitive part of the second condition implies the scalar condition

$$\Box_K\Big(g(\vec{x})e^{A-\Phi}\Big)+4f_0 f'(y)=0\,. \tag{5.20}$$

This is a necessary, but not sufficient condition since we must also enforce that the primitive part of the first term vanishes. For $f'(y)=0$ we see that we require a Laplacian Eigenfunction, but since the only Eigenfunctions on a Kähler manifold are constant it follows that the bracketed term is constant. We shall refrain from imposing any of these restrictions for the moment and proceed with the remaining Bianchi identities.

For H we find the two conditions

$$0=\Big(2f''(y)+4f'(y)^2\Big)J_K+d_4 d_4^c\Big(g(\vec{x})e^{\Phi-A}\Big)\,, \tag{5.21}$$

$$0=\Big(t_1''(y)+4f'(y)t_1'(y)\Big)\Sigma\wedge J_K-\Big[d_4\Big(e^{3(\Phi-A)}h^{(1,1)}(\vec{x},y)\Big)+t_1(y)d_4^c\Big(g(\vec{x})e^{\Phi-A}\Big)\Big]\wedge\nu_2\,. \tag{5.22}$$

In the massive case these are implied by the Bianchi identity conditions for g_2, but in the massless case they are generically not.

Finally let us consider the Bianchi identity for the R-R 4-form. There are three conditions, the first two are

$$\Big[t_1(y)\,d_4\Big(g(\vec{x})\,e^{A-\Phi}I\cdot\Sigma\Big)+t_1'(y)\,\Sigma\wedge d_4^c\Big(g(\vec{x})\,e^{3A}\Big)\Big]\wedge J=0\,, \tag{5.23}$$

$$\begin{aligned}&g(\vec{x})\Big[2f'(y)\,t_1'(y)\,e^{A-\Phi}+e^{-4f(y)}\partial_y\Big(e^{4f(y)+A-\Phi}t_1'(y)\Big)\Big]\,I\cdot\Sigma\wedge J_K\\&=\Big[t_1(y)\,d_4 g(\vec{x})^2-g(\vec{x})\,h^{(1,1)}(\vec{x},y)\,d_4^c\Big(e^{2(\Phi-A)}\Big)\Big]\wedge\nu_2\,.\end{aligned} \tag{5.24}$$

whilst the final condition from the Bianchi identity gives the 4-form equation

$$\begin{aligned}0=&\frac{e^{-4f(y)}}{g(\vec{x})^2}\partial_y\Big(e^{-4A}\big(2f'(y)+\partial_y(5A-\Phi)\big)\Big)\frac{1}{2}J_K\wedge J_K+\Big[e^{4(\Phi-A)}\big(h^{(1,1)}(\vec{x},y)\big)^2+t_1(y)^2 g(\vec{x})^2\Big]\nu_2\wedge\nu_2\\&-e^{-4f(y)}d_4 d_4^c\Big(g(\vec{x})^{-1}e^{\Phi-5A}\Big)\wedge J_K+2t_1'(y)^2 g(\vec{x})e^{A-\Phi}\Sigma\wedge I\cdot\Sigma\wedge J_K\\&+\frac{e^{-4f(y)}}{4m^2}\Big(d_4P_K\wedge d_4P_K+2d_4P_K\wedge d_4d_4^c\log g(\vec{x})+d_4d_4^c\log g(\vec{x})\wedge d_4d_4^c\log g(\vec{x})\Big)\,.\end{aligned} \tag{5.25}$$

We may rewrite this as a scalar equation:

$$\begin{aligned}&\Box_K R_K - \frac{1}{2}R_K^2 + R^{mn}R_{mn} + 2\Box_K\Box_K \log g(\vec{x}) + 2R\Box_K \log g(\vec{x}) - 4R^{mn}\nabla_m\nabla_n \log g(\vec{x})\\&- 2(\Box_K \log g(\vec{x}))^2 + 2\nabla^m\nabla^n \log g(\vec{x})\Big(\nabla_m\nabla_n \log g(\vec{x}) - I_n{}^r\nabla_r\nabla_s \log g(\vec{x}) I^s{}_m\Big)\\&= 8m^2 e^{4f(y)}\Big[2t_1'(y)^2 g(\vec{x})e^{A-\Phi}|\Sigma|^2 + \frac{2e^{-4f(y)}}{g(\vec{x})^2}\partial_y\Big(e^{-4A}f'(y)\Big)\\&\quad - \Big[e^{4(\Phi-A)}\Big(h^{(1,1)}(\vec{x},y)\Big)^2 + t_1(y)^2 g(\vec{x})^2\Big]|\nu_2|^2\Big]\end{aligned} \tag{5.26}$$

Above we have presented results for a general Kähler base with a generic one-form Σ and its closed primitive two-form field-strength ν_2. We now want to solve the conditions explicitly whilst keeping the generality of our ansatz by not inserting any particular Kähler metric. A useful restriction is to consider four-dimensional toric metrics, this preempts our later discussion. For four-dimensional toric metrics the one-form Σ is d_4^c-closed, in fact it can be written as $d_4^c s$ for some function s. In addition we can introduce symplectic coordinates. This allows us to split the form equations depending on the number of legs along the torus coordinates of the toric action. For example Σ has legs only along the angular coordinates and none along the non-angular coordinates. In addition, note that since ν_2 is primitive it is anti-self-dual and therefore $\nu_2 \wedge \nu_2 \neq 0$.

5.1 Massive GK geometries

In the previous section we have further reduced the conditions for a solution to exist with a warped Kähler metric. In this section we will solve these conditions explicitly, focussing first on the massive case. From (5.18) we see that necessarily

$$d_4\Big(e^{\Phi-A}h^{(1,1)}(\vec{x},y)\Big) = 0\,, \tag{5.27}$$

whilst (5.19) implies

$$\partial_y h^{(1,1)}(\vec{x},y) + 4f'(y)\,h^{(1,1)}(\vec{x},y) = 0\,, \qquad t_1'(y)\,d_4^c\Big(g(\vec{x})e^{A-\Phi}\Big)\wedge\Sigma = 0\,. \tag{5.28}$$

From (5.18) we also have

$$t_1(y)d_4^c(g(\vec{x})e^{A-\Phi})\wedge\nu_2 = f_0 t_1'(y)\Sigma\wedge J_K\,, \tag{5.29}$$

and combining with the previous condition we find that we should impose the two conditions

$$t_1'(y) = 0\,, \qquad d_4\Big(g(\vec{x})\,e^{A-\Phi}\Big) = 0\,. \tag{5.30}$$

From (5.17) it then follows that $f'(y) = 0$ and therefore we have

$$e^{2(A-\Phi)} = 2f_0 y + c(\vec{x})\,. \tag{5.31}$$

It follows from (5.30) that $d_4 c(\vec{x}) = d_4 g(\vec{x}) = 0$ and the only non-trivial remaining conditions fix $h^{(1,1)}(\vec{x},y)$ to be constant. It remains to solve the final scalar constraint in (5.26). The condition reduces to

$$\Box_K R_K - \tfrac{1}{2}R_K^2 + R^{mn}R_{mn} = -8m^2 e^{4f}\Big[(2f_0 y + b_1)^{-2}\Big(h^{(1,1)}\Big)^2 + t_1^2 g^2\Big]|\nu_2|^2\,. \tag{5.32}$$

First note that the left-hand side is the same master equation components that govern GK geometries [9], whilst the right-hand side acts as a flux source term. Since the left-hand side is independent of y it follows that we must set $h^{(1,1)} = 0$.

The final solution is

$$ds^2 = e^{2A}\Big[ds^2(\mathrm{AdS}_3) + \frac{1}{4m^2}(d\psi + P_K)^2 + (d\varphi + t_1\Sigma)^2 + e^{\Phi-5A}\Big(e^{\Phi-A}dy^2 + ds^2(\mathcal{M}_4)\Big)\Big] \tag{5.33}$$

where

$$e^{-4A} = \frac{R_K}{8m^2}\sqrt{2f_0 y + c}\,, \qquad e^{-2\Phi} = \frac{(2f_0 y + c)^{5/4}\sqrt{R_K}}{2\sqrt{2}m}\,, \tag{5.34}$$

where the metric on $\mathcal{M}_4$ is Kähler and satisfies the master equation

$$\Box_K R_K - \tfrac{1}{2}R_K^2 + R^{mn}R_{mn} = 8m^2 t_1^2 |d\Sigma|^2\,. \tag{5.35}$$

The solution is supported by the magnetic fluxes

$$\begin{aligned} &H = 0\,, \quad f_0\,, \quad f_2 = 0\,, \\ &f_4 = t_1 D\varphi\wedge dy\wedge\nu_2 + \frac{1}{4m^2}\Big((d\psi + P_K)\wedge dP_K - \frac{1}{2}\star_4 dR_K\Big)\wedge dy\,, \\ &f_6 = -\frac{1}{4m^2}(d\psi + P_K)\wedge D\varphi\wedge J\wedge dP_K\,. \end{aligned} \tag{5.36}$$

We have presented a general class of solution above determined by solving the master equation, (5.35), for a four-dimensional Kähler base. Note that there is a large similarity between the geometry here and the so-called GK geometries [9] that appear in AdS_3 solutions of Type IIB [7] and AdS_2 solutions of 11d supergravity [8]. One of the advances made in investigating these solutions is the construction of an extremal problem that determines the central charge (IIB)/free energy (11d) of the solution using just the topology of the manifold and without the need for an explicit metric, see [6]. Given the close connection we can also define an extremal problem for our setup, and the first in massive Type IIA.

5.2 The extremal problem

In the remainder of this section we will set $t_1 = 0$. To put the solution into a more amenable parametrisation for the extremal problem we first perform a few redefinitions. First define a new length scale $L = m^{-1}$ and redefine

$$\begin{aligned} &ds^2(\mathrm{AdS}_3) = L^2 ds^2(\mathrm{AdS}_3)^{\mathrm{unit}}\,, \quad ds^2(\mathcal{M}_4) = L^2 ds^2(\mathcal{B}_4)\,, \quad \varphi = L\hat{\varphi}\,, \quad f_0 = \frac{\hat{f}_0}{L}\,, \\ &y = L\frac{-c + h_8(\hat{y})^2}{2\hat{f}_0}\,, \quad h_8(\hat{y}) = \hat{f}_0\hat{y} + \hat{c}\,. \end{aligned} \tag{5.37}$$

It is also useful to define

$$e^B = \frac{R_B}{8}\,, \quad \eta = \frac{1}{2}(d\psi + P)\,, \tag{5.38}$$

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with R_B the (dimensionless) Ricci scalar of the base metric $\mathcal{B}_4$. With these redefinitions the metric becomes

$$L^{-2}ds^2 = \frac{e^{-B/2}}{\sqrt{h_8(\hat{y})}}\Big[ds^2(\mathrm{AdS}_3)^{\mathrm{unit}} + \eta^2 + d\hat{\varphi}^2 + e^B\Big(ds^2(\mathcal{B}_4) + h_8(\hat{y})d\hat{y}^2\Big)\Big]\,. \tag{5.39}$$

We have dropped the subscript "K" on the 1-form P, and all forms will be the defined on $\mathcal{B}_4$ henceforth. Recall that $dP = \rho$ with ρ the Ricci-form of $\mathcal{B}_4$. The metric is then in a form similar to GK geometries, and satisfies the same master equation, this time in four dimensions. The 1-form η is the 1-form dual to R-symmetry Killing vector $\partial_\psi \equiv \xi$ and in keeping with the notation in [6] we call this the R-symmetry vector. We define Y_5 to be the manifold consisting of the U(1) R-symmetry direction fibered over the base $\mathcal{B}_4$. This is the five-dimensional version of a GK geometry.

We can identify the linear function $h_8(\hat{y})$ as the warp factor of D8-branes which are arrayed at fixed points along the $\hat{y}$ line interval and wrap the remaining directions. After the redefinitions the non-trivial dilaton and magnetic fluxes take the form

$$e^{-2\Phi} = h_8(\hat{y})^{5/2}e^{B/2}\,, \tag{5.40}$$

$$L\,f_0 = \partial_{\hat{y}}h_8(\hat{y})\,, \tag{5.41}$$

$$L^{-3}f_4 = \frac{h_8(\hat{y})}{2}\Big(\eta\wedge\rho - \frac{1}{4}\star_4 dR\Big)\wedge d\hat{y}\,, \tag{5.42}$$

$$L^{-5}f_6 = -\frac{1}{2}\eta\wedge\rho\wedge J\wedge d\hat{\varphi}\,. \tag{5.43}$$

For the solution to be well-defined it remains to fix the period of $\hat{y}$ and quantise the fluxes correctly. Supersymmetry and the equations of motion impose that $h_8(\hat{y})$ is a linear function with first derivative the Romans mass. The function $h_8(\hat{y})$ must be continuous but need only be piecewise smooth, in particular we may allow for jumps in the Romans mass in different patches of the line interval, let there be p such jumps. Note that we require $p \geq 1$ for the space to close, if there is no jump the space is non-compact. We can then bound the line interval between two zeroes of $h_8(\hat{y})$. Without loss of generality we may take the smaller root to be at 0 and the second to be at some strictly positive root $\hat{y}_{p+1}$ which caps the space. At the two end-points the degeneration of the metric shows that the space is capped off by O8-planes.

The most general h_8 one may construct is[9]

$$h_8(\hat{y}) = \frac{2\pi\ell_s}{L}\begin{cases}\hat{f}_0^{(0)}\hat{y} & 0\leq\hat{y}\leq\hat{y}_1\,,\\ \quad\vdots & \\ \hat{f}_0^{(i)}(\hat{y}-\hat{y}_i)+c^{(i)} & \hat{y}_i\leq\hat{y}\leq\hat{y}_{i+1}\,,\\ \quad\vdots & \\ \hat{f}_0^{(p)}(\hat{y}-\hat{y}_p)+c^{(p)} & \hat{y}_p\leq\hat{y}\leq\hat{y}_{p+1}\,,\end{cases} \tag{5.44}$$

[9] We include an overall factor depending on the string length for later.

subject to the continuity conditions

$$\hat{f}_0^{(i)}\left(\hat{y}_{i+1}-\hat{y}_i\right)+c^{(i)}=c^{(i+1)}\,, \tag{5.45}$$

$$\hat{f}_0^{(p)}\left(\hat{y}_{p+1}-\hat{y}_p\right)+c^{(p)}=0\,. \tag{5.46}$$

Note that this defines the $c^{(i)}$ iteratively as

$$c^{(i+1)}=\sum_{k=0}^{i}\hat{f}_0^{(k)}\left(\hat{y}_{k+1}-\hat{y}_k\right)\,, \tag{5.47}$$

and a constraint on $\hat{y}_{p+1}$,

$$0=\sum_{k=0}^{p}\hat{f}_0^{(k)}\left(\hat{y}_{k+1}-\hat{y}_k\right)\,. \tag{5.48}$$

In addition we require that $h_8(\hat{y})$ defines a convex curve, this implies

$$\hat{f}_0^{(i)}>\hat{f}_0^{(i+1)}\,,\qquad \forall i \tag{5.49}$$

and guarantees that only D8-branes appear in the bulk as opposed to O8-planes. There are $2p+1$ free parameters, the $p+1$ constants $\hat{f}_0^{(i)}$ and the p locations of a jump y_i, $1\leq i\leq p$. The final end-point is fixed by the choice of this data. In addition as we will see soon this data is further constrained by flux quantisation.

We now want to rephrase the problem of computing the central charge and performing flux quantisation as an extremal problem following [6]. Note that the geometry Y_5 considered here is precisely the $n=2$ version of the theory considered in [6, 9]. Our solution is determined by a $2p+1$-dimensional charge vector containing the D8-brane information, the $\hat{f}_0^{(i)}$ and $\hat{y}_i$ parameters, and a base Y_5 which is of GK type [9] for $n=2$. As such the extremal problem will make use of existing results in the literature, in particular in [6] and the followups [10, 12–17, 78]. For clarity we will review these results repurposed to our problem.

We first fix the complex cone $C(Y_5)$ and endow it with a nowhere-zero closed holomorphic three-form and holomorphic $\mathrm{U}(1)^s$ action. We pick a basis of the $\mathrm{U}(1)^s$ action where the holomorphic volume form has charge 2 under the first basis vector and is uncharged under the remaining $s-1$ vectors. The R-symmetry vector may then be written as

$$\xi=\sum_{I=1}^{s}b_I\partial_{\psi_I}\,. \tag{5.50}$$

The vector $\vec{b}\equiv(b_1,b_2,b_3)$ parametrises the choice of R-symmetry vector, and is subject to $b_1=2$ which should be imposed at the end. We may define the 5d supersymmetric action

$$S_{\mathrm{SUSY}}[\xi,J]=\int_{Y_5}\eta\wedge\rho\wedge J\,, \tag{5.51}$$

which is a functional of the choice of R-symmetry and Kähler metric. Note that it depends only on the cohomology class and not the explicit representative. The master equation (5.35) (with $t_1=0$) may be integrated to obtain

$$0=\int_{Y_5}\eta\wedge\rho\wedge\rho\,, \tag{5.52}$$

which is a necessary condition for the equations of motion to be solved. With the above constraint and the assumption that the cohomology condition

$$H^2(Y_5, \mathbb{R}) \simeq H^2_{\mathcal{B}_4}(\mathcal{F}_\xi)/[\rho] \,, \tag{5.53}$$

holds true, where $\mathcal{F}_\xi$ is the transverse foliation of the R-symmetry vector, we may define a consistent quantisation of the fluxes.

The quantisation condition for the magnetic fluxes is

$$2\pi\ell_s f_0 = n \in \mathbb{Z} \,, \quad \frac{1}{(2\pi\ell_s)^3}\int_{\sigma_{a,i}} f_4 = M_{a,i} \in \mathbb{Z} \,, \quad \frac{1}{(2\pi\ell_s)^5}\int_{\Sigma_A} f_6 = N_A \in \mathbb{Z} \,, \tag{5.54}$$

with $\sigma_{a,i}$ all four-cycles and Σ_A all six-cycles in the geometry. The quantisation condition for the Romans mass implies

$$\hat{f}_0^{(i)} \in \mathbb{Z} \,, \tag{5.55}$$

justifying our choice of normalisation earlier. For the magnetic 6-form flux there is a single relevant six-cycle consisting of Y_5 and the U(1) direction parametrised by $\hat{\varphi}$. Let $\hat{\varphi}$ have period $2\pi\ell_s l_\varphi/L$, then the quantisation condition reads

$$\frac{L^4 l_\varphi}{2(2\pi\ell_s)^4}\int_{Y_5} \eta\wedge\rho\wedge J = N \in \mathbb{Z} \,, \tag{5.56}$$

where we have dropped a total derivative term. This should be understood as the number of D2-branes in the geometry probed by the cone over Y_5 and smeared along the circle. We will turn to evaluating this integral using the results in [6] shortly. The final flux quantisation condition we must consider is the quantisation of the magnetic 4-form flux. The relevant four-cycles in the geometry consist of the union of line-segments with three-cycles in Y_5:

$$\sigma_{a,i} = [\hat{y}_i, \hat{y}_{i+1}] \times \hat{\sigma}_a \,, \tag{5.57}$$

with $\hat{\sigma}_a$ giving a basis of three-cycles in Y_5. In total there are $(p+1)\times b_3(Y_5)$ such cycles to consider. The quantisation condition becomes

$$\left[\frac{L^2}{2(2\pi\ell_s)^2}\int_{\hat{\sigma}_a}\eta\wedge\rho\right]\times\left[\frac{L}{2\pi\ell_s}\int_{\hat{y}_i}^{\hat{y}_{i+1}} h_8(\hat{y})d\hat{y}\right] = M_{a,i}\in\mathbb{Z} \,, \tag{5.58}$$

where we have split the terms suggestively. The first integral may again be computed using the toric formulae in [6] as we will explain shortly. The second may be integrated to give:

$$\begin{aligned}\frac{L}{(2\pi\ell_s)}\int_{\hat{y}_i}^{\hat{y}_{i+1}} h_8(\hat{y})d\hat{y} &= \frac{1}{2}(\hat{y}_{i+1}-\hat{y}_i)(c^{(i+1)}+c^{(i)})\\ &= \frac{1}{2}(\hat{y}_{i+1}-\hat{y}_i)\left[\hat{f}_0^{(i)}(\hat{y}_{i+1}-\hat{y}_i) + 2\sum_{k=0}^{i-1}\hat{f}_0^{(k)}(\hat{y}_{k+1}-\hat{y}_k)\right]\end{aligned} \tag{5.59}$$

To satisfy this quantisation condition let us define

$$\frac{L^2}{2(2\pi\ell_s)^2}\int_{\hat{\sigma}_a}\eta\wedge\rho = M_a \,, \tag{5.60}$$

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and

$$\frac{L}{2\pi\ell_s}\int_{\hat{y}_i}^{\hat{y}_{i+1}} h_8(\hat{y})d\hat{y} = n_i\,, \tag{5.61}$$

so that

$$M_{a,i} = M_a n_i \in \mathbb{Z}\,. \tag{5.62}$$

The simplest possibility to satisfy the condition is to take both M_a and n_i integer, however we may take the more general choice of fixing $M_a \in \mathbb{Z}$ and

$$n_i \in \mathbb{Z}/\mathrm{gcd}(M_a)\,. \tag{5.63}$$

We see that imposing flux quantisation splits into a part dependent on the geometry of Y_5 and a second part dependent only on the D8-brane parameters. We then need to use the results in [6, 17] to evaluate the following integrals for flux quantisation

$$\frac{L^4 l_\varphi}{2(2\pi\ell_s)^4}\int_{Y_5}\eta\wedge\rho\wedge J = N \in \mathbb{Z}\,, \tag{5.64}$$

$$\frac{L^2}{2(2\pi\ell_s)^2}\int_{\hat{\sigma}_a}\eta\wedge\rho = M_a \in \mathbb{Z}\,. \tag{5.65}$$

Next observe that the central charge of the solution can be obtained by using the Brown-Henneaux formula [80] giving

$$\begin{aligned} c &= \frac{48\pi^2 L^8}{(2\pi\ell_s)^8}\int_{M_7} h_8(\hat{y})e^B\frac{1}{2}\eta\wedge J^2\wedge d\hat{\varphi}\wedge d\hat{y} \\ &= \frac{12\pi^2 L^8}{(2\pi\ell_s)^8}\int_{M_7} h_8(\hat{y})\eta\wedge J\wedge\rho\wedge d\hat{\varphi}\wedge d\hat{y}\,, \end{aligned} \tag{5.66}$$

where in the last line we have used the properties of the Ricci-form. In terms of the supersymmetric 5d action this is

$$c = \frac{12\pi^2 L^7 l_\varphi}{(2\pi\ell_s)^7} S_{\mathrm{SUSY}}\int h_8(\hat{y})d\hat{y}\,. \tag{5.67}$$

Strictly this is off-shell expression for the central charge as we have not satisfied the equations of motion yet.

Once the constraint equation (5.52) and flux quantisation have been imposed we can then extremise the supersymmetric action S_{SUSY} over the choice of R-symmetry vector and Kähler parameters. The central charge is then

$$c = \frac{12\pi^2 L^7 l_\varphi}{(2\pi\ell_s)^7} S_{\mathrm{SUSY}}\Big|_{\text{on-shell}}\int h_8(\hat{y})d\hat{y}\,. \tag{5.68}$$

Using the flux quantisation we may rewrite this as

$$c = \frac{48\pi^2 N M_1}{\int_{\hat{\sigma}_1}\eta\wedge\rho}\sum_{i=0}^{p} n_i\,, \tag{5.69}$$

where we picked a particular flux number M_1. Note the similarity with the expression in [6]. The presence of the D8-branes lead to a deformation of the central charge by the n_i dependent piece. One must still impose that the flux parameters M_a are related as

$$\frac{M_1}{M_{a\neq 1}} = \frac{\int_{\hat{\sigma}_1} \eta \wedge \rho}{\int_{\hat{\sigma}_a} \eta \wedge \rho} \,. \tag{5.70}$$

We now want to evaluate the final integrals for flux quantisation. We may directly use the expressions in [6] if we assume that Y_5 is toric or the more elegant expressions using the master volume in [17]. This was later extended to toric manifolds fibered over a Kähler base in [13] and one could in principle use their results for an S^3 fibered over a general Riemann surface or an S^1 over a 4d Kähler base.[10] Rather than presenting these more complicated cases we will present the simpler case when Y_5 is toric for completeness.

If Y_5 is toric it means we have a holomorphic $\mathrm{U}(1)^s$ action. This defines a set of vectors v_a, $a = 1, \ldots, d$ which are inward pointing normals to the facets of the polyhedral cone and define the geometry. We refer the reader to [79] for a more detailed exposition of toric geometry. One can define the master volume

$$\mathcal{V} \equiv \int_{Y_5} \frac{1}{2} \eta \wedge J \wedge J \,. \tag{5.71}$$

The Kähler form may be expanded in a basis C_a of basic representatives of $H^2_{\mathcal{B}_4}(\mathcal{F}_\xi)$, see [17, 79], where the two-forms C_a are Poincaré dual to the restriction of toric divisors of the cone $C(Y_5)$, as

$$[J] = -2\pi \sum_{a=1}^{d} \lambda_a C_a \,. \tag{5.72}$$

Only $d-3$ of the C_a are independent [79], and thus only $d-3$ of the Kähler parameters λ_a will appear in the expressions. The Ricci form can be expanded similarly as

$$\rho = 2\pi \sum_{a=1}^{d} C_a \,. \tag{5.73}$$

The master volume of Y_5 can then be determined in terms of the toric data as

$$\mathcal{V}\left(\vec{b}, \lambda_a, \vec{v}_a\right) = \frac{(2\pi)^3}{2} \sum_{a=1}^{d} \lambda_a \frac{\lambda_{a-1}\left(\vec{v}_a, \vec{v}_{a+1}, \vec{b}\right) - \lambda_a\left(\vec{v}_{a-1}, \vec{v}_{a+1}, \vec{b}\right) + \lambda_{a+1}\left(\vec{v}_{a-1}, \vec{v}_a, \vec{b}\right)}{\left(\vec{v}_{a-1}, \vec{v}_a, \vec{b}\right)\left(\vec{v}_a, \vec{v}_{a+1}, \vec{b}\right)} \,. \tag{5.74}$$

[10] Strictly these should be free of orbifold singularities.

It then follows that the integrals we needed for flux quantisation and the constraint equation may be determined in terms of the master volume as

$$\int_{Y_5} \eta \wedge \rho \wedge \rho = \sum_{a,b=1}^{d} \frac{\partial^2 \mathcal{V}}{\partial \lambda_a \partial \lambda_b}\,, \tag{5.75}$$

$$\int_{Y_5} \eta \wedge \rho \wedge J = -\sum_{a=1}^{d} \frac{\partial \mathcal{V}}{\partial \lambda_a}\,, \tag{5.76}$$

$$\int_{\hat{\sigma}_a} \eta \wedge \rho = \frac{1}{2\pi} \sum_{b=1}^{d} \frac{\partial^2 \mathcal{V}}{\partial \lambda_a \partial \lambda_b}\,. \tag{5.77}$$

With the above expressions we may perform the quantisation of the fluxes, extremise the action and obtain the central charge. First we determine the constraint. Note that since the master volume is quadratic in λ_a the constraint

$$\sum_{a,b=1}^{d} \frac{\partial^2 \mathcal{V}}{\partial \lambda_a \partial \lambda_b} = 0\,, \tag{5.78}$$

is independent of λ and must be solved for b_3 (or b_2). We can solve for the flux parameter N in terms of one of the λ_a,

$$N = -\frac{2\,(2\pi \ell_s)^4}{L^4 l_\varphi} \sum_{a=1}^{d} \frac{\partial \mathcal{V}}{\partial \lambda_a} \tag{5.79}$$

Next the fluxes M_a are given by

$$M_a = \frac{2\,(2\pi \ell_s)^2}{2\pi L^2} \sum_{b=1}^{d} \frac{\partial^2 \mathcal{V}}{\partial \lambda_a \partial \lambda_b}\,, \tag{5.80}$$

and are independent of the λ_a. We must now impose that

$$M_1 \sum_{b=1}^{d} \frac{\partial^2 \mathcal{V}}{\partial \lambda_a \partial \lambda_b} = M_a \sum_{b=1}^{d} \frac{\partial^2 \mathcal{V}}{\partial \lambda_1 \partial \lambda_b}\,, \tag{5.81}$$

for all a. This imposes $b_3(Y_5) - 1$ constraints for a single free variable. If $b_3(Y_5) = 1$ we retain the free variable b_2 which is fixed by extremising the trial central charge

$$c = \frac{96\pi^3 N M_1}{\sum_{b=1}^{d} \frac{\partial^2 \mathcal{V}}{\partial \lambda_1 \partial \lambda_b}} \sum_{i=0}^{p} n_i\,, \tag{5.82}$$

over the remaining parameter b_2.[11] If $b_3(Y_5) = 2$ then we may use (5.81) to fix b_2 in terms of the flux parameters, and then there is nothing left to extremise. For $b_3(Y_5) > 2$ not only is there nothing to extremise and b_2 is fixed in terms of the fluxes once again but the fluxes M_a also satisfy a non-trivial constraint. This agrees with the analysis in [6] of 7d GK geometries with a T^2 factor.

[11]This is independent of the λ's again.

Brane	$\mathbb{R}^{1,1}$		$C(Y_5)$						U(1)	I
							C			
D2	×	×	•	•	•	•	•	•	–	×
D4	×	×	–	–	–	–	×	×	×	•
D8	×	×	×	×	×	×	×	×	×	•

Table 1. The brane configuration arising from our setup. The D2-branes are located at the tip of the cone over Y_5, smeared over the circle and lie along the line interval. The D4-branes wrap a two-cycle in Y_5, denoted by C in the table, along with the circle and are located at the special points of the line interval. Finally the D8-branes are located at the distinguished points of the line interval.

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Brane construction

The gravity solution suggests the brane realisation given in table 1. This consists of a stack of N D2-branes probed by the cone over Y_5, with flavour D4 and D8-branes located at the distinguished points along the line interval. One should think of this brane construction as giving rise to a quiver field theory consisting of subquivers joined together and flavoured by D8-branes. Each subquiver is given by the field theory living on a stack of N D2-branes probed by $C(Y_5)$, smeared along a circle and wrapping a finite length line interval. It would be interesting to construct such a dual field theory in the future.

Another interesting point to highlight is the connection of these geometries to the F-theory ones discussed in [11, 43, 44] and for which an extremal problem was derived in [10] by making use of M/F-duality. Dualising along the U(1) to Type IIB the D8-branes become 7-branes and the Romans mass becomes a non-trivial axion. The metric is then of the class discussed in section 4.2.2 of [11] and should give a local description of the base of an elliptically fibered K3 surface or equivalently locally is the hyper-Kähler manifold

$$ds^2\left(\text{HK}\right) = h_8\left(\hat{y}\right)\left(dy^2 + dx_1^2 + dx_2^2\right) + \frac{1}{h_8\left(\hat{y}\right)}\left(d\hat{\varphi} + h_8'\left(\hat{y}\right)x_1 dx_2\right)^2 . \tag{5.83}$$

It would be interesting to map the extremal problem here into the one considered in [10].

6 Geometric conditions for arbitrary extended chiral supersymmetry

In this section we give necessary and sufficient conditions for AdS_3 solutions of Type II supergravity to preserve arbitrary extended chiral supersymmetry — i.e. $\mathcal{N} = (n, 0)$ for $2 < n \leq 8$ (The case of $\mathcal{N} = (8, 0)$ is maximal [81]). As we shall see, these conditions are actually implied by the $\mathcal{N} = (2, 0)$ conditions of section 2.

A solution preserving $\mathcal{N} = (n, 0)$ supersymmetry must support two n-tuplets of Majorana spinors on M_7, $\chi^I_{1,2}$ for $I = 1, 2, \ldots, n$. In terms of these one can define $\frac{n}{2}(n-1)$ independent $\mathcal{N} = 2$ sub-sectors, the same number of independent components as a $\dim(n)$ anti-symmetric matrix. Each of these sub-sectors must obey the conditions of section 2 for potentially differing $(\xi, \tilde{\xi}, c, \Psi_\pm)$. However, by exploiting constant $\text{GL}(n,\mathbb{R})$ transformations of $\chi^I_{1,2}$ one can take them to obey

$$\chi_1^{I\dagger}\chi_1^J + \chi_2^{I\dagger}\chi_2^J = 2e^A\delta^{IJ} , \qquad \chi_1^{I\dagger}\chi_1^J - \chi_2^{I\dagger}\chi_2^J = ce^{-A}\delta^{IJ} , \tag{6.1}$$

without loss of generality — so that they share the same c. In terms of these one can define the real 1-form valued $n \times n$ anti-symmetric matrices

$$\xi^{IJ} = -i\left(\chi_1^I \gamma_a \chi_1^J \mp \chi_2^I \gamma_a \chi_2^J\right) \mathrm{e}^a, \quad \tilde{\xi}^{IJ} = -i\left(\chi_1^I \gamma_a \chi_1^J \pm \chi_2^I \gamma_a \chi_2^J\right) \mathrm{e}^a\,, \tag{6.2}$$

where the vectors dual to the components of ξ^{IJ} are all Killing vectors with respect to the entire solution under which $\chi^I_{1,2}$ are charged. The necessary and sufficient conditions for $\mathcal{N} = (n, 0)$ supersymmetry can then be expressed covariantly as

$$d\tilde{\xi}^{IJ} = \iota_{\xi^{IJ}} H \qquad e^{3A} h_0 = -mc\,, \tag{6.3a}$$

$$d_H\left(e^{A-\Phi}\Psi_\mp^{(IJ)}\right) = \mp\frac{c}{16}\delta^{IJ} f_\pm\,, \tag{6.3b}$$

$$d_H\left(e^{2A-\Phi}\Psi_\pm^{(IJ)}\right) \mp 2me^{A-\Phi}\Psi_\mp^{(IJ)} = \frac{1}{8}e^{3A} \star_7 \lambda f_\pm \delta^{IJ}\,, \tag{6.3c}$$

$$d_H\left(e^{-\Phi}\Psi_\pm^{[IJ]}\right) = \frac{1}{16}\left(\tilde{\xi}^{IJ} \wedge + \iota_{\xi^{IJ}}\right) f_\pm\,, \tag{6.3d}$$

$$d_H\left(e^{3A-\Phi}\Psi_\mp^{[IJ]}\right) = \mp e^{3A-\Phi} h_0 \Psi_\pm^{[IJ]} \pm \frac{1}{16}\left(\tilde{\xi}^{IJ} \wedge + \iota_{\xi^{IJ}}\right) e^{3A} \star_7 \lambda f_\pm\,, \tag{6.3e}$$

$$\left(\Psi_\mp^{(IJ)}, f_\mp\right)_7 = \mp\frac{1}{2}\delta^{IJ}\left(m + \frac{1}{4}e^{-A} c h_0\right) e^{-\Phi} \mathrm{vol}\left(\mathrm{M}_7\right)\,, \tag{6.3f}$$

which we should stress contain many redundant expressions. What considering the $\frac{n}{2}(n-1)$ independent $\mathcal{N} = (2, 0)$ sub-sectors does not tell us however is the following

1. How many of the $\frac{n}{2}(n-1)$ Killing vectors dual to ξ^{IJ} are independent.
2. How $\chi^I_{1,2}$ transform under every component of ξ^{IJ}.

To be clear we do know how the Killing vector associated to each $\mathcal{N} = (2, 0)$ sub-sector acts on the spinors that make up that sector — what we don't know is how they act on the remaining spinors of the n-tuplet. If we were considering for instance AdS_4 solutions with extended supersymmetry, which have superconformal group $\mathrm{OSp}(n|4)$ and spinors transforming in the $\mathbf{n}$ of the $\mathrm{SO}(n)$ R-symmetry, it would be clear that ξ^{IJ} should contain all $\frac{n}{2}(n-1)$ independent SO(n) Killing vectors (and possibly some additional flavour isometries). The structure of the chiral superconformal algebras for AdS_3 with extended supersymmetry is however more rich than the higher dimensional cases [82]. There exists the analogous possibility of $\mathrm{OSp}(n|2)$ with spinors in the fundamental of the $\mathrm{SO}(n)$ R-symmetry, however there are several other options. Several of these have R-symmetry groups of dimension less than $\frac{n}{2}(n-1)$, for small $\mathcal{N} = (4, 0)$ for instance it is SU(2). The algebras that are consistent with AdS_3 solutions (the simple Lie super-algebras) can be classified in terms of the Lie algebra of their R-symmetry $\mathfrak{g}$ and a corresponding representation $\rho_{\mathfrak{g}}$ [83], which $\chi^I_{1,2}$ should transform in under $\mathfrak{g}$. There should thus exist a real basis of $\mathfrak{g}$, $T^a_{\mathfrak{g}}$ for $a = 1, \ldots, \dim(\mathfrak{g})$, in the representation $\rho_{\mathfrak{g}}$ such that

$$\mathcal{L}_{K^a_{\mathfrak{g}}} \chi^I_{1,2} = (T^a_{\mathfrak{g}})^{IJ} \chi^J_{1,2}\,, \tag{6.4}$$

where $K^a_{\mathfrak{g}}$ are the Killing vectors of $\mathfrak{g}$. This leads us to make the mild conjecture that the different possibilities for super-conformal algebras can be distinguished by decomposing

$$\xi^{IJ} = -8mc^a K^a_{\mathfrak{g}} \left(T^a_{\mathfrak{g}}\right)^{IJ} + \xi^{IJ}_0\,, \qquad \mathcal{L}_{\xi^{IJ}_0}\chi^L_{1,2} = 0\,, \tag{6.5}$$

where ξ^{IJ}_0 are some additional flavour (or uncharged) isometries in M_7 that we cannot exclude the possibility of. Here c^a are a set of constants one needs to keep arbitrary for consistency with large $\mathcal{N} = (4,0)$ which depends on a continuous parameter — we expect $c_a = 1$ in all other cases. Note we are assuming conventions where

$$\left[T^a_{\mathfrak{g}}, T^b_{\mathfrak{g}}\right] = -f^{abc}T^c_{\mathfrak{g}}\,, \qquad \text{Tr}\left(T^a_{\mathfrak{g}}T^b_{\mathfrak{g}}\right) = -\frac{n}{4}\delta^{ab}\,, \qquad \left[K^a_{\mathfrak{g}}, K^b_{\mathfrak{g}}\right] = f^{abc}K^c_{\mathfrak{g}}\,, \tag{6.6}$$

which is the reason for the $-8m$ in the first expression in (6.5). We have explicitly checked this proposal for the classes of small $\mathcal{N} = (4,0)$ solutions in [38] and [59], the large $\mathcal{N} = (4,0)$ solutions in [36] and the $\mathcal{N} = (3,0)$ solutions in [37]. In these cases one finds $\xi^{IJ}_0 = 0$, though we are aware of some examples for which this is not the case (see the discussion about a priori isometries in [59]).

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A AdS$_3$ spinors and bi-linears

In this appendix we will give some details on the spinors and bi-linears of AdS$_3$, which supplements the following appendix.

AdS$_3$ is a maximally symmetric space with global SO(2,2) = SL(2)$_+\times$SL(2)$_-$ symmetry and Ricci tensor Ricci(AdS$_3$) $= -2m^2 g(\text{AdS}_3)$. It comes equipped with Killing spinors charged under SL(2)$_\pm$ defined through the Killing spinor equation

$$\nabla_\mu\zeta_\pm = \pm\frac{m}{2}\gamma^{(3)}_\mu\zeta_\pm\,, \tag{A.1}$$

where in this work we will be interested in solution preserving SL(2)$_+$ specifically. A particular parameterisation of AdS$_3$ is given by the vielbein

$$\text{e}^0 = e^{mr}dt\,, \qquad \text{e}^1 = e^{mr}dx\,, \qquad \text{e}^2 = dr\,, \tag{A.2}$$

and in terms of this, one can show that ζ_+ decomposes to two independent components as

$$\zeta_+ = c_1\zeta^P + c_2\zeta^C\,, \qquad \zeta^P = \begin{pmatrix} e^{\frac{m}{2}r} \\ 0 \end{pmatrix}, \qquad \zeta^C = e^{\frac{m}{2}r}\begin{pmatrix} m(t+x)e^{\frac{m}{2}r} \\ e^{-\frac{m}{2}r} \end{pmatrix}. \tag{A.3}$$

where c_1, c_2 are constants, ζ^P is the Poincaré (or spacetime) supercharge and ζ^C is the conformal supercharge — together realising $\mathcal{N} = (1,0)$ superconformal symmetry. Note we have taken $\gamma^{(3)}_\mu = (i\sigma_2, \sigma_1, \sigma_3)_\mu$ here. In terms of these one can define the $\text{SL}(2)_+$ doublet

$$\zeta^I = \begin{pmatrix} \zeta^P \\ \zeta^C \end{pmatrix}, \tag{A.4}$$

which gives rise to a matrix of bi-linears of the form

$$\zeta^I \otimes \bar\zeta^J = \frac{1}{2}\begin{pmatrix} v_1\wedge(1-u) & 1+u-\frac{1}{2}v_1\wedge v_2 - \text{vol}(\text{AdS}_3) \\ -1+u-\frac{1}{2}v_1\wedge v_2 + \text{vol}(\text{AdS}_3) & v_2\wedge(1+u) \end{pmatrix} \tag{A.5}$$

where in terms of (A.2) the various 1-forms that appear here are

$$v_1 = e^{2mr}(dt-dx)\,, \qquad v_2 = m^2(t+x)v_1 + 2m(t+x)dr + dt + dx\,, \qquad u = dr + m(t+x)v_1\,. \tag{A.6}$$

These obey the following simple identities

$$dv_1 = -2mv_1\wedge u\,, \quad dv_2 = 2mv_2\wedge u\,, \quad du = -mv_1\wedge v_2\,, \quad v_1\wedge v_2\wedge u = 2\text{vol}(\text{AdS}_3)\,, \tag{A.7}$$

which imply that $d(\zeta^I\otimes\bar\zeta^J) = 2m(\zeta^I\otimes\bar\zeta^J)_2$ and $\zeta^I\otimes\bar\zeta^J\wedge\text{vol}(\text{AdS}_3) = -(\zeta^I\otimes\bar\zeta^J)_3$. Further one can show that these 1-forms obey the following conditions under the Lie derivative and interior product

$$\begin{aligned} &\nabla_{(\mu}(u_i)_{\nu)} = 0\,, \quad \mathcal{L}_{u_i}u_j = 2mf_{ijk}u_k\,, \qquad u_i = (\frac{1}{2}(v_1+v_2), \frac{1}{2}(v_1-v_2), u)_i\,, \\ &\langle v_1, v_1\rangle = \langle v_2, v_2\rangle = \langle v_1, u\rangle = \langle v_2, u\rangle = 0\,, \qquad \langle v_1, v_2\rangle = -2\,, \qquad \langle u, u\rangle = 1\,. \end{aligned} \tag{A.8}$$

where f_{ijk} are the structure constants of SL(2), i.e. the Lie alebra of SL(2) is spanned by $\tau_i = \frac{1}{2}(i\sigma_2, \sigma_1, \sigma_3)_i$ which are such that $[\tau_i, \tau_j] = f_{ijk}\tau_k$. We thus have that $((v_1)^\mu\partial_\mu, (v_2)^\mu\partial_\mu)$ are null Killing vectors, $(u_1)^\mu\partial_\mu$ is a space-like Killing vector and $((\zeta^I\otimes\bar\zeta^J)_1)^\mu\partial_\mu$ is a symmetric matrix containing the three independent Killing vectors of SL(2). It also follows that

$$\iota_{(\zeta^I\otimes\bar\zeta^J)_1}\text{vol}\,(\text{AdS}_3) = -\left(\zeta^I\otimes\bar\zeta^J\right)_2\,. \tag{A.9}$$

For the following appendix it will be useful to decompose (A.5) as

$$\zeta^I\otimes\bar\zeta^J = \frac{1}{2}\left(\delta^{IJ}\psi^{(0)} + (\sigma_1)^{IJ}\psi^{(1)} + i(\sigma_2)^{IJ}\psi^{(2)} + (\sigma_3)^{IJ}\psi^{(3)}\right), \tag{A.10}$$

where everything here is real and $\Sigma^A = (\mathbb{I}, \sigma_1, i\sigma_2, \sigma_3)^A$ is such that $(\Sigma^A)^{IJ}(\Sigma^B)^{IJ} = 2\delta^{AB}$. Then defining a time-like Killing vector $k^\mu\partial_\mu$ through its dual 1-form as

$$k = u_1 = \frac{1}{2}(v_1+v_2)\,, \tag{A.11}$$

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one has that

$$\mathcal{L}_k\left(\psi^{(0)}+i\psi^{(2)}\right)=0\,,\qquad \mathcal{L}_v\left(\psi^{(1)}+i\psi^{(3)}\right)=2mi\left(\psi^{(1)}+i\psi^{(3)}\right)\,,$$
$$\iota_k\left(\psi^{(0)}+i\psi^{(2)}\right)=i\left(\psi^{(0)}_2+i\psi^{(2)}_0\right)\,,\qquad \iota_k\left(\psi^{(1)}+i\psi^{(3)}\right)=i\left(\psi^{(1)}_1+i\psi^{(3)}_1\right)\,,$$
$$k\wedge\left(\psi^{(0)}+i\psi^{(2)}\right)=i\left(\psi^{(0)}_1+i\psi^{(2)}_3\right)\,,\qquad k\wedge\left(\psi^{(1)}+i\psi^{(3)}\right)=i(\psi^{(1)}_2+i\psi^{(3)}_2)\,,\quad \text{(A.12)}$$

i.e. $\psi^{0,2}$ are singlets under $v^\mu\partial_\mu$ while $\psi^{1,3}$ are charged. These expressions will be important for identifying the 7d bi-linears charged under the U(1) R-symmetry in the following appendix. Note that we also have

$$\mathcal{L}_k\zeta^I=m\epsilon^{IJ}\zeta^J\,, \qquad \text{(A.13)}$$

providing a map between the two supercharges contained in ζ^I.

B Detailed derivation of geometric conditions for $\mathcal{N}=(2,0)$ AdS$_3$

In this appendix we will give a detailed derivation of the necessary and sufficient conditions for an AdS$_3$ solution of Type II supergravity to preserve $\mathcal{N}=(2,0)$ supersymmetry; we will make use of an existing classification for totally generic Type II solutions [84].

A solution of Type II supergravity preserving SO(2,2) in terms of an AdS$_3$ factor can in general be written in the form

$$ds^2=e^{2A}ds^2(\text{AdS}_3)+ds^2(\text{M}_7)\,,\quad H^{(10\text{d})}=e^{3A}h_0\text{vol}(\text{AdS}_3)+H\,,$$
$$F=f_\pm+e^{3A}\text{vol}(\text{AdS}_3)\wedge\star_7\lambda(f_\pm)\,,$$

where $e^{2A}, H, f_\pm$ and the dilaton Φ have support on M$_7$ alone and $e^{3A}h_0$ is a constant. Here $f_+=f_0+f_2+f_4+f_6$ should be taken in IIA and $f_-=f_1+f_3+f_5+f_7$ should be taken in IIB; this convention for upper/lower signs will be used throughout. For such a solution to preserve $\mathcal{N}=(2,0)$ supersymmetry its $d=10$ Majorana-Weyl Killing spinors should decompose as

$$\epsilon_1=\sum_{I=1}^2\zeta^I\otimes\theta_+\otimes\chi^I_1,\quad \epsilon_2=\sum_{I=1}^2\zeta^I\otimes\theta_\mp\otimes\chi^I_2\,, \qquad \text{(B.1)}$$

where ζ^I are a doublet of SL(2)$_+$ Killing spinors on AdS$_3$ that are Majorana, $\chi^I_{1,2}$ are independent doublets of Majorana spinors on M$_7$ and $\theta_\pm$ are the auxiliary vectors one always needs when decomposing an even-dimensional space in terms of two odd ones; they parameterise the $d=10$ chirality indicated by $\pm$. The astute reader will note that by identifying ζ^I appearing in (B.1) with the SL(2) doublet of the previous section we are only manifestly preserving two real supercharges, where as $\mathcal{N}=(2,0)$ superconformal symmetry preserves four. The resolution to this naive paradox is that (A.13) ensures that there are another two supercharges any solution consistent with (B.1) must also be consistent with, i.e. $\epsilon_{1,2}$ as defined above, but for $\zeta^I \to \epsilon^{IJ}\zeta^J$ (that is unless $m=0$ sending AdS$_3$ $\to$Mink$_3$). We shall take the $d=10$ gamma matrices to decompose as

$$\Gamma_\mu = e^A\gamma_\mu^{(3)} \otimes \sigma_3 \otimes \mathbb{I}_8\,, \qquad \Gamma_a = \mathbb{I}_2 \otimes \sigma_1 \otimes \gamma_a\,, \tag{B.2}$$

where $\gamma_\mu^{(3)}$ are the real Cliff(1,2) gamma matrices of the previous appendix and γ_a are a set of gamma matrices on M$_7$ such that $i\gamma_{1234567}=1$. The chirality matrix and intertwiner defining Majorana conjugation can then be taken to be

$$\hat{\Gamma} = -\mathbb{I}_2 \otimes \sigma_2 \otimes \mathbb{I}_8\,, \qquad B^{(10)} = \mathbb{I}_2 \otimes \sigma_3 \otimes B\,, \qquad B\gamma_a B^{-1} = -\gamma_a^*\,, \quad BB^* = 1\,. \tag{B.3}$$

Then given that we must have $\epsilon^c_{1,2} = B^{(10)}\epsilon^*_{1,2} = \epsilon_{1,2}$ we can without loss of generality take ζ^I to be the real doublet of AdS$_3$ spinors in the previous appendix and

$$\theta_\pm = \frac{1}{2}\begin{pmatrix} 1 \\ \mp i \end{pmatrix}. \tag{B.4}$$

From this point we begin to make use of [84] which give necessary and sufficient geometric conditions for supersymmetry of any Type II solution, the fundamental objects are the following bi-linears in ten dimensions

$$K = \frac{1}{64}(\bar{\epsilon}_1\Gamma_M\epsilon_1 + \bar{\epsilon}_2\Gamma_M\epsilon_2)dX^M\,, \quad \tilde{K} = \frac{1}{64}(\bar{\epsilon}_1\Gamma_M\epsilon_1 - \bar{\epsilon}_2\Gamma_M\epsilon_2)dX^M\,, \qquad \Psi^{(10\text{d})} = \epsilon_1 \otimes \bar{\epsilon}_2\,. \tag{B.5}$$

The first necessary condition we consider is that $K^M\partial_M$ is a Killing vector under which both the bosonic supergravity fields and $\Psi^{(10\text{d})}$ are singlets. Given our ansatz we find

$$K = \frac{1}{32}\left(e^A\left(\zeta^I \otimes \bar{\zeta}^J\right)_1\left(\chi_1^{J\dagger}\chi_1^I + \chi_2^{J\dagger}\chi_2^I\right) - \xi\right), \quad \tilde{K} = \frac{1}{32}\left(e^A\left(\zeta^I \otimes \bar{\zeta}^J\right)_1\left(\chi_1^{J\dagger}\chi_1^I - \chi_2^{J\dagger}\chi_2^I\right) - \tilde{\xi}\right), \tag{B.6}$$

for the real $d=7$ one-forms

$$\xi = -i\left(\chi_1^1\gamma_a\chi_1^2 \mp \chi_2^1\gamma_a\chi_2^2\right)\text{e}^a, \qquad \tilde{\xi} = -i\left(\chi_1^1\gamma_a\chi_1^2 \pm \chi_2^1\gamma_a\chi_2^2\right)\text{e}^a\,, \tag{B.7}$$

where e^a is a vielbein on M$_7$. Imposing $\nabla_{(M}K_{N)} = \mathcal{L}_K\Phi = \mathcal{L}_K F = \mathcal{L}_K H^{(10\text{d})} = 0$ and making use of the fact that $(\zeta^I \otimes \bar{\zeta}^J)_1$is a matrix containing 1-forms dual to Killing vectors on AdS$_3$, we find the $d=7$ conditions

$$\begin{aligned} \nabla_{(a}\xi_{b)} = 0, \quad \mathcal{L}_\xi A = \mathcal{L}_\xi\Phi = \mathcal{L}_\xi f_\pm = \mathcal{L}_\xi H = 0\,, \\ d(e^{-A}\left(\left(\chi_1^{(I\dagger}\chi_1^{J)} + \chi_2^{(I\dagger}\chi_2^{J)}\right)\right) = 0\,, \end{aligned} \tag{B.8}$$

necessary follow — i.e. $\xi^a\partial_a$ is a Killing vector with respect to all bosonic supergravity fields and

$$\chi_1^{I\dagger}\chi_1^J + \chi_2^{I\dagger}\chi_2^J = e^A c_+^{IJ}\,, \tag{B.9}$$

for c_+^{IJ} a symmetric (this follows because $\chi_{1,2}^I$ are Majorana) constant matrix such that $c^{11}, c^{22} > 0$. The next necessary condition we consider is $d\tilde{K} = \iota_K H^{(10\text{d})}$ which gives rise to the $d=7$ conditions

$$d\tilde{\xi} = \iota_\xi H\,, \qquad \chi_1^{I\dagger}\chi_1^J - \chi_2^{I\dagger}\chi_2^J = e^{-A}c_-^{IJ}\,, \qquad 2mc_-^{IJ} = -c_+^{IJ}e^{3A}h_0\,, \tag{B.10}$$

for c_-^{IJ} another symmetric constant matrix. We can now use a constant GL(2,$\mathbb{R}$) of $\chi^I_{1,2}$ (which can be absorbed with a corresponding inverse transformation of ζ^I in (B.1)) to fix

$$c_+^{IJ} = 2\delta^{IJ}\,, \qquad c_-^{IJ} = \delta^{IJ} c \qquad \Rightarrow \qquad e^{3A} h_0 = -mc\,, \tag{B.11}$$

without loss of generality. This refines (B.6) as

$$K = \frac{1}{32}\left(2e^{2A} k - \xi\right)\,, \quad \tilde{K} = \frac{1}{32}\left(ck - \tilde{\xi}\right)\,, \tag{B.12}$$

for $k^\mu \partial_\mu$ the time-like Killing vector on AdS$_3$ defined in the previous appendix. Now we turn our attention to the bi-linear $\Psi^{(10\text{d})}$ which must obey the necessary condition

$$\left(d - H^{(10\text{d})}\wedge\right)\left(e^{-\Phi}\Psi^{(10\text{d})}\right) = -\left(\tilde{K}\wedge + \iota_K\right) F\,. \tag{B.13}$$

We find that the objects appearing here decompose as

$$\begin{aligned} 2\Psi^{(10\text{d})} &= \left(\zeta^I \otimes \bar{\zeta}^J\right)_0 \left(\Psi^{[IJ]}\right)_\pm \mp \left(\zeta^I \otimes \bar{\zeta}^J\right)_3 \wedge \left(e^{3A}\Psi^{[IJ]}\right)_\mp \\ &\quad + \left(\zeta^I \otimes \bar{\zeta}^J\right)_2 \wedge \left(e^{2A}\Psi^{(IJ)}\right)_\pm \mp \left(\zeta^I \otimes \bar{\zeta}^J\right)_1 \wedge \left(e^{A}\Psi^{(IJ)}\right)_\mp \end{aligned} \tag{B.14}$$

$$\begin{aligned} -\left(\tilde{K}\wedge + \iota_K\right) F &= \frac{1}{32}\epsilon^{IJ}\left(\left(\zeta^I \otimes \bar{\zeta}^J\right)_0 \left(\tilde{\xi}\wedge + \iota_\xi\right) f + e^{3A}\left(\zeta^I \otimes \bar{\zeta}^J\right)_3 \wedge \left(\tilde{\xi}\wedge + \iota_\xi\right) \star_7 \lambda f_\pm\right) \\ &\quad + \frac{1}{16} e^{3A}\delta^{IJ}\left[-\frac{c}{2}\left(\zeta^I \otimes \bar{\zeta}^J\right)_1 \wedge f_\pm + \left(\zeta^I \otimes \bar{\zeta}^J\right)_2 \wedge \star_7 \lambda f_\pm\right]\,, \end{aligned}$$

where we define the $d = 7$ matrix bi-linear

$$\Psi^{IJ} \equiv \chi_1^I \otimes \chi_2^{J\dagger}\,. \tag{B.15}$$

Plugging this into (B.13) yields the $d = 7$ differential bi-linear constraints

$$d_H\left(e^{A-\Phi}\Psi_\mp^{(IJ)}\right) = \mp\frac{c}{16}\delta^{IJ} f_\pm\,, \tag{B.16a}$$

$$d_H\left(e^{2A-\Phi}\Psi_\pm^{(IJ)}\right) \mp 2me^{A-\Phi}\Psi_\mp^{(IJ)} = \frac{1}{8}e^{3A}\star_7 \lambda f_\pm \delta^{IJ}\,, \tag{B.16b}$$

$$d_H\left(e^{-\Phi}\Psi_\pm^{[IJ]}\right) = \frac{1}{16}\epsilon^{IJ}\left(\tilde{\xi}\wedge + \iota_\xi\right) f_\pm\,, \tag{B.16c}$$

$$d_H\left(e^{3A-\Phi}\Psi_\mp^{[IJ]}\right) = \mp e^{3A-\Phi} h_0 \Psi_\pm^{[IJ]} \pm \frac{1}{16}\epsilon^{IJ}\left(\tilde{\xi}\wedge + \iota_\xi\right) e^{3A}\star_7 \lambda f_\pm\,, \tag{B.16d}$$

where we note that the (11) and (22) components of (B.16a)–(B.16b) reproduce the differential $\mathcal{N} = (1,0)$ conditions presented in [58], as they should. The conditions derived thus far are not sufficient for supersymmetry to hold, for that one must also solve the pairing constraints, namely (3.1c)-(3.1d) of [84]. Generically these are the hardest conditions to deal with, however in this case we can rely on earlier AdS$_3$ work for $\mathcal{N} = (1,0)$ solutions [35, 44, 58]

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(respectively the original work, first to use these conventions and first to generalise to $c \neq 0$) which informs us that these conditions are implied by

$$(\Psi^{11}_{\mp}, f_{\mp})_7 = (\Psi^{22}_{\mp}, f_{\mp})_7 = \mp\frac{1}{2}\left(m + \frac{1}{4}e^{-A}ch_0\right)e^{-\Phi}\text{vol}(\text{M}_7)\,, \tag{B.17}$$

however one can show that the difference of these conditions is actually implied by the trace of (B.16b) — a similar outcome was found for $\mathcal{N} = (1,1)$ AdS_3 and the steps to show this are analogous (see appendix C of [58]). Further one can show that $(\Psi^{(1)}_{\mp}, f_{\mp})_7 = (\Psi^{(3)}_{\mp}, f_{\mp})_7 = 0$ allowing one to write the pairing constraints in a covariant fashion as

$$(\Psi^{(IJ)}_{\mp}, f_{\mp})_7 = \mp\frac{1}{2}\delta^{IJ}\left(m + \frac{1}{4}e^{-A}ch_0\right)e^{-\Phi}\text{vol}(\text{M}_7)\,, \tag{B.18}$$

where only the trace of this contains non-trivial information. We have now derived a necessary and sufficient set of conditions for $\mathcal{N} = (2,0)$ supersymmetry, these can however be refined somewhat: it is well known that the $\mathcal{N} = (2,0)$ AdS_3 solutions come equipped with a U(1) R-symmetry under which the spinors $\chi^I_{1,2}$ should be charged — having established that $\xi^a\partial_a$ is necessarily a Killing vector, clearly it is this that should be identified with that U(1). Indeed a consequence of supersymmetry is that $\mathcal{L}_K\epsilon_{1,2} = 0$, and since K is spanned by k and ξ with ζ^I transforming non triviality under the former, clearly $\chi^I_{1,2}$ must transform under $\xi^a\partial_a$ for (B.1) to be consistent. We find

$$\mathcal{L}_\xi\chi^I_{1,2} = -2m\epsilon^{IJ}\chi^J_{1,2}\,, \tag{B.19}$$

i.e. $\chi^I_{1,2}$ are SO(2) doublets as expected. As the matrix bi-linear Ψ^{IJ} is a tensor product of SO(2) doublets it should decompose into irreducible representations of SO(2) as $\mathbf{2}\otimes\mathbf{2} = \mathbf{1}\oplus\mathbf{1}\oplus\mathbf{2}$, as such Ψ^{IJ} should contain both singlet and doublet contributions. To see this it is helpful to decompose

$$\Psi^{IJ} = \frac{e^A}{2}\left(\delta^{IJ}\Psi^{(0)} + \sigma_1^{IJ}\Psi^{(1)} + i\sigma_2^{IJ}\Psi^{(2)} + \sigma_3^{IJ}\Psi^{(3)}\right)\,, \quad \Psi^{(0,1,2,3)} = \Psi^{(0,1,2,3)}_+ + i\Psi^{(0,1,2,3)}_-\,, \tag{B.20}$$

where $\Psi^{(0,1,2,3)}_\pm$ are real. We mentioned before that $\Psi^{(10\text{d})}$ should be a singlet with respect to K, given that the AdS_3 bi-linears transform non-trivially under k as in (A.12), it follows that

$$\mathcal{L}_K\Psi^{(10)} = 0 \qquad \Rightarrow \qquad \mathcal{L}_\xi\Psi^{(0)} = \mathcal{L}_\xi\Psi^{(2)} = 0\,, \quad \mathcal{L}_\xi(\Psi^{(1)} + i\Psi^{(3)}) = -4im(\Psi^{(1)} + i\Psi^{(3)})\,, \tag{B.21}$$

so it is only $\Psi^{(1,3)}$ that are charged under the U(1) R-symmetry. Another useful condition that follows when supersymmetry holds is that $(\iota_K + \tilde{K}\wedge)\Psi^{(10\text{d})} = 0$, which one can show implies the following conditions on the $d = 7$ bi-linears

$$\begin{aligned}
\left(\iota_\xi + \tilde{\xi}\wedge\right)\left(\Psi^{(0)}_\mp + i\Psi^{(2)}_\mp\right) &= \mp ice^{-A}\left(\Psi^{(0)}_+ + i\Psi^{(2)}_\pm\right)\,,\\
\left(\iota_\xi + \tilde{\xi}\wedge\right)\left(\Psi^{(1)}_\mp + i\Psi^{(3)}_\mp\right) &= \mp 2ie^{A}\left(\Psi^{(1)}_+ + i\Psi^{(3)}_\pm\right)\,,\\
\left(\iota_\xi + \tilde{\xi}\wedge\right)\left(\Psi^{(0)}_\pm + i\Psi^{(2)}_\pm\right) &= \pm 2ie^{A}\left(\Psi^{(0)}_\mp + i\Psi^{(2)}_\mp\right)\,,\\
\left(\iota_\xi + \tilde{\xi}\wedge\right)\left(\Psi^{(1)}_\pm + i\Psi^{(3)}_\pm\right) &= \pm ice^{-A}\left(\Psi^{(1)}_\mp + i\Psi^{(3)}_\mp\right)\,.
\end{aligned} \tag{B.22}$$

Additionally $\langle K, \tilde{K}\rangle = 0$ must hold, as $K, \tilde{K}$ are proportional to the sum and difference of 2 null vectors, this implies

$$\langle \xi, \tilde{\xi}\rangle = 2c\,. \tag{B.23}$$

From these identities one can show that (B.16d) and the $\Psi^{(0)}_{\pm}$ dependent contribution to (B.16a) are implied by the other conditions in general.

This concludes our derivation of the necessary and sufficient geometric conditions for $\mathcal{N} = (2,0)$ AdS$_3$ solutions of Type II supergravity, we summarise our results and present them in a compact fashion in section 2.

C SU(2)-structure and torsion classes in six dimensions

An SU(2)-structure in six dimensions is defined by a $(1,0)$-form z a real $(1,1)$-form j_2 and a $(2,0)$-form ω_2 which obey the relations

$$j_2\wedge\omega_2 = 0\,,\quad j_2\wedge j_2 = \frac{1}{2}\omega_2\wedge\overline{\omega}_2\,,\quad \omega_2\wedge\omega_2 = 0\,, \tag{C.1}$$

with z orthogonal to j_2 and ω_2. As such an SU(2)-structure indicates an obvious decomposition of the $d = 6$ metric as

$$ds^2(\text{M}_6) = \frac{i}{2}z\overline{z} + ds^2(\text{M}_4)\,. \tag{C.2}$$

There always exists a canonical frame $(\text{e}^1, \text{e}^2, \text{e}^3, \text{e}^4)$ of M_4 such that

$$j_2 = \text{e}^1\wedge\text{e}^2+\text{e}^3\wedge\text{e}^4,\qquad \omega_2 = (\text{e}^1+i\text{e}^2)\wedge(\text{e}^3+i\text{e}^4)\quad\Rightarrow\quad \frac{1}{2}j_2\wedge j_2 = \text{e}^{1234} = \text{vol}(\text{M}_4)\,, \tag{C.3}$$

which makes clear that $(j_2, \text{Re}\omega_2, \text{Im}\omega_2)$ span the real self-dual 2-forms on M_4.

In the main text it will be useful to know the torsion classes for an SU(2)-structure in six dimensions, these can be computed with group theory given that the torsion classes should form irreducible representations of SU(2), here we will take a different approach and exploit a canonical frame in $d = 6$.

The aim is to decompose the exterior derivatives of (z, j_2, ω_2) in terms of objects with useful properties under the wedge product and hodge dual — these will in fact turn out to be irreducible representations of SU(2), but this fact is somewhat auxiliary to this usefulness. The first step is to introduce some complex primitive (i.e. $j_2\wedge T_i = 0$) (1,1)-forms T_i, holomorphic 1-forms V_i and complex functions S_i (here $i\in\mathbb{N}$). In the canonical frame these are expressed in a basis of the following

$$V_i:\ \left(\text{e}^1+i\text{e}^2, \text{e}^3+i\text{e}^4\right),\qquad T_i:\ \left(\text{e}^1\wedge\text{e}^2-\text{e}^3\wedge\text{e}^4, \text{e}^1\wedge\text{e}^3+\text{e}^2\wedge\text{e}^4, \text{e}^1\wedge\text{e}^4-\text{e}^2\wedge\text{e}^3\right). \tag{C.4}$$

It is then a simple matter to confirm that one can express generic n-forms on M_4 $X^{(4)}_n$ in a basis of the forms introduced so far as

$$\begin{aligned} X^{(4)}_1 &= V_1+\overline{V}_2\,, & X^{(4)}_2 &= S_1\omega_2+S_2\overline{\omega}_2+S_3 j_2+T_1\,,\\ X^{(4)}_3 &= \left(V_3+\overline{V}_4\right)\wedge j_2\,, & X^{(4)}_4 &= S_5 j_2\wedge j_2\,. \end{aligned} \tag{C.5}$$

Relevant to the torsion class are generic complex 2 and 3-forms on M_6, which can thus be decomposed as

$$X_2^{(6)} = S_1\omega_2 + S_2\overline{\omega}_2 + S_3 j_2 + T_1 + S_4 z\wedge\overline{z} + (V_1+\overline{V}_2)\wedge z + (V_3+\overline{V}_4)\wedge\overline{z}\,, \tag{C.6}$$

$$X_3^{(6)} = (V_5+\overline{V}_6)\wedge j_2 + z\wedge(S_5\omega_2+S_6\overline{\omega}_2+S_7 j_2+T_2) + \overline{z}\wedge(S_8\omega_2+S_9\overline{\omega}_2+S_{10}j_2+T_3) + (V_7+\overline{V}_8)\wedge z\wedge\overline{z}\,.$$

However, $dj_2, d\omega_2$ are not generic, they must also be consistent with the exterior derivatives of (C.1) which forces some terms to be equal. We find in general that the torsion classes take the form

$$\begin{aligned} dz &= S_1\omega_2+S_2\overline{\omega}_2+S_3 j_2+T_1+S_4 z\wedge\overline{z}+z\wedge W_1+\overline{z}\wedge W_2\,,\\ d\omega_2 &= \left(V_1+\overline{V}_2\right)\wedge j_2+z\wedge(S_5\omega_2+S_6 j_2+T_2)+\overline{z}\wedge(S_7\omega_2+S_8 j_2+T_3)+i\left(\iota_{\overline{V}_3}\omega_2\right)\wedge z\wedge\overline{z}\,,\\ dj_2 &= V_4\wedge j_2+V_3\wedge z\wedge\overline{z}+z\wedge\left(-\frac{1}{2}\overline{S}_8\omega_2-\frac{1}{2}S_6\overline{\omega}_2+\frac{1}{2}\left(S_5+\overline{S}_7\right)j_2+T_4\right)+\text{c.c}\,, \end{aligned} \tag{C.7}$$

where W_i are simply complex 1-forms on M_4 that can each be expressed in terms of two holomorphic 1-forms if we wish. Note this gives the correct counting 8 scalars, 8 holomorphic 1-forms and 4 primitive (1,1)-forms.

As promised, by making use of the canonical frame one can compute many nice identities the SU(2)-structure forms and their torsion classes must obey — for instance

$$\begin{aligned} &T_i\wedge j_2 = T_i\wedge\omega_2 = V_i\wedge\omega_2 = 0,\\ &\star_6 V_i = \frac{1}{2}V_i\wedge j_2\wedge z\wedge\overline{z}, \qquad \star_6\left(V_i\wedge j_2\right) = \frac{1}{2}V_i\wedge z\wedge\overline{z},\\ &\star_6 j_2 = \frac{i}{2}j_2\wedge z\wedge\overline{z}, \qquad \star_6\omega_2 = \frac{i}{2}\omega_2\wedge z\wedge\overline{z}, \qquad \star_6 T_i = -\frac{i}{2}T_i\wedge z\wedge\overline{z},\\ &\left(\iota_{\overline{V}_i}\omega_2\right)\wedge j_2 = -i\overline{V}_i\wedge\omega_2\,, \qquad \left(\iota_{\overline{V}_i}\omega_2\right)\wedge\overline{\omega}_2 = 4i\overline{V}_i\wedge j_2\,, \end{aligned} \tag{C.8}$$

which we make use of in the main text.

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