

Fano 3-folds, reflexive polytopes and brane brick models

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ABSTRACT: Reflexive polytopes in n dimensions have attracted much attention both in mathematics and theoretical physics due to their connection to Fano n -folds and mirror symmetry. This work focuses on the 18 regular reflexive polytopes corresponding to smooth Fano 3-folds. For the first time, we show that all 18 regular reflexive polytopes have corresponding $2d$ $(0, 2)$ gauge theories realized by brane brick models. These $2d$ gauge theories can be considered as the worldvolume theories of D1-branes probing the toric Calabi-Yau 4-singularities whose toric diagrams are given by the associated regular reflexive polytopes. The generators of the mesonic moduli space of the brane brick models are shown to form a lattice of generators due to the charges under the rank 3 mesonic flavor symmetry. It is shown that the lattice of generators is the exact polar dual reflexive polytope to the corresponding toric diagram of the brane brick model. This duality not only highlights the close relationship between the geometry and $2d$ gauge theory, but also opens up pathways towards new discoveries in relation to reflexive polytopes and brane brick models.

KEYWORDS: D-Branes, Differential and Algebraic Geometry, Supersymmetric Gauge Theory

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1 Introduction

The study of worldvolume theories of D-branes probing Calabi-Yau singularities has been immensely fruitful in the past [1–4]. More recently, interest has grown in studying the setup of D1-branes probing Calabi-Yau 4-fold singularities as a pathway towards a better understanding of $2d$ $\mathcal{N} = (0, 2)$ gauge theories [5–11]. When the Calabi-Yau 4-fold is toric, it was shown that the corresponding gauge theories are endowed with additional structural features. In this case, the map between the Calabi-Yau geometry and the gauge theory is considerably simplified by what we now call as brane brick models, which are Type IIA configurations that are connected to the D1-brane at the Calabi-Yau singularity via T-duality [5, 6]. One of the main virtues of these brane setups is that both the gauge theory and the underlying geometry can be easily determined from them.

There are various approaches for deriving the brane brick models associated to a given toric Calabi-Yau 4-fold, including partial resolution [5, 6], mirror symmetry [8] and the topological B-model [12, 13]. In addition, more efficient algorithms that apply to large classes of toric geometries, such as orbifold reduction, $3d$ printing and Calabi-Yau products, have been developed [14–16]. Using these methods, brane brick models corresponding to a variety of toric Calabi-Yau 4-folds have been found. Interestingly, some of these examples involve $2d$ $(0, 2)$ gauge theories that exhibit novel gauge theory phenomena such as triality [17, 18], which through brane brick models obtained a brane picture interpretation as well as a geometrical interpretation through Calabi-Yau mirror symmetry [7, 8].

A particular class of toric Calabi-Yau 4-folds however has not yet been systematically studied in the context of brane brick models and the corresponding $2d$ supersymmetric gauge theories. This class of toric Calabi-Yau 4-folds is realized as complex cones over Gorenstein Fano varieties that are constructed from a special set of lattice polytopes known as reflexive polytopes. This class of polytopes is special because they come in pairs that are related by a polar duality, which only exists between reflexive polytopes. Due to this property, reflexive polytopes have a long history in string theory, beginning with their introduction through mirror symmetry [19–23] in the study of string theory compactifications on Calabi-Yau manifolds. Batyrev and Borisov [22–26] pioneered the systematic search for mirror paired Calabi-Yau manifolds that are realized as hypersurfaces in toric varieties given by pairs of dual reflexive polytopes. Figure 1 shows an example of two dual reflexive polytopes in dimension 3.

Thanks to the tour de force search for reflexive polytopes by Kreuzer and Skarke [19–21], we know today how many of them exist up to lattice dimension 4. Our interest lies in

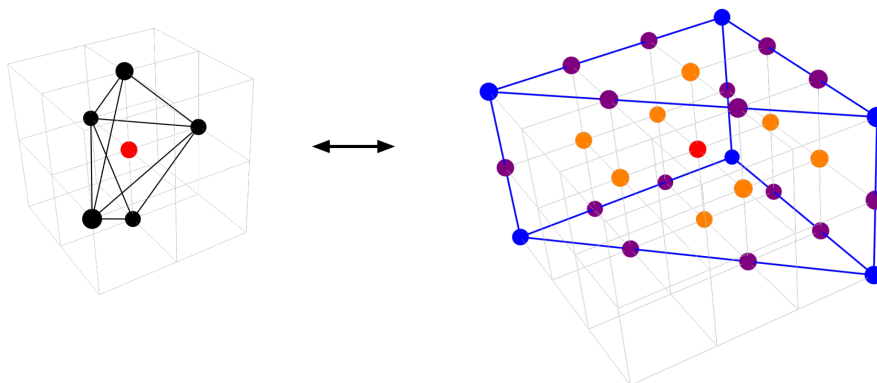


Figure 1. The toric diagram for $M^{3,2} (\mathbb{P}^2 \times \mathbb{P}^1)$ and the corresponding dual polytope. Both are reflexive polytopes in dimension 3 with the single interior point highlighted in red.

lattice dimension 3, where there are 4,319 reflexive polytopes up to $GL(3, \mathbb{Z})$ equivalence. The corresponding toric varieties are known as Fano 3-folds. In order to study brane brick models corresponding to these reflexive polytopes, we are interested in the associated complex cones — the non-compact toric Calabi-Yau 4-folds.

In this work, we focus on reflexive polytopes in dimension 3 that are also regular. A reflexive polytope is regular if every cone in the fan defined by the polytope has generators that form part of a \mathbb{Z} -basis. Amongst the 4,319 reflexive polytopes in dimension 3, there are a manageable subset of 18 polytopes that are both reflexive and regular. A property of regular reflexive polytopes is that their associated Gorenstein Fano varieties are smooth. The fact that the polytopes are reflexive and regular has an interesting consequence for the corresponding brane brick models that we aim to explore as part of this work.

One of the main aims of this work is to find first the brane brick models and hence the corresponding $2d$ $(0, 2)$ gauge theories for toric Calabi-Yau 4-folds whose toric diagrams are the 18 regular reflexive polytopes in dimension 3. Because the toric diagrams are regular and reflexive, GLSM fields associated to brick matchings in the brane brick model either refer to the extremal corner points or the single interior point of the toric diagram. Moreover, the toric Calabi-Yau 4-fold given by the regular reflexive polytopes is expected to be the mesonic moduli space of the $2d$ $(0, 2)$ gauge theories represented by the brane brick models. The spectrum of mesonic gauge invariant operators is therefore intricately lined to the GLSM fields corresponding to the points in the regular reflexive toric diagram.

In order to study this correspondence, we calculate the full spectrum of mesonic gauge invariant operators by obtaining the generating function known as the Hilbert series [27–29]. Using plethystics [30, 31], we are able to identify the generating set of mesonic gauge invariant operators for each brane brick model associated to a regular reflexive polytope. As it is the case for any mesonic gauge invariant operator, the generators carry charges under the global symmetry of the gauge theory, which can be obtained from the isometry of the corresponding toric Calabi-Yau 4-fold. The mesonic flavor symmetry, which is part of the global symmetry, assigns charges to the generators that can be scaled to be integer valued. This enables us to represent each set of mesonic flavor charges carried by a generator as a point on a \mathbb{Z}^3

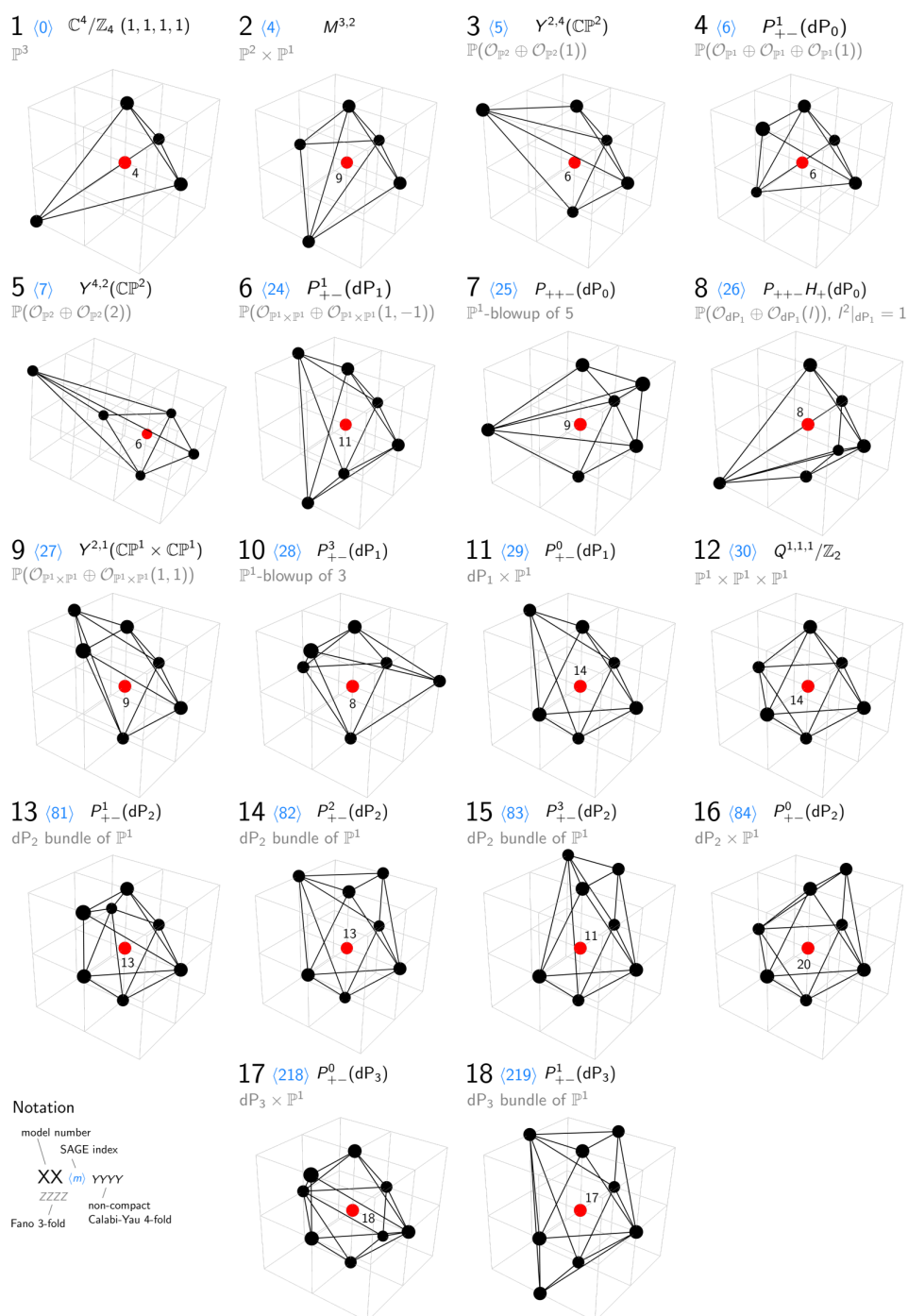


Figure 2. The 18 regular reflexive polytopes in dimension 3 corresponding to toric non-compact Calabi-Yau 4-folds and corresponding smooth Fano 3-folds.

lattice. The convex hull of the set points associated to the set of generators of the mesonic moduli space forms a convex lattice polygon, which we call as the generator lattice.

One of the main results of this work is that for all 18 brane brick models corresponding to toric Calabi-Yau 4-folds whose toric diagram is a regular reflexive polygon, the corresponding mesonic moduli spaces have a generator lattice which is the polar dual to the toric diagram of the Calabi-Yau 4-fold. This correspondence between the toric diagrams and the generator lattices of the mesonic moduli space is exact because on both sides of the correspondence we are dealing with reflexive polytopes on a \mathbb{Z}^3 lattice. Our work illustrates this correspondence explicitly for all 18 regular reflexive polytopes corresponding to the 18 smooth Fano 3-folds for the first time, whilst making simultaneously a connection to $2d$ $(0, 2)$ gauge theories realized by brane brick models.

Our work is structured as follows. In section 2, we give a brief overview of reflexive polytopes and talk about the construction of Fano n -folds. The appearance of the corresponding non-compact toric Calabi-Yau 4-folds is discussed. Section 3 continues with a review on brane brick models and the realization of $2d$ $(0, 2)$ gauge theories as worldvolume theories of D1-brane probing toric Calabi-Yau 4-folds. We discuss the characterization of the mesonic moduli spaces of brane brick models in terms of Hilbert series and the construction of generator lattices. The following sections then discuss the brane brick models for the 18 regular reflexive polytopes and illustrate how the toric diagrams are polar duals to the corresponding generator lattices. We conclude our work with a short summary of its main results.

2 18 smooth Fano 3-folds and toric Calabi-Yau 4-folds

Let us begin with a short review about the particular set of lattice polytopes known as reflexive polytopes and the role they play in our work.

2.1 Reflexive polytopes

Reflexive polytopes form a special subset of *lattice polytopes* Δ_n in \mathbb{Z}^n . We call the convex hull of a finite number of points in \mathbb{Z}^n a lattice polytope Δ_n in dimension n . The vertices of the lattice polytope form the set $\Delta_n \cap \mathbb{Z}^n$. In this work, we call these convex lattice polytopes Δ_n also *toric diagrams*.

A lattice polytope is called *reflexive* if the dual polytope (sometimes also known as the *polar polytope*) given by

$$\Delta_n^\circ = \{ \mathbf{v} \in \mathbb{Z}^n \mid \mathbf{m} \cdot \mathbf{n} \geq -1 \ \forall \mathbf{m} \in \Delta_n \}, \tag{2.1}$$

is also a lattice polytope in \mathbb{Z}^n . Reflexive polytopes and their duals have a unique interior lattice point at the origin $(0, \dots, 0) \in \mathbb{Z}^n$.

For a given n , there is a finite number of such reflexive polytopes up to $\text{GL}(n, \mathbb{Z})$ equivalence and this makes the problem of classifying these reflexive polytopes an interesting combinatorial problem. For $n = 2$, it is straightforward to identify the 16 reflexive polygons up to $\text{GL}(2, \mathbb{Z})$ equivalence. For higher n , the problem of classifying all reflexive polytopes becomes significantly more complicated. Due to Kreuzer-Skarke [19–21], the number of reflexive polytopes is known to be 4,319 in $n = 3$ and 473,800 in $n = 4$ as summarized in table 1.

d	Number of Polytopes	Number of Regular Polytopes
1	1	1
2	16	5
3	4,319	18
4	473,800,776	124

Table 1. The number of distinct reflexive polytopes and distinct regular reflexive polytopes in dimension $d \leq 4$ [19–21].

A subset of reflexive polytopes in each dimension n is known to be *regular*. A polytope is called regular if every cone in the fan has generators that form part of a \mathbb{Z} -basis. That means, for instance in dimension $n = 2$, the boundary edges of the polygons do not contain any internal points. For dimension $n = 3$, the boundary faces of regular polytopes are all triangles that do not contain internal points on their boundary edges and in the interior of the triangles. There are exactly 18 regular reflexive polytopes in dimension $n = 3$ as shown in table 1. Batyrev-Borisov [22–26] studied these reflexive polytopes Δ_n in n dimensions in order to construct new families of smooth Calabi-Yau hypersurfaces in toric varieties given by Δ_n . To be more precise, if Δ_n is reflexive, it corresponds to a *Gorenstein toric Fano variety* $X(\Delta_n)$. We here also call $X(\Delta_n)$ a toric *Fano n -fold*, which is smooth if Δ_n is regular in addition to being reflexive. The duality between reflexive polytopes that relates dual families of smooth Fano n -folds $X(\Delta_n)$ has been shown to be *mirror symmetry* [19–23].

Every reflexive polytope Δ_n is also associated with a non-compact toric Calabi-Yau $(n+1)$ -fold, which is basically the affine cone over the base $X(\Delta_n)$. As discussed in [32], many properties of the Fano n -folds can be related to geometrical properties of the corresponding toric Calabi-Yau $(n + 1)$ -folds. The following section summarizes both constructions for given Δ_n , their geometrical properties and the notation that we use to identify them.

2.2 Fano n -folds and Calabi-Yau $(n + 1)$ -folds

The following section gives a brief summary on Fano n -folds and non-compact toric Calabi-Yau $(n + 1)$ -folds that correspond to reflexive polytopes Δ_n .

Fano n -folds. Given a lattice polytope Δ_n in dimension n , one can construct a compact toric variety $X(\Delta_n)$ of complex dimension n [33, 34]. From the lattice polytope, one can construct the normal fan $\Sigma(\Delta_n)$ as the positive hull of the n -cones over all the faces of Δ_n . Given the fan Σ from the polytope Δ_n , the corresponding compact toric variety $X(\Sigma)$ follows by gluing in the standard way the affine varieties of each of the cones in Σ . If the lattice polytope Δ_n is reflexive, $X(\Sigma)$ is known as a Gorenstein toric Fano variety or short as a Fano n -fold [35]. Furthermore, if Δ_n is regular, meaning every cone in the fan $\Sigma(\Delta_n)$ has generators that form part of a \mathbb{Z} -basis, $X(\Sigma)$ is also known to be smooth [34].

As discussed in section 2.1, there are in dimension $n = 3$ precisely 18 distinct reflexive polytopes that are regular. Each of these 18 polytopes corresponds to a smooth toric Fano 3-fold $X(\Delta_3)$. In this work, we concentrate on these 18 reflexive polytopes and

Model	Polytope	E	Calabi-Yau 4-fold	Fano 3-fold
1	$\langle 0 \rangle$	4	$\mathbb{C}^4/\mathbb{Z}_4 (1, 1, 1, 1)$	\mathbb{P}^3
2	$\langle 4 \rangle$	5	$M^{3,2}, Y^{2,3}(\mathbb{C}\mathbb{P}^2), P_{+-}^0(\text{dP}_0)$	$\mathbb{P}^2 \times \mathbb{P}^1$
3	$\langle 5 \rangle$	5	$Y^{2,4}(\mathbb{C}\mathbb{P}^2)$	$\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$
4	$\langle 6 \rangle$	5	$P_{+-}^1(\text{dP}_0)$	$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$
5	$\langle 7 \rangle$	5	$Y^{2,5}(\mathbb{C}\mathbb{P}^2)$	$\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2))$
6	$\langle 24 \rangle$	6	$P_{+-}^1(\text{dP}_1)$	$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, -1))$
7	$\langle 25 \rangle$	6	$P_{++-}(\text{dP}_0)$	\mathbb{P}^1 -blowup of 5
8	$\langle 26 \rangle$	6	$P_{++-}H_+(\text{dP}_0)$	$\mathbb{P}(\mathcal{O}_{\text{dP}_1} \oplus \mathcal{O}_{\text{dP}_1}(l)), l^2 _{\text{dP}_1} = 1$
9	$\langle 27 \rangle$	6	$Y^{2,1}(\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1), P_{+-}^2(\text{dP}_1)$	$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1))$
10	$\langle 28 \rangle$	6	$P_{+-}^3(\text{dP}_1)$	\mathbb{P}^1 -blowup of 3
11	$\langle 29 \rangle$	6	$P_{+-}^0(\text{dP}_1), P_{+-}^1(F_0)$	$\text{dP}_1 \times \mathbb{P}^1$
12	$\langle 30 \rangle$	6	$Q^{1,1,1}/\mathbb{Z}_2, P_{+-}^0(F_0)$	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$
13	$\langle 81 \rangle$	7	$P_{+-}^1(\text{dP}_2)$	dP_2 bundle of \mathbb{P}^1
14	$\langle 82 \rangle$	7	$P_{+-}^2(\text{dP}_2)$	dP_2 bundle of \mathbb{P}^1
15	$\langle 83 \rangle$	7	$P_{+-}^3(\text{dP}_2)$	dP_2 bundle of \mathbb{P}^1
16	$\langle 84 \rangle$	7	$P_{+-}^0(\text{dP}_2)$	$\text{dP}_2 \times \mathbb{P}^1$
17	$\langle 218 \rangle$	8	$P_{+-}^0(\text{dP}_3)$	$\text{dP}_3 \times \mathbb{P}^1$
18	$\langle 219 \rangle$	8	$P_{+-}^1(\text{dP}_3)$	dP_3 bundle of \mathbb{P}^1

Table 2. The names of smooth Fano 3-folds and toric Calabi-Yau 4-folds corresponding to the 18 regular reflexive polytopes in dimension 3. E is the number of external points of the reflexive polytope.

the corresponding toric Fano 3-folds. Figure 2 shows the 18 reflexive polytopes with the corresponding names of toric Fano 3-folds. Table 2 also summarizes the names for the 18 toric Fano 3-folds with the corresponding SAGE polytope index [36].

Non-compact Calabi-Yau $(n + 1)$ -folds. A lattice polytope Δ_n can also be associated to a non-compact toric Calabi-Yau $(n + 1)$ -fold. The complex cone over the toric variety $X(\Delta_n)$ associated to the lattice polytope Δ_n is an affine toric Calabi-Yau $(n + 1)$ -fold \mathcal{X}_{n+1} of complex dimension $n + 1$.

Let us give a brief overview on the construction of \mathcal{X}_{n+1} . First, let us explicitly define the polyhedral cones σ that are generated by the vertices of $X(\Delta_n)$. Given that the vertices of $X(\Delta_n)$ are in \mathbb{Z}^n , the origin $N := (0, \dots, 0) \in \mathbb{Z}^{n+1}$ is taken to be the apex of the polytope in \mathbb{Z}^n . Then the vectors \mathbf{u}_i from this apex to the vertices of $X(\Delta_n)$ generate the cone σ as follows

$$\sigma = \left\{ \sum_{\mathbf{u}_i} \lambda_i \mathbf{u}_i \mid \lambda_{\mathbf{u}_i} \geq 0 \right\} \subset N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}. \tag{2.2}$$

Like the polytope Δ_n having a dual following (2.1), the cone also has a dual σ^\vee that lives in $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ where $M := \text{hom}(N, \mathbb{Z})$. The dual cone σ^\vee is given by

$$\sigma^\vee = \{ \mathbf{v} \in M_{\mathbb{R}} \mid \mathbf{v} \cdot \mathbf{u} \geq 0 \quad \forall \mathbf{u} \in \sigma \} . \tag{2.3}$$

Following the definition of the dual cone σ^\vee , \mathcal{X}_{n+1} can be identified as the maximal spectrum of the group algebra generated by the lattice points covered by σ^\vee in M . This is given by

$$\mathcal{X}_{n+1} \simeq \text{Spec}(\mathbb{C}[\sigma^\vee \cap M]) . \tag{2.4}$$

Note that the end-points of the vectors generating the cone are all co-hyperplanar. This guarantees that \mathcal{X}_{n+1} is a Gorenstein singularity and hence relates to a Calabi-Yau manifold.

While \mathcal{X}_{n+1} can be considered as the complex cone over the Gorenstein Fano variety $X(\Delta_n)$, as a toric non-compact Calabi-Yau $(n + 1)$ -fold it is also a real cone over a Sasaki-Einstein manifold Y of real dimension $2n + 1$. The Sasaki-Einstein manifold Y is a Riemannian manifold where the cone over Y has a metric of the form

$$ds^2(\mathcal{X}_{n+1}) = dr^2 + r^2 ds^2(Y) , \tag{2.5}$$

which is Ricci-flat and Kähler. The metrics of large classes of Sasaki-Einstein manifolds were found in the past [37–39]. Moreover, these Sasaki-Einstein manifolds have a volume. The volume has a minimum non-zero point that determines the Reeb vector field for the corresponding Sasaki-Einstein manifold [40]. The volume functional that is used for the minimization process can be written in terms of the Reeb vector field. In [32], it was shown that the minimum volume of Sasaki-Einstein manifolds corresponding to toric Calabi-Yau manifolds and obtained from reflexive polytopes up to dimension $n = 4$ are related to topological quantities such as the Chern number and the Euler number of the toric varieties.

In this work, we concentrate on the 18 regular reflexive polytopes in dimension $n = 3$ and the corresponding non-compact toric Calabi-Yau 4-folds. The 18 polytopes with the names of the corresponding Calabi-Yau 4-folds are shown in figure 2 and listed in table 2. Note that the toric Calabi-Yau 4-folds corresponding to regular reflexive polytopes Δ_3 and smooth Fano 3-folds are part of one or multiple large families of toric Calabi-Yau 4-folds. Accordingly, the naming convention for the toric Calabi-Yau 4-folds follows the naming conventions for these large families, which are as follows:

- *Abelian orbifolds of \mathbb{C}^4 .* Abelian orbifolds of \mathbb{C}^4 take the general form $\mathbb{C}^4/\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \mathbb{Z}_{n_3}$ where the order of the orbifold is $N = n_1 n_2 n_3$. For each \mathbb{Z}_{n_i} factor, there is an integer 4 vector (a_1, a_2, a_3, a_4) that determines the orbifold \mathbb{Z}_{n_i} -action on \mathbb{C}^4 . Given that \mathbb{C}^4 has coordinates z_k , the orbifold \mathbb{Z}_{n_i} -action $(a_{i1}, a_{i2}, a_{i3}, a_{i4})$ acts on the coordinates as follows

$$z_k \mapsto \omega_i^{a_{ik}} z_k , \tag{2.6}$$

where $(\omega_i)^{n_i} = 1$. The corresponding convex lattice polytopes Δ_3 for Abelian orbifolds of \mathbb{C}^4 are 3-dimensional lattice tetrahedra with polytope volume N , where the volume of \mathbb{C}^4 is normalized to 1. Distinct Abelian orbifolds of \mathbb{C}^4 for a given order N were counted and classified in [41, 42]. Since all Abelian orbifolds of \mathbb{C}^4 are toric Calabi-Yau

4-folds, we label the Calabi-Yau geometries with the orbifold name containing the orbifold action on \mathbb{C}^4 . In this work, the Abelian orbifold of the form $\mathbb{C}^4/\mathbb{Z}_4$ with action $(1, 1, 1, 1)$ is the only example where the corresponding polytope Δ_4 is a regular reflexive polytope. In fact, the toric diagram for $\mathbb{C}^4/\mathbb{Z}_4$ with action $(1, 1, 1, 1)$ is the unique regular lattice tetrahedron with a single internal point as shown as Model 1 with polytope index $\langle 0 \rangle$ in figure 2 and table 2.

- $M^{3,2}$ and $Q^{1,1,1}/\mathbb{Z}_2$. $M^{3,2}$ and $Q^{1,1,1}$ are Sasaki-Einstein 7-manifolds [43] whose metric is explicitly known. They are higher-dimensional generalizations of the famous $T^{1,1}$ Sasaki-Einstein 5-manifold and were some of the few known Sasaki-Einstein 7-manifolds before the discovery of some infinite families. The cone over $M^{3,2}$ is a toric Calabi-Yau 4-fold which has a regular reflexive polytope as its toric diagram. In figure 2 and table 2, this Calabi-Yau 4-fold is labeled as Model 2. In addition, even though $M^{3,2}$ refers to the Sasaki-Einstein 7-manifold, we use the name interchangeably also for the corresponding Calabi-Yau cone. The cone over $Q^{1,1,1}$ is a toric Calabi-Yau 4-fold whose toric diagram is the smallest lattice octahedron in \mathbb{Z}^3 . The non-trivial \mathbb{Z}_2 orbifold on this Calabi-Yau 4-fold has the regular lattice octahedron with a single internal point as its toric diagram as shown as Model 12 in figure 2 and table 2. We refer to this Calabi-Yau 4-fold as $Q^{1,1,1}/\mathbb{Z}_2$.
- $Y^{p,k}(\mathbb{CP}^2)$ and $Y^{p,k}(\mathbb{CP}^1 \times \mathbb{CP}^1)$ families. It was shown in [37, 38] that in all odd dimensions there are infinite families of Sasaki-Einstein manifolds. In particular, [39] shows that for any positive curvature Kähler-Einstein manifold B_{2m} there is an infinite class of Sasaki-Einstein $(2m + 3)$ -manifolds. In the case for Sasaki-Einstein 7-manifolds, two infinite classes labeled by two integers p and k were found with B_4 being \mathbb{CP}^2 and $\mathbb{CP}^1 \times \mathbb{CP}^1$. These are known as $Y^{p,k}(\mathbb{CP}^2)$ and $Y^{p,k}(\mathbb{CP}^1 \times \mathbb{CP}^1)$ and the cone over these Sasaki-Einstein 7-manifolds is a toric Calabi-Yau 4-fold. The coordinates of the toric diagrams of these Calabi-Yau 4-folds can be written in terms of the integers p and k as illustrated in figure 3. In the particular cases of $Y^{2,3}(\mathbb{CP}^2)$, $Y^{2,4}(\mathbb{CP}^2)$, $Y^{2,5}(\mathbb{CP}^2)$ and $Y^{2,1}(\mathbb{CP}^1 \times \mathbb{CP}^1)$, the corresponding toric diagrams are regular reflexive polytopes as shown in figure 2 and table 2. We refer to these as Models 2, 3, 5 and 9 respectively and also refer to the toric Calabi-Yau 4-folds by the names of the corresponding Sasaki-Einstein 7-manifolds.
- *Calabi-Yau 4-folds with a Calabi-Yau 3-fold base.* In this work, we introduce names for toric Calabi-Yau 4-folds whose toric diagrams relate to toric diagrams of toric Calabi-Yau 3-folds. Given the toric diagram of a toric Calabi-Yau 3-fold, it can be placed on the plane $z = 0$ in a \mathbb{Z}^3 lattice. By adding additional lattice points above or below the $z = 0$ plane, the convex hull of all the points forms a 3-dimensional convex lattice polytope corresponding to a toric Calabi-Yau 4-fold. For example, if one adds a single lattice point at height $z = 1$ above the toric diagram of the Calabi-Yau 3-fold (CY_3) at height $z = 0$, the resulting toric diagram will correspond to the toric Calabi-Yau 4-fold of the form $\mathbb{C} \times CY_3$.

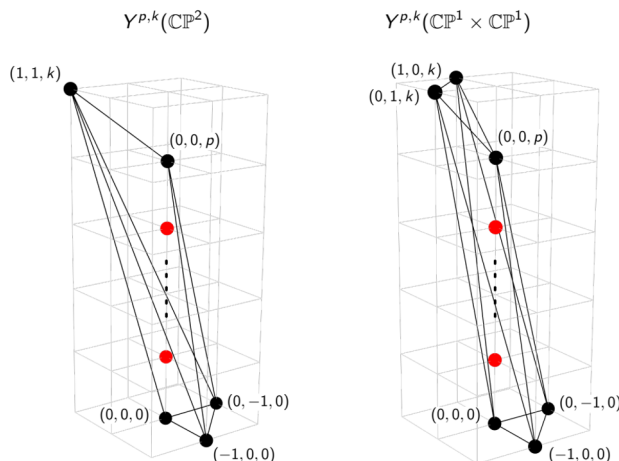


Figure 3. Toric diagrams corresponding to the $Y^{p,k}(\mathbb{C}\mathbb{P}^2)$ and $Y^{p,k}(\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1)$ families of toric Calabi-Yau 4-folds.

In order to refer to toric Calabi-Yau 4-folds with reflexive polytopes Δ_3 as their toric diagrams, we add two lattice points to a toric diagram Δ_2 at height $z = 0$. One point is added above Δ_2 at height $z = 1$ and another point is added below Δ_2 at height $z = -1$. If Δ_2 is a reflexive polygon in 2-dimension then the resulting Δ_3 from this construction is also a reflexive polygon. We denote such toric Calabi-Yau 4-folds as $P_{+-}^i(\text{CY}_3)$, where i labels the $\text{GL}(3, \mathbb{Z})$ -distinct combinations of adding a lattice point at height $z = 1$ above and at height $z = -1$ below the toric diagram of CY_3 . Figure 4 shows how Models 2 and 4 are respectively $P_{+-}^0(\text{dP}_0)$ and $P_{+-}^1(\text{dP}_0)$ and accordingly refer to examples of toric diagrams of Calabi-Yau 4-folds obtained from the toric diagram of dP_0 , which refers to both the 0-th del Pezzo surface and the 3-dimensional Calabi-Yau cone over it.

Figure 4 shows a generalization of this construction with Model 7, where starting from the toric diagram of dP_0 , two lattice points are added at height $z = 1$ and one at height $z = -1$, giving a regular reflexive polytope for a toric Calabi-Yau 4-fold that we call $P_{++-}(\text{dP}_0)$. The last example in figure 4 is a case when starting with the toric diagram of dP_0 , two lattice points are added at height $z = 1$, one added at height $z = -1$ and one external point of the toric diagram of dP_0 lifted to height $z = 1$. We refer to the toric Calabi-Yau 4-fold of the resulting regular reflexive polytope as $P_{++-}H_+(\text{dP}_0)$, where the added H_+ indicated the lifting of an external point of the toric diagram of dP_0 .

In this work, most toric Calabi-Yau 4-folds whose toric diagrams are regular reflexive lattice polytopes are of the form $P_{+-}^i(\text{CY}_3)$. For some cases, a given toric Calabi-Yau 4-fold $P_{+-}^i(\text{CY}_3)$ is the same as adding lattice points to another toric Calabi-Yau 3-fold CY'_3 , i.e. $P_{+-}^i(\text{CY}'_3)$. An example would be $P_{+-}^1(F_0)$, which is the same as $P_{+-}^0(\text{dP}_1)$ as illustrated in figure 5. Here, F_0 refers to both the zeroth Hirzebruch surface and the Calabi-Yau cone over it similar to the first del Pezzo surface dP_1 .

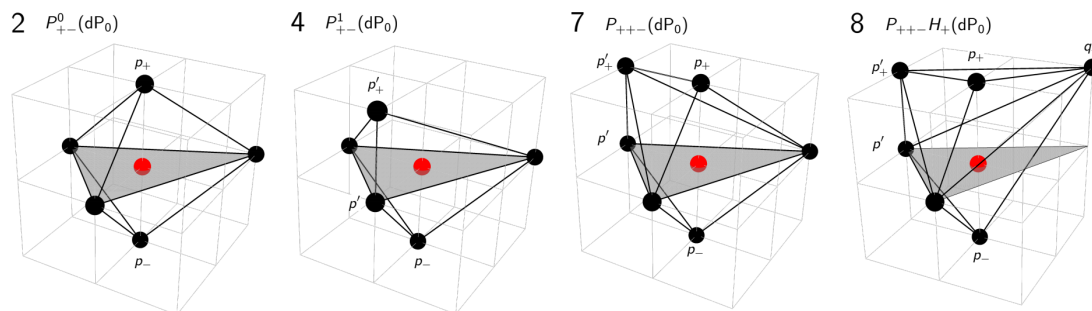


Figure 4. Naming convention for toric Calabi-Yau 4-folds that are related to toric Calabi-Yau 3-folds. The red lattice points indicate the unique internal point of the regular reflexive polytopes.

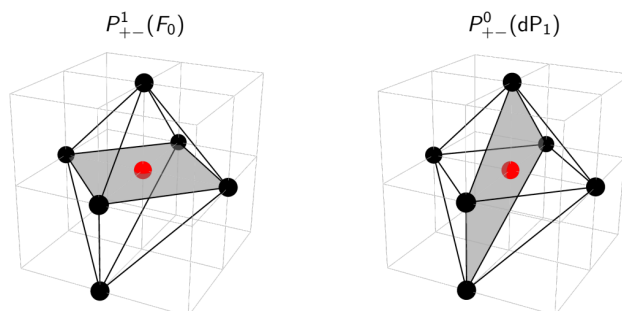


Figure 5. The toric diagram for $P_{+-}^1(F_0)$ is the same as $P_{+-}^0(dP_1)$.

3 Brane brick models and their mesonic moduli space

In the following section, we summarize the construction of brane brick models that correspond to $2d$ $(0, 2)$ supersymmetric gauge theories corresponding to toric Calabi-Yau 4-folds. We discuss in particular the construction of the mesonic moduli space \mathcal{M}^{mes} of these $2d$ $(0, 2)$ supersymmetric gauge theories that exhibits interesting features for the case when the toric diagram for the corresponding Calabi-Yau 4-folds is a regular reflexive polytope. For completeness, a brief review of brane brick models is presented here and the reader is referred to [5, 6] for more details.

3.1 $2d$ $(0, 2)$ theories and toric Calabi-Yau 4-folds

Quiver. The gauge symmetry and matter content of the $2d$ $(0, 2)$ theories are encoded in a generalized quiver diagram. The generalized quiver diagram contains two types of fields in the bifundamental or adjoint representation of the $U(N_i)$ gauge groups. The fields are either chiral X_{ij} , represented by directed black arrows in the quiver diagram, or Fermi Λ_{ij} , represented by unoriented red lines in the quiver diagram. Fermi fields are not assigned an orientation, as illustrated in figure 6, due to the $\Lambda_{ij} \leftrightarrow \bar{\Lambda}_{ji}$ symmetry of $2d$ $(0, 2)$ theories. The nodes of the quiver correspond to the $U(N_i)$ gauge groups of the $2d$ $(0, 2)$ theory. Figure 7 shows an example of a quiver diagram for the $2d$ $(0, 2)$ theory corresponding to the Abelian orbifold of the form $\mathbb{C}^4/\mathbb{Z}_2(1, 1, 1, 1)$.

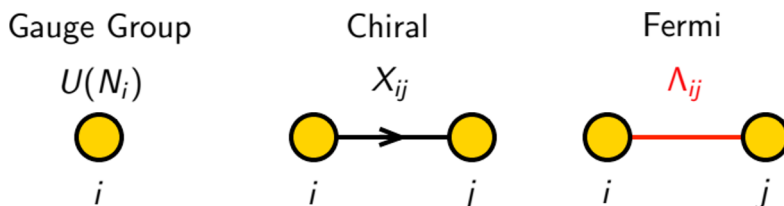


Figure 6. Quiver representation of $U(N_i)$ gauge groups, chiral and Fermi fields. The subindices of field variables indicate the gauge nodes under which they transform. Fermi fields are not assigned an orientation due to the $\Lambda_{ij} \leftrightarrow \bar{\Lambda}_{ji}$ symmetry.

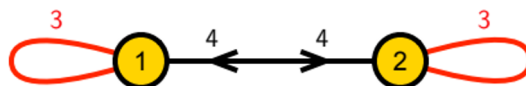


Figure 7. Quiver for $\mathbb{C}^4/\mathbb{Z}_2$ (1, 1, 1, 1). The numbers next to edges correspond to the multiplicity of field between the adjacent nodes in the quiver diagram.

In this work, we focus on the case when all ranks of the gauge groups are equal, i.e. $N_i = N$. This simplifies the non-abelian $SU(N_i)^2$ anomaly cancellation conditions on the quiver [5, 6].¹ It states that for each quiver node i , the number of connected arrows corresponding to chiral fields n_i^X and Fermi fields n_i^F need to satisfy the following condition

$$n_i^X - n_i^F = 2. \tag{3.1}$$

Adjoint chiral or Fermi fields contribute 2 to n_i^X and n_i^F , respectively.

Binomial J - and E -terms. Every Fermi field Λ_{ij} corresponds to a pair of holomorphic functions of chiral fields, which we call $E_{ij}(X)$ and $J_{ji}(X)$. These functions are restricted to be binomial in the case when the probed Calabi-Yau 4-fold is toric. This is linked to the fact that ideals defining toric varieties are binomial prime ideals, which are precisely defined by the J - and E -terms of the $2d$ (0, 2) theory as we will see below in the discussion on moduli spaces. For now, it is noted that this restriction was called the *toric condition* in [5, 6] and implies the following general form of the J - and E -terms,

$$J_{ji} = J_{ji}^+ - J_{ji}^-, \quad E_{ij} = E_{ij}^+ - E_{ij}^-, \tag{3.2}$$

where J_{ji}^\pm and E_{ij}^\pm are holomorphic monomials in chiral fields.

As we will see in the following sections, this work will identify for each toric Calabi-Yau 4-fold whose toric diagram is a regular reflexive polygon a corresponding $2d$ (0, 2) theory, which is the worldvolume theory of the D1-brane probing the Calabi-Yau 4-fold. For each of the 18 regular reflexive polytopes, we will uniquely identify the corresponding $2d$ (0, 2) theory in terms of its J - and E -terms and its quiver diagram.

¹For this class of $2d$ (0, 2) theories living on D1-branes probing Calabi-Yau singularities, Abelian gauge anomalies are cancelled by a generalized Green-Schwarz mechanism through interactions with bulk RR-field [44].

	0	1	2	3	4	5	6	7	8	9	
D4	×	×	×	·	×	·	×	·	·	·	
NS5	×	×	—————			Σ	—————			·	·

Table 3. Brane brick models are Type IIA configurations where D4-branes are suspended from an NS5-brane that wraps a holomorphic surface Σ . This configuration is T-dual to the D1-brane probing the toric Calabi-Yau 4-fold corresponding to Σ .

3.2 Brane brick models

The $2d$ $(0, 2)$ worldvolume theory of probe $D1$ -branes on toric Calabi-Yau 4-folds can be represented in terms of a T-dual Type IIA brane configuration known as a *brane brick model* [5, 6]. Brane brick models are powerful tools to study $2d$ $(0, 2)$ theories and corresponding toric Calabi-Yau 4-folds because they combine field theory information and information about the Calabi-Yau geometry in a single representation. Accordingly, they play an analogous role for $2d$ $(0, 2)$ theories and toric Calabi-Yau 4-folds as brane tilings do for $4d$ $\mathcal{N} = 1$ theories and toric Calabi-Yau 3-folds [4, 45, 46].

Brane configuration. A brane brick model is a Type IIA brane configuration of $D4$ -branes wrapping a 3-torus T^3 and suspended from a NS5-brane wrapping a holomorphic surface Σ . The holomorphic surface Σ is the zero locus of the Newton polynomial corresponding to the toric Calabi-Yau 4-fold,

$$\sum_{(a,b,c) \in V} c_{(a,b,c)} x^a y^b z^c = 0, \tag{3.3}$$

where $c_{(a,b,c)}$ take values in \mathbb{C}^* and V is the set of lattice points in the toric diagram Δ_3 of the toric Calabi-Yau 4-fold. The intersection between the holomorphic surface Σ and the 3-torus T^3 is precisely where the $D4$ -brane meets the NS5-brane as summarized in table 3.

The intersections create a tessellation of the 3-torus T^3 which we call as the *brane brick model*. For simplicity, we can replace Σ by its simpler skeleton diagram that consists of $2d$ faces that indicate the locations of the NS5-brane wrapping Σ . These $2d$ faces separate T^3 into $3d$ polytopes filled by $D4$ -branes. We call these $3d$ polytopes as *bricks*.

Dictionary. The brane brick model on T^3 consists of the following fundamental components:

- *Bricks.* Bricks are 3-dimensional polytopes that tessellate the 3-torus T^3 . Each brick corresponds to a $U(N_i)$ gauge group of the $2d$ theory. As the 3-dimensional generalization of a brane interval, its interior indicates the location of the $D4$ -branes suspended between the NS5-brane wrapping Σ .
- *Faces.* The 2-dimensional brick faces are even-sided and can be oriented systematically along their boundary edges such they are either *oriented* or *unoriented*. Oriented and unoriented faces correspond to bifundamental (or adjoint) chiral and Fermi fields, respectively. Faces corresponding to Fermi fields are 4-sided. The two bricks adjacent

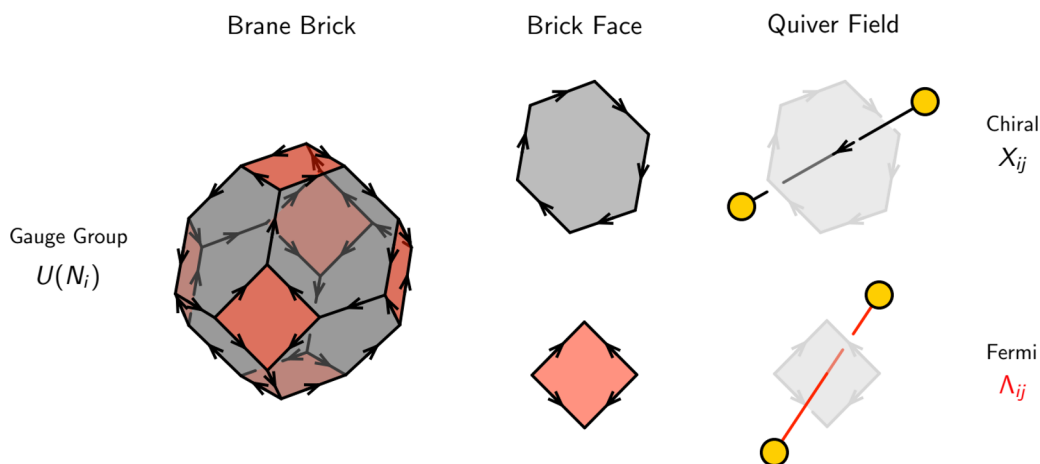


Figure 8. $2d$ brick faces in the brane brick model can be systematically oriented along their boundary edges such that oriented even-sided faces correspond to chiral fields and unoriented 4-sided faces correspond to Fermi fields.

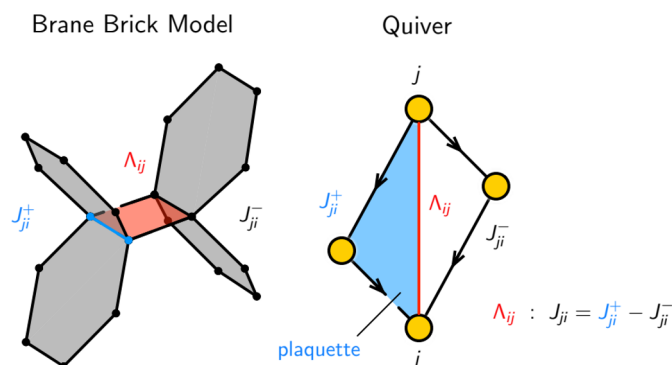


Figure 9. Edges in the brane brick model correspond to monomials in the J - and E -terms of the $2d$ theory. The corresponding collection of chiral fields with the associated Fermi field forms what we call a *plaquette* (blue). Each Fermi face with its four edges is associated with four plaquettes that form the J - and E -terms corresponding to the Fermi field.

to a given face correspond to the gauge groups under which the corresponding field transforms. Figure 8 shows how the two different types of faces correspond to chiral and Fermi fields in the $2d$ theory.

- *Edges.* Edges in a brane brick model are always adjacent to a Fermi face and a number of chiral faces. We call the collection of chiral fields and the Fermi field associated to the faces that coincide at a given edge a *plaquette*, as illustrated in figure 9. The chiral fields in a plaquette form one of the monomials J_{ji}^\pm and E_{ij}^\pm in the J - and E -terms of the $2d$ theory. Every Fermi face with its four edges is associated with four plaquettes that form the J - and E -terms corresponding to the Fermi field. Opposite edges of a 4-sided Fermi face form a J - (or E -) term while the other pair of opposite edges form a E - (or J -) term, as illustrated in figure 9.

Brane Brick Model	Gauge Theory
Brick	Gauge group
Oriented face between bricks i and j	Chiral field in the bifundamental representation of nodes i and j (adjoint for $i = j$)
Unoriented square face between bricks i and j	Fermi field in the bifundamental representation of nodes i and j (adjoint for $i = j$)
Edge	Plaquette encoding a monomial in a J - or E -term

Table 4. Dictionary between brane brick models and $2d$ gauge theories.

The complete dictionary between the $2d$ gauge theory and the brane brick model is summarized in table 4. For a thorough discussion of these constructions and generalizations beyond T^3 see [6, 47].

In this work, we will identify one brane brick model representing a $2d$ $(0, 2)$ theory for each of the toric Calabi-Yau 4-folds which have regular reflexive polytopes as their toric diagrams.² We will identify the J - and E -terms as well as the quiver diagram which define the brane brick model as well as the corresponding $2d$ $(0, 2)$ theory.

3.3 The mesonic moduli space and the forward algorithm

The Calabi-Yau 4-fold geometry that is probed by the D1-branes is related to the vacuum moduli space of the $2d$ gauge theory that lives on the worldvolume of the D1-branes. We call this particular vacuum moduli space the *mesonic moduli space* \mathcal{M}^{mes} of the brane brick model. The mesonic moduli space is the toric Calabi-Yau 4-fold probed by the D1-brane when we consider the Abelian $2d$ theory with $U(1)$ gauge groups. As usual, for $2d$ theories we have in mind the classical moduli spaces, which should be regarded, in the spirit of the Born-Oppenheimer approximation, as target spaces of non-linear sigma models.

Mesonic moduli space \mathcal{M}^{mes} . The mesonic moduli space is determined by the J - and E -term constraints as well as the D -term constraints of the $2d$ $(0, 2)$ theory. Let us summarize the properties and definition of the mesonic moduli space as follows:

- The J - and E -terms of the Abelian $2d$ $(0, 2)$ theory form a binomial ideal of the following form

$$\mathcal{I}_{J=0, E=0} = \langle J_{ji}^+ - J_{ji}^-, E_{ij}^+ - E_{ij}^- = 0 \rangle, \tag{3.4}$$

where as discussed in section 3.1 J_{ji}^\pm and E_{ij}^\pm are monomials in chiral fields. We note that the quotient ring of the form

$$R_X = \mathbb{C}[X_{ij}] / \mathcal{I}_{J=0, E=0} \tag{3.5}$$

²Generically, multiple brane brick models can be associated to a given toric Calabi-Yau 4-fold (see e.g. [14]). Such brane brick models correspond to IR equivalent $2d$ $(0, 2)$ gauge theories related by triality [18]. An exhaustive classification of the triality dual phases for each of the Calabi-Yau 4-folds we consider is beyond the scope of this paper.

captures the essence of a toric variety X and we call it the coordinate ring of X . We note that

$$\mathcal{F}^b = \text{Spec}(\mathbb{C}[X_{ij}]/\mathcal{I}_{J=0,E=0}). \tag{3.6}$$

We call \mathcal{F}^b the *master space* of the corresponding brane brick model.³

Usually, the ideal $\mathcal{I}_{J=0,E=0}$ is reducible into irreducible components, which are known as primary ideals. Taking the largest of these primary ideals in (3.6), we obtain the *coherent component* of the master space which we denote as $^{\text{Irr}}\mathcal{F}^b$. In the following discussion, we will concentrate on the coherent component of the master space and for simplicity use $^{\text{Irr}}\mathcal{F}^b$ and \mathcal{F}^b interchangeably for the Abelian $2d$ $(0, 2)$ theories.

- The *mesonic moduli space* of the one D1-brane theory is related to the master space $^{\text{Irr}}\mathcal{F}^b$. It is obtained by quotienting out the $U(1)^G$ gauge charges, where G is the number of gauge groups in the brane brick model. It is important to note that an overall $U(1)$ decouples, giving a total of $U(1)^{G-1}$ independent charges. Accordingly, the mesonic moduli space takes the following form

$$\mathcal{M}^{\text{mes}} = ^{\text{Irr}}\mathcal{F}^b / U(1)^{G-1}. \tag{3.7}$$

The mesonic moduli space \mathcal{M}^{mes} is a toric Calabi-Yau 4-fold for brane brick models. It is exactly the same Calabi-Yau that is probed by the D1-branes whose worldvolume theory is the $2d$ $(0, 2)$ theory given by our brane brick model.

The dimension of the mesonic moduli space \mathcal{M}^{mes} can be derived by starting with the number of chiral fields n^χ . The J - and E -terms impose $n^F - 3$ independent constraints where n^F is the number of Fermi fields in the brane brick model. Further restrictions come from demanding invariance under the $G - 1$ independent gauge charges and the anomaly cancellation condition from (3.1) that sets $n^\chi - n^F = G$. By combining all these constraints, one obtains the dimension of the mesonic moduli space \mathcal{M}^{mes} to be $n^\chi - (n^F - 3) - (G - 1) = 4$ as expected.

In the following work, we concentrate on brane brick models and corresponding $2d$ $(0, 2)$ theories whose mesonic moduli space \mathcal{M}^{mes} is a toric Calabi-Yau 4-fold that has a regular reflexive lattice polytope as its toric diagram. In order to illustrate how the toric diagram can be systematically obtained from the brane brick model, a brief review on the *forward algorithm* [6] which translates the $2d$ gauge theory information into toric data is given below.

Forward algorithm. The forward algorithm for brane brick models involves the following steps:

- *K-matrix.* Let us denote by X_m with $m = 1, \dots, n^\chi$ the chiral fields of the brane brick model. The space of solutions of the J - and E -terms of the brane brick model can be expressed in terms of $G + 3$ independent chiral fields, which we label by

³The concept of master space was introduced in [48, 49] for $4d$ $\mathcal{N} = 1$ gauge theories but, as mentioned above, can be naturally generalized to supersymmetric gauge theories in other dimensions [5].

v_k . Accordingly, with G being the number of gauge groups, the chiral fields can be expressed as follows

$$X_m = \prod_k v_k^{K_{mk}}, \tag{3.8}$$

where $m = 1, \dots, n^\chi$ and $k = 1, \dots, G + 3$. K is a $n^\chi \times (G + 3)$ -dimensional matrix that encodes the relations from the vanishing J - and E -terms of the brane brick model.

- *P-matrix and brick matchings.* We note that the K -matrix can contain negative integers as entries meaning that chiral fields X_m in the brane brick model sometimes are expressed in terms of negative powers of independent fields v_k . In order to avoid negative entries in K , we define a new matrix T to be the space of vectors dual to K as follows

$$K \cdot T \geq 0. \tag{3.9}$$

Using T , we can now express in terms of the independent chiral fields v_k a new set of fields p_α as follows

$$v_k = \prod_\alpha p_\alpha^{T_{k\alpha}}, \tag{3.10}$$

where $\alpha = 1, \dots, c$. The p_α are interpreted as GLSM fields [50] in the toric description of the mesonic moduli space \mathcal{M}^{mes} of the brane brick model.

Using (3.8) and (3.10), all chiral fields X_m of the brane brick model can be expressed in terms of GLSM fields as follows

$$X_m = \prod_\alpha p_\alpha^{P_{m\alpha}} \tag{3.11}$$

where the $(n^\chi \times c)$ -dimensional P -matrix is given by

$$P_{n^\chi \times c} = K_{n^\chi \times (G+3)} \cdot T_{(G+3) \times c}. \tag{3.12}$$

The labels of the matrices above indicate their dimensions. We also note that the entries of the P -matrix are strictly greater or equal to zero.

In the brane brick model, the GLSM fields encoded in the P -matrix in terms of chiral fields have an additional combinatorial meaning. The collection of chiral fields associated to a GLSM field is a special collection of fields that cover every plaquette in the brane brick model exactly once [6]. We call this collection of fields also a *brick matching*.

- *Q_{JE} -matrix.* The relations between chiral fields of the brane brick model are given by the J - and E -terms. These relations can be expressed in terms of a collection of $U(1)$ -charges to the GLSM fields that form a new basis of fields parameterizing the J - and E -terms. These charges are given by a $((c - (G + 3)) \times c)$ -dimensional charge matrix called the Q_{JE} -matrix, which is the kernel of the P -matrix as follows

$$(Q_{JE})_{(c-(G+3)) \times c} = \ker(P). \tag{3.13}$$

- *Q_D -matrix.* In order to obtain the mesonic moduli space of the abelian $2d$ theory corresponding to a brane brick model, we have to impose the D -terms. Accordingly, we need to express the $U(1)$ gauge charges on chiral fields as $U(1)$ charges on the GLSM fields. The $U(1)$ gauge charges on chiral fields are given by the $(G \times n^\chi)$ -dimensional quiver incidence matrix d , where G is the number of nodes in the quiver. Because all chiral fields are either in the bifundamental or adjoint representation, the incidence matrix satisfies

$$\sum_a d_{ai} = 0, \tag{3.14}$$

where $a = 1, \dots, G$ and $i = 1, \dots, n^\chi$. Hence, only $G - 1$ rows of the incidence matrix d are independent allowing us to reduce it to a $((G - 1) \times n^\chi)$ -dimensional matrix \bar{d} . Using the reduced incidence matrix \bar{d} , the $U(1)$ charges on the GLSM fields p_α fields can be summarized in a $((G - 1) \times c)$ -dimensional matrix Q_D . The Q_D -matrix can be obtained from the following relation

$$\bar{d}_{(G-1) \times n^\chi} = (Q_D)_{(G-1) \times c} \cdot P_{c \times n^\chi}^t. \tag{3.15}$$

- *Toric diagram.* The Q_{JE} - and Q_D - matrices contain the $U(1)$ -charges on the GLSM fields which are associated with the vanishing J - and E -terms as well as the D -terms of the brane brick model, respectively. The charge matrices Q_{JE} and Q_D can be combined into the following total charge matrix

$$(Q_t)_{(c-4) \times c} = \left((Q_{JE})_{(c-(G+3)) \times c}, (Q_D)_{(G-1) \times c} \right). \tag{3.16}$$

The total charge matrix Q_t is $((c - 4) \times c)$ -dimensional where c is the number of GLSM fields.

The kernel of the total charge matrix Q_t is $(4 \times c)$ -dimensional,

$$G_t = \ker(Q_t), \tag{3.17}$$

and encodes the toric diagram of the toric Calabi-Yau 4-fold that is the mesonic moduli space \mathcal{M}^{mes} of the brane brick model. Every column of G_t corresponds to a GLSM field and a brick matching of the brane brick model and determines the position of a point in the \mathbb{Z}^4 -lattice. The convex hull of the points forms the toric diagram Δ of the toric Calabi-Yau 4-fold. One can find a suitable $GL(4, \mathbb{Z})$ transformation of the coordinates of the points encoded in G_t such that all points lie on the 3-dimensional hyperplane in \mathbb{Z}^4 , giving us the 3-dimensional lattice polytope Δ . Multiple GLSM fields may be mapped to the same point in the toric diagram.

One of the aims of our work is to identify for each toric diagram which is a regular reflexive lattice polytope a corresponding brane brick model. As a result, we expect to be able to identify 18 distinct brane brick models whose mesonic moduli space is a toric Calabi-Yau 4-fold with a toric diagram that is one of the 18 regular reflexive polytopes.

The definition of the mesonic moduli space \mathcal{M}^{mes} in (3.7) and the formula for the toric diagram encoded in G_t in (3.17) both refer to the same toric Calabi-Yau 4-fold associated to a brane brick model. Before we proceed, it is interesting to point out that for the $2d$ $(0, 2)$ gauge theories associated to toric Calabi-Yau 4-folds, the forward algorithm often results in additional GLSM fields that we call as *extra GLSM fields*.

Extra GLSM fields. In some cases, the forward algorithm leads to extra GLSM fields in the P -matrix in (3.12). These in turn manifest themselves as additional points in the toric diagram given by the G_t -matrix in (3.17). These points lie outside the 3-dimensional hyperplane of the 3-dimensional toric diagram.

It is important to note that these extra GLSM fields are *redundant* for the description of the mesonic moduli space \mathcal{M}^{mes} , meaning that the toric diagram without the points corresponding to the extra GLSM fields is the correct toric diagram for the toric Calabi-Yau 4-fold. This is because these extra GLSM fields correspond to an over-parameterization of the mesonic moduli space \mathcal{M}^{mes} [5]. While normally, the mesonic moduli space \mathcal{M}^{mes} is parameterized by mesonic gauge invariant operators formed by the chiral fields, which form the spectrum of operators for the quotient in (3.7), the presence or absence of the extra GLSM fields does not affect the spectrum of gauge invariant operators. In fact, if we describe the mesonic moduli space \mathcal{M}^{mes} in terms of mesonic gauge invariant operators that generate the entire spectrum of operators as well as their defining relations, then the extra GLSM fields do not affect the generators as well as the defining relations amongst them, leaving the mesonic moduli space \mathcal{M}^{mes} unaffected.

The *Hilbert series* [27–29] is an important tool to characterize the spectrum of gauge invariant operators and hence the moduli space of a gauge theory. In the following sections, we will show whenever extra GLSM fields are present, that their removal does not affect the algebraic properties of the mesonic moduli space \mathcal{M}^{mes} of a brane brick model using the Hilbert series. As a result, the following toric diagrams that we calculate using the forward algorithm will only contain points corresponding to normal GLSM fields and have all points corresponding to extra GLSM fields removed. Let us in the following section discuss the calculation of Hilbert series for the mesonic moduli space \mathcal{M}^{mes} and how it can be used to characterize the algebraic structure of \mathcal{M}^{mes} for brane brick models.

3.4 Hilbert series and plethystics

Hilbert series. The mesonic moduli space \mathcal{M}^{mes} of a brane brick model is the space of gauge invariant operators under J - and E -term charges Q_{JE} and D -term charges Q_D . The *Hilbert series* is a generating function that counts gauge invariant operators [27–29] of a moduli space. It contains information about the moduli space generators and the defining relations that they form amongst themselves. For charges $Q = (Q_{JE}, Q_D)$, the Hilbert series for the mesonic moduli space $\mathcal{M} = \mathcal{M}^{\text{mes}}$ is given by the *Molien integral*

$$g_1(y_\alpha; \mathcal{M}) = \prod_{i=1}^{|Q|} \oint_{|z_i|=1} \frac{dz_i}{2\pi i z_i} \prod_{\alpha=1}^c \frac{1}{1 - y_\alpha \prod_{j=1}^{|Q|} z_j^{Q_{j\alpha}}}, \quad (3.18)$$

where c is the number of brick matchings corresponding to the GLSM fields of the brane brick model and $|Q|$ is the number of rows in the total charge matrix Q_t .

The fugacity y_α counts GLSM fields of the brane brick model and we set it to be $y_\alpha = t_i$ if the GLSM field corresponds to an extremal point p_i of the toric diagram of the Calabi-Yau 4-fold. Furthermore, we set the fugacity to be $y_\alpha = y_{s_m}$ if it corresponds to brick matchings s_m corresponding to the single internal point of the reflexive toric diagram.

Finally, we set $y_\alpha = y_{o_k}$ if the GLSM field is an extra GLSM field that does not contribute to the algebraic structure of the moduli space. In fact, in the following discussion of brane brick models corresponding to regular reflexive polytopes, we see that setting the fugacities for extra GLSM fields $y_{o_k} = 1$ does not change the algebraic structure of the mesonic moduli space captured by the Hilbert series. This will be indicated in the Hilbert series calculation when needed.

Plethystics. The moduli space is specified by its generators and defining relations formed by the generators. In order to obtain information about the generators and relations amongst them, we make use of the *plethystic logarithm* of the Hilbert series [30, 31]. The plethystic logarithm takes the form

$$\text{PL} [g_1(y_\alpha; \mathcal{M})] = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log [g_1(y_\alpha^k; \mathcal{M})] , \tag{3.19}$$

where $\mu(k)$ is the Möbius function. If the expansion of the plethystic logarithm is *finite*, the corresponding moduli space is known to be a *complete intersection* generated by a finite number of generators subject to a finite number of relations. The first positive terms of the finite expansion correspond to the counting of generators while the following negative terms correspond to the counting of the defining relations formed amongst them. On the other hand, if the expansion of the plethystic logarithm is infinite, the moduli space is known to be a *non-complete intersection*. The first positive terms of the expansion again refer to generators of the moduli space while all higher order terms refer to relations amongst generators and relations amongst relations known as *syzygies*.

In the following work, the aim will be to identify the generators of the mesonic moduli space of brane brick models corresponding to toric Calabi-Yau 4-folds whose toric diagram is one of the 18 regular reflexive lattice polytopes in 3 dimensions. Using the plethystic logarithm of the Hilbert series of the mesonic moduli space, which is in terms of fugacities of the GLSM fields corresponding to brick matchings of the brane brick model, we are able to write the generators of the mesonic moduli space in terms of GLSM fields. Since the GLSM fields are themselves related to the chiral fields of the corresponding $2d$ gauge theory, which we know are subject to the J - and E -term constraints, we are not only able to write the generators of the mesonic moduli space in terms of chiral fields, but also reconstruct the different J - and E -term equivalent expressions of the generators in terms of chiral fields.

3.5 Duality between generator lattices and toric diagrams

Global symmetries. From the isometry of the mesonic moduli space, which is a toric Calabi-Yau 4-fold, we can obtain the global symmetry of the brane brick model which contains $U(1)^4$. A linear combination of the four $U(1)$'s relates to the R -symmetry of $(0, 2)$ supersymmetry. The non-R $U(1)^3$ symmetry is known as the *mesonic flavor symmetry*. For some mesonic moduli spaces, the mesonic flavor symmetry is enhanced to a non-abelian group. In table 5, we summarize the global symmetries of the 18 brane brick models corresponding to regular reflexive polytopes. The Hilbert series of the mesonic moduli space can be refined in terms of fugacities that count the global symmetry charges carried by

Model	Global Symmetry
1	$SU(4) \times U(1)$
2	$SU(3) \times SU(2) \times U(1)$
3	$SU(3) \times U(1) \times U(1)$
4	$SU(2) \times SU(2) \times U(1) \times U(1)$
5	$SU(3) \times U(1) \times U(1)$
6	$SU(2) \times SU(2) \times U(1) \times U(1)$
7	$SU(2) \times U(1) \times U(1) \times U(1)$
8	$SU(2) \times U(1) \times U(1) \times U(1)$
9	$SU(2) \times SU(2) \times U(1) \times U(1)$
10	$SU(2) \times U(1) \times U(1) \times U(1)$
11	$SU(2) \times SU(2) \times U(1) \times U(1)$
12	$SU(2) \times SU(2) \times SU(2) \times U(1)$
13	$SU(2) \times U(1) \times U(1) \times U(1)$
14	$SU(2) \times U(1) \times U(1) \times U(1)$
15	$SU(2) \times U(1) \times U(1) \times U(1)$
16	$SU(2) \times U(1) \times U(1) \times U(1)$
17	$SU(2) \times U(1) \times U(1) \times U(1)$
18	$SU(2) \times U(1) \times U(1) \times U(1)$

Table 5. The global symmetries for the 18 models that can be obtained from the isometries of the toric Calabi-Yau 4-folds.

each of the gauge invariant operators. This can be done by mapping the fugacities counting GLSM fields y_α into fugacities that count the global symmetry charges (x_1, x_2, x_3, t) . Here, it is important to note that only the GLSM fields corresponding to extremal points in the toric diagram carry non-zero charges while all other GLSM fields carry no charges under the global symmetry.

In the following discussion, without loss of generality, we restrict ourselves to fugacities in the Hilbert series that count the mesonic flavor charges (x_1, x_2, x_3) and an additional fugacity t_α that instead of the R -symmetry charge counts the degree in GLSM fields p_α corresponding to the extremal toric points for each gauge invariant operator. This selection of independent fugacities $(x_1, x_2, x_3, t_\alpha)$ for each gauge invariant operator counted by the Hilbert series encodes the rank 4 global symmetry of the brane brick model. Furthermore, this choice of fugacities enables us to identify for each gauge invariant operator of the mesonic moduli space their mesonic flavor charges through fugacities (x_1, x_2, x_3) whilst still containing information about the composition of each gauge invariant operator in terms of GLSM fields through the fugacities t_α .

Lattice of generators. The lattice of generators is formed by the mesonic flavor charges carried by the generators of the mesonic moduli space. The non-R mesonic flavor symmetry

has rank 3 and the lattice that is formed by the charges is 3-dimensional. By a suitable global scaling of the mesonic charges, the charges carried by the gauge invariant operators can be made to be in \mathbb{Z}^3 . There is only a finite number of generators for each mesonic moduli space of a brane brick model. Each integer 3-vector of mesonic flavor charges associated to a generator can be considered as the coordinates of a point on the \mathbb{Z}^3 lattice. The set of lattice points for the generators of the mesonic moduli space forms a convex polytope, which we refer from now on as the *generator lattice* of the brane brick model.

Duality between generator lattices and toric diagrams.

The generator lattice of a brane brick model is the dual of the toric diagram. If the toric diagram is a reflexive polytope, then by duality of reflexive polytopes as stated in (2.1), the generator lattice is also a convex lattice polytope that is reflexive. Accordingly, we are closing the circle of our discussion by stating that for brane brick models with toric diagrams that are reflexive polytopes, the corresponding mesonic moduli space has generator lattices which are the reflexive dual of the toric diagrams up to $GL(3, \mathbb{Z})$ isomorphism.

In the following sections, brane brick models corresponding to all 18 regular reflexive polytopes are classified. Furthermore, it is shown that the mesonic moduli spaces of the brane brick models have generator lattices that are reflexive polytopes which are reflexive dual to the toric diagrams of the brane brick models. In order to show this, we calculate the Hilbert series for each brane brick model and through plethystics write down the set of generators with their mesonic flavor charges and chiral field content. This is the first time *all* Fano 3-folds and the associated Calabi-Yau 4-folds corresponding to the 18 regular reflexive polytopes have been systematically associated to quiver gauge theories realized in string theory.

4 Model 1: $\mathbb{C}^4/\mathbb{Z}_4$ (1, 1, 1, 1) [\mathbb{P}^3 , $\langle 0 \rangle$]

Model 1 corresponds to the Abelian orbifold of the form $\mathbb{C}^4/\mathbb{Z}_4$ (1, 1, 1, 1). The corresponding brane brick model has the quiver in figure 10 and the *J*- and *E*-terms are given as follows

$$\begin{array}{l}
 \begin{array}{cc}
 J & E \\
 \Lambda_{13}^1 : Z_{34}Y_{41} - Y_{34}Z_{41} & P_{12}X_{23} - X_{12}P_{23} \\
 \Lambda_{13}^2 : X_{34}Z_{41} - Z_{34}X_{41} & P_{12}Y_{23} - Y_{12}P_{23} \\
 \Lambda_{13}^3 : Y_{34}X_{41} - X_{34}Y_{41} & P_{12}Z_{23} - Z_{12}P_{23} \\
 \Lambda_{24}^1 : Z_{41}Y_{12} - Y_{41}Z_{12} & P_{23}X_{34} - X_{23}P_{34} \\
 \Lambda_{24}^2 : X_{41}Z_{12} - Z_{41}X_{12} & P_{23}Y_{34} - Y_{23}P_{34} \\
 \Lambda_{24}^3 : Y_{41}X_{12} - X_{41}Y_{12} & P_{23}Z_{34} - Z_{23}P_{34} \\
 \Lambda_{31}^1 : Z_{12}Y_{23} - Y_{12}Z_{23} & P_{34}X_{41} - X_{34}P_{41} \\
 \Lambda_{31}^2 : X_{12}Z_{23} - Z_{12}X_{23} & P_{34}Y_{41} - Y_{34}P_{41} \\
 \Lambda_{31}^3 : Y_{12}X_{23} - X_{12}Y_{23} & P_{34}Z_{41} - Z_{34}P_{41} \\
 \Lambda_{42}^1 : Z_{23}Y_{34} - Y_{23}Z_{34} & P_{41}X_{12} - X_{41}P_{12} \\
 \Lambda_{42}^2 : X_{23}Z_{34} - Z_{23}X_{34} & P_{41}Y_{12} - Y_{41}P_{12} \\
 \Lambda_{42}^3 : Y_{23}X_{34} - X_{23}Y_{34} & P_{41}Z_{12} - Z_{41}P_{12}
 \end{array}
 \end{array} . \tag{4.1}$$

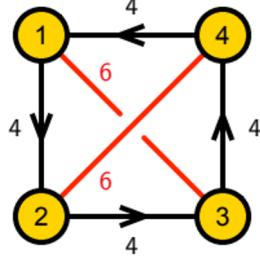


Figure 10. Quiver for Model 1.

Through the forward algorithm, the P -matrix of Model 1 can be calculated as follows

$$P = \left(\begin{array}{c|cccc|cccc} & p_1 & p_2 & p_3 & p_4 & s_1 & s_2 & s_3 & s_4 \\ \hline P_{12} & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ P_{23} & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ P_{34} & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ P_{41} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline X_{12} & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ X_{23} & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ X_{34} & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ X_{41} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline Y_{12} & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ Y_{23} & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ Y_{34} & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ Y_{41} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ \hline Z_{12} & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ Z_{23} & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ Z_{34} & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ Z_{41} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right). \quad (4.2)$$

The J - and E -term charges are given by

$$Q_{JE} = \left(\begin{array}{cccc|cccc} p_1 & p_2 & p_3 & p_4 & s_1 & s_2 & s_3 & s_4 \\ \hline 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \end{array} \right), \quad (4.3)$$

and the D -term charges are given by

$$Q_D = \left(\begin{array}{cccc|cccc} p_1 & p_2 & p_3 & p_4 & s_1 & s_2 & s_3 & s_4 \\ \hline 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{array} \right). \quad (4.4)$$

The toric diagram of Model 1 can be found to be as follows,

$$G_t = \left(\begin{array}{cccc|cccc} p_1 & p_2 & p_3 & p_4 & s_1 & s_2 & s_3 & s_4 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \end{array} \right). \quad (4.5)$$

The toric diagram with brick matching labels is shown in figure 11.

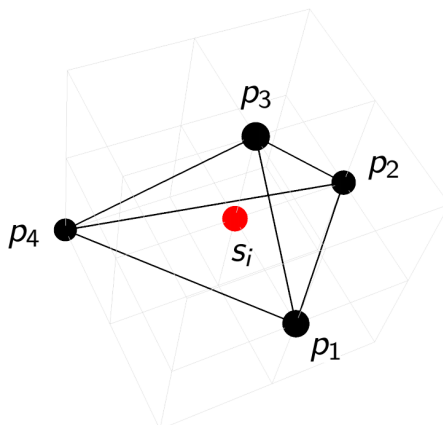


Figure 11. Toric diagram for Model 1.

	SU(4)	U(1)	fugacity
p_1	(+1,0,0)	r_1	t_1
p_2	(-1,+1,0)	r_2	t_2
p_3	(0,-1,+1)	r_3	t_3
p_4	(0,0,-1)	r_4	t_4

Table 6. Global symmetry charges on the extremal brick matchings p_i of Model 1.

Using the Molien integral formula, the Hilbert series of the mesonic moduli space of Model 1 is found as

$$g_1(t_i, y_s; \mathcal{M}_1) = \frac{P(t_i, y_s; \mathcal{M}_1)}{(1 - y_s t_1^4)(1 - y_s t_2^4)(1 - y_s t_3^4)(1 - y_s t_4^4)}, \quad (4.6)$$

where t_i are the fugacities for the extremal brick matchings p_i . y_s counts the brick matching product $s_1 s_2 s_3 s_4$ corresponding to the single internal point of the toric diagram of Model 1. The explicit numerator $P(t_i, y_s; \mathcal{M}_1)$ of the Hilbert series is given in the appendix section A.1.

By setting $t_i = t$ for all i and $y_s = 1$, the unrefined Hilbert series takes the following form

$$g_1(t, 1; \mathcal{M}_1) = \frac{1 + 31t^4 + 31t^8 + t^{12}}{(1 - t^4)^4}, \quad (4.7)$$

where the palindromic numerator indicates that the mesonic moduli space is Calabi-Yau.

The global symmetry of Model 1 and the charges on the extremal brick matchings under the global symmetry are summarized in table 6. Using the following fugacity map,

$$t = t_1^{1/4} t_2^{1/4} t_3^{1/4} t_4^{1/4}, \quad x_1 = \frac{t_1^{3/4}}{t_2^{1/4} t_3^{1/4} t_4^{1/4}}, \quad x_2 = \frac{t_1^{1/2} t_2^{1/2}}{t_3^{1/2} t_4^{1/2}}, \quad x_3 = \frac{t_1^{1/4} t_2^{1/4} t_3^{1/4}}{t_4^{3/4}}, \quad (4.8)$$

the Hilbert series for Model 1 can be rewritten in terms of characters of irreducible representations of $SU(4)$, the mesonic flavor symmetry of Model 1, as follows

$$g_1(t, x_i; \mathcal{M}_1) = \sum_{n=0}^{\infty} [4n, 0, 0] t^{4n}. \quad (4.9)$$

Here, $[n_1, n_2, n_3] = [n_1, n_2, n_3]_{SU(4)}$ is the character of the irreducible representation of $SU(4)$ labeled by the highest weight n_1, n_2, n_3 . The corresponding plethystic logarithm is

$$PL[g_1(t, x_i; \mathcal{M}_1)] = [4, 0, 0] t^4 - ([4, 2, 0] + [0, 4, 0]) t^8 + \dots, \quad (4.10)$$

where the mesonic moduli space is identified as a non-complete intersection. The set of generators transform under the $[4, 0, 0]$ representation of the mesonic flavor symmetry. Using the following fugacity map

$$\tilde{t} = t_1^{1/4} t_2^{1/4} t_3^{1/4} t_4^{1/4}, \quad \tilde{x}_1 = \frac{t_1}{t_4}, \quad \tilde{x}_2 = \frac{t_4}{t_3}, \quad \tilde{x}_3 = \frac{t_2}{t_4}, \quad (4.11)$$

the mesonic flavor charges on the gauge invariant operators become \mathbb{Z} -valued. The generators in terms of brick matchings and their corresponding individual \mathbb{Z}^3 -charges are summarized in table 7. The generator lattice as shown in table 7 is a convex lattice polytope, which is reflexive. It is the dual of the toric diagram of Model 1 in figure 11. For completeness, table 8 shows the generators of Model 1 in terms of chiral fields with the corresponding mesonic flavor charges.

5 Model 2: $M^{3,2} [\mathbb{P}^2 \times \mathbb{P}^1, \langle 4 \rangle]$

Model 2 corresponds to the Calabi-Yau cone over the $M^{3,2}$ surface. The corresponding brane brick model has the quiver in figure 12 and the J - and E -terms are given as follows

$$\begin{array}{ll}
 & J \\
 \Lambda_{12}^1 : & X_{23} X_{31} - Z_{23} Y_{31} \\
 \Lambda_{12}^2 : & Y_{23} Y_{31} - X_{23} Z_{31} \\
 \Lambda_{12}^3 : & Z_{23} Z_{31} - Y_{23} X_{31} \\
 \Lambda_{26}^1 : & X_{64} X_{42} - Z_{64} Y_{42} \\
 \Lambda_{26}^2 : & Y_{64} Y_{42} - X_{64} Z_{42} \\
 \Lambda_{26}^3 : & Z_{64} Z_{42} - Y_{64} X_{42} \\
 \Lambda_{26}^4 : & X_{64} R_{42} - Z_{64} S_{42} \\
 \Lambda_{26}^5 : & Y_{64} S_{42} - X_{64} T_{42} \\
 \Lambda_{26}^6 : & Z_{64} T_{42} - Y_{64} R_{42} \\
 \Lambda_{34}^1 : & X_{42} X_{23} - Z_{42} Y_{23} \\
 \Lambda_{34}^2 : & Y_{42} Y_{23} - X_{42} Z_{23} \\
 \Lambda_{34}^3 : & Z_{42} Z_{23} - Y_{42} X_{23} \\
 \Lambda_{34}^4 : & R_{42} X_{23} - T_{42} Y_{23} \\
 \Lambda_{34}^5 : & S_{42} Y_{23} - R_{42} Z_{23} \\
 \Lambda_{34}^6 : & T_{42} Z_{23} - S_{42} X_{23} \\
 \Lambda_{45}^1 : & X_{56} X_{64} - Z_{56} Y_{64} \\
 \Lambda_{45}^2 : & Y_{56} Y_{64} - X_{56} Z_{64} \\
 \Lambda_{45}^3 : & Z_{56} Z_{64} - Y_{56} X_{64}
 \end{array}
 \begin{array}{ll}
 & E \\
 P_{14} X_{42} - Q_{14} R_{42} \\
 P_{14} Y_{42} - Q_{14} S_{42} \\
 P_{14} Z_{42} - Q_{14} T_{42} \\
 P_{25} X_{56} - X_{23} P_{36} \\
 P_{25} Y_{56} - Y_{23} P_{36} \\
 P_{25} Z_{56} - Z_{23} P_{36} \\
 X_{23} Q_{36} - Q_{25} X_{56} \\
 Y_{23} Q_{36} - Q_{25} Y_{56} \\
 Z_{23} Q_{36} - Q_{25} Z_{56} \\
 P_{36} X_{64} - X_{31} P_{14} \\
 P_{36} Y_{64} - Y_{31} P_{14} \\
 P_{36} Z_{64} - Z_{31} P_{14} \\
 X_{31} Q_{14} - Q_{36} X_{64} \\
 Y_{31} Q_{14} - Q_{36} Y_{64} \\
 Z_{31} Q_{14} - Q_{36} Z_{64} \\
 R_{42} Q_{25} - X_{42} P_{25} \\
 S_{42} Q_{25} - Y_{42} P_{25} \\
 T_{42} Q_{25} - Z_{42} P_{25}
 \end{array}. \quad (5.1)$$

generator	$SU(4)_{(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)}$
$p_1^4 s$	(3, 1, -1)
$p_1^3 p_2 s$	(2, 1, 0)
$p_1^2 p_2^2 s$	(1, 1, 1)
$p_1 p_2^3 s$	(0, 1, 2)
$p_2^4 s$	(-1, 1, 3)
$p_1^3 p_3 s$	(2, 0, -1)
$p_1^2 p_2 p_3 s$	(1, 0, 0)
$p_1 p_2^2 p_3 s$	(0, 0, 1)
$p_2^3 p_3 s$	(-1, 0, 2)
$p_1^2 p_3^2 s$	(1, -1, -1)
$p_1 p_2 p_3^2 s$	(0, -1, 0)
$p_2^2 p_3^2 s$	(-1, -1, 1)
$p_1 p_3^3 s$	(0, -2, -1)
$p_2 p_3^3 s$	(-1, -2, 0)
$p_3^4 s$	(-1, -3, -1)
$p_1^3 p_4 s$	(2, 1, -1)
$p_1^2 p_2 p_4 s$	(1, 1, 0)
$p_1 p_2^2 p_4 s$	(0, 1, 1)
$p_2^3 p_4 s$	(-1, 1, 2)
$p_1^2 p_3 p_4 s$	(1, 0, -1)
$p_1 p_2 p_3 p_4 s$	(0, 0, 0)
$p_2^2 p_3 p_4 s$	(-1, 0, 1)
$p_1 p_3^2 p_4 s$	(0, -1, -1)
$p_2 p_3^2 p_4 s$	(-1, -1, 0)
$p_3^3 p_4 s$	(-1, -2, -1)
$p_1^2 p_4^2 s$	(1, 1, -1)
$p_1 p_2 p_4^2 s$	(0, 1, 0)
$p_2^2 p_4^2 s$	(-1, 1, 1)
$p_1 p_3 p_4^2 s$	(0, 0, -1)
$p_2 p_3 p_4^2 s$	(-1, 0, 0)
$p_3^2 p_4^2 s$	(-1, -1, -1)
$p_1 p_4^3 s$	(0, 1, -1)
$p_2 p_4^3 s$	(-1, 1, 0)
$p_3 p_4^3 s$	(-1, 0, -1)
$p_4^4 s$	(-1, 1, -1)

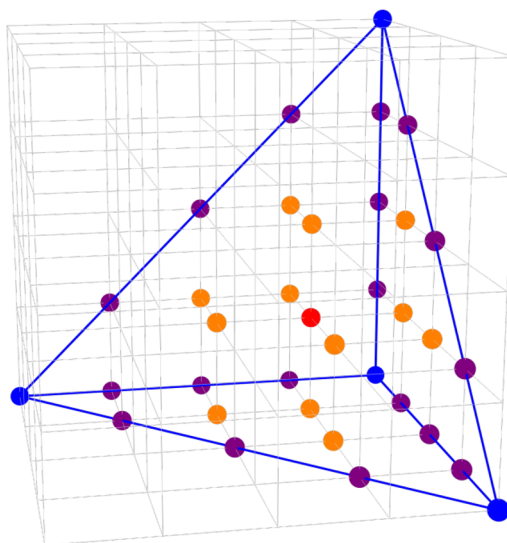


Table 7. The generators and lattice of generators of the mesonic moduli space of Model 1 in terms of brick matchings with the corresponding flavor charges.

generator	$SU(4)_{(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)}$
$P_{12}P_{23}P_{34}P_{41}$	(3, 1, -1)
$X_{12}X_{23}X_{34}X_{41}$	(2, 1, 0)
$Y_{12}Y_{23}Y_{34}Y_{41}$	(1, 1, 1)
$Z_{12}Z_{23}Z_{34}Z_{41}$	(0, 1, 2)
$P_{12}P_{23}P_{34}X_{41} = P_{12}P_{23}X_{34}P_{41} = P_{12}X_{23}P_{41}P_{34} = X_{12}P_{23}P_{34}P_{41}$	(-1, 1, 3)
$P_{12}P_{23}P_{34}Y_{41} = P_{12}P_{23}Y_{34}P_{41} = P_{12}Y_{23}P_{41}P_{34} = Y_{12}P_{23}P_{34}P_{41}$	(2, 0, -1)
$P_{12}P_{23}P_{34}Z_{41} = P_{12}P_{23}Z_{34}P_{41} = P_{12}Z_{23}P_{41}P_{34} = Z_{12}P_{23}P_{34}P_{41}$	(1, 0, 0)
$P_{12}X_{23}X_{34}X_{41} = X_{12}P_{23}X_{34}X_{41} = X_{12}P_{34}X_{41}X_{23} = X_{12}X_{23}X_{34}P_{41}$	(0, 0, 1)
$P_{12}Y_{23}Y_{34}Y_{41} = Y_{12}P_{23}Y_{34}Y_{41} = Y_{12}P_{34}Y_{41}Y_{23} = Y_{12}Y_{23}Y_{34}P_{41}$	(-1, 0, 2)
$P_{12}Z_{23}Z_{34}Z_{41} = Z_{12}P_{23}Z_{34}Z_{41} = Z_{12}P_{34}Z_{41}Z_{23} = Z_{12}Z_{23}Z_{34}P_{41}$	(1, -1, -1)
$X_{12}X_{23}X_{34}Y_{41} = X_{12}X_{23}Y_{34}X_{41} = X_{12}Y_{23}X_{41}X_{34} = Y_{12}X_{23}X_{34}X_{41}$	(0, -1, 0)
$X_{12}X_{23}X_{34}Z_{41} = X_{12}X_{23}Z_{34}X_{41} = X_{12}Z_{23}X_{41}X_{34} = Z_{12}X_{23}X_{34}X_{41}$	(-1, -1, 1)
$X_{12}Y_{23}Y_{34}Y_{41} = Y_{12}X_{23}Y_{34}Y_{41} = Y_{12}X_{34}Y_{41}Y_{23} = Y_{12}Y_{23}Y_{34}X_{41}$	(0, -2, -1)
$X_{12}Z_{23}Z_{34}Z_{41} = Z_{12}X_{23}Z_{34}Z_{41} = Z_{12}X_{34}Z_{41}Z_{23} = Z_{12}Z_{23}Z_{34}X_{41}$	(-1, -2, 0)
$Y_{12}Y_{23}Y_{34}Z_{41} = Y_{12}Y_{23}Z_{34}Y_{41} = Y_{12}Z_{23}Y_{41}Y_{34} = Z_{12}Y_{23}Y_{34}Y_{41}$	(-1, -3, -1)
$Y_{12}Z_{23}Z_{34}Z_{41} = Z_{12}Y_{23}Z_{34}Z_{41} = Z_{12}Y_{34}Z_{41}Z_{23} = Z_{12}Z_{23}Z_{34}Y_{41}$	(2, 1, -1)
$P_{12}P_{23}X_{34}X_{41} = P_{12}X_{23}P_{34}X_{41} = P_{12}X_{23}X_{34}P_{41} = X_{12}P_{23}P_{34}X_{41} = X_{12}P_{41}P_{23}X_{34} = X_{12}X_{23}P_{34}P_{41}$	(1, 1, 0)
$P_{12}P_{23}Y_{34}Y_{41} = P_{12}Y_{23}P_{34}Y_{41} = P_{12}Y_{23}Y_{34}P_{41} = Y_{12}P_{23}P_{34}Y_{41} = Y_{12}P_{41}P_{23}Y_{34} = Y_{12}Y_{23}P_{34}P_{41}$	(0, 1, 1)
$P_{12}P_{23}Z_{34}Z_{41} = P_{12}Z_{23}P_{34}Z_{41} = P_{12}Z_{23}Z_{34}P_{41} = Z_{12}P_{23}P_{34}Z_{41} = Z_{12}P_{41}P_{23}Z_{34} = Z_{12}Z_{23}P_{34}P_{41}$	(-1, 1, 2)
$X_{12}X_{23}Y_{34}Y_{41} = X_{12}Y_{23}X_{34}Y_{41} = X_{12}Y_{23}Y_{34}X_{41} = Y_{12}X_{23}X_{34}Y_{41} = Y_{12}X_{41}X_{23}Y_{34} = Y_{12}Y_{23}X_{34}X_{41}$	(1, 0, -1)
$X_{12}X_{23}Z_{34}Z_{41} = X_{12}Z_{23}X_{34}Z_{41} = X_{12}Z_{23}Z_{34}X_{41} = Z_{12}X_{23}X_{34}Z_{41} = Z_{12}X_{41}X_{23}Z_{34} = Z_{12}Z_{23}X_{34}X_{41}$	(0, 0, 0)
$Y_{12}Y_{23}Z_{34}Z_{41} = Y_{12}Z_{23}Y_{34}Z_{41} = Y_{12}Z_{23}Z_{34}Y_{41} = Z_{12}Y_{23}Y_{34}Z_{41} = Z_{12}Y_{41}Y_{23}Z_{34} = Z_{12}Z_{23}Y_{34}Y_{41}$	(-1, 0, 1)
$P_{12}P_{23}X_{34}Y_{41} = P_{12}P_{23}Y_{34}X_{41} = P_{12}X_{23}P_{34}Y_{41} = P_{12}Y_{23}X_{41}P_{34} = P_{12}X_{23}Y_{34}P_{41} = P_{12}Y_{23}X_{34}P_{41}$ $= X_{12}P_{23}P_{34}Y_{41} = Y_{12}P_{23}P_{34}X_{41} = X_{12}P_{41}P_{23}Y_{34} = Y_{12}P_{23}X_{34}P_{41} = X_{12}Y_{23}P_{34}P_{41} = Y_{12}X_{23}P_{34}P_{41}$	(-1, -1, 0)
$P_{12}P_{23}X_{34}Z_{41} = P_{12}P_{23}Z_{34}X_{41} = P_{12}X_{23}P_{34}Z_{41} = P_{12}Z_{23}X_{41}P_{34} = P_{12}X_{23}Z_{34}P_{41} = P_{12}Z_{23}X_{34}P_{41}$ $= X_{12}P_{23}P_{34}Z_{41} = Z_{12}P_{23}P_{34}X_{41} = X_{12}P_{41}P_{23}Z_{34} = Z_{12}P_{23}X_{34}P_{41} = X_{12}Z_{23}P_{34}P_{41} = Z_{12}X_{23}P_{34}P_{41}$	(-1, -2, -1)
$P_{12}P_{23}Y_{34}Z_{41} = P_{12}P_{23}Z_{34}Y_{41} = P_{12}Y_{23}P_{34}Z_{41} = P_{12}Z_{23}Y_{41}P_{34} = P_{12}Y_{23}Z_{34}P_{41} = P_{12}Z_{23}Y_{34}P_{41}$ $= Y_{12}P_{23}P_{34}Z_{41} = Z_{12}P_{23}P_{34}Y_{41} = Y_{12}P_{41}P_{23}Z_{34} = Z_{12}P_{23}Y_{34}P_{41} = Y_{12}Z_{23}P_{34}P_{41} = Z_{12}Y_{23}P_{34}P_{41}$	(1, 1, -1)
$P_{12}X_{23}X_{34}Y_{41} = P_{12}X_{23}Y_{34}X_{41} = P_{12}Y_{23}X_{41}X_{34} = X_{12}P_{23}X_{34}Y_{41} = X_{12}P_{23}Y_{34}X_{41} = Y_{12}P_{23}X_{34}X_{41}$ $= X_{12}P_{34}Y_{41}X_{23} = X_{12}Y_{23}P_{34}X_{41} = Y_{12}X_{23}P_{34}X_{41} = X_{12}X_{23}Y_{34}P_{41} = X_{12}Y_{23}X_{34}P_{41} = Y_{12}X_{34}X_{23}P_{41}$	(0, 1, 0)
$P_{12}X_{23}X_{34}Z_{41} = P_{12}X_{23}Z_{34}X_{41} = P_{12}Z_{23}X_{41}X_{34} = X_{12}P_{23}X_{34}Z_{41} = X_{12}P_{23}Z_{34}X_{41} = Z_{12}P_{23}X_{34}X_{41}$ $= X_{12}P_{34}Z_{41}X_{23} = X_{12}Z_{23}P_{34}X_{41} = Z_{12}X_{23}P_{34}X_{41} = X_{12}X_{23}Z_{34}P_{41} = X_{12}Z_{23}X_{34}P_{41} = Z_{12}X_{34}X_{23}P_{41}$	(-1, 1, 1)
$P_{12}X_{23}Y_{34}Y_{41} = P_{12}Y_{23}X_{34}Y_{41} = P_{12}Y_{23}Y_{34}X_{41} = X_{12}P_{23}Y_{34}Y_{41} = Y_{12}P_{23}X_{34}Y_{41} = Y_{12}X_{41}P_{23}Y_{34}$ $= X_{12}P_{34}Y_{41}Y_{23} = Y_{12}X_{23}P_{34}Y_{41} = Y_{12}Y_{23}P_{34}X_{41} = X_{12}Y_{23}Y_{34}P_{41} = Y_{12}X_{23}Y_{34}P_{41} = Y_{12}Y_{23}X_{34}P_{41}$	(0, 0, -1)
$P_{12}X_{23}Z_{34}Z_{41} = P_{12}Z_{23}X_{34}Z_{41} = P_{12}Z_{23}Z_{34}X_{41} = X_{12}P_{23}Z_{34}Z_{41} = Z_{12}P_{23}X_{34}Z_{41} = Z_{12}X_{41}P_{23}Z_{34}$ $= X_{12}P_{34}Z_{41}Z_{23} = Z_{12}X_{23}P_{34}Z_{41} = Z_{12}Z_{23}P_{34}X_{41} = X_{12}Z_{23}Z_{34}P_{41} = Z_{12}X_{23}Z_{34}P_{41} = Z_{12}Z_{23}X_{34}P_{41}$	(-1, 0, 0)
$P_{12}Y_{23}Y_{34}Z_{41} = P_{12}Y_{23}Z_{34}Y_{41} = P_{12}Z_{23}Y_{41}Y_{34} = Y_{12}P_{23}Y_{34}Z_{41} = Y_{12}P_{23}Z_{34}Y_{41} = Z_{12}P_{23}Y_{34}Y_{41}$ $= Y_{12}P_{34}Z_{41}Y_{23} = Y_{12}Z_{23}P_{34}Y_{41} = Z_{12}Y_{23}P_{34}Y_{41} = Y_{12}Y_{23}Z_{34}P_{41} = Y_{12}Z_{23}Y_{34}P_{41} = Z_{12}Y_{34}Y_{23}P_{41}$	(-1, -1, -1)
$P_{12}Y_{23}Z_{34}Z_{41} = P_{12}Z_{23}Y_{34}Z_{41} = P_{12}Z_{23}Z_{34}Y_{41} = Y_{12}P_{23}Z_{34}Z_{41} = Z_{12}P_{23}Y_{34}Z_{41} = Z_{12}Y_{41}P_{23}Z_{34}$ $= Y_{12}P_{34}Z_{41}Z_{23} = Z_{12}Y_{23}P_{34}Z_{41} = Z_{12}Z_{23}P_{34}Y_{41} = Y_{12}Z_{23}Z_{34}P_{41} = Z_{12}Y_{23}Z_{34}P_{41} = Z_{12}Z_{23}Y_{34}P_{41}$	(0, 1, -1)
$X_{12}X_{23}Y_{34}Z_{41} = X_{12}X_{23}Z_{34}Y_{41} = X_{12}Y_{23}X_{34}Z_{41} = X_{12}Z_{23}Y_{41}X_{34} = X_{12}Y_{23}Z_{34}X_{41} = X_{12}Z_{23}Y_{34}X_{41}$ $= Y_{12}X_{23}X_{34}Z_{41} = Z_{12}X_{23}X_{34}Y_{41} = Y_{12}X_{41}X_{23}Z_{34} = Z_{12}X_{23}Y_{34}X_{41} = Y_{12}Z_{23}X_{34}X_{41} = Z_{12}Y_{23}X_{34}X_{41}$	(-1, 1, 0)
$X_{12}Y_{23}Y_{34}Z_{41} = X_{12}Y_{23}Z_{34}Y_{41} = X_{12}Z_{23}Y_{41}Y_{34} = Y_{12}X_{23}Y_{34}Z_{41} = Y_{12}X_{23}Z_{34}Y_{41} = Z_{12}X_{23}Y_{34}Y_{41}$ $= Y_{12}X_{34}Z_{41}Y_{23} = Y_{12}Z_{23}X_{34}Y_{41} = Z_{12}Y_{23}X_{34}Y_{41} = Y_{12}Y_{23}Z_{34}X_{41} = Y_{12}Z_{23}Y_{34}X_{41} = Z_{12}Y_{34}Y_{23}X_{41}$	(-1, 0, -1)
$X_{12}Y_{23}Z_{34}Z_{41} = X_{12}Z_{23}Y_{34}Z_{41} = X_{12}Z_{23}Z_{34}Y_{41} = Y_{12}X_{23}Z_{34}Z_{41} = Z_{12}X_{23}Y_{34}Z_{41} = Z_{12}Y_{41}X_{23}Z_{34}$ $= Y_{12}X_{34}Z_{41}Z_{23} = Z_{12}Y_{23}X_{34}Z_{41} = Z_{12}Z_{23}X_{34}Y_{41} = Y_{12}Z_{23}Z_{34}X_{41} = Z_{12}Y_{23}Z_{34}X_{41} = Z_{12}Z_{23}Y_{34}X_{41}$	(-1, 1, -1)
$P_{12}X_{23}Y_{34}Z_{41} = P_{12}X_{23}Z_{34}Y_{41} = P_{12}Y_{23}X_{34}Z_{41} = P_{12}Z_{23}Y_{41}X_{34} = P_{12}Y_{23}Z_{34}X_{41} = P_{12}Z_{23}Y_{34}X_{41}$ $= X_{12}P_{23}Y_{34}Z_{41} = X_{12}P_{23}Z_{34}Y_{41} = Y_{12}P_{23}X_{34}Z_{41} = Z_{12}P_{23}X_{34}Y_{41} = Y_{12}X_{41}P_{23}Z_{34} = Z_{12}P_{23}Y_{34}X_{41}$	
$= X_{12}P_{34}Z_{41}Y_{23} = X_{12}Z_{23}P_{34}Y_{41} = Y_{12}X_{23}P_{34}Z_{41} = Z_{12}X_{23}P_{34}Y_{41} = Y_{12}Z_{23}P_{34}X_{41} = Z_{12}Y_{23}P_{34}X_{41}$	

Table 8. The generators in terms of bifundamental chiral fields for Model 1.

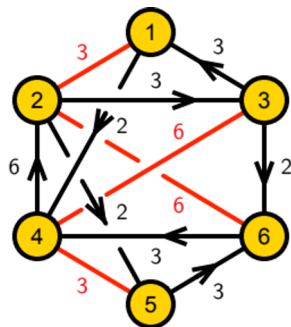


Figure 12. Quiver for Model 2.

The P -matrix of Model 2 takes the form

$$P = \left(\begin{array}{c|cccccc|cccccccc} & p_1 & p_2 & p_3 & p_4 & p_5 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 \\ \hline P_{14} & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ P_{25} & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ P_{36} & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ \hline Q_{14} & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ Q_{25} & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ Q_{36} & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ \hline R_{42} & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ S_{42} & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ T_{42} & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline X_{23} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ X_{31} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ X_{42} & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ X_{56} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ X_{64} & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline Y_{23} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ Y_{31} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ Y_{42} & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ Y_{56} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ Y_{64} & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline Z_{23} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ Z_{31} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ Z_{42} & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ Z_{56} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ Z_{64} & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right). \tag{5.2}$$

The J - and E -term charges are given by

$$Q_{JE} = \left(\begin{array}{c|cccccc|cccccccc} & p_1 & p_2 & p_3 & p_4 & p_5 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 \\ \hline & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 1 & 1 \\ & 1 & 1 & -1 & -1 & -1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 2 & 2 \\ & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 0 & 1 & 0 \\ & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array} \right), \tag{5.3}$$

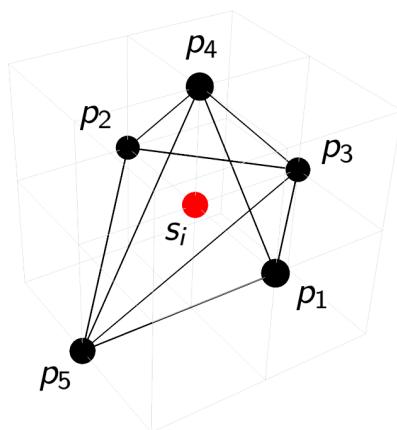


Figure 13. Toric diagram for Model 2.

and the D -term charges are given by

$$Q_D = \left(\begin{array}{ccccc|cccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 \\ 0 & 0 & -1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \end{array} \right). \quad (5.4)$$

The toric diagram of Model 2 is found to be as follows

$$G_t = \left(\begin{array}{ccccc|cccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right). \quad (5.5)$$

The toric diagram with brick matching labels is shown in figure 13.

Using the Molien integral formula, the Hilbert series of the mesonic moduli space of Model 2 is found to be as follows

$$g_1(t_i, y_s; \mathcal{M}_2) = \frac{P(t_i, y_s; \mathcal{M}_2)}{(1 - y_s t_1^2 t_3^3)(1 - y_s t_2^2 t_3^3)(1 - y_s t_1^2 t_4^3)(1 - y_s t_2^2 t_4^3)} \times \frac{1}{(1 - y_s t_1^2 t_5^3)(1 - y_s t_2^2 t_5^3)}, \quad (5.6)$$

where t_i are the fugacities for the extremal brick matchings p_i . y_s counts the product of brick matchings $s_1 \dots s_9$ corresponding to the single internal point of the toric diagram of Model 2. The explicit numerator $P(t_i, y_s; \mathcal{M}_2)$ of the Hilbert series is given in the appendix section A.2.

By setting $t_i = t$ for all i and $y_s = 1$, the unrefined Hilbert series takes the following form

$$g_1(t, 1; \mathcal{M}_2) = \frac{1 + 26t^5 + 26t^{10} + t^{15}}{(1 - t^5)^4}, \quad (5.7)$$

where the palindromic numerator indicates that the mesonic moduli space is Calabi-Yau.

	$SU(3)_{(x_1, x_2)}$	$SU(2)_z$	$U(1)$	fugacity
p_1	(0, 0)	+1	r_1	t_1
p_2	(0, 0)	-1	r_2	t_2
p_3	(+1, 0)	0	r_3	t_3
p_4	(-1, +1)	0	r_4	t_4
p_5	(0, -1)	0	r_5	t_5

Table 9. Global symmetry charges on the extremal brick matchings p_i of Model 2.

The global symmetry of Model 2 and the charges on the extremal brick matchings under the global symmetry are summarized in table 9. Using the following fugacity map,

$$t = t_5 x_2, \quad x_1 = \frac{t_3^2 t_4 t_5}{t_1^2 t_2^2}, \quad x_2 = \frac{t_3 t_4}{t_1 t_2}, \quad z = \frac{t_5 x_2}{t_2}, \quad (5.8)$$

the Hilbert series for Model 2 can be rewritten in terms of characters of irreducible representations of $SU(3) \times SU(2)$, the mesonic flavor symmetry of Model 2, as follows

$$g_1(t, x_i, z; \mathcal{M}_2) = \sum_{n=0}^{\infty} [3n, 0; 2n] t^{5n}, \quad (5.9)$$

where $[m_1, m_2; n] = [m_1, m_2]_{SU(3)_{(x_1, x_2)}} [n]_{SU(2)_z}$. The corresponding plethystic logarithm is

$$\begin{aligned} \text{PL}[g_1(t, x_i, z; \mathcal{M}_2)] &= [3, 0; 2] t^5 - ([6, 0; 0] + [4, 1; 2] + [2, 2; 4] + \\ &+ [2, 2; 0] + [0, 3; 2]) t^{10} + \dots, \end{aligned} \quad (5.10)$$

where we see that the mesonic moduli space is a non-complete intersection.

The set of generators transform under the $[3, 0; 2]$ representation of the mesonic flavor symmetry. Using the following fugacity map

$$\tilde{t} = t_3^{1/3} t_4^{1/3} t_5^{1/3}, \quad \tilde{x}_1 = \frac{t_3}{t_4}, \quad \tilde{x}_2 = \frac{t_3}{t_5}, \quad \tilde{z} = \frac{t_3^{2/3} t_4^{2/3} t_5^{2/3}}{t_2^2} \quad (5.11)$$

the mesonic flavor charges on the gauge invariant operators become \mathbb{Z} -valued. The generators in terms of brick matchings and their corresponding rescaled mesonic flavor charges are summarized in table 10.

The generator lattice as shown in table 10 is a convex lattice polytope, which is reflexive. It is the dual of the toric diagram of Model 2 shown in figure 13. For completeness, table 11 shows the generators of Model 2 in terms of chiral fields with the corresponding mesonic flavor charges.

generator	$SU(3)_{(\hat{x}_1, \hat{x}_2)}$	$SU(2)_{\hat{z}}$
$p_1^2 p_3^3 s$	(1, 1)	1
$p_1 p_2 p_3^3 s$	(1, 1)	0
$p_2^2 p_3^3 s$	(1, 1)	-1
$p_1^2 p_3^2 p_4 s$	(0, 1)	1
$p_1 p_2 p_3^2 p_4 s$	(0, 1)	0
$p_2^2 p_3^2 p_4 s$	(0, 1)	-1
$p_1^2 p_3 p_4^2 s$	(-1, 1)	1
$p_1 p_2 p_3 p_4^2 s$	(-1, 1)	0
$p_2^2 p_3 p_4^2 s$	(-1, 1)	-1
$p_1^2 p_4^3 s$	(-2, 1)	1
$p_1 p_2 p_4^3 s$	(-2, 1)	0
$p_2^2 p_4^3 s$	(-2, 1)	-1
$p_1^2 p_3^2 p_5 s$	(1, 0)	1
$p_1 p_2 p_3^2 p_5 s$	(1, 0)	0
$p_2^2 p_3^2 p_5 s$	(1, 0)	-1
$p_1^2 p_3 p_4 p_5 s$	(0, 0)	1
$p_1 p_2 p_3 p_4 p_5 s$	(0, 0)	0
$p_2^2 p_3 p_4 p_5 s$	(0, 0)	-1
$p_1^2 p_4^2 p_5 s$	(-1, 0)	1
$p_1 p_2 p_4^2 p_5 s$	(-1, 0)	0
$p_2^2 p_4^2 p_5 s$	(-1, 0)	-1
$p_1^2 p_3 p_5^2 s$	(1, -1)	1
$p_1 p_2 p_3 p_5^2 s$	(1, -1)	0
$p_2^2 p_3 p_5^2 s$	(1, -1)	-1
$p_1^2 p_4 p_5^2 s$	(0, -1)	1
$p_1 p_2 p_4 p_5^2 s$	(0, -1)	0
$p_2^2 p_4 p_5^2 s$	(0, -1)	-1
$p_1^2 p_5^3 s$	(1, -2)	1
$p_1 p_2 p_5^3 s$	(1, -2)	0
$p_2^2 p_5^3 s$	(1, -2)	-1

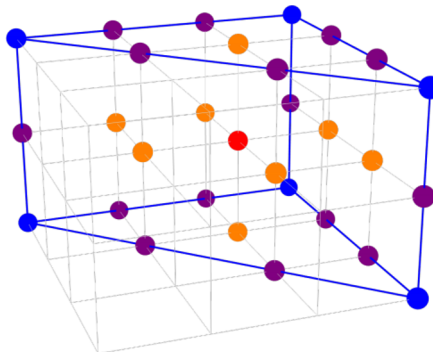


Table 10. The generators and lattice of generators of the mesonic moduli space of Model 2 in terms of brick matchings with the corresponding flavor charges.

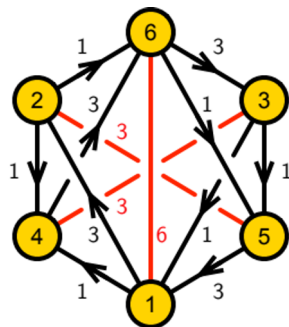


Figure 14. Quiver for Model 3.

6 Model 3: $Y^{2,4}(\mathbb{CP}^2)$ [$\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$, $\langle 5 \rangle$]

Model 3 corresponds to one of the $Y^{p,k}(\mathbb{CP}^2)$ models, $Y^{2,4}(\mathbb{CP}^2)$. The corresponding brane brick model has the quiver in figure 14 and the J - and E -terms are given as follows

$$\begin{array}{l}
 \Lambda_{16}^1 : X_{63}P_{35}Z_{51} - Z_{63}P_{35}X_{51} \quad P_{12}X_{26} - Q_{14}X_{46} \\
 \Lambda_{16}^2 : P_{63}P_{35}X_{51} - X_{63}P_{35}Y_{51} \quad X_{12}X_{26} - Q_{14}Y_{46} \\
 \Lambda_{16}^3 : P_{63}P_{35}Z_{51} - Z_{63}P_{35}Y_{51} \quad Q_{14}Z_{46} - Y_{12}X_{26} \\
 \Lambda_{43}^1 : P_{35}Z_{51}Q_{14} - Q_{31}X_{12}P_{24} \quad X_{46}X_{63} - Z_{46}P_{63} \\
 \Lambda_{43}^2 : P_{35}X_{51}Q_{14} - Q_{31}Y_{12}P_{24} \quad Y_{46}P_{63} - X_{46}Z_{63} \\
 \Lambda_{43}^3 : P_{35}Y_{51}Q_{14} - Q_{31}P_{12}P_{24} \quad Z_{46}Z_{63} - Y_{46}X_{63} \\
 \Lambda_{52}^1 : P_{24}X_{46}Q_{65} - X_{26}P_{63}P_{35} \quad X_{51}X_{12} - Z_{51}Y_{12} \\
 \Lambda_{52}^2 : P_{24}Y_{46}Q_{65} - X_{26}Z_{63}P_{35} \quad Y_{51}Y_{12} - X_{51}P_{12} \\
 \Lambda_{52}^3 : P_{24}Z_{46}Q_{65} - X_{26}X_{63}P_{35} \quad Z_{51}P_{12} - Y_{51}X_{12} \\
 \Lambda_{61}^1 : X_{12}P_{24}X_{46} - P_{12}P_{24}Y_{46} \quad X_{63}Q_{31} - Q_{65}X_{51} \\
 \Lambda_{61}^2 : Y_{12}P_{24}Y_{46} - X_{12}P_{24}Z_{46} \quad P_{63}Q_{31} - Q_{65}Y_{51} \\
 \Lambda_{61}^3 : P_{12}P_{24}Z_{46} - Y_{12}P_{24}X_{46} \quad Z_{63}Q_{31} - Q_{65}Z_{51}
 \end{array} \quad . \quad (6.1)$$

Using the forward algorithm, we can calculate the P -matrix of Model 3. It takes the form

$$P = \left(\begin{array}{c|cccc|cccc|cccc|cc}
 & p_1 & p_2 & p_3 & p_4 & p_5 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & o_1 & o_2 \\
 \hline
 P_{12} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
 P_{24} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
 P_{35} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
 P_{63} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
 \hline
 Q_{14} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
 Q_{31} & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
 Q_{65} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
 \hline
 X_{12} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
 X_{26} & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 X_{46} & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
 X_{51} & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
 X_{63} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
 \hline
 Y_{12} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
 Y_{46} & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
 Y_{51} & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
 \hline
 Z_{4,6} & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
 Z_{51} & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
 Z_{63} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1
 \end{array} \right) \quad . \quad (6.2)$$

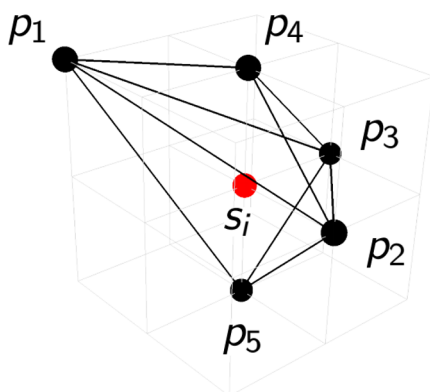


Figure 15. Toric diagram for Model 3.

The J - and E -term charges are given by

$$Q_{JE} = \left(\begin{array}{ccccc|cccccc|cc} p_1 & p_2 & p_3 & p_4 & p_5 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & o_1 & o_2 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 1 \end{array} \right), \quad (6.3)$$

and the Q_D -matrix takes the form

$$Q_D = \left(\begin{array}{ccccc|cccccc|cc} p_1 & p_2 & p_3 & p_4 & p_5 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & o_1 & o_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array} \right). \quad (6.4)$$

The toric diagram of Model 3 is given by the G_t -matrix,

$$G_t = \left(\begin{array}{ccccc|cccccc|cc} p_1 & p_2 & p_3 & p_4 & p_5 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & o_1 & o_2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array} \right). \quad (6.5)$$

Figure 15 shows the toric diagram of Model 3 with brick matching labels.

Using the Molien integral formula, the Hilbert series of the mesonic moduli space of Model 3 is found to be as follows

$$g_1(t_i, y_s; \mathcal{M}_3) = \frac{P(t_i, y_s; \mathcal{M}_3)}{(1 - y_s y_o^3 t_1^4 t_4^2)(1 - y_s y_o^3 t_2^4 t_4^2)(1 - y_s y_o^3 t_3^4 t_4^2)(1 - y_s y_o t_1^2 t_5^2)} \times \frac{1}{(1 - y_s y_o t_2^2 t_5^2)(1 - y_s y_o t_3^2 t_5^2)}, \quad (6.6)$$

where t_i are the fugacities for the extremal brick matchings p_i . y_s counts the product of brick matchings $s_1 \dots s_6$ corresponding to the single internal point of the toric diagram of Model 3.

	$SU(3)_{(x_1, x_2)}$	$U(1)_b$	$U(1)$	fugacity
p_1	(1, 0)	0	r_1	t_1
p_2	(-1, +1)	0	r_2	t_2
p_3	(0, -1)	0	r_3	t_3
p_4	(0, 0)	+1	r_4	t_4
p_5	(0, 0)	-1	r_5	t_5

Table 12. Global symmetry charges on the extremal brick matchings p_i of Model 3.

Additionally, y_o counts the product of extra GLSM fields $o_1 o_2$. The explicit numerator of the Hilbert series $P(t_i, y_s; \mathcal{M}_3)$ is given in the appendix section A.3. We note that by setting $y_o = 1$, the characterization of the mesonic moduli space by the Hilbert series does not change. This implies that the extra GLSM fields indeed are an over-parameterization of the moduli space as expected.

By setting $t_i = t$ for all extremal brick matching fugacities, and $y_s = 1$ and $y_o = 1$ for all other fugacities, the unrefined Hilbert series takes the following form

$$g_1(t, 1, 1; \mathcal{M}_3) = \frac{1}{(1-t^4)^3(1-t^6)^3} \times (1+3t^4+10t^5+12t^6-9t^9-26t^{10}+6t^{11}+3t^{12}+3t^{13}+6t^{14}-26t^{15}-9t^{16}+12t^{19}+10t^{20}+3t^{21}+t^{25}), \quad (6.7)$$

where the palindromic numerator indicates that the mesonic moduli space is Calabi-Yau.

The global symmetry of Model 3 and the charges on the extremal brick matchings under the global symmetry are summarized in table 12. Using the following fugacity map,

$$t = t_3 x_2, \quad x_1 = \frac{t_4 t_5}{t_2 t_3}, \quad x_2 = \frac{t_2 x_1^2}{t_1}, \quad b = \frac{t_4 x_1}{t_1}, \quad (6.8)$$

the Hilbert series for Model 3 can be rewritten in terms of characters of irreducible representations of $SU(3) \times U(1)$, the mesonic flavor symmetry of Model 3, as follows

$$g_1(t, x_i, b; \mathcal{M}_3) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \left[[2n_1 + 3n_2, 0] b^{-2n_1} t^{4n_1+5n_2} + [4n_1 + 3n_2 + 4, 0] b^{2n_1+2} t^{6n_1+5n_2+6} \right], \quad (6.9)$$

where $[m_1, m_2] = [m_1, m_2]_{SU(3)_{(x_1, x_2)}}$ and b is the fugacity for the $U(1)$ factor of the mesonic flavor symmetry. The corresponding plethystic logarithm is

$$\begin{aligned} \text{PL}[g_1(t, x_i, b; \mathcal{M}_3)] &= [2, 0] b^{-2} t^4 + [3, 0] t^5 + [4, 0] b^2 t^6 \\ &\quad - [0, 2] b^{-4} t^8 - ([3, 1] b^{-2} + [1, 2] b^{-2}) t^9 - ([6, 0] + [4, 1] + 2[2, 2]) t^{10} \\ &\quad - ([5, 1] b^2 + [3, 2] b^2 + [1, 3] b^2) t^{11} - ([4, 2] b^4 + [0, 4] b^4) t^{12} + \dots, \end{aligned} \quad (6.10)$$

where we see that the mesonic moduli space is a non-complete intersection. The generators form 3 sets which respectively transform as $[2, 0] b^{-2}$, $[3, 0]$ and $[4, 0] b^2$ of the mesonic flavor symmetry. Using the following fugacity map

$$\tilde{t} = t_4^{1/2} t_5^{1/2}, \quad \tilde{x}_1 = \frac{t_4^{3/2} t_5^{3/2}}{t_2^2 t_3}, \quad \tilde{x}_2 = \frac{t_4^{3/2} t_5^{3/2}}{t_2 t_3^2}, \quad \tilde{b} = \frac{t_4^2}{t_2 t_3}, \quad (6.11)$$

the mesonic flavor charges on the gauge invariant operators become \mathbb{Z} -valued. The generators in terms of brick matchings and their corresponding rescaled mesonic flavor charges are summarized in table 13. The generator lattice as shown in table 13 is a convex lattice polytope, which is reflexive. It is the dual of the toric diagram of Model 3 shown in figure 15. Note that the 3 sets of generators transforming under $[2, 0]b^{-2}$, $[3, 0]$ and $[4, 0]b^2$ of the mesonic flavor symmetry form the 3 layers of the generator lattice in table 13. For completeness, table 14 shows the generators of Model 2 in terms of chiral fields with the corresponding mesonic flavor charges.

7 Model 4: $P_{+-}^1(\text{dP}_0)$ [$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$, $\langle 6 \rangle$]

Model 4 corresponds to the Calabi-Yau 4-fold $P_{+-}^1(\text{dP}_0)$. The corresponding brane brick model has the quiver in figure 16 and the J - and E -terms are given as follows

$$\begin{array}{ll}
 & J \\
 \Lambda_{12} : & Z_{23}Z_{31} - Y_{23}X_{31} \\
 \Lambda_{15}^1 : & X_{53}X_{31} - Z_{56}Y_{61} \\
 \Lambda_{15}^2 : & Y_{56}Y_{61} - X_{53}Z_{31} \\
 \Lambda_{26}^1 : & Y_{61}Q_{14}Y_{42} - X_{64}Z_{42} \\
 \Lambda_{26}^2 : & Z_{64}Z_{42} - Y_{61}Q_{14}X_{42} \\
 \Lambda_{26}^3 : & Y_{61}P_{14}Y_{42} - X_{64}T_{42} \\
 \Lambda_{26}^4 : & Z_{64}T_{42} - Y_{61}P_{14}X_{42} \\
 \Lambda_{26}^5 : & X_{64}X_{42} - Z_{64}Y_{42} \\
 \Lambda_{34}^1 : & X_{42}Q_{25}X_{53} - Z_{42}Y_{23} \\
 \Lambda_{34}^2 : & Z_{42}Z_{23} - Y_{42}Q_{25}X_{53} \\
 \Lambda_{34}^3 : & X_{42}P_{25}X_{53} - T_{42}Y_{23} \\
 \Lambda_{34}^4 : & T_{42}Z_{23} - Y_{42}P_{25}X_{53} \\
 \Lambda_{34}^5 : & Y_{42}Y_{23} - X_{42}Z_{23} \\
 \Lambda_{45} : & Z_{56}Z_{64} - Y_{56}X_{64} \\
 & E \\
 & P_{14}Z_{42} - Q_{14}T_{42} \\
 & P_{14}X_{42}Q_{25} - Q_{14}X_{42}P_{25} \\
 & P_{14}Y_{42}Q_{25} - Q_{14}Y_{42}P_{25} \\
 & P_{25}Y_{56} - Y_{23}P_{36} \\
 & P_{25}Z_{56} - Z_{23}P_{36} \\
 & Y_{23}Q_{36} - Q_{25}Y_{56} \\
 & Z_{23}Q_{36} - Q_{25}Z_{56} \\
 & P_{25}X_{53}Q_{36} - Q_{25}X_{53}P_{36} \\
 & P_{36}X_{64} - X_{31}P_{14} \\
 & P_{36}Z_{64} - Z_{31}P_{14} \\
 & X_{31}Q_{14} - Q_{36}X_{64} \\
 & Z_{31}Q_{14} - Q_{36}Z_{64} \\
 & P_{36}Y_{61}Q_{14} - Q_{36}Y_{61}P_{14} \\
 & T_{42}Q_{25} - Z_{42}P_{25}
 \end{array} . \tag{7.1}$$

Using the forward algorithm, we are able to obtain the brick matchings for Model 4. They are summarized in the P -matrix which takes the form

$$P = \left(\begin{array}{c|cccccc|cccccc|cc}
 & p_1 & p_2 & p_3 & p_4 & p_5 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & o_1 & o_2 \\
 \hline
 P_{14} & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
 P_{25} & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
 P_{36} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
 \hline
 Q_{14} & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
 Q_{25} & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
 Q_{36} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
 \hline
 T_{42} & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
 X_{31} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
 X_{42} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 X_{53} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
 X_{64} & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 \hline
 Y_{23} & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
 Y_{42} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 Y_{56} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
 Y_{61} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
 \hline
 Z_{23} & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
 Z_{31} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
 Z_{42} & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
 Z_{56} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
 Z_{64} & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0
 \end{array} \right) . \tag{7.2}$$

generator	$SU(3)_{(\tilde{x}_1, \tilde{x}_2)}$	$U(1)_{\tilde{b}}$
$p_1^2 p_5^2 s o$	(1, 1)	-1
$p_1 p_2 p_5^2 s o$	(0, 1)	-1
$p_2^2 p_5^2 s o$	(-1, 1)	-1
$p_1 p_3 p_5^2 s o$	(1, 0)	-1
$p_2 p_3 p_5^2 s o$	(0, 0)	-1
$p_3^2 p_5^2 s o$	(1, -1)	-1
$p_1^3 p_4 p_5 s o^2$	(1, 1)	0
$p_1^2 p_2 p_4 p_5 s o^2$	(0, 1)	0
$p_1 p_2^2 p_4 p_5 s o^2$	(-1, 1)	0
$p_2^3 p_4 p_5 s o^2$	(-2, 1)	0
$p_1^2 p_3 p_4 p_5 s o^2$	(1, 0)	0
$p_1 p_2 p_3 p_4 p_5 s o^2$	(0, 0)	0
$p_2^2 p_3 p_4 p_5 s o^2$	(-1, 0)	0
$p_1 p_3^2 p_4 p_5 s o^2$	(1, -1)	0
$p_2 p_3^2 p_4 p_5 s o^2$	(0, -1)	0
$p_3^3 p_4 p_5 s o^2$	(1, -2)	0
$p_1^4 p_4^2 s o^3$	(1, 1)	1
$p_1^3 p_2 p_4^2 s o^3$	(0, 1)	1
$p_1^2 p_2^2 p_4^2 s o^3$	(-1, 1)	1
$p_1 p_2^3 p_4^2 s o^3$	(-2, 1)	1
$p_2^4 p_4^2 s o^3$	(-3, 1)	1
$p_1^3 p_3 p_4^2 s o^3$	(1, 0)	1
$p_1^2 p_2 p_3 p_4^2 s o^3$	(0, 0)	1
$p_1 p_2^2 p_3 p_4^2 s o^3$	(-1, 0)	1
$p_2^3 p_3 p_4^2 s o^3$	(-2, 0)	1
$p_1^2 p_3^2 p_4^2 s o^3$	(1, -1)	1
$p_1 p_2 p_3^2 p_4^2 s o^3$	(0, -1)	1
$p_2^2 p_3^2 p_4^2 s o^3$	(-1, -1)	1
$p_1 p_3^3 p_4^2 s o^3$	(1, -2)	1
$p_2 p_3^3 p_4^2 s o^3$	(0, -2)	1
$p_3^4 p_4^2 s o^3$	(1, -3)	1

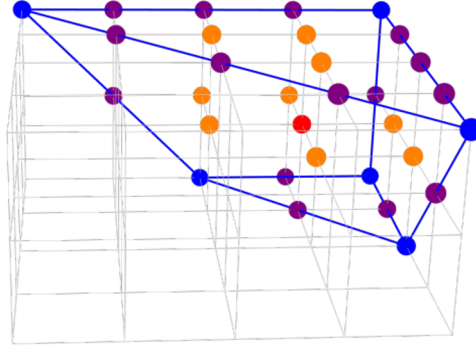


Table 13. The generators and lattice of generators of the mesonic moduli space of Model 3 in terms of brick matchings with the corresponding flavor charges.

generator	$SU(3)_{(\tilde{x}_1, \tilde{x}_2)}$	$U(1)_{\tilde{b}}$
$Q_{14}Z_{46}X_{63}Q_{31} = Q_{14}Z_{46}Q_{65}X_{51} = Y_{12}X_{26}X_{63}Q_{31} = Y_{12}X_{26}Q_{65}X_{51}$	(1, 1)	-1
$Q_{14}Y_{46}X_{63}Q_{31} = Q_{14}Z_{46}Z_{63}Q_{31} = Q_{14}Y_{46}Q_{65}X_{51} = Q_{14}Z_{46}Q_{65}Z_{51} = X_{12}X_{26}X_{63}Q_{31} = Y_{12}X_{26}Z_{63}Q_{31}$ $= X_{12}X_{26}Q_{65}X_{51} = Y_{12}X_{26}Q_{65}Z_{51}$	(0, 1)	-1
$Q_{14}Y_{46}Z_{63}Q_{31} = Q_{14}Y_{46}Q_{65}Z_{51} = X_{12}X_{26}Z_{63}Q_{31} = X_{12}X_{26}Q_{65}Z_{51}$	(-1, 1)	-1
$P_{12}X_{26}X_{63}Q_{31} = P_{12}X_{26}Q_{65}X_{51} = Q_{14}Z_{46}P_{63}Q_{31} = Y_{12}X_{26}P_{63}Q_{31} = Q_{14}X_{46}X_{63}Q_{31} = Q_{14}X_{46}Q_{65}X_{51}$ $= Q_{14}Z_{46}Q_{65}Y_{51} = Y_{12}X_{26}Q_{65}Y_{51}$	(1, 0)	-1
$P_{12}X_{26}Z_{63}Q_{31} = P_{12}X_{26}Q_{65}Z_{51} = Q_{14}Y_{46}P_{63}Q_{31} = X_{12}X_{26}P_{63}Q_{31} = Q_{14}X_{46}Z_{63}Q_{31} = Q_{14}X_{46}Q_{65}Z_{51}$ $= Q_{14}Y_{46}Q_{65}Y_{51} = X_{12}X_{26}Q_{65}Y_{51}$	(0, 0)	-1
$P_{12}X_{26}P_{63}Q_{31} = P_{12}X_{26}Q_{65}Y_{51} = Q_{14}X_{46}P_{63}Q_{31} = Q_{14}X_{46}Q_{65}Y_{51}$	(1, -1)	-1
$Y_{12}P_{24}Z_{46}X_{63}Q_{31} = Y_{12}P_{24}Z_{46}Q_{65}X_{51} = Q_{14}Z_{46}X_{63}P_{35}X_{51} = Y_{12}X_{26}X_{63}P_{35}X_{51}$	(1, 1)	0
$X_{12}P_{24}Z_{46}X_{63}Q_{31} = Y_{12}P_{24}Y_{46}X_{63}Q_{31} = Y_{12}P_{24}Z_{46}Z_{63}Q_{31} = X_{12}P_{24}Z_{46}Q_{65}X_{51} = Y_{12}P_{24}Z_{46}Q_{65}Z_{51}$ $= Q_{14}Y_{46}X_{63}P_{35}X_{51} = Q_{14}Z_{46}Z_{63}P_{35}X_{51} = Q_{14}Z_{46}X_{63}P_{35}Z_{51} = X_{12}X_{26}X_{63}P_{35}X_{51} = Y_{12}X_{26}Z_{63}P_{35}X_{51} = Y_{12}X_{26}X_{63}P_{35}Z_{51}$	(0, 1)	0
$X_{12}P_{24}Y_{46}X_{63}Q_{31} = X_{12}P_{24}Z_{46}Z_{63}Q_{31} = Y_{12}P_{24}Y_{46}Z_{63}Q_{31} = X_{12}P_{24}Y_{46}Q_{65}X_{51} = X_{12}P_{24}Z_{46}Q_{65}Z_{51} = Y_{12}P_{24}Y_{46}Q_{65}Z_{51}$ $= Q_{14}Y_{46}Z_{63}P_{35}X_{51} = Q_{14}Y_{46}X_{63}P_{35}Z_{51} = Q_{14}Z_{46}Z_{63}P_{35}Z_{51} = X_{12}X_{26}Z_{63}P_{35}X_{51} = X_{12}X_{26}X_{63}P_{35}Z_{51} = Y_{12}X_{26}Z_{63}P_{35}Z_{51}$	(-1, 1)	0
$X_{12}P_{24}Y_{46}Z_{63}Q_{31} = X_{12}P_{24}Y_{46}Q_{65}Z_{51} = Q_{14}Y_{46}Z_{63}P_{35}Z_{51} = X_{12}X_{26}Z_{63}P_{35}Z_{51} = X_{12}X_{26}X_{63}P_{35}Z_{51} = Y_{12}X_{26}Z_{63}P_{35}Z_{51}$ $= X_{12}P_{24}Z_{46}P_{63}Q_{31} = X_{12}P_{24}Z_{46}Q_{65}X_{51} = Q_{14}Y_{46}Z_{63}P_{35}Z_{51} = X_{12}X_{26}Z_{63}P_{35}Z_{51} = X_{12}X_{26}X_{63}P_{35}Z_{51}$	(-2, 1)	0
$P_{12}P_{24}Z_{46}X_{63}Q_{31} = P_{12}P_{24}Z_{46}Q_{65}X_{51} = P_{12}X_{26}X_{63}P_{35}X_{51} = Y_{12}P_{24}Z_{46}P_{63}Q_{31} = Y_{12}P_{24}X_{46}X_{63}Q_{31} = Y_{12}P_{24}X_{46}Q_{65}X_{51}$ $= Y_{12}P_{24}Z_{46}Q_{65}Y_{51} = Q_{14}Z_{46}P_{63}P_{35}X_{51} = Y_{12}X_{26}P_{63}P_{35}X_{51} = Q_{14}X_{46}X_{63}P_{35}X_{51} = Q_{14}Z_{46}X_{63}P_{35}Y_{51} = Y_{12}X_{26}X_{63}P_{35}Y_{51}$	(1, 0)	0
$P_{12}P_{24}Y_{46}X_{63}Q_{31} = P_{12}P_{24}Z_{46}Z_{63}Q_{31} = P_{12}P_{24}Y_{46}Q_{65}X_{51} = P_{12}P_{24}Z_{46}Q_{65}Z_{51} = P_{12}X_{26}Z_{63}P_{35}X_{51} = P_{12}X_{26}X_{63}P_{35}Z_{51}$ $= X_{12}P_{24}Z_{46}P_{63}Q_{31} = Y_{12}P_{24}X_{46}X_{63}Q_{31} = X_{12}P_{24}X_{46}Z_{63}Q_{31} = X_{12}P_{24}X_{46}Q_{65}X_{51} = X_{12}X_{26}Z_{63}P_{35}Y_{51}$ $= Y_{12}P_{24}X_{46}Q_{65}Z_{51} = Y_{12}P_{24}Y_{46}Q_{65}Y_{51} = Q_{14}Y_{46}P_{63}P_{35}X_{51} = Q_{14}Z_{46}P_{63}P_{35}Z_{51} = X_{12}X_{26}P_{63}P_{35}X_{51} = Y_{12}X_{26}P_{63}P_{35}Z_{51}$ $= Q_{14}X_{46}Z_{63}P_{35}X_{51} = Q_{14}X_{46}X_{63}P_{35}Y_{51} = Q_{14}Y_{46}Z_{63}P_{35}Y_{51} = Q_{14}Z_{46}Z_{63}P_{35}Y_{51} = X_{12}X_{26}X_{63}P_{35}Y_{51} = Y_{12}X_{26}Z_{63}P_{35}Y_{51}$	(-1, 0)	0
$P_{12}P_{24}Y_{46}Z_{63}Q_{31} = P_{12}P_{24}Y_{46}Q_{65}Z_{51} = P_{12}X_{26}Z_{63}P_{35}Z_{51} = X_{12}P_{24}Y_{46}P_{63}Q_{31} = X_{12}P_{24}X_{46}Z_{63}Q_{31} = X_{12}P_{24}X_{46}Q_{65}Z_{51}$ $= X_{12}P_{24}Z_{46}P_{63}Q_{31} = Q_{14}Y_{46}P_{63}P_{35}Z_{51} = X_{12}X_{26}P_{63}P_{35}Z_{51} = Q_{14}X_{46}Z_{63}P_{35}Z_{51} = Q_{14}Y_{46}Z_{63}P_{35}Y_{51} = X_{12}X_{26}Z_{63}P_{35}Y_{51}$ $= Q_{14}X_{46}Z_{63}P_{35}Y_{51} = Q_{14}X_{46}X_{63}P_{35}Y_{51} = Q_{14}Y_{46}Z_{63}P_{35}Y_{51} = Q_{14}Z_{46}Z_{63}P_{35}Y_{51} = X_{12}X_{26}X_{63}P_{35}Y_{51} = Y_{12}X_{26}Z_{63}P_{35}Y_{51}$	(1, -1)	0
$P_{12}P_{24}X_{46}P_{63}Q_{31} = Y_{12}P_{24}X_{46}Q_{65}Y_{51} = Q_{14}X_{46}P_{63}P_{35}X_{51} = Q_{14}Z_{46}P_{63}P_{35}Y_{51} = Y_{12}X_{26}P_{63}P_{35}Y_{51} = Q_{14}X_{46}X_{63}P_{35}Y_{51}$ $= Y_{12}P_{24}X_{46}P_{63}Q_{31} = P_{12}P_{24}X_{46}Z_{63}Q_{31} = P_{12}P_{24}X_{46}Q_{65}Z_{51} = P_{12}P_{24}Y_{46}Q_{65}Y_{51} = P_{12}X_{26}P_{63}P_{35}Z_{51} = P_{12}X_{26}Z_{63}P_{35}Y_{51}$ $= X_{12}P_{24}X_{46}P_{63}Q_{31} = X_{12}P_{24}X_{46}Q_{65}Y_{51} = Q_{14}X_{46}P_{63}P_{35}Z_{51} = Q_{14}Y_{46}Z_{63}P_{35}Y_{51} = X_{12}X_{26}P_{63}P_{35}Y_{51} = Q_{14}X_{46}Z_{63}P_{35}Y_{51}$	(0, -1)	0
$P_{12}P_{24}X_{46}P_{63}Q_{31} = P_{12}P_{24}X_{46}Q_{65}Y_{51} = P_{12}X_{26}P_{63}P_{35}Z_{51} = Q_{14}Y_{46}P_{63}P_{35}Y_{51} = X_{12}X_{26}P_{63}P_{35}Y_{51} = Q_{14}X_{46}Z_{63}P_{35}Y_{51}$ $= P_{12}P_{24}X_{46}P_{63}Q_{31} = P_{12}P_{24}X_{46}Q_{65}Y_{51} = P_{12}X_{26}P_{63}P_{35}Z_{51} = Q_{14}Y_{46}P_{63}P_{35}Y_{51} = X_{12}X_{26}P_{63}P_{35}Y_{51} = Q_{14}X_{46}Z_{63}P_{35}Y_{51}$	(1, -2)	0
$Y_{12}P_{24}Z_{46}X_{63}P_{35}X_{51}$	(1, 1)	1
$X_{12}P_{24}Z_{46}X_{63}P_{35}X_{51} = Y_{12}P_{24}Y_{46}X_{63}P_{35}X_{51} = Y_{12}P_{24}Z_{46}Z_{63}P_{35}X_{51} = Y_{12}P_{24}Z_{46}X_{63}P_{35}Z_{51}$	(0, 1)	1
$X_{12}P_{24}Y_{46}X_{63}P_{35}X_{51} = X_{12}P_{24}Z_{46}Z_{63}P_{35}X_{51} = X_{12}P_{24}Z_{46}X_{63}P_{35}Z_{51} = Y_{12}P_{24}Y_{46}Z_{63}P_{35}X_{51} = Y_{12}P_{24}Y_{46}X_{63}P_{35}Z_{51} = Y_{12}P_{24}Z_{46}Z_{63}P_{35}Z_{51}$	(-1, 1)	1
$X_{12}P_{24}Y_{46}Z_{63}P_{35}X_{51} = X_{12}P_{24}Y_{46}X_{63}P_{35}Z_{51} = X_{12}P_{24}Z_{46}Z_{63}P_{35}Z_{51} = Y_{12}P_{24}Y_{46}Z_{63}P_{35}Y_{51} = Y_{12}P_{24}Z_{46}Z_{63}P_{35}Y_{51}$	(-2, 1)	1
$X_{12}P_{24}Y_{46}Z_{63}P_{35}Z_{51}$	(-3, 1)	1
$P_{12}P_{24}Z_{46}X_{63}P_{35}X_{51} = Y_{12}P_{24}Z_{46}P_{63}P_{35}X_{51} = Y_{12}P_{24}Z_{46}X_{63}P_{35}Z_{51} = Y_{12}P_{24}Z_{46}X_{63}P_{35}Y_{51}$	(1, 0)	1
$P_{12}P_{24}Y_{46}X_{63}P_{35}X_{51} = P_{12}P_{24}Y_{46}Z_{63}P_{35}X_{51} = P_{12}P_{24}Y_{46}X_{63}P_{35}Z_{51} = X_{12}P_{24}Y_{46}P_{63}P_{35}X_{51} = Y_{12}P_{24}Z_{46}P_{63}P_{35}Z_{51}$ $= X_{12}P_{24}X_{46}X_{63}P_{35}X_{51} = X_{12}P_{24}Z_{46}X_{63}P_{35}Y_{51} = Y_{12}P_{24}X_{46}X_{63}P_{35}Z_{51} = Y_{12}P_{24}Y_{46}X_{63}P_{35}Y_{51} = Y_{12}P_{24}Z_{46}Z_{63}P_{35}Y_{51}$	(0, 0)	1
$P_{12}P_{24}Y_{46}Z_{63}P_{35}X_{51} = P_{12}P_{24}Y_{46}X_{63}P_{35}Z_{51} = P_{12}P_{24}Z_{46}Z_{63}P_{35}X_{51} = X_{12}P_{24}Y_{46}Z_{63}P_{35}X_{51} = X_{12}P_{24}Z_{46}Z_{63}P_{35}Z_{51} = Y_{12}P_{24}Y_{46}Z_{63}P_{35}Y_{51}$ $= X_{12}P_{24}Y_{46}Z_{63}P_{35}Z_{51} = X_{12}P_{24}X_{46}X_{63}P_{35}X_{51} = X_{12}P_{24}Z_{46}X_{63}P_{35}Y_{51} = X_{12}P_{24}Y_{46}X_{63}P_{35}Y_{51} = Y_{12}P_{24}Z_{46}Z_{63}P_{35}Y_{51}$	(-1, 0)	1
$P_{12}P_{24}Y_{46}Z_{63}P_{35}Z_{51} = X_{12}P_{24}Y_{46}Z_{63}P_{35}Y_{51} = X_{12}P_{24}Z_{46}Z_{63}P_{35}Y_{51} = X_{12}P_{24}Y_{46}Z_{63}P_{35}Y_{51} = Y_{12}P_{24}Z_{46}Z_{63}P_{35}Y_{51}$ $= X_{12}P_{24}Y_{46}Z_{63}P_{35}X_{51} = X_{12}P_{24}X_{46}X_{63}P_{35}Z_{51} = X_{12}P_{24}Z_{46}Z_{63}P_{35}Z_{51} = X_{12}P_{24}Y_{46}Z_{63}P_{35}Z_{51} = Y_{12}P_{24}Z_{46}Z_{63}P_{35}Z_{51}$	(-2, 0)	1
$P_{12}P_{24}Z_{46}P_{63}P_{35}X_{51} = P_{12}P_{24}Z_{46}X_{63}P_{35}X_{51} = P_{12}P_{24}Z_{46}Z_{63}P_{35}X_{51} = Y_{12}P_{24}Z_{46}P_{63}P_{35}X_{51} = Y_{12}P_{24}Z_{46}X_{63}P_{35}Z_{51}$ $= Y_{12}P_{24}Z_{46}Z_{63}P_{35}Z_{51} = X_{12}P_{24}Z_{46}P_{63}P_{35}Z_{51} = X_{12}P_{24}Z_{46}X_{63}P_{35}Y_{51} = X_{12}P_{24}Z_{46}Z_{63}P_{35}Y_{51} = Y_{12}P_{24}Z_{46}P_{63}P_{35}Y_{51}$ $= Y_{12}P_{24}Z_{46}X_{63}P_{35}Y_{51} = Y_{12}P_{24}Z_{46}Z_{63}P_{35}Y_{51} = X_{12}P_{24}Z_{46}P_{63}P_{35}Y_{51} = X_{12}P_{24}Z_{46}X_{63}P_{35}Y_{51} = X_{12}P_{24}Z_{46}Z_{63}P_{35}Y_{51}$	(1, -1)	1
$P_{12}P_{24}Y_{46}P_{63}P_{35}X_{51} = P_{12}P_{24}Y_{46}Z_{63}P_{35}X_{51} = P_{12}P_{24}Y_{46}X_{63}P_{35}Z_{51} = P_{12}P_{24}Z_{46}Z_{63}P_{35}X_{51} = P_{12}P_{24}Y_{46}Z_{63}P_{35}Z_{51} = P_{12}P_{24}Y_{46}X_{63}P_{35}Y_{51}$ $= P_{12}P_{24}Z_{46}Z_{63}P_{35}Y_{51} = X_{12}P_{24}Y_{46}P_{63}P_{35}X_{51} = X_{12}P_{24}Z_{46}X_{63}P_{35}Z_{51} = X_{12}P_{24}Y_{46}Z_{63}P_{35}Z_{51} = X_{12}P_{24}Y_{46}Z_{63}P_{35}Y_{51}$ $= X_{12}P_{24}Y_{46}Z_{63}P_{35}Y_{51} = X_{12}P_{24}X_{46}X_{63}P_{35}X_{51} = X_{12}P_{24}Z_{46}X_{63}P_{35}Y_{51} = X_{12}P_{24}Y_{46}X_{63}P_{35}Y_{51} = Y_{12}P_{24}Z_{46}Z_{63}P_{35}Y_{51}$	(0, -1)	1
$P_{12}P_{24}Y_{46}Z_{63}P_{35}Z_{51} = P_{12}P_{24}Y_{46}X_{63}P_{35}Y_{51} = P_{12}P_{24}Z_{46}Z_{63}P_{35}Z_{51} = X_{12}P_{24}Y_{46}Z_{63}P_{35}X_{51} = X_{12}P_{24}Z_{46}Z_{63}P_{35}X_{51}$ $= X_{12}P_{24}Y_{46}Z_{63}P_{35}Z_{51} = X_{12}P_{24}X_{46}X_{63}P_{35}Z_{51} = X_{12}P_{24}Z_{46}X_{63}P_{35}Y_{51} = X_{12}P_{24}Y_{46}X_{63}P_{35}Y_{51} = Y_{12}P_{24}Z_{46}Z_{63}P_{35}Y_{51}$	(-1, -1)	1
$P_{12}P_{24}X_{46}P_{63}P_{35}X_{51} = P_{12}P_{24}X_{46}Z_{63}P_{35}X_{51} = P_{12}P_{24}X_{46}Q_{65}Y_{51} = P_{12}P_{24}X_{46}P_{63}P_{35}Z_{51} = Y_{12}P_{24}X_{46}P_{63}P_{35}Y_{51}$ $= P_{12}P_{24}X_{46}P_{63}P_{35}Z_{51} = P_{12}P_{24}X_{46}Z_{63}P_{35}Y_{51} = P_{12}P_{24}X_{46}Q_{65}Y_{51} = P_{12}P_{24}X_{46}P_{63}P_{35}Y_{51} = Y_{12}P_{24}X_{46}P_{63}P_{35}Y_{51}$	(1, -2)	1
$P_{12}P_{24}X_{46}P_{63}P_{35}Z_{51} = P_{12}P_{24}X_{46}Z_{63}P_{35}Z_{51} = P_{12}P_{24}X_{46}Q_{65}Y_{51} = P_{12}P_{24}X_{46}P_{63}P_{35}Y_{51} = Y_{12}P_{24}X_{46}P_{63}P_{35}Y_{51}$	(0, -2)	1
$P_{12}P_{24}X_{46}P_{63}P_{35}Y_{51}$	(1, -3)	1

Table 14. The generators in terms of bifundamental chiral fields for Model 3.

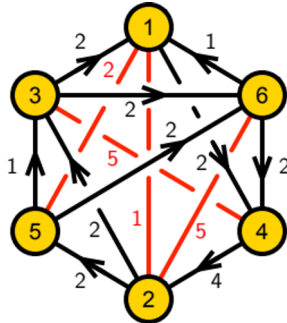


Figure 16. Quiver for Model 4.

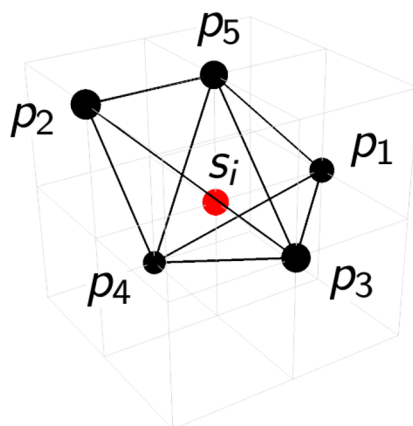


Figure 17. Toric diagram for Model 4.

The J - and E -term charges are given by

$$Q_{JE} = \left(\begin{array}{ccccc|cccccc|cc} p_1 & p_2 & p_3 & p_4 & p_5 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & o_1 & o_2 \\ \hline 1 & 1 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{array} \right), \quad (7.3)$$

and the D -term charges are given by

$$Q_D = \left(\begin{array}{ccccc|cccccc|cc} p_1 & p_2 & p_3 & p_4 & p_5 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & o_1 & o_2 \\ \hline 0 & 0 & -1 & -1 & -1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \end{array} \right). \quad (7.4)$$

The toric diagram of Model 4 is given by

$$G_t = \left(\begin{array}{ccccc|cccccc|cc} p_1 & p_2 & p_3 & p_4 & p_5 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & o_1 & o_2 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array} \right), \quad (7.5)$$

where figure 17 shows the toric diagram with brick matching labels.

Using the Molien integral formula, the Hilbert series of the mesonic moduli space of Model 4 is found to be as follows

$$g_1(t_i, y_s, y_o; \mathcal{M}_4) = \frac{P(t_i, y_s, y_o; \mathcal{M}_4)}{(1 - y_s y_o t_1 t_3^3)(1 - y_s y_o t_2 t_3^3)(1 - y_s y_o t_1 t_4^3)(1 - y_s y_o t_2 t_4^3)} \times \frac{1}{(1 - y_s y_o^4 t_1^4 t_5^3)(1 - y_s y_o^4 t_2^4 t_5^3)}, \quad (7.6)$$

where t_i are the fugacities for the extremal brick matchings p_i . y_s counts the product of brick matchings $s_1 \dots s_6$ corresponding to the single internal point of the toric diagram of Model 4. Additionally, the fugacity y_o corresponds to the product $o_1 o_2$ of extra GLSM fields. The

	SU(2) _x	SU(2) _y	U(1) _b	U(1)	fugacity
p_1	+1	0	+1	r_1	t_1
p_2	-1	0	+1	r_2	t_2
p_3	0	+1	0	r_3	t_3
p_4	0	-1	0	r_4	t_4
p_5	0	0	-2	r_5	t_5

Table 15. Global symmetry charges on the extremal brick matchings p_i of Model 4.

explicit numerator $P(t_i, y_s, y_o; \mathcal{M}_4)$ of the Hilbert series is given in the appendix section A.4. We note that setting the fugacity $y_o = 1$ does not change the overall characterization of the mesonic moduli space by the Hilbert series, indicating that the extra GLSM fields, as expected, correspond to an over-parameterization of the moduli space.

By setting $t_i = t$ for the fugacities of the extremal brick matchings, and all other fugacities to $y_s = 1$ and $y_o = 1$, the unrefined Hilbert series takes the following form

$$g_1(t, 1, 1; \mathcal{M}_4) = \frac{1}{(1-t^4)^4(1-t^7)^2} \times (1 + 4t^4 + 9t^5 + 8t^6 + 3t^7 - 5t^8 - 12t^9 - 7t^{10} - 4t^{11} + 3t^{12} + 3t^{13} - 4t^{14} - 7t^{15} - 12t^{16} - 5t^{17} + 3t^{18} + 8t^{19} + 9t^{20} + 4t^{21} + t^{25}), \quad (7.7)$$

where the palindromic numerator indicates that the mesonic moduli space is Calabi-Yau.

The global symmetry of Model 4 and the charges on the extremal brick matchings under the global symmetry are summarized in table 15. Using the following fugacity map,

$$t = \frac{t_1 t_2 t_5}{t_3 t_4}, \quad x = \frac{t_1^{1/2}}{t_2^{1/2}}, \quad y = \frac{t_1 t_2 t_5}{t_3 t_4^2}, \quad b = \frac{t_3 t_4}{t_1^{1/2} t_2^{1/2} t_5}, \quad (7.8)$$

the Hilbert series for Model 4 can be rewritten in terms of characters of irreducible representations of $SU(2) \times SU(2) \times U(1)$. In terms of fugacities of the mesonic flavor symmetry of Model 4, the Hilbert series becomes

$$g_1(t, x, y, b; \mathcal{M}_4) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \left[[n_1 + 2n_2; 3n_1 + 2n_2] b^{n_1} t^{4n_1+5n_2} + [3n_1 + 2n_2 + 3; n_1 + 2n_2 + 1] b^{-n_1-1} t^{6n_1+5n_2+6} + [3n_1 + 7n_2 + 7; n_1 + n_2 + 1] b^{-n_1-3n_2-3} t^{6n_1+13n_2+13} + [4n_1 + 7n_2 + 4; n_2] b^{-2n_1-3n_2-2} t^{7n_1+13n_2+7} \right], \quad (7.9)$$

where $[m; n] = [m]_{SU(2)_x} [n]_{SU(2)_y}$. The fugacity b counts charges under the U(1) factor of the mesonic flavor symmetry.

The corresponding plethystic logarithm is

$$\begin{aligned}
 \text{PL}[g_1(t, x, y, b; \mathcal{M}_4)] = & [1; 3]bt^4 + [2; 2]t^5 + [3; 1]b^{-1}t^6 + [4; 0]b^{-2}t^7 \\
 & - ([0; 4] + [2; 2] + 1)b^2t^8 - ([1; 5] + [3; 3] + [1; 3] + [3; 1] + [1; 1])bt^9 \\
 & - ([4; 4] + [2; 4] + [4; 2] + [0; 4] + 2[2; 2] + [4; 0] + 1)t^{10} \\
 & - ([5; 3] + 2[3; 3] + [5; 1] + [1; 3] + [3; 1] + [1; 1])b^{-1}t^{11} \\
 & - ([6; 2] + [4; 2] + 2[2; 2] + [4; 0] + 1)b^{-2}t^{12} + \dots, \tag{7.10}
 \end{aligned}$$

where we see that the mesonic moduli space is a non-complete intersection. The generators form 4 sets that transform under $[1; 3]b$, $[2; 2]$, $[3; 1]b^{-1}$ and $[4; 0]b^{-2}$ of the mesonic flavor symmetry of Model 4, respectively. Using the following fugacity map

$$\tilde{t} = t_3^{1/2}t_4^{1/2}, \quad \tilde{x} = \frac{t_3}{t_4}, \quad \tilde{y} = \frac{t_2^2t_5}{t_3^{3/2}t_4^{3/2}}, \quad \tilde{b} = \frac{t_3^{1/2}t_4^{3/2}}{t_2t_5}, \tag{7.11}$$

the mesonic flavor charges on the gauge invariant operators become \mathbb{Z} -valued. The generators in terms of brick matchings and their corresponding rescaled mesonic flavor charges are summarized in table 16. The generator lattice as shown in table 16 is a convex lattice polytope, which is reflexive. It is the dual of the toric diagram of Model 4 shown in figure 17. We also note that the 4 layers of points in the generator lattice in table 16 corresponds to the 4 sets of generators that transform respectively under $[1; 3]b$, $[2; 2]$, $[3; 1]b^{-1}$ and $[4; 0]b^{-2}$ of the mesonic flavor symmetry. For completeness, table 17 and table 18 show the generators of Model 4 in terms of chiral fields with the corresponding mesonic flavor charges.

8 Model 5: $Y^{2,5}(\mathbb{CP}^2)$ [$\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)), \langle 7 \rangle$]

Model 5 corresponds to one of the $Y^{p,k}(\mathbb{CP}^2)$ models, $Y^{2,5}(\mathbb{CP}^2)$. The corresponding brane brick model has the quiver in figure 18 and the J - and E -terms are given as follows

$$\begin{array}{ll}
 & J \\
 \Lambda_{12}^1 : & X_{25}X_{51} - Q_{25}Z_{51} \\
 \Lambda_{12}^2 : & X_{25}R_{51} - P_{25}Z_{51} \\
 \Lambda_{16}^1 : & X_{64}Q_{42}Y_{21} - X_{65}X_{51} \\
 \Lambda_{16}^2 : & X_{65}Z_{51} - X_{64}X_{42}Y_{21} \\
 \Lambda_{16}^3 : & X_{64}P_{42}Y_{21} - X_{65}R_{51} \\
 \Lambda_{21}^1 : & X_{14}X_{42} - X_{13}X_{32} \\
 \Lambda_{23}^1 : & X_{36}X_{64}Q_{42} - Q_{36}X_{64}X_{42} \\
 \Lambda_{23}^2 : & X_{36}X_{64}P_{42} - P_{36}X_{64}X_{42} \\
 \Lambda_{32}^1 : & Y_{21}X_{13} - X_{25}Y_{53} \\
 \Lambda_{35}^1 : & Z_{51}Q_{13} - X_{51}X_{13} \\
 \Lambda_{35}^2 : & Z_{51}P_{13} - R_{51}X_{13} \\
 \Lambda_{45}^1 : & X_{51}X_{14} - Y_{53}Q_{36}X_{64} \\
 \Lambda_{45}^2 : & Y_{53}X_{36}X_{64} - Z_{51}X_{14} \\
 \Lambda_{45}^3 : & R_{51}X_{14} - Y_{53}P_{36}X_{64} \\
 \Lambda_{53}^1 : & X_{32}X_{25} - X_{36}X_{65}
 \end{array}
 \quad
 \begin{array}{ll}
 & E \\
 P_{13}X_{32} - X_{14}P_{42} \\
 X_{14}Q_{42} - Q_{13}X_{32} \\
 P_{13}X_{36} - X_{13}P_{36} \\
 P_{13}Q_{36} - Q_{13}P_{36} \\
 X_{13}Q_{36} - Q_{13}X_{36} \\
 P_{25}X_{51} - Q_{25}R_{51} \\
 P_{25}Y_{53} - Y_{21}P_{13} \\
 Y_{21}Q_{13} - Q_{25}Y_{53} \\
 P_{36}X_{64}Q_{42} - Q_{36}X_{64}P_{42} \\
 P_{36}X_{65} - X_{32}P_{25} \\
 X_{32}Q_{25} - Q_{36}X_{65} \\
 P_{42}X_{25} - X_{42}P_{25} \\
 P_{42}Q_{25} - Q_{42}P_{25} \\
 X_{42}Q_{25} - Q_{42}X_{25} \\
 R_{51}Q_{13} - X_{51}P_{13}
 \end{array}. \tag{8.1}$$

generator	$SU(2)_{\bar{x}}$	$SU(2)_{\bar{y}}$	$U(1)_{\bar{b}}$
$p_1 p_3^3 so$	2	0	1
$p_2 p_3^3 so$	2	1	1
$p_1 p_3^2 p_4 so$	1	0	1
$p_2 p_3^2 p_4 so$	1	1	1
$p_1 p_3 p_4^2 so$	0	0	1
$p_2 p_3 p_4^2 so$	0	1	1
$p_1 p_4^3 so$	-1	0	1
$p_2 p_4^3 so$	-1	1	1
$p_1^2 p_3^2 p_5 so^2$	1	-1	0
$p_1 p_2 p_3^2 p_5 so^2$	1	0	0
$p_2^2 p_3^2 p_5 so^2$	1	1	0
$p_1^2 p_3 p_4 p_5 so^2$	0	-1	0
$p_1 p_2 p_3 p_4 p_5 so^2$	0	0	0
$p_2^2 p_3 p_4 p_5 so^2$	0	1	0
$p_1^2 p_4^2 p_5 so^2$	-1	-1	0
$p_1 p_2 p_4^2 p_5 so^2$	-1	0	0
$p_2^2 p_4^2 p_5 so^2$	-1	1	0
$p_1^3 p_3 p_5^2 so^3$	0	-2	-1
$p_1^2 p_2 p_3 p_5^2 so^3$	0	-1	-1
$p_1 p_2^2 p_3 p_5^2 so^3$	0	0	-1
$p_2^3 p_3 p_5^2 so^3$	0	1	-1
$p_1^3 p_4 p_5^2 so^3$	-1	-2	-1
$p_1^2 p_2 p_4 p_5^2 so^3$	-1	-1	-1
$p_1 p_2^2 p_4 p_5^2 so^3$	-1	0	-1
$p_2^3 p_4 p_5^2 so^3$	-1	1	-1
$p_1^4 p_5^3 so^4$	-1	-3	-2
$p_1^3 p_2 p_5^3 so^4$	-1	-2	-2
$p_1^2 p_2^2 p_5^3 so^4$	-1	-1	-2
$p_1 p_2^3 p_5^3 so^4$	-1	0	-2
$p_2^4 p_5^3 so^4$	-1	1	-2

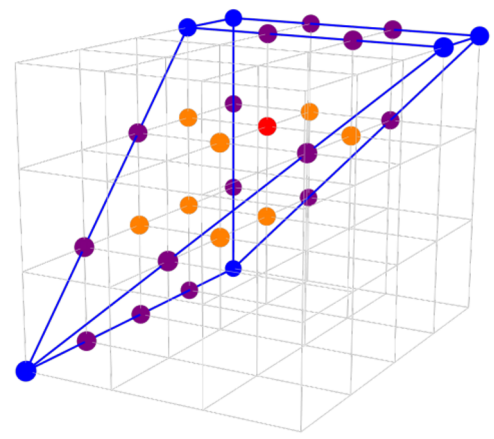


Table 16. The generators and lattice of generators of the mesonic moduli space of Model 4 in terms of brick matchings with the corresponding flavor charges.

generator	SU(2) _{\bar{x}}	SU(2) _{\bar{y}}	U(1) _{\bar{b}}
$P_{14}Y_{42}Z_{23}X_{31} = P_{25}Z_{56}X_{64}Y_{42} = Z_{23}P_{36}X_{64}Y_{42}$	2	0	1
$Q_{14}Y_{42}Z_{23}X_{31} = Q_{25}Z_{56}X_{64}Y_{42} = Z_{23}Q_{36}X_{64}Y_{42}$	2	1	1
$P_{14}X_{42}Z_{23}X_{31} = P_{14}Y_{42}Y_{23}X_{31} = P_{14}Y_{42}Z_{23}Z_{31} = P_{25}Z_{56}X_{64}X_{42} = P_{25}Y_{56}X_{64}Y_{42} = P_{25}Z_{56}Z_{64}Y_{42}$ $= Z_{23}P_{36}X_{64}X_{42} = Y_{23}P_{36}X_{64}Y_{42} = Z_{23}P_{36}Z_{64}Y_{42}$	1	0	1
$Q_{14}X_{42}Z_{23}X_{31} = Q_{14}Y_{42}Y_{23}X_{31} = Q_{14}Y_{42}Z_{23}Z_{31} = Q_{25}Z_{56}X_{64}X_{42} = Q_{25}Y_{56}X_{64}Y_{42} = Q_{25}Z_{56}Z_{64}Y_{42}$ $= Z_{23}Q_{36}X_{64}X_{42} = Y_{23}Q_{36}X_{64}Y_{42} = Z_{23}Q_{36}Z_{64}Y_{42}$	1	1	1
$P_{14}X_{42}Y_{23}X_{31} = P_{14}X_{42}Z_{23}Z_{31} = P_{14}Y_{42}Y_{23}Z_{31} = P_{25}Y_{56}X_{64}X_{42} = P_{25}Z_{56}Z_{64}X_{42} = P_{25}Y_{56}Z_{64}Y_{42}$ $= Y_{23}P_{36}X_{64}X_{42} = Z_{23}P_{36}Z_{64}X_{42} = Y_{23}P_{36}Z_{64}Y_{42}$	0	0	1
$Q_{14}X_{42}Y_{23}X_{31} = Q_{14}X_{42}Z_{23}Z_{31} = Q_{14}Y_{42}Y_{23}Z_{31} = Q_{25}Y_{56}X_{64}X_{42} = Q_{25}Z_{56}Z_{64}X_{42} = Q_{25}Y_{56}Z_{64}Y_{42}$ $= Y_{23}Q_{36}X_{64}X_{42} = Z_{23}Q_{36}Z_{64}X_{42} = Y_{23}Q_{36}Z_{64}Y_{42}$	0	1	1
$P_{14}X_{42}Y_{23}Z_{31} = P_{25}Y_{56}Z_{64}X_{42} = Y_{23}P_{36}Z_{64}X_{42}$	-1	0	1
$Q_{14}X_{42}Y_{23}Z_{31} = Q_{25}Y_{56}Z_{64}X_{42} = Y_{23}Q_{36}Z_{64}X_{42}$	-1	1	1
$P_{14}T_{42}Z_{23}X_{31} = P_{25}Z_{56}X_{64}T_{42} = Z_{23}P_{36}X_{64}T_{42} = P_{14}Y_{42}P_{25}X_{53}X_{31} = P_{14}Y_{42}P_{25}Z_{56}Y_{61}$ $= P_{14}Y_{42}Z_{23}P_{36}Y_{61} = P_{25}X_{53}P_{36}X_{64}Y_{42}$	1	-1	0
$P_{14}Z_{42}Z_{23}X_{31} = P_{25}Z_{56}X_{64}Z_{42} = Z_{23}P_{36}X_{64}Z_{42} = Q_{14}T_{42}Z_{23}X_{31} = Q_{25}Z_{56}X_{64}T_{42} = Z_{23}Q_{36}X_{64}T_{42}$ $= P_{14}Y_{42}Q_{25}X_{53}X_{31} = P_{14}Y_{42}Q_{25}Z_{56}Y_{61} = P_{14}Y_{42}Z_{23}Q_{36}Y_{61} = Q_{14}Y_{42}P_{25}X_{53}X_{31} = Q_{14}Y_{42}P_{25}Z_{56}Y_{61}$ $= P_{25}X_{53}Q_{36}X_{64}Y_{42} = Q_{14}Y_{42}Z_{23}P_{36}Y_{61} = Q_{25}X_{53}P_{36}X_{64}Y_{42}$	1	0	0
$Q_{14}Z_{42}Z_{23}X_{31} = Q_{25}Z_{56}X_{64}Z_{42} = Z_{23}Q_{36}X_{64}Z_{42} = Q_{14}Y_{42}Q_{25}X_{53}X_{31} = Q_{14}Y_{42}Q_{25}Z_{56}Y_{61}$ $= Q_{14}Y_{42}Z_{23}Q_{36}Y_{61} = Q_{25}X_{53}Q_{36}X_{64}Y_{42}$	1	1	0

Table 17. The generators in terms of bifundamental chiral fields for Model 4 (*Part 1*).

Using the forward algorithm, we are able to obtain the brick matchings for Model 5. They are summarized in the P -matrix which takes the form

$$P = \left(\begin{array}{c|cccccc|cccccc|cccccc}
 & p_1 & p_2 & p_3 & p_4 & p_5 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 \\
 \hline
 P_{13} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
 P_{25} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
 P_{36} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
 P_{42} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
 \hline
 Q_{13} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
 Q_{25} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
 Q_{36} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
 Q_{42} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
 \hline
 R_{51} & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
 \hline
 X_{13} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
 X_{14} & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
 X_{25} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
 X_{32} & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
 X_{36} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
 X_{42} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
 X_{51} & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
 X_{64} & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 X_{65} & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
 \hline
 Y_{21} & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
 Y_{53} & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
 \hline
 Z_{51} & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1
 \end{array} \right). \tag{8.2}$$

generator	$SU(2)_{\tilde{x}}$	$SU(2)_{\tilde{y}}$	$U(1)_{\tilde{b}}$
$P_{14}T_{42}Y_{23}X_{31} = P_{14}T_{42}Z_{23}Z_{31} = P_{25}Y_{56}X_{64}T_{42} = P_{25}Z_{56}Z_{64}T_{42} = Y_{23}P_{36}X_{64}T_{42} = Z_{23}P_{36}Z_{64}T_{42}$ $= P_{14}X_{42}P_{25}X_{53}X_{31} = P_{14}X_{42}P_{25}Z_{56}Y_{61} = P_{14}Y_{42}P_{25}X_{53}Z_{31} = P_{14}Y_{42}P_{25}Y_{56}Y_{61} = P_{14}X_{42}Z_{23}P_{36}Y_{61}$ $= P_{14}Y_{42}Y_{23}P_{36}Y_{61} = P_{25}X_{53}P_{36}X_{64}X_{42} = P_{25}X_{53}P_{36}Z_{64}Y_{42}$	0	-1	0
$P_{14}Z_{42}Y_{23}X_{31} = P_{14}Z_{42}Z_{23}Z_{31} = P_{25}Y_{56}X_{64}Z_{42} = P_{25}Z_{56}Z_{64}Z_{42} = Y_{23}P_{36}X_{64}Z_{42} = Z_{23}P_{36}Z_{64}Z_{42}$ $= Q_{14}T_{42}Y_{23}X_{31} = Q_{14}T_{42}Z_{23}Z_{31} = Q_{25}Y_{56}X_{64}T_{42} = Q_{25}Z_{56}Z_{64}T_{42} = Y_{23}Q_{36}X_{64}T_{42} = Z_{23}Q_{36}Z_{64}T_{42}$ $= P_{14}X_{42}Q_{25}X_{53}X_{31} = P_{14}X_{42}Q_{25}Z_{56}Y_{61} = P_{14}Y_{42}Q_{25}X_{53}Z_{31} = P_{14}Y_{42}Q_{25}Y_{56}Y_{61} = P_{14}X_{42}Z_{23}Q_{36}Y_{61}$ $= P_{14}Y_{42}Y_{23}Q_{36}Y_{61} = Q_{14}X_{42}P_{25}X_{53}X_{31} = Q_{14}X_{42}P_{25}Z_{56}Y_{61} = Q_{14}Y_{42}P_{25}X_{53}Z_{31} = Q_{14}Y_{42}P_{25}Y_{56}Y_{61}$ $= P_{25}X_{53}Q_{36}X_{64}X_{42} = P_{25}X_{53}Q_{36}Z_{64}Y_{42} = Q_{14}X_{42}Z_{23}P_{36}Y_{61} = Q_{14}Y_{42}Y_{23}P_{36}Y_{61} = Q_{25}X_{53}P_{36}X_{64}X_{42}$ $= Q_{25}X_{53}P_{36}Z_{64}Y_{42}$	0	0	0
$Q_{14}Z_{42}Y_{23}X_{31} = Q_{14}Z_{42}Z_{23}Z_{31} = Q_{25}Y_{56}X_{64}Z_{42} = Q_{25}Z_{56}Z_{64}Z_{42} = Y_{23}Q_{36}X_{64}Z_{42} = Z_{23}Q_{36}Z_{64}Z_{42}$ $= Q_{14}X_{42}Q_{25}X_{53}X_{31} = Q_{14}X_{42}Q_{25}Z_{56}Y_{61} = Q_{14}Y_{42}Q_{25}X_{53}Z_{31} = Q_{14}Y_{42}Q_{25}Y_{56}Y_{61} = Q_{14}X_{42}Z_{23}Q_{36}Y_{61}$ $= Q_{14}Y_{42}Y_{23}Q_{36}Y_{61} = Q_{25}X_{53}Q_{36}X_{64}X_{42} = Q_{25}X_{53}Q_{36}Z_{64}Y_{42}$	0	1	0
$P_{14}T_{42}Z_{23}Z_{31} = P_{25}Y_{56}Z_{64}T_{42} = Y_{23}P_{36}Z_{64}T_{42} = P_{14}X_{42}P_{25}X_{53}Z_{31} = P_{14}X_{42}P_{25}Y_{56}Y_{61}$ $= P_{14}X_{42}Y_{23}P_{36}Y_{61} = P_{25}X_{53}P_{36}Z_{64}X_{42}$	-1	-1	0
$P_{14}Z_{42}Y_{23}Z_{31} = P_{25}Y_{56}Z_{64}Z_{42} = Y_{23}P_{36}Z_{64}Z_{42} = Q_{14}T_{42}Y_{23}Z_{31} = Q_{25}Y_{56}Z_{64}T_{42} = Y_{23}Q_{36}Z_{64}T_{42}$ $= P_{14}X_{42}Q_{25}X_{53}Z_{31} = P_{14}X_{42}Q_{25}Y_{56}Y_{61} = P_{14}X_{42}Y_{23}Q_{36}Y_{61} = Q_{14}X_{42}P_{25}X_{53}Z_{31} = Q_{14}X_{42}P_{25}Y_{56}Y_{61}$ $= P_{25}X_{53}Q_{36}Z_{64}X_{42} = Q_{14}X_{42}Y_{23}P_{36}Y_{61} = Q_{25}X_{53}P_{36}Z_{64}X_{42}$	-1	0	0
$Q_{14}Z_{42}Y_{23}Z_{31} = Q_{25}Y_{56}Z_{64}Z_{42} = Y_{23}Q_{36}Z_{64}Z_{42} = Q_{14}X_{42}Q_{25}X_{53}Z_{31} = Q_{14}X_{42}Q_{25}Y_{56}Y_{61}$ $= Q_{14}X_{42}Y_{23}Q_{36}Y_{61} = Q_{25}X_{53}Q_{36}Z_{64}X_{42}$	-1	1	0
$P_{14}Y_{42}P_{25}X_{53}P_{36}Y_{61} = P_{14}T_{42}P_{25}X_{53}X_{31} = P_{14}T_{42}P_{25}Z_{56}Y_{61} = P_{14}T_{42}Z_{23}P_{36}Y_{61} = P_{25}X_{53}P_{36}X_{64}T_{42}$ $P_{14}Y_{42}P_{25}X_{53}Q_{36}Y_{61} = P_{14}Y_{42}Q_{25}X_{53}P_{36}Y_{61} = Q_{14}Y_{42}P_{25}X_{53}P_{36}Y_{61} = P_{14}Z_{42}P_{25}X_{53}X_{31}$ $= P_{14}Z_{42}P_{25}Z_{56}Y_{61} = P_{14}Z_{42}Z_{23}P_{36}Y_{61} = P_{14}T_{42}Q_{25}X_{53}X_{31} = P_{14}T_{42}Q_{25}Z_{56}Y_{61} = P_{14}T_{42}Z_{23}Q_{36}Y_{61}$ $= P_{25}X_{53}P_{36}X_{64}Z_{42} = Q_{14}T_{42}P_{25}X_{53}X_{31} = Q_{14}T_{42}P_{25}Z_{56}Y_{61} = P_{25}X_{53}Q_{36}X_{64}T_{42} = Q_{14}T_{42}Z_{23}P_{36}Y_{61}$ $= Q_{25}X_{53}P_{36}X_{64}T_{42}$	0	-2	-1
$P_{14}Y_{42}Q_{25}X_{53}Q_{36}Y_{61} = Q_{14}Y_{42}P_{25}X_{53}Q_{36}Y_{61} = Q_{14}Y_{42}Q_{25}X_{53}P_{36}Y_{61} = P_{14}Z_{42}Q_{25}X_{53}X_{31}$ $= P_{14}Z_{42}Q_{25}Z_{56}Y_{61} = P_{14}Z_{42}Z_{23}Q_{36}Y_{61} = Q_{14}Z_{42}P_{25}X_{53}X_{31} = Q_{14}Z_{42}P_{25}Z_{56}Y_{61} = P_{25}X_{53}Q_{36}X_{64}Z_{42}$ $= Q_{14}Z_{42}Z_{23}P_{36}Y_{61} = Q_{25}X_{53}P_{36}X_{64}Z_{42} = Q_{14}T_{42}Q_{25}X_{53}X_{31} = Q_{14}T_{42}Q_{25}Z_{56}Y_{61} = Q_{14}T_{42}Z_{23}Q_{36}Y_{61}$ $= Q_{25}X_{53}Q_{36}X_{64}T_{42}$	0	0	-1
$Q_{14}Y_{42}Q_{25}X_{53}Q_{36}Y_{61} = Q_{14}Z_{42}Q_{25}X_{53}X_{31} = Q_{14}Z_{42}Q_{25}Z_{56}Y_{61} = Q_{14}Z_{42}Z_{23}Q_{36}Y_{61} = Q_{25}X_{53}Q_{36}X_{64}Z_{42}$ $P_{14}X_{42}P_{25}X_{53}P_{36}Y_{61} = P_{14}T_{42}P_{25}X_{53}Z_{31} = P_{14}T_{42}P_{25}Y_{56}Y_{61} = P_{14}T_{42}Y_{23}P_{36}Y_{61} = P_{25}X_{53}P_{36}Z_{64}T_{42}$ $P_{14}X_{42}P_{25}X_{53}Q_{36}Y_{61} = P_{14}X_{42}Q_{25}X_{53}P_{36}Y_{61} = Q_{14}X_{42}P_{25}X_{53}P_{36}Y_{61} = P_{14}Z_{42}P_{25}X_{53}Z_{31}$ $= P_{14}Z_{42}P_{25}Y_{56}Y_{61} = P_{14}Z_{42}Y_{23}P_{36}Y_{61} = P_{14}T_{42}Q_{25}X_{53}Z_{31} = P_{14}T_{42}Q_{25}Y_{56}Y_{61} = P_{14}T_{42}Y_{23}Q_{36}Y_{61}$ $= P_{25}X_{53}P_{36}Z_{64}Z_{42} = Q_{14}T_{42}P_{25}X_{53}Z_{31} = Q_{14}T_{42}P_{25}Y_{56}Y_{61} = P_{25}X_{53}Q_{36}Z_{64}T_{42} = Q_{14}T_{42}Y_{23}P_{36}Y_{61}$ $= Q_{25}X_{53}P_{36}Z_{64}T_{42}$	-1	-2	-1
$P_{14}X_{42}Q_{25}X_{53}Q_{36}Y_{61} = Q_{14}X_{42}P_{25}X_{53}Q_{36}Y_{61} = Q_{14}X_{42}Q_{25}X_{53}P_{36}Y_{61} = P_{14}Z_{42}Q_{25}X_{53}Z_{31}$ $= P_{14}Z_{42}Q_{25}Z_{56}Y_{61} = P_{14}Z_{42}Y_{23}Q_{36}Y_{61} = Q_{14}Z_{42}P_{25}X_{53}Z_{31} = Q_{14}Z_{42}P_{25}Z_{56}Y_{61} = P_{25}X_{53}Q_{36}Z_{64}Z_{42}$ $= Q_{14}Z_{42}Y_{23}P_{36}Y_{61} = Q_{25}X_{53}P_{36}Z_{64}Z_{42} = Q_{14}T_{42}Q_{25}X_{53}Z_{31} = Q_{14}T_{42}Q_{25}Y_{56}Y_{61} = Q_{14}T_{42}Z_{23}Q_{36}Y_{61}$ $= Q_{25}X_{53}Q_{36}Z_{64}T_{42}$	-1	0	-1
$Q_{14}X_{42}Q_{25}X_{53}Q_{36}Y_{61} = Q_{14}Z_{42}Q_{25}X_{53}Z_{31} = Q_{14}Z_{42}Q_{25}Y_{56}Y_{61} = Q_{14}Z_{42}Y_{23}Q_{36}Y_{61} = Q_{25}X_{53}Q_{36}Z_{64}Z_{42}$ $P_{14}T_{42}P_{25}X_{53}P_{36}Y_{61}$	-1	1	-1
$P_{14}Z_{42}P_{25}X_{53}P_{36}Y_{61} = P_{14}T_{42}P_{25}X_{53}Q_{36}Y_{61} = P_{14}T_{42}Q_{25}X_{53}P_{36}Y_{61} = Q_{14}T_{42}P_{25}X_{53}P_{36}Y_{61}$ $P_{14}Z_{42}P_{25}X_{53}Q_{36}Y_{61} = P_{14}Z_{42}Q_{25}X_{53}P_{36}Y_{61} = P_{14}T_{42}Q_{25}X_{53}Q_{36}Y_{61} = Q_{14}Z_{42}P_{25}X_{53}P_{36}Y_{61}$ $= Q_{14}T_{42}P_{25}X_{53}Q_{36}Y_{61} = Q_{14}T_{42}Q_{25}X_{53}P_{36}Y_{61}$	-1	-3	-2
$P_{14}Z_{42}P_{25}X_{53}Q_{36}Y_{61} = P_{14}Z_{42}Q_{25}X_{53}P_{36}Y_{61} = P_{14}T_{42}Q_{25}X_{53}Q_{36}Y_{61} = Q_{14}Z_{42}P_{25}X_{53}P_{36}Y_{61}$ $= Q_{14}T_{42}P_{25}X_{53}Q_{36}Y_{61} = Q_{14}T_{42}Q_{25}X_{53}P_{36}Y_{61}$	-1	-2	-2
$P_{14}Z_{42}P_{25}X_{53}Q_{36}Y_{61} = P_{14}Z_{42}Q_{25}X_{53}P_{36}Y_{61} = P_{14}T_{42}Q_{25}X_{53}Q_{36}Y_{61} = Q_{14}Z_{42}P_{25}X_{53}P_{36}Y_{61}$ $= Q_{14}T_{42}P_{25}X_{53}Q_{36}Y_{61} = Q_{14}T_{42}Q_{25}X_{53}P_{36}Y_{61}$	-1	-1	-2
$P_{14}Z_{42}Q_{25}X_{53}Q_{36}Y_{61} = Q_{14}Z_{42}P_{25}X_{53}Q_{36}Y_{61} = Q_{14}Z_{42}Q_{25}X_{53}P_{36}Y_{61} = Q_{14}T_{42}Q_{25}X_{53}Q_{36}Y_{61}$ $Q_{14}Z_{42}Q_{25}X_{53}Q_{36}Y_{61}$	-1	0	-2
$Q_{14}Z_{42}Q_{25}X_{53}Q_{36}Y_{61} = Q_{14}Z_{42}Q_{25}X_{53}Z_{31} = Q_{14}Z_{42}Q_{25}Y_{56}Y_{61} = Q_{14}Z_{42}Y_{23}Q_{36}Y_{61} = Q_{25}X_{53}Q_{36}Z_{64}Z_{42}$ $P_{14}T_{42}P_{25}X_{53}P_{36}Y_{61}$	-1	1	-2

Table 18. The generators in terms of bifundamental chiral fields for Model 4 (*Part 2*).

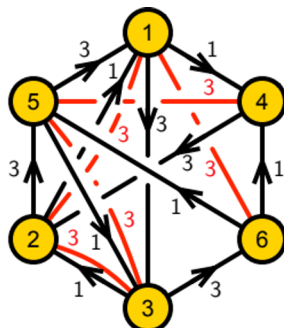


Figure 18. Quiver for Model 5.

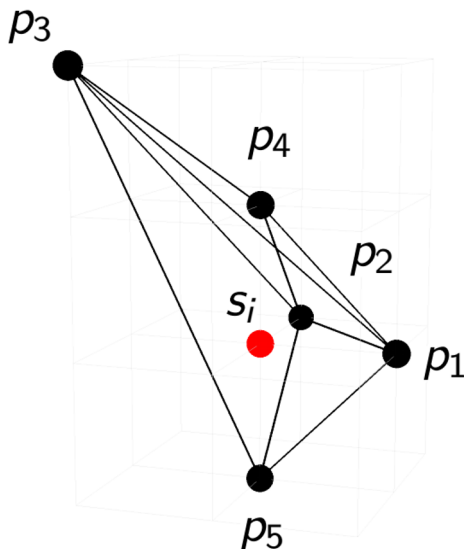


Figure 19. Toric diagram for Model 5.

The J - and E -term charges are given by

$$Q_{JE} = \left(\begin{array}{ccccc|cccccc|cccccc} p_1 & p_2 & p_3 & p_4 & p_5 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 \\ 1 & 1 & 1 & 0 & 2 & 0 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right), \tag{8.3}$$

and the D -term charges are given by

$$Q_D = \left(\begin{array}{ccccc|cccccc|cccccc} p_1 & p_2 & p_3 & p_4 & p_5 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right). \tag{8.4}$$

The toric diagram of Model 5 is given by

$$G_t = \left(\begin{array}{ccccc|cccccc|cccccc} p_1 & p_2 & p_3 & p_4 & p_5 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{array} \right), \tag{8.5}$$

where figure 19 shows the toric diagram with brick matching labels.

	$SU(3)_{(x_1, x_2)}$	$U(1)_b$	$U(1)$	fugacity
p_1	(1, 0)	0	r_1	t_1
p_2	(-1, +1)	0	r_2	t_2
p_3	(0, -1)	0	r_3	t_3
p_4	(0, 0)	+1	r_4	t_4
p_5	(0, 0)	-1	r_5	t_5

Table 19. Global symmetry charges on the extremal brick matchings p_i of Model 5.

The Hilbert series of the mesonic moduli space of Model 5 takes the form

$$g_1(t_i, y_s, y_o; \mathcal{M}_5) = \frac{P(t_i, y_s, y_o; \mathcal{M}_5)}{(1 - y_s y_o^3 t_1^5 t_4^2)(1 - y_s y_o^3 t_2^5 t_4^2)(1 - y_s y_o^3 t_3^5 t_4^2)(1 - y_s y_o t_1 t_5^2)} \times \frac{1}{(1 - y_s y_o t_2 t_5^2)(1 - y_s y_o t_3 t_5^2)}, \quad (8.6)$$

where t_i are the fugacities for the extremal brick matchings p_i . y_s counts the product $s_1 \dots s_6$ and y_o counts the product $o_1 \dots o_6$. The explicit numerator $P(t_i, y_s, y_o; \mathcal{M}_5)$ of the Hilbert series is given in the appendix section A.5. We note that setting the fugacity $y_o = 1$ does not change the overall characterization of the mesonic moduli space by the Hilbert series, indicating that the extra GLSM fields, as expected, correspond to an over-parameterization of the moduli space.

By setting $t_i = t$ for the fugacities of the extremal brick matchings, and all other fugacities to $y_s = 1$ and $y_o = 1$, the unrefined Hilbert series takes the following form

$$g_1(t, 1, 1; \mathcal{M}_5) = \frac{1}{(1 - t^3)^3(1 - t^7)^3} \times (1 + 10t^5 + 18t^7 - 15t^8 - 35t^{10} + 6t^{11} + 15t^{12} + 15t^{13} + 6t^{14} - 35t^{15} - 15t^{17} + 18t^{18} + 10t^{20} + t^{25}), \quad (8.7)$$

where the palindromic numerator indicates that the mesonic moduli space is Calabi-Yau.

The global symmetry of Model 5 and the charges on the extremal brick matchings under the global symmetry are summarized in table 19. Using the following fugacity map,

$$t = \frac{t_4^2 t_5^2}{t_1 t_2 t_3}, \quad x_1 = \frac{t_4 t_5}{t_2 t_3}, \quad x_2 = \frac{t_4^2 t_5^2}{t_1 t_2 t_3^2}, \quad b = \frac{t_4^2 t_5^2}{t_1 t_2 t_3}, \quad (8.8)$$

the Hilbert series for Model 5 can be rewritten in terms of characters of irreducible representations of $SU(3) \times U(1)$. In terms of fugacities of the mesonic flavor symmetry of Model 5, the Hilbert series becomes

$$g_1(t, x_i, b; \mathcal{M}_5) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \left[[n_1 + 3n_2, 0] b^{-2n_1} t^{3n_1 + 5n_2} + [5n_1 + 3n_2 + 5, 0] b^{2n_1 + 2} t^{7n_1 + 5n_2 + 7} \right], \quad (8.9)$$

where $[m_1, m_2] = [m_1, m_2]_{\text{SU}(3)_{(x_1, x_2)}}$. The fugacity b counts charges under the $U(1)$ factor of the mesonic flavor symmetry. The corresponding plethystic logarithm is

$$\begin{aligned} \text{PL}[g_1(t, x_i, b; \mathcal{M}_5)] &= [1, 0]b^{-2}t^3 + [3, 0]t^5 + [5, 0]b^2t^7 \\ &\quad - [2, 1]b^{-2}t^8 - ([6, 0] + [4, 1] + [2, 2])t^{10} + [2, 0]b^{-4}t^{11} + \dots, \end{aligned} \quad (8.10)$$

where we see that the mesonic moduli space is a non-complete intersection.

The generators form 3 sets that transform under $[1, 0]b^{-2}$, $[3, 0]$ and $[5, 0]b^2$ of the mesonic flavor symmetry of Model 5, respectively. Using the following fugacity map

$$\tilde{t} = t_4^{1/2}t_5^{1/2}, \quad \tilde{x}_1 = \frac{t_4^{3/2}t_5^{3/2}}{t_2^2t_3}, \quad \tilde{x}_2 = \frac{t_4^{3/2}t_5^{3/2}}{t_2t_3^2}, \quad \tilde{b} = \frac{t_4^3t_5}{t_2^2t_3^2}, \quad (8.11)$$

the mesonic flavor charges on the gauge invariant operators become \mathbb{Z} -valued. The generators in terms of brick matchings and their corresponding rescaled mesonic flavor charges are summarized in table 20.

The generator lattice as shown in table 20 is a convex lattice polytope, which is reflexive. It is the dual of the toric diagram of Model 5 shown in figure 19. We also note that the 3 layers of points in the generator lattice in table 20 corresponds to the 3 sets that transform under $[1, 0]b^{-2}$, $[3, 0]$ and $[5, 0]b^2$ of the mesonic flavor symmetry. For completeness, table 21 and table 22 show the generators of Model 5 in terms of chiral fields with the corresponding mesonic flavor charges.

9 Model 6: $P_{+-}^1(\text{dP}_1) [\mathbb{P}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, -1)), \langle 24 \rangle]$

Model 6 corresponds to the Calabi-Yau 4-fold $P_{+-}^1(\text{dP}_1)$. The corresponding brane brick model has the quiver in figure 20 and the J - and E -terms are given as follows

$$\begin{array}{ll} & J \\ \Lambda_{17}^1 : & X_{78}Y_{86}Y_{61} - X_{72}Q_{26}X_{61} \\ \Lambda_{17}^2 : & X_{72}Q_{26}Z_{61} - X_{78}X_{86}Y_{61} \\ \Lambda_{17}^3 : & X_{78}Y_{86}R_{61} - X_{72}P_{26}X_{61} \\ \Lambda_{17}^4 : & X_{72}P_{26}Z_{61} - X_{78}X_{86}R_{61} \\ \Lambda_{18}^1 : & X_{86}X_{61} - Y_{86}Z_{61} \\ \Lambda_{21}^1 : & X_{13}X_{34}Y_{42} - Y_{13}X_{34}X_{42} \\ \Lambda_{25}^1 : & X_{54}X_{42} - X_{57}X_{72} \\ \Lambda_{25}^2 : & Y_{57}X_{72} - X_{54}Y_{42} \\ \Lambda_{36}^1 : & Z_{61}Y_{13} - X_{61}X_{13} \\ \Lambda_{38}^1 : & Y_{86}Y_{61}X_{13} - X_{86}Y_{61}Y_{13} \\ \Lambda_{38}^2 : & Y_{86}R_{61}X_{13} - X_{86}R_{61}Y_{13} \\ \Lambda_{46}^1 : & X_{61}Q_{15}X_{54} - Y_{61}Y_{13}X_{34} \\ \Lambda_{46}^2 : & Y_{61}X_{13}X_{34} - Z_{61}Q_{15}X_{54} \\ \Lambda_{46}^3 : & X_{61}P_{15}X_{54} - R_{61}Y_{13}X_{34} \\ \Lambda_{46}^4 : & R_{61}X_{13}X_{34} - Z_{61}P_{15}X_{54} \\ \Lambda_{65}^1 : & X_{57}X_{78}Y_{86} - Y_{57}X_{78}X_{86} \end{array} \quad \begin{array}{ll} & E \\ P_{15}X_{57} - X_{13}P_{37} \\ P_{15}Y_{57} - Y_{13}P_{37} \\ X_{13}Q_{37} - Q_{15}X_{57} \\ Y_{13}Q_{37} - Q_{15}Y_{57} \\ P_{15}X_{54}Q_{48} - Q_{15}X_{54}P_{48} \\ P_{26}Y_{61} - Q_{26}R_{61} \\ P_{26}X_{61}Q_{15} - Q_{26}X_{61}P_{15} \\ P_{26}Z_{61}Q_{15} - Q_{26}Z_{61}P_{15} \\ P_{37}X_{72}Q_{26} - Q_{37}X_{72}P_{26} \\ P_{37}X_{78} - X_{34}P_{48} \\ X_{34}Q_{48} - Q_{37}X_{78} \\ P_{48}X_{86} - X_{42}P_{26} \\ P_{48}Y_{86} - Y_{42}P_{26} \\ X_{42}Q_{26} - Q_{48}X_{86} \\ Y_{42}Q_{26} - Q_{48}Y_{86} \\ R_{61}Q_{15} - Y_{61}P_{15} \end{array} . \quad (9.1)$$

generator	$SU(3)_{(\hat{x}_1, \hat{x}_2)}$	$U(1)_{\hat{b}}$
$p_1 p_5^2 so$	(1, 1)	-1
$p_2 p_5^2 so$	(0, 1)	-1
$p_3 p_5^2 so$	(1, 0)	-1
$p_1^3 p_4 p_5 so^2$	(1, 1)	0
$p_1^2 p_2 p_4 p_5 so^2$	(0, 1)	0
$p_1 p_2^2 p_4 p_5 so^2$	(-1, 1)	0
$p_2^3 p_4 p_5 so^2$	(-2, 1)	0
$p_1^2 p_3 p_4 p_5 so^2$	(1, 0)	0
$p_1 p_2 p_3 p_4 p_5 so^2$	(0, 0)	0
$p_2^2 p_3 p_4 p_5 so^2$	(-1, 0)	0
$p_1 p_3^2 p_4 p_5 so^2$	(1, -1)	0
$p_2 p_3^2 p_4 p_5 so^2$	(0, -1)	0
$p_3^3 p_4 p_5 so^2$	(1, -2)	0
$p_1^5 p_4^2 so^3$	(1, 1)	1
$p_1^4 p_2 p_4^2 so^3$	(0, 1)	1
$p_1^3 p_2^2 p_4^2 so^3$	(-1, 1)	1
$p_1^2 p_2^3 p_4^2 so^3$	(-2, 1)	1
$p_1 p_2^4 p_4^2 so^3$	(-3, 1)	1
$p_2^5 p_4^2 so^3$	(-4, 1)	1
$p_1^4 p_3 p_4^2 so^3$	(1, 0)	1
$p_1^3 p_2 p_3 p_4^2 so^3$	(0, 0)	1
$p_1^2 p_2^2 p_3 p_4^2 so^3$	(-1, 0)	1
$p_1 p_2^3 p_3 p_4^2 so^3$	(-2, 0)	1
$p_2^4 p_3 p_4^2 so^3$	(-3, 0)	1
$p_1^3 p_3^2 p_4^2 so^3$	(1, -1)	1
$p_1^2 p_2 p_3^2 p_4^2 so^3$	(0, -1)	1
$p_1 p_2^2 p_3^2 p_4^2 so^3$	(-1, -1)	1
$p_2^3 p_3^2 p_4^2 so^3$	(-2, -1)	1
$p_1^2 p_3^3 p_4^2 so^3$	(1, -2)	1
$p_1 p_2 p_3^3 p_4^2 so^3$	(0, -2)	1
$p_2^2 p_3^3 p_4^2 so^3$	(-1, -2)	1
$p_1 p_3^4 p_4^2 so^3$	(1, -3)	1
$p_2 p_3^4 p_4^2 so^3$	(0, -3)	1
$p_3^5 p_4^2 so^3$	(1, -4)	1

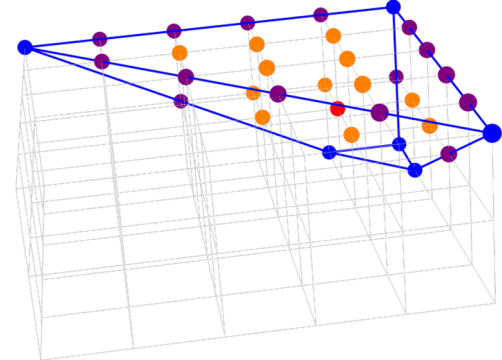


Table 20. The generators and lattice of generators of the mesonic moduli space of Model 5 in terms of brick matchings with the corresponding flavor charges.

generator	$SU(3)_{(\bar{x}_1, \bar{x}_2)}$	$U(1)_b$
$P_{13}X_{32}Y_{21} = P_{25}Y_{53}X_{32} = P_{36}X_{65}Y_{53} = X_{14}P_{42}Y_{21}$	(1, 1)	-1
$Q_{13}X_{32}Y_{21} = Q_{25}Y_{53}X_{32} = Q_{36}X_{65}Y_{53} = X_{14}Q_{42}Y_{21}$	(0, 1)	-1
$X_{13}X_{32}Y_{21} = X_{14}X_{42}Y_{21} = X_{25}Y_{53}X_{32} = X_{36}X_{65}Y_{53}$	(1, 0)	-1
$P_{13}P_{36}X_{64}P_{42}Y_{21} = P_{25}Y_{53}P_{36}X_{64}P_{42} = P_{13}X_{32}P_{25}R_{51} = P_{13}P_{36}X_{65}R_{51} = X_{14}P_{42}P_{25}R_{51}$	(1, 1)	0
$P_{13}P_{36}X_{64}Q_{42}Y_{21} = P_{13}Q_{36}X_{64}P_{42}Y_{21} = P_{25}Y_{53}P_{36}X_{64}Q_{42} = P_{25}Y_{53}Q_{36}X_{64}P_{42} = Q_{13}P_{36}X_{64}P_{42}Y_{21}$ $= Q_{25}Y_{53}P_{36}X_{64}P_{42} = P_{13}X_{32}P_{25}X_{51} = P_{13}P_{36}X_{65}X_{51} = P_{13}X_{32}Q_{25}R_{51} = P_{13}Q_{36}X_{65}R_{51}$ $= X_{14}P_{42}P_{25}X_{51} = Q_{13}X_{32}P_{25}R_{51} = X_{14}Q_{42}P_{25}R_{51} = Q_{13}P_{36}X_{65}R_{51} = X_{14}P_{42}Q_{25}R_{51}$	(0, 1)	0
$P_{13}Q_{36}X_{64}Q_{42}Y_{21} = P_{25}Y_{53}Q_{36}X_{64}Q_{42} = Q_{13}P_{36}X_{64}Q_{42}Y_{21} = Q_{25}Y_{53}P_{36}X_{64}Q_{42} = Q_{13}Q_{36}X_{64}P_{42}Y_{21}$ $= Q_{25}Y_{53}Q_{36}X_{64}P_{42} = P_{13}X_{32}Q_{25}X_{51} = P_{13}Q_{36}X_{65}X_{51} = Q_{13}X_{32}P_{25}X_{51} = X_{14}Q_{42}P_{25}X_{51}$ $= Q_{13}P_{36}X_{65}X_{51} = X_{14}P_{42}Q_{25}X_{51} = Q_{13}X_{32}Q_{25}R_{51} = Q_{13}Q_{36}X_{65}R_{51} = X_{14}Q_{42}Q_{25}R_{51}$	(-1, 1)	0
$Q_{13}Q_{36}X_{64}Q_{42}Y_{21} = Q_{25}Y_{53}Q_{36}X_{64}Q_{42} = Q_{13}X_{32}Q_{25}X_{51} = Q_{13}Q_{36}X_{65}X_{51} = X_{14}Q_{42}Q_{25}X_{51}$	(-2, 1)	0
$P_{13}P_{36}X_{64}X_{42}Y_{21} = P_{13}X_{36}X_{64}P_{42}Y_{21} = P_{25}Y_{53}P_{36}X_{64}X_{42} = P_{25}Y_{53}X_{36}X_{64}P_{42} = X_{13}P_{36}X_{64}P_{42}Y_{21}$ $= X_{25}Y_{53}P_{36}X_{64}P_{42} = P_{13}X_{32}P_{25}Z_{51} = P_{13}P_{36}X_{65}Z_{51} = P_{13}X_{32}X_{25}R_{51} = P_{13}X_{36}X_{65}R_{51}$ $= X_{14}P_{42}P_{25}Z_{51} = X_{13}X_{32}P_{25}R_{51} = X_{14}X_{42}P_{25}R_{51} = X_{13}P_{36}X_{65}R_{51} = X_{14}P_{42}X_{25}R_{51}$	(0, 0)	0
$P_{13}Q_{36}X_{64}X_{42}Y_{21} = P_{13}X_{36}X_{64}Q_{42}Y_{21} = P_{25}Y_{53}Q_{36}X_{64}X_{42} = P_{25}Y_{53}X_{36}X_{64}Q_{42} = Q_{13}P_{36}X_{64}X_{42}Y_{21}$ $= Q_{25}Y_{53}P_{36}X_{64}X_{42} = X_{13}P_{36}X_{64}Q_{42}Y_{21} = X_{25}Y_{53}P_{36}X_{64}Q_{42} = Q_{13}X_{36}X_{64}P_{42}Y_{21} = Q_{25}Y_{53}X_{36}X_{64}P_{42}$ $= X_{13}Q_{36}X_{64}P_{42}Y_{21} = X_{25}Y_{53}Q_{36}X_{64}P_{42} = P_{13}X_{32}Q_{25}Z_{51} = P_{13}Q_{36}X_{65}Z_{51} = P_{13}X_{32}X_{25}X_{51}$ $= P_{13}X_{36}X_{65}X_{51} = Q_{13}X_{32}P_{25}Z_{51} = X_{14}Q_{42}P_{25}Z_{51} = X_{13}X_{32}P_{25}X_{51} = X_{14}X_{42}P_{25}X_{51} = Q_{13}P_{36}X_{65}Z_{51}$ $= X_{13}P_{36}X_{65}X_{51} = X_{14}P_{42}Q_{25}Z_{51} = X_{14}P_{42}X_{25}X_{51} = Q_{13}X_{32}X_{25}R_{51} = Q_{13}X_{36}X_{65}R_{51} = X_{13}X_{32}Q_{25}R_{51}$ $= X_{14}X_{42}Q_{25}R_{51} = X_{13}Q_{36}X_{65}R_{51} = X_{14}Q_{42}X_{25}R_{51}$	(-1, 0)	0
$Q_{13}Q_{36}X_{64}X_{42}Y_{21} = Q_{13}X_{36}X_{64}Q_{42}Y_{21} = Q_{25}Y_{53}Q_{36}X_{64}X_{42} = Q_{25}Y_{53}X_{36}X_{64}Q_{42} = X_{13}Q_{36}X_{64}Q_{42}Y_{21}$ $= X_{25}Y_{53}Q_{36}X_{64}Q_{42} = Q_{13}X_{32}Q_{25}Z_{51} = Q_{13}Q_{36}X_{65}Z_{51} = Q_{13}X_{32}X_{25}X_{51} = Q_{13}X_{36}X_{65}X_{51}$ $= X_{14}Q_{42}Q_{25}Z_{51} = X_{13}X_{32}Q_{25}X_{51} = X_{14}X_{42}Q_{25}X_{51} = X_{13}Q_{36}X_{65}X_{51} = X_{14}Q_{42}X_{25}X_{51}$	(1, -1)	0
$P_{13}X_{36}X_{64}X_{42}Y_{21} = P_{25}Y_{53}X_{36}X_{64}X_{42} = X_{13}P_{36}X_{64}X_{42}Y_{21} = X_{25}Y_{53}P_{36}X_{64}X_{42} = X_{13}X_{36}X_{64}P_{42}Y_{21}$ $= X_{25}Y_{53}X_{36}X_{64}P_{42} = P_{13}X_{32}X_{25}Z_{51} = P_{13}X_{36}X_{65}Z_{51} = X_{13}X_{32}P_{25}Z_{51} = X_{14}X_{42}P_{25}Z_{51}$ $= X_{13}P_{36}X_{65}Z_{51} = X_{14}P_{42}X_{25}Z_{51} = X_{13}X_{32}X_{25}R_{51} = X_{13}X_{36}X_{65}R_{51} = X_{14}X_{42}X_{25}R_{51}$	(0, -1)	0
$Q_{13}X_{36}X_{64}X_{42}Y_{21} = Q_{25}Y_{53}X_{36}X_{64}X_{42} = X_{13}Q_{36}X_{64}X_{42}Y_{21} = X_{25}Y_{53}Q_{36}X_{64}X_{42} = X_{13}X_{36}X_{64}Q_{42}Y_{21}$ $= X_{25}Y_{53}Q_{36}X_{64}Q_{42} = Q_{13}X_{32}X_{25}Z_{51} = Q_{13}X_{36}X_{65}Z_{51} = X_{13}X_{32}Q_{25}Z_{51} = X_{14}X_{42}Q_{25}Z_{51}$ $= X_{13}Q_{36}X_{65}Z_{51} = X_{14}Q_{42}X_{25}Z_{51} = X_{13}X_{32}X_{25}R_{51} = X_{13}X_{36}X_{65}R_{51} = X_{14}X_{42}X_{25}R_{51}$	(1, -2)	0
$X_{13}X_{36}X_{64}X_{42}Y_{21} = X_{25}Y_{53}X_{36}X_{64}X_{42} = X_{13}X_{32}X_{25}Z_{51} = X_{13}X_{36}X_{65}Z_{51} = X_{14}X_{42}X_{25}Z_{51}$	(1, 1)	1
$P_{13}P_{36}X_{64}P_{42}P_{25}R_{51}$	(1, 1)	1
$P_{13}P_{36}X_{64}P_{42}P_{25}X_{51} = P_{13}P_{36}X_{64}Q_{42}P_{25}R_{51} = P_{13}Q_{36}X_{64}P_{42}P_{25}R_{51} = P_{13}P_{36}X_{64}P_{42}Q_{25}R_{51}$ $= Q_{13}P_{36}X_{64}P_{42}P_{25}R_{51}$	(0, 1)	1
$P_{13}P_{36}X_{64}Q_{42}P_{25}X_{51} = P_{13}Q_{36}X_{64}P_{42}P_{25}X_{51} = P_{13}Q_{36}X_{64}Q_{42}P_{25}R_{51} = P_{13}P_{36}X_{64}P_{42}Q_{25}X_{51}$ $= P_{13}P_{36}X_{64}Q_{42}Q_{25}R_{51} = P_{13}Q_{36}X_{64}P_{42}Q_{25}R_{51} = Q_{13}P_{36}X_{64}P_{42}P_{25}X_{51} = Q_{13}P_{36}X_{64}Q_{42}P_{25}R_{51}$ $= Q_{13}Q_{36}X_{64}P_{42}P_{25}R_{51} = Q_{13}P_{36}X_{64}P_{42}Q_{25}R_{51}$	(-1, 1)	1
$P_{13}Q_{36}X_{64}Q_{42}P_{25}X_{51} = P_{13}P_{36}X_{64}Q_{42}Q_{25}X_{51} = P_{13}Q_{36}X_{64}P_{42}Q_{25}X_{51} = P_{13}Q_{36}X_{64}Q_{42}Q_{25}R_{51}$ $= Q_{13}P_{36}X_{64}Q_{42}P_{25}X_{51} = Q_{13}Q_{36}X_{64}P_{42}P_{25}X_{51} = Q_{13}Q_{36}X_{64}Q_{42}P_{25}R_{51} = Q_{13}P_{36}X_{64}P_{42}Q_{25}R_{51}$ $= Q_{13}P_{36}X_{64}Q_{42}Q_{25}R_{51} = Q_{13}Q_{36}X_{64}P_{42}Q_{25}R_{51}$	(-2, 1)	1
$P_{13}Q_{36}X_{64}Q_{42}Q_{25}R_{51} = Q_{13}Q_{36}X_{64}P_{42}Q_{25}R_{51} = Q_{13}P_{36}X_{64}P_{42}Q_{25}R_{51}$	(-3, 1)	1
$Q_{13}Q_{36}X_{64}Q_{42}Q_{25}X_{51}$	(-4, 1)	1
$P_{13}P_{36}X_{64}P_{42}P_{25}Z_{51} = P_{13}P_{36}X_{64}X_{42}P_{25}R_{51} = P_{13}X_{36}X_{64}P_{42}P_{25}R_{51} = P_{13}P_{36}X_{64}P_{42}X_{25}R_{51}$ $= X_{13}P_{36}X_{64}P_{42}P_{25}R_{51}$	(1, 0)	1
$P_{13}P_{36}X_{64}Q_{42}P_{25}Z_{51} = P_{13}P_{36}X_{64}X_{42}P_{25}X_{51} = P_{13}Q_{36}X_{64}P_{42}P_{25}Z_{51} = P_{13}X_{36}X_{64}P_{42}P_{25}X_{51}$ $= P_{13}Q_{36}X_{64}X_{42}P_{25}R_{51} = P_{13}X_{36}X_{64}Q_{42}P_{25}R_{51} = P_{13}P_{36}X_{64}P_{42}Q_{25}Z_{51} = P_{13}P_{36}X_{64}P_{42}X_{25}X_{51}$ $= P_{13}P_{36}X_{64}X_{42}Q_{25}R_{51} = P_{13}P_{36}X_{64}Q_{42}X_{25}R_{51} = P_{13}X_{36}X_{64}P_{42}Q_{25}R_{51} = P_{13}Q_{36}X_{64}P_{42}X_{25}R_{51}$ $= Q_{13}P_{36}X_{64}P_{42}P_{25}Z_{51} = X_{13}P_{36}X_{64}P_{42}P_{25}X_{51} = Q_{13}P_{36}X_{64}X_{42}P_{25}R_{51} = X_{13}P_{36}X_{64}Q_{42}P_{25}R_{51}$ $= Q_{13}X_{36}X_{64}P_{42}P_{25}R_{51} = X_{13}Q_{36}X_{64}P_{42}P_{25}R_{51} = Q_{13}P_{36}X_{64}P_{42}X_{25}R_{51} = X_{13}P_{36}X_{64}P_{42}Q_{25}R_{51}$	(0, 0)	1

Table 21. The generators in terms of bifundamental chiral fields for Model 5 (*Part 1*).

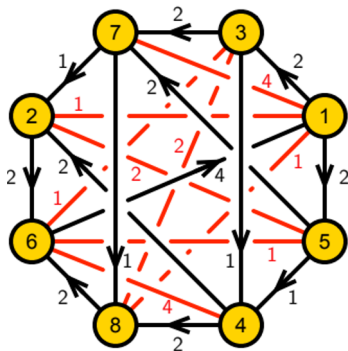


Figure 20. Quiver for Model 6.

Following the forward algorithm, we obtain the brick matchings. They are summarized in the P -matrix, which takes the form

$$P = \left(\begin{array}{c|cccccccc|cccccccc|cccc}
 & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & s_{10} & s_{11} & o_1 & o_2 & o_3 & o_4 \\
 \hline
 P_{15} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
 P_{26} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
 P_{37} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
 P_{48} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
 \hline
 Q_{15} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
 Q_{26} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
 Q_{37} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
 Q_{48} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
 \hline
 R_{61} & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
 Z_{61} & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
 \hline
 X_{13} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
 X_{34} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
 X_{42} & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
 X_{54} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
 X_{57} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
 X_{61} & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
 X_{72} & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
 X_{78} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
 X_{86} & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 \hline
 Y_{13} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
 Y_{42} & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
 Y_{57} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
 Y_{61} & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
 Y_{86} & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
 \end{array} \right). \tag{9.2}$$

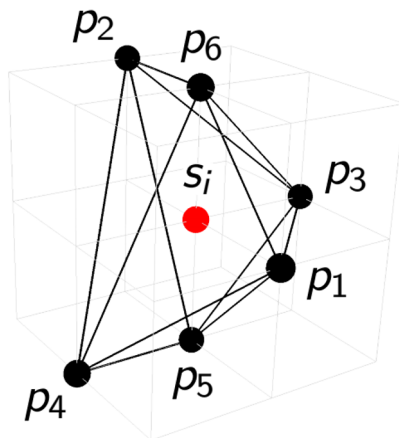


Figure 21. Toric diagram for Model 6.

The J - and E -term charges are given by

$$Q_{JE} = \left(\begin{array}{cccccc|cccccccc|cccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & s_{10} & s_{11} & o_1 & o_2 & o_3 & o_4 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & -1 & -1 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right), \quad (9.3)$$

and the D -term charges are given by

$$Q_D = \left(\begin{array}{cccccc|cccccccc|cccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & s_{10} & s_{11} & o_1 & o_2 & o_3 & o_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right). \quad (9.4)$$

The toric diagram of Model 6 is given by

$$G_t = \left(\begin{array}{cccccc|cccccccc|cccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & s_{10} & s_{11} & o_1 & o_2 & o_3 & o_4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & 1 \end{array} \right), \quad (9.5)$$

where figure 21 shows the toric diagram with brick matching labels.

	SU(2) _x	SU(2) _y	U(1) _b	U(1)	fugacity
p_1	+1	0	0	r_1	t_1
p_2	-1	0	0	r_2	t_2
p_3	0	+1	0	r_3	t_3
p_4	0	-1	0	r_4	t_4
p_5	0	0	+1	r_5	t_5
p_6	0	0	-1	r_6	t_6

Table 23. Global symmetry charges on the extremal brick matchings p_i of Model 6.

The Hilbert series of the mesonic moduli space of Model 6 takes the form

$$\begin{aligned}
 g_1(t_i, y_s, y_{o_1}, y_{o_2}; \mathcal{M}_6) &= \frac{P(t_i, y_s, y_{o_1}, y_{o_2}; \mathcal{M}_6)}{(1 - y_s y_{o_1}^3 y_{o_2} t_1 t_3^3 t_5^2)(1 - y_s y_{o_1}^3 y_{o_2} t_2 t_3^3 t_5^2)(1 - y_s y_{o_1}^3 y_{o_2} t_1 t_4^3 t_5^2)} \\
 &\times \frac{1}{(1 - y_s y_{o_1}^3 y_{o_2} t_2 t_4^3 t_5^2)(1 - y_s y_{o_1} y_{o_2}^3 t_1^3 t_3 t_6^2)(1 - y_s y_{o_1} y_{o_2}^3 t_2^3 t_3 t_6^2)} \\
 &\times \frac{1}{(1 - y_s y_{o_1} y_{o_2}^3 t_1^3 t_4 t_6^2)(1 - y_s y_{o_1} y_{o_2}^3 t_2^3 t_4 t_6^2)}, \tag{9.6}
 \end{aligned}$$

where t_i are the fugacities for the extremal brick matchings p_i . y_s counts the brick matching product $s_1 \dots s_{11}$ corresponding to the single internal point of the toric diagram of Model 6. Additionally, y_{o_1} and y_{o_2} count the products of extra GLSM fields $o_1 o_2$ and $o_3 o_4$, respectively. The explicit numerator $P(t_i, y_s, y_{o_1}, y_{o_2}; \mathcal{M}_6)$ of the Hilbert series is given in the appendix section A.6. We note that setting the fugacities $y_{o_1} = 1$ and $y_{o_2} = 1$ does not change the overall characterization of the mesonic moduli space by the Hilbert series, indicating that the extra GLSM fields, as expected, correspond to an over-parameterization of the moduli space.

By setting $t_i = t$ for the fugacities of the extremal brick matchings, and all other fugacities to $y_s = 1$, $y_{o_1} = 1$ and $y_{o_2} = 1$, the unrefined Hilbert series takes the following form

$$g_1(t, 1, 1, 1; \mathcal{M}_6) = \frac{1 + 21t^6 + 21t^{12} + t^{18}}{(1 - t^6)^4}, \tag{9.7}$$

where the palindromic numerator indicates that the mesonic moduli space is Calabi-Yau.

The global symmetry of Model 6 and the charges on the extremal brick matchings under the global symmetry are summarized in table 23. We can use the following fugacity map,

$$t = t_1^{1/2} t_2^{1/2}, \quad x = \frac{t_1^{1/2}}{t_2^{1/2}}, \quad y = \frac{t_3}{t_1^{1/2} t_2^{1/2}}, \quad b = \frac{t_1^{1/2} t_2^{1/2}}{t_6}, \tag{9.8}$$

where $t_3 = t_5 t_6 t_4^{-1}$ and $t_1 = t_5 t_6 t_2^{-1}$, in order to rewrite the Hilbert series for Model 6 in terms of characters of irreducible representations of $SU(2) \times SU(2) \times U(1)$. In terms of

fugacities of the mesonic flavor symmetry of Model 6, the Hilbert series becomes

$$\begin{aligned}
 g_1(t, x, y, b; \mathcal{M}_6) = & \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \left[[3n_1 + 2n_2; n_1 + 2n_2] b^{-2n_1} t^{6n_1+6n_2} \right. \\
 & + [3n_1 + n_2 + 1; 5n_1 + 3n_2 + 3] b^{2n_1+2n_2+2} t^{12n_1+6n_2+6} \\
 & + [3n_1 + 5n_2 + 3; 5n_1 + 7n_2 + 5] b^{2n_1+2n_2+2} t^{12n_1+18n_2+12} \\
 & + [7n_1 + 5n_2 + 5; 9n_1 + 7n_2 + 7] b^{2n_1+2n_2+2} t^{24n_1+18n_2+18} \\
 & \left. + [7n_1 + 2n_2 + 7; 9n_1 + 2n_2 + 9] b^{2n_1+2} t^{24n_1+6n_2+24} \right], \quad (9.9)
 \end{aligned}$$

where $[m; n] = [m]_{\text{SU}(2)_x} [n]_{\text{SU}(2)_y}$. The fugacity b counts charges under the $U(1)$ factor of the mesonic flavor symmetry. We can write the Hilbert series also in highest weight form, where each highest weight n of a character is counted by its own fugacity μ to give

$$h_1(t, \mu_1, \mu_2, b; \mathcal{M}_6) = \frac{1 + \mu_1^2 \mu_2^2 t^6}{(1 - \mu_1^3 \mu_2 b^{-2} t^6)(1 - \mu_1 \mu_2^3 b^2 t^6)}, \quad (9.10)$$

where $\mu_1^m \mu_2^n \sim [m]_{\text{SU}(2)_x} [n]_{\text{SU}(2)_y}$. The plethystic logarithm of the Hilbert series in (9.9) is

$$\begin{aligned}
 \text{PL}[g_1(t, x, y, b; \mathcal{M}_6)] = & ([1; 3]b^2 + [2; 2] + [3; 1]b^{-2})t^6 - (1 + b^{-4} + b^4 + [0; 4] \\
 & + [0; 4]b^4 + [1; 1]b^{-2} + [1; 1]b^2 + [1; 3]b^{-2} + [1; 3]b^2 + [1; 5]b^2 + 2[2; 2] + [2; 2]b^{-4} \\
 & + [2; 2]b^4 + [2; 4] + [3; 1]b^{-2} + [3; 1]b^2 + [3; 3]b^{-2} + [3; 3]b^2 + [4; 0] + [4; 0]b^{-4} \\
 & + [4; 2] + [4; 4] + [5; 1]b^{-2})t^{12} + \dots, \quad (9.11)
 \end{aligned}$$

where $[m; n] = [m]_{\text{SU}(2)_x} [n]_{\text{SU}(2)_y}$. From the plethystic logarithm, we see that the mesonic moduli space is a non-complete intersection.

The generators form 3 sets that transform under $[1; 3]b^2$, $[2; 2]$ and $[3; 1]b^{-2}$ of the mesonic flavor symmetry of Model 6, respectively. Using the following fugacity map

$$\tilde{t} = t_5^{1/2} t_6^{1/2}, \quad \tilde{x} = \frac{t_1}{t_2}, \quad \tilde{y} = \frac{t_3}{t_4}, \quad \tilde{b} = \frac{t_2 t_4}{t_6^2} \quad (9.12)$$

the mesonic flavor charges on the gauge invariant operators become \mathbb{Z} -valued. The generators in terms of brick matchings and their corresponding rescaled mesonic flavor charges are summarized in table 24. The generator lattice as shown in table 24 is a convex lattice polytope, which is reflexive. It is the dual of the toric diagram of Model 6 shown in figure 21. We also note that the 3 layers of points in the generator lattice in table 24 corresponds to the 3 sets that transform under $[1, 0]b^{-2}$, $[3, 0]$ and $[5, 0]b^2$ of the mesonic flavor symmetry. For completeness, table 25 and table 26 show the generators of Model 6 in terms of chiral fields with the corresponding mesonic flavor charges.

generator	SU(2) _{\bar{x}}	SU(2) _{\bar{y}}	U(1) _{\bar{b}}
$p_1 p_3^3 p_5^2 \text{ } so_1^3 o_2$	1	2	1
$p_2 p_3^3 p_5^2 \text{ } so_1^3 o_2$	0	2	1
$p_1 p_3^2 p_4 p_5^2 \text{ } so_1^3 o_2$	1	1	1
$p_2 p_3^2 p_4 p_5^2 \text{ } so_1^3 o_2$	0	1	1
$p_1 p_3 p_4^2 p_5^2 \text{ } so_1^3 o_2$	1	0	1
$p_2 p_3 p_4^2 p_5^2 \text{ } so_1^3 o_2$	0	0	1
$p_1 p_4^3 p_5^2 \text{ } so_1^3 o_2$	1	-1	1
$p_2 p_4^3 p_5^2 \text{ } so_1^3 o_2$	0	-1	1
$p_1^2 p_3^2 p_5 p_6 \text{ } so_1^2 o_2^2$	1	1	0
$p_1 p_2 p_3^2 p_5 p_6 \text{ } so_1^2 o_2^2$	0	1	0
$p_2^2 p_3^2 p_5 p_6 \text{ } so_1^2 o_2^2$	-1	1	0
$p_1^2 p_3 p_4 p_5 p_6 \text{ } so_1^2 o_2^2$	1	0	0
$p_1 p_2 p_3 p_4 p_5 p_6 \text{ } so_1^2 o_2^2$	0	0	0
$p_2^2 p_3 p_4 p_5 p_6 \text{ } so_1^2 o_2^2$	-1	0	0
$p_1^2 p_4^2 p_5 p_6 \text{ } so_1^2 o_2^2$	1	-1	0
$p_1 p_2 p_4^2 p_5 p_6 \text{ } so_1^2 o_2^2$	0	-1	0
$p_2^2 p_4^2 p_5 p_6 \text{ } so_1^2 o_2^2$	-1	-1	0
$p_1^3 p_3 p_6^2 \text{ } so_1 o_2^3$	1	0	-1
$p_1^2 p_2 p_3 p_6^2 \text{ } so_1 o_2^3$	0	0	-1
$p_1 p_2^2 p_3 p_6^2 \text{ } so_1 o_2^3$	-1	0	-1
$p_3^3 p_3 p_6^2 \text{ } so_1 o_2^3$	-2	0	-1
$p_1^3 p_4 p_6^2 \text{ } so_1 o_2^3$	1	-1	-1
$p_1^2 p_2 p_4 p_6^2 \text{ } so_1 o_2^3$	0	-1	-1
$p_1 p_2^2 p_4 p_6^2 \text{ } so_1 o_2^3$	-1	-1	-1
$p_2^3 p_4 p_6^2 \text{ } so_1 o_2^3$	-2	-1	-1

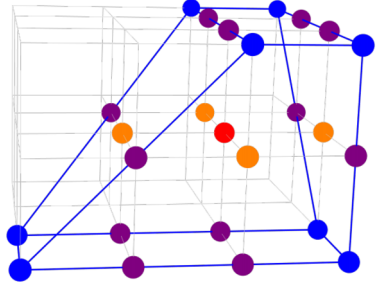


Table 24. The generators and lattice of generators of the mesonic moduli space of Model 6 in terms of brick matchings with the corresponding flavor charges.

generator	SU(2) _{\tilde{x}}	SU(2) _{\tilde{y}}	U(1) _{\tilde{z}}
$P_{15}X_{57}X_{78}X_{86}Z_{61} = X_{13}X_{34}X_{42}P_{26}Z_{61} = X_{13}P_{37}X_{78}X_{86}Z_{61} = X_{13}X_{34}P_{48}X_{86}Z_{61}$	1	2	1
$Q_{15}X_{57}X_{78}X_{86}Z_{61} = X_{13}X_{34}X_{42}Q_{26}Z_{61} = X_{13}Q_{37}X_{78}X_{86}Z_{61} = X_{13}X_{34}Q_{48}X_{86}Z_{61}$	0	2	1
$P_{15}X_{57}X_{78}X_{86}X_{61} = P_{15}X_{57}X_{78}Y_{86}Z_{61} = P_{15}Y_{57}X_{78}X_{86}Z_{61} = X_{13}X_{34}X_{42}P_{26}X_{61} = X_{13}X_{34}Y_{42}P_{26}Z_{61}$ $= Y_{13}X_{34}X_{42}P_{26}Z_{61} = X_{13}P_{37}X_{78}X_{86}X_{61} = X_{13}P_{37}X_{78}Y_{86}Z_{61} = Y_{13}P_{37}X_{78}X_{86}Z_{61} = X_{13}X_{34}P_{48}X_{86}X_{61}$ $= X_{13}X_{34}P_{48}Y_{86}Z_{61} = Y_{13}X_{34}P_{48}X_{86}Z_{61}$	1	1	1
$Q_{15}X_{57}X_{78}X_{86}X_{61} = Q_{15}X_{57}X_{78}Y_{86}Z_{61} = Q_{15}Y_{57}X_{78}X_{86}Z_{61} = X_{13}X_{34}X_{42}Q_{26}X_{61} = X_{13}X_{34}Y_{42}Q_{26}Z_{61}$ $= Y_{13}X_{34}X_{42}Q_{26}Z_{61} = X_{13}Q_{37}X_{78}X_{86}X_{61} = X_{13}Q_{37}X_{78}Y_{86}Z_{61} = Y_{13}Q_{37}X_{78}X_{86}Z_{61} = X_{13}X_{34}Q_{48}X_{86}X_{61}$ $= X_{13}X_{34}Q_{48}Y_{86}Z_{61} = Y_{13}X_{34}Q_{48}X_{86}Z_{61}$	0	1	1
$P_{15}X_{57}X_{78}Y_{86}X_{61} = P_{15}Y_{57}X_{78}X_{86}X_{61} = P_{15}Y_{57}X_{78}Y_{86}Z_{61} = X_{13}X_{34}Y_{42}P_{26}X_{61} = Y_{13}X_{34}X_{42}P_{26}X_{61}$ $= Y_{13}X_{34}Y_{42}P_{26}Z_{61} = X_{13}P_{37}X_{78}Y_{86}X_{61} = Y_{13}P_{37}X_{78}X_{86}X_{61} = Y_{13}P_{37}X_{78}Y_{86}Z_{61} = X_{13}X_{34}P_{48}Y_{86}X_{61}$ $= Y_{13}X_{34}P_{48}X_{86}X_{61} = Y_{13}X_{34}P_{48}Y_{86}Z_{61}$	1	0	1
$Q_{15}X_{57}X_{78}Y_{86}X_{61} = Q_{15}Y_{57}X_{78}X_{86}X_{61} = Q_{15}Y_{57}X_{78}Y_{86}Z_{61} = X_{13}X_{34}Y_{42}Q_{26}X_{61} = Y_{13}X_{34}X_{42}Q_{26}X_{61}$ $= Y_{13}X_{34}Y_{42}Q_{26}Z_{61} = X_{13}Q_{37}X_{78}Y_{86}X_{61} = Y_{13}Q_{37}X_{78}X_{86}X_{61} = Y_{13}Q_{37}X_{78}Y_{86}Z_{61} = X_{13}X_{34}Q_{48}Y_{86}X_{61}$ $= Y_{13}X_{34}Q_{48}X_{86}X_{61} = Y_{13}X_{34}Q_{48}Y_{86}Z_{61}$	0	0	1
$P_{15}Y_{57}X_{78}Y_{86}X_{61} = Y_{13}X_{34}Y_{42}P_{26}X_{61} = Y_{13}P_{37}X_{78}Y_{86}X_{61} = Y_{13}X_{34}P_{48}Y_{86}X_{61}$	1	-1	1
$Q_{15}Y_{57}X_{78}Y_{86}X_{61} = Y_{13}X_{34}Y_{42}Q_{26}X_{61} = Y_{13}Q_{37}X_{78}Y_{86}X_{61} = Y_{13}X_{34}Q_{48}Y_{86}X_{61}$	0	-1	1
$P_{15}X_{54}X_{42}P_{26}Z_{61} = P_{15}X_{54}Y_{72}P_{26}Z_{61} = P_{15}X_{54}P_{48}X_{86}Z_{61} = P_{15}X_{57}X_{78}X_{86}R_{61} = X_{13}P_{37}X_{72}P_{26}Z_{61}$ $= X_{13}X_{34}X_{42}P_{26}R_{61} = X_{13}P_{37}X_{78}X_{86}R_{61} = X_{13}X_{34}P_{48}X_{86}R_{61}$	1	1	0
$P_{15}X_{54}X_{42}Q_{26}Z_{61} = P_{15}X_{57}X_{72}Q_{26}Z_{61} = P_{15}X_{54}Q_{48}X_{86}Z_{61} = P_{15}X_{57}X_{78}X_{86}Y_{61} = Q_{15}X_{54}X_{42}P_{26}Z_{61}$ $= Q_{15}X_{57}X_{72}P_{26}Z_{61} = X_{13}Q_{37}X_{72}P_{26}Z_{61} = X_{13}X_{34}X_{42}P_{26}Y_{61} = X_{13}P_{37}X_{72}Q_{26}Z_{61} = X_{13}P_{37}X_{78}X_{86}Y_{61}$ $= Q_{15}X_{54}P_{48}X_{86}Z_{61} = X_{13}X_{34}P_{48}X_{86}Y_{61} = Q_{15}X_{57}X_{78}X_{86}R_{61} = X_{13}X_{34}X_{42}Q_{26}R_{61} = X_{13}Q_{37}X_{78}X_{86}R_{61}$ $= X_{13}X_{34}Q_{48}X_{86}R_{61}$	0	1	0
$Q_{15}X_{54}X_{42}Q_{26}Z_{61} = Q_{15}X_{57}X_{72}Q_{26}Z_{61} = Q_{15}X_{54}Q_{48}X_{86}Z_{61} = Q_{15}X_{57}X_{78}X_{86}Y_{61} = X_{13}Q_{37}X_{72}Q_{26}Z_{61}$ $= X_{13}X_{34}X_{42}Q_{26}Y_{61} = X_{13}Q_{37}X_{78}X_{86}Y_{61} = X_{13}X_{34}Q_{48}X_{86}Y_{61}$	-1	1	0
$P_{15}X_{54}X_{42}P_{26}X_{61} = P_{15}X_{54}Y_{42}P_{26}Z_{61} = P_{15}X_{57}X_{72}P_{26}X_{61} = P_{15}Y_{57}X_{72}P_{26}Z_{61} = P_{15}X_{54}P_{48}X_{86}X_{61}$ $= P_{15}X_{54}P_{48}Y_{86}Z_{61} = P_{15}X_{57}X_{78}Y_{86}R_{61} = P_{15}Y_{57}X_{78}X_{86}R_{61} = X_{13}P_{37}X_{72}P_{26}X_{61} = Y_{13}P_{37}X_{72}P_{26}Z_{61}$ $= X_{13}X_{34}Y_{42}P_{26}R_{61} = Y_{13}X_{34}X_{42}P_{26}R_{61} = X_{13}P_{37}X_{78}Y_{86}R_{61} = Y_{13}P_{37}X_{78}X_{86}R_{61} = X_{13}X_{34}P_{48}Y_{86}R_{61}$ $= Y_{13}X_{34}P_{48}X_{86}R_{61}$	1	0	0

Table 25. The generators in terms of bifundamental chiral fields for Model 6 (*Part 1*).

10 Model 7: $P_{++-}(\text{dP}_0)$ [\mathbb{P}^1 -blowup of 5, $\langle 25 \rangle$]

Model 7 corresponds to the Calabi-Yau 4-fold $P_{++-}(\text{dP}_0)$. The corresponding brane brick model has the quiver in figure 22 and the J - and E -terms are given as follows

$$\begin{array}{lll}
 & J & E \\
 \Lambda_{25}^1 : & X_{51}X_{12} - Z_{51}Y_{12} & P_{24}X_{48}Q_{85} - X_{28}P_{83}P_{35} \\
 \Lambda_{25}^2 : & Y_{51}Y_{12} - X_{51}Z_{17}P_{72} & P_{24}Y_{48}Q_{85} - Y_{23}P_{35} \\
 \Lambda_{25}^3 : & Z_{51}Z_{17}P_{72} - Y_{51}X_{12} & P_{24}Z_{48}Q_{85} - Z_{23}P_{35} \\
 \Lambda_{31}^1 : & Y_{12}Y_{23} - X_{12}Z_{23} & P_{35}Y_{51} - Y_{36}P_{61} \\
 \Lambda_{62}^1 : & X_{28}X_{86} - Z_{23}Y_{36} & P_{61}X_{12} - X_{67}P_{72} \\
 \Lambda_{62}^2 : & Y_{23}Y_{36} - X_{28}Z_{86} & P_{61}Y_{12} - Y_{67}P_{72} \\
 \Lambda_{64}^1 : & X_{48}X_{86} - Z_{48}P_{83}Y_{36} & X_{67}Q_{74} - Q_{61}X_{12}P_{24} \\
 \Lambda_{64}^2 : & Y_{48}P_{83}Y_{36} - X_{48}Z_{86} & Y_{67}Q_{74} - Q_{61}Y_{12}P_{24} \\
 \Lambda_{64}^3 : & Z_{48}Z_{86} - Y_{48}X_{86} & P_{61}Z_{17}Q_{74} - Q_{61}Z_{17}P_{72}P_{24} \\
 \Lambda_{73}^1 : & Y_{36}Y_{67} - P_{35}X_{51}Z_{17} & P_{72}Y_{23} - Q_{74}Y_{48}P_{83} \\
 \Lambda_{73}^2 : & P_{35}Z_{51}Z_{17} - Y_{36}X_{67} & P_{72}Z_{23} - Q_{74}Z_{48}P_{83} \\
 \Lambda_{78}^1 : & X_{86}X_{67} - Z_{86}Y_{67} & P_{72}X_{28} - Q_{74}X_{48} \\
 \Lambda_{81}^1 : & X_{12}P_{24}X_{48} - Z_{17}P_{72}P_{24}Y_{48} & X_{86}Q_{61} - Q_{85}X_{51} \\
 \Lambda_{81}^2 : & Y_{12}P_{24}Y_{48} - X_{12}P_{24}Z_{48} & P_{83}Y_{36}Q_{61} - Q_{85}Y_{51} \\
 \Lambda_{81}^3 : & Z_{17}P_{72}P_{24}Z_{48} - Y_{12}P_{24}X_{48} & Z_{86}Q_{61} - Q_{85}Z_{51} \\
 \Lambda_{81}^4 : & X_{12}X_{28} - Z_{17}Q_{74}Y_{48} & P_{83}P_{35}X_{51} - X_{86}P_{61} \\
 \Lambda_{81}^5 : & Z_{17}Q_{74}Z_{48} - Y_{12}X_{28} & P_{83}P_{35}Z_{51} - Z_{86}P_{61}
 \end{array} \quad (10.1)$$

generator	SU(2) _x	SU(2) _y	U(1) _b
$P_{15}X_{54}X_{42}Q_{26}X_{61} = P_{15}X_{54}Y_{42}Q_{26}Z_{61} = P_{15}X_{57}X_{72}Q_{26}X_{61} = P_{15}Y_{57}X_{72}Q_{26}Z_{61} = P_{15}X_{54}Q_{48}X_{86}X_{61}$ $= P_{15}X_{54}Q_{48}Y_{86}Z_{61} = P_{15}X_{57}X_{78}Y_{86}Y_{61} = P_{15}Y_{57}X_{78}X_{86}Y_{61} = Q_{15}X_{54}X_{42}P_{26}X_{61} = Q_{15}X_{54}Y_{42}P_{26}Z_{61}$ $= Q_{15}X_{57}X_{72}P_{26}X_{61} = Q_{15}Y_{57}X_{72}P_{26}Z_{61} = X_{13}Q_{37}X_{72}P_{26}X_{61} = Y_{13}Q_{37}X_{72}P_{26}Z_{61} = X_{13}X_{34}Y_{42}P_{26}Y_{61}$ $= Y_{13}X_{34}X_{42}P_{26}Y_{61} = X_{13}P_{37}X_{72}Q_{26}X_{61} = Y_{13}P_{37}X_{72}Q_{26}Z_{61} = X_{13}P_{37}X_{78}Y_{86}Y_{61} = Y_{13}P_{37}X_{78}X_{86}Y_{61}$ $= Q_{15}X_{54}P_{48}X_{86}X_{61} = Q_{15}X_{54}P_{48}Y_{86}Z_{61} = X_{13}X_{34}P_{48}Y_{86}Y_{61} = Y_{13}X_{34}P_{48}X_{86}Y_{61} = Q_{15}X_{57}X_{78}Y_{86}R_{61}$ $= Q_{15}Y_{57}X_{78}X_{86}R_{61} = X_{13}X_{34}Y_{42}Q_{26}R_{61} = Y_{13}X_{34}X_{42}Q_{26}R_{61} = X_{13}Q_{37}X_{78}Y_{86}R_{61} = Y_{13}Q_{37}X_{78}X_{86}R_{61}$ $= X_{13}X_{34}Q_{48}Y_{86}R_{61} = Y_{13}X_{34}Q_{48}X_{86}R_{61}$	0	0	0
$Q_{15}X_{54}X_{42}Q_{26}X_{61} = Q_{15}X_{54}Y_{42}Q_{26}Z_{61} = Q_{15}X_{57}X_{72}Q_{26}X_{61} = Q_{15}Y_{57}X_{72}Q_{26}Z_{61} = Q_{15}X_{54}Q_{48}X_{86}X_{61}$ $= Q_{15}X_{54}Q_{48}Y_{86}Z_{61} = Q_{15}X_{57}X_{78}Y_{86}Y_{61} = Q_{15}Y_{57}X_{78}X_{86}Y_{61} = X_{13}Q_{37}X_{72}Q_{26}X_{61} = Y_{13}Q_{37}X_{72}Q_{26}Z_{61}$ $= X_{13}X_{34}Y_{42}Q_{26}Y_{61} = Y_{13}X_{34}X_{42}Q_{26}Y_{61} = X_{13}Q_{37}X_{78}Y_{86}Y_{61} = Y_{13}Q_{37}X_{78}X_{86}Y_{61} = X_{13}X_{34}Q_{48}Y_{86}Y_{61}$ $= Y_{13}X_{34}Q_{48}X_{86}Y_{61}$	-1	0	0
$P_{15}X_{54}Y_{42}P_{26}X_{61} = P_{15}Y_{57}X_{72}P_{26}X_{61} = P_{15}X_{54}P_{48}Y_{86}X_{61} = P_{15}Y_{57}X_{78}Y_{86}R_{61} = Y_{13}P_{37}X_{72}P_{26}X_{61}$ $= Y_{13}X_{34}Y_{42}P_{26}R_{61} = Y_{13}P_{37}X_{78}Y_{86}R_{61} = Y_{13}X_{34}P_{48}Y_{86}R_{61}$	1	-1	0
$P_{15}X_{54}Y_{42}Q_{26}X_{61} = P_{15}Y_{57}X_{72}Q_{26}X_{61} = P_{15}X_{54}Q_{48}Y_{86}X_{61} = P_{15}Y_{57}X_{78}Y_{86}Y_{61} = Q_{15}X_{54}Y_{42}P_{26}X_{61}$ $= Q_{15}Y_{57}X_{72}P_{26}X_{61} = Y_{13}Q_{37}X_{72}P_{26}X_{61} = Y_{13}X_{34}Y_{42}P_{26}Y_{61} = Y_{13}P_{37}X_{72}Q_{26}X_{61} = Y_{13}P_{37}X_{78}Y_{86}Y_{61}$ $= Q_{15}X_{54}P_{48}Y_{86}X_{61} = Y_{13}X_{34}P_{48}Y_{86}Y_{61} = Q_{15}Y_{57}X_{78}Y_{86}R_{61} = Y_{13}X_{34}Y_{42}Q_{26}R_{61} = Y_{13}Q_{37}X_{78}Y_{86}R_{61}$ $= Y_{13}X_{34}Q_{48}Y_{86}R_{61}$	0	-1	0
$Q_{15}X_{54}Y_{42}Q_{26}X_{61} = Q_{15}Y_{57}X_{72}Q_{26}X_{61} = Q_{15}X_{54}Q_{48}Y_{86}X_{61} = Q_{15}Y_{57}X_{78}Y_{86}Y_{61} = Y_{13}Q_{37}X_{72}Q_{26}X_{61}$ $= Y_{13}X_{34}Y_{42}Q_{26}Y_{61} = Y_{13}Q_{37}X_{78}Y_{86}Y_{61} = Y_{13}X_{34}Q_{48}Y_{86}Y_{61}$	-1	-1	0
$P_{15}X_{54}X_{42}P_{26}R_{61} = P_{15}X_{57}X_{72}P_{26}R_{61} = P_{15}X_{54}P_{48}X_{86}R_{61} = X_{13}P_{37}X_{72}P_{26}R_{61}$	1	0	-1
$P_{15}X_{54}X_{42}P_{26}Y_{61} = P_{15}X_{57}X_{72}P_{26}Y_{61} = P_{15}X_{54}P_{48}X_{86}Y_{61} = P_{15}X_{54}X_{42}Q_{26}R_{61} = P_{15}X_{57}X_{72}Q_{26}R_{61}$ $= P_{15}X_{54}Q_{48}X_{86}R_{61} = X_{13}P_{37}X_{72}P_{26}Y_{61} = Q_{15}X_{54}X_{42}P_{26}R_{61} = Q_{15}X_{57}X_{72}P_{26}R_{61} = X_{13}Q_{37}X_{72}P_{26}R_{61}$ $= X_{13}P_{37}X_{72}Q_{26}R_{61} = Q_{15}X_{54}P_{48}X_{86}R_{61}$	0	0	-1
$P_{15}X_{54}X_{42}Q_{26}Y_{61} = P_{15}X_{57}X_{72}Q_{26}Y_{61} = P_{15}X_{54}Q_{48}X_{86}Y_{61} = Q_{15}X_{54}X_{42}P_{26}Y_{61} = Q_{15}X_{57}X_{72}P_{26}Y_{61}$ $= X_{13}Q_{37}X_{72}P_{26}Y_{61} = X_{13}P_{37}X_{72}Q_{26}Y_{61} = Q_{15}X_{54}P_{48}X_{86}Y_{61} = Q_{15}X_{54}X_{42}Q_{26}R_{61} = Q_{15}X_{57}X_{72}Q_{26}R_{61}$ $= Q_{15}X_{54}Q_{48}X_{86}R_{61} = X_{13}Q_{37}X_{72}Q_{26}R_{61}$	-1	0	-1
$Q_{15}X_{54}X_{42}Q_{26}Y_{61} = Q_{15}X_{57}X_{72}Q_{26}Y_{61} = Q_{15}X_{54}Q_{48}X_{86}Y_{61} = X_{13}Q_{37}X_{72}Q_{26}Y_{61}$	-2	0	-1
$P_{15}X_{54}Y_{42}P_{26}R_{61} = P_{15}Y_{57}X_{72}P_{26}R_{61} = P_{15}X_{54}P_{48}Y_{86}R_{61} = Y_{13}P_{37}X_{72}P_{26}R_{61}$	1	-1	-1
$P_{15}X_{54}Y_{42}P_{26}Y_{61} = P_{15}Y_{57}X_{72}P_{26}Y_{61} = P_{15}X_{54}P_{48}Y_{86}Y_{61} = P_{15}X_{54}Y_{42}Q_{26}R_{61} = P_{15}Y_{57}X_{72}Q_{26}R_{61}$ $= P_{15}X_{54}Q_{48}Y_{86}R_{61} = Y_{13}P_{37}X_{72}P_{26}Y_{61} = Q_{15}X_{54}Y_{42}P_{26}R_{61} = Q_{15}Y_{57}X_{72}P_{26}R_{61} = Y_{13}Q_{37}X_{72}P_{26}R_{61}$ $= Y_{13}P_{37}X_{72}Q_{26}R_{61} = Q_{15}X_{54}P_{48}Y_{86}R_{61}$	0	-1	-1
$P_{15}X_{54}Y_{42}Q_{26}Y_{61} = P_{15}Y_{57}X_{72}Q_{26}Y_{61} = P_{15}X_{54}Q_{48}Y_{86}Y_{61} = Q_{15}X_{54}Y_{42}P_{26}Y_{61} = Q_{15}Y_{57}X_{72}P_{26}Y_{61}$ $= Y_{13}Q_{37}X_{72}P_{26}Y_{61} = Y_{13}P_{37}X_{72}Q_{26}Y_{61} = Q_{15}X_{54}P_{48}Y_{86}Y_{61} = Q_{15}X_{54}Y_{42}Q_{26}R_{61} = Q_{15}Y_{57}X_{72}Q_{26}R_{61}$ $= Q_{15}X_{54}Q_{48}Y_{86}R_{61} = Y_{13}Q_{37}X_{72}Q_{26}R_{61}$	-1	-1	-1
$Q_{15}X_{54}Y_{42}Q_{26}Y_{61} = Q_{15}Y_{57}X_{72}Q_{26}Y_{61} = Q_{15}X_{54}Q_{48}Y_{86}Y_{61} = Y_{13}Q_{37}X_{72}Q_{26}Y_{61}$	-2	-1	-1

Table 26. The generators in terms of bifundamental chiral fields for Model 6 (*Part 2*).

The P -matrix for Model 7 is

$$P = \begin{pmatrix}
 & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & o_7 & o_8 \\
 P_{24} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
 P_{35} & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
 P_{61} & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 P_{72} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
 P_{83} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
 Q_{61} & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
 Q_{74} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
 Q_{85} & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
 X_{12} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
 X_{28} & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
 X_{48} & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 1 & 1 & 1 & 0 & 1 & 0 \\
 X_{51} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
 X_{67} & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
 X_{86} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 Y_{12} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
 Y_{23} & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
 Y_{36} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
 Y_{48} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
 Y_{51} & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 0 \\
 Y_{67} & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
 Z_{17} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
 Z_{23} & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
 Z_{48} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
 Z_{51} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
 Z_{86} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
 \end{pmatrix}. \tag{10.2}$$

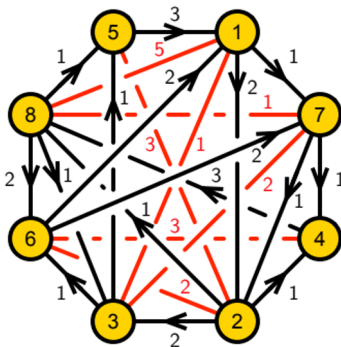


Figure 22. Quiver for Model 7.

The J - and E -term charges are given by

$$Q_{JE} = \left(\begin{array}{cccccc|cccccc|cccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & o_7 & o_8 \\ \hline 1 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right), \tag{10.3}$$

and the D -term charges are given by

$$Q_D = \left(\begin{array}{cccccc|cccccc|cccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & o_7 & o_8 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right). \tag{10.4}$$

The toric diagram of Model 7 is given by

$$G_t = \left(\begin{array}{cccccc|cccccc|cccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & o_7 & o_8 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{array} \right), \tag{10.5}$$

where figure 23 shows the toric diagram with brick matching labels.

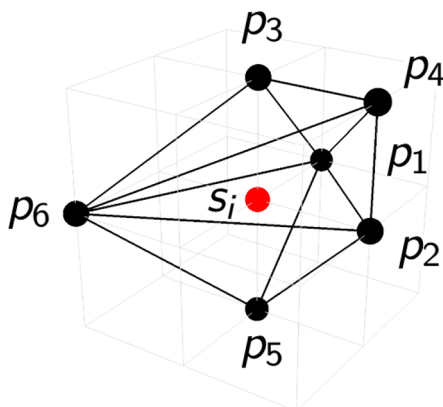


Figure 23. Toric diagram for Model 7.

The Hilbert series of the mesonic moduli space of Model 7 takes the form

$$\begin{aligned}
 g_1(t_i, y_s, y_{o_1}, y_{o_2}, y_{o_3}; \mathcal{M}_7) &= \frac{P(t_i, y_s, y_{o_1}, y_{o_2}, y_{o_3}; \mathcal{M}_7)}{(1 - y_s y_{o_1}^2 y_{o_2} y_{o_3}^3 t_1^3 t_3^2 t_4)(1 - y_s y_{o_1}^5 y_{o_2}^4 y_{o_3}^3 t_2^3 t_3^2 t_4^4)} \\
 &\times \frac{1}{(1 - y_s y_{o_1} y_{o_2} y_{o_3}^2 t_1^3 t_3 t_5)(1 - y_s y_{o_1} y_{o_2}^2 y_{o_3} t_1^2 t_2 t_5^2)(1 - y_s y_{o_1}^3 y_{o_2}^4 y_{o_3} t_2^3 t_4^2 t_5^2)} \\
 &\times \frac{1}{(1 - y_s y_{o_1} y_{o_2}^2 y_{o_3} t_2 t_5^2 t_6^2)(1 - y_s y_{o_1}^2 y_{o_2} y_{o_3}^3 t_3^2 t_4 t_6^3)(1 - y_s y_{o_1} y_{o_2} y_{o_3}^2 t_3 t_5 t_6^3)}, \quad (10.6)
 \end{aligned}$$

where t_i are the fugacities for the extremal brick matchings p_i . y_s counts the brick matching product $s_1 \dots s_9$ corresponding to the single internal point of the toric diagram of Model 7. Additionally, y_{o_1} , y_{o_2} and y_{o_3} count the products of extra GLSM fields $o_1 o_2 o_3$, $o_4 o_5 o_6$ and $o_7 o_8$, respectively. The explicit numerator $P(t_i, y_s, y_{o_1}, y_{o_2}, y_{o_3}; \mathcal{M}_7)$ of the Hilbert series is given in the appendix section A.7. We note that setting the fugacities $y_{o_1} = 1$, $y_{o_2} = 1$ and $y_{o_3} = 1$ does not change the overall characterization of the mesonic moduli space by the Hilbert series, indicating that the extra GLSM fields, as expected, correspond to an over-parameterization of the moduli space.

By setting $t_i = t$ for the fugacities of the extremal brick matchings, and all other fugacities to $y_s = 1$, $y_{o_1} = 1$, $y_{o_2} = 1$ and $y_{o_3} = 1$, the unrefined Hilbert series takes the following form

$$\begin{aligned}
 g_1(t, 1, 1, 1, 1; \mathcal{M}_7) &= \frac{1}{(1 - t^5)^3 (1 - t^6)^2 (1 - t^7) (1 - t^9)} \\
 &\times (1 + 4t^5 + 7t^6 + 5t^7 + 3t^8 - 13t^{11} - 14t^{12} - 8t^{13} - 4t^{14} - 2t^{15} + t^{16} + 2t^{17} \\
 &\quad + 6t^{18} - 6t^{19} - 2t^{20} - t^{21} + 2t^{22} + 4t^{23} + 8t^{24} + 14t^{25} + 13t^{26} - 3t^{29} - 5t^{30} \\
 &\quad - 7t^{31} - 4t^{32} - t^{37}), \quad (10.7)
 \end{aligned}$$

where the palindromic numerator indicates that the mesonic moduli space is Calabi-Yau.

The global symmetry of Model 6 and the charges on the extremal brick matchings under

	SU(2) _x	U(1) _{b₁}	U(1) _{b₂}	U(1)	fugacity
p_1	+1	0	0	r_1	t_1
p_2	0	+1	0	r_2	t_2
p_3	0	-1	0	r_3	t_3
p_4	0	0	+1	r_4	t_4
p_5	0	0	-1	r_5	t_5
p_6	-1	0	0	r_6	t_6

Table 27. Global symmetry charges on the extremal brick matchings p_i of Model 7.

the global symmetry are summarized in table 27. We can use the following fugacity map,

$$t = \frac{t_1 t_6}{t_4^{1/2} t_5^{1/2}}, \quad x = \frac{t_1}{t_4^{1/2} t_5^{1/2}}, \quad b_1 = \frac{t_2}{t_4^{1/2} t_5^{1/2}}, \quad b_2 = \frac{t_4^{1/2}}{t_5^{1/2}}, \quad (10.8)$$

where $t_2 t_3 = t_4 t_5$ and $t_1 t_6 = t_4 t_5$, in order to rewrite the Hilbert series for Model 6 in terms of characters of irreducible representations of SU(2) × U(1) × U(1). In terms of fugacities of the mesonic flavor symmetry of Model 7, the Hilbert series in highest weight form can be written as

$$\begin{aligned}
 h_1(t, \mu, b_1, b_2; \mathcal{M}_7) &= \frac{1}{(1 - b_1^3 t^7)(1 - b_1 b_2^4 t^9)(1 - \mu^2 b_1 b_2^{-2} t^5)} \\
 &\times \frac{1}{(1 - \mu^3 b_1^{-1} b_2^{-1} t^5)(1 - \mu^3 b_1^{-2} b_2 t^6)} \times (1 + \mu b_1^2 b_2^{-1} t^6 + \mu^2 t^6 + \mu b_1 b_2 t^7 \\
 &+ \mu^2 b_1^{-1} b_2^2 t^7 + b_1^2 b_2^2 t^8 + \mu b_2^3 t^8 - \mu^4 b_1 b_2^{-2} t^{11} - \mu^5 b_1^{-1} b_2^{-1} t^{11} - \mu^3 b_1^2 b_2^{-1} t^{12} \\
 &- 2\mu^4 t^{12} - \mu^5 b_1^{-2} b_2 t^{12} - \mu^2 b_1^3 t^{13} - 2\mu^3 b_1 b_2 t^{13} - \mu^4 b_1^{-1} b_2^2 t^{13} - \mu^2 b_1^2 b_2^2 t^{14} \\
 &- \mu^3 b_2^3 t^{14} - \mu b_1^3 b_2^3 t^{15} + \mu^7 b_1^{-1} b_2^{-1} t^{17} + \mu^5 b_1^2 b_2^{-1} t^{18} + \mu^6 t^{18} + \mu^5 b_1 b_2 t^{19} \\
 &+ \mu^7 b_1 b_2 t^{25} + \mu^6 b_1^2 b_2^2 t^{26}), \quad (10.9)
 \end{aligned}$$

where $\mu^m \sim [m]_{\text{SU}(2)_x}$. Note that in highest weight form, the fugacity μ counts the highest weight of the irreducible representations of SU(2). Additionally, the fugacities b_1 and b_2 count the charges under the U(1) factors of the mesonic flavor symmetry.

The plethystic logarithm of the Hilbert series takes the following form

$$\begin{aligned}
 \text{PL}[g_1(t, x, b_1, b_2; \mathcal{M}_7)] &= ([2]b_1 b_2^{-2} + [3]b_1^{-1} b_2^{-1})t^5 + ([1]b_1^2 b_2^{-1} + [2] + [3]b_1^{-2} b_2^1)t^6 \\
 &+ (b_1^3 + [1]b_1 b_2 + [2]b_1^{-1} b_2^2)t^7 + (b_1^2 b_2^2 + [1]b_2^3)t^8 + b_1 b_2^4 t^9 \\
 &- ([3]b_2^{-3} + b_1^2 b_2^{-4} + [1]b_2^{-3} + [2]b_1^{-2} b_2^{-2})t^{10} - ([4]b_1 b_2^{-2} + [5]b_1^{-1} b_2^{-1} + [1]b_1^3 b_2^{-3} \\
 &+ 2[2]b_1 b_2^{-2} + 2[3]b_1^{-1} b_2^{-1} + [4]b_1^{-3} + b_1 b_2^{-2} + 2[1]b_1^{-1} b_2^{-1} + [2]b_1^{-3} + b_1^{-3})t^{11} \\
 &- ([2]b_1^4 b_2^{-2} + 2[3]b_1^2 b_2^{-1} + 3[4] + [5]b_1^{-2} b_2 + 2[1]b_1^2 b_2^{-1} + 3[2] + 2[3]b_1^{-2} b_2 + 2 \\
 &+ 2[1]b_1^{-2} b_2 + [2]b_1^{-4} b_2^2)t^{12} - (2[2]b_1^3 + 4[3]b_1 b_2 + 2[4]b_1^{-1} b_2^2 + b_1^3 + 3[1]b_1 b_2 \\
 &+ 3[2]b_1^{-1} b_2^2 + [3]b_1^{-3} b_2^3 + b_1^{-1} b_2^2 + [1]b_1^{-3} b_2^3)t^{13} - ([1]b_1^4 b_2 + 4[2]b_1^2 b_2^2 + 3[3]b_2^3 \\
 &+ [4]b_1^{-2} b_2^4 + b_1^2 b_2^2 + 2[1]b_2^3 + [2]b_1^{-2} b_2^4 + b_1^{-1} b_2^4)t^{14} + \dots, \quad (10.10)
 \end{aligned}$$

generator	$SU(2)_{\tilde{x}}$	$U(1)_{\tilde{b}_1}$	$U(1)_{\tilde{b}_2}$
$p_1^3 p_3 p_5 \text{ } so_1 o_2 o_3^2$	2	0	-1
$p_1^2 p_2 p_5^2 \text{ } so_1 o_2^2 o_3$	1	1	0
$p_1^2 p_3 p_5 p_6 \text{ } so_1 o_2 o_3^2$	1	0	-1
$p_1 p_2 p_5^2 p_6 \text{ } so_1 o_2^2 o_3$	0	1	0
$p_1 p_3 p_5 p_6^2 \text{ } so_1 o_2 o_3^2$	0	0	-1
$p_2 p_5^2 p_6^2 \text{ } so_1 o_2^2 o_3$	-1	1	0
$p_3 p_5 p_6^3 \text{ } so_1 o_2 o_3^2$	-1	0	-1
$p_1^3 p_3^2 p_4 \text{ } so_1^2 o_2 o_3^3$	2	-1	-1
$p_1^2 p_2 p_3 p_4 p_5 \text{ } so_1^2 o_2^2 o_3^2$	1	0	0
$p_1 p_2^2 p_4 p_5^2 \text{ } so_1^2 o_3^2 o_3$	0	1	1
$p_1^2 p_3^2 p_4 p_6 \text{ } so_1^2 o_2 o_3^3$	1	-1	-1
$p_1 p_2 p_3 p_4 p_5 p_6 \text{ } so_1^2 o_2^2 o_3^2$	0	0	0
$p_2^2 p_4 p_5^2 p_6 \text{ } so_1^2 o_3^2 o_3$	-1	1	1
$p_1 p_3^2 p_4 p_6^2 \text{ } so_1^2 o_2 o_3^3$	0	-1	-1
$p_2 p_3 p_4 p_5 p_6^2 \text{ } so_1^2 o_2^2 o_3^2$	-1	0	0
$p_3^2 p_4 p_6^3 \text{ } so_1^2 o_2 o_3^3$	-1	-1	-1
$p_1^2 p_2 p_3^2 p_4^2 \text{ } so_1^3 o_2^2 o_3^3$	1	-1	0
$p_1 p_2^2 p_3 p_4^2 p_5 \text{ } so_1^3 o_2^2 o_3^2$	0	0	1
$p_3^2 p_4^2 p_5^2 \text{ } so_1^3 o_2^4 o_3$	-1	1	2
$p_1 p_2 p_3^2 p_4^2 p_6 \text{ } so_1^3 o_2^2 o_3^3$	0	-1	0
$p_2^2 p_3 p_4^2 p_5 p_6 \text{ } so_1^3 o_2^2 o_3^2$	-1	0	1
$p_2 p_3^2 p_4^2 p_6^2 \text{ } so_1^3 o_2^2 o_3^3$	-1	-1	0
$p_1 p_2^2 p_3^2 p_4^3 \text{ } so_1^4 o_2^3 o_3^3$	0	-1	1
$p_3^2 p_3 p_4^3 p_5 \text{ } so_1^4 o_2^4 o_3^2$	-1	0	2
$p_2^2 p_3^2 p_4^3 p_6 \text{ } so_1^4 o_2^3 o_3^3$	-1	-1	1
$p_3^2 p_3^2 p_4^4 \text{ } so_1^5 o_2^4 o_3^3$	-1	-1	2

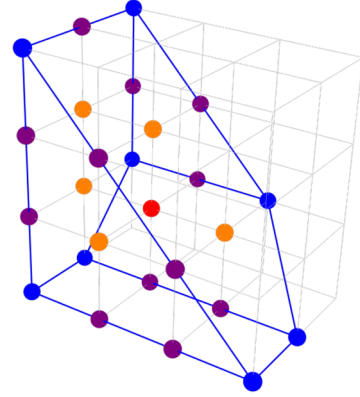


Table 28. The generators and lattice of generators of the mesonic moduli space of Model 7 in terms of brick matchings with the corresponding flavor charges.

where $[m] = [m]_{SU(2)_x}$. From the plethystic logarithm, we see that the mesonic moduli space is a non-complete intersection.

Using the following fugacity map

$$\tilde{t} = t_4^{1/2} t_5^{1/2}, \quad \tilde{x} = \frac{t_4 t_5}{t_6^2}, \quad \tilde{b}_1 = \frac{t_5^{3/2}}{t_3 t_4^{1/2}}, \quad \tilde{b}_2 = \frac{t_4^{3/2} t_5^{1/2}}{t_3 t_6}, \quad (10.11)$$

where $t_1 t_6 = t_2 t_3 = t_4 t_5$, the mesonic flavor charges on the gauge invariant operators become \mathbb{Z} -valued. The generators in terms of brick matchings and their corresponding rescaled mesonic flavor charges are summarized in table 28. The generator lattice as shown in table 28 is a convex lattice polytope, which is reflexive. It is the dual of the toric diagram of Model 7 shown in figure 23. For completeness, table 29 and table 30 show the generators of Model 7 in terms of chiral fields with the corresponding mesonic flavor charges.

generator	$SU(2)_{\bar{x}}$	$U(1)_{\bar{b}_1}$	$U(1)_{\bar{b}_2}$
$X_{12}Y_{23}P_{35}Z_{51} = Y_{48}Z_{86}X_{67}Q_{74} = X_{12}P_{24}Y_{48}Z_{86}Q_{61} = X_{12}P_{24}Y_{48}Q_{85}Z_{51}$	2	0	-1
$Z_{17}Q_{74}Y_{48}Z_{86}Q_{61} = Z_{17}Q_{74}Y_{48}Q_{85}Z_{51} = X_{12}X_{28}Z_{86}Q_{61} = X_{12}Y_{23}Y_{36}Q_{61} = X_{12}X_{28}Q_{85}Z_{51}$	1	1	0
$X_{12}Y_{23}P_{35}X_{51} = X_{12}Z_{23}P_{35}Z_{51} = Y_{12}Y_{23}P_{35}Z_{51} = Y_{48}X_{86}X_{67}Q_{74} = Z_{48}Z_{86}X_{67}Q_{74} = Y_{48}Z_{86}Y_{67}Q_{74}$ $= X_{12}P_{24}Y_{48}X_{86}Q_{61} = X_{12}P_{24}Z_{48}Z_{86}Q_{61} = Y_{12}P_{24}Y_{48}Z_{86}Q_{61} = X_{12}P_{24}Y_{48}Q_{85}X_{51} = X_{12}P_{24}Z_{48}Q_{85}Z_{51}$ $= Y_{12}P_{24}Y_{48}Q_{85}Z_{51}$	1	0	-1
$Z_{17}Q_{74}Y_{48}X_{86}Q_{61} = Z_{17}Q_{74}Z_{48}Z_{86}Q_{61} = Z_{17}Q_{74}Y_{48}Q_{85}X_{51} = Z_{17}Q_{74}Z_{48}Q_{85}Z_{51} = X_{12}X_{28}X_{86}Q_{61}$ $= X_{12}Z_{23}Y_{36}Q_{61} = Y_{12}X_{28}Z_{86}Q_{61} = Y_{12}Y_{23}Y_{36}Q_{61} = X_{12}X_{28}Q_{85}X_{51} = Y_{12}X_{28}Q_{85}Z_{51}$	0	1	0
$X_{12}Z_{23}P_{35}X_{51} = Y_{12}Y_{23}P_{35}X_{51} = Y_{12}Z_{23}P_{35}Z_{51} = Z_{48}X_{86}X_{67}Q_{74} = Y_{48}X_{86}Y_{67}Q_{74} = Z_{48}Z_{86}Y_{67}Q_{74}$ $= X_{12}P_{24}Z_{48}X_{86}Q_{61} = Y_{12}P_{24}Y_{48}X_{86}Q_{61} = Y_{12}P_{24}Z_{48}Z_{86}Q_{61} = X_{12}P_{24}Z_{48}Q_{85}X_{51} = Y_{12}P_{24}Y_{48}Q_{85}X_{51}$ $= Y_{12}P_{24}Z_{48}Q_{85}Z_{51}$	0	0	-1
$Z_{17}Q_{74}Z_{48}X_{86}Q_{61} = Z_{17}Q_{74}Z_{48}Q_{85}X_{51} = Y_{12}X_{28}X_{86}Q_{61} = Y_{12}Z_{23}Y_{36}Q_{61} = Y_{12}X_{28}Q_{85}X_{51}$	-1	1	0
$Y_{12}Z_{23}P_{35}X_{51} = Z_{48}X_{86}Y_{67}Q_{74} = Y_{12}P_{24}Z_{48}X_{86}Q_{61} = Y_{12}P_{24}Z_{48}Q_{85}X_{51}$	-1	0	-1
$X_{12}P_{24}Y_{48}P_{83}P_{35}Z_{51} = X_{12}P_{24}Y_{48}Z_{86}P_{61} = P_{24}Y_{48}Z_{86}X_{67}P_{72}$	2	-1	-1
$Z_{17}P_{72}P_{24}Y_{48}Z_{86}Q_{61} = Z_{17}P_{72}P_{24}Y_{48}Q_{85}Z_{51} = X_{12}P_{24}Y_{48}P_{83}Y_{36}Q_{61} = Z_{17}Q_{74}Y_{48}P_{83}P_{35}Z_{51}$ $= X_{12}Y_{23}P_{35}Y_{51} = X_{12}X_{28}Z_{86}P_{61} = X_{12}Y_{23}Y_{36}P_{61} = X_{28}Z_{86}X_{67}P_{72} = Y_{23}Y_{36}X_{67}P_{72} = X_{48}Z_{86}X_{67}Q_{74}$ $= X_{12}P_{24}X_{48}Z_{86}Q_{61} = X_{12}P_{24}X_{48}Q_{85}Z_{51} = X_{12}P_{24}Y_{48}Q_{85}Y_{51} = Z_{17}P_{72}Y_{23}P_{35}Z_{51} = X_{12}X_{28}P_{83}P_{35}Z_{51}$ $= Z_{17}Q_{74}Y_{48}Z_{86}P_{61} = Y_{36}X_{67}Q_{74}Y_{48}P_{83}$	1	0	0
$X_{12}X_{28}Q_{85}Y_{51} = Z_{17}Q_{74}Y_{48}P_{83}Y_{36}Q_{61} = Z_{17}P_{72}X_{28}Z_{86}Q_{61} = Z_{17}P_{72}Y_{23}Y_{36}Q_{61} = Z_{17}P_{72}X_{28}Q_{85}Z_{51}$ $= X_{12}X_{28}P_{83}Y_{36}Q_{61} = Z_{17}Q_{74}X_{48}Z_{86}Q_{61} = Z_{17}Q_{74}X_{48}Q_{85}Z_{51} = Z_{17}Q_{74}Y_{48}Q_{85}Y_{51}$	0	1	1
$X_{12}P_{24}Y_{48}P_{83}P_{35}X_{51} = X_{12}P_{24}Z_{48}P_{83}P_{35}Z_{51} = Y_{12}P_{24}Y_{48}P_{83}P_{35}Z_{51} = X_{12}P_{24}Y_{48}X_{86}P_{61}$ $= X_{12}P_{24}Z_{48}Z_{86}P_{61} = Y_{12}P_{24}Y_{48}Z_{86}P_{61} = P_{24}Y_{48}X_{86}X_{67}P_{72} = P_{24}Z_{48}Z_{86}X_{67}P_{72} = P_{24}Y_{48}Z_{86}Y_{67}P_{72}$ $Z_{17}P_{72}P_{24}Y_{48}X_{86}Q_{61} = Z_{17}P_{72}P_{24}Z_{48}Z_{86}Q_{61} = Z_{17}P_{72}P_{24}Y_{48}Q_{85}X_{51} = Z_{17}P_{72}P_{24}Z_{48}Q_{85}Z_{51}$ $= X_{12}P_{24}Z_{48}P_{83}Y_{36}Q_{61} = Y_{12}P_{24}Y_{48}P_{83}Y_{36}Q_{61} = Z_{17}Q_{74}Y_{48}P_{83}P_{35}X_{51} = Z_{17}Q_{74}Z_{48}P_{83}P_{35}Z_{51}$ $= X_{12}Z_{23}P_{35}Y_{51} = Y_{12}Y_{23}P_{35}Y_{51} = X_{12}X_{28}X_{86}P_{61} = X_{12}Z_{23}Y_{36}P_{61} = Y_{12}X_{28}Z_{86}P_{61} = Y_{12}Y_{23}Y_{36}P_{61}$ $= X_{28}X_{86}X_{67}P_{72} = X_{28}Z_{86}Y_{67}P_{72} = Z_{23}Y_{36}X_{67}P_{72} = Y_{23}Y_{36}Y_{67}P_{72} = X_{48}X_{86}X_{67}Q_{74} = X_{48}Z_{86}Y_{67}Q_{74}$ $= X_{12}P_{24}X_{48}X_{86}Q_{61} = Y_{12}P_{24}X_{48}Z_{86}Q_{61} = X_{12}P_{24}X_{48}Q_{85}X_{51} = X_{12}P_{24}Z_{48}Q_{85}Y_{51} = Y_{12}P_{24}X_{48}Q_{85}Z_{51}$ $= Y_{12}P_{24}Y_{48}Q_{85}Y_{51} = Z_{17}P_{72}Y_{23}P_{35}X_{51} = Z_{17}P_{72}Z_{23}P_{35}Z_{51} = X_{12}X_{28}P_{83}P_{35}X_{51} = Y_{12}X_{28}P_{83}P_{35}Z_{51}$ $= Z_{17}Q_{74}Y_{48}X_{86}P_{61} = Z_{17}Q_{74}Z_{48}Z_{86}P_{61} = Y_{36}X_{67}Q_{74}Z_{48}P_{83} = Y_{36}Y_{67}Q_{74}Y_{48}P_{83}$	1	-1	-1
$Y_{12}X_{28}Q_{85}Y_{51} = Z_{17}Q_{74}Z_{48}P_{83}Y_{36}Q_{61} = Z_{17}P_{72}X_{28}X_{86}Q_{61} = Z_{17}P_{72}Z_{23}Y_{36}Q_{61} = Z_{17}P_{72}X_{28}Q_{85}X_{51}$ $= Y_{12}X_{28}P_{83}Y_{36}Q_{61} = Z_{17}Q_{74}X_{48}X_{86}Q_{61} = Z_{17}Q_{74}X_{48}Q_{85}X_{51} = Z_{17}Q_{74}Z_{48}Q_{85}Y_{51}$	0	0	0
$X_{12}P_{24}Z_{48}P_{83}P_{35}X_{51} = Y_{12}P_{24}Y_{48}P_{83}P_{35}X_{51} = Y_{12}P_{24}Z_{48}P_{83}P_{35}Z_{51} = X_{12}P_{24}Z_{48}X_{86}P_{61}$ $= Y_{12}P_{24}Y_{48}X_{86}P_{61} = Y_{12}P_{24}Z_{48}Z_{86}P_{61} = P_{24}Z_{48}X_{86}X_{67}P_{72} = P_{24}Y_{48}X_{86}Y_{67}P_{72} = P_{24}Z_{48}Z_{86}Y_{67}P_{72}$ $Z_{17}P_{72}P_{24}Z_{48}X_{86}Q_{61} = Z_{17}P_{72}P_{24}Z_{48}Q_{85}X_{51} = Y_{12}P_{24}Z_{48}P_{83}Y_{36}Q_{61} = Z_{17}Q_{74}Z_{48}P_{83}P_{35}X_{51}$ $= Y_{12}Z_{23}P_{35}Y_{51} = Y_{12}X_{28}X_{86}P_{61} = Y_{12}Z_{23}Y_{36}P_{61} = X_{28}X_{86}Y_{67}P_{72} = Z_{23}Y_{36}Y_{67}P_{72} = X_{48}X_{86}Y_{67}Q_{74}$ $= Y_{12}P_{24}X_{48}X_{86}Q_{61} = Y_{12}P_{24}X_{48}Q_{85}X_{51} = Y_{12}P_{24}Z_{48}Q_{85}Y_{51} = Z_{17}P_{72}Z_{23}P_{35}X_{51}$ $= Y_{12}X_{28}P_{83}P_{35}X_{51} = Z_{17}Q_{74}Z_{48}X_{86}P_{61} = Y_{36}Y_{67}Q_{74}Z_{48}P_{83}$	-1	1	1
$Y_{12}X_{28}Q_{85}Y_{51} = Z_{17}Q_{74}Z_{48}P_{83}Y_{36}Q_{61} = Z_{17}P_{72}X_{28}X_{86}Q_{61} = Z_{17}P_{72}Z_{23}Y_{36}Q_{61} = Z_{17}P_{72}X_{28}Q_{85}X_{51}$ $= Y_{12}X_{28}P_{83}Y_{36}Q_{61} = Z_{17}Q_{74}X_{48}X_{86}Q_{61} = Z_{17}Q_{74}X_{48}Q_{85}X_{51} = Z_{17}Q_{74}Z_{48}Q_{85}Y_{51}$	0	-1	-1
$X_{12}P_{24}Z_{48}P_{83}P_{35}X_{51} = Y_{12}P_{24}Y_{48}P_{83}P_{35}X_{51} = Y_{12}P_{24}Z_{48}P_{83}P_{35}Z_{51} = X_{12}P_{24}Z_{48}X_{86}P_{61}$ $= Y_{12}P_{24}Y_{48}X_{86}P_{61} = Y_{12}P_{24}Z_{48}Z_{86}P_{61} = P_{24}Z_{48}X_{86}X_{67}P_{72} = P_{24}Y_{48}X_{86}Y_{67}P_{72} = P_{24}Z_{48}Z_{86}Y_{67}P_{72}$ $Z_{17}P_{72}P_{24}Z_{48}X_{86}Q_{61} = Z_{17}P_{72}P_{24}Z_{48}Q_{85}X_{51} = Y_{12}P_{24}Z_{48}P_{83}Y_{36}Q_{61} = Z_{17}Q_{74}Z_{48}P_{83}P_{35}X_{51}$ $= Y_{12}Z_{23}P_{35}Y_{51} = Y_{12}X_{28}X_{86}P_{61} = Y_{12}Z_{23}Y_{36}P_{61} = X_{28}X_{86}Y_{67}P_{72} = Z_{23}Y_{36}Y_{67}P_{72} = X_{48}X_{86}Y_{67}Q_{74}$ $= Y_{12}P_{24}X_{48}X_{86}Q_{61} = Y_{12}P_{24}X_{48}Q_{85}X_{51} = Y_{12}P_{24}Z_{48}Q_{85}Y_{51} = Z_{17}P_{72}Z_{23}P_{35}X_{51}$ $= Y_{12}X_{28}P_{83}P_{35}X_{51} = Z_{17}Q_{74}Z_{48}X_{86}P_{61} = Y_{36}Y_{67}Q_{74}Z_{48}P_{83}$	-1	0	0

Table 29. The generators in terms of bifundamental chiral fields for Model 7 (*Part 1*).

generator	$SU(3)_{(\bar{x}_1, \bar{x}_2)}$	$U(1)_{\bar{b}}$
$Y_{12}P_{24}Z_{48}P_{83}P_{35}X_{51} = Y_{12}P_{24}Z_{48}X_{86}P_{61} = P_{24}Z_{48}X_{86}Y_{67}P_{72}$	-1	-1
$Z_{17}P_{72}P_{24}Y_{48}P_{83}P_{35}Z_{51} = X_{12}P_{24}X_{48}Z_{86}P_{61} = P_{24}X_{48}Z_{86}X_{67}P_{72} = X_{12}P_{24}X_{48}P_{83}P_{35}Z_{51}$	1	-1
$= X_{12}P_{24}Y_{48}P_{83}P_{35}Y_{51} = Z_{17}P_{72}P_{24}Y_{48}Z_{86}P_{61} = X_{12}P_{24}Y_{48}P_{83}Y_{36}P_{61} = P_{24}Y_{48}P_{83}Y_{36}X_{67}P_{72}$	0	0
$Z_{17}P_{72}P_{24}Y_{48}P_{83}Y_{36}Q_{61} = Z_{17}P_{72}P_{24}X_{48}Z_{86}Q_{61} = Z_{17}P_{72}P_{24}X_{48}Q_{85}Z_{51} = Z_{17}P_{72}P_{24}Y_{48}Q_{85}Y_{51}$	0	0
$= X_{12}P_{24}X_{48}P_{83}Y_{36}Q_{61} = Z_{17}P_{72}X_{28}P_{83}P_{35}Z_{51} = Z_{17}Q_{74}X_{48}P_{83}P_{35}Z_{51} = Z_{17}Q_{74}Y_{48}P_{83}P_{35}Y_{51}$		1
$= Z_{17}Q_{74}Y_{48}P_{83}Y_{36}P_{61} = X_{12}P_{24}X_{48}Q_{85}Y_{51} = Z_{17}P_{72}Y_{23}P_{35}Y_{51} = X_{12}X_{28}P_{83}P_{35}Y_{51}$		
$= Z_{17}P_{72}X_{28}Z_{86}P_{61} = Z_{17}P_{72}Y_{23}Y_{36}P_{61} = X_{12}X_{28}P_{83}Y_{36}P_{61} = Z_{17}Q_{74}X_{48}Z_{86}P_{61} = X_{28}P_{83}Y_{36}X_{67}P_{72}$		
$= Y_{36}X_{67}Q_{74}X_{48}P_{83}$		
$Z_{17}P_{72}X_{28}Q_{85}Y_{51} = Z_{17}Q_{74}X_{48}Q_{85}Y_{51} = Z_{17}P_{72}X_{28}P_{83}Y_{36}Q_{61} = Z_{17}Q_{74}X_{48}P_{83}Y_{36}Q_{61}$	-1	1
$Z_{17}P_{72}P_{24}Y_{48}P_{83}P_{35}X_{51} = Z_{17}P_{72}P_{24}Z_{48}P_{83}P_{35}Z_{51} = X_{12}P_{24}X_{48}X_{86}P_{61} = Y_{12}P_{24}X_{48}Z_{86}P_{61}$	0	-1
$= P_{24}X_{48}X_{86}X_{67}P_{72} = P_{24}X_{48}Z_{86}Y_{67}P_{72} = X_{12}P_{24}X_{48}P_{83}P_{35}X_{51} = X_{12}P_{24}Z_{48}P_{83}P_{35}Y_{51}$		
$= Y_{12}P_{24}X_{48}P_{83}P_{35}Z_{51} = Y_{12}P_{24}Y_{48}P_{83}P_{35}Y_{51} = Z_{17}P_{72}P_{24}Y_{48}X_{86}P_{61} = Z_{17}P_{72}P_{24}Z_{48}Z_{86}P_{61}$		
$= X_{12}P_{24}Z_{48}P_{83}Y_{36}P_{61} = Y_{12}P_{24}Y_{48}P_{83}Y_{36}P_{61} = P_{24}Z_{48}P_{83}Y_{36}X_{67}P_{72} = P_{24}Y_{48}P_{83}Y_{36}Y_{67}P_{72}$		
$Z_{17}P_{72}P_{24}Z_{48}P_{83}P_{35}Q_{61} = Z_{17}P_{72}P_{24}X_{48}X_{86}Q_{61} = Z_{17}P_{72}P_{24}X_{48}Q_{85}X_{51} = Z_{17}P_{72}P_{24}Z_{48}Q_{85}Y_{51}$	-1	0
$= Y_{12}P_{24}X_{48}P_{83}Y_{36}Q_{61} = Z_{17}P_{72}X_{28}P_{83}P_{35}X_{51} = Z_{17}Q_{74}X_{48}P_{83}P_{35}X_{51} = Z_{17}Q_{74}Z_{48}P_{83}P_{35}Y_{51}$		1
$= Z_{17}Q_{74}Z_{48}P_{83}Y_{36}P_{61} = Y_{12}P_{24}X_{48}Q_{85}Y_{51} = Z_{17}P_{72}Z_{23}P_{35}Y_{51} = Y_{12}X_{28}P_{83}P_{35}Y_{51}$		
$= Z_{17}P_{72}X_{28}X_{86}P_{61} = Z_{17}P_{72}Z_{23}Y_{36}P_{61} = Y_{12}X_{28}P_{83}Y_{36}P_{61} = Z_{17}Q_{74}X_{48}X_{86}P_{61}$		
$= X_{28}P_{83}Y_{36}Y_{67}P_{72} = Y_{36}Y_{67}Q_{74}X_{48}P_{83}$		
$Z_{17}P_{72}P_{24}Z_{48}P_{83}P_{35}X_{51} = Y_{12}P_{24}X_{48}X_{86}P_{61} = P_{24}X_{48}X_{86}Y_{67}P_{72} = Y_{12}P_{24}X_{48}P_{83}P_{35}X_{51}$	-1	-1
$= Y_{12}P_{24}Z_{48}P_{83}P_{35}Y_{51} = Z_{17}P_{72}P_{24}Z_{48}X_{86}P_{61} = Y_{12}P_{24}Z_{48}P_{83}Y_{36}P_{61} = P_{24}Z_{48}P_{83}Y_{36}Y_{67}P_{72}$		0
$Z_{17}P_{72}P_{24}X_{48}P_{83}P_{35}Z_{51} = Z_{17}P_{72}P_{24}Y_{48}P_{83}P_{35}Y_{51} = Z_{17}P_{72}P_{24}Y_{48}P_{83}Y_{36}P_{61}$	0	-1
$= X_{12}P_{24}X_{48}P_{83}P_{35}Y_{51} = Z_{17}P_{72}P_{24}X_{48}Z_{86}P_{61} = X_{12}P_{24}X_{48}P_{83}Y_{36}P_{61} = P_{24}X_{48}P_{83}Y_{36}X_{67}P_{72}$		1
$Z_{17}P_{72}P_{24}X_{48}P_{83}Y_{36}Q_{61} = Z_{17}P_{72}P_{24}X_{48}Q_{85}Y_{51} = Z_{17}P_{72}X_{28}P_{83}P_{35}Y_{51} = Z_{17}Q_{74}X_{48}P_{83}P_{35}Y_{51}$	-1	0
$= Z_{17}P_{72}X_{28}P_{83}Y_{36}P_{61} = Z_{17}Q_{74}X_{48}P_{83}Y_{36}P_{61}$		2
$Z_{17}P_{72}P_{24}X_{48}P_{83}P_{35}X_{51} = Z_{17}P_{72}P_{24}Z_{48}P_{83}P_{35}Y_{51} = Z_{17}P_{72}P_{24}Z_{48}P_{83}Y_{36}P_{61}$	-1	-1
$= Y_{12}P_{24}X_{48}P_{83}P_{35}Y_{51} = Z_{17}P_{72}P_{24}X_{48}X_{86}P_{61} = Y_{12}P_{24}X_{48}P_{83}Y_{36}P_{61} = P_{24}X_{48}P_{83}Y_{36}Y_{67}P_{72}$		1
$Z_{17}P_{72}P_{24}X_{48}P_{83}P_{35}Y_{51} = Z_{17}P_{72}P_{24}Z_{48}P_{83}Y_{36}P_{61}$	-1	-1
$Z_{17}P_{72}P_{24}X_{48}P_{83}P_{35}Z_{51} = Z_{17}P_{72}P_{24}X_{48}P_{83}Y_{36}P_{61}$	-1	-1

Table 30. The generators in terms of bifundamental chiral fields for Model 7 (*Part 2*).

11 Model 8: $P_{+-}H_+(dP_0)$ [$\mathbb{P}(\mathcal{O}_{dP_1} \oplus \mathcal{O}_{dP_1}(l)), l^2|_{dP_1} = 1, \langle 26 \rangle$]

Model 8 corresponds to toric Calabi-Yau 4-fold $P_{+-}H_+(dP_0)$. The corresponding brane brick model has the quiver in figure 24 and the J - and E -terms are given as follows

$$\begin{array}{ll}
 & J \\
 \Lambda_{16}^1 : & X_{67}Y_{75}Q_{58}X_{81} - X_{62}X_{21} \\
 \Lambda_{16}^2 : & X_{67}Y_{75}P_{58}X_{81} - R_{62}X_{21} \\
 \Lambda_{17}^1 : & X_{72}X_{21} - Y_{75}Y_{58}X_{81} \\
 \Lambda_{24}^1 : & Q_{47}X_{72} - X_{46}X_{62} \\
 \Lambda_{24}^2 : & P_{47}X_{72} - X_{46}R_{62} \\
 \Lambda_{28}^1 : & X_{86}X_{62} - X_{81}Q_{14}Y_{42} \\
 \Lambda_{28}^2 : & X_{81}X_{13}X_{34}Y_{42} - X_{86}X_{67}X_{72} \\
 \Lambda_{28}^3 : & X_{86}R_{62} - X_{81}P_{14}Y_{42} \\
 \Lambda_{32}^1 : & Y_{25}X_{53} - X_{21}X_{13} \\
 \Lambda_{37}^1 : & Y_{75}Q_{58}X_{81}X_{13} - X_{72}Q_{25}X_{53} \\
 \Lambda_{37}^2 : & Y_{75}P_{58}X_{81}X_{13} - X_{72}P_{25}X_{53} \\
 \Lambda_{45}^1 : & Q_{58}X_{81}X_{13}X_{34} - Y_{58}X_{81}Q_{14} \\
 \Lambda_{45}^2 : & P_{58}X_{81}X_{13}X_{34} - Y_{58}X_{81}P_{14} \\
 \Lambda_{54}^1 : & X_{46}X_{67}Y_{75} - Y_{42}Y_{25} \\
 \Lambda_{56}^1 : & X_{62}Y_{25} - X_{67}X_{72}Q_{25} \\
 \Lambda_{56}^2 : & R_{62}Y_{25} - X_{67}X_{72}P_{25} \\
 \Lambda_{65}^1 : & Y_{58}X_{86} - X_{53}X_{34}X_{46}
 \end{array}
 \qquad
 \begin{array}{ll}
 & E \\
 P_{14}X_{46} - X_{13}P_{36} \\
 X_{13}Q_{36} - Q_{14}X_{46} \\
 P_{14}Q_{47} - Q_{14}P_{47} \\
 P_{25}X_{53}X_{34} - X_{21}P_{14} \\
 X_{21}Q_{14} - Q_{25}X_{53}X_{34} \\
 P_{25}Y_{58} - Y_{25}P_{58} \\
 P_{25}Q_{58} - Q_{25}P_{58} \\
 Y_{25}Q_{58} - Q_{25}Y_{58} \\
 P_{36}X_{62} - Q_{36}R_{62} \\
 P_{36}X_{67} - X_{34}P_{47} \\
 X_{34}Q_{47} - Q_{36}X_{67} \\
 P_{47}Y_{75} - Y_{42}P_{25} \\
 Y_{42}Q_{25} - Q_{47}Y_{75} \\
 P_{58}X_{81}Q_{14} - Q_{58}X_{81}P_{14} \\
 P_{58}X_{86} - X_{53}P_{36} \\
 X_{53}Q_{36} - Q_{58}X_{86} \\
 R_{62}Q_{25} - X_{62}P_{25}
 \end{array}
 \qquad (11.1)$$

The J - and E -term charges are given by

$$Q_{JE} = \left(\begin{array}{cccccc|cccccc|cccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & o_7 & o_8 & o_9 & o_{10} & o_{11} & o_{12} & o_{13} & o_{14} & o_{15} \\ 1 & 1 & 2 & 3 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & -1 & 0 \end{array} \right), \tag{11.3}$$

and the D -term charges are given by

$$Q_D = \left(\begin{array}{cccccc|cccccc|cccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & o_7 & o_8 & o_9 & o_{10} & o_{11} & o_{12} & o_{13} & o_{14} & o_{15} \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{array} \right). \tag{11.4}$$

The toric diagram of Model 8 is given by

$$G_t = \left(\begin{array}{cccccc|cccccc|cccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & o_7 & o_8 & o_9 & o_{10} & o_{11} & o_{12} & o_{13} & o_{14} & o_{15} \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & -1 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array} \right), \tag{11.5}$$

where figure 25 shows the toric diagram with brick matching labels.

The Hilbert series of the mesonic moduli space of Model 8 takes the form

$$g_1(t_i, y_s, y_{o_1}, y_{o_2}, y_{o_3}; \mathcal{M}_8) = \frac{P(t_i, y_s, y_{o_1}, y_{o_2}, y_{o_3}; \mathcal{M}_8)}{(1 - y_s y_{o_1}^2 y_{o_2} y_{o_3} t_1 t_4^2 t_5)(1 - y_s y_{o_1}^2 y_{o_2} y_{o_3} t_2 t_4^2 t_5)} \times \frac{1}{(1 - y_s y_{o_1}^3 y_{o_2}^2 y_{o_3} t_3 t_4^2 t_5)(1 - y_s y_{o_1} y_{o_2} y_{o_3}^2 t_1^3 t_4 t_6)(1 - y_s y_{o_1} y_{o_2} y_{o_3}^2 t_3^2 t_4 t_6)} \times \frac{1}{(1 - y_s y_{o_1} y_{o_2}^2 y_{o_3}^3 t_1^4 t_3 t_6^2)(1 - y_s y_{o_1} y_{o_2}^2 y_{o_3}^3 t_2^4 t_3 t_6^2)(1 - y_s y_{o_1}^5 y_{o_2}^6 y_{o_3}^3 t_3^5 t_4^2 t_6^2)}, \tag{11.6}$$

where t_i are the fugacities for the extremal brick matchings p_i . y_s counts the brick matching product $s_1 \dots s_8$ corresponding to the single internal point of the toric diagram of Model 8.

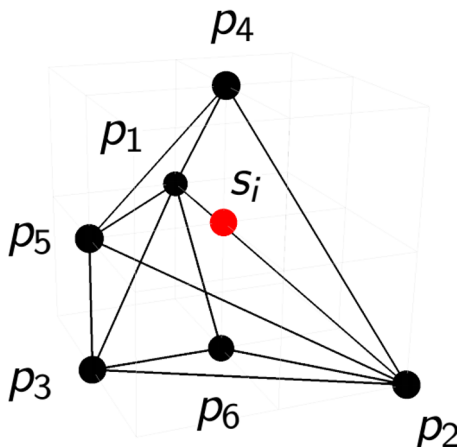


Figure 25. Toric diagram for Model 8.

Additionally, y_{o_1} , y_{o_2} and y_{o_3} count the products of extra GLSM fields $o_1 o_2$, $o_3 o_4 o_5 o_6$ and $o_7 \dots o_{15}$, respectively. The explicit numerator $P(t_i, y_s, y_{o_1}, y_{o_2}, y_{o_3}; \mathcal{M}_8)$ of the Hilbert series is given in the appendix section A.8. We note that setting the fugacities $y_{o_1} = 1$, $y_{o_2} = 1$ and $y_{o_3} = 1$ does not change the overall characterization of the mesonic moduli space by the Hilbert series, indicating that the extra GLSM fields, as expected, correspond to an over-parameterization of the moduli space.

By setting $t_i = t$ for the fugacities of the extremal brick matchings, and all other fugacities to $y_s = 1$, $y_{o_1} = 1$, $y_{o_2} = 1$ and $y_{o_3} = 1$, the unrefined Hilbert series takes the following form

$$\begin{aligned}
 g_1(t, 1, 1, 1; \mathcal{M}_8) &= \frac{(1-t)^3}{(1-t^4)^2(1-t^5)^2(1-t^7)^2(1-t^{11})} \\
 &\times (1 + 3t + 6t^2 + 10t^3 + 15t^4 + 24t^5 + 40t^6 + 68t^7 + 113t^8 + 175t^9 + 253t^{10} \\
 &\quad + 336t^{11} + 409t^{12} + 462t^{13} + 490t^{14} + 496t^{15} + 493t^{16} + 491t^{17} + 493t^{18} \\
 &\quad + 496t^{19} + 490t^{20} + 462t^{21} + 409t^{22} + 336t^{23} + 253t^{24} + 175t^{25} + 113t^{26} \\
 &\quad + 68t^{27} + 40t^{28} + 24t^{29} + 15t^{30} + 10t^{31} + 6t^{32} + 3t^{33} + t^{34}), \tag{11.7}
 \end{aligned}$$

where the palindromic numerator shows that the mesonic moduli space is Calabi-Yau.

The global symmetry of Model 8 and the charges on the extremal brick matchings under the global symmetry are summarized in table 31. We can use the following fugacity map,

$$t = t_1^{1/2} t_2^{1/2}, \quad x = \frac{t_1^{1/2}}{t_2^{1/2}}, \quad b_1 = \frac{t_1^{1/2} t_2^{1/2}}{t_4^{1/2}}, \quad b_2 = \frac{t_1^{1/2} t_2^{1/2} t_5}{t_3 t_4}, \tag{11.8}$$

where $t_3 t_4 = t_5 t_6$ and $t_1 t_2 = t_5 t_6$, in order to rewrite the Hilbert series for Model 8 in terms of characters of irreducible representations of $SU(2) \times U(1) \times U(1)$. In highest weight form,

	SU(2) _x	U(1) _{b₁}	U(1) _{b₂}	U(1)	fugacity
p_1	+1	0	0	r_1	t_1
p_2	-1	0	0	r_2	t_2
p_3	0	+1	0	r_3	t_3
p_4	0	-1	0	r_4	t_4
p_5	0	0	+1	r_5	t_5
p_6	0	0	-1	r_6	t_6

Table 31. Global symmetry charges on the extremal brick matchings p_i of Model 8.

the Hilbert series of Model 8 can be written as

$$\begin{aligned}
 h_1(t, \mu, b_1, b_2; \mathcal{M}_8) &= \frac{1}{(1 - \mu b_1^{-2} b_2 t^4)(1 - \mu^3 b_1^{-1} b_2^{-1} t^5)(1 - b_1^{-1} b_2^2 t^5)} \\
 &\times \frac{1}{(1 - \mu^4 b_1 b_2^{-2} t^7)(1 - b_1^5 b_2^2 t^{11})} \times (1 + \mu^2 t^6 + \mu b_1 b_2 t^7 + \mu^3 b_1^2 b_2^{-1} t^8 + b_1^2 b_2^2 t^8 \\
 &+ \mu^2 b_1^3 t^9 - \mu^3 b_1^{-2} b_2 t^{10} + \mu b_1^4 b_2 t^{10} - \mu^5 b_1^{-1} b_2^{-1} t^{11} - \mu^2 b_1^{-1} b_2^2 t^{11} - 2\mu^4 t^{12} \\
 &- \mu b_2^3 t^{12} - \mu^6 b_1 b_2^{-2} t^{13} - 2\mu^3 b_1 b_2 t^{13} - \mu^5 b_1^2 b_2^{-1} t^{14} - \mu^2 b_1^2 b_2^2 t^{14} - \mu^4 b_1^3 t^{15} \\
 &- \mu b_1^3 b_2^3 t^{15} + \mu^5 b_1^{-2} b_2 t^{16} + \mu^7 b_1^{-1} b_2^{-1} t^{17} + \mu^4 b_1^{-1} b_2^2 t^{17} + \mu^6 t^{18} + \mu^5 b_1 b_2 t^{19} \\
 &- \mu^6 b_1^3 t^{21} + \mu^7 b_1 b_2 t^{25} + \mu^6 b_1^2 b_2^2 t^{26}), \tag{11.9}
 \end{aligned}$$

where $\mu^m \sim [m]_{\text{SU}(2)_x}$. Here in highest weight form, the fugacity μ counts the highest weight of irreducible representations of SU(2).

The plethystic logarithm of the Hilbert series takes the form

$$\begin{aligned}
 \text{PL}[g_1(t, x, b_1, b_2; \mathcal{M}_8)] &= [1]b_1^{-2}b_2t^4 + ([3]b_1^{-1}b_2^{-1} + b_1^{-1}b_2^2)t^5 + [2]t^6 \\
 &+ ([4]b_1b_2^{-2} + [1]b_1b_2)t^7 + ([3]b_1^2b_2^{-1} + b_1^2b_2^2)t^8 + ([2]b_1^3 + [2]b_1^{-3})t^9 + [1]b_1^4b_2t^{10} \\
 &- ([3]b_1^{-2}b_2 + [2]b_1^{-2}b_2^{-2} + [1]b_1^{-2}b_2)t^{10} - (b_1^5b_2^2 + [5]b_1^{-1}b_2^{-1} + 2[3]b_1^{-1}b_2^{-1} \\
 &+ [2]b_1^{-1}b_2^2 + b[1]b_1^{-1}b_2^{-1} + b_1^{-1}b_2^2)t^{11} - ([5]b_2^{-3} + 3[4] + [3]b_2^{-3} + 2[2] + [1]b_2^3 \\
 &+ [1]b_2^{-3} + 1)t^{12} - ([6]b_1b_2^{-2} + 2[4]b_1b_2^{-2} + 3[3]b_1b_2 + 2[2]b_1b_2^{-2} + 2[1]b_1b_2 \\
 &+ b_1b_2^{-2})t^{13} - (2[5]b_1^2b_2^{-1} + [4]b_1^2b_2^{-4} + 3[3]b_1^2b_2^{-1} + 3[2]b_1^2b_2^2 + 2[1]b_1^2b_2^{-1} + b_1^2b_2^2 \\
 &+ b_1^2b_2^{-4})t^{14} + \dots, \tag{11.10}
 \end{aligned}$$

where $[m] = [m]_{\text{SU}(2)_x}$. From the plethystic logarithm, we see that the mesonic moduli space is a non-complete intersection.

Using the following fugacity map

$$\tilde{t} = t_5^{1/2} t_6^{1/2}, \quad \tilde{x} = \frac{t_5 t_6}{t_2^2}, \quad \tilde{b}_1 = \frac{t_2 t_5^{1/2}}{t_4 t_6^{1/2}}, \quad \tilde{b}_2 = \frac{t_2 t_4^2}{t_5 t_6^2}, \tag{11.11}$$

where $t_1 t_2 = t_3 t_4 = t_5 t_6$, the mesonic flavor charges on the gauge invariant operators become \mathbb{Z} -valued. The generators in terms of brick matchings and their corresponding rescaled

generator	$SU(2)_{\tilde{x}}$	$U(1)_{\tilde{b}_1}$	$U(1)_{\tilde{b}_2}$
$p_1 p_4^2 p_5 \ so_1^2 o_2 o_3$	1	0	1
$p_2 p_4^2 p_5 \ so_1^2 o_2 o_3$	0	0	1
$p_3 p_4^2 p_5^2 \ so_1^3 o_2^2 o_3$	1	1	1
$p_1^3 p_4 p_6 \ so_1 o_2 o_3^2$	1	-1	0
$p_1^2 p_2 p_4 p_6 \ so_1 o_2 o_3^2$	0	-1	0
$p_1 p_2^2 p_4 p_6 \ so_1 o_2 o_3^2$	-1	-1	0
$p_2^3 p_4 p_6 \ so_1 o_2 o_3^2$	-2	-1	0
$p_1^2 p_3 p_4 p_5 p_6 \ so_1^2 o_2^2 o_3^2$	1	0	0
$p_1 p_2 p_3 p_4 p_5 p_6 \ so_1^2 o_2^2 o_3^2$	0	0	0
$p_2^2 p_3 p_4 p_5 p_6 \ so_1^2 o_2^2 o_3^2$	-1	0	0
$p_1 p_2^2 p_4 p_5^2 p_6 \ so_1^3 o_2^3 o_3^2$	1	1	0
$p_2 p_2^2 p_4 p_5^2 p_6 \ so_1^3 o_2^3 o_3^2$	0	1	0
$p_1^4 p_3 p_6^2 \ so_1 o_2^2 o_3^3$	1	-1	-1
$p_1^3 p_2 p_3 p_6^2 \ so_1 o_2^2 o_3^3$	0	-1	-1
$p_1^2 p_2^2 p_3 p_6^2 \ so_1 o_2^2 o_3^3$	-1	-1	-1
$p_1 p_2^3 p_3 p_6^2 \ so_1 o_2^2 o_3^3$	-2	-1	-1
$p_2^4 p_3 p_6^2 \ so_1 o_2^2 o_3^3$	-3	-1	-1
$p_3^3 p_4 p_5^3 p_6 \ so_1^4 o_2^4 o_3^2$	1	2	0
$p_1^3 p_2^2 p_3 p_5 p_6^2 \ so_1^2 o_2^3 o_3^3$	1	0	-1
$p_1^2 p_2 p_2^2 p_3 p_5 p_6^2 \ so_1^2 o_2^3 o_3^3$	0	0	-1
$p_1 p_2^2 p_2^2 p_3 p_5 p_6^2 \ so_1^2 o_2^3 o_3^3$	-1	0	-1
$p_2^3 p_2^2 p_3 p_5 p_6^2 \ so_1^2 o_2^3 o_3^3$	-2	0	-1
$p_1^2 p_3^3 p_5^2 p_6^2 \ so_1^3 o_2^4 o_3^3$	1	1	-1
$p_1 p_2 p_2^3 p_3^2 p_5^2 p_6^2 \ so_1^3 o_2^4 o_3^3$	0	1	-1
$p_2^2 p_3^3 p_5^2 p_6^2 \ so_1^3 o_2^4 o_3^3$	-1	1	-1
$p_1 p_3^4 p_5^3 p_6^2 \ so_1^4 o_2^5 o_3^3$	1	2	-1
$p_2 p_3^4 p_5^3 p_6^2 \ so_1^4 o_2^5 o_3^3$	0	2	-1
$p_3^5 p_5^4 p_6^2 \ so_1^5 o_2^6 o_3^3$	1	3	-1

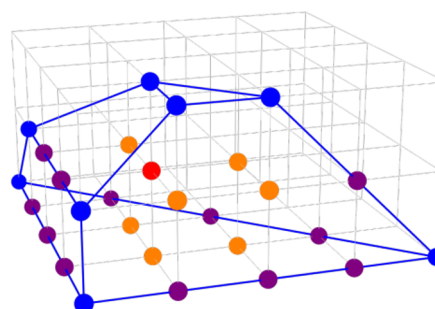


Table 32. The generators and lattice of generators of the mesonic moduli space of Model 8 in terms of brick matchings with the corresponding flavor charges.

mesonic flavor charges are summarized in table 32. The generator lattice as shown in table 32 is a convex lattice polytope, which is reflexive. It is the dual of the toric diagram of Model 8 shown in figure 25. For completeness, table 33 and table 34 show the generators of Model 8 in terms of chiral fields with the corresponding mesonic flavor charges.

generator	SU(2) $_{\bar{x}}$	U(1) $_{\delta_1}$	U(1) $_{\delta_2}$
$P_{14}Y_{42}X_{21} = P_{25}X_{53}X_{34}Y_{42} = P_{36}X_{67}Y_{75}X_{53} = X_{34}P_{47}Y_{75}X_{53} = P_{58}X_{86}X_{67}Y_{75}$	1	0	1
$Q_{14}Y_{42}X_{21} = Q_{25}X_{53}X_{34}Y_{42} = Q_{36}X_{67}Y_{75}X_{53} = X_{34}Q_{47}Y_{75}X_{53} = Q_{58}X_{86}X_{67}Y_{75}$	0	0	1
$X_{34}X_{46}X_{67}Y_{75}X_{53} = X_{13}X_{34}Y_{42}X_{21} = Y_{25}X_{53}X_{34}Y_{42} = Y_{58}X_{86}X_{67}Y_{75}$	1	1	1
$P_{25}X_{53}P_{36}R_{62} = P_{25}P_{58}X_{86}R_{62} = P_{14}Y_{42}P_{25}P_{58}X_{81} = P_{14}P_{47}Y_{75}P_{58}X_{81}$	1	-1	0
$P_{25}X_{53}P_{36}X_{62} = P_{25}P_{58}X_{86}X_{62} = P_{25}X_{53}Q_{36}R_{62} = P_{25}Q_{58}X_{86}R_{62} = Q_{25}X_{53}P_{36}R_{62} = Q_{25}P_{58}X_{86}R_{62}$	0	-1	0
$= P_{14}Y_{42}P_{25}Q_{58}X_{81} = P_{14}P_{47}Y_{75}Q_{58}X_{81} = P_{14}Y_{42}Q_{25}P_{58}X_{81} = P_{14}Q_{47}Y_{75}P_{58}X_{81} = Q_{14}Y_{42}P_{25}P_{58}X_{81}$			
$= Q_{14}P_{47}Y_{75}P_{58}X_{81}$			
$P_{25}X_{53}Q_{36}X_{62} = P_{25}Q_{58}X_{86}X_{62} = Q_{25}X_{53}P_{36}X_{62} = Q_{25}P_{58}X_{86}X_{62} = Q_{25}X_{53}Q_{36}R_{62} = Q_{25}Q_{58}X_{86}R_{62}$	-1	-1	0
$= P_{14}Y_{42}Q_{25}Q_{58}X_{81} = P_{14}Q_{47}Y_{75}Q_{58}X_{81} = Q_{14}Y_{42}P_{25}Q_{58}X_{81} = Q_{14}P_{47}Y_{75}Q_{58}X_{81} = Q_{14}Y_{42}Q_{25}P_{58}X_{81}$			
$= Q_{14}Q_{47}Y_{75}P_{58}X_{81}$			
$Q_{25}X_{53}Q_{36}X_{62} = Q_{25}Q_{58}X_{86}X_{62} = Q_{14}Y_{42}Q_{25}Q_{58}X_{81} = Q_{14}Q_{47}Y_{75}Q_{58}X_{81}$	-2	-1	0
$P_{14}X_{46}X_{67}Y_{75}P_{58}X_{81} = X_{13}X_{34}Y_{42}P_{25}P_{58}X_{81} = X_{13}P_{36}X_{67}Y_{75}P_{58}X_{81} = X_{13}X_{34}P_{47}Y_{75}P_{58}X_{81}$	1	0	0
$= P_{14}P_{47}X_{72}X_{21} = P_{14}X_{46}R_{62}X_{21} = P_{25}Y_{58}X_{86}R_{62} = X_{13}P_{36}R_{62}X_{21} = Y_{25}X_{53}P_{36}R_{62} = Y_{25}P_{58}X_{86}R_{62}$			
$= P_{14}Y_{42}P_{25}Y_{58}X_{81} = P_{14}P_{47}Y_{75}Y_{58}X_{81} = P_{14}Y_{42}Y_{25}P_{58}X_{81} = P_{25}X_{53}P_{36}X_{67}X_{72} = P_{25}X_{53}X_{34}P_{47}X_{72}$			
$= P_{25}P_{58}X_{86}X_{67}X_{72} = P_{25}X_{53}X_{34}X_{46}R_{62}$			
$P_{14}X_{46}X_{67}Y_{75}Q_{58}X_{81} = X_{13}X_{34}Y_{42}P_{25}Q_{58}X_{81} = X_{13}P_{36}X_{67}Y_{75}Q_{58}X_{81} = X_{13}X_{34}P_{47}Y_{75}Q_{58}X_{81}$	0	0	0
$= Q_{14}X_{46}X_{67}Y_{75}P_{58}X_{81} = X_{13}X_{34}Y_{42}Q_{25}P_{58}X_{81} = X_{13}Q_{36}X_{67}Y_{75}P_{58}X_{81} = X_{13}X_{34}Q_{47}Y_{75}P_{58}X_{81}$			
$= P_{14}Q_{47}X_{72}X_{21} = P_{14}X_{46}X_{62}X_{21} = P_{25}Y_{58}X_{86}X_{62} = X_{13}P_{36}X_{62}X_{21} = Y_{25}X_{53}P_{36}X_{62} = Q_{14}P_{47}X_{72}X_{21}$			
$= Y_{25}P_{58}X_{86}X_{62} = Q_{14}X_{46}R_{62}X_{21} = Q_{25}Y_{58}X_{86}R_{62} = X_{13}Q_{36}R_{62}X_{21} = Y_{25}X_{53}Q_{36}R_{62} = Y_{25}Q_{58}X_{86}R_{62}$			
$= P_{14}Y_{42}Q_{25}Y_{58}X_{81} = P_{14}Q_{47}Y_{75}Y_{58}X_{81} = P_{14}Y_{42}Y_{25}Q_{58}X_{81} = Q_{14}Y_{42}P_{25}Y_{58}X_{81} = P_{25}X_{53}Q_{36}X_{67}X_{72}$			
$= P_{25}X_{53}X_{34}P_{47}X_{72} = P_{25}Q_{58}X_{86}X_{67}X_{72} = P_{25}X_{53}X_{34}X_{46}X_{62} = Q_{25}X_{53}P_{36}X_{67}X_{72} = Q_{14}P_{47}Y_{75}Y_{58}X_{81}$			
$= Q_{25}X_{53}X_{34}P_{47}X_{72} = Q_{14}Y_{42}Y_{25}P_{58}X_{81} = Q_{25}P_{58}X_{86}X_{67}X_{72} = Q_{25}X_{53}X_{34}X_{46}R_{62}$			
$Q_{14}X_{46}X_{67}Y_{75}Q_{58}X_{81} = X_{13}X_{34}Y_{42}Q_{25}Q_{58}X_{81} = X_{13}Q_{36}X_{67}Y_{75}Q_{58}X_{81} = X_{13}X_{34}Q_{47}Y_{75}Q_{58}X_{81}$	-1	0	0
$= Q_{14}Q_{47}X_{72}X_{21} = Q_{14}X_{46}X_{62}X_{21} = Q_{25}Y_{58}X_{86}X_{62} = X_{13}Q_{36}X_{62}X_{21} = Y_{25}X_{53}Q_{36}X_{62} = Y_{25}Q_{58}X_{86}X_{62}$			
$= Q_{14}Y_{42}Q_{25}Y_{58}X_{81} = Q_{14}Q_{47}Y_{75}Y_{58}X_{81} = Q_{14}Y_{42}Y_{25}Q_{58}X_{81} = Q_{25}X_{53}Q_{36}X_{67}X_{72} = Q_{25}X_{53}X_{34}Q_{47}X_{72}$			
$= Q_{25}Q_{58}X_{86}X_{67}X_{72} = Q_{25}X_{53}X_{34}X_{46}X_{62}$			
$Y_{25}Y_{58}X_{86}R_{62} = X_{13}X_{34}X_{46}X_{67}Y_{75}P_{58}X_{81} = P_{14}X_{46}X_{67}Y_{75}Y_{58}X_{81} = X_{13}X_{34}Y_{42}P_{25}Y_{58}X_{81}$	1	1	0
$= P_{25}X_{53}X_{34}X_{46}X_{67}X_{72} = X_{13}P_{36}X_{67}Y_{75}Y_{58}X_{81} = X_{13}X_{34}P_{47}Y_{75}Y_{58}X_{81} = X_{13}X_{34}Y_{42}Y_{25}P_{58}X_{81}$			
$= P_{14}X_{46}X_{67}X_{72}X_{21} = P_{14}Y_{42}Y_{25}Y_{58}X_{81} = P_{25}Y_{58}X_{86}X_{67}X_{72} = X_{13}P_{36}X_{67}X_{72}X_{21} = Y_{25}X_{53}P_{36}X_{67}X_{72}$			
$= X_{13}X_{34}P_{47}X_{72}X_{21} = Y_{25}X_{53}X_{34}P_{47}X_{72} = Y_{25}P_{58}X_{86}X_{67}X_{72} = X_{13}X_{34}X_{46}R_{62}X_{21} = Y_{25}X_{53}X_{34}X_{46}R_{62}$			
$Y_{25}Y_{58}X_{86}X_{62} = X_{13}X_{34}X_{46}X_{67}Y_{75}Q_{58}X_{81} = Q_{14}X_{46}X_{67}Y_{75}Y_{58}X_{81} = X_{13}X_{34}Y_{42}Q_{25}Y_{58}X_{81}$	0	1	0
$= Q_{25}X_{53}X_{34}X_{46}X_{67}X_{72} = X_{13}Q_{36}X_{67}Y_{75}Y_{58}X_{81} = X_{13}X_{34}Q_{47}Y_{75}Y_{58}X_{81} = X_{13}X_{34}Y_{42}Y_{25}Q_{58}X_{81}$			
$= Q_{14}X_{46}X_{67}X_{72}X_{21} = Q_{14}Y_{42}Y_{25}Y_{58}X_{81} = Q_{25}Y_{58}X_{86}X_{67}X_{72} = X_{13}Q_{36}X_{67}X_{72}X_{21} = Y_{25}X_{53}Q_{36}X_{67}X_{72}$			
$= X_{13}X_{34}Q_{47}X_{72}X_{21} = Y_{25}X_{53}X_{34}Q_{47}X_{72} = Y_{25}Q_{58}X_{86}X_{67}X_{72} = X_{13}X_{34}X_{46}X_{62}X_{21} = Y_{25}X_{53}X_{34}X_{46}X_{62}$			
$P_{14}P_{47}X_{72}P_{25}P_{58}X_{81} = P_{14}X_{46}R_{62}P_{25}P_{58}X_{81} = X_{13}P_{36}R_{62}P_{25}P_{58}X_{81}$	1	-1	-1
$P_{14}P_{47}X_{72}P_{25}Q_{58}X_{81} = P_{14}Q_{47}X_{72}P_{25}P_{58}X_{81} = P_{14}X_{46}X_{62}P_{25}P_{58}X_{81} = P_{14}X_{46}R_{62}P_{25}Q_{58}X_{81}$	0	-1	-1
$= P_{14}P_{47}X_{72}Q_{25}P_{58}X_{81} = P_{14}X_{46}R_{62}Q_{25}P_{58}X_{81} = X_{13}P_{36}X_{62}P_{25}P_{58}X_{81} = X_{13}P_{36}R_{62}P_{25}Q_{58}X_{81}$			
$= Q_{14}P_{47}X_{72}P_{25}P_{58}X_{81} = Q_{14}X_{46}R_{62}P_{25}P_{58}X_{81} = X_{13}Q_{36}R_{62}P_{25}P_{58}X_{81} = X_{13}P_{36}R_{62}Q_{25}P_{58}X_{81}$			
$P_{14}Q_{47}X_{72}P_{25}Q_{58}X_{81} = P_{14}X_{46}X_{62}P_{25}Q_{58}X_{81} = P_{14}P_{47}X_{72}Q_{25}Q_{58}X_{81} = P_{14}Q_{47}X_{72}Q_{25}P_{58}X_{81}$	-1	-1	-1
$= P_{14}X_{46}X_{62}Q_{25}P_{58}X_{81} = P_{14}X_{46}R_{62}Q_{25}Q_{58}X_{81} = X_{13}P_{36}X_{62}P_{25}Q_{58}X_{81} = Q_{14}P_{47}X_{72}P_{25}Q_{58}X_{81}$			
$= Q_{14}Q_{47}X_{72}P_{25}P_{58}X_{81} = Q_{14}X_{46}X_{62}P_{25}P_{58}X_{81} = X_{13}Q_{36}X_{62}P_{25}P_{58}X_{81} = Q_{14}X_{46}R_{62}P_{25}Q_{58}X_{81}$			
$= X_{13}Q_{36}R_{62}P_{25}Q_{58}X_{81} = X_{13}P_{36}X_{62}Q_{25}P_{58}X_{81} = X_{13}P_{36}R_{62}Q_{25}Q_{58}X_{81} = Q_{14}P_{47}X_{72}Q_{25}P_{58}X_{81}$			
$= Q_{14}X_{46}R_{62}Q_{25}P_{58}X_{81} = X_{13}Q_{36}R_{62}Q_{25}P_{58}X_{81}$			

Table 33. The generators in terms of bifundamental chiral fields for Model 8 (*Part 1*).

generator	SU(2) $_{\bar{x}}$	U(1) $_{\bar{b}_1}$	U(1) $_{\bar{b}_2}$
$P_{14}Q_{47}X_{72}Q_{25}Q_{58}X_{81} = P_{14}X_{46}X_{62}Q_{25}Q_{58}X_{81} = Q_{14}Q_{47}X_{72}P_{25}Q_{58}X_{81} = Q_{14}X_{46}X_{62}P_{25}Q_{58}X_{81}$	-2	-1	-1
$= X_{13}Q_{36}X_{62}P_{25}Q_{58}X_{81} = X_{13}P_{36}X_{62}Q_{25}Q_{58}X_{81} = Q_{14}P_{47}X_{72}Q_{25}Q_{58}X_{81} = Q_{14}Q_{47}X_{72}Q_{25}P_{58}X_{81}$			
$= Q_{14}X_{46}X_{62}Q_{25}P_{58}X_{81} = X_{13}Q_{36}X_{62}Q_{25}P_{58}X_{81} = Q_{14}X_{46}R_{62}Q_{25}Q_{58}X_{81} = X_{13}Q_{36}R_{62}Q_{25}Q_{58}X_{81}$			
$Q_{14}Q_{47}X_{72}Q_{25}Q_{58}X_{81} = P_{14}X_{46}X_{62}Q_{25}Q_{58}X_{81} = X_{13}Q_{36}X_{62}Q_{25}Q_{58}X_{81}$	-3	-1	-1
$Y_{25}Y_{58}X_{86}X_{67}X_{72} = X_{13}X_{34}X_{46}X_{67}Y_{75}Y_{58}X_{81} = X_{13}X_{34}X_{46}X_{67}X_{72}X_{21} = X_{13}X_{34}Y_{42}Y_{25}Y_{58}X_{81}$	1	2	0
$= Y_{25}X_{53}X_{34}X_{46}X_{67}X_{72}$			
$P_{14}X_{46}X_{67}X_{72}P_{25}P_{58}X_{81} = X_{13}P_{36}X_{67}X_{72}P_{25}P_{58}X_{81} = X_{13}X_{34}P_{47}X_{72}P_{25}P_{58}X_{81}$	1	0	-1
$= X_{13}X_{34}X_{46}R_{62}P_{25}P_{58}X_{81} = P_{14}P_{47}X_{72}P_{25}Y_{58}X_{81} = P_{14}X_{46}R_{62}P_{25}Y_{58}X_{81}$			
$= P_{14}P_{47}X_{72}Y_{25}P_{58}X_{81} = P_{14}X_{46}R_{62}Y_{25}P_{58}X_{81} = X_{13}P_{36}R_{62}Y_{25}P_{58}X_{81} = X_{13}P_{36}R_{62}Y_{25}P_{58}X_{81}$			
$P_{14}X_{46}X_{67}X_{72}P_{25}Q_{58}X_{81} = P_{14}X_{46}X_{67}X_{72}Q_{25}P_{58}X_{81} = X_{13}P_{36}X_{67}X_{72}P_{25}Q_{58}X_{81}$	0	0	-1
$= X_{13}X_{34}P_{47}X_{72}P_{25}Q_{58}X_{81} = Q_{14}X_{46}X_{67}X_{72}P_{25}P_{58}X_{81} = X_{13}Q_{36}X_{67}X_{72}P_{25}P_{58}X_{81}$			
$= X_{13}X_{34}Q_{47}X_{72}P_{25}P_{58}X_{81} = X_{13}X_{34}X_{46}X_{62}P_{25}P_{58}X_{81} = X_{13}X_{34}X_{46}R_{62}P_{25}Q_{58}X_{81}$			
$= X_{13}P_{36}X_{67}X_{72}Q_{25}P_{58}X_{81} = X_{13}X_{34}P_{47}X_{72}Q_{25}P_{58}X_{81} = X_{13}X_{34}X_{46}R_{62}Q_{25}P_{58}X_{81}$			
$= P_{14}Q_{47}X_{72}P_{25}Y_{58}X_{81} = P_{14}X_{46}X_{62}P_{25}Y_{58}X_{81} = P_{14}P_{47}X_{72}Q_{25}Y_{58}X_{81} = P_{14}P_{47}X_{72}Y_{25}Q_{58}X_{81}$			
$= P_{14}Q_{47}X_{72}Y_{25}P_{58}X_{81} = P_{14}X_{46}X_{62}Y_{25}P_{58}X_{81} = P_{14}X_{46}R_{62}Q_{25}Y_{58}X_{81} = P_{14}X_{46}R_{62}Y_{25}Q_{58}X_{81}$			
$= X_{13}P_{36}X_{62}P_{25}Y_{58}X_{81} = Q_{14}P_{47}X_{72}P_{25}Y_{58}X_{81} = Q_{14}X_{46}R_{62}P_{25}Y_{58}X_{81} = X_{13}Q_{36}R_{62}P_{25}Y_{58}X_{81}$			
$= X_{13}P_{36}X_{62}Y_{25}P_{58}X_{81} = X_{13}P_{36}R_{62}Q_{25}Y_{58}X_{81} = X_{13}P_{36}R_{62}Y_{25}Q_{58}X_{81} = Q_{14}P_{47}X_{72}Y_{25}P_{58}X_{81}$			
$= Q_{14}X_{46}R_{62}Y_{25}P_{58}X_{81} = X_{13}Q_{36}R_{62}Y_{25}P_{58}X_{81}$			
$P_{14}X_{46}X_{67}X_{72}Q_{25}Q_{58}X_{81} = Q_{14}X_{46}X_{67}X_{72}P_{25}Q_{58}X_{81} = X_{13}Q_{36}X_{67}X_{72}P_{25}Q_{58}X_{81}$	-1	0	-1
$= X_{13}X_{34}Q_{47}X_{72}P_{25}Q_{58}X_{81} = X_{13}X_{34}X_{46}X_{62}P_{25}Q_{58}X_{81} = X_{13}P_{36}X_{67}X_{72}Q_{25}Q_{58}X_{81}$			
$= X_{13}X_{34}P_{47}X_{72}Q_{25}Q_{58}X_{81} = Q_{14}X_{46}X_{67}X_{72}Q_{25}P_{58}X_{81} = X_{13}Q_{36}X_{67}X_{72}Q_{25}P_{58}X_{81}$			
$= X_{13}X_{34}Q_{47}X_{72}Q_{25}P_{58}X_{81} = X_{13}X_{34}X_{46}X_{62}Q_{25}P_{58}X_{81} = X_{13}X_{34}X_{46}R_{62}Q_{25}Q_{58}X_{81}$			
$= P_{14}Q_{47}X_{72}Q_{25}Y_{58}X_{81} = P_{14}X_{46}X_{62}Q_{25}Y_{58}X_{81} = P_{14}Q_{47}X_{72}Y_{25}Q_{58}X_{81} = P_{14}X_{46}X_{62}Y_{25}Q_{58}X_{81}$			
$= Q_{14}Q_{47}X_{72}Y_{25}Y_{58}X_{81} = Q_{14}X_{46}X_{62}Y_{25}Y_{58}X_{81} = X_{13}Q_{36}X_{62}Y_{25}Y_{58}X_{81} = X_{13}P_{36}X_{62}Q_{25}Y_{58}X_{81}$			
$= X_{13}P_{36}X_{62}Y_{25}Q_{58}X_{81} = Q_{14}P_{47}X_{72}Y_{25}Y_{58}X_{81} = Q_{14}P_{47}X_{72}Y_{25}Q_{58}X_{81} = Q_{14}Q_{47}X_{72}Y_{25}P_{58}X_{81}$			
$= Q_{14}X_{46}X_{62}Y_{25}P_{58}X_{81} = X_{13}Q_{36}X_{62}Y_{25}P_{58}X_{81} = Q_{14}X_{46}R_{62}Q_{25}Y_{58}X_{81} = Q_{14}X_{46}R_{62}Y_{25}Q_{58}X_{81}$			
$= X_{13}Q_{36}R_{62}Q_{25}Y_{58}X_{81} = X_{13}Q_{36}R_{62}Y_{25}Q_{58}X_{81}$			
$Q_{14}X_{46}X_{67}X_{72}Q_{25}Q_{58}X_{81} = X_{13}Q_{36}X_{67}X_{72}Q_{25}Q_{58}X_{81} = X_{13}X_{34}Q_{47}X_{72}Q_{25}Q_{58}X_{81}$	-2	0	-1
$= X_{13}X_{34}X_{46}X_{62}Q_{25}Q_{58}X_{81} = Q_{14}Q_{47}X_{72}Q_{25}Y_{58}X_{81} = Q_{14}X_{46}X_{62}Q_{25}Y_{58}X_{81}$			
$= Q_{14}Q_{47}X_{72}Y_{25}Q_{58}X_{81} = Q_{14}X_{46}X_{62}Y_{25}Q_{58}X_{81} = X_{13}Q_{36}X_{62}Q_{25}Y_{58}X_{81} = X_{13}Q_{36}X_{62}Y_{25}Q_{58}X_{81}$			
$X_{13}X_{34}X_{46}X_{67}X_{72}P_{25}P_{58}X_{81} = P_{14}P_{47}X_{72}Y_{25}Y_{58}X_{81} = P_{14}X_{46}R_{62}Y_{25}Y_{58}X_{81}$	1	1	-1
$= X_{13}P_{36}R_{62}Y_{25}Y_{58}X_{81} = P_{14}X_{46}X_{67}X_{72}P_{25}Y_{58}X_{81} = P_{14}X_{46}X_{67}X_{72}Y_{25}P_{58}X_{81}$			
$= X_{13}P_{36}X_{67}X_{72}P_{25}Y_{58}X_{81} = X_{13}X_{34}P_{47}X_{72}P_{25}Y_{58}X_{81} = X_{13}X_{34}X_{46}R_{62}P_{25}Y_{58}X_{81}$			
$= X_{13}P_{36}X_{67}X_{72}Y_{25}P_{58}X_{81} = X_{13}X_{34}P_{47}X_{72}Y_{25}P_{58}X_{81} = X_{13}X_{34}X_{46}R_{62}Y_{25}P_{58}X_{81}$			
$X_{13}X_{34}X_{46}X_{67}X_{72}P_{25}Q_{58}X_{81} = X_{13}X_{34}X_{46}X_{67}X_{72}Q_{25}P_{58}X_{81} = P_{14}Q_{47}X_{72}Y_{25}Y_{58}X_{81}$	0	1	-1
$= P_{14}X_{46}X_{62}Y_{25}Y_{58}X_{81} = X_{13}P_{36}X_{62}Y_{25}Y_{58}X_{81} = Q_{14}P_{47}X_{72}Y_{25}Y_{58}X_{81} = Q_{14}X_{46}R_{62}Y_{25}Y_{58}X_{81}$			
$= X_{13}Q_{36}R_{62}Y_{25}Y_{58}X_{81} = P_{14}X_{46}X_{67}X_{72}Q_{25}Y_{58}X_{81} = P_{14}X_{46}X_{67}X_{72}Y_{25}Q_{58}X_{81}$			
$= Q_{14}X_{46}X_{67}X_{72}P_{25}Y_{58}X_{81} = X_{13}Q_{36}X_{67}X_{72}P_{25}Y_{58}X_{81} = X_{13}X_{34}Q_{47}X_{72}P_{25}Y_{58}X_{81}$			
$= X_{13}X_{34}X_{46}X_{62}P_{25}Y_{58}X_{81} = X_{13}P_{36}X_{67}X_{72}Q_{25}Y_{58}X_{81} = X_{13}P_{36}X_{67}X_{72}Y_{25}Q_{58}X_{81}$			
$= X_{13}X_{34}P_{47}X_{72}Q_{25}Y_{58}X_{81} = X_{13}X_{34}P_{47}X_{72}Y_{25}Q_{58}X_{81} = Q_{14}X_{46}X_{67}X_{72}Y_{25}P_{58}X_{81}$			
$= X_{13}Q_{36}X_{67}X_{72}Y_{25}P_{58}X_{81} = X_{13}X_{34}Q_{47}X_{72}Y_{25}P_{58}X_{81} = X_{13}X_{34}X_{46}X_{62}Y_{25}P_{58}X_{81}$			
$= X_{13}X_{34}X_{46}R_{62}Q_{25}Y_{58}X_{81} = X_{13}X_{34}X_{46}R_{62}Y_{25}Q_{58}X_{81}$			
$X_{13}X_{34}X_{46}X_{67}X_{72}Q_{25}Q_{58}X_{81} = P_{14}Q_{47}X_{72}Y_{25}Y_{58}X_{81} = Q_{14}X_{46}X_{62}Y_{25}Y_{58}X_{81}$	-1	1	-1
$= X_{13}Q_{36}X_{62}Y_{25}Y_{58}X_{81} = Q_{14}X_{46}X_{67}X_{72}Q_{25}Y_{58}X_{81} = Q_{14}X_{46}X_{67}X_{72}Y_{25}Q_{58}X_{81}$			
$= X_{13}Q_{36}X_{67}X_{72}Q_{25}Y_{58}X_{81} = X_{13}X_{34}Q_{47}X_{72}Q_{25}Y_{58}X_{81} = X_{13}X_{34}X_{46}X_{62}Q_{25}Y_{58}X_{81}$			
$= X_{13}Q_{36}X_{67}X_{72}Y_{25}Q_{58}X_{81} = X_{13}X_{34}Q_{47}X_{72}Y_{25}Q_{58}X_{81} = X_{13}X_{34}X_{46}X_{62}Y_{25}Q_{58}X_{81}$			
$X_{13}X_{34}X_{46}X_{67}X_{72}P_{25}Y_{58}X_{81} = X_{13}X_{34}X_{46}X_{67}X_{72}Y_{25}P_{58}X_{81} = P_{14}X_{46}X_{67}X_{72}Y_{25}Y_{58}X_{81}$	1	2	-1
$= X_{13}P_{36}X_{67}X_{72}Y_{25}Y_{58}X_{81} = X_{13}X_{34}P_{47}X_{72}Y_{25}Y_{58}X_{81} = X_{13}X_{34}X_{46}R_{62}Y_{25}Y_{58}X_{81}$			
$X_{13}X_{34}X_{46}X_{67}X_{72}Q_{25}Y_{58}X_{81} = X_{13}X_{34}X_{46}X_{67}X_{72}Q_{25}Q_{58}X_{81} = Q_{14}X_{46}X_{67}X_{72}Y_{25}Y_{58}X_{81}$	0	2	-1
$= X_{13}Q_{36}X_{67}X_{72}Y_{25}Y_{58}X_{81} = X_{13}X_{34}Q_{47}X_{72}Y_{25}Y_{58}X_{81} = X_{13}X_{34}X_{46}X_{62}Y_{25}Y_{58}X_{81}$			
$X_{13}X_{34}X_{46}X_{67}X_{72}Y_{25}Y_{58}X_{81}$	1	3	-1

Table 34. The generators in terms of bifundamental chiral fields for Model 8 (Part 2).

12 Model 9: $Y^{2,1}(\mathbb{CP}^1 \times \mathbb{CP}^1)$ [$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1))$, $\langle 27 \rangle$]

Model 9 corresponds to the $Y^{p,k}(\mathbb{CP}^1 \times \mathbb{CP}^1)$ class of toric Calabi-Yau 4-folds. It is $Y^{2,1}(\mathbb{CP}^1 \times \mathbb{CP}^1)$ and the corresponding brane brick model has the quiver in figure 26. The J - and E -terms are

$$\begin{array}{ll}
 & J \\
 \Lambda_{17}^1 : & X_{74}Q_{48}Y_{82}Y_{21} - X_{72}Q_{26}R_{61} \\
 \Lambda_{17}^2 : & X_{72}Q_{26}T_{61} - X_{74}Q_{48}X_{82}Y_{21} \\
 \Lambda_{17}^3 : & X_{74}P_{48}Y_{82}Y_{21} - X_{72}P_{26}R_{61} \\
 \Lambda_{17}^4 : & X_{72}P_{26}T_{61} - X_{74}P_{48}X_{82}Y_{21} \\
 \Lambda_{18}^1 : & X_{82}Q_{26}R_{61} - Y_{82}Q_{26}T_{61} \\
 \Lambda_{18}^2 : & X_{82}P_{26}R_{61} - Y_{82}P_{26}T_{61} \\
 \Lambda_{25}^1 : & X_{58}X_{82} - X_{57}X_{72} \\
 \Lambda_{25}^2 : & X_{57}X_{74}Q_{48}Y_{82} - Y_{57}X_{74}Q_{48}X_{82} \\
 \Lambda_{25}^3 : & Y_{57}X_{72} - X_{58}Y_{82} \\
 \Lambda_{25}^4 : & X_{57}X_{74}P_{48}Y_{82} - Y_{57}X_{74}P_{48}X_{82} \\
 \Lambda_{36}^1 : & T_{61}Y_{13} - R_{61}X_{13} \\
 \Lambda_{38}^1 : & Y_{82}Y_{21}X_{13} - X_{82}Y_{21}Y_{13} \\
 \Lambda_{46}^1 : & R_{61}X_{14} - Y_{65}Y_{57}X_{74} \\
 \Lambda_{46}^2 : & Y_{65}X_{57}X_{74} - T_{61}X_{14} \\
 & E \\
 & P_{15}X_{57} - X_{13}P_{37} \\
 & P_{15}Y_{57} - Y_{13}P_{37} \\
 & X_{13}Q_{37} - Q_{15}X_{57} \\
 & Y_{13}Q_{37} - Q_{15}Y_{57} \\
 & P_{15}X_{58} - X_{14}P_{48} \\
 & X_{14}Q_{48} - Q_{15}X_{58} \\
 & P_{26}R_{61}Q_{15} - Q_{26}R_{61}P_{15} \\
 & P_{26}Y_{65} - Y_{21}P_{15} \\
 & P_{26}T_{61}Q_{15} - Q_{26}T_{61}P_{15} \\
 & Y_{21}Q_{15} - Q_{26}Y_{65} \\
 & P_{37}X_{72}Q_{26} - Q_{37}X_{72}P_{26} \\
 & P_{37}X_{74}Q_{48} - Q_{37}X_{74}P_{48} \\
 & P_{48}X_{82}Q_{26} - Q_{48}X_{82}P_{26} \\
 & P_{48}Y_{82}Q_{26} - Q_{48}Y_{82}P_{26}
 \end{array} \quad (12.1)$$

Following the forward algorithm, we obtain the brick matchings for Model 9. The brick matchings are summarized in the P -matrix, which takes the form

$$P = \begin{pmatrix}
 & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & o_7 & o_8 & o_9 & o_{10} \\
 P_{15} & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\
 P_{26} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
 P_{37} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 2 & 0 \\
 P_{48} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
 Q_{15} & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\
 Q_{26} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
 Q_{37} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 2 & 0 \\
 Q_{48} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
 R_{61} & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
 T_{61} & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
 X_{13} & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 2 \\
 X_{14} & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
 X_{57} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
 X_{58} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
 X_{72} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 X_{74} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 X_{82} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
 Y_{13} & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 2 \\
 Y_{21} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 Y_{57} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
 Y_{65} & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
 Y_{82} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1
 \end{pmatrix} \quad (12.2)$$

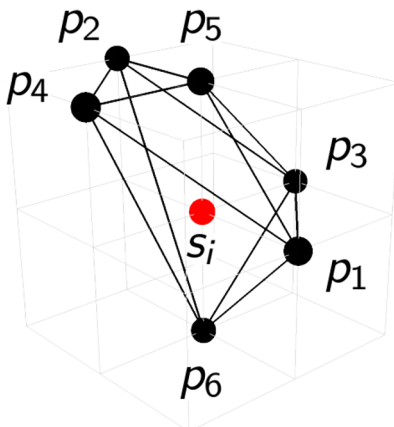


Figure 27. Toric diagram for Model 9.

The Hilbert series of the mesonic moduli space of Model 9 takes the form

$$\begin{aligned}
 g_1(t_i, y_s, y_{o_1}, y_{o_2}; \mathcal{M}_9) &= \frac{P(t_i, y_s, y_{o_1}, y_{o_2}; \mathcal{M}_9)}{(1 - y_s y_{o_1}^3 y_{o_2}^4 t_1^3 t_3^3 t_5^2)(1 - y_s y_{o_1}^3 y_{o_2}^4 t_2^3 t_3^3 t_5^2)} \\
 &\times \frac{1}{(1 - y_s y_{o_1}^3 y_{o_2}^4 t_1^3 t_4^3 t_5^2)(1 - y_s y_{o_1}^3 y_{o_2}^4 t_2^3 t_4^3 t_5^2)(1 - y_s y_{o_1} y_{o_2}^2 t_1 t_3 t_6^2)} \\
 &\times \frac{1}{(1 - y_s y_{o_1} y_{o_2}^2 t_2 t_3 t_6^2)(1 - y_s y_{o_1} y_{o_2}^2 t_1 t_4 t_6^2)(1 - y_s y_{o_1} y_{o_2}^2 t_2 t_4 t_6^2)}, \tag{12.6}
 \end{aligned}$$

where t_i are the fugacities for the extremal brick matchings p_i . y_s counts the brick matching product $s_1 \dots s_9$ corresponding to the single internal point of the toric diagram of Model 9. Additionally, y_{o_1} and y_{o_2} count the products of extra GLSM fields $o_1 \dots o_9$ and $o_9 o_{10}$, respectively. The explicit numerator $P(t_i, y_s, y_{o_1}, y_{o_2}; \mathcal{M}_9)$ of the Hilbert series is given in the appendix section A.9. We note that setting the fugacities $y_{o_1} = 1$ and $y_{o_2} = 1$ does not change the overall characterization of the mesonic moduli space by the Hilbert series, indicating that the extra GLSM fields, as expected, correspond to an over-parameterization of the moduli space.

By setting $t_i = t$ for the fugacities of the extremal brick matchings, and all other fugacities to $y_s = 1$, $y_{o_1} = 1$ and $y_{o_2} = 1$, the unrefined Hilbert series takes the following form

$$\begin{aligned}
 g_1(t, 1, 1, 1; \mathcal{M}_9) &= \frac{1}{(1 - t^2)(1 - t^8)^3} \times (1 - t^2 + 4t^4 + 5t^6 + 13t^8 - 6t^{10} + 13t^{12} \\
 &\quad + 5t^{14} + 4t^{16} - t^{18} + t^{20}), \tag{12.7}
 \end{aligned}$$

where the palindromic numerator indicates that the mesonic moduli space is Calabi-Yau.

The global symmetry of Model 9 and the charges on the extremal brick matchings under the global symmetry are summarized in table 35. We can use the following fugacity map,

$$t = t_1^{1/2} t_2^{1/2}, \quad x = \frac{t_1^{1/2}}{t_2^{1/2}}, \quad y = \frac{t_3}{t_1^{1/2} t_2^{1/2}}, \quad b = \frac{t_1^{1/2} t_2^{1/2}}{t_6}, \tag{12.8}$$

	SU(2) _x	SU(2) _y	U(1) _b	U(1)	fugacity
p_1	+1	0	0	r_1	t_1
p_2	-1	0	0	r_2	t_2
p_3	0	+1	0	r_3	t_3
p_4	0	-1	0	r_4	t_4
p_5	0	0	+1	r_5	t_5
p_6	0	0	-1	r_6	t_6

Table 35. Global symmetry charges on the extremal brick matchings p_i of Model 9.

where $t_3t_4 = t_5t_6$ and $t_1t_2 = t_5t_6$, in order to rewrite the Hilbert series for Model 8 in terms of characters of irreducible representations of $SU(2) \times SU(2) \times U(1)$. In highest weight form, the Hilbert series of Model 9 can be written as

$$h_1(t, \mu_1, \mu_2, b; \mathcal{M}_9) = \frac{1 + \mu_1^2 \mu_2^2 t^6}{(1 - \mu_1 \mu_2 b^{-2} t^4)(1 - \mu_1^3 \mu_2^3 b^2 t^8)}, \tag{12.9}$$

where $\mu_1^m \mu_2^n \sim [m]_{SU(2)_x} [n]_{SU(2)_y}$. Here in highest weight form, the fugacities μ_1 and μ_2 count the highest weight of irreducible representations of $SU(2)_x \times SU(2)_y$.

The plethystic logarithm of the Hilbert series takes the form

$$\begin{aligned} g_1(t, x, y, b; \mathcal{M}_9) &= [1; 1]b^{-2}t^4 + [2; 2]t^6 + [3; 3]b^2t^8 - b^{-4}t^8 \\ &\quad - ([1; 3]b^{-2} + [3; 1]b^{-2} + [1; 1]b^{-2})t^{10} - ([4; 4] + [2; 4] + [4; 2] + [0; 4] \\ &\quad + [4; 0] + 2[2; 2] + 1)t^{12} - [3; 5]b^2t^{14} + \dots, \end{aligned} \tag{12.10}$$

where $[m; n] = [m]_{SU(2)_x} [n]_{SU(2)_y}$. From the plethystic logarithm, we see that the mesonic moduli space is a non-complete intersection.

The generators form 3 sets that transform under $[1; 1]b^{-2}$, $[2; 2]$ and $[3; 3]b^2$ of the mesonic flavor symmetry of Model 9, respectively. Using the following fugacity map

$$\tilde{t} = t_5^{1/2} t_6^{1/2}, \quad \tilde{x} = \frac{t_1}{t_2}, \quad \tilde{y} = \frac{t_3}{t_4}, \quad \tilde{b} = \frac{t_5^2}{t_2 t_4}, \tag{12.11}$$

where $t_1t_2 = t_3t_4 = t_5t_6$, the mesonic flavor charges on the gauge invariant operators become \mathbb{Z} -valued. The generators in terms of brick matchings and their corresponding rescaled mesonic flavor charges are summarized in table 36. The generator lattice as shown in table 36 is a convex lattice polytope, which is reflexive. It is the dual of the toric diagram of Model 9 shown in figure 27. We also note that the 3 layers of points in the generator lattice in table 36 corresponds to the 3 sets that transform under $[1; 1]b^{-2}$, $[2; 2]$ and $[3; 3]b^2$ of the mesonic flavor symmetry. For completeness, table 37 and table 38 show the generators of Model 9 in terms of chiral fields with the corresponding mesonic flavor charges.

generator	$SU(2)_{\bar{x}}$	$SU(2)_{\bar{y}}$	$U(1)_{\bar{b}}$
$p_1 p_3 p_6^2 so_1 o_2^2$	1	1	-1
$p_2 p_3 p_6^2 so_1 o_2^2$	0	1	-1
$p_1 p_4 p_6^2 so_1 o_2^2$	1	0	-1
$p_2 p_4 p_6^2 so_1 o_2^2$	0	0	-1
$p_1^2 p_3^2 p_5 p_6 so_1^2 o_2^3$	1	1	0
$p_1 p_2 p_3^2 p_5 p_6 so_1^2 o_2^3$	0	1	0
$p_2^2 p_3^2 p_5 p_6 so_1^2 o_2^3$	-1	1	0
$p_1^2 p_3 p_4 p_5 p_6 so_1^2 o_2^3$	1	0	0
$p_1 p_2 p_3 p_4 p_5 p_6 so_1^2 o_2^3$	0	0	0
$p_2^2 p_3 p_4 p_5 p_6 so_1^2 o_2^3$	-1	0	0
$p_1^2 p_4 p_5 p_6 so_1^2 o_2^3$	1	-1	0
$p_1 p_2 p_4^2 p_5 p_6 so_1^2 o_2^3$	0	-1	0
$p_2^2 p_4^2 p_5 p_6 so_1^2 o_2^3$	-1	-1	0
$p_1^3 p_3^3 p_5^2 so_1^3 o_2^4$	1	1	1
$p_1^2 p_2 p_3^3 p_5^2 so_1^3 o_2^4$	0	1	1
$p_1 p_2^2 p_3^3 p_5^2 so_1^3 o_2^4$	-1	1	1
$p_2^3 p_3^3 p_5^2 so_1^3 o_2^4$	-2	1	1
$p_1^3 p_3^2 p_4 p_5^2 so_1^3 o_2^4$	1	0	1
$p_1^2 p_2 p_3^2 p_4 p_5^2 so_1^3 o_2^4$	0	0	1
$p_1 p_2^2 p_3^2 p_4 p_5^2 so_1^3 o_2^4$	-1	0	1
$p_2^3 p_3^2 p_4 p_5^2 so_1^3 o_2^4$	-2	0	1
$p_1^3 p_3 p_4^2 p_5^2 so_1^3 o_2^4$	1	-1	1
$p_1^2 p_2 p_3 p_4^2 p_5^2 so_1^3 o_2^4$	0	-1	1
$p_1 p_2^2 p_3 p_4^2 p_5^2 so_1^3 o_2^4$	-1	-1	1
$p_2^3 p_3 p_4^2 p_5^2 so_1^3 o_2^4$	-2	-1	1
$p_1^3 p_4^3 p_5^2 so_1^3 o_2^4$	1	-2	1
$p_1^2 p_2 p_4^3 p_5^2 so_1^3 o_2^4$	0	-2	1
$p_1 p_2^2 p_4^3 p_5^2 so_1^3 o_2^4$	-1	-2	1
$p_2^3 p_4^3 p_5^2 so_1^3 o_2^4$	-2	-2	1

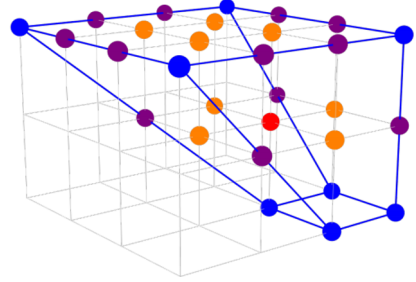


Table 36. The generators and lattice of generators of the mesonic moduli space of Model 9 in terms of brick matchings with the corresponding flavor charges.

generator	$SU(2)_{\bar{x}}$	$SU(2)_{\bar{y}}$	$U(1)_{\bar{b}}$
$P_{15}X_{58}Y_{82}Y_{21} = P_{15}Y_{57}X_{72}Y_{21} = P_{26}Y_{65}X_{58}Y_{82} = P_{26}Y_{65}Y_{57}X_{72} = Y_{13}P_{37}X_{72}Y_{21} = X_{14}P_{48}Y_{82}Y_{21}$	1	1	-1
$Q_{15}X_{58}Y_{82}Y_{21} = Q_{15}Y_{57}X_{72}Y_{21} = Q_{26}Y_{65}X_{58}Y_{82} = Q_{26}Y_{65}Y_{57}X_{72} = Y_{13}Q_{37}X_{72}Y_{21} = X_{14}Q_{48}Y_{82}Y_{21}$	0	1	-1
$P_{15}X_{57}X_{72}Y_{21} = P_{15}X_{58}X_{82}Y_{21} = P_{26}Y_{65}X_{57}X_{72} = P_{26}Y_{65}X_{58}X_{82} = X_{13}P_{37}X_{72}Y_{21} = X_{14}P_{48}X_{82}Y_{21}$	1	0	-1
$Q_{15}X_{57}X_{72}Y_{21} = Q_{15}X_{58}X_{82}Y_{21} = Q_{26}Y_{65}X_{57}X_{72} = Q_{26}Y_{65}X_{58}X_{82} = X_{13}Q_{37}X_{72}Y_{21} = X_{14}Q_{48}X_{82}Y_{21}$	0	0	-1
$P_{15}Y_{57}X_{74}P_{48}Y_{82}Y_{21} = P_{26}Y_{65}Y_{57}X_{74}P_{48}Y_{82} = Y_{13}P_{37}X_{74}P_{48}Y_{82}Y_{21} = P_{15}X_{58}Y_{82}P_{26}R_{61}$ $= P_{15}Y_{57}X_{72}P_{26}R_{61} = Y_{13}P_{37}X_{72}P_{26}R_{61} = X_{14}P_{48}Y_{82}P_{26}R_{61}$	1	1	0
$P_{15}Y_{57}X_{74}Q_{48}Y_{82}Y_{21} = P_{26}Y_{65}Y_{57}X_{74}Q_{48}Y_{82} = Y_{13}P_{37}X_{74}Q_{48}Y_{82}Y_{21} = Q_{15}Y_{57}X_{74}P_{48}Y_{82}Y_{21}$ $= Q_{26}Y_{65}Y_{57}X_{74}P_{48}Y_{82} = Y_{13}Q_{37}X_{74}P_{48}Y_{82}Y_{21} = P_{15}X_{58}Y_{82}Q_{26}R_{61} = P_{15}Y_{57}X_{72}Q_{26}R_{61}$ $= Q_{15}X_{58}Y_{82}P_{26}R_{61} = Q_{15}Y_{57}X_{72}P_{26}R_{61} = Y_{13}Q_{37}X_{72}P_{26}R_{61} = X_{14}Q_{48}Y_{82}P_{26}R_{61} = Y_{13}P_{37}X_{72}Q_{26}R_{61}$ $= X_{14}P_{48}Y_{82}Q_{26}R_{61}$	0	1	0
$Q_{15}Y_{57}X_{74}Q_{48}Y_{82}Y_{21} = Q_{26}Y_{65}Y_{57}X_{74}Q_{48}Y_{82} = Y_{13}Q_{37}X_{74}Q_{48}Y_{82}Y_{21} = Q_{15}X_{58}Y_{82}Q_{26}R_{61}$ $= Q_{15}Y_{57}X_{72}Q_{26}R_{61} = Y_{13}Q_{37}X_{72}Q_{26}R_{61} = X_{14}Q_{48}Y_{82}Q_{26}R_{61}$	-1	1	0
$P_{15}X_{57}X_{74}P_{48}Y_{82}Y_{21} = P_{15}Y_{57}X_{74}P_{48}X_{82}Y_{21} = P_{26}Y_{65}X_{57}X_{74}P_{48}Y_{82} = P_{26}Y_{65}Y_{57}X_{74}P_{48}X_{82}$ $= X_{13}P_{37}X_{74}P_{48}Y_{82}Y_{21} = Y_{13}P_{37}X_{74}P_{48}X_{82}Y_{21} = P_{15}X_{57}X_{72}P_{26}R_{61} = P_{15}X_{58}X_{82}P_{26}R_{61}$ $= P_{15}X_{58}Y_{82}P_{26}T_{61} = P_{15}Y_{57}X_{72}P_{26}T_{61} = X_{13}P_{37}X_{72}P_{26}R_{61} = Y_{13}P_{37}X_{72}P_{26}T_{61} = X_{14}P_{48}X_{82}P_{26}R_{61}$ $= X_{14}P_{48}Y_{82}P_{26}T_{61}$	1	0	0
$P_{15}X_{57}X_{74}Q_{48}Y_{82}Y_{21} = P_{15}Y_{57}X_{74}Q_{48}X_{82}Y_{21} = P_{26}Y_{65}X_{57}X_{74}Q_{48}Y_{82} = P_{26}Y_{65}Y_{57}X_{74}Q_{48}X_{82}$ $= X_{13}P_{37}X_{74}Q_{48}Y_{82}Y_{21} = Y_{13}P_{37}X_{74}Q_{48}X_{82}Y_{21} = Q_{15}X_{57}X_{74}P_{48}Y_{82}Y_{21} = Q_{15}Y_{57}X_{74}P_{48}X_{82}Y_{21}$ $= Q_{26}Y_{65}X_{57}X_{74}P_{48}Y_{82} = Q_{26}Y_{65}Y_{57}X_{74}P_{48}X_{82} = X_{13}Q_{37}X_{74}P_{48}Y_{82}Y_{21} = Y_{13}Q_{37}X_{74}P_{48}X_{82}Y_{21}$ $= P_{15}X_{57}X_{72}Q_{26}R_{61} = P_{15}X_{58}X_{82}Q_{26}R_{61} = P_{15}X_{58}Y_{82}Q_{26}T_{61} = P_{15}Y_{57}X_{72}Q_{26}T_{61} = Q_{15}X_{57}X_{72}P_{26}R_{61}$ $= Q_{15}X_{58}X_{82}P_{26}R_{61} = Q_{15}X_{58}Y_{82}P_{26}T_{61} = Q_{15}Y_{57}X_{72}P_{26}T_{61} = X_{13}Q_{37}X_{72}P_{26}R_{61} = Y_{13}Q_{37}X_{72}P_{26}T_{61}$ $= X_{14}Q_{48}X_{82}P_{26}R_{61} = X_{14}Q_{48}Y_{82}P_{26}T_{61} = X_{13}P_{37}X_{72}Q_{26}R_{61} = Y_{13}P_{37}X_{72}Q_{26}T_{61} = X_{14}P_{48}X_{82}Q_{26}R_{61}$ $= X_{14}P_{48}Y_{82}Q_{26}T_{61}$	0	0	0
$Q_{15}X_{57}X_{74}Q_{48}Y_{82}Y_{21} = Q_{15}Y_{57}X_{74}Q_{48}X_{82}Y_{21} = Q_{26}Y_{65}X_{57}X_{74}Q_{48}Y_{82} = Q_{26}Y_{65}Y_{57}X_{74}Q_{48}X_{82}$ $= X_{13}Q_{37}X_{74}Q_{48}Y_{82}Y_{21} = Y_{13}Q_{37}X_{74}Q_{48}X_{82}Y_{21} = Q_{15}X_{57}X_{74}P_{48}Y_{82}Y_{21} = Q_{15}Y_{57}X_{74}P_{48}X_{82}Y_{21}$ $= Q_{26}Y_{65}X_{57}X_{74}P_{48}Y_{82} = Q_{26}Y_{65}Y_{57}X_{74}P_{48}X_{82} = X_{13}Q_{37}X_{74}P_{48}Y_{82}Y_{21} = Y_{13}Q_{37}X_{74}P_{48}X_{82}Y_{21}$ $= P_{15}X_{57}X_{72}Q_{26}R_{61} = P_{15}X_{58}X_{82}Q_{26}R_{61} = P_{15}X_{58}Y_{82}Q_{26}T_{61} = P_{15}Y_{57}X_{72}Q_{26}T_{61} = Q_{15}X_{57}X_{72}P_{26}R_{61}$ $= Q_{15}X_{58}X_{82}P_{26}R_{61} = Q_{15}X_{58}Y_{82}P_{26}T_{61} = Q_{15}Y_{57}X_{72}P_{26}T_{61} = X_{13}Q_{37}X_{72}P_{26}R_{61} = Y_{13}Q_{37}X_{72}P_{26}T_{61}$ $= X_{14}Q_{48}X_{82}P_{26}R_{61} = X_{14}Q_{48}Y_{82}P_{26}T_{61} = X_{13}P_{37}X_{72}Q_{26}R_{61} = Y_{13}P_{37}X_{72}Q_{26}T_{61} = X_{14}P_{48}X_{82}Q_{26}R_{61}$ $= X_{14}P_{48}Y_{82}Q_{26}T_{61}$	-1	0	0
$P_{15}X_{57}X_{74}P_{48}X_{82}Y_{21} = P_{26}Y_{65}X_{57}X_{74}P_{48}X_{82} = X_{13}P_{37}X_{74}P_{48}X_{82}Y_{21} = P_{15}X_{57}X_{72}P_{26}T_{61}$ $= P_{15}X_{58}X_{82}P_{26}T_{61} = X_{13}P_{37}X_{72}P_{26}T_{61} = X_{14}P_{48}X_{82}P_{26}T_{61}$	0	-1	0
$Q_{15}X_{57}X_{74}Q_{48}X_{82}Y_{21} = Q_{26}Y_{65}X_{57}X_{74}Q_{48}X_{82} = X_{13}Q_{37}X_{74}Q_{48}X_{82}Y_{21} = Q_{15}X_{57}X_{72}Q_{26}T_{61}$ $= Q_{15}X_{58}X_{82}Q_{26}T_{61} = X_{13}Q_{37}X_{72}Q_{26}T_{61} = X_{14}Q_{48}X_{82}Q_{26}T_{61}$	-1	-1	0
$P_{15}Y_{57}X_{74}P_{48}Y_{82}P_{26}R_{61} = Y_{13}P_{37}X_{74}P_{48}Y_{82}P_{26}R_{61}$	1	1	1
$P_{15}Y_{57}X_{74}Q_{48}Y_{82}P_{26}R_{61} = P_{15}Y_{57}X_{74}P_{48}Y_{82}Q_{26}R_{61} = Y_{13}P_{37}X_{74}Q_{48}Y_{82}P_{26}R_{61}$ $= Q_{15}Y_{57}X_{74}P_{48}Y_{82}P_{26}R_{61} = Y_{13}Q_{37}X_{74}P_{48}Y_{82}P_{26}R_{61} = Y_{13}P_{37}X_{74}P_{48}Y_{82}Q_{26}R_{61}$	0	1	1
$P_{15}Y_{57}X_{74}Q_{48}Y_{82}Q_{26}R_{61} = Q_{15}Y_{57}X_{74}Q_{48}Y_{82}P_{26}R_{61} = Y_{13}Q_{37}X_{74}Q_{48}Y_{82}P_{26}R_{61}$ $= Y_{13}P_{37}X_{74}Q_{48}Y_{82}Q_{26}R_{61} = Q_{15}Y_{57}X_{74}P_{48}Y_{82}Q_{26}R_{61} = Y_{13}Q_{37}X_{74}P_{48}Y_{82}Q_{26}R_{61}$	-1	1	1
$Q_{15}Y_{57}X_{74}Q_{48}Y_{82}Q_{26}R_{61} = Y_{13}Q_{37}X_{74}Q_{48}Y_{82}Q_{26}R_{61}$	-2	1	1
$P_{15}X_{57}X_{74}P_{48}Y_{82}P_{26}R_{61} = P_{15}Y_{57}X_{74}P_{48}X_{82}P_{26}R_{61} = P_{15}Y_{57}X_{74}P_{48}Y_{82}P_{26}T_{61}$ $= X_{13}P_{37}X_{74}P_{48}Y_{82}P_{26}R_{61} = Y_{13}P_{37}X_{74}P_{48}X_{82}P_{26}R_{61} = Y_{13}P_{37}X_{74}P_{48}Y_{82}P_{26}T_{61}$	1	0	1
$P_{15}X_{57}X_{74}Q_{48}Y_{82}P_{26}R_{61} = P_{15}Y_{57}X_{74}Q_{48}X_{82}P_{26}R_{61} = P_{15}Y_{57}X_{74}Q_{48}Y_{82}P_{26}T_{61}$ $= P_{15}X_{57}X_{74}P_{48}Y_{82}Q_{26}R_{61} = P_{15}Y_{57}X_{74}P_{48}X_{82}Q_{26}R_{61} = P_{15}Y_{57}X_{74}P_{48}Y_{82}Q_{26}T_{61}$ $= X_{13}P_{37}X_{74}Q_{48}Y_{82}P_{26}R_{61} = Y_{13}P_{37}X_{74}Q_{48}X_{82}P_{26}R_{61} = Y_{13}P_{37}X_{74}Q_{48}Y_{82}P_{26}T_{61}$ $= Q_{15}X_{57}X_{74}P_{48}Y_{82}P_{26}R_{61} = Q_{15}Y_{57}X_{74}P_{48}X_{82}P_{26}R_{61} = Q_{15}Y_{57}X_{74}P_{48}Y_{82}P_{26}T_{61}$ $= X_{13}Q_{37}X_{74}P_{48}Y_{82}P_{26}R_{61} = Y_{13}Q_{37}X_{74}P_{48}X_{82}P_{26}R_{61} = Y_{13}Q_{37}X_{74}P_{48}Y_{82}P_{26}T_{61}$ $= X_{13}P_{37}X_{74}P_{48}Y_{82}Q_{26}R_{61} = Y_{13}P_{37}X_{74}P_{48}X_{82}Q_{26}R_{61} = Y_{13}P_{37}X_{74}P_{48}Y_{82}Q_{26}T_{61}$	0	0	1

Table 37. The generators in terms of bifundamental chiral fields for Model 9 (*Part 1*).

13 Model 10: $P_{+-}^3(\text{dP}_1)$ [\mathbb{P}^1 -blowup of 3, $\langle 28 \rangle$]

Model 10 corresponds to the toric Calabi-Yau 4-fold $P_{+-}^3(\text{dP}_1)$. The corresponding brane brick model has the quiver in figure 28 and the J - and E -terms are

$$\begin{array}{ll}
 & J \\
 \Lambda_{17}^1 : & X_{74}Q_{48}Y_{82}Y_{21} - X_{72}Q_{26}X_{61} \\
 \Lambda_{17}^2 : & X_{72}Z_{21} - X_{74}X_{42}Y_{21} \\
 \Lambda_{17}^3 : & X_{74}P_{48}Y_{82}Y_{21} - X_{72}P_{26}X_{61} \\
 \Lambda_{18}^1 : & X_{86}X_{61} - Y_{82}Z_{21} \\
 \Lambda_{25}^1 : & X_{54}X_{42} - X_{57}X_{72} \\
 \Lambda_{25}^2 : & X_{57}X_{74}Q_{48}Y_{82} - Y_{53}Q_{37}X_{74}X_{42} \\
 \Lambda_{25}^3 : & Y_{53}Q_{37}X_{72} - X_{54}Q_{48}Y_{82} \\
 \Lambda_{25}^4 : & X_{57}X_{74}P_{48}Y_{82} - Y_{53}P_{37}X_{74}X_{42} \\
 \Lambda_{25}^5 : & Y_{53}P_{37}X_{72} - X_{54}P_{48}Y_{82} \\
 \Lambda_{36}^1 : & Z_{65}Y_{53} - X_{61}X_{13} \\
 \Lambda_{38}^1 : & Y_{82}Y_{21}X_{13} - X_{86}Y_{65}Y_{53} \\
 \Lambda_{46}^1 : & X_{61}Q_{15}X_{54} - Y_{65}Y_{53}Q_{37}X_{74} \\
 \Lambda_{46}^2 : & Y_{65}X_{57}X_{74} - Z_{65}X_{54} \\
 \Lambda_{46}^3 : & X_{61}P_{15}X_{54} - Y_{65}Y_{53}P_{37}X_{74} \\
 & E \\
 & P_{15}X_{57} - X_{13}P_{37} \\
 & P_{15}Y_{53}Q_{37} - Q_{15}Y_{53}P_{37} \\
 & X_{13}Q_{37} - Q_{15}X_{57} \\
 & P_{15}X_{54}Q_{48} - Q_{15}X_{54}P_{48} \\
 & P_{26}X_{61}Q_{15} - Q_{26}X_{61}P_{15} \\
 & P_{26}Y_{65} - Y_{21}P_{15} \\
 & P_{26}Z_{65} - Z_{21}P_{15} \\
 & Y_{21}Q_{15} - Q_{26}Y_{65} \\
 & Z_{21}Q_{15} - Q_{26}Z_{65} \\
 & P_{37}X_{72}Q_{26} - Q_{37}X_{72}P_{26} \\
 & P_{37}X_{74}Q_{48} - Q_{37}X_{74}P_{48} \\
 & P_{48}X_{86} - X_{42}P_{26} \\
 & P_{48}Y_{82}Q_{26} - Q_{48}Y_{82}P_{26} \\
 & X_{42}Q_{26} - Q_{48}X_{86}
 \end{array} \quad . \quad (13.1)$$

Following the forward algorithm, we obtain the brick matchings for Model 10. The brick matchings are summarized in the P -matrix, which takes the form

$$P = \left(\begin{array}{c|cccccccc|cccccccc|cccccccc}
 & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & o_7 & o_8 & o_9 \\
 \hline
 P_{15} & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
 P_{26} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
 P_{37} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
 P_{48} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
 \hline
 Q_{15} & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
 Q_{26} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
 Q_{37} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
 Q_{48} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
 \hline
 X_{13} & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
 X_{42} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
 X_{54} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 X_{57} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
 X_{61} & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 X_{72} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 X_{74} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 X_{86} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
 \hline
 Y_{21} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 Y_{53} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
 Y_{65} & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 Y_{82} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
 \hline
 Z_{21} & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
 Z_{65} & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1
 \end{array} \right) . \quad (13.2)$$

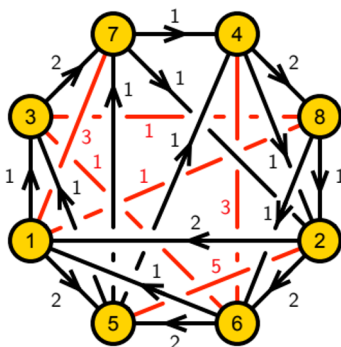


Figure 28. Quiver for Model 10.

The J - and E -term charges are given by

$$Q_{JE} = \left(\begin{array}{cccccc|cccccc|cccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & o_7 & o_8 & o_9 \\ \hline 1 & 1 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & 1 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right), \quad (13.3)$$

and the D -term charges are given by

$$Q_D = \left(\begin{array}{cccccc|cccccc|cccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & o_7 & o_8 & o_9 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right). \quad (13.4)$$

The toric diagram of Model 10 is given by

$$G_t = \left(\begin{array}{cccccc|cccccc|cccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & o_7 & o_8 & o_9 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & -1 & 0 \\ 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array} \right), \quad (13.5)$$

where figure 29 shows the toric diagram with brick matching labels.

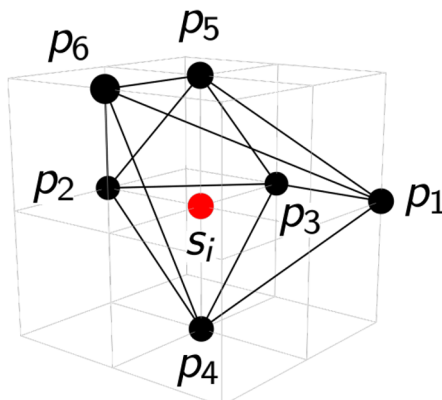


Figure 29. Toric diagram for Model 10.

The Hilbert series of the mesonic moduli space of Model 10 takes the form

$$\begin{aligned}
 g_1(t_i, y_s, y_{o_1}, y_{o_2}; \mathcal{M}_{10}) &= \frac{P(t_i, y_s, y_{o_1}, y_{o_2}; \mathcal{M}_{10})}{(1 - y_s y_{o_1}^2 y_{o_2} t_1^2 t_3 t_4^2)(1 - y_s y_{o_1}^2 y_{o_2} t_2^2 t_3 t_4^2)} \\
 &\times \frac{1}{(1 - y_s y_{o_1}^4 y_{o_2}^3 t_1^4 t_3^3 t_5^2)(1 - y_s y_{o_1}^4 y_{o_2}^3 t_2^4 t_3^3 t_5^2)(1 - y_s y_{o_1} y_{o_2} t_1 t_4^2 t_6^2)} \\
 &\times \frac{1}{(1 - y_s y_{o_1} y_{o_2} t_2 t_4^2 t_6^2)(1 - y_s y_{o_1} y_{o_2}^3 t_1 t_5^2 t_6^3)(1 - y_s y_{o_1} y_{o_2}^3 t_2 t_5^2 t_6^3)}, \quad (13.6)
 \end{aligned}$$

where t_i are the fugacities for the extremal brick matchings p_i . y_s counts the brick matching product $s_1 \dots s_8$ corresponding to the single internal point of the toric diagram of Model 10. Additionally, y_{o_1} and y_{o_2} count the products of extra GLSM fields $o_1 \dots o_5$ and $o_6 \dots o_9$, respectively. The explicit numerator $P(t_i, y_s, y_{o_1}, y_{o_2}; \mathcal{M}_{10})$ of the Hilbert series is given in the appendix section A.10.

By setting $t_i = t$ for the fugacities of the extremal brick matchings, and all other fugacities to $y_s = 1$, $y_{o_1} = 1$ and $y_{o_2} = 1$, the unrefined Hilbert series takes the following form

$$\begin{aligned}
 g_1(t, 1, 1, 1; \mathcal{M}_{10}) &= \frac{(1 - t^2)^2 (1 - t^3)^2}{(1 - t^4)^2 (1 - t^5)^2 (1 - t^6)^2 (1 - t^9)^2} \times (1 + 2t^2 + 2t^3 + 3t^4 \\
 &+ 7t^5 + 10t^6 + 19t^7 + 27t^8 + 41t^9 + 51t^{10} + 62t^{11} + 76t^{12} + 73t^{13} + 85t^{14} \\
 &+ 85t^{15} + 81t^{16} + 85t^{17} + 85t^{18} + 73t^{19} + 76t^{20} + 62t^{21} + 51t^{22} + 41t^{23} \\
 &+ 27t^{24} + 19t^{25} + 10t^{26} + 7t^{27} + 3t^{28} + 2t^{29} + 2t^{30} + t^{32}), \quad (13.7)
 \end{aligned}$$

where the palindromic numerator indicates that the mesonic moduli space is Calabi-Yau.

The global symmetry of Model 10 and the charges on the extremal brick matchings under the global symmetry are summarized in table 39. We can use the following fugacity map,

$$t = t_1^{1/2} t_2^{1/2}, \quad x = \frac{t_1^{1/2}}{t_2^{1/2}}, \quad y = \frac{t_1^{1/2} t_2^{1/2}}{t_4}, \quad b = \frac{t_1^{1/2} t_2^{1/2}}{t_6}, \quad (13.8)$$

where $t_3 t_4 = t_5 t_6$ and $t_1 t_2 = t_5 t_6$, in order to rewrite the Hilbert series for Model 10 in terms of characters of irreducible representations of $SU(2) \times U(1) \times U(1)$.

	SU(2) _x	U(1) _{b₁}	U(1) _{b₂}	U(1)	fugacity
p_1	+1	0	0	r_1	t_1
p_2	-1	0	0	r_2	t_2
p_3	0	+1	0	r_3	t_3
p_4	0	-1	0	r_4	t_4
p_5	0	0	+1	r_5	t_5
p_6	0	0	-1	r_6	t_6

Table 39. Global symmetry charges on the extremal brick matchings p_i of Model 10.

In highest weight form, the Hilbert series of Model 10 can be written as

$$\begin{aligned}
 h_1(t, \mu_1, \mu_2, b; \mathcal{M}_{10}) &= \frac{1}{(1 - \mu b_1^{-2} b_2^{-1} t^4)(1 - \mu^2 b_1^{-1} t^5)(1 - \mu b_2^{-1} t^6)(1 - \mu^4 b_1^3 b_2^2 t^9)} \\
 &\times (1 + \mu b_1^{-1} b_2^{-1} t^5 + \mu^2 t^6 + \mu^2 b_1 t^7 + \mu^3 b_1 b_2 t^7 + \mu^3 b_1^2 b_2 t^8 - \mu^3 b_1^{-2} b_2^{-1} t^{10} - \mu^4 b_1^{-1} t^{11} \\
 &\quad - \mu^3 b_1^{-1} b_2^{-1} t^{11} - \mu^4 t^{12} - \mu^5 b_1 b_2 t^{13} - \mu^6 t^{18}), \tag{13.9}
 \end{aligned}$$

where $\mu^m \sim [m]_{\text{SU}(2)_x}$. Here in highest weight form, the fugacity μ counts the highest weight of irreducible representations of SU(2)_x.

The plethystic logarithm of the Hilbert series takes the form

$$\begin{aligned}
 \text{PL}[g_1(t, x, b_1, b_2; \mathcal{M}_{10})] &= [1]b_1^{-2}b_2^{-1}t^4 + ([2]b_1^{-1} + [1]b_1^{-1}b_2^{-1})t^5 + ([2] + [1]b_2^{-1})t^6 \\
 &\quad + ([3]b_1b_2 + [2]b_1)t^7 + [3]b_1^2b_2t^8 + [4]b_1^3b_2^2t^9 \\
 &\quad - ([1]b_1^{-3}b_2^{-1} + b_1^{-3}b_2^{-2})t^9 - ([3]b_1^{-2}b_2^{-1} + [2]b_1^{-2}b_2^{-2} + 2[1]b_1^{-2}b_2^{-1} + b_1^{-2}b_2^{-2} \\
 &\quad + b_1^{-2})t^{10} - ([4]b_1^{-1} + 2[3]b_1^{-1}b_2^{-1} + 2[2]b_1^{-1} + 3[1]b_1^{-1}b_2^{-1} + b_1^{-1}b_2^{-2} + b_1^{-1})t^{11} \\
 &\quad - (3[4] + [3]b_2 + [3]b_2^{-1} + 3[2] + [1]b_2 + 2[1]b_2^{-1} + 2)t^{12} - (2[5]b_1b_2 + 2[4]b_1 \\
 &\quad + 3[3]b_1b_2 + 3[2]b_1 + 2[1]b_1b_2 + [1]b_1b_2^{-1} + b_1)t^{13} - ([6]b_1^2b_2^2 + 2[5]b_1^2b_2 + [4]b_1^2b_2^2 \\
 &\quad + [4]b_1^2 + 3[3]b_1^2b_2 + 2[2]b_1^2b_2^2 + [2]b_1^2 + 2[1]b_1^2b_2 + b_1^2)t^{14} + \dots, \tag{13.10}
 \end{aligned}$$

where $[m] = [m]_{\text{SU}(2)}$. From the plethystic logarithm, we see that the mesonic moduli space is a non-complete intersection.

Using the following fugacity map

$$\tilde{t} = t_5^{1/2} t_6^{1/2}, \quad \tilde{x} = \frac{t_1}{t_2}, \quad \tilde{b}_1 = \frac{t_5^{1/2} t_6^{1/2}}{t_4}, \quad \tilde{b}_2 = \frac{t_5^{3/2} t_6^{1/2}}{t_2 t_4}, \tag{13.11}$$

where $t_1 t_2 = t_3 t_4 = t_5 t_6$, the mesonic flavor charges on the gauge invariant operators become \mathbb{Z} -valued. The generators in terms of brick matchings and their corresponding rescaled mesonic flavor charges are summarized in table 40. The generator lattice as shown in table 40 is a convex lattice polytope, which is reflexive. It is the dual of the toric diagram of Model 10 shown in figure 29. For completeness, table 41 and table 42 show the generators of Model 10 in terms of chiral fields with the corresponding mesonic flavor charges.

generator	SU(2) $_{\tilde{x}}$	U(1) $_{\tilde{b}_1}$	U(1) $_{\tilde{b}_2}$
$p_2^2 p_3 p_4^2 so_1^2 o_2$	-1	-1	0
$p_2 p_4^2 p_6 so_1 o_2$	0	-1	-1
$p_2^3 p_3^2 p_4 p_5 so_1^3 o_2^2$	-2	0	1
$p_2^2 p_3 p_4 p_5 p_6 so_1^2 o_2^2$	-1	0	0
$p_2 p_4 p_5 p_6^2 so_1 o_2^2$	0	0	-1
$p_1 p_2 p_3 p_4^2 so_1^2 o_2$	0	-1	0
$p_1 p_4^2 p_6 so_1 o_2$	1	-1	-1
$p_2^4 p_3^2 p_5^2 so_1^4 o_2^3$	-3	1	2
$p_2^3 p_3^2 p_5^2 p_6 so_1^3 o_2^3$	-2	1	1
$p_2^2 p_3 p_5^2 p_6^2 so_1^2 o_2^3$	-1	1	0
$p_1 p_2^2 p_3^2 p_4 p_5 so_1^3 o_2^2$	-1	0	1
$p_2 p_5^2 p_6^3 so_1 o_2^3$	0	1	-1
$p_1 p_2 p_3 p_4 p_5 p_6 so_1^2 o_2^2$	0	0	0
$p_1 p_4 p_5 p_6^2 so_1 o_2^2$	1	0	-1
$p_1^2 p_3 p_4^2 so_1^2 o_2$	1	-1	0
$p_1 p_2^3 p_3^3 p_5^2 so_1^4 o_2^3$	-2	1	2
$p_1 p_2^2 p_3^2 p_5^2 p_6 so_1^3 o_2^3$	-1	1	1
$p_1 p_2 p_3 p_5^2 p_6^2 so_1^2 o_2^3$	0	1	0
$p_1^2 p_2 p_3^2 p_4 p_5 so_1^3 o_2^2$	0	0	1
$p_1 p_5^2 p_6^3 so_1 o_2^3$	1	1	-1
$p_1^2 p_3 p_4 p_5 p_6 so_1^2 o_2^2$	1	0	0
$p_1^2 p_2 p_3^3 p_5^2 so_1^4 o_2^3$	-1	1	2
$p_1^2 p_2 p_3^2 p_5^2 p_6 so_1^3 o_2^3$	0	1	1
$p_1^2 p_3 p_5^2 p_6^2 so_1^2 o_2^3$	1	1	0
$p_1^3 p_3 p_4 p_5 so_1^3 o_2^2$	1	0	1
$p_1^3 p_2 p_3^3 p_5^2 so_1^4 o_2^3$	0	1	2
$p_1^3 p_3^2 p_5^2 p_6 so_1^3 o_2^3$	1	1	1
$p_1^4 p_3^3 p_5^2 so_1^4 o_2^3$	1	1	2

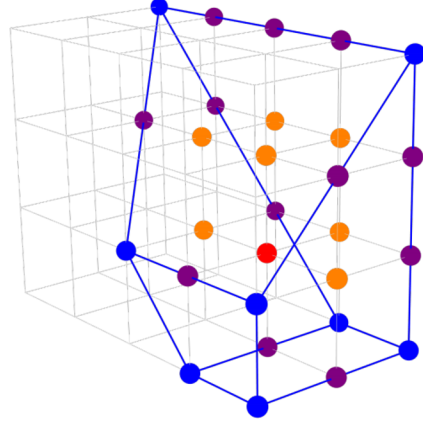


Table 40. The generators and lattice of generators of the mesonic moduli space of Model 10 in terms of brick matchings with the corresponding flavor charges.

generator	SU(2) $_{\tilde{x}}$	U(1) $_{\tilde{b}_1}$	U(1) $_{\tilde{b}_2}$
$Q_{15} Y_{53} Q_{37} X_{72} Y_{21} = Q_{15} X_{54} Q_{48} Y_{82} Y_{21} = Q_{26} Y_{65} Y_{53} Q_{37} X_{72} = Q_{26} Y_{65} X_{54} Q_{48} Y_{82}$	-1	-1	0
$Q_{15} X_{54} X_{42} Y_{21} = Q_{15} X_{57} X_{72} Y_{21} = Q_{26} Y_{65} X_{54} X_{42} = Q_{26} Y_{65} X_{57} X_{72} = X_{13} Q_{37} X_{72} Y_{21} = Q_{48} X_{86} Y_{65} X_{54}$	0	-1	-1
$Q_{15} Y_{53} Q_{37} X_{72} Q_{26} X_{61} = Q_{15} X_{54} Q_{48} Y_{82} Q_{26} X_{61} = Q_{15} Y_{53} Q_{37} X_{74} Q_{48} Y_{82} Y_{21} = Q_{26} Y_{65} Y_{53} Q_{37} X_{74} Q_{48} Y_{82}$	-2	0	1
$Q_{15} Y_{53} Q_{37} X_{74} X_{42} Y_{21} = Q_{15} X_{57} X_{74} Q_{48} Y_{82} Y_{21} = Q_{26} Y_{65} Y_{53} Q_{37} X_{74} X_{42} = Q_{26} Y_{65} X_{57} X_{74} Q_{48} Y_{82}$	-1	0	0
$= X_{13} Q_{37} X_{74} Q_{48} Y_{82} Y_{21} = Q_{37} X_{74} Q_{48} X_{86} Y_{65} Y_{53} = Q_{15} X_{54} X_{42} Q_{26} X_{61} = Q_{15} X_{57} X_{72} Q_{26} X_{61}$			
$= Q_{15} Y_{53} Q_{37} X_{72} Z_{21} = Q_{15} X_{54} Q_{48} X_{86} X_{61} = Q_{15} X_{54} Q_{48} Y_{82} Z_{21} = X_{13} Q_{37} X_{72} Q_{26} X_{61} = Q_{26} Z_{65} Y_{53} Q_{37} X_{72}$			
$= Q_{26} Z_{65} X_{54} Q_{48} Y_{82}$			
$Q_{15} X_{57} X_{74} X_{42} Y_{21} = Q_{26} Y_{65} X_{57} X_{74} X_{42} = X_{13} Q_{37} X_{74} X_{42} Y_{21} = Q_{48} X_{86} Y_{65} X_{57} X_{74} = Q_{15} X_{54} X_{42} Z_{21}$	0	0	-1
$= Q_{15} X_{57} X_{72} Z_{21} = Q_{26} Z_{65} X_{54} X_{42} = Q_{26} Z_{65} X_{57} X_{72} = X_{13} Q_{37} X_{72} Z_{21} = Q_{48} X_{86} Z_{65} X_{54}$			
$P_{15} Y_{53} Q_{37} X_{72} Y_{21} = P_{15} X_{54} Q_{48} Y_{82} Y_{21} = P_{26} Y_{65} Y_{53} Q_{37} X_{72} = P_{26} Y_{65} X_{54} Q_{48} Y_{82} = Q_{15} Y_{53} P_{37} X_{72} Y_{21}$	0	-1	0
$= Q_{26} Y_{65} Y_{53} P_{37} X_{72} = Q_{15} X_{54} P_{48} Y_{82} Y_{21} = Q_{26} Y_{65} X_{54} P_{48} Y_{82}$			
$P_{15} X_{54} X_{42} Y_{21} = P_{15} X_{57} X_{72} Y_{21} = P_{26} Y_{65} X_{54} X_{42} = P_{26} Y_{65} X_{57} X_{72} = X_{13} P_{37} X_{72} Y_{21} = P_{48} X_{86} Y_{65} X_{54}$	1	-1	-1
$Q_{15} Y_{53} Q_{37} X_{74} Q_{48} Y_{82} Q_{26} X_{61}$	-3	1	2
$Q_{15} Y_{53} Q_{37} X_{74} X_{42} Q_{26} X_{61} = Q_{15} X_{57} X_{74} Q_{48} Y_{82} Q_{26} X_{61} = Q_{15} Y_{53} Q_{37} X_{74} Q_{48} X_{86} X_{61}$	-2	1	1
$= Q_{15} Y_{53} Q_{37} X_{74} Q_{48} Y_{82} Z_{21} = X_{13} Q_{37} X_{74} Q_{48} Y_{82} Q_{26} X_{61} = Q_{26} Z_{65} Y_{53} Q_{37} X_{74} Q_{48} Y_{82}$			

Table 41. The generators in terms of bifundamental chiral fields for Model 10 (Part 1).

14 Model 11: $P_{+-}^0(\text{dP}_1)$ [$\text{dP}_1 \times \mathbb{P}^1$, $\langle 29 \rangle$]

Model 11 corresponds to the toric Calabi-Yau 4-fold $P_{+-}^0(\text{dP}_1)$. The corresponding brane brick model has the quiver in figure 30 and the J - and E -terms are

$$\begin{array}{ll}
 & J \qquad \qquad \qquad E \\
 \Lambda_{16}^1 : & Y_{68}X_{81} - X_{67}X_{71} \quad P_{15}X_{56} - X_{12}P_{26} \\
 \Lambda_{16}^2 : & X_{67}Y_{71} - X_{68}X_{81} \quad P_{15}Y_{56} - Y_{12}P_{26} \\
 \Lambda_{16}^3 : & X_{68}X_{87}X_{71} - Y_{68}X_{87}Y_{71} \quad P_{15}Z_{56} - Z_{12}P_{26} \\
 \Lambda_{16}^4 : & Y_{68}S_{81} - X_{67}S_{71} \quad X_{12}Q_{26} - Q_{15}X_{56} \\
 \Lambda_{16}^5 : & X_{67}T_{71} - X_{68}S_{81} \quad Y_{12}Q_{26} - Q_{15}Y_{56} \\
 \Lambda_{16}^6 : & X_{68}X_{87}S_{71} - Y_{68}X_{87}T_{71} \quad Z_{12}Q_{26} - Q_{15}Z_{56} \\
 \Lambda_{27}^1 : & Y_{71}Y_{12} - X_{71}X_{12} \quad P_{26}X_{67} - X_{23}P_{37} \\
 \Lambda_{27}^2 : & T_{71}Y_{12} - S_{71}X_{12} \quad X_{23}Q_{37} - Q_{26}X_{67} \\
 \Lambda_{28}^1 : & X_{87}X_{71}Z_{12} - X_{81}Y_{12} \quad P_{26}X_{68} - X_{24}P_{48} \\
 \Lambda_{28}^2 : & X_{81}X_{12} - X_{87}Y_{71}Z_{12} \quad P_{26}Y_{68} - Y_{24}P_{48} \\
 \Lambda_{28}^3 : & X_{87}S_{71}Z_{12} - S_{81}Y_{12} \quad X_{24}Q_{48} - Q_{26}X_{68} \\
 \Lambda_{28}^4 : & S_{81}X_{12} - X_{87}T_{71}Z_{12} \quad Y_{24}Q_{48} - Q_{26}Y_{68} \\
 \Lambda_{31}^1 : & Z_{12}X_{24}X_{43} - X_{12}X_{23} \quad P_{37}X_{71} - Q_{37}S_{71} \\
 \Lambda_{31}^2 : & Y_{12}X_{23} - Z_{12}Y_{24}X_{43} \quad P_{37}Y_{71} - Q_{37}T_{71} \\
 \Lambda_{41} : & X_{12}Y_{24} - Y_{12}X_{24} \quad P_{48}X_{81} - Q_{48}S_{81} \\
 \Lambda_{47}^1 : & X_{71}Z_{12}X_{24} - Y_{71}Z_{12}Y_{24} \quad P_{48}X_{87} - X_{43}P_{37} \\
 \Lambda_{47}^2 : & S_{71}Z_{12}X_{24} - T_{71}Z_{12}Y_{24} \quad X_{43}Q_{37} - Q_{48}X_{87} \\
 \Lambda_{75}^1 : & Z_{56}X_{68}X_{87} - X_{56}X_{67} \quad S_{71}Q_{15} - X_{71}P_{15} \\
 \Lambda_{75}^2 : & Y_{56}X_{67} - Z_{56}Y_{68}X_{87} \quad T_{71}Q_{15} - Y_{71}P_{15} \\
 \Lambda_{85} : & X_{56}Y_{68} - Y_{56}X_{68} \quad S_{81}Q_{15} - X_{81}P_{15}
 \end{array} \tag{14.1}$$

Following the forward algorithm, we obtain the brick matchings for Model 11. The brick matchings are summarized in the P -matrix, which takes the form

$$P = \begin{pmatrix}
 & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & s_{10} & s_{11} & s_{12} & s_{13} & s_{14} & o_1 & o_2 & o_3 & o_4 \\
 P_{15} & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 P_{26} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
 P_{37} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
 P_{48} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
 Q_{15} & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 Q_{26} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
 Q_{37} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
 Q_{48} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
 S_{71} & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
 S_{81} & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
 T_{71} & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
 X_{12} & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
 X_{23} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 X_{24} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 X_{43} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
 X_{56} & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 1 \\
 X_{67} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
 X_{68} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
 X_{71} & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
 X_{81} & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
 X_{87} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
 Y_{12} & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
 Y_{24} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
 Y_{56} & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 1 & 1 \\
 Y_{68} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
 Y_{71} & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
 Z_{12} & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 Z_{56} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
 \end{pmatrix}. \tag{14.2}$$

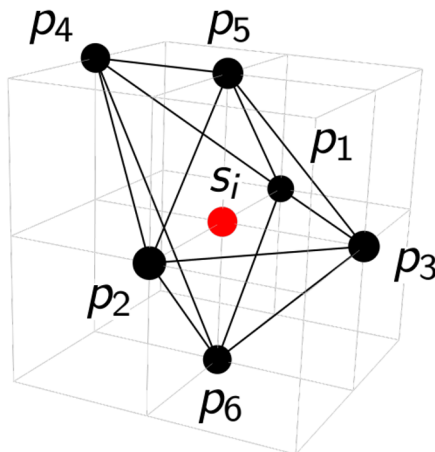


Figure 31. Toric diagram for Model 11.

The Hilbert series of the mesonic moduli space of Model 11 takes the form

$$g_1(t_i, y_s, y_o; \mathcal{M}_{11}) = \frac{P(t_i, y_s, y_o; \mathcal{M}_{11})}{(1 - y_s y_o^3 t_1^2 t_3^2 t_5^2)(1 - y_s y_o^3 t_2^2 t_3^2 t_5^2)(1 - y_s y_o^3 t_1^2 t_4^2 t_5^2)(1 - y_s y_o^3 t_2^2 t_4^2 t_5^2)} \times \frac{1}{(1 - y_s y_o t_1^2 t_3 t_6^2)(1 - y_s y_o t_2^2 t_3 t_6^2)(1 - y_s y_o t_1^2 t_4 t_6^2)(1 - y_s y_o t_2^2 t_4 t_6^2)}, \quad (14.6)$$

where t_i are the fugacities for the extremal brick matchings p_i . y_s counts the brick matching product $s_1 \dots s_{14}$ corresponding to the single internal point of the toric diagram of Model 11. Additionally, y_o counts the product of extra GLSM fields $o_1 \dots o_4$. The explicit numerator $P(t_i, y_s, y_o; \mathcal{M}_{11})$ of the Hilbert series is given in the appendix section A.11. We note that setting the fugacity $y_o = 1$ does not change the overall characterization of the mesonic moduli space by the Hilbert series, indicating that the extra GLSM fields, as expected, correspond to an over-parameterization of the moduli space.

By setting $t_i = t$ for the fugacities of the extremal brick matchings, and all other fugacities to $y_s = 1$ and $y_o = 1$, the unrefined Hilbert series takes the following form

$$g_1(t, 1, 1, 1; \mathcal{M}_{11}) = \frac{(1-t)^2}{(1-t^5)^3(1-t^7)^3} \times (1 + 2t + 3t^2 + 4t^3 + 5t^4 + 9t^5 + 22t^6 + 44t^7 + 66t^8 + 88t^9 + 110t^{10} + 125t^{11} + 120t^{12} + 118t^{13} + 118t^{14} + 118t^{15} + 120t^{16} + 125t^{17} + 110t^{18} + 88t^{19} + 66t^{20} + 44t^{21} + 22t^{22} + 9t^{23} + 5t^{24} + 4t^{25} + 3t^{26} + 2t^{27} + t^{28}), \quad (14.7)$$

where the palindromic numerator indicates that the mesonic moduli space is Calabi-Yau.

The global symmetry of Model 11 and the charges on the extremal brick matchings under the global symmetry are summarized in table 43. We can use the following fugacity map,

$$t = t_1^{1/2} t_2^{1/2}, \quad x = \frac{t_1^{1/2}}{t_2^{1/2}}, \quad y = \frac{t_3^{1/2}}{t_4^{1/2}}, \quad b = \frac{t_1^{1/2} t_2^{1/2}}{t_6}, \quad (14.8)$$

	SU(2) _x	SU(2) _y	U(1) _b	U(1)	fugacity
p_1	+1	0	0	r_1	t_1
p_2	-1	0	0	r_2	t_2
p_3	0	+1	0	r_3	t_3
p_4	0	-1	0	r_4	t_4
p_5	0	0	+1	r_5	t_5
p_6	0	0	-1	r_6	t_6

Table 43. Global symmetry charges on the extremal brick matchings p_i of Model 11.

where $t_3t_4 = t_5t_6$ and $t_1t_2 = t_5t_6$, in order to rewrite the Hilbert series for Model 11 in terms of characters of irreducible representations of $SU(2) \times SU(2) \times U(1)$. The character expansion of the Hilbert series is

$$g_1(t, x, y, b; \mathcal{M}_{11}) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \left[[2n_1+2n_2; n_1+2n_2] b^{-2n_1} t^{5n_1+6n_2} + [2n_1+2n_2+2; 2n_1+3n_2+3] b^{2n_1+2} t^{6n_1+7n_2+7} \right], \quad (14.9)$$

where $[m; n] = [m]_{SU(2)_x} [n]_{SU(2)_y}$.

In highest weight form, the Hilbert series of Model 11 can be written as

$$h_1(t, \mu_1, \mu_2, b; \mathcal{M}_{11}) = \frac{(1 + \mu_1^2 \mu_2^2 t^6)}{(1 - \mu_1^2 \mu_2 b^{-2} t^5)(1 - \mu_1^2 \mu_2^3 b^2 t^7)}, \quad (14.10)$$

where $\mu_1^m \sim [m]_{SU(2)_x}$ and $\mu_2^n \sim [n]_{SU(2)_y}$. Here in highest weight form, the fugacities μ_1 and μ_2 count the highest weight of irreducible representations of $SU(2)_x \times SU(2)_y$.

The plethystic logarithm of the Hilbert series takes the form

$$\begin{aligned} \text{PL}[g_1(t, x, y, b; \mathcal{M}_{11})] &= [2; 1]b^{-2}t^5 + [2; 2]t^6 + [2; 3]b^2t^7 \\ &\quad - ([0; 2]b^{-4} + [2; 0]b^{-4})t^{10} - ([0; 1]b^{-2} + [0; 3]b^{-2} + [2; 1]b^{-2} + [2; 3]b^{-2} \\ &\quad + [4; 1]b^{-2})t^{11} + (1 + [0; 2] + 2[0; 4] + 2[2; 2] + [2; 4] + [4; 0] + [4; 2] \\ &\quad + [4; 4])t^{12} - ([0; 1]b^2 + [0; 3]b^2 + [0; 5]b^2 + [2; 1]b^2 + [2; 3]b^2 + [2; 5]b^2 \\ &\quad + [4; 1]b^2 + [4; 3]b^2)t^{13} - ([0; 2]b^4 + [0; 6]b^4 + [2; 0]b^4 + [2; 4]b^4 + [4; 2]b^4)t^{14} \\ &\quad + \dots, \end{aligned} \quad (14.11)$$

where $[m; n] = [m]_{SU(2)_x} [n]_{SU(2)_y}$. From the plethystic logarithm, we see that the mesonic moduli space is a non-complete intersection.

The generators form 3 sets that transform under $[2; 1]b^{-2}$, $[2; 2]$ and $[2; 3]b^2$ of the mesonic flavor symmetry of Model 11, respectively. Using the following fugacity map

$$\tilde{t} = t_5^{1/2} t_6^{1/2}, \quad \tilde{x} = \frac{t_1}{t_2}, \quad \tilde{y} = \frac{t_3}{t_4}, \quad \tilde{b} = \frac{t_4 t_5^{1/2}}{t_6^{3/2}}, \quad (14.12)$$

where $t_1t_2 = t_3t_4 = t_5t_6$, the mesonic flavor charges on the gauge invariant operators become \mathbb{Z} -valued. The generators in terms of brick matchings and their corresponding rescaled mesonic flavor charges are summarized in table 44. The generator lattice as shown in

generator	SU(2) _{\bar{x}}	SU(2) _{\bar{y}}	U(1) _{\bar{b}}
$p_1^2 p_3 p_6^2 so$	1	0	-1
$p_1 p_2 p_3 p_6^2 so$	0	0	-1
$p_2^2 p_3 p_6^2 so$	-1	0	-1
$p_1^2 p_4 p_6^2 so$	1	-1	-1
$p_1 p_2 p_4 p_6^2 so$	0	-1	-1
$p_2^2 p_4 p_6^2 so$	-1	-1	-1
$p_1^2 p_3^2 p_5 p_6 so^2$	1	1	0
$p_1 p_2 p_3^2 p_5 p_6 so^2$	0	1	0
$p_2^2 p_3^2 p_5 p_6 so^2$	-1	1	0
$p_1^2 p_3 p_4 p_5 p_6 so^2$	1	0	0
$p_1 p_2 p_3 p_4 p_5 p_6 so^2$	0	0	0
$p_2^2 p_3 p_4 p_5 p_6 so^2$	-1	0	0
$p_1^2 p_4^2 p_5 p_6 so^2$	1	-1	0
$p_1 p_2 p_4^2 p_5 p_6 so^2$	0	-1	0
$p_2^2 p_4^2 p_5 p_6 so^2$	-1	-1	0
$p_1^2 p_3^3 p_5^2 so^3$	1	2	1
$p_1 p_2 p_3^3 p_5^2 so^3$	0	2	1
$p_2^2 p_3^3 p_5^2 so^3$	-1	2	1
$p_1^2 p_3^2 p_4 p_5^2 so^3$	1	1	1
$p_1 p_2 p_3^2 p_4 p_5^2 so^3$	0	1	1
$p_2^2 p_3^2 p_4 p_5^2 so^3$	-1	1	1
$p_1^2 p_3 p_4^2 p_5^2 so^3$	1	0	1
$p_1 p_2 p_3 p_4^2 p_5^2 so^3$	0	0	1
$p_2^2 p_3 p_4^2 p_5^2 so^3$	-1	0	1
$p_1^2 p_4 p_5^2 so^3$	1	-1	1
$p_1 p_2 p_4 p_5^2 so^3$	0	-1	1
$p_2^2 p_4 p_5^2 so^3$	-1	-1	1

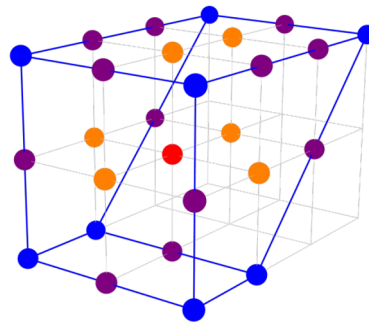


Table 44. The generators and lattice of generators of the mesonic moduli space of Model 11 in terms of brick matchings with the corresponding flavor charges.

generator	SU(2) _{\bar{x}}	SU(2) _{\bar{y}}	U(1) _{\bar{b}}
$P_{15} Z_{56} X_{67} S_{71} = P_{15} Z_{56} Y_{68} S_{81} = Z_{12} P_{26} X_{67} S_{71} = Z_{12} P_{26} Y_{68} S_{81} = Z_{12} X_{23} P_{37} S_{71} = Z_{12} Y_{24} P_{48} S_{81}$	1	0	-1
$P_{15} Z_{56} X_{67} X_{71} = P_{15} Z_{56} Y_{68} X_{81} = Z_{12} P_{26} X_{67} X_{71} = Z_{12} P_{26} Y_{68} X_{81} = Z_{12} X_{23} P_{37} X_{71} = Z_{12} Y_{24} P_{48} X_{81}$	0	0	-1
$Q_{15} Z_{56} X_{67} S_{71} = Q_{15} Z_{56} Y_{68} S_{81} = Z_{12} Q_{26} X_{67} S_{71} = Z_{12} Q_{26} Y_{68} S_{81} = Z_{12} X_{23} Q_{37} S_{71} = Z_{12} Y_{24} Q_{48} S_{81}$	-1	0	-1
$Q_{15} Z_{56} X_{67} X_{71} = Q_{15} Z_{56} Y_{68} X_{81} = Z_{12} Q_{26} X_{67} X_{71} = Z_{12} Q_{26} Y_{68} X_{81} = Z_{12} X_{23} Q_{37} X_{71} = Z_{12} Y_{24} Q_{48} X_{81}$	1	-1	-1
$P_{15} Z_{56} X_{68} S_{81} = P_{15} Z_{56} X_{67} T_{71} = Z_{12} P_{26} X_{68} S_{81} = Z_{12} P_{26} X_{67} T_{71} = Z_{12} X_{23} P_{37} T_{71} = Z_{12} X_{24} P_{48} S_{81}$	1	-1	-1
$P_{15} Z_{56} X_{67} Y_{71} = P_{15} Z_{56} X_{68} X_{81} = Z_{12} P_{26} X_{67} Y_{71} = Z_{12} P_{26} X_{68} X_{81} = Z_{12} X_{23} P_{37} Y_{71} = Z_{12} X_{24} P_{48} X_{81}$	0	-1	-1
$Q_{15} Z_{56} X_{68} S_{81} = Q_{15} Z_{56} X_{67} T_{71} = Z_{12} Q_{26} X_{68} S_{81} = Z_{12} Q_{26} X_{67} T_{71} = Z_{12} X_{23} Q_{37} T_{71} = Z_{12} X_{24} Q_{48} S_{81}$	-1	-1	-1
$Q_{15} Z_{56} X_{67} Y_{71} = Q_{15} Z_{56} X_{68} X_{81} = Z_{12} Q_{26} X_{67} Y_{71} = Z_{12} Q_{26} X_{68} X_{81} = Z_{12} X_{23} Q_{37} Y_{71} = Z_{12} X_{24} Q_{48} X_{81}$	-1	-1	-1
$P_{15} Z_{56} Y_{68} X_{87} S_{71} = Z_{12} P_{26} Y_{68} X_{87} S_{71} = Z_{12} Y_{24} X_{43} P_{37} S_{71} = Z_{12} Y_{24} P_{48} X_{87} S_{71} = P_{15} Y_{56} X_{67} S_{71}$	1	1	0
$= P_{15} Y_{56} Y_{68} S_{81} = Y_{12} P_{26} X_{67} S_{71} = Y_{12} P_{26} Y_{68} S_{81} = Y_{12} X_{23} P_{37} S_{71} = Y_{12} Y_{24} P_{48} S_{81}$			

Table 45. The generators in terms of bifundamental chiral fields for Model 11 (*Part 1*).

table 44 is a convex lattice polytope, which is reflexive. It is the dual of the toric diagram of Model 11 shown in figure 31. We also note that the 3 layers of points in the generator lattice in table 44 corresponds to the 3 sets that transform under $[2; 1]b^{-2}$, $[2; 2]$ and $[2; 3]b^2$ of the mesonic flavor symmetry. For completeness, table 45 and table 46 show the generators of Model 11 in terms of chiral fields with the corresponding mesonic flavor charges.

15 Model 12: $Q^{1,1,1}/\mathbb{Z}_2$ [$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, $\langle 30 \rangle$]

Model 12 corresponds to the toric Calabi-Yau 4-fold $Q^{1,1,1}/\mathbb{Z}_2$. The corresponding brane brick model has the quiver in figure 32 and the J - and E -terms are

$$\begin{array}{ll}
 & J \\
 \Lambda_{12}^1 : & Y_{23}Y_{34}X_{41} - X_{23}Y_{34}Y_{41} \\
 \Lambda_{12}^2 : & X_{23}X_{34}Y_{41} - Y_{23}X_{34}X_{41} \\
 \Lambda_{27}^1 : & X_{78}Y_{85}Y_{52} - Y_{78}Y_{85}X_{52} \\
 \Lambda_{27}^2 : & Y_{78}X_{85}X_{52} - X_{78}X_{85}Y_{52} \\
 \Lambda_{27}^3 : & X_{78}Y_{85}S_{52} - Y_{78}Y_{85}R_{52} \\
 \Lambda_{27}^4 : & Y_{78}X_{85}R_{52} - X_{78}X_{85}S_{52} \\
 \Lambda_{38}^1 : & Y_{85}Y_{52}X_{23} - X_{85}Y_{52}Y_{23} \\
 \Lambda_{38}^2 : & X_{85}X_{52}Y_{23} - Y_{85}X_{52}X_{23} \\
 \Lambda_{38}^3 : & Y_{85}S_{52}X_{23} - X_{85}S_{52}Y_{23} \\
 \Lambda_{38}^4 : & X_{85}R_{52}Y_{23} - Y_{85}R_{52}X_{23} \\
 \Lambda_{45}^1 : & X_{52}Y_{23}Y_{34} - Y_{52}Y_{23}X_{34} \\
 \Lambda_{45}^2 : & Y_{52}X_{23}X_{34} - X_{52}X_{23}Y_{34} \\
 \Lambda_{45}^3 : & R_{52}Y_{23}Y_{34} - S_{52}Y_{23}X_{34} \\
 \Lambda_{45}^4 : & S_{52}X_{23}X_{34} - R_{52}X_{23}Y_{34} \\
 \Lambda_{56}^1 : & Y_{67}Y_{78}X_{85} - X_{67}Y_{78}Y_{85} \\
 \Lambda_{56}^2 : & X_{67}X_{78}Y_{85} - Y_{67}X_{78}X_{85}
 \end{array}
 \begin{array}{ll}
 & E \\
 P_{15}X_{52} - & Q_{15}R_{52} \\
 P_{15}Y_{52} - & Q_{15}S_{52} \\
 P_{26}X_{67} - & X_{23}P_{37} \\
 P_{26}Y_{67} - & Y_{23}P_{37} \\
 X_{23}Q_{37} - & Q_{26}X_{67} \\
 Y_{23}Q_{37} - & Q_{26}Y_{67} \\
 P_{37}X_{78} - & X_{34}P_{48} \\
 P_{37}Y_{78} - & Y_{34}P_{48} \\
 X_{34}Q_{48} - & Q_{37}X_{78} \\
 Y_{34}Q_{48} - & Q_{37}Y_{78} \\
 P_{48}X_{85} - & X_{41}P_{15} \\
 P_{48}Y_{85} - & Y_{41}P_{15} \\
 X_{41}Q_{15} - & Q_{48}X_{85} \\
 Y_{41}Q_{15} - & Q_{48}Y_{85} \\
 R_{52}Q_{26} - & X_{52}P_{26} \\
 S_{52}Q_{26} - & Y_{52}P_{26}
 \end{array}
 \tag{15.1}$$

Using the forward algorithm, we can find the brick matchings for Model 12. The resulting brick matchings are summarized in the P -matrix, which takes the following form

$$P = \begin{pmatrix}
 & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & s_{10} & s_{11} & s_{12} & s_{13} & s_{14} \\
 P_{15} & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 P_{26} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
 P_{37} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
 P_{48} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
 Q_{15} & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 Q_{26} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
 Q_{37} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
 Q_{48} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
 R_{52} & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 S_{52} & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 X_{23} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 X_{34} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
 X_{41} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
 X_{52} & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 X_{67} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
 X_{78} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
 X_{85} & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
 Y_{23} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 Y_{34} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
 Y_{41} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
 Y_{52} & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 Y_{67} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
 Y_{78} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
 Y_{85} & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
 \end{pmatrix}.
 \tag{15.2}$$

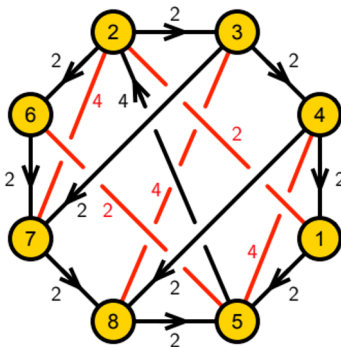


Figure 32. Quiver for Model 12.

The J - and E -term charges are given by

$$Q_{JE} = \left(\begin{array}{cccccc|cccccccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & s_{10} & s_{11} & s_{12} & s_{13} & s_{14} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \end{array} \right), \quad (15.3)$$

and the D -term charges are given by

$$Q_D = \left(\begin{array}{cccccc|cccccccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & s_{10} & s_{11} & s_{12} & s_{13} & s_{14} \\ \hline 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right). \quad (15.4)$$

The toric diagram of Model 12 is given by

$$G_t = \left(\begin{array}{cccccc|cccccccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & s_{10} & s_{11} & s_{12} & s_{13} & s_{14} \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right), \quad (15.5)$$

where figure 33 shows the toric diagram with brick matching labels.

The Hilbert series of the mesonic moduli space of Model 12 takes the form

$$g_1(t_i, y_s; \mathcal{M}_{12}) = \frac{P(t_i, y_s; \mathcal{M}_{12})}{(1 - y_s t_1^2 t_3^2 t_5^2)(1 - y_s t_2^2 t_3^2 t_5^2)(1 - y_s t_1^2 t_4^2 t_5^2)(1 - y_s t_2^2 t_4^2 t_5^2)} \times \frac{1}{(1 - y_s t_1^2 t_3^2 t_6^2)(1 - y_s t_2^2 t_3^2 t_6^2)(1 - y_s t_1^2 t_4^2 t_6^2)(1 - y_s t_2^2 t_4^2 t_6^2)}, \quad (15.6)$$

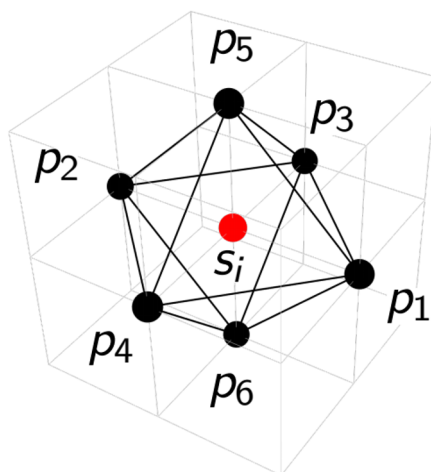


Figure 33. Toric diagram for Model 12.

	SU(2) _x	SU(2) _y	SU(2) _z	U(1)	fugacity
p_1	+1	0	0	r_1	t_1
p_2	-1	0	0	r_2	t_2
p_3	0	+1	0	r_3	t_3
p_4	0	-1	0	r_4	t_4
p_5	0	0	+1	r_5	t_5
p_6	0	0	-1	r_6	t_6

Table 47. Global symmetry charges on the extremal brick matchings p_i of Model 12.

where t_i are the fugacities for the extremal brick matchings p_i . y_s counts the brick matching product $s_1 \dots s_{14}$ corresponding to the single internal point of the toric diagram of Model 12. The explicit numerator $P(t_i, y_s; \mathcal{M}_{12})$ of the Hilbert series is given in the appendix section [A.12](#).

By setting $t_i = t$ and $y_s = 1$, the unrefined Hilbert series takes the following form

$$g_1(t, 1; \mathcal{M}_{12}) = \frac{1 + 19t^6 - 63t^{12} + 43t^{18} + 43t^{24} - 63t^{30} + 19t^{36} + t^{42}}{(1 - t^6)^8}, \quad (15.7)$$

where the palindromic numerator indicates that the mesonic moduli space is Calabi-Yau.

The global symmetry of Model 12 and the charges on the extremal brick matchings under the global symmetry are summarized in table [47](#). We can use the following fugacity map,

$$t = t_1^{1/2} t_2^{1/2}, \quad x = \frac{t_1^{1/2}}{t_2^{1/2}}, \quad y = \frac{t_3^{1/2}}{t_4^{1/2}}, \quad z = \frac{t_5^{1/2}}{t_6^{1/2}}, \quad (15.8)$$

where $t_1 t_2 = t_3 t_4 = t_5 t_6$, in order to rewrite the Hilbert series for Model 12 in terms of characters of irreducible representations of $SU(2) \times SU(2) \times SU(2)$. The character expansion of the Hilbert series is

$$g_1(t, x, y, z; \mathcal{M}_{12}) = \sum_{n=0}^{\infty} [2n; 2n; 2n] t^{6n}, \tag{15.9}$$

where $[m_1; m_2; m_3] = [m_1]_{SU(2)_x} [m_2]_{SU(2)_y} [m_3]_{SU(2)_z}$. In highest weight form, the Hilbert series of Model 12 can be written as

$$h_1(t, \mu_1, \mu_2, \mu_3; \mathcal{M}_{12}) = \frac{1}{(1 - \mu_1^2 \mu_2^2 \mu_3^2 t^6)}, \tag{15.10}$$

where $\mu_1^{m_1} \mu_2^{m_2} \mu_3^{m_3} \sim [m_1]_{SU(2)_x} [m_2]_{SU(2)_y} [m_3]_{SU(2)_z}$. Here in highest weight form, the fugacities μ_1 , μ_2 and μ_3 count the highest weights of irreducible representations of $SU(2)_x \times SU(2)_y \times SU(2)_z$.

The plethystic logarithm of the Hilbert series takes the form

$$\begin{aligned} \text{PL}[g_1(t, x, y, z; \mathcal{M}_{12})] &= [2; 2; 2] t^6 - (1 + [0; 0; 4] + [0; 2; 2] + [0; 4; 0] \\ &\quad + [0; 4; 4] + [2; 0; 2] + [2; 2; 0] + [2; 2; 4] + [2; 4; 2] + [4; 0; 0] + [4; 0; 4] \\ &\quad + [4; 2; 2] + [4; 4; 0]) t^{12} + \dots, \end{aligned} \tag{15.11}$$

where $[k; m; n] = [k]_{SU(2)_x} [m]_{SU(2)_y} [n]_{SU(2)_z}$. From the plethystic logarithm, we see that the mesonic moduli space is a non-complete intersection.

We note that all generators transform together in the adjoint representation of each of the $SU(2)$ factors of the mesonic flavor symmetry of Model 12. Using the following fugacity map

$$\tilde{t} = t_5^{1/2} t_6^{1/2}, \quad \tilde{x} = \frac{t_1}{t_2}, \quad \tilde{y} = \frac{t_3}{t_4}, \quad \tilde{z} = \frac{t_5}{t_6}, \tag{15.12}$$

where $t_1 t_2 = t_3 t_4 = t_5 t_6$, the mesonic flavor charges on the gauge invariant operators become \mathbb{Z} -valued. The generators in terms of brick matchings and their corresponding rescaled mesonic flavor charges are summarized in table 48. The generator lattice as shown in table 48 is a convex lattice polytope, which is reflexive. It is the dual of the toric diagram of Model 12 shown in figure 33. It is also interesting to note that the generator lattice forms layers in any of the 3 planar directions of \mathbb{Z}^3 . Each layer of points in the generator lattice forms a subset of generators that transforms under the adjoint representations of a pair of $SU(2)$ factors of the mesonic flavor symmetry of Model 12. For completeness, we also refer to table 49 and table 50 that show the generators of Model 12 in terms of chiral fields with the corresponding mesonic flavor charges.

generator	$SU(2)_{\tilde{x}}$	$SU(2)_{\tilde{y}}$	$SU(2)_{\tilde{z}}$
$p_1^2 p_3^2 p_5^2 s$	1	1	1
$p_1 p_2 p_3^2 p_5^2 s$	0	1	1
$p_2^2 p_3^2 p_5^2 s$	-1	1	1
$p_1^2 p_3 p_4 p_5^2 s$	1	0	1
$p_1 p_2 p_3 p_4 p_5^2 s$	0	0	1
$p_2^2 p_3 p_4 p_5^2 s$	-1	0	1
$p_1^2 p_4^2 p_5^2 s$	1	-1	1
$p_1 p_2 p_4^2 p_5^2 s$	0	-1	1
$p_2^2 p_4^2 p_5^2 s$	-1	-1	1
$p_1^2 p_3^2 p_5 p_6 s$	1	1	0
$p_1 p_2 p_3^2 p_5 p_6 s$	0	1	0
$p_2^2 p_3^2 p_5 p_6 s$	-1	1	0
$p_1^2 p_3 p_4 p_5 p_6 s$	1	0	0
$p_1 p_2 p_3 p_4 p_5 p_6 s$	0	0	0
$p_2^2 p_3 p_4 p_5 p_6 s$	-1	0	0
$p_1^2 p_4^2 p_5 p_6 s$	1	-1	0
$p_1 p_2 p_4^2 p_5 p_6 s$	0	-1	0
$p_2^2 p_4^2 p_5 p_6 s$	-1	-1	0
$p_1^2 p_3^2 p_6^2 s$	1	1	-1
$p_1 p_2 p_3^2 p_6^2 s$	0	1	-1
$p_2^2 p_3^2 p_6^2 s$	-1	1	-1
$p_1^2 p_3 p_4 p_6^2 s$	1	0	-1
$p_1 p_2 p_3 p_4 p_6^2 s$	0	0	-1
$p_2^2 p_3 p_4 p_6^2 s$	-1	0	-1
$p_1^2 p_4^2 p_6^2 s$	1	-1	-1
$p_1 p_2 p_4^2 p_6^2 s$	0	-1	-1
$p_2^2 p_4^2 p_6^2 s$	-1	-1	-1

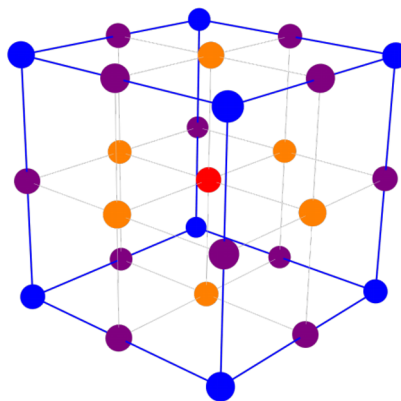


Table 48. The generators and lattice of generators of the mesonic moduli space of Model 12 in terms of brick matchings with the corresponding flavor charges.

generator	SU(2) _{\hat{x}}	SU(2) _{\hat{y}}	SU(2) _{\hat{z}}
$P_{15}R_{52}X_{23}X_{34}X_{41} = P_{26}X_{67}X_{78}X_{85}R_{52} = X_{23}P_{37}X_{78}X_{85}R_{52} = X_{23}X_{34}P_{48}X_{85}R_{52}$	1	1	1
$P_{15}X_{52}X_{23}X_{34}X_{41} = P_{26}X_{67}X_{78}X_{85}X_{52} = X_{23}P_{37}X_{78}X_{85}X_{52} = X_{23}X_{34}P_{48}X_{85}X_{52} = Q_{15}R_{52}X_{23}X_{34}X_{41}$	0	1	1
$= Q_{26}X_{67}X_{78}X_{85}R_{52} = X_{23}Q_{37}X_{78}X_{85}R_{52} = X_{23}X_{34}Q_{48}X_{85}R_{52}$			
$Q_{15}X_{52}X_{23}X_{34}X_{41} = Q_{26}X_{67}X_{78}X_{85}X_{52} = X_{23}Q_{37}X_{78}X_{85}X_{52} = X_{23}X_{34}Q_{48}X_{85}X_{52}$	-1	1	1
$P_{15}R_{52}X_{23}Y_{34}X_{41} = P_{15}S_{52}X_{23}X_{34}X_{41} = P_{26}X_{67}Y_{78}X_{85}R_{52} = P_{26}X_{67}X_{78}X_{85}S_{52} = X_{23}P_{37}Y_{78}X_{85}R_{52}$	1	0	1
$= X_{23}P_{37}X_{78}X_{85}S_{52} = X_{23}Y_{34}P_{48}X_{85}R_{52} = X_{23}X_{34}P_{48}X_{85}S_{52}$			
$P_{15}Y_{52}X_{23}X_{34}X_{41} = P_{15}X_{52}X_{23}Y_{34}X_{41} = P_{26}X_{67}Y_{78}X_{85}X_{52} = P_{26}X_{67}X_{78}X_{85}Y_{52} = X_{23}P_{37}Y_{78}X_{85}X_{52}$	0	0	1
$= X_{23}P_{37}X_{78}X_{85}Y_{52} = X_{23}X_{34}P_{48}X_{85}Y_{52} = X_{23}Y_{34}P_{48}X_{85}X_{52} = Q_{15}R_{52}X_{23}Y_{34}X_{41} = Q_{15}S_{52}X_{23}X_{34}X_{41}$			
$= Q_{26}X_{67}Y_{78}X_{85}R_{52} = Q_{26}X_{67}X_{78}X_{85}S_{52} = X_{23}Q_{37}Y_{78}X_{85}R_{52} = X_{23}Q_{37}X_{78}X_{85}S_{52} = X_{23}Y_{34}Q_{48}X_{85}R_{52}$			
$= X_{23}X_{34}Q_{48}X_{85}S_{52}$			
$Q_{15}Y_{52}X_{23}X_{34}X_{41} = Q_{15}X_{52}X_{23}Y_{34}X_{41} = Q_{26}X_{67}Y_{78}X_{85}X_{52} = Q_{26}X_{67}X_{78}X_{85}Y_{52} = X_{23}Q_{37}Y_{78}X_{85}X_{52}$	-1	0	1
$= X_{23}Q_{37}X_{78}X_{85}Y_{52} = X_{23}X_{34}Q_{48}X_{85}Y_{52} = X_{23}Y_{34}Q_{48}X_{85}X_{52}$			
$P_{15}S_{52}X_{23}Y_{34}X_{41} = P_{26}X_{67}Y_{78}X_{85}S_{52} = X_{23}P_{37}Y_{78}X_{85}S_{52} = X_{23}Y_{34}P_{48}X_{85}S_{52}$	1	-1	1
$P_{15}Y_{52}X_{23}Y_{34}X_{41} = P_{26}X_{67}Y_{78}X_{85}Y_{52} = X_{23}P_{37}Y_{78}X_{85}Y_{52} = X_{23}Y_{34}P_{48}X_{85}Y_{52} = Q_{15}S_{52}X_{23}Y_{34}X_{41}$	0	-1	1
$= Q_{26}X_{67}Y_{78}X_{85}S_{52} = X_{23}Q_{37}Y_{78}X_{85}S_{52} = X_{23}Y_{34}Q_{48}X_{85}S_{52}$			
$Q_{15}Y_{52}X_{23}Y_{34}X_{41} = Q_{26}X_{67}Y_{78}X_{85}Y_{52} = X_{23}Q_{37}Y_{78}X_{85}Y_{52} = X_{23}Y_{34}Q_{48}X_{85}Y_{52}$	-1	-1	1
$P_{15}R_{52}X_{23}X_{34}Y_{41} = P_{15}R_{52}Y_{23}X_{34}X_{41} = P_{26}X_{67}X_{78}Y_{85}R_{52} = P_{26}Y_{67}X_{78}X_{85}R_{52} = X_{23}P_{37}X_{78}Y_{85}R_{52}$	1	1	0
$= Y_{23}P_{37}X_{78}X_{85}R_{52} = X_{23}X_{34}P_{48}Y_{85}R_{52} = Y_{23}X_{34}P_{48}X_{85}R_{52}$			
$P_{15}X_{52}X_{23}X_{34}Y_{41} = P_{15}X_{52}Y_{23}X_{34}X_{41} = P_{26}X_{67}X_{78}Y_{85}X_{52} = P_{26}Y_{67}X_{78}X_{85}X_{52} = X_{23}P_{37}X_{78}Y_{85}X_{52}$	0	1	0
$= Y_{23}P_{37}X_{78}X_{85}X_{52} = X_{23}X_{34}P_{48}Y_{85}X_{52} = Y_{23}X_{34}P_{48}X_{85}X_{52} = Q_{15}R_{52}X_{23}X_{34}Y_{41} = Q_{15}R_{52}Y_{23}X_{34}X_{41}$			
$= Q_{26}X_{67}X_{78}Y_{85}R_{52} = Q_{26}Y_{67}X_{78}X_{85}R_{52} = X_{23}Q_{37}X_{78}Y_{85}R_{52} = Y_{23}Q_{37}X_{78}X_{85}R_{52} = X_{23}X_{34}Q_{48}Y_{85}R_{52}$			
$= Y_{23}X_{34}Q_{48}X_{85}R_{52}$			
$Q_{15}X_{52}X_{23}X_{34}Y_{41} = Q_{15}X_{52}Y_{23}X_{34}X_{41} = Q_{26}X_{67}X_{78}Y_{85}X_{52} = Q_{26}Y_{67}X_{78}X_{85}X_{52} = X_{23}Q_{37}X_{78}Y_{85}X_{52}$	-1	1	0
$= Y_{23}Q_{37}X_{78}X_{85}X_{52} = X_{23}X_{34}Q_{48}Y_{85}X_{52} = Y_{23}X_{34}Q_{48}X_{85}X_{52}$			
$P_{15}R_{52}X_{23}Y_{34}Y_{41} = P_{15}R_{52}Y_{23}Y_{34}X_{41} = P_{15}S_{52}X_{23}X_{34}Y_{41} = P_{15}S_{52}Y_{23}X_{34}X_{41} = P_{26}X_{67}Y_{78}Y_{85}R_{52}$	1	0	0
$= P_{26}Y_{67}Y_{78}X_{85}R_{52} = P_{26}X_{67}X_{78}Y_{85}S_{52} = P_{26}Y_{67}X_{78}X_{85}S_{52} = X_{23}P_{37}Y_{78}Y_{85}R_{52} = Y_{23}P_{37}Y_{78}X_{85}R_{52}$			
$= X_{23}P_{37}X_{78}Y_{85}S_{52} = Y_{23}P_{37}X_{78}X_{85}S_{52} = X_{23}Y_{34}P_{48}Y_{85}R_{52} = Y_{23}Y_{34}P_{48}X_{85}R_{52} = X_{23}X_{34}P_{48}Y_{85}S_{52}$			
$= Y_{23}X_{34}P_{48}X_{85}S_{52}$			
$P_{15}Y_{52}X_{23}X_{34}Y_{41} = P_{15}X_{52}X_{23}Y_{34}Y_{41} = P_{15}Y_{52}Y_{23}X_{34}X_{41} = P_{15}X_{52}Y_{23}Y_{34}X_{41} = P_{26}X_{67}Y_{78}Y_{85}X_{52}$	0	0	0
$= P_{26}Y_{67}Y_{78}X_{85}X_{52} = P_{26}X_{67}X_{78}Y_{85}Y_{52} = P_{26}Y_{67}X_{78}X_{85}Y_{52} = X_{23}P_{37}Y_{78}Y_{85}X_{52} = X_{23}P_{37}X_{78}Y_{85}Y_{52}$			
$= Y_{23}P_{37}Y_{78}X_{85}X_{52} = Y_{23}P_{37}X_{78}X_{85}Y_{52} = X_{23}X_{34}P_{48}Y_{85}Y_{52} = X_{23}Y_{34}P_{48}Y_{85}X_{52} = Y_{23}X_{34}P_{48}X_{85}Y_{52}$			
$= Y_{23}Y_{34}P_{48}X_{85}X_{52} = Q_{15}R_{52}X_{23}Y_{34}Y_{41} = Q_{15}R_{52}Y_{23}Y_{34}X_{41} = Q_{15}S_{52}X_{23}X_{34}Y_{41} = Q_{15}S_{52}Y_{23}X_{34}X_{41}$			
$= Q_{26}X_{67}Y_{78}Y_{85}R_{52} = Q_{26}Y_{67}Y_{78}X_{85}R_{52} = Q_{26}X_{67}X_{78}Y_{85}S_{52} = Q_{26}Y_{67}X_{78}X_{85}S_{52} = X_{23}Q_{37}Y_{78}Y_{85}R_{52}$			
$= Y_{23}Q_{37}Y_{78}X_{85}R_{52} = X_{23}Q_{37}X_{78}Y_{85}S_{52} = Y_{23}Q_{37}X_{78}X_{85}S_{52} = X_{23}Y_{34}Q_{48}Y_{85}R_{52} = Y_{23}Y_{34}Q_{48}X_{85}R_{52}$			
$= X_{23}X_{34}Q_{48}Y_{85}S_{52} = Y_{23}X_{34}Q_{48}X_{85}S_{52}$			
$Q_{15}Y_{52}X_{23}X_{34}Y_{41} = Q_{15}X_{52}X_{23}Y_{34}Y_{41} = Q_{15}Y_{52}Y_{23}X_{34}X_{41} = Q_{15}X_{52}Y_{23}Y_{34}X_{41} = P_{26}X_{67}Y_{78}Y_{85}X_{52}$	-1	0	0
$= Q_{26}Y_{67}Y_{78}X_{85}X_{52} = Q_{26}X_{67}X_{78}Y_{85}Y_{52} = Q_{26}Y_{67}X_{78}X_{85}Y_{52} = X_{23}Q_{37}Y_{78}Y_{85}X_{52} = X_{23}Q_{37}X_{78}Y_{85}Y_{52}$			
$= Y_{23}Q_{37}Y_{78}X_{85}X_{52} = Y_{23}Q_{37}X_{78}X_{85}Y_{52} = X_{23}X_{34}Q_{48}Y_{85}Y_{52} = X_{23}Y_{34}Q_{48}Y_{85}X_{52} = Y_{23}X_{34}Q_{48}X_{85}Y_{52}$			
$= Y_{23}Y_{34}Q_{48}X_{85}X_{52}$			

Table 49. The generators in terms of bifundamental chiral fields for Model 12 (*Part 1*).

generator	$SU(2)_{\bar{x}}$	$SU(2)_{\bar{y}}$	$SU(2)_{\bar{z}}$
$P_{15}S_{52}X_{23}Y_{34}Y_{41} = P_{15}S_{52}Y_{23}Y_{34}X_{41} = P_{26}X_{67}Y_{78}Y_{85}S_{52} = P_{26}Y_{67}Y_{78}X_{85}S_{52} = X_{23}P_{37}Y_{78}Y_{85}S_{52}$ $= Y_{23}P_{37}Y_{78}X_{85}S_{52} = X_{23}Y_{34}P_{48}Y_{85}S_{52} = Y_{23}Y_{34}P_{48}X_{85}S_{52}$	1	-1	0
$P_{15}Y_{52}X_{23}Y_{34}Y_{41} = P_{15}Y_{52}Y_{23}Y_{34}X_{41} = P_{26}X_{67}Y_{78}Y_{85}Y_{52} = P_{26}Y_{67}Y_{78}X_{85}Y_{52} = X_{23}P_{37}Y_{78}Y_{85}Y_{52}$ $= Y_{23}P_{37}Y_{78}X_{85}Y_{52} = X_{23}Y_{34}P_{48}Y_{85}Y_{52} = Y_{23}Y_{34}P_{48}X_{85}Y_{52} = Q_{15}S_{52}X_{23}Y_{34}Y_{41} = Q_{15}S_{52}Y_{23}Y_{34}X_{41}$ $= Q_{26}X_{67}Y_{78}Y_{85}S_{52} = Q_{26}Y_{67}Y_{78}X_{85}S_{52} = X_{23}Q_{37}Y_{78}Y_{85}S_{52} = Y_{23}Q_{37}Y_{78}X_{85}S_{52} = X_{23}Y_{34}Q_{48}Y_{85}S_{52}$ $= Y_{23}Y_{34}Q_{48}X_{85}S_{52}$	0	-1	0
$Q_{15}Y_{52}X_{23}Y_{34}Y_{41} = Q_{15}Y_{52}Y_{23}Y_{34}X_{41} = Q_{26}X_{67}Y_{78}Y_{85}Y_{52} = Q_{26}Y_{67}Y_{78}X_{85}Y_{52} = X_{23}Q_{37}Y_{78}Y_{85}Y_{52}$ $= Y_{23}Q_{37}Y_{78}X_{85}Y_{52} = X_{23}Y_{34}Q_{48}Y_{85}Y_{52} = Y_{23}Y_{34}Q_{48}X_{85}Y_{52}$	-1	-1	0
$P_{15}R_{52}Y_{23}X_{34}Y_{41} = P_{26}Y_{67}X_{78}Y_{85}R_{52} = Y_{23}P_{37}X_{78}Y_{85}R_{52} = Y_{23}X_{34}P_{48}Y_{85}R_{52}$	1	1	-1
$P_{15}X_{52}Y_{23}X_{34}Y_{41} = P_{26}Y_{67}X_{78}Y_{85}X_{52} = Y_{23}P_{37}X_{78}Y_{85}X_{52} = Y_{23}X_{34}P_{48}Y_{85}X_{52} = Q_{15}R_{52}Y_{23}X_{34}Y_{41}$ $= Q_{26}Y_{67}X_{78}Y_{85}R_{52} = Y_{23}Q_{37}X_{78}Y_{85}R_{52} = Y_{23}X_{34}Q_{48}Y_{85}R_{52}$	0	1	-1
$Q_{15}X_{52}Y_{23}X_{34}Y_{41} = Q_{26}Y_{67}X_{78}Y_{85}X_{52} = Y_{23}Q_{37}X_{78}Y_{85}X_{52} = Y_{23}X_{34}Q_{48}Y_{85}X_{52}$	-1	1	-1
$P_{15}R_{52}Y_{23}Y_{34}Y_{41} = P_{15}S_{52}Y_{23}X_{34}Y_{41} = P_{26}Y_{67}Y_{78}Y_{85}R_{52} = P_{26}Y_{67}X_{78}Y_{85}S_{52} = Y_{23}P_{37}Y_{78}Y_{85}R_{52}$ $= Y_{23}P_{37}X_{78}Y_{85}S_{52} = Y_{23}Y_{34}P_{48}Y_{85}R_{52} = Y_{23}X_{34}P_{48}Y_{85}S_{52}$	1	0	-1
$P_{15}Y_{52}Y_{23}X_{34}Y_{41} = P_{15}X_{52}Y_{23}Y_{34}Y_{41} = P_{26}Y_{67}Y_{78}Y_{85}X_{52} = P_{26}Y_{67}X_{78}Y_{85}Y_{52} = Y_{23}P_{37}Y_{78}Y_{85}X_{52}$ $= Y_{23}P_{37}X_{78}Y_{85}Y_{52} = Y_{23}X_{34}P_{48}Y_{85}Y_{52} = Y_{23}Y_{34}P_{48}Y_{85}X_{52} = Q_{15}R_{52}Y_{23}Y_{34}Y_{41} = Q_{15}S_{52}Y_{23}X_{34}Y_{41}$ $= Q_{26}Y_{67}Y_{78}Y_{85}R_{52} = Q_{26}Y_{67}X_{78}Y_{85}S_{52} = Y_{23}Q_{37}Y_{78}Y_{85}R_{52} = Y_{23}Q_{37}X_{78}Y_{85}S_{52} = Y_{23}Y_{34}Q_{48}Y_{85}R_{52}$ $= Y_{23}X_{34}Q_{48}Y_{85}S_{52}$	0	0	-1
$Q_{15}Y_{52}Y_{23}X_{34}Y_{41} = Q_{15}X_{52}Y_{23}Y_{34}Y_{41} = Q_{26}Y_{67}Y_{78}Y_{85}X_{52} = Q_{26}Y_{67}X_{78}Y_{85}Y_{52} = Y_{23}Q_{37}Y_{78}Y_{85}X_{52}$ $= Y_{23}Q_{37}X_{78}Y_{85}Y_{52} = Y_{23}X_{34}Q_{48}Y_{85}Y_{52} = Y_{23}Y_{34}Q_{48}Y_{85}X_{52}$	-1	0	-1
$P_{15}S_{52}Y_{23}Y_{34}Y_{41} = P_{26}Y_{67}Y_{78}Y_{85}S_{52} = Y_{23}P_{37}Y_{78}Y_{85}S_{52} = Y_{23}Y_{34}P_{48}Y_{85}S_{52}$	1	-1	-1
$P_{15}Y_{52}Y_{23}Y_{34}Y_{41} = P_{26}Y_{67}Y_{78}Y_{85}Y_{52} = Y_{23}P_{37}Y_{78}Y_{85}Y_{52} = Y_{23}Y_{34}P_{48}Y_{85}Y_{52} = Q_{15}S_{52}Y_{23}Y_{34}Y_{41}$ $= Q_{26}Y_{67}Y_{78}Y_{85}S_{52} = Y_{23}Q_{37}Y_{78}Y_{85}S_{52} = Y_{23}Y_{34}Q_{48}Y_{85}S_{52}$	0	-1	-1
$Q_{15}Y_{52}Y_{23}Y_{34}Y_{41} = Q_{26}Y_{67}Y_{78}Y_{85}Y_{52} = Y_{23}Q_{37}Y_{78}Y_{85}Y_{52} = Y_{23}Y_{34}Q_{48}Y_{85}Y_{52}$	-1	-1	-1

Table 50. The generators in terms of bifundamental chiral fields for Model 12 (*Part 2*).

16 Model 13: $P_{+-}^1(\text{dP}_2)$ [dP_2 bundle of \mathbb{P}^1 , $\langle 81 \rangle$]

Model 13 corresponds to the toric Calabi-Yau 4-fold $P_{+-}^1(\text{dP}_2)$. The corresponding brane brick model has the quiver in figure 34 and the J - and E -terms are

$$\begin{array}{ll}
 & J \\
 \Lambda_{18}^1 : & X_{89}Y_{97}X_{75}Q_{5.10}X_{10.1} - X_{82}Q_{27}X_{71} \\
 \Lambda_{18}^2 : & X_{89}Y_{97}X_{75}P_{5.10}X_{10.1} - X_{82}P_{27}X_{71} \\
 \Lambda_{19}^1 : & X_{97}X_{71} - Y_{97}Y_{7.10}X_{10.1} \\
 \Lambda_{26}^1 : & X_{64}X_{42} - X_{68}X_{82} \\
 \Lambda_{2.10}^1 : & X_{10.1}X_{13}X_{34}Y_{42} - X_{10.3}X_{34}X_{42} \\
 \Lambda_{2.10}^2 : & X_{10.3}Q_{38}X_{82} - X_{10.1}Q_{16}X_{64}Y_{42} \\
 \Lambda_{2.10}^3 : & X_{10.3}P_{38}X_{82} - X_{10.1}P_{16}X_{64}Y_{42} \\
 \Lambda_{37}^1 : & Y_{7.10}X_{10.3} - X_{71}X_{13} \\
 \Lambda_{39}^1 : & Y_{97}X_{75}Q_{5.10}X_{10.1}X_{13} - X_{97}X_{75}Q_{5.10}X_{10.3} \\
 \Lambda_{39}^2 : & Y_{97}X_{75}P_{5.10}X_{10.1}X_{13} - X_{97}X_{75}P_{5.10}X_{10.3} \\
 \Lambda_{47}^1 : & X_{71}Q_{16}X_{64} - X_{75}Q_{5.10}X_{10.3}X_{34} \\
 \Lambda_{47}^2 : & X_{75}Q_{5.10}X_{10.1}X_{13}X_{34} - Y_{7.10}X_{10.1}Q_{16}X_{64} \\
 \Lambda_{47}^3 : & X_{71}P_{16}X_{64} - X_{75}P_{5.10}X_{10.3}X_{34} \\
 \Lambda_{47}^4 : & X_{75}P_{5.10}X_{10.1}X_{13}X_{34} - Y_{7.10}X_{10.1}P_{16}X_{64} \\
 \Lambda_{56}^1 : & X_{68}X_{89}Y_{97}X_{75} - X_{64}Y_{42}Y_{25} \\
 \Lambda_{58}^1 : & X_{82}Y_{25} - X_{89}X_{97}X_{75}
 \end{array}
 \qquad
 \begin{array}{ll}
 & E \\
 P_{16}X_{68} - X_{13}P_{38} & \\
 X_{13}Q_{38} - Q_{16}X_{68} & \\
 P_{16}X_{64}Q_{49} - Q_{16}X_{64}P_{49} & \\
 P_{27}X_{71}Q_{16} - Q_{27}X_{71}P_{16} & \\
 P_{27}X_{75}Q_{5.10} - Q_{27}X_{75}P_{5.10} & \\
 P_{27}Y_{7.10} - Y_{25}P_{5.10} & \\
 Y_{25}Q_{5.10} - Q_{27}Y_{7.10} & \\
 P_{38}X_{82}Q_{27} - Q_{38}X_{82}P_{27} & \\
 P_{38}X_{89} - X_{34}P_{49} & \\
 X_{34}Q_{49} - Q_{38}X_{89} & \\
 P_{49}X_{97} - X_{42}P_{27} & \\
 P_{49}Y_{97} - Y_{42}P_{27} & \\
 X_{42}Q_{27} - Q_{49}X_{97} & \\
 Y_{42}Q_{27} - Q_{49}Y_{97} & \\
 P_{5.10}X_{10.1}Q_{16} - Q_{5.10}X_{10.1}P_{16} & \\
 P_{5.10}X_{10.3}Q_{38} - Q_{5.10}X_{10.3}P_{38} &
 \end{array}
 \quad . \quad (16.1)$$

Using the forward algorithm, we can obtain the brick matchings for Model 13. The brick matchings are summarized in the P -matrix, which takes the form

$$P = \left(\begin{array}{c|cccccccccccccccc|cccccccccccccccc}
 & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & s_{10} & s_{11} & s_{12} & s_{13} & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & o_7 & o_8 & o_9 & o_{10} & o_{11} & o_{12} & o_{13} & o_{14} \\
 \hline
 P_{16} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 P_{27} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
 P_{38} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
 P_{49} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
 P_{5.10} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
 \hline
 Q_{16} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 Q_{27} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
 Q_{38} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
 Q_{49} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
 Q_{5.10} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
 \hline
 X_{13} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 X_{34} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 X_{42} & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 X_{64} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
 X_{68} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
 X_{71} & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 X_{75} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
 X_{82} & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
 X_{89} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
 X_{97} & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
 X_{10.1} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 X_{10.3} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 \hline
 Y_{25} & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 Y_{42} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 Y_{7.10} & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 Y_{97} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
 \end{array} \right) . \quad (16.2)$$

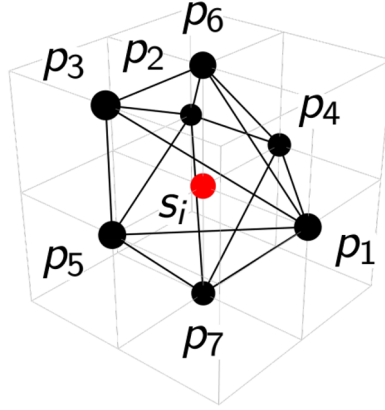


Figure 35. Toric diagram for Model 13.

The Hilbert series of the mesonic moduli space of Model 13 takes the form

$$\begin{aligned}
 g_1(t_i, y_s, y_{o_1}, y_{o_2}, y_{o_3}, y_{o_4}; \mathcal{M}_{13}) &= \frac{P(t_i, y_s, y_{o_1}, y_{o_2}, y_{o_3}, y_{o_4}; \mathcal{M}_{13})}{(1 - y_s y_{o_1}^2 y_{o_2} y_{o_3}^3 y_{o_4}^3 t_1^3 t_3 t_4^2 t_6^2)(1 - y_s y_{o_1}^2 y_{o_2} y_{o_3}^3 y_{o_4}^3 t_2^3 t_3 t_4^2 t_6^2)} \\
 &\times \frac{1}{(1 - y_s y_{o_1}^4 y_{o_2}^3 y_{o_3}^3 y_{o_4} t_1 t_3^3 t_5^2 t_6^2)(1 - y_s y_{o_1}^4 y_{o_2}^3 y_{o_3}^3 y_{o_4} t_2 t_3^3 t_5^2 t_6^2)(1 - y_s y_{o_1} y_{o_2} y_{o_3}^2 y_{o_4}^3 t_1^3 t_4^2 t_6 t_7)} \\
 &\times \frac{1}{(1 - y_s y_{o_1} y_{o_2} y_{o_3}^2 y_{o_4}^3 t_2^3 t_4^2 t_6 t_7)(1 - y_s y_{o_1} y_{o_2}^2 y_{o_3} y_{o_4}^2 t_1^2 t_4 t_5 t_7^2)(1 - y_s y_{o_1} y_{o_2}^2 y_{o_3} y_{o_4}^2 t_2^2 t_4 t_5 t_7^2)} \\
 &\times \frac{1}{(1 - y_s y_{o_1}^2 y_{o_2}^3 y_{o_3} y_{o_4} t_1 t_3 t_5^2 t_7^2)(1 - y_s y_{o_1}^2 y_{o_2}^3 y_{o_3} y_{o_4} t_2 t_3 t_5^2 t_7^2)}, \tag{16.6}
 \end{aligned}$$

where t_i are the fugacities for the extremal brick matchings p_i . y_s counts the brick matching product $s_1 \dots s_{13}$ corresponding to the single internal point of the toric diagram of Model 13. Additionally, $y_{o_1}, y_{o_2}, y_{o_3}$ and y_{o_4} count the products of extra GLSM fields $o_1 o_2, o_3 o_4$ and $o_5 \dots o_8$ and $o_9 \dots o_{14}$, respectively. The explicit numerator $P(t_i, y_s, y_{o_1}, y_{o_2}, y_{o_3}, y_{o_4}; \mathcal{M}_{13})$ of the Hilbert series is given in the appendix section A.13. We note that setting the fugacities $y_{o_1} = 1, \dots, y_{o_4} = 1$ does not change the overall characterization of the mesonic moduli space by the Hilbert series, indicating that the extra GLSM fields, as expected, correspond to an over-parameterization of the moduli space.

By setting $t_i = t$ for the fugacities of the extremal brick matchings, and all other fugacities to 1, the unrefined Hilbert series takes the following form

$$\begin{aligned}
 g_1(t, 1, 1, 1, 1, 1; \mathcal{M}_{13}) &= \frac{(1-t)(1-t^2)^3}{(1-t^6)^3(1-t^7)^2(1-t^8)^3} \times (1 + t + 4t^2 + 4t^3 + 10t^4 \\
 &+ 10t^5 + 22t^6 + 29t^7 + 56t^8 + 77t^9 + 128t^{10} + 170t^{11} + 254t^{12} + 311t^{13} \\
 &+ 412t^{14} + 464t^{15} + 567t^{16} + 594t^{17} + 684t^{18} + 669t^{19} + 742t^{20} + 694t^{21} \\
 &+ 742t^{22} + 669t^{23} + 684t^{24} + 594t^{25} + 567t^{26} + 464t^{27} + 412t^{28} + 311t^{29} \\
 &+ 254t^{30} + 170t^{31} + 128t^{32} + 77t^{33} + 56t^{34} + 29t^{35} + 22t^{36} + 10t^{37} + 10t^{38} \\
 &+ 4t^{39} + 4t^{40} + t^{41} + t^{42}), \tag{16.7}
 \end{aligned}$$

where the palindromic numerator indicates that the mesonic moduli space is Calabi-Yau.

	$SU(2)_x$	$U(1)_{b_1}$	$U(1)_{b_2}$	$U(1)$	fugacity
p_1	+1	0	0	r_1	t_1
p_2	-1	0	0	r_2	t_2
p_3	0	+1	0	r_3	t_3
p_4	0	-1	0	r_4	t_4
p_5	0	0	+1	r_5	t_5
p_6	0	0	-1	r_6	t_6
p_7	0	0	0	r_7	t_7

Table 51. Global symmetry charges on the extremal brick matchings p_i of Model 13.

The global symmetry of Model 13 and the charges on the extremal brick matchings under the global symmetry are summarized in table 51. We can use the following fugacity map,

$$t = t_7, \quad x = \frac{t_7}{t_2}, \quad b_1 = \frac{t_7}{t_4}, \quad b_2 = \frac{t_5}{t_7}, \quad (16.8)$$

where $t_1 t_2 = t_3 t_4 = t_5 t_6 = t_7^2$, in order to rewrite the Hilbert series for Model 13 in terms of characters of irreducible representations of $SU(2) \times U(1) \times U(1)$.

The highest weight form of the Hilbert series of Model 13 is

$$\begin{aligned}
 h_1(t, \mu, b_1, b_2; \mathcal{M}_{13}) &= \frac{1}{(1 - \mu^2 b_1^{-1} b_2 t^6)(1 - \mu b_1 b_2^2 t^6)(1 - \mu^3 b_1^{-2} b_2^{-1} t^7)(1 - \mu^3 b_1^{-1} b_2^{-2} t^8)} \\
 &\times \frac{1}{(1 - \mu b_1^3 t^8)} \times (1 + \mu^2 t^7 + \mu b_1^2 b_2 t^7 + \mu^2 b_1 b_2^{-1} t^8 - \mu^4 b_1^{-1} b_2 t^{13} - \mu^3 b_1 b_2^2 t^{13} - 2\mu^4 t^{14} \\
 &- \mu^5 b_1^{-2} b_2^{-1} t^{14} - \mu^3 b_1^2 b_2 t^{14} - \mu^5 b_1^{-1} b_2^{-2} t^{15} - \mu^4 b_1 b_2^{-1} t^{15} + \mu^6 b_1^{-1} b_2 t^{20} + \mu^6 t^{21} \\
 &+ \mu^7 b_1^{-2} b_2^{-1} t^{21} + \mu^8 t^{28}), \quad (16.9)
 \end{aligned}$$

where $\mu^m \sim [m]_{SU(2)_x}$. Here in highest weight form, the fugacity μ counts the highest weight of irreducible representations of $SU(2)_x$. The fugacities b_1 and b_2 count charges under the two $U(1)$ factors of the mesonic flavor symmetry.

The plethystic logarithm of the Hilbert series takes the form

$$\begin{aligned}
 \text{PL}[g_1(t, x, b_1, b_2; \mathcal{M}_{13})] &= ([1]b_1 b_2^2 + [2])t^6 + ([1]b_1^2 b_2 + [2]b_1 b_2^{-1} + [3]b_2^{-3})t^7 \\
 &+ ([1]b_1^3 + [2]b_1^2 b_2^{-2} + [3]b_1 b_2^{-4})t^8 \\
 &- (1 + [1]b_1 b_2^2)t^{12} - (b_1 b_2^{-1} + b_1^3 b_2^3 + [1]b_2^{-3} + 2[1]b_1^2 b_2 + 2[2]b_1 b_2^{-1} + [3]b_2^{-3} \\
 &+ [3]b_1^2 b_2 + [4]b_1 b_2^{-1})t^{13} - (2b_1^2 b_2^{-2} + b_1^4 b_2^2 + 3[1]b_1^3 + 2[1]b_1 b_2^{-4} + [2]b_2^{-6} \\
 &+ 3[2]b_1^2 b_2^{-2} + [2]b_1^4 b_2^2 + 2[3]b_1^3 + 2[3]b_1 b_2^{-4} + 3[4]b_1^2 b_2^{-2} + [5]b_1 b_2^{-4})t^{14} \\
 &- (b_1 b_2^{-7} + b_1^3 b_2^{-3} + b_1^5 b_2 + 2[1]b_1^2 b_2^{-5} + 2[1]b_1^4 b_2^{-1} + [2]b_1 b_2^{-7} + 3[2]b_1^3 b_2^{-3} \\
 &+ 2[3]b_1^2 b_2^{-5} + [3]b_1^4 b_2^{-1} + [4]b_1 b_2^{-7} + 2[4]b_1^3 b_2^{-3} + [5]b_1^2 b_2^{-5})t^{15} - (b_1^4 b_2^{-4} \\
 &+ [1]b_1^3 b_2^{-6} + [1]b_1^5 b_2^{-2} + [2]b_1^2 b_2^{-8} + [2]b_1^4 b_2^{-4} + [3]b_1^3 b_2^{-6} + [4]b_1^4 b_2^{-4})t^{16} + \dots, \quad (16.10)
 \end{aligned}$$

where $[m] = [m]_{SU(2)_x}$. From the plethystic logarithm, we see that the mesonic moduli space is a non-complete intersection.

generator	$SU(2)_{\tilde{x}}$	$U(1)_{\tilde{b}_1}$	$U(1)_{\tilde{b}_2}$
$p_1^2 p_4 p_5 p_7^2 \ so_1 o_2^2 o_3 o_4^2$	0	-1	1
$p_1 p_2 p_4 p_5 p_7^2 \ so_1 o_2^2 o_3 o_4^2$	0	0	1
$p_2^2 p_4 p_5 p_7^2 \ so_1 o_2^2 o_3 o_4^2$	0	1	1
$p_1 p_3 p_5^2 p_7^2 \ so_1^2 o_2^3 o_3 o_4$	1	0	1
$p_2 p_3 p_5^2 p_7^2 \ so_1^2 o_2^3 o_3 o_4$	1	1	1
$p_1^3 p_4^2 p_6 p_7 \ so_1 o_2 o_3^2 o_4^3$	-1	-2	0
$p_1^2 p_2 p_4^2 p_6 p_7 \ so_1 o_2 o_3^2 o_4^3$	-1	-1	0
$p_1 p_2^2 p_4^2 p_6 p_7 \ so_1 o_2 o_3^2 o_4^3$	-1	0	0
$p_2^2 p_4^2 p_6 p_7 \ so_1 o_2 o_3^2 o_4^3$	-1	1	0
$p_1^2 p_3 p_4 p_5 p_6 p_7 \ so_1^2 o_2^2 o_3^2 o_4^2$	0	-1	0
$p_1 p_2 p_3 p_4 p_5 p_6 p_7 \ so_1^2 o_2^2 o_3^2 o_4^2$	0	0	0
$p_2^2 p_3 p_4 p_5 p_6 p_7 \ so_1^2 o_2^2 o_3^2 o_4^2$	0	1	0
$p_1 p_3^2 p_5^2 p_6 p_7 \ so_1^3 o_2^3 o_3^2 o_4$	1	0	0
$p_2 p_3^2 p_5^2 p_6 p_7 \ so_1^3 o_2^3 o_3^2 o_4$	1	1	0
$p_1^3 p_3 p_4^2 p_6^2 \ so_1^2 o_2 o_3^2 o_4^3$	-1	-2	-1
$p_1^2 p_2 p_3 p_4^2 p_6^2 \ so_1^2 o_2 o_3^2 o_4^3$	-1	-1	-1
$p_1 p_2^2 p_3 p_4^2 p_6^2 \ so_1^2 o_2 o_3^2 o_4^3$	-1	0	-1
$p_2^2 p_3 p_4^2 p_6^2 \ so_1^2 o_2 o_3^2 o_4^3$	-1	1	-1
$p_1^2 p_3^2 p_4 p_5 p_6^2 \ so_1^3 o_2^2 o_3^2 o_4^2$	0	-1	-1
$p_1 p_2 p_3^2 p_4 p_5 p_6^2 \ so_1^3 o_2^2 o_3^2 o_4^2$	0	0	-1
$p_2^2 p_3^2 p_4 p_5 p_6^2 \ so_1^3 o_2^2 o_3^2 o_4^2$	0	1	-1
$p_1 p_3^3 p_5^2 p_6^2 \ so_1^4 o_2^3 o_3^2 o_4$	1	0	-1
$p_2 p_3^3 p_5^2 p_6^2 \ so_1^4 o_2^3 o_3^2 o_4$	1	1	-1

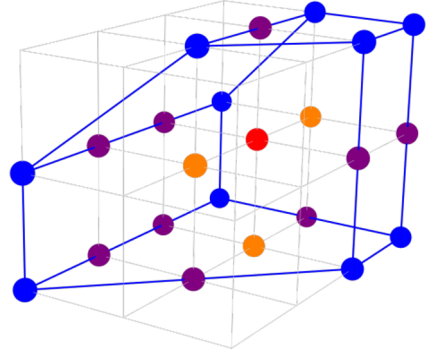


Table 52. The generators and lattice of generators of the mesonic moduli space of Model 13 in terms of brick matchings with the corresponding flavor charges.

By using the following fugacity map

$$\tilde{t} = t_5^{1/2} t_6^{1/2}, \quad \tilde{x} = \frac{t_5^3}{t_2 t_7^2}, \quad \tilde{b}_1 = \frac{t_2^2 t_7}{t_5^{3/2} t_6^{3/2}}, \quad \tilde{b}_2 = \frac{t_7^2}{t_5 t_6}, \quad (16.11)$$

where $t_1^2 t_2^2 t_7^2 = t_5^3 t_6^3$, $t_4^2 t_5 = t_6 t_7^2$ and $t_3^2 t_7^2 = t_5^3 t_6$, the mesonic flavor charges on the gauge invariant operators become \mathbb{Z} -valued. The generators in terms of brick matchings and their corresponding rescaled mesonic flavor charges are summarized in table 52. The generator lattice as shown in table 52 is a convex lattice polytope, which is reflexive. It is the dual of the toric diagram of Model 13 shown in figure 35. For completeness, table 53 and table 54 show the generators of Model 13 in terms of chiral fields with the corresponding mesonic flavor charges.

generator	SU(2) $_{\bar{x}}$	U(1) $_{b_1}$	U(1) $_{b_2}$
$P_{16}X_{64}Y_{42}P_{27}X_{71} = P_{16}X_{64}P_{49}Y_{97}X_{71} = P_{27}X_{75}P_{5.10}X_{10.3}X_{34}Y_{42} = P_{38}X_{89}Y_{97}X_{75}P_{5.10}X_{10.3}$ $= X_{34}P_{49}Y_{97}X_{75}P_{5.10}X_{10.3}$	0	-1	1
$P_{16}X_{64}Y_{42}Q_{27}X_{71} = P_{16}X_{64}Q_{49}Y_{97}X_{71} = Q_{16}X_{64}Y_{42}P_{27}X_{71} = Q_{16}X_{64}P_{49}Y_{97}X_{71}$ $= P_{27}X_{75}Q_{5.10}X_{10.3}X_{34}Y_{42} = P_{38}X_{89}Y_{97}X_{75}Q_{5.10}X_{10.3} = X_{34}P_{49}Y_{97}X_{75}Q_{5.10}X_{10.3} = Q_{27}X_{75}P_{5.10}X_{10.3}X_{34}Y_{42}$ $= Q_{38}X_{89}Y_{97}X_{75}P_{5.10}X_{10.3} = X_{34}Q_{49}Y_{97}X_{75}P_{5.10}X_{10.3}$	0	0	1
$Q_{16}X_{64}Y_{42}Q_{27}X_{71} = Q_{16}X_{64}Q_{49}Y_{97}X_{71} = Q_{27}X_{75}Q_{5.10}X_{10.3}X_{34}Y_{42} = Q_{38}X_{89}Y_{97}X_{75}Q_{5.10}X_{10.3}$ $= X_{34}Q_{49}Y_{97}X_{75}Q_{5.10}X_{10.3}$	0	1	1
$P_{16}X_{68}X_{89}Y_{97}X_{71} = X_{13}X_{34}Y_{42}P_{27}X_{71} = P_{27}Y_{7.10}X_{10.3}X_{34}Y_{42} = X_{13}P_{38}X_{89}Y_{97}X_{71} = P_{38}X_{89}Y_{97}Y_{7.10}X_{10.3}$ $= X_{13}X_{34}P_{49}Y_{97}X_{71} = X_{34}P_{49}Y_{97}Y_{7.10}X_{10.3} = Y_{25}P_{5.10}X_{10.3}X_{34}Y_{42}$	1	0	1
$Q_{16}X_{68}X_{89}Y_{97}X_{71} = X_{13}X_{34}Y_{42}Q_{27}X_{71} = Q_{27}Y_{7.10}X_{10.3}X_{34}Y_{42} = X_{13}Q_{38}X_{89}Y_{97}X_{71} = Q_{38}X_{89}Y_{97}Y_{7.10}X_{10.3}$ $= X_{13}X_{34}Q_{49}Y_{97}X_{71} = X_{34}Q_{49}Y_{97}Y_{7.10}X_{10.3} = Y_{25}Q_{5.10}X_{10.3}X_{34}Y_{42}$	1	1	1
$P_{27}X_{75}P_{5.10}X_{10.3}P_{38}X_{82} = P_{16}X_{64}Y_{42}P_{27}X_{75}P_{5.10}X_{10.1} = P_{16}X_{64}P_{49}Y_{97}X_{75}P_{5.10}X_{10.1}$	-1	-2	0
$P_{27}X_{75}Q_{5.10}X_{10.3}P_{38}X_{82} = P_{27}X_{75}P_{5.10}X_{10.3}Q_{38}X_{82} = Q_{27}X_{75}P_{5.10}X_{10.3}P_{38}X_{82}$ $= P_{16}X_{64}Y_{42}P_{27}X_{75}Q_{5.10}X_{10.1} = P_{16}X_{64}P_{49}Y_{97}X_{75}Q_{5.10}X_{10.1} = P_{16}X_{64}Y_{42}Q_{27}X_{75}P_{5.10}X_{10.1}$ $= P_{16}X_{64}Q_{49}Y_{97}X_{75}P_{5.10}X_{10.1} = Q_{16}X_{64}Y_{42}P_{27}X_{75}P_{5.10}X_{10.1} = Q_{16}X_{64}P_{49}Y_{97}X_{75}P_{5.10}X_{10.1}$	-1	-1	0
$P_{27}X_{75}Q_{5.10}X_{10.3}P_{38}X_{82} = Q_{27}X_{75}Q_{5.10}X_{10.3}P_{38}X_{82} = Q_{27}X_{75}P_{5.10}X_{10.3}Q_{38}X_{82}$ $= P_{16}X_{64}Y_{42}Q_{27}X_{75}Q_{5.10}X_{10.1} = P_{16}X_{64}Q_{49}Y_{97}X_{75}Q_{5.10}X_{10.1} = Q_{16}X_{64}Y_{42}P_{27}X_{75}Q_{5.10}X_{10.1}$ $= Q_{16}X_{64}P_{49}Y_{97}X_{75}Q_{5.10}X_{10.1} = Q_{16}X_{64}Y_{42}Q_{27}X_{75}P_{5.10}X_{10.1} = Q_{16}X_{64}Q_{49}Y_{97}X_{75}P_{5.10}X_{10.1}$ $Q_{27}X_{75}Q_{5.10}X_{10.3}Q_{38}X_{82} = Q_{16}X_{64}Y_{42}Q_{27}X_{75}Q_{5.10}X_{10.1} = Q_{16}X_{64}Q_{49}Y_{97}X_{75}Q_{5.10}X_{10.1}$	-1	0	0
$P_{16}X_{68}X_{89}Y_{97}X_{75}P_{5.10}X_{10.1} = X_{13}X_{34}Y_{42}P_{27}X_{75}P_{5.10}X_{10.1} = X_{13}P_{38}X_{89}Y_{97}X_{75}P_{5.10}X_{10.1}$ $= X_{13}X_{34}P_{49}Y_{97}X_{75}P_{5.10}X_{10.1} = P_{16}X_{64}X_{42}P_{27}X_{71} = P_{16}X_{68}X_{82}P_{27}X_{71} = P_{16}X_{64}P_{49}X_{97}X_{71}$ $= X_{13}P_{38}X_{82}P_{27}X_{71} = P_{27}Y_{7.10}X_{10.3}P_{38}X_{82} = Y_{25}P_{5.10}X_{10.3}P_{38}X_{82} = P_{16}X_{64}Y_{42}P_{27}Y_{7.10}X_{10.1}$ $= P_{16}X_{64}P_{49}Y_{97}Y_{7.10}X_{10.1} = P_{16}X_{64}Y_{42}Y_{25}P_{5.10}X_{10.1} = P_{27}X_{75}P_{5.10}X_{10.3}X_{34}X_{42} = P_{38}X_{89}X_{97}X_{75}P_{5.10}X_{10.3}$ $= X_{34}P_{49}X_{97}X_{75}P_{5.10}X_{10.3}$	-1	1	0
$P_{16}X_{68}X_{89}Y_{97}X_{75}P_{5.10}X_{10.1} = X_{13}X_{34}Y_{42}P_{27}X_{75}P_{5.10}X_{10.1} = X_{13}P_{38}X_{89}Y_{97}X_{75}P_{5.10}X_{10.1}$ $= X_{13}X_{34}P_{49}Y_{97}X_{75}P_{5.10}X_{10.1} = Q_{16}X_{68}X_{89}Y_{97}X_{75}P_{5.10}X_{10.1} = X_{13}X_{34}Y_{42}Q_{27}X_{75}P_{5.10}X_{10.1}$ $= X_{13}Q_{38}X_{89}Y_{97}X_{75}P_{5.10}X_{10.1} = X_{13}X_{34}Q_{49}Y_{97}X_{75}P_{5.10}X_{10.1} = P_{16}X_{64}X_{42}Q_{27}X_{71} = P_{16}X_{68}X_{82}Q_{27}X_{71}$ $= P_{16}X_{64}Q_{49}X_{97}X_{71} = Q_{16}X_{64}X_{42}P_{27}X_{71} = Q_{16}X_{68}X_{82}P_{27}X_{71} = X_{13}Q_{38}X_{82}P_{27}X_{71} = P_{27}Y_{7.10}X_{10.3}Q_{38}X_{82}$ $= X_{13}P_{38}X_{82}Q_{27}X_{71} = Q_{27}Y_{7.10}X_{10.3}P_{38}X_{82} = Y_{25}Q_{5.10}X_{10.3}P_{38}X_{82} = Q_{16}X_{64}P_{49}X_{97}X_{71} = Y_{25}P_{5.10}X_{10.3}Q_{38}X_{82}$ $= P_{16}X_{64}Y_{42}Q_{27}Y_{7.10}X_{10.1} = P_{16}X_{64}Q_{49}Y_{97}Y_{7.10}X_{10.1} = P_{16}X_{64}Y_{42}Y_{25}Q_{5.10}X_{10.1} = Q_{16}X_{64}Y_{42}P_{27}Y_{7.10}X_{10.1}$ $= P_{27}X_{75}Q_{5.10}X_{10.3}X_{34}X_{42} = P_{38}X_{89}X_{97}X_{75}Q_{5.10}X_{10.3} = Q_{16}X_{64}P_{49}Y_{97}Y_{7.10}X_{10.1} = X_{34}P_{49}X_{97}X_{75}Q_{5.10}X_{10.3}$ $= Q_{16}X_{64}Y_{42}Y_{25}P_{5.10}X_{10.1} = Q_{27}X_{75}P_{5.10}X_{10.3}X_{34}X_{42} = Q_{38}X_{89}X_{97}X_{75}P_{5.10}X_{10.3} = X_{34}Q_{49}X_{97}X_{75}P_{5.10}X_{10.3}$ $Q_{16}X_{68}X_{89}Y_{97}X_{75}Q_{5.10}X_{10.1} = X_{13}X_{34}Y_{42}Q_{27}X_{75}Q_{5.10}X_{10.1} = X_{13}Q_{38}X_{89}Y_{97}X_{75}Q_{5.10}X_{10.1}$ $= X_{13}X_{34}Q_{49}Y_{97}X_{75}Q_{5.10}X_{10.1} = Q_{16}X_{64}X_{42}Q_{27}X_{71} = Q_{16}X_{68}X_{82}Q_{27}X_{71} = Q_{16}X_{64}Q_{49}X_{97}X_{71}$ $= X_{13}Q_{38}X_{82}Q_{27}X_{71} = Q_{27}Y_{7.10}X_{10.3}Q_{38}X_{82} = Y_{25}Q_{5.10}X_{10.3}Q_{38}X_{82} = Q_{16}X_{64}Y_{42}Q_{27}Y_{7.10}X_{10.1}$ $= Q_{16}X_{64}Q_{49}Y_{97}Y_{7.10}X_{10.1} = Q_{16}X_{64}Y_{42}Y_{25}Q_{5.10}X_{10.1} = Q_{27}X_{75}Q_{5.10}X_{10.3}X_{34}X_{42} = Q_{38}X_{89}X_{97}X_{75}Q_{5.10}X_{10.3}$ $= X_{34}Q_{49}X_{97}X_{75}Q_{5.10}X_{10.3}$	0	0	0
$P_{16}X_{68}X_{89}Y_{97}Y_{7.10}X_{10.1} = X_{13}X_{34}Y_{42}P_{27}Y_{7.10}X_{10.1} = X_{13}P_{38}X_{89}Y_{97}Y_{7.10}X_{10.1} = X_{13}X_{34}P_{49}Y_{97}Y_{7.10}X_{10.1}$ $= X_{13}X_{34}Y_{42}Y_{25}P_{5.10}X_{10.1} = P_{16}X_{68}X_{89}X_{97}X_{71} = X_{13}X_{34}X_{42}P_{27}X_{71} = P_{27}Y_{7.10}X_{10.3}X_{34}X_{42}$ $= X_{13}P_{38}X_{89}X_{97}X_{71} = P_{38}X_{89}X_{97}Y_{7.10}X_{10.3} = X_{13}X_{34}P_{49}X_{97}X_{71} = X_{34}P_{49}X_{97}Y_{7.10}X_{10.3} = Y_{25}P_{5.10}X_{10.3}X_{34}X_{42}$	1	0	0

Table 53. The generators in terms of bifundamental chiral fields for Model 13 (*Part 1*).

generator	SU(2) _{\bar{x}}	U(1) _{\bar{b}_1}	U(1) _{\bar{b}_2}
$Q_{16}X_{68}X_{89}Y_{97}Y_{7.10}X_{10.1} = X_{13}X_{34}Y_{42}Q_{27}Y_{7.10}X_{10.1} = X_{13}Q_{38}X_{89}Y_{97}Y_{7.10}X_{10.1} = X_{13}X_{34}Q_{49}Y_{97}Y_{7.10}X_{10.1}$ $= X_{13}X_{34}Y_{42}Y_{25}Q_{5.10}X_{10.1} = Q_{16}X_{68}X_{89}X_{97}X_{71} = X_{13}X_{34}X_{42}Q_{27}X_{71} = Q_{27}Y_{7.10}X_{10.3}X_{34}X_{42}$ $= X_{13}Q_{38}X_{89}X_{97}X_{71} = Q_{38}X_{89}X_{97}Y_{710}X_{103} = X_{13}X_{34}Q_{49}X_{97}X_{71} = X_{34}Q_{49}X_{97}Y_{7.10}X_{10.3} = Y_{25}Q_{5.10}X_{10.3}X_{34}X_{42}$	1	1	0
$P_{16}X_{64}X_{42}P_{27}X_{75}P_{5.10}X_{10.1} = P_{16}X_{68}X_{82}P_{27}X_{75}P_{5.10}X_{10.1} = P_{16}X_{64}P_{49}X_{97}X_{75}P_{5.10}X_{10.1}$ $= X_{13}P_{38}X_{82}P_{27}X_{75}P_{5.10}X_{10.1}$	-1	-2	-1
$P_{16}X_{64}X_{42}P_{27}X_{75}Q_{5.10}X_{10.1} = P_{16}X_{68}X_{82}P_{27}X_{75}Q_{5.10}X_{10.1} = P_{16}X_{64}P_{49}X_{97}X_{75}Q_{5.10}X_{10.1}$ $= P_{16}X_{64}X_{42}Q_{27}X_{75}P_{5.10}X_{10.1} = P_{16}X_{68}X_{82}Q_{27}X_{75}P_{5.10}X_{10.1} = P_{16}X_{64}Q_{49}X_{97}X_{75}P_{5.10}X_{10.1}$ $= X_{13}P_{38}X_{82}P_{27}X_{75}Q_{5.10}X_{10.1} = Q_{16}X_{64}X_{42}P_{27}X_{75}P_{5.10}X_{10.1} = Q_{16}X_{68}X_{82}P_{27}X_{75}P_{5.10}X_{10.1}$ $= X_{13}Q_{38}X_{82}P_{27}X_{75}P_{5.10}X_{10.1} = X_{13}P_{38}X_{82}Q_{27}X_{75}P_{5.10}X_{10.1} = Q_{16}X_{64}P_{49}X_{97}X_{75}P_{5.10}X_{10.1}$ $P_{16}X_{64}X_{42}Q_{27}X_{75}Q_{5.10}X_{10.1} = P_{16}X_{68}X_{82}Q_{27}X_{75}Q_{5.10}X_{10.1} = P_{16}X_{64}Q_{49}X_{97}X_{75}Q_{5.10}X_{10.1}$ $= Q_{16}X_{64}X_{42}P_{27}X_{75}Q_{5.10}X_{10.1} = Q_{16}X_{68}X_{82}P_{27}X_{75}Q_{5.10}X_{10.1} = X_{13}Q_{38}X_{82}P_{27}X_{75}Q_{5.10}X_{10.1}$ $= X_{13}P_{38}X_{82}Q_{27}X_{75}Q_{5.10}X_{10.1} = Q_{16}X_{64}P_{49}X_{97}X_{75}Q_{5.10}X_{10.1} = Q_{16}X_{64}X_{42}Q_{27}X_{75}P_{5.10}X_{10.1}$ $= Q_{16}X_{68}X_{82}Q_{27}X_{75}P_{5.10}X_{10.1} = Q_{16}X_{64}Q_{49}X_{97}X_{75}P_{5.10}X_{10.1} = X_{13}Q_{38}X_{82}Q_{27}X_{75}P_{5.10}X_{10.1}$ $Q_{16}X_{64}X_{42}Q_{27}X_{75}Q_{5.10}X_{10.1} = Q_{16}X_{68}X_{82}Q_{27}X_{75}Q_{5.10}X_{10.1} = Q_{16}X_{64}Q_{49}X_{97}X_{75}Q_{5.10}X_{10.1}$ $= X_{13}Q_{38}X_{82}Q_{27}X_{75}Q_{5.10}X_{10.1}$	-1	-1	-1
$P_{16}X_{68}X_{89}X_{97}X_{75}P_{5.10}X_{10.1} = X_{13}X_{34}X_{42}P_{27}X_{75}P_{510}X_{101} = X_{13}P_{38}X_{89}X_{97}X_{75}P_{5.10}X_{10.1}$ $= X_{13}X_{34}P_{49}X_{97}X_{75}P_{5.10}X_{10.1} = P_{16}X_{64}X_{42}P_{27}Y_{7.10}X_{10.1} = P_{16}X_{68}X_{82}P_{27}Y_{710}X_{101}$ $= P_{16}X_{64}P_{49}X_{97}Y_{7.10}X_{10.1} = P_{16}X_{64}X_{42}Y_{25}P_{5.10}X_{10.1} = P_{16}X_{68}X_{82}Y_{25}P_{5.10}X_{10.1} = X_{13}P_{38}X_{82}P_{27}Y_{7.10}X_{101}$ $= X_{13}P_{38}X_{82}Y_{25}P_{510}X_{101}$	0	-1	-1
$P_{16}X_{68}X_{89}X_{97}X_{75}Q_{5.10}X_{10.1} = X_{13}X_{34}X_{42}P_{27}X_{75}Q_{5.10}X_{10.1} = X_{13}P_{38}X_{89}X_{97}X_{75}Q_{5.10}X_{10.1}$ $= X_{13}X_{34}P_{49}X_{97}X_{75}Q_{5.10}X_{10.1} = Q_{16}X_{68}X_{89}X_{97}X_{75}P_{5.10}X_{10.1} = X_{13}X_{34}X_{42}Q_{27}X_{75}P_{5.10}X_{10.1}$ $= X_{13}Q_{38}X_{89}X_{97}X_{75}P_{5.10}X_{10.1} = X_{13}X_{34}Q_{49}X_{97}X_{75}P_{5.10}X_{10.1} = P_{16}X_{64}X_{42}Q_{27}Y_{7.10}X_{10.1}$ $= P_{16}X_{68}X_{82}Q_{27}Y_{7.10}X_{10.1} = P_{16}X_{64}Q_{49}X_{97}Y_{7.10}X_{10.1} = P_{16}X_{64}X_{42}Y_{25}Q_{5.10}X_{10.1} = P_{16}X_{68}X_{82}Y_{25}Q_{5.10}X_{10.1}$ $= Q_{16}X_{64}X_{42}P_{27}Y_{7.10}X_{10.1} = Q_{16}X_{68}X_{82}P_{27}Y_{7.10}X_{10.1} = X_{13}Q_{38}X_{82}P_{27}Y_{7.10}X_{10.1} = X_{13}P_{38}X_{82}Q_{27}Y_{7.10}X_{10.1}$ $= X_{13}P_{38}X_{82}Y_{25}Q_{5.10}X_{10.1} = Q_{16}X_{64}P_{49}X_{97}Y_{7.10}X_{10.1} = Q_{16}X_{64}X_{42}Y_{25}P_{5.10}X_{10.1} = Q_{16}X_{68}X_{82}Y_{25}P_{5.10}X_{10.1}$ $= X_{13}Q_{38}X_{82}Y_{25}P_{5.10}X_{10.1}$	0	0	-1
$Q_{16}X_{68}X_{89}X_{97}X_{75}Q_{5.10}X_{10.1} = X_{13}X_{34}X_{42}Q_{27}X_{75}Q_{5.10}X_{10.1} = X_{13}Q_{38}X_{89}X_{97}X_{75}Q_{5.10}X_{10.1}$ $= X_{13}X_{34}Q_{49}X_{97}X_{75}Q_{5.10}X_{10.1} = Q_{16}X_{64}X_{42}Q_{27}Y_{7.10}X_{10.1} = Q_{16}X_{68}X_{82}Q_{27}Y_{7.10}X_{10.1}$ $= Q_{16}X_{64}Q_{49}X_{97}Y_{7.10}X_{10.1} = Q_{16}X_{64}X_{42}Y_{25}Q_{5.10}X_{10.1} = Q_{16}X_{68}X_{82}Y_{25}Q_{5.10}X_{10.1} = X_{13}Q_{38}X_{82}Q_{27}Y_{7.10}X_{10.1}$ $= X_{13}Q_{38}X_{82}Y_{25}Q_{5.10}X_{10.1}$	0	1	-1
$P_{16}X_{68}X_{89}X_{97}Y_{7.10}X_{10.1} = X_{13}X_{34}X_{42}P_{27}Y_{7.10}X_{10.1} = X_{13}P_{38}X_{89}X_{97}Y_{7.10}X_{10.1} = X_{13}X_{34}P_{49}X_{97}Y_{7.10}X_{10.1}$ $= X_{13}X_{34}X_{42}Y_{25}P_{5.10}X_{10.1}$	1	0	-1
$Q_{16}X_{68}X_{89}X_{97}Y_{7.10}X_{10.1} = X_{13}X_{34}X_{42}Q_{27}Y_{7.10}X_{10.1} = X_{13}Q_{38}X_{89}X_{97}Y_{7.10}X_{10.1} = X_{13}X_{34}Q_{49}X_{97}Y_{7.10}X_{10.1}$ $= X_{13}X_{34}X_{42}Y_{25}Q_{5.10}X_{10.1}$	1	1	-1

Table 54. The generators in terms of bifundamental chiral fields for Model 13 (*Part 2*).

17 Model 14: $P_{+-}^2(\mathrm{dP}_2)$ [dP_2 bundle of \mathbb{P}^1 , $\langle 82 \rangle$]

Model 14 corresponds to the toric Calabi-Yau 4-fold $P_{+-}^2(\mathrm{dP}_2)$. The corresponding brane brick model has the quiver in figure 36 and the J - and E -terms are

$$\begin{array}{lll}
 & J & E \\
 \Lambda_{18}^1 : & X_{84}Q_{49}Y_{92}X_{25}X_{51} - X_{82}Q_{27}X_{71} & P_{16}X_{68} - X_{13}P_{38} \\
 \Lambda_{18}^2 : & X_{84}P_{49}Y_{92}X_{25}X_{51} - X_{82}P_{27}X_{71} & X_{13}Q_{38} - Q_{16}X_{68} \\
 \Lambda_{19}^1 : & X_{92}Q_{27}X_{71} - Y_{92}Q_{27}Y_{75}X_{51} & P_{16}X_{69} - X_{14}P_{49} \\
 \Lambda_{19}^2 : & X_{92}P_{27}X_{71} - Y_{92}P_{27}Y_{75}X_{51} & X_{14}Q_{49} - Q_{16}X_{69} \\
 \Lambda_{26}^1 : & X_{69}X_{92} - X_{68}X_{82} & P_{27}X_{71}Q_{16} - Q_{27}X_{71}P_{16} \\
 \Lambda_{2,10}^1 : & X_{10.6}X_{68}X_{84}Q_{49}Y_{92} - X_{10.8}X_{84}Q_{49}X_{92} & P_{27}X_{7.10} - X_{25}P_{5.10} \\
 \Lambda_{2,10}^2 : & X_{10.8}X_{82} - X_{10.6}X_{69}Y_{92} & P_{27}Y_{75}Q_{5.10} - Q_{27}Y_{75}P_{5.10} \\
 \Lambda_{2,10}^3 : & X_{10.6}X_{68}X_{84}P_{49}Y_{92} - X_{10.8}X_{84}P_{49}X_{92} & X_{25}Q_{5.10} - Q_{27}X_{7.10} \\
 \Lambda_{37}^1 : & Y_{75}X_{53} - X_{71}X_{13} & P_{38}X_{82}Q_{27} - Q_{38}X_{82}P_{27} \\
 \Lambda_{39}^1 : & Y_{92}X_{25}X_{51}X_{13} - X_{92}X_{25}X_{53} & P_{38}X_{84}Q_{49} - Q_{38}X_{84}P_{49} \\
 \Lambda_{47}^1 : & X_{71}X_{14} - X_{7.10}X_{10.8}X_{84} & P_{49}X_{92}Q_{27} - Q_{49}X_{92}P_{27} \\
 \Lambda_{47}^2 : & X_{7.10}X_{10.6}X_{68}X_{84} - Y_{75}X_{51}X_{14} & P_{49}Y_{92}Q_{27} - Q_{49}Y_{92}P_{27} \\
 \Lambda_{56}^1 : & X_{68}X_{84}Q_{49}Y_{92}X_{25} - X_{69}Y_{92}Q_{27}Y_{75} & P_{5.10}X_{10.6} - X_{51}P_{16} \\
 \Lambda_{56}^2 : & X_{68}X_{84}P_{49}Y_{92}X_{25} - X_{69}Y_{92}P_{27}Y_{75} & X_{51}Q_{16} - Q_{5.10}X_{10.6} \\
 \Lambda_{58}^1 : & X_{82}Q_{27}Y_{75} - X_{84}Q_{49}X_{92}X_{25} & P_{5.10}X_{10.8} - X_{53}P_{38} \\
 \Lambda_{58}^2 : & X_{82}P_{27}Y_{75} - X_{84}P_{49}X_{92}X_{25} & X_{53}Q_{38} - Q_{5.10}X_{10.8}
 \end{array} \tag{17.1}$$

The brick matchings are summarized in the P -matrix, which takes the form

$$P = \begin{pmatrix}
 \begin{array}{c|cccccccccccc}
 & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & s_{10} & s_{11} & s_{12} & s_{13} \\
 P_{15} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
 P_{27} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 P_{38} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
 P_{49} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
 P_{5,10} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 \hline
 Q_{16} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 Q_{27} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
 Q_{38} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
 Q_{49} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
 Q_{5,10} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
 \hline
 X_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
 X_{14} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
 X_{25} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 X_{51} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
 X_{53} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
 X_{68} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
 X_{69} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
 X_{71} & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
 X_{7.10} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 X_{82} & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
 X_{84} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
 X_{92} & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
 X_{10.6} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
 X_{10.8} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
 \hline
 Y_{75} & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
 Y_{92} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
 \end{array}
 \end{pmatrix} \tag{17.2}$$

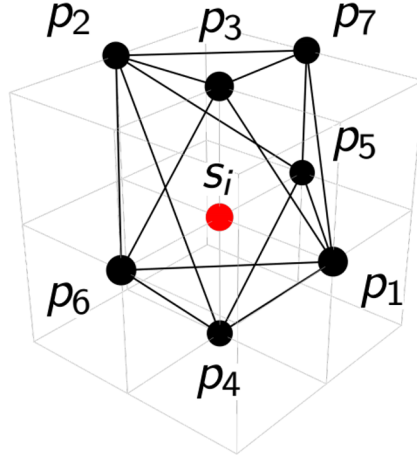


Figure 37. Toric diagram for Model 14.

The Hilbert series of the mesonic moduli space of Model 14 takes the form

$$\begin{aligned}
 g_1(t_i, y_s, y_{o_1}, y_{o_2}, y_{o_3}, y_{o_4}; \mathcal{M}_{14}) &= \frac{P(t_i, y_s, y_{o_1}, y_{o_2}, y_{o_3}, y_{o_4}; \mathcal{M}_{14})}{(1 - y_s y_{o_1} y_{o_2}^2 y_{o_3} y_{o_4}^2 t_1 t_4^2 t_5 t_6)(1 - y_s y_{o_1} y_{o_2}^2 y_{o_3} y_{o_4}^2 t_2 t_4^2 t_5 t_6)} \\
 &\times \frac{1}{(1 - y_s y_{o_1} y_{o_2} y_{o_3}^2 y_{o_4}^3 t_1^2 t_3 t_4 t_6^2)(1 - y_s y_{o_1} y_{o_2} y_{o_3}^2 y_{o_4}^3 t_2^2 t_3 t_4 t_6^2)(1 - y_s y_{o_1}^2 y_{o_2}^3 y_{o_3} y_{o_4}^2 t_1 t_4^2 t_5^2 t_7)} \\
 &\times \frac{1}{(1 - y_s y_{o_1}^2 y_{o_2}^3 y_{o_3} y_{o_4}^2 t_2 t_4^2 t_5^2 t_7)(1 - y_s y_{o_1}^2 y_{o_2}^3 y_{o_3}^4 t_1^3 t_3^2 t_6^2 t_7)(1 - y_s y_{o_1}^2 y_{o_2} y_{o_3}^3 y_{o_4}^4 t_2^3 t_3^2 t_6^2 t_7)} \\
 &\times \frac{1}{(1 - y_s y_{o_1}^4 y_{o_2}^3 y_{o_3}^4 t_1^3 t_3^2 t_5^2 t_7^3)(1 - y_s y_{o_1}^4 y_{o_2}^3 y_{o_3}^4 t_2^3 t_3^2 t_5^2 t_7^3)}, \tag{17.6}
 \end{aligned}$$

where t_i are the fugacities for the extremal brick matchings p_i . y_s counts the brick matching product $s_1 \dots s_{13}$ corresponding to the single internal point of the toric diagram of Model 14. Additionally, y_{o_1} counts $o_1 \dots o_6$, y_{o_2} counts $o_7 \dots o_9$, y_{o_3} counts $o_{10} \dots o_{20}$ and y_{o_4} counts $o_{21} \dots o_{24}$. The explicit numerator $P(t_i, y_s, y_{o_1}, y_{o_2}, y_{o_3}, y_{o_4}; \mathcal{M}_{14})$ of the Hilbert series is given in the appendix section A.14. We note that setting the fugacities $y_{o_1} = 1, \dots, y_{o_4} = 1$ does not change the overall characterization of the mesonic moduli space by the Hilbert series, indicating that the extra GLSM fields, as expected, correspond to an over-parameterization of the moduli space.

By setting $t_i = t$ for the fugacities of the extremal brick matchings, and all other fugacities to 1, the unrefined Hilbert series takes the following form

$$\begin{aligned}
 g_1(t, 1, 1, 1, 1, 1; \mathcal{M}_{14}) &= \frac{(1 - t^2)^4}{(1 - t)(1 - t^6)^2(-1 + t^8)^2(1 - t^{10})^2} \times (1 - t + 3t^2 \\
 &\quad - 3t^3 + 6t^4 - 4t^5 + 11t^6 - 4t^7 + 20t^8 - 4t^9 + 34t^{10} - 6t^{11} + 53t^{12} - 14t^{13} \\
 &\quad + 73t^{14} - 26t^{15} + 91t^{16} - 34t^{17} + 97t^{18} - 34t^{19} + 91t^{20} - 26t^{21} + 73t^{22} \\
 &\quad - 14t^{23} + 53t^{24} - 6t^{25} + 34t^{26} - 4t^{27} + 20t^{28} - 4t^{29} + 11t^{30} - 4t^{31} + 6t^{32} \\
 &\quad - 3t^{33} + 3t^{34} - t^{35} + t^{36}), \tag{17.7}
 \end{aligned}$$

where the palindromic numerator indicates that the mesonic moduli space is Calabi-Yau.

	SU(2) _x	U(1) _{b₁}	U(1) _{b₂}	U(1)	fugacity
p_1	+1	0	0	r_1	t_1
p_2	-1	0	0	r_2	t_2
p_3	0	+1	0	r_3	t_3
p_4	0	-1	0	r_4	t_4
p_5	0	0	+1	r_5	t_5
p_6	0	0	-1	r_6	t_6
p_7	0	0	0	r_7	t_7

Table 55. Global symmetry charges on the extremal brick matchings p_i of Model 14.

The global symmetry of Model 14 and the charges on the extremal brick matchings under the global symmetry are summarized in table 55. We can use the following fugacity map,

$$t = t_7, \quad x = \frac{t_7}{t_2}, \quad b_1 = \frac{t_7}{t_4}, \quad b_2 = \frac{t_5}{t_7}, \quad (17.8)$$

where $t_1 t_2 = t_3 t_4 = t_5 t_6 = t_7^2$, in order to rewrite the Hilbert series for Model 14 in terms of characters of irreducible representations of $SU(2) \times U(1) \times U(1)$.

The character expansion of the Hilbert series of Model 14 takes the following form

$$\begin{aligned}
 g_1(t, x, b_1, b_2; \mathcal{M}_{14}) = & 1 + [1]b_1^{-2}t^5 + ([1]b_1^{-2}b_2^2 + [2]b_2^{-2})t^6 + [2]t^7 + ([2]b_2^2 + [3]b_1^2b_2^{-2})t^8 \\
 & + [3]b_1^2t^9 + ([2]b_1^{-4} + [3]b_1^2b_2^2)t^{10} + ([2]b_1^{-4}b_2^2 + [3]b_1^{-2}b_2^{-2})t^{11} + ([2]b_1^{-4}b_2^4 + [3]b_1^{-2} \\
 & + [4]b_2^{-4})t^{12} + ([3]b_1^{-2}b_2^2 + [4]b_2^{-2})t^{13} + ([3]b_1^{-2}b_2^4 + [4] + [5]b_1^2b_2^{-4})t^{14} + ([3]b_1^{-6} \\
 & + [4]b_2^2 + [5]b_1^2b_2^{-2})t^{15} + ([3]b_1^{-6}b_2^2 + [4]b_1^{-4}b_2^{-2} + [4]b_2^4 + [5]b_1^2 + [6]b_1^4b_2^{-4})t^{16} \\
 & + ([3]b_1^{-6}b_2^4 + [4]b_1^{-4} + [5]b_1^{-2}b_2^{-4} + [5]b_1^2b_2^2 + [6]b_1^4b_2^{-2})t^{17} + ([3]b_1^{-6}b_2^6 + [4]b_1^{-4}b_2^2 \\
 & + [5]b_1^{-2}b_2^{-2} + [5]b_1^2b_2^4 + [6]b_1^4 + [6]b_2^{-6})t^{18} + ([4]b_1^{-4}b_2^4 + [5]b_1^{-2} + [6]b_2^{-4} \\
 & + [6]b_1^4b_2^2)t^{19} + ([4]b_1^{-8} + [4]b_1^{-4}b_2^6 + [5]b_1^{-2}b_2^2 + [6]b_2^{-2} + [6]b_1^4b_2^4 + [7]b_1^2b_2^{-6})t^{20} \\
 & + \dots, \quad (17.9)
 \end{aligned}$$

where $[m] = [m]_{SU(2)_x}$. The fugacities b_1 and b_2 count charges under the two $U(1)$ factors of the mesonic flavor symmetry.

The plethystic logarithm of the Hilbert series takes the form

$$\begin{aligned}
 \text{PL}[g_1(t, x, b_1, b_2; \mathcal{M}_{14})] = & [1]b_1^{-2}t^5 + ([1]b_1^{-2}b_2^2 + [2]b_2^{-2})t^6 + [2]t^7 \\
 & + ([2]b_2^2 + [3]b_1^2b_2^{-2})t^8 + [3]b_1^2t^9 + [3]b_1^2b_2^2t^{10} \\
 & - (b_1^{-4}b_2^2 + [1]b_1^{-2}b_2^{-2})t^{11} - (b_2^{-4} + 2[1]b_1^{-2} + [3]b_1^{-2})t^{12} - (b_2^{-2} + 2[1]b_1^{-2}b_2^2 \\
 & + 2[2]b_2^{-2} + [3]b_1^{-2}b_2^2 + [4]b_2^{-2})t^{13} - (2 + [1]b_1^2b_2^{-4} + [1]b_1^{-2}b_2^4 + 3[2] + [3]b_1^2b_2^{-4} \\
 & + 3[4])t^{14} - (b_2^2 + [1]2b_1^2b_2^{-2} + 3[2]b_2^2 + 2[3]b_1^2b_2^{-2} + 2[4]b_2^2 + [5]b_1^2b_2^{-2})t^{15} \\
 & - (b_2^4 + 3[1]b_1^2 + [2]b_1^4b_2^{-4} + [2]b_2^4 + 3[3]b_1^2 + [4]b_2^4 + 2[5]b_1^2)t^{16} \\
 & - (b_1^4b_2^{-2} + 2[1]b_1^2b_2^2 + [2]b_1^4b_2^{-2} + 2[3]b_1^2b_2^2 + [4]b_1^4b_2^{-2} + [5]b_1^2b_2^2)t^{17} + \dots, \quad (17.10)
 \end{aligned}$$

where $[m] = [m]_{SU(2)_x}$. From the plethystic logarithm, we see that the mesonic moduli space is a non-complete intersection.

generator	$SU(2)_{\tilde{x}}$	$U(1)_{\tilde{b}_1}$	$U(1)_{\tilde{b}_2}$
$p_1 p_4^2 p_5 p_6 \text{ } so_1 o_2^2 o_3 o_4^2$	0	-1	0
$p_2 p_4^2 p_5 p_6 \text{ } so_1 o_2^2 o_3 o_4^2$	-1	-1	0
$p_1^2 p_3 p_4 p_6^2 \text{ } so_1 o_2 o_3^2 o_4^3$	1	0	-1
$p_1 p_2 p_3 p_4 p_6^2 \text{ } so_1 o_2 o_3^2 o_4^3$	0	0	-1
$p_2^2 p_3 p_4 p_6^2 \text{ } so_1 o_2 o_3^2 o_4^3$	-1	0	-1
$p_1 p_4^2 p_5^2 p_7 \text{ } so_1^2 o_2^3 o_3 o_4^2$	0	-1	1
$p_2 p_4^2 p_5^2 p_7 \text{ } so_1^2 o_2^3 o_3 o_4^2$	-1	-1	1
$p_1^2 p_3 p_4 p_5 p_6 p_7 \text{ } so_1^2 o_2^2 o_3^2 o_4^3$	1	0	0
$p_1 p_2 p_3 p_4 p_5 p_6 p_7 \text{ } so_1^2 o_2^2 o_3^2 o_4^3$	0	0	0
$p_2^2 p_3 p_4 p_5 p_6 p_7 \text{ } so_1^2 o_2^2 o_3^2 o_4^3$	-1	0	0
$p_1^3 p_3^2 p_6^2 p_7 \text{ } so_1^2 o_2^3 o_3^2 o_4^4$	2	1	-1
$p_1^2 p_2 p_3^2 p_6^2 p_7 \text{ } so_1^2 o_2^3 o_3^2 o_4^4$	1	1	-1
$p_1 p_2^2 p_3^2 p_6^2 p_7 \text{ } so_1^2 o_2^3 o_3^2 o_4^4$	0	1	-1
$p_2^3 p_3^2 p_6^2 p_7 \text{ } so_1^2 o_2^3 o_3^2 o_4^4$	-1	1	-1
$p_1^2 p_3 p_4 p_5^2 p_7^2 \text{ } so_1^3 o_2^3 o_3^2 o_4^3$	1	0	1
$p_1 p_2 p_3 p_4 p_5^2 p_7^2 \text{ } so_1^3 o_2^3 o_3^2 o_4^3$	0	0	1
$p_2^2 p_3 p_4 p_5^2 p_7^2 \text{ } so_1^3 o_2^3 o_3^2 o_4^3$	-1	0	1
$p_1^3 p_3^2 p_5 p_6 p_7^2 \text{ } so_1^3 o_2^3 o_3^2 o_4^4$	2	1	0
$p_1^2 p_2 p_3^2 p_5 p_6 p_7^2 \text{ } so_1^3 o_2^3 o_3^2 o_4^4$	1	1	0
$p_1 p_2^2 p_3^2 p_5 p_6 p_7^2 \text{ } so_1^3 o_2^3 o_3^2 o_4^4$	0	1	0
$p_2^2 p_3^2 p_5 p_6 p_7^2 \text{ } so_1^3 o_2^3 o_3^2 o_4^4$	-1	1	0
$p_1^3 p_3^2 p_5^2 p_7^3 \text{ } so_1^4 o_2^3 o_3^2 o_4^4$	2	1	1
$p_1^2 p_2 p_3^2 p_5^2 p_7^3 \text{ } so_1^4 o_2^3 o_3^2 o_4^4$	1	1	1
$p_1 p_2^2 p_3^2 p_5^2 p_7^3 \text{ } so_1^4 o_2^3 o_3^2 o_4^4$	0	1	1
$p_2^3 p_3^2 p_5^2 p_7^3 \text{ } so_1^4 o_2^3 o_3^2 o_4^4$	-1	1	1

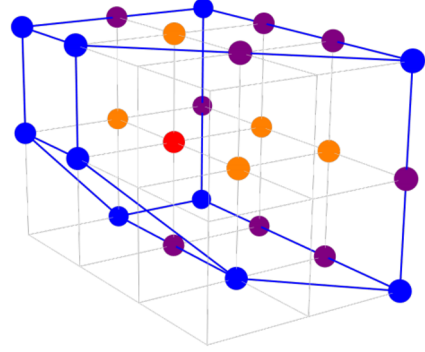


Table 56. The generators and lattice of generators of the mesonic moduli space of Model 14 in terms of brick matchings with the corresponding flavor charges.

By using the following fugacity map

$$\tilde{t} = t_7, \quad \tilde{x} = \frac{t_7^2}{t_2^2}, \quad \tilde{b}_1 = \frac{t_2 t_7}{t_4^2}, \quad \tilde{b}_2 = \frac{t_7^2}{t_6^2}, \quad (17.11)$$

where $t_1 t_2 = t_3 t_4 = t_5 t_6 = t_7^3$, the mesonic flavor charges on the gauge invariant operators become \mathbb{Z} -valued. The generators in terms of brick matchings and their corresponding rescaled mesonic flavor charges are summarized in table 56. The generator lattice as shown in table 56 is a convex lattice polytope, which is reflexive. It is the dual of the toric diagram of Model 14 shown in figure 37. For completeness, table 57 and table 58 show the generators of Model 14 in terms of chiral fields with the corresponding mesonic flavor charges.

generator	SU(2) _{\bar{x}}	U(1) _{\bar{b}_1}	U(1) _{\bar{b}_2}
$P_{27}X_{7.10}X_{10.8}X_{82} = X_{25}X_{53}P_{38}X_{82} = X_{25}P_{5.10}X_{10.8}X_{82} = P_{16}X_{69}Y_{92}X_{25}X_{51} = P_{27}X_{7.10}X_{10.6}X_{69}Y_{92}$ $= X_{14}P_{49}Y_{92}X_{25}X_{51} = X_{25}P_{5.10}X_{10.6}X_{69}Y_{92}$	0	-1	0
$Q_{27}X_{7.10}X_{10.8}X_{82} = X_{25}X_{53}Q_{38}X_{82} = X_{25}Q_{5.10}X_{10.8}X_{82} = Q_{16}X_{69}Y_{92}X_{25}X_{51} = Q_{27}X_{7.10}X_{10.6}X_{69}Y_{92}$ $= X_{14}Q_{49}Y_{92}X_{25}X_{51} = X_{25}Q_{5.10}X_{10.6}X_{69}Y_{92}$	-1	-1	0
$P_{16}X_{69}Y_{92}P_{27}X_{71} = X_{14}P_{49}Y_{92}P_{27}X_{71} = P_{27}X_{7.10}X_{10.8}X_{84}P_{49}Y_{92} = X_{25}X_{53}P_{38}X_{84}P_{49}Y_{92}$ $= X_{25}P_{5.10}X_{10.8}X_{84}P_{49}Y_{92}$	1	0	-1
$P_{16}X_{69}Y_{92}Q_{27}X_{71} = Q_{16}X_{69}Y_{92}P_{27}X_{71} = X_{14}Q_{49}Y_{92}P_{27}X_{71} = X_{14}P_{49}Y_{92}Q_{27}X_{71}$ $= P_{27}X_{7.10}X_{10.8}X_{84}Q_{49}Y_{92} = X_{25}X_{53}P_{38}X_{84}Q_{49}Y_{92} = Q_{27}X_{7.10}X_{10.8}X_{84}P_{49}Y_{92} = X_{25}X_{53}Q_{38}X_{84}P_{49}Y_{92}$ $= X_{25}Q_{5.10}X_{10.8}X_{84}P_{49}Y_{92} = X_{25}P_{5.10}X_{10.8}X_{84}Q_{49}Y_{92}$	0	0	-1
$Q_{16}X_{69}Y_{92}Q_{27}X_{71} = X_{14}Q_{49}Y_{92}Q_{27}X_{71} = Q_{27}X_{7.10}X_{10.8}X_{84}Q_{49}Y_{92} = X_{25}X_{53}Q_{38}X_{84}Q_{49}Y_{92}$ $= X_{25}Q_{5.10}X_{10.8}X_{84}Q_{49}Y_{92}$	-1	0	-1
$P_{16}X_{68}X_{82}X_{25}X_{51} = P_{16}X_{69}X_{92}X_{25}X_{51} = P_{27}X_{7.10}X_{10.6}X_{68}X_{82} = P_{27}X_{7.10}X_{10.6}X_{69}X_{92} = X_{13}P_{38}X_{82}X_{25}X_{51}$ $= X_{14}P_{49}X_{92}X_{25}X_{51} = X_{25}P_{5.10}X_{10.6}X_{68}X_{82} = X_{25}P_{5.10}X_{10.6}X_{69}X_{92}$	0	-1	1
$Q_{16}X_{68}X_{82}X_{25}X_{51} = Q_{16}X_{69}X_{92}X_{25}X_{51} = Q_{27}X_{7.10}X_{10.6}X_{68}X_{82} = Q_{27}X_{7.10}X_{10.6}X_{69}X_{92} = X_{13}Q_{38}X_{82}X_{25}X_{51}$ $= X_{14}Q_{49}X_{92}X_{25}X_{51} = X_{25}Q_{5.10}X_{10.6}X_{68}X_{82} = X_{25}Q_{5.10}X_{10.6}X_{69}X_{92}$	-1	-1	1
$P_{16}X_{68}X_{84}P_{49}Y_{92}X_{25}X_{51} = P_{27}X_{7.10}X_{10.6}X_{68}X_{84}P_{49}Y_{92} = X_{13}P_{38}X_{84}P_{49}Y_{92}X_{25}X_{51}$ $= X_{25}P_{5.10}X_{10.6}X_{68}X_{84}P_{49}Y_{92} = P_{16}X_{68}X_{82}P_{27}X_{71} = P_{16}X_{69}X_{92}P_{27}X_{71} = X_{13}P_{38}X_{82}P_{27}X_{71}$ $= P_{27}Y_{75}X_{53}P_{38}X_{82} = X_{14}P_{49}X_{92}P_{27}X_{71} = P_{27}Y_{75}P_{5.10}X_{10.8}X_{82} = P_{16}X_{69}Y_{92}P_{27}Y_{75}X_{51}$ $= X_{14}P_{49}Y_{92}P_{27}Y_{75}X_{51} = P_{27}X_{7.10}X_{10.8}X_{84}P_{49}X_{92} = P_{27}Y_{75}P_{5.10}X_{10.6}X_{69}Y_{92} = X_{25}X_{53}P_{38}X_{84}P_{49}X_{92}$ $= X_{25}P_{5.10}X_{10.8}X_{84}P_{49}X_{92}$	1	0	0
$P_{16}X_{68}X_{84}Q_{49}Y_{92}X_{25}X_{51} = P_{27}X_{7.10}X_{10.6}X_{68}X_{84}Q_{49}Y_{92} = X_{13}P_{38}X_{84}Q_{49}Y_{92}X_{25}X_{51}$ $= Q_{16}X_{68}X_{84}P_{49}Y_{92}X_{25}X_{51} = Q_{27}X_{7.10}X_{10.6}X_{68}X_{84}P_{49}Y_{92} = X_{13}Q_{38}X_{84}P_{49}Y_{92}X_{25}X_{51}$ $= X_{25}Q_{5.10}X_{10.6}X_{68}X_{84}P_{49}Y_{92} = X_{25}P_{5.10}X_{10.6}X_{68}X_{84}Q_{49}Y_{92} = P_{16}X_{68}X_{82}Q_{27}X_{71} = P_{16}X_{69}X_{92}Q_{27}X_{71}$ $= Q_{16}X_{68}X_{82}P_{27}X_{71} = Q_{16}X_{69}X_{92}P_{27}X_{71} = X_{13}Q_{38}X_{82}P_{27}X_{71} = P_{27}Y_{75}X_{53}Q_{38}X_{82} = X_{14}Q_{49}X_{92}P_{27}X_{71}$ $= P_{27}Y_{75}Q_{5.10}X_{10.8}X_{82} = X_{13}P_{38}X_{82}Q_{27}X_{71} = Q_{27}Y_{75}X_{53}P_{38}X_{82} = X_{14}P_{49}X_{92}Q_{27}X_{71} = Q_{27}Y_{75}P_{5.10}X_{10.8}X_{82}$ $= P_{16}X_{69}Y_{92}Q_{27}Y_{75}X_{51} = Q_{16}X_{69}Y_{92}P_{27}Y_{75}X_{51} = X_{14}Q_{49}Y_{92}P_{27}Y_{75}X_{51} = P_{27}X_{7.10}X_{10.8}X_{84}Q_{49}X_{92}$ $= P_{27}Y_{75}Q_{5.10}X_{10.6}X_{69}Y_{92} = X_{25}X_{53}P_{38}X_{84}Q_{49}X_{92} = X_{14}P_{49}Y_{92}Q_{27}Y_{75}X_{51} = Q_{27}X_{7.10}X_{10.8}X_{84}P_{49}X_{92}$ $= X_{25}X_{53}Q_{38}X_{84}P_{49}X_{92} = X_{25}Q_{5.10}X_{10.8}X_{84}P_{49}X_{92} = Q_{27}Y_{75}P_{5.10}X_{10.6}X_{69}Y_{92} = X_{25}P_{5.10}X_{10.8}X_{84}Q_{49}X_{92}$ $Q_{16}X_{68}X_{84}Q_{49}Y_{92}X_{25}X_{51} = Q_{27}X_{7.10}X_{10.6}X_{68}X_{84}Q_{49}Y_{92} = X_{13}Q_{38}X_{84}Q_{49}Y_{92}X_{25}X_{51}$ $= X_{25}Q_{5.10}X_{10.6}X_{68}X_{84}Q_{49}Y_{92} = Q_{16}X_{68}X_{82}Q_{27}X_{71} = Q_{16}X_{69}X_{92}Q_{27}X_{71} = X_{13}Q_{38}X_{82}Q_{27}X_{71}$ $= Q_{27}Y_{75}X_{53}Q_{38}X_{82} = X_{14}Q_{49}X_{92}Q_{27}X_{71} = Q_{27}Y_{75}Q_{5.10}X_{10.8}X_{82} = Q_{16}X_{69}Y_{92}Q_{27}Y_{75}X_{51}$ $= X_{14}Q_{49}Y_{92}Q_{27}Y_{75}X_{51} = Q_{27}X_{7.10}X_{10.8}X_{84}Q_{49}X_{92} = Q_{27}Y_{75}Q_{5.10}X_{10.6}X_{69}Y_{92} = X_{25}X_{53}Q_{38}X_{84}Q_{49}X_{92}$ $= X_{25}Q_{5.10}X_{10.8}X_{84}Q_{49}X_{92}$	0	0	0
$Q_{16}X_{68}X_{84}Q_{49}Y_{92}X_{25}X_{51} = Q_{27}X_{7.10}X_{10.6}X_{68}X_{84}Q_{49}Y_{92} = X_{13}Q_{38}X_{84}Q_{49}Y_{92}X_{25}X_{51}$ $= X_{25}Q_{5.10}X_{10.6}X_{68}X_{84}Q_{49}Y_{92} = Q_{16}X_{68}X_{82}Q_{27}X_{71} = Q_{16}X_{69}X_{92}Q_{27}X_{71}$ $= X_{13}Q_{38}X_{82}P_{27}X_{71} = P_{27}Y_{75}X_{53}Q_{38}X_{82} = X_{14}Q_{49}X_{92}P_{27}X_{71}$ $= P_{27}Y_{75}Q_{5.10}X_{10.8}X_{82} = X_{13}P_{38}X_{82}Q_{27}X_{71} = Q_{27}Y_{75}X_{53}P_{38}X_{82} = X_{14}P_{49}X_{92}Q_{27}X_{71} = Q_{27}Y_{75}P_{5.10}X_{10.8}X_{82}$ $= P_{16}X_{69}Y_{92}Q_{27}Y_{75}X_{51} = Q_{16}X_{69}Y_{92}P_{27}Y_{75}X_{51} = X_{14}Q_{49}Y_{92}P_{27}Y_{75}X_{51} = P_{27}X_{7.10}X_{10.8}X_{84}Q_{49}X_{92}$ $= P_{27}Y_{75}Q_{5.10}X_{10.6}X_{69}Y_{92} = X_{25}X_{53}P_{38}X_{84}Q_{49}X_{92} = X_{14}P_{49}Y_{92}Q_{27}Y_{75}X_{51} = Q_{27}X_{7.10}X_{10.8}X_{84}P_{49}X_{92}$ $= X_{25}X_{53}Q_{38}X_{84}P_{49}X_{92} = X_{25}Q_{5.10}X_{10.8}X_{84}P_{49}X_{92} = Q_{27}Y_{75}P_{5.10}X_{10.6}X_{69}Y_{92} = X_{25}P_{5.10}X_{10.8}X_{84}Q_{49}X_{92}$ $Q_{16}X_{68}X_{84}Q_{49}Y_{92}X_{25}X_{51} = Q_{27}X_{7.10}X_{10.6}X_{68}X_{84}Q_{49}Y_{92} = X_{13}Q_{38}X_{84}Q_{49}Y_{92}X_{25}X_{51}$ $= X_{25}Q_{5.10}X_{10.6}X_{68}X_{84}Q_{49}Y_{92} = Q_{16}X_{68}X_{82}Q_{27}X_{71} = Q_{16}X_{69}X_{92}Q_{27}X_{71} = X_{13}Q_{38}X_{82}Q_{27}X_{71}$ $= Q_{27}Y_{75}X_{53}Q_{38}X_{82} = X_{14}Q_{49}X_{92}Q_{27}X_{71} = Q_{27}Y_{75}Q_{5.10}X_{10.8}X_{82} = Q_{16}X_{69}Y_{92}Q_{27}Y_{75}X_{51}$ $= X_{14}Q_{49}Y_{92}Q_{27}Y_{75}X_{51} = Q_{27}X_{7.10}X_{10.8}X_{84}Q_{49}X_{92} = Q_{27}Y_{75}Q_{5.10}X_{10.6}X_{69}Y_{92} = X_{25}X_{53}Q_{38}X_{84}Q_{49}X_{92}$ $= X_{25}Q_{5.10}X_{10.8}X_{84}Q_{49}X_{92}$	-1	0	0

Table 57. The generators in terms of bifundamental chiral fields for Model 14 (*Part 1*).

18 Model 15: $P_{+-}^3(\text{dP}_2)$ [dP_2 bundle of \mathbb{P}^1 , $\langle 83 \rangle$]

Model 15 corresponds to the toric Calabi-Yau 4-fold $P_{+-}^3(\text{dP}_2)$. The corresponding brane brick model has the quiver in figure 38 and the J - and E -terms are

	J	E	
Λ_{18}^1 :	$X_{84}Y_{42}X_{25}X_{51} - X_{82}X_{21}$	$P_{16}X_{63}Q_{38} - Q_{16}X_{63}P_{38}$	(18.1)
Λ_{19}^1 :	$X_{92}X_{21} - Y_{97}Y_{75}X_{51}$	$P_{16}X_{64}Q_{49} - Q_{16}X_{64}P_{49}$	
Λ_{26}^1 :	$X_{64}Q_{49}X_{92} - X_{63}Q_{38}X_{82}$	$P_{27}X_{76} - X_{21}P_{16}$	
Λ_{26}^2 :	$X_{64}P_{49}X_{92} - X_{63}P_{38}X_{82}$	$X_{21}Q_{16} - Q_{27}X_{76}$	
$\Lambda_{2,10}^1$:	$X_{10.6}X_{63}Q_{38}X_{84}Y_{42} - X_{10.8}X_{84}Q_{49}X_{92}$	$P_{27}X_{7,10} - X_{25}P_{5,10}$	
$\Lambda_{2,10}^2$:	$X_{10.8}X_{82} - X_{10.6}X_{64}Y_{42}$	$P_{27}Y_{75}Q_{5,10} - Q_{27}Y_{75}P_{5,10}$	
$\Lambda_{2,10}^3$:	$X_{10.6}X_{63}P_{38}X_{84}Y_{42} - X_{10.8}X_{84}P_{49}X_{92}$	$X_{25}Q_{5,10} - Q_{27}X_{7,10}$	
Λ_{37}^1 :	$Y_{75}X_{53} - X_{76}X_{63}$	$P_{38}X_{82}Q_{27} - Q_{38}X_{82}P_{27}$	
Λ_{39}^1 :	$Y_{97}X_{7,10}X_{10.6}X_{63} - X_{92}X_{25}X_{53}$	$P_{38}X_{84}Q_{49} - Q_{38}X_{84}P_{49}$	
Λ_{47}^1 :	$X_{76}X_{64} - X_{7,10}X_{10.8}X_{84}$	$P_{49}X_{92}Q_{27} - Q_{49}X_{92}P_{27}$	
Λ_{47}^2 :	$X_{7,10}X_{10.6}X_{63}Q_{38}X_{84} - Y_{75}Q_{5,10}X_{10.6}X_{64}$	$P_{49}Y_{97} - Y_{42}P_{27}$	
Λ_{47}^3 :	$X_{7,10}X_{10.6}X_{63}P_{38}X_{84} - Y_{75}X_{51}P_{16}X_{64}$	$Y_{42}Q_{27} - Q_{49}Y_{97}$	
Λ_{56}^1 :	$X_{63}Q_{38}X_{84}Y_{42}X_{25} - X_{64}Y_{42}Q_{27}Y_{75}$	$P_{5,10}X_{10.6} - X_{51}P_{16}$	
Λ_{56}^2 :	$X_{63}P_{38}X_{84}Y_{42}X_{25} - X_{64}P_{49}Y_{97}Y_{75}$	$X_{51}Q_{16} - Q_{5,10}X_{10.6}$	
Λ_{58}^1 :	$X_{82}Q_{27}Y_{75} - X_{84}Q_{49}X_{92}X_{25}$	$P_{5,10}X_{10.8} - X_{53}P_{38}$	
Λ_{58}^2 :	$X_{82}P_{27}Y_{75} - X_{84}P_{49}X_{92}X_{25}$	$X_{53}Q_{38} - Q_{5,10}X_{10.8}$	

Using the forward algorithm, we are able to obtain the brick matchings for Model 15. The brick matchings are summarized in the P -matrix, which takes the form

$$P = \begin{pmatrix}
 \begin{array}{c|cccccccccccccccccccccccccccc}
 & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & s_{10} & s_{11} & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & o_7 & o_8 & o_9 & o_{10} & o_{11} & o_{12} & o_{13} & o_{14} & o_{15} & o_{16} & o_{17} & o_{18} & o_{19} & o_{20} & o_{21} & o_{22} & o_{23} & o_{24} & o_{25} & o_{26} & o_{27} & o_{28} & o_{29} \\
 P_{16} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 0 & 0 \\
 P_{27} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
 P_{38} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
 P_{49} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
 P_{5,10} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
 Q_{16} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 0 & 0 \\
 Q_{27} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
 Q_{38} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
 Q_{49} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
 Q_{5,10} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
 X_{21} & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 2 & 1 & 1 & 1 & 1 & 0 & 0 & 2 & 2 \\
 X_{25} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
 X_{51} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\
 X_{53} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
 X_{63} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 X_{64} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 X_{76} & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 X_{7,10} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 X_{82} & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
 X_{84} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
 X_{92} & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 X_{10,6} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
 X_{10,8} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
 Y_{42} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
 Y_{75} & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
 Y_{97} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 \end{array}
 \end{pmatrix}. \tag{18.2}$$

	SU(2) _x	U(1) _{b₁}	U(1) _{b₂}	U(1)	fugacity
p_1	+1	0	0	r_1	t_1
p_2	-1	0	0	r_2	t_2
p_3	0	+1	0	r_3	t_3
p_4	0	-1	0	r_4	t_4
p_5	0	0	+1	r_5	t_5
p_6	0	0	-1	r_6	t_6
p_7	0	0	0	r_7	t_7

Table 59. Global symmetry charges on the extremal brick matchings p_i of Model 15.

The Hilbert series of the mesonic moduli space of Model 15 is given by

$$\begin{aligned}
 g_1(t_i, y_s, y_{o_1}, y_{o_2}, y_{o_3}, y_{o_4}; \mathcal{M}_{15}) &= \frac{P(t_i, y_s, y_{o_1}, y_{o_2}, y_{o_3}, y_{o_4}; \mathcal{M}_{15})}{(1 - y_s y_{o_1} y_{o_2} y_{o_3}^2 y_{o_4}^2 t_1 t_4^2 t_5 t_6)} \\
 &\times \frac{1}{(1 - y_s y_{o_1} y_{o_2} y_{o_3}^2 y_{o_4}^2 t_2 t_4^2 t_5 t_6)(1 - y_s y_{o_1} y_{o_2}^2 y_{o_3} y_{o_4}^2 t_1 t_3 t_4 t_6^2)(1 - y_s y_{o_1} y_{o_2}^2 y_{o_3} y_{o_4}^2 t_2 t_3 t_4 t_6^2)} \\
 &\times \frac{1}{(1 - y_s y_{o_1}^2 y_{o_2} y_{o_3}^3 y_{o_4}^3 t_1^2 t_4^2 t_5^2 t_7)(1 - y_s y_{o_1}^2 y_{o_2} y_{o_3}^3 y_{o_4}^3 t_2^2 t_4^2 t_5^2 t_7)(1 - y_s y_{o_1}^2 y_{o_2}^3 y_{o_3} y_{o_4}^3 t_1^2 t_3^2 t_6^2 t_7)} \\
 &\times \frac{1}{(1 - y_s y_{o_1}^2 y_{o_2}^3 y_{o_3}^3 y_{o_4}^3 t_2^2 t_3^2 t_6^2 t_7)(1 - y_s y_{o_1}^4 y_{o_2}^3 y_{o_3}^3 y_{o_4}^5 t_1^4 t_3^2 t_5^2 t_7^3)(1 - y_s y_{o_1}^4 y_{o_2}^3 y_{o_3}^3 y_{o_4}^5 t_2^4 t_3^2 t_5^2 t_7^3)},
 \end{aligned} \tag{18.6}$$

where t_i are the fugacities for the extremal brick matchings p_i . Additionally, y_s counts $s_1 \dots s_{13}$, y_{o_1} counts $o_1 \dots o_{13}$, y_{o_2} counts $o_{14} \dots o_{19}$, y_{o_3} counts $o_{20} \dots o_{24}$ and y_{o_4} counts $o_{25} \dots o_{29}$. The explicit numerator $P(t_i, y_s, y_{o_1}, y_{o_2}, y_{o_3}, y_{o_4}; \mathcal{M}_{15})$ of the Hilbert series is given in the appendix section A.15. We note that setting the fugacities $y_{o_1} = 1, \dots, y_{o_4} = 1$ does not change the overall characterization of the mesonic moduli space by the Hilbert series, indicating that the extra GLSM fields, as expected, correspond to an over-parameterization of the moduli space.

By setting $t_i = t$ for the fugacities of the extremal brick matchings, and all other fugacities to 1, the unrefined Hilbert series takes the following form

$$\begin{aligned}
 g_1(t, 1, 1, 1, 1, 1; \mathcal{M}_{15}) &= \frac{(1-t)(1-t^2)(1-t^3)}{(1-t^5)^3(1-t^7)^2(1-t^{11})^2} \times (1 + t + 2t^2 + 3t^3 + 4t^4 \\
 &+ 6t^5 + 8t^6 + 17t^7 + 20t^8 + 38t^9 + 48t^{10} + 70t^{11} + 76t^{12} + 105t^{13} + 99t^{14} \\
 &+ 124t^{15} + 116t^{16} + 133t^{17} + 118t^{18} + 141t^{19} + 118t^{20} + 133t^{21} + 116t^{22} \\
 &+ 124t^{23} + 99t^{24} + 105t^{25} + 76t^{26} + 70t^{27} + 48t^{28} + 38t^{29} + 20t^{30} + 17t^{31} \\
 &+ 8t^{32} + 6t^{33} + 4t^{34} + 3t^{35} + 2t^{36} + t^{37} + t^{38}),
 \end{aligned} \tag{18.7}$$

where the palindromic numerator indicates that the mesonic moduli space is Calabi-Yau.

The global symmetry of Model 15 and the charges on the extremal brick matchings under the global symmetry are summarized in table 59. We can use the following fugacity map,

$$t = t_7, \quad x = \frac{t_7}{t_2}, \quad b_1 = \frac{t_7}{t_4}, \quad b_2 = \frac{t_5}{t_7}, \tag{18.8}$$

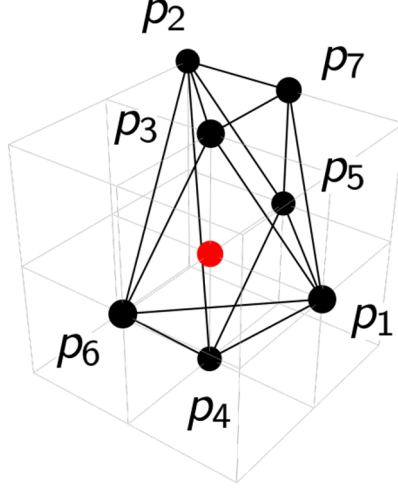


Figure 39. Toric diagram for Model 15.

where $t_1 t_2 = t_3 t_4 = t_5 t_6 = t_7^2$, in order to rewrite the Hilbert series for Model 15 in terms of characters of irreducible representations of $SU(2) \times U(1) \times U(1)$.

The highest weight form of the Hilbert series of Model 15 is

$$\begin{aligned}
 h_1(t, \mu, b_1, b_2; \mathcal{M}_{15}) &= \frac{1}{(1 - \mu b_1^{-2} t^5)(1 - \mu b_2^{-2} t^5)(1 - \mu^2 b_1^2 b_2^{-2} t^7)(1 - \mu^2 b_1^{-2} b_2^2 t^7)} \\
 &\times \frac{1}{(1 - \mu^4 b_1^2 b_2^2 t^{11})} \times (1 + \mu^2 t^7 + \mu^3 b_1^2 t^9 + \mu^3 b_2^2 t^9 - \mu^3 b_1^{-2} t^{12} - \mu^3 b_2^{-2} t^{12} - 2\mu^4 t^{14} \\
 &\quad - \mu^4 b_1^2 b_2^{-2} t^{14} - \mu^4 b_1^{-2} b_2^2 t^{14} - \mu^5 b_1^2 t^{16} - \mu^5 b_2^2 t^{16} + \mu^5 b_1^{-2} t^{19} + \mu^5 b_2^{-2} t^{19} + \mu^6 t^{21} \\
 &\quad + \mu^8 t^{28}), \tag{18.9}
 \end{aligned}$$

where $\mu^m \sim [m]_{SU(2)_x}$. Here in highest weight form, the fugacity μ counts the highest weight of irreducible representations of $SU(2)_x$. The fugacities b_1 and b_2 count charges under the two $U(1)$ factors of the mesonic flavor symmetry.

The plethystic logarithm of the Hilbert series takes the form

$$\begin{aligned}
 \text{PL}[g_1(t, x, b_1, b_2; \mathcal{M}_{15})] &= ([1]b_2^{-2} + [1]b_1^{-2})t^5 + ([2]b_1^2 b_2^{-2} + [2] + [2]b_1^{-2} b_2^2)t^7 \\
 &\quad + b_1^{-2} b_2^{-2} t^{10} - [4]b_1^2 b_2^2 t^{11} - ([3]b_2^{-2} + [1]b_1^2 b_2^{-4} + [3]b_1^{-2} + 2[1]b_2^{-2} \\
 &\quad + 2[1]b_1^{-2} + [1]b_1^{-4} b_2^2)t^{12} - ([4]b_1^2 b_2^{-2} + 3[4] + 2[2]b_1^2 b_2^{-2} + b_1^4 b_2^{-4} + [4]b_1^{-2} b_2^2 \\
 &\quad + 3[2] + b_1^2 b_2^{-2} + 2[2]b_1^{-2} b_2^2 + 2 + b_1^{-2} b_2^2 + b_1^{-4} b_2^4)t^{14} - (2[5]b_1^2 + [3]b_1^4 b_2^{-2} \\
 &\quad + 2[5]b_2^2 + 3[3]b_1^2 + [1]b_1^4 b_2^{-2} + 3[3]b_2^2 + 2[1]b_1^2 + [3]b_1^{-2} b_2^4 + 2[1]b_2^2 \\
 &\quad + [1]b_1^{-2} b_2^4)t^{16} + \dots, \tag{18.10}
 \end{aligned}$$

where $[m] = [m]_{SU(2)_x}$. From the plethystic logarithm, we see that the mesonic moduli space is a non-complete intersection.

By using the following fugacity map

$$\tilde{t} = t_7, \quad \tilde{x} = \frac{t_7^2}{t_2^2}, \quad \tilde{b}_1 = \frac{t_2 t_7}{t_4^2}, \quad \tilde{b}_2 = \frac{t_2 t_7}{t_6^2}, \tag{18.11}$$

generator	$SU(2)_{\tilde{x}}$	$U(1)_{\tilde{b}_1}$	$U(1)_{\tilde{b}_2}$
$p_1 p_4^2 p_5 p_6 \ so_1 o_2 o_3^2 o_4^2$	0	-1	0
$p_2 p_4^2 p_5 p_6 \ so_1 o_2 o_3^2 o_4^2$	-1	-1	0
$p_1 p_3 p_4 p_6^2 \ so_1 o_2^2 o_3 o_4^2$	0	0	-1
$p_2 p_3 p_4 p_6^2 \ so_1 o_2^2 o_3 o_4^2$	-1	0	-1
$p_1^2 p_4^2 p_5^2 p_7 \ so_1^2 o_2 o_3^3 o_4^3$	1	-1	1
$p_1 p_2 p_4^2 p_5^2 p_7 \ so_1^2 o_2 o_3^3 o_4^3$	0	-1	1
$p_2^2 p_4^2 p_5^2 p_7 \ so_1^2 o_2 o_3^3 o_4^3$	-1	-1	1
$p_1^2 p_3 p_4 p_5 p_6 p_7 \ so_1^2 o_2^2 o_3^3 o_4^3$	1	0	0
$p_1 p_2 p_3 p_4 p_5 p_6 p_7 \ so_1^2 o_2^2 o_3^3 o_4^3$	0	0	0
$p_2^2 p_3 p_4 p_5 p_6 p_7 \ so_1^2 o_2^2 o_3^3 o_4^3$	-1	0	0
$p_1^2 p_3^2 p_6^2 p_7 \ so_1^2 o_2^3 o_3 o_4^3$	1	1	-1
$p_1 p_2 p_3^2 p_6^2 p_7 \ so_1^2 o_2^3 o_3 o_4^3$	0	1	-1
$p_2^2 p_3^2 p_6^2 p_7 \ so_1^2 o_2^3 o_3 o_4^3$	-1	1	-1
$p_1^3 p_3 p_4 p_5^2 p_7^2 \ so_1^3 o_2^2 o_3^3 o_4^4$	2	0	1
$p_1^2 p_2 p_3 p_4 p_5^2 p_7^2 \ so_1^3 o_2^2 o_3^3 o_4^4$	1	0	1
$p_1 p_2^2 p_3 p_4 p_5^2 p_7^2 \ so_1^3 o_2^2 o_3^3 o_4^4$	0	0	1
$p_2^3 p_3 p_4 p_5^2 p_7^2 \ so_1^3 o_2^2 o_3^3 o_4^4$	-1	0	1
$p_1^3 p_3^2 p_5 p_6 p_7^2 \ so_1^3 o_2^3 o_3^2 o_4^4$	2	1	0
$p_1^2 p_2 p_3^2 p_5 p_6 p_7^2 \ so_1^3 o_2^3 o_3^2 o_4^4$	1	1	0
$p_1 p_2^2 p_3^2 p_5 p_6 p_7^2 \ so_1^3 o_2^3 o_3^2 o_4^4$	0	1	0
$p_2^3 p_3^2 p_5 p_6 p_7^2 \ so_1^3 o_2^3 o_3^2 o_4^4$	-1	1	0
$p_1^4 p_3^2 p_5^2 p_7^3 \ so_1^4 o_2^3 o_3^3 o_4^5$	3	1	1
$p_1^3 p_2 p_3^2 p_5^2 p_7^3 \ so_1^4 o_2^3 o_3^3 o_4^5$	2	1	1
$p_1^2 p_2^2 p_3^2 p_5^2 p_7^3 \ so_1^4 o_2^3 o_3^3 o_4^5$	1	1	1
$p_1 p_2^3 p_3^2 p_5^2 p_7^3 \ so_1^4 o_2^3 o_3^3 o_4^5$	0	1	1
$p_2^4 p_3^2 p_5^2 p_7^3 \ so_1^4 o_2^3 o_3^3 o_4^5$	-1	1	1

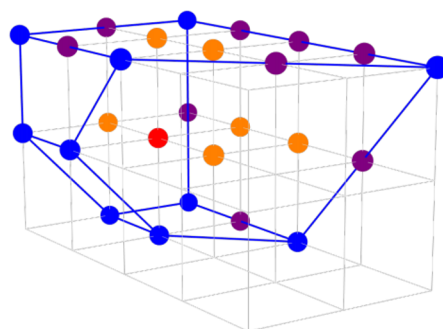


Table 60. The generators and lattice of generators of the mesonic moduli space of Model 15 in terms of brick matchings with the corresponding flavor charges.

where $t_1 t_2 = t_3 t_4 = t_5 t_6 = t_7^2$, the mesonic flavor charges on the gauge invariant operators become \mathbb{Z} -valued. The generators in terms of brick matchings and their corresponding rescaled mesonic flavor charges are summarized in table 60. The generator lattice as shown in table 60 is a convex lattice polytope, which is reflexive. It is the dual of the toric diagram of Model 15 shown in figure 39. For completeness, table 61 and table 62 show the generators of Model 15 in terms of chiral fields with the corresponding mesonic flavor charges.

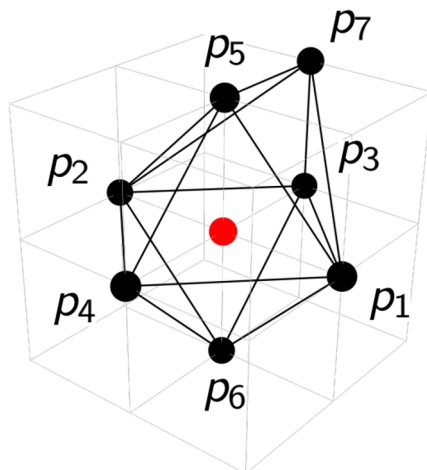


Figure 41. Toric diagram for Model 16.

The Hilbert series of the mesonic moduli space of Model 16 is

$$\begin{aligned}
 g_1(t_i, y_s, y_{o_1}, y_{o_2}, y_{o_3}; \mathcal{M}_{16}) &= \frac{P(t_i, y_s, y_{o_1}, y_{o_2}, y_{o_3}; \mathcal{M}_{16})}{(1 - y_s y_{o_1} y_{o_2} y_{o_3}^2 t_1^2 t_4^2 t_5 t_6)(1 - y_s y_{o_1} y_{o_2} y_{o_3}^2 t_2^2 t_4^2 t_5 t_6)} \\
 &\times \frac{1}{(1 - y_s y_{o_1}^2 y_{o_2} y_{o_3} t_1^2 t_3 t_4 t_6^2)(1 - y_s y_{o_1}^2 y_{o_2} y_{o_3} t_2^2 t_3 t_4 t_6^2)(1 - y_s y_{o_1} y_{o_2}^2 y_{o_3}^3 t_1^2 t_4^2 t_5 t_6)} \\
 &\times \frac{1}{(1 - y_s y_{o_1} y_{o_2}^2 y_{o_3}^3 t_2^2 t_4^2 t_5^2 t_7)(1 - y_s y_{o_1}^3 y_{o_2}^2 y_{o_3} t_1^2 t_3^2 t_6^2 t_7)(1 - y_s y_{o_1}^3 y_{o_2}^2 y_{o_3} t_2^2 t_3^2 t_6^2 t_7)} \\
 &\times \frac{1}{(1 - y_s y_{o_1}^3 y_{o_2}^4 y_{o_3}^3 t_1^2 t_3^2 t_5^2 t_7^3)(1 - y_s y_{o_1}^3 y_{o_2}^4 y_{o_3}^3 t_2^2 t_3^2 t_5^2 t_7^3)}, \tag{19.6}
 \end{aligned}$$

where t_i are the fugacities for the extremal brick matchings p_i . Furthermore, y_s counts $s_1 \dots s_{20}$, y_{o_1} counts $o_1 \dots o_5$, y_{o_2} counts $o_6 \dots o_{10}$ and y_{o_3} counts $o_{11} \dots o_{15}$. The explicit numerator $P(t_i, y_s, y_{o_1}, y_{o_2}, y_{o_3}; \mathcal{M}_{16})$ of the Hilbert series is given in the appendix section A.16. We note that setting the fugacities $y_{o_1} = 1, \dots, y_{o_3} = 1$ does not change the overall characterization of the mesonic moduli space by the Hilbert series, indicating that the extra GLSM fields, as expected, correspond to an over-parameterization of the moduli space.

By setting $t_i = t$ for the fugacities of the extremal brick matchings, and all other fugacities to 1, the unrefined Hilbert series takes the following form

$$\begin{aligned}
 g_1(t, 1, 1, 1, 1, 1; \mathcal{M}_{16}) &= \frac{(1-t)^2(1-t^3)^2}{(1-t^6)^3(1-t^7)^2(1-t^9)^2} \times (1+t+t^2+3t^3+3t^4+3t^5 \\
 &+ 9t^6+16t^7+22t^8+33t^9+47t^{10}+59t^{11}+75t^{12}+83t^{13}+91t^{14}+105t^{15} \\
 &+ 102t^{16}+104t^{17}+116t^{18}+104t^{19}+102t^{20}+105t^{21}+91t^{22}+83t^{23}+75t^{24} \\
 &+ 59t^{25}+47t^{26}+33t^{27}+22t^{28}+16t^{29}+9t^{30}+3t^{31}+3t^{32}+3t^{33}+t^{34}+t^{35} \\
 &+ t^{36}), \tag{19.7}
 \end{aligned}$$

where the palindromic numerator indicates that the mesonic moduli space is Calabi-Yau.

	SU(2) _x	U(1) _{b₁}	U(1) _{b₂}	U(1)	fugacity
p ₁	+1	0	0	r ₁	t ₁
p ₂	-1	0	0	r ₂	t ₂
p ₃	0	+1	0	r ₃	t ₃
p ₄	0	-1	0	r ₄	t ₄
p ₅	0	0	+1	r ₅	t ₅
p ₆	0	0	-1	r ₆	t ₆
p ₇	0	0	0	r ₇	t ₇

Table 63. Global symmetry charges on the extremal brick matchings p_i of Model 16.

The global symmetry of Model 16 and the charges on the extremal brick matchings under the global symmetry are summarized in table 63. We can use the following fugacity map,

$$t = t_7, \quad x = \frac{t_7}{t_2}, \quad b_1 = \frac{t_7}{t_4}, \quad b_2 = \frac{t_5}{t_7}, \quad (19.8)$$

where $t_1 t_2 = t_3 t_4 = t_5 t_6 = t_7^2$, in order to rewrite the Hilbert series for Model 16 in terms of characters of irreducible representations of $SU(2) \times U(1) \times U(1)$.

The highest weight form of the Hilbert series of Model 16 is

$$\begin{aligned}
 h_1(t, \mu, b_1, b_2; \mathcal{M}_{16}) &= \frac{1}{(1 - \mu^2 b_1^{-2} t^6)(1 - \mu^2 b_2^{-2} t^6)(1 - \mu^2 b_1^2 b_2^{-2} t^7)(1 - \mu^2 b_1^{-2} b_2^2 t^7)} \\
 &\times \frac{1}{(1 - \mu^2 b_1^2 b_2^2 t^9)} \times (1 + \mu^2 t^7 + \mu^2 b_1^2 t^8 + \mu^2 b_2^2 t^8 - \mu^4 b_1^{-2} t^{13} - \mu^4 b_2^{-2} t^{13} - 2\mu^4 t^{14} \\
 &\quad - \mu^4 b_1^2 b_2^{-2} t^{14} - \mu^4 b_1^{-2} b_2^2 t^{14} - \mu^4 b_1^2 t^{15} - \mu^4 b_2^2 t^{15} + \mu^6 b_1^{-2} t^{20} + \mu^6 b_2^{-2} t^{20} + \mu^6 t^{21} \\
 &\quad + \mu^8 t^{28}), \quad (19.9)
 \end{aligned}$$

where $\mu^m \sim [m]_{SU(2)_x}$. Here in highest weight form, the fugacity μ counts the highest weight of irreducible representations of $SU(2)_x$. The fugacities b_1 and b_2 count charges under the two U(1) factors of the mesonic flavor symmetry.

The plethystic logarithm of the Hilbert series takes the form

$$\begin{aligned}
 \text{PL}[g_1(t, x, b_1, b_2; \mathcal{M}_{16})] &= ([2]b_2^{-2} + [2]b_1^{-2})t^6 + ([2]b_1^2 b_2^{-2} + [2] + [2]b_1^{-2} b_2^2)t^7 \\
 &\quad + ([2]b_1^2 + [2]b_2^2)t^8 + [2]b_1^2 b_2^2 t^9 - ([2]b_1^{-2} b_2^{-2} + b_2^{-4} + b_1^{-2} b_2^{-2} + b_1^{-4})t^{12} \\
 &\quad - ([4]b_2^{-2} + [2]b_1^2 b_2^{-4} + [4]b_1^{-2} + 2[2]b_2^{-2} + b_1^2 b_2^{-4} + 2[2]b_1^{-2} + 2b_2^{-2} + [2]b_1^{-4} b_2^2 \\
 &\quad + 2b_1^{-2} + b_1^{-4} b_2^2)t^{13} - ([4]b_1^2 b_2^{-2} + 3[4] + 2[2]b_1^2 b_2^{-2} + b_1^4 b_2^{-4} + [4]b_1^{-2} b_2^2 + 3[2] \\
 &\quad + 2b_1^2 b_2^{-2} + 2[2]b_1^{-2} b_2^2 + 4 + 2b_1^{-2} b_2^2 + b_1^{-4} b_2^4)t^{14} - (2[4]b_1^2 + [2]b_1^4 b_2^{-2} + 2[4]b_2^2 \\
 &\quad + 3[2]b_1^2 + b_1^4 b_2^{-2} + 3[2]b_2^2 + 3b_1^2 + [2]b_1^{-2} b_2^4 + 3b_2^2 + b_1^{-2} b_2^4)t^{15} - ([4]b_1^4 + [4]b_1^2 b_2^2 \\
 &\quad + [2]b_1^4 + [4]b_2^4 + 2[2]b_1^2 b_2^2 + 2b_1^4 + [2]b_2^4 + 2b_1^2 b_2^2 + 2b_2^4)t^{16} - ([2]b_1^4 b_2^2 + [2]b_1^2 b_2^4 \\
 &\quad + b_1^4 b_2^2 + b_1^2 b_2^4)t^{17} - ([2]b_1^{-2} b_2^{-4} + [2]b_1^{-4} b_2^{-2} + b_1^{-2} b_2^{-4} + b_1^{-4} b_2^{-2})t^{18} + \dots, \quad (19.10)
 \end{aligned}$$

where $[m] = [m]_{SU(2)_x}$. From the plethystic logarithm, we see that the mesonic moduli space is a non-complete intersection.

generator	$SU(2)_{\tilde{x}}$	$U(1)_{\tilde{b}_1}$	$U(1)_{\tilde{b}_2}$
$p_1^2 p_4^2 p_5 p_6 so_1 o_2 o_3^2 t^6$	1	-1	0
$p_1 p_2 p_4^2 p_5 p_6 so_1 o_2 o_3^2 t^6$	0	-1	0
$p_2^2 p_4^2 p_5 p_6 so_1 o_2 o_3^2 t^6$	-1	-1	0
$p_1^2 p_3 p_4 p_6^2 so_1^2 o_2 o_3 t^6$	1	0	-1
$p_1 p_2 p_3 p_4 p_6^2 so_1^2 o_2 o_3 t^6$	0	0	-1
$p_2^2 p_3 p_4 p_6^2 so_1^2 o_2 o_3 t^6$	-1	0	-1
$p_1^2 p_4^2 p_5^2 p_7 so_1 o_2^2 o_3^3 t^7$	1	-1	1
$p_1 p_2 p_4^2 p_5^2 p_7 so_1 o_2^2 o_3^3 t^7$	0	-1	1
$p_2^2 p_4^2 p_5^2 p_7 so_1 o_2^2 o_3^3 t^7$	-1	-1	1
$p_1^2 p_3 p_4 p_5 p_6 p_7 so_1^2 o_2^2 o_3 t^7$	1	0	0
$p_1 p_2 p_3 p_4 p_5 p_6 p_7 so_1^2 o_2^2 o_3 t^7$	0	0	0
$p_2^2 p_3 p_4 p_5 p_6 p_7 so_1^2 o_2^2 o_3 t^7$	-1	0	0
$p_1^2 p_3^2 p_6^2 p_7 so_1^3 o_2^2 o_3 t^7$	1	1	-1
$p_1 p_2 p_3^2 p_6^2 p_7 so_1^3 o_2^2 o_3 t^7$	0	1	-1
$p_2^2 p_3^2 p_6^2 p_7 so_1^3 o_2^2 o_3 t^7$	-1	1	-1
$p_1^2 p_3 p_4 p_5^2 p_7^2 so_1^2 o_2^3 o_3^2 t^8$	1	0	1
$p_1 p_2 p_3 p_4 p_5^2 p_7^2 so_1^2 o_2^3 o_3^2 t^8$	0	0	1
$p_2^2 p_3 p_4 p_5^2 p_7^2 so_1^2 o_2^3 o_3^2 t^8$	-1	0	1
$p_1^2 p_3^2 p_5 p_6 p_7^2 so_1^3 o_2^3 o_3^2 t^8$	1	1	0
$p_1 p_2 p_3^2 p_5 p_6 p_7^2 so_1^3 o_2^3 o_3^2 t^8$	0	1	0
$p_2^2 p_3^2 p_5 p_6 p_7^2 so_1^3 o_2^3 o_3^2 t^8$	-1	1	0
$p_1^2 p_3^2 p_5^2 p_7^3 so_1^3 o_2^4 o_3^3 t^9$	1	1	1
$p_1 p_2 p_3^2 p_5^2 p_7^3 so_1^3 o_2^4 o_3^3 t^9$	0	1	1
$p_2^2 p_3^2 p_5^2 p_7^3 so_1^3 o_2^4 o_3^3 t^9$	-1	1	1

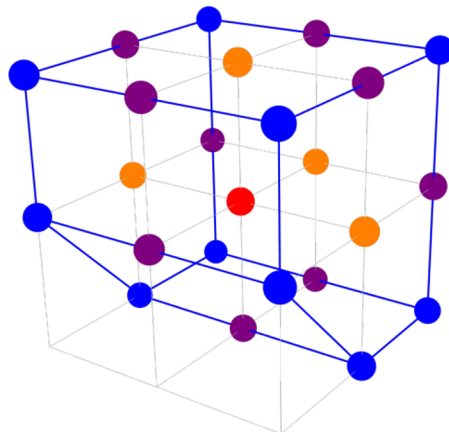


Table 64. The generators and lattice of generators of the mesonic moduli space of Model 16 in terms of brick matchings with the corresponding flavor charges.

By using the following fugacity map

$$\tilde{t} = t_7, \quad \tilde{x} = \frac{t_7^2}{t_2^2}, \quad \tilde{b}_1 = \frac{t_7^2}{t_4^2}, \quad \tilde{b}_2 = \frac{t_7^2}{t_6^2}, \quad (19.11)$$

where $t_1 t_2 = t_3 t_4 = t_5 t_6 = t_7^2$, the mesonic flavor charges on the gauge invariant operators become \mathbb{Z} -valued. The generators in terms of brick matchings and their corresponding rescaled mesonic flavor charges are summarized in table 64. The generator lattice as shown in table 64 is a convex lattice polytope, which is reflexive. It is the dual of the toric diagram of Model 16 shown in figure 41. For completeness, table 65 and table 66 show the generators of Model 16 in terms of chiral fields with the corresponding mesonic flavor charges.

generator	SU(2) $_{\tilde{x}}$	U(1) $_{\tilde{b}_1}$	U(1) $_{\tilde{b}_2}$
$P_{27}R_{75}X_{53}X_{32} = P_{38}X_{87}R_{75}X_{53} = P_{5.10}X_{10.8}X_{87}R_{75} = P_{16}X_{69}Y_{97}R_{75}X_{51} = X_{14}Y_{42}P_{27}R_{75}X_{51}$ $= X_{14}P_{49}Y_{97}R_{75}X_{51} = P_{5.10}X_{10.6}X_{69}Y_{97}R_{75}$	1	-1	0
$P_{27}X_{75}X_{53}X_{32} = P_{38}X_{87}X_{75}X_{53} = P_{5.10}X_{10.8}X_{87}X_{75} = Q_{27}R_{75}X_{53}X_{32} = Q_{38}X_{87}R_{75}X_{53} = Q_{5.10}X_{10.8}X_{87}R_{75}$ $= P_{16}X_{69}Y_{97}X_{75}X_{51} = X_{14}Y_{42}P_{27}X_{75}X_{51} = X_{14}P_{49}Y_{97}X_{75}X_{51} = P_{5.10}X_{10.6}X_{69}Y_{97}X_{75} = Q_{16}X_{69}Y_{97}R_{75}X_{51}$ $= X_{14}Y_{42}Q_{27}R_{75}X_{51} = X_{14}Q_{49}Y_{97}R_{75}X_{51} = Q_{5.10}X_{10.6}X_{69}Y_{97}R_{75}$	0	-1	0
$Q_{27}X_{75}X_{53}X_{32} = Q_{38}X_{87}X_{75}X_{53} = Q_{5.10}X_{10.8}X_{87}X_{75} = Q_{16}X_{69}Y_{97}X_{75}X_{51} = X_{14}Y_{42}Q_{27}X_{75}X_{51}$ $= X_{14}Q_{49}Y_{97}X_{75}X_{51} = Q_{5.10}X_{10.6}X_{69}Y_{97}X_{75}$	-1	-1	0
$P_{16}X_{69}Y_{97}Y_{71} = X_{14}Y_{42}P_{27}Y_{71} = X_{14}P_{49}Y_{97}Y_{71} = P_{27}R_{75}X_{53}X_{34}Y_{42} = P_{38}X_{89}Y_{97}R_{75}X_{53}$ $= X_{34}P_{49}Y_{97}R_{75}X_{53} = P_{5.10}X_{10.8}X_{89}Y_{97}R_{75}$	1	0	-1
$P_{16}X_{69}Y_{97}X_{71} = X_{14}Y_{42}P_{27}X_{71} = X_{14}P_{49}Y_{97}X_{71} = Q_{16}X_{69}Y_{97}Y_{71} = X_{14}Y_{42}Q_{27}Y_{71} = X_{14}Q_{49}Y_{97}Y_{71}$ $= P_{27}X_{75}X_{53}X_{34}Y_{42} = P_{38}X_{89}Y_{97}X_{75}X_{53} = X_{34}P_{49}Y_{97}X_{75}X_{53} = P_{5.10}X_{10.8}X_{89}Y_{97}X_{75} = Q_{27}R_{75}X_{53}X_{34}Y_{42}$ $= Q_{38}X_{89}Y_{97}R_{75}X_{53} = X_{34}Q_{49}Y_{97}R_{75}X_{53} = Q_{5.10}X_{10.8}X_{89}Y_{97}R_{75}$	0	0	-1
$Q_{16}X_{69}Y_{97}X_{71} = X_{14}Y_{42}Q_{27}X_{71} = X_{14}Q_{49}Y_{97}X_{71} = Q_{27}X_{75}X_{53}X_{34}Y_{42} = Q_{38}X_{89}Y_{97}X_{75}X_{53}$ $= X_{34}Q_{49}Y_{97}X_{75}X_{53} = Q_{5.10}X_{10.8}X_{89}Y_{97}X_{75}$	-1	0	-1
$P_{16}X_{68}X_{87}R_{75}X_{51} = P_{16}X_{69}X_{97}R_{75}X_{51} = X_{13}X_{32}P_{27}R_{75}X_{51} = X_{14}X_{42}P_{27}R_{75}X_{51} = X_{13}P_{38}X_{87}R_{75}X_{51}$ $= X_{14}P_{49}X_{97}R_{75}X_{51} = P_{5.10}X_{10.6}X_{68}X_{87}R_{75} = P_{5.10}X_{10.6}X_{69}X_{97}R_{75}$	1	-1	1

Table 65. The generators in terms of bifundamental chiral fields for Model 16 (*Part 1*).

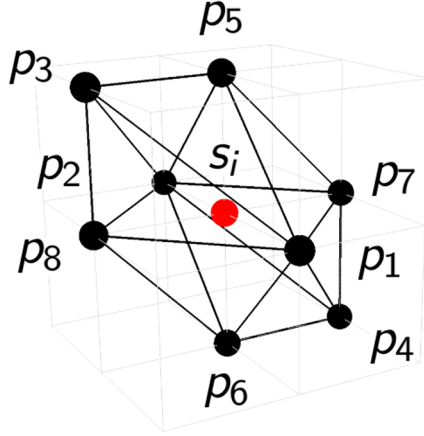


Figure 43. Toric diagram for Model 17.

The Hilbert series of the mesonic moduli space of Model 17 is

$$\begin{aligned}
 g_1(t_i, y_s, y_{o_1}, y_{o_2}, y_{o_3}, y_{o_4}, y_{o_5}, y_{o_6}; \mathcal{M}_{17}) &= \frac{P(t_i, y_s, y_{o_1}, y_{o_2}, y_{o_3}, y_{o_4}, y_{o_5}, y_{o_6}; \mathcal{M}_{17})}{(1 - y_s y_{o_1}^3 y_{o_2} y_{o_3}^3 y_{o_4} y_{o_5}^2 y_{o_6}^2 t_1^2 t_3 t_4 t_5^2 t_7^2)} \\
 &\times \frac{1}{(1 - y_s y_{o_1}^3 y_{o_2} y_{o_3}^3 y_{o_4} y_{o_5}^2 y_{o_6}^2 t_2^2 t_3 t_4 t_5^2 t_7^2)} (1 - y_s y_{o_1}^3 y_{o_2} y_{o_3}^3 y_{o_4} y_{o_5}^2 y_{o_6}^2 t_1^2 t_4^2 t_5 t_6 t_7^2) \\
 &\times \frac{1}{(1 - y_s y_{o_1}^3 y_{o_2} y_{o_3}^3 y_{o_4} y_{o_5}^2 y_{o_6}^2 t_2^2 t_4^2 t_5 t_6 t_7^2)} (1 - y_s y_{o_1}^2 y_{o_2}^2 y_{o_3}^3 y_{o_4} y_{o_5} y_{o_6}^3 t_1^2 t_3^2 t_5^2 t_7 t_8) \\
 &\times \frac{1}{(1 - y_s y_{o_1}^2 y_{o_2}^2 y_{o_3}^3 y_{o_4} y_{o_5} y_{o_6}^3 t_2^2 t_3^2 t_5^2 t_7 t_8)} (1 - y_s y_{o_1}^2 y_{o_2}^2 y_{o_3}^3 y_{o_4} y_{o_5} y_{o_6}^3 t_1^2 t_4^2 t_6^2 t_7 t_8) \\
 &\times \frac{1}{(1 - y_s y_{o_1}^2 y_{o_2}^2 y_{o_3}^3 y_{o_4} y_{o_5} y_{o_6}^3 t_2^2 t_4^2 t_6^2 t_7 t_8)} (1 - y_s y_{o_1} y_{o_2}^3 y_{o_3}^3 y_{o_4} y_{o_5} y_{o_6}^3 t_1^2 t_3^2 t_5 t_6 t_8^2) \\
 &\times \frac{1}{(1 - y_s y_{o_1} y_{o_2}^3 y_{o_3}^3 y_{o_4} y_{o_5} y_{o_6}^3 t_2^2 t_3^2 t_5 t_6 t_8^2)} (1 - y_s y_{o_1} y_{o_2}^3 y_{o_3}^3 y_{o_4} y_{o_5} y_{o_6}^3 t_1^2 t_3 t_4 t_6^2 t_8^2) \\
 &\times \frac{1}{(1 - y_s y_{o_1} y_{o_2}^3 y_{o_3}^3 y_{o_4} y_{o_5} y_{o_6}^3 t_2^2 t_3 t_4 t_6^2 t_8^2)}, \tag{20.6}
 \end{aligned}$$

where t_i are the fugacities for the extremal brick matchings p_i . Additionally, y_s counts $s_1 \dots s_{18}$, y_{o_1} counts $o_1 o_2 o_3$, y_{o_2} counts $o_4 o_5 o_6$, y_{o_3} counts $o_7 o_8 o_9$, y_{o_4} counts $o_{10} o_{11} o_{12}$, y_{o_5} counts $o_{13} o_{14} o_{15}$ and y_{o_6} counts $o_{16} o_{17} o_{18}$. The explicit numerator $P(t_i, y_s, y_{o_1}, \dots, y_{o_6}; \mathcal{M}_{17})$ of the Hilbert series is given in the appendix section A.17. We note that setting the fugacities $y_{o_1} = 1, \dots, y_{o_6} = 1$ does not change the overall characterization of the mesonic moduli space by the Hilbert series, indicating that the extra GLSM fields, as expected, correspond to an over-parameterization of the moduli space.

By setting $t_i = t$ for the fugacities of the extremal brick matchings, and all other fugacities to 1, the unrefined Hilbert series takes the following form

$$g_1(t, 1, 1, 1, 1, 1, 1, 1; \mathcal{M}_{17}) = \frac{1 + 17t^8 + 17t^{16} + t^{24}}{(1 - t^8)^4}, \tag{20.7}$$

where the palindromic numerator indicates that the mesonic moduli space is Calabi-Yau.

	SU(2) _x	U(1) _{b₁}	U(1) _{b₂}	U(1)	fugacity
p_1	+1	0	0	r_1	t_1
p_2	-1	0	0	r_2	t_2
p_3	0	+1	0	r_3	t_3
p_4	0	-1	0	r_4	t_4
p_5	0	0	+1	r_5	t_5
p_6	0	0	-1	r_6	t_6
p_7	0	0	0	r_7	t_7
p_8	0	0	0	r_8	t_8

Table 67. Global symmetry charges on the extremal brick matchings p_i of Model 17.

The global symmetry of Model 17 and the charges on the extremal brick matchings under the global symmetry are summarized in table 67. We can use the following fugacity map,

$$t = t_7 = t_8, \quad x = \frac{t_8}{t_2}, \quad b_1 = \frac{t_8}{t_4}, \quad b_2 = \frac{t_5}{t_8}, \quad (20.8)$$

where $t_1 t_2 = t_3 t_4 = t_5 t_6 = t_8^2$, in order to rewrite the Hilbert series for Model 17 in terms of characters of irreducible representations of $SU(2) \times U(1) \times U(1)$.

The highest weight form of the Hilbert series of Model 17 is

$$\begin{aligned}
 h_1(t, \mu, b_1, b_2; \mathcal{M}_{17}) &= \frac{1}{(1 - \mu^2 b_1^{-2} t^8)(1 - \mu^2 b_2^{-2} t^8)(1 - \mu^2 b_1^{-2} b_2^{-2} t^8)(1 - b_1^2 \mu^2 t^8)} \\
 &\times \frac{1}{(1 - b_2^2 \mu^2 t^8)(1 - b_1^2 b_2^2 \mu^2 t^8)} \times (1 + \mu^2 t^8 - 2\mu^4 t^{16} - \mu^4 b_1^{-2} t^{16} - \mu^4 b_1^2 t^{16} - \mu^4 b_2^{-2} t^{16} \\
 &\quad - \mu^4 b_1^{-2} b_2^{-2} t^{16} - \mu^4 b_2^2 t^{16} - \mu^4 b_1^2 b_2^2 t^{16} + 2\mu^6 t^{24} + \mu^6 b_1^{-2} t^{24} + \mu^6 b_1^2 t^{24} + \mu^6 b_2^{-2} t^{24} \\
 &\quad + \mu^6 b_1^{-2} b_2^{-2} t^{24} + \mu^6 b_2^2 t^{24} + \mu^6 b_1^2 b_2^2 t^{24} - \mu^8 t^{32} - \mu^{10} t^{40}), \quad (20.9)
 \end{aligned}$$

where $\mu^m \sim [m]_{SU(2)_x}$. Here in highest weight form, the fugacity μ counts the highest weight of irreducible representations of $SU(2)_x$. The fugacities b_1 and b_2 count charges under the two U(1) factors of the mesonic flavor symmetry.

The plethystic logarithm of the Hilbert series is

$$\begin{aligned}
 \text{PL}[g_1(t, x, b_1, b_2; \mathcal{M}_{17})] &= ([2] + [2]b_1^{-2} + [2]b_1^2 + [2]b_2^{-2} + [2]b_1^{-2}b_2^{-2} + [2]b_2^2 + [2]b_1^2b_2^2)t^8 \\
 &\quad - (b_1^{-4} + 4 + 2b_1^{-2} + 2b_1^2 + b_1^4 + b_2^{-4} + b_1^{-4}b_2^{-4} + b_1^{-2}b_2^{-4} + 2b_2^{-2} + b_1^{-4}b_2^{-2} + 2b_1^{-2}b_2^{-2} \\
 &\quad + b_1^2b_2^{-2} + 2b_2^2 + b_1^{-2}b_2^2 + 2b_1^2b_2^2 + b_1^4b_2^2 + b_2^4 + b_1^2b_2^4 + b_1^4b_2^4 + 3[2] + 2[2]b_1^{-2} + 2[2]b_1^2 \\
 &\quad + [2]b_1^{-2}b_2^{-4} + 2[2]b_2^{-2} + [2]b_1^{-4}b_2^{-2} + 2[2]b_1^{-2}b_2^{-2} + [2]b_1^2b_2^{-2} + 2[2]b_2^2 + [2]b_1^{-2}b_2^2 \\
 &\quad + 2[2]b_1^2b_2^2 + [2]b_1^4b_2^2 + [2]b_1^2b_2^4 + 3[4] + [4]b_1^{-2} + [4]b_1^2 + [4]b_2^{-2} + [4]b_1^{-2}b_2^{-2} + [4]b_2^2 \\
 &\quad + [4]b_1^2b_2^2)t^{16} + \dots, \quad (20.10)
 \end{aligned}$$

where $[m] = [m]_{SU(2)_x}$. From the plethystic logarithm, we see that the mesonic moduli space is a non-complete intersection.

generator	$SU(2)_{\tilde{x}}$	$U(1)_{\tilde{b}_1}$	$U(1)_{\tilde{b}_2}$
$p_1^2 p_3 p_4 p_5^2 p_7^2 s o_1^3 o_2 o_3^3 o_4 o_5^2 o_6^2$	1	0	1
$p_1 p_2 p_3 p_4 p_5^2 p_7^2 s o_1^3 o_2 o_3^3 o_4 o_5^2 o_6^2$	0	0	1
$p_2^2 p_3 p_4 p_5^2 p_7^2 s o_1^3 o_2 o_3^3 o_4 o_5^2 o_6^2$	-1	0	1
$p_1^2 p_4^2 p_5 p_6 p_7^2 s o_1^3 o_2 o_3^3 o_4^2 o_5^3 o_6$	1	-1	0
$p_1 p_2 p_4^2 p_5 p_6 p_7^2 s o_1^3 o_2 o_3^3 o_4^2 o_5^3 o_6$	0	-1	0
$p_2^2 p_4^2 p_5 p_6 p_7^2 s o_1^3 o_2 o_3^3 o_4^2 o_5^3 o_6$	-1	-1	0
$p_1^2 p_3^2 p_5^2 p_7 p_8 s o_1^2 o_2^2 o_3^3 o_4 o_5 o_6^3$	1	1	1
$p_1 p_2 p_3^2 p_5^2 p_7 p_8 s o_1^2 o_2^2 o_3^3 o_4 o_5 o_6^3$	0	1	1
$p_2^2 p_3^2 p_5^2 p_7 p_8 s o_1^2 o_2^2 o_3^3 o_4 o_5 o_6^3$	-1	1	1
$p_1^2 p_3 p_4 p_5 p_6 p_7 p_8 s o_1^2 o_2^2 o_3^3 o_4^2 o_5^2 o_6^2$	1	0	0
$p_1 p_2 p_3 p_4 p_5 p_6 p_7 p_8 s o_1^2 o_2^2 o_3^3 o_4^2 o_5^2 o_6^2$	0	0	0
$p_2^2 p_3 p_4 p_5 p_6 p_7 p_8 s o_1^2 o_2^2 o_3^3 o_4^2 o_5^2 o_6^2$	-1	0	0
$p_1^2 p_4^2 p_6^2 p_7 p_8 s o_1^2 o_2^2 o_3 o_4^3 o_5^3 o_6$	1	-1	-1
$p_1 p_2 p_4^2 p_6^2 p_7 p_8 s o_1^2 o_2^2 o_3 o_4^3 o_5^3 o_6$	0	-1	-1
$p_2^2 p_4^2 p_6^2 p_7 p_8 s o_1^2 o_2^2 o_3 o_4^3 o_5^3 o_6$	-1	-1	-1
$p_1^2 p_3^2 p_5 p_6 p_8^2 s o_1 o_2^3 o_3^3 o_4^2 o_5 o_6^3$	1	1	0
$p_1 p_2 p_3^2 p_5 p_6 p_8^2 s o_1 o_2^3 o_3^3 o_4^2 o_5 o_6^3$	0	1	0
$p_2^2 p_3^2 p_5 p_6 p_8^2 s o_1 o_2^3 o_3^3 o_4^2 o_5 o_6^3$	-1	1	0
$p_1^2 p_3 p_4 p_6^2 p_8^2 s o_1 o_2^3 o_3 o_4^3 o_5^2 o_6^2$	1	0	-1
$p_1 p_2 p_3 p_4 p_6^2 p_8^2 s o_1 o_2^3 o_3 o_4^3 o_5^2 o_6^2$	0	0	-1
$p_2^2 p_3 p_4 p_6^2 p_8^2 s o_1 o_2^3 o_3 o_4^3 o_5^2 o_6^2$	-1	0	-1

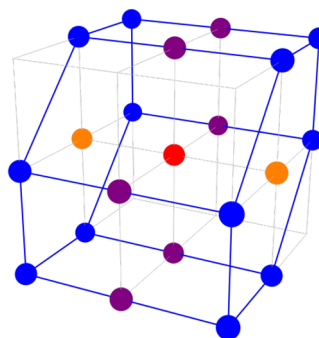


Table 68. The generators and lattice of generators of the mesonic moduli space of Model 17 in terms of brick matchings with the corresponding flavor charges.

By using the following fugacity map

$$\tilde{t} = t_7 = t_8, \quad \tilde{x} = \frac{t_8^2}{t_2^2}, \quad \tilde{b}_1 = \frac{t_8^2}{t_4^2}, \quad \tilde{b}_2 = \frac{t_8^2}{t_6^2}, \tag{20.11}$$

where $t_1 t_2 = t_3 t_4 = t_5 t_6 = t_8^2$, the mesonic flavor charges on the gauge invariant operators become \mathbb{Z} -valued. The generators in terms of brick matchings and their corresponding rescaled mesonic flavor charges are summarized in table 68. The generator lattice as shown in table 68 is a convex lattice polytope, which is reflexive. It is the dual of the toric diagram of Model 17 shown in figure 43. For completeness, table 69 and table 70 show the generators of Model 17 in terms of chiral fields with the corresponding mesonic flavor charges.

generator	$SU(2)_E$	$U(1)_{b_1}$	$U(1)_{b_2}$
$P_{17}X_{7,10}R_{10,5}P_{5,11}S_{11,6}X_{61} = X_{14}P_{4,10}R_{10,5}P_{5,11}S_{11,6}X_{61} = P_{5,11}S_{11,6}P_{6,12}X_{12,7}X_{7,10}R_{10,5} = P_{17}X_{7,10}R_{10,3}X_{32}P_{28}S_{86}X_{61}$ $= P_{17}X_{7,10}R_{10,3}P_{39}X_{98}S_{86}X_{61} = X_{14}P_{4,10}R_{10,3}X_{32}P_{28}S_{86}X_{61} = P_{28}S_{86}P_{6,12}X_{12,7}X_{7,10}R_{10,3}X_{32}$ $= X_{14}P_{4,10}R_{10,3}P_{39}X_{98}S_{86}X_{61} = P_{39}X_{98}S_{86}P_{6,12}X_{12,7}X_{7,10}R_{10,3}$	1	0	1
$P_{17}X_{7,10}R_{10,5}Q_{5,11}S_{11,6}X_{61} = X_{14}P_{4,10}R_{10,5}Q_{5,11}S_{11,6}X_{61} = Q_{17}X_{7,10}R_{10,5}P_{5,11}S_{11,6}X_{61} = X_{14}Q_{4,10}R_{10,5}P_{5,11}S_{11,6}X_{61}$ $= P_{5,11}S_{11,6}Q_{6,12}X_{12,7}X_{7,10}R_{10,5} = Q_{5,11}S_{11,6}P_{6,12}X_{12,7}X_{7,10}R_{10,5} = P_{17}X_{7,10}R_{10,3}X_{32}Q_{28}S_{86}X_{61}$ $= P_{17}X_{7,10}R_{10,3}Q_{39}X_{98}S_{86}X_{61} = Q_{17}X_{7,10}R_{10,3}X_{32}P_{28}S_{86}X_{61} = X_{14}Q_{4,10}R_{10,3}X_{32}P_{28}S_{86}X_{61}$ $= P_{28}S_{86}Q_{6,12}X_{12,7}X_{7,10}R_{10,3}X_{32} = Q_{17}X_{7,10}R_{10,3}P_{39}X_{98}S_{86}X_{61} = X_{14}Q_{4,10}R_{10,3}P_{39}X_{98}S_{86}X_{61}$ $= P_{39}X_{98}S_{86}Q_{6,12}X_{12,7}X_{7,10}R_{10,3} = X_{14}P_{4,10}R_{10,3}X_{32}Q_{28}S_{86}X_{61} = X_{14}P_{4,10}R_{10,3}Q_{39}X_{98}S_{86}X_{61}$ $= Q_{28}S_{86}P_{6,12}X_{12,7}X_{7,10}R_{10,3}X_{32} = Q_{39}X_{98}S_{86}P_{6,12}X_{12,7}X_{7,10}R_{10,3}$	0	0	1
$Q_{17}X_{7,10}R_{10,5}Q_{5,11}S_{11,6}X_{61} = X_{14}Q_{4,10}R_{10,5}Q_{5,11}S_{11,6}X_{61} = Q_{5,11}S_{11,6}Q_{6,12}X_{12,7}X_{7,10}R_{10,5}$ $= Q_{17}X_{7,10}R_{10,3}X_{32}Q_{28}S_{86}X_{61} = Q_{17}X_{7,10}R_{10,3}Q_{39}X_{98}S_{86}X_{61} = X_{14}Q_{4,10}R_{10,3}X_{32}Q_{28}S_{86}X_{61}$ $= Q_{28}S_{86}Q_{6,12}X_{12,7}X_{7,10}R_{10,3}X_{32} = X_{14}Q_{4,10}R_{10,3}Q_{39}X_{98}S_{86}X_{61} = Q_{39}X_{98}S_{86}Q_{6,12}X_{12,7}X_{7,10}R_{10,3}$	-1	0	1
$P_{17}X_{7,10}R_{10,3}X_{32}P_{28}S_{81} = P_{17}X_{7,10}R_{10,3}P_{39}X_{98}S_{81} = X_{14}P_{4,10}R_{10,3}X_{32}P_{28}S_{81} = X_{14}P_{4,10}R_{10,3}P_{39}X_{98}S_{81}$ $= P_{17}X_{7,10}R_{10,3}P_{39}X_{9,11}S_{11,6}X_{61} = P_{17}X_{7,10}R_{10,3}X_{35}P_{5,11}S_{11,6}X_{61} = X_{14}P_{4,10}R_{10,3}P_{39}X_{9,11}S_{11,6}X_{61}$ $= P_{39}X_{9,11}S_{11,6}P_{6,12}X_{12,7}X_{7,10}R_{10,3} = X_{14}P_{4,10}R_{10,3}X_{35}P_{5,11}S_{11,6}X_{61} = X_{35}P_{5,11}S_{11,6}P_{6,12}X_{12,7}X_{7,10}R_{10,3}$	1	-1	0
$P_{17}X_{7,10}R_{10,3}X_{32}Q_{28}S_{81} = P_{17}X_{7,10}R_{10,3}Q_{39}X_{98}S_{81} = Q_{17}X_{7,10}R_{10,3}X_{32}P_{28}S_{81} = X_{14}Q_{4,10}R_{10,3}X_{32}P_{28}S_{81}$ $= Q_{17}X_{7,10}R_{10,3}P_{39}X_{98}S_{81} = X_{14}Q_{4,10}R_{10,3}P_{39}X_{98}S_{81} = X_{14}P_{4,10}R_{10,3}X_{32}Q_{28}S_{81} = X_{14}P_{4,10}R_{10,3}Q_{39}X_{98}S_{81}$ $= P_{17}X_{7,10}R_{10,3}Q_{39}X_{9,11}S_{11,6}X_{61} = P_{17}X_{7,10}R_{10,3}X_{35}Q_{5,11}S_{11,6}X_{61} = Q_{17}X_{7,10}R_{10,3}P_{39}X_{9,11}S_{11,6}X_{61}$ $= X_{14}Q_{4,10}R_{10,3}P_{39}X_{9,11}S_{11,6}X_{61} = P_{39}X_{9,11}S_{11,6}Q_{6,12}X_{12,7}X_{7,10}R_{10,3} = X_{14}P_{4,10}R_{10,3}Q_{39}X_{9,11}S_{11,6}X_{61}$ $= X_{14}P_{4,10}R_{10,3}X_{35}Q_{5,11}S_{11,6}X_{61} = Q_{17}X_{7,10}R_{10,3}X_{35}P_{5,11}S_{11,6}X_{61} = X_{14}Q_{4,10}R_{10,3}X_{35}P_{5,11}S_{11,6}X_{61}$ $= X_{35}P_{5,11}S_{11,6}Q_{6,12}X_{12,7}X_{7,10}R_{10,3} = Q_{39}X_{9,11}S_{11,6}P_{6,12}X_{12,7}X_{7,10}R_{10,3} = X_{35}Q_{5,11}S_{11,6}P_{6,12}X_{12,7}X_{7,10}R_{10,3}$	0	-1	0
$Q_{17}X_{7,10}R_{10,3}X_{32}Q_{28}S_{81} = Q_{17}X_{7,10}R_{10,3}Q_{39}X_{98}S_{81} = X_{14}Q_{4,10}R_{10,3}X_{32}Q_{28}S_{81} = X_{14}Q_{4,10}R_{10,3}Q_{39}X_{98}S_{81}$ $= Q_{17}X_{7,10}R_{10,3}Q_{39}X_{9,11}S_{11,6}X_{61} = Q_{17}X_{7,10}R_{10,3}X_{35}Q_{5,11}S_{11,6}X_{61} = X_{14}Q_{4,10}R_{10,3}Q_{39}X_{9,11}S_{11,6}X_{61}$ $= Q_{39}X_{9,11}S_{11,6}Q_{6,12}X_{12,7}X_{7,10}R_{10,3} = X_{14}Q_{4,10}R_{10,3}X_{35}Q_{5,11}S_{11,6}X_{61} = X_{35}Q_{5,11}S_{11,6}P_{6,12}X_{12,7}X_{7,10}R_{10,3}$	-1	-1	0
$P_{17}R_{73}X_{32}P_{28}S_{86}X_{61} = P_{17}R_{73}P_{39}X_{98}S_{86}X_{61} = P_{28}S_{86}P_{6,12}X_{12,7}R_{73}X_{32} = P_{39}X_{98}S_{86}P_{6,12}X_{12,7}R_{73}$ $= P_{17}X_{7,10}R_{10,5}X_{52}P_{28}S_{86}X_{61} = P_{17}X_{7,10}R_{10,5}P_{5,11}X_{11,8}S_{86}X_{61} = X_{14}P_{4,10}R_{10,5}X_{52}P_{28}S_{86}X_{61}$ $= P_{28}S_{86}P_{6,12}X_{12,7}X_{7,10}R_{10,5}X_{52} = X_{14}P_{4,10}R_{10,5}P_{5,11}X_{11,8}S_{86}X_{61} = P_{5,11}X_{11,8}S_{86}P_{6,12}X_{12,7}X_{7,10}R_{10,5}$	1	1	1

Table 69. The generators in terms of bifundamental chiral fields for Model 17 (*Part 1*).

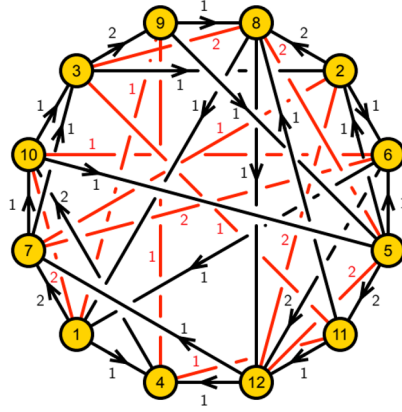


Figure 44. Quiver for Model 18.

The J - and E -term charges are given by

$$Q_{JE} = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & s_{10} & s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} & s_{17} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} & e_{11} & e_{12} & e_{13} & e_{14} & e_{15} & e_{16} & e_{17} & e_{18} & e_{19} & e_{20} & e_{21} & e_{22} & e_{23} & e_{24} & e_{25} & e_{26} & e_{27} & e_{28} & e_{29} & e_{30} & e_{31} & e_{32} & e_{33} & e_{34} & e_{35} & e_{36} & e_{37} & e_{38} & e_{39} \end{pmatrix}, \tag{21.3}$$

and the D -term charges are given by

$$Q_D = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & s_{10} & s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} & s_{17} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} & e_{11} & e_{12} & e_{13} & e_{14} & e_{15} & e_{16} & e_{17} & e_{18} & e_{19} & e_{20} & e_{21} & e_{22} & e_{23} & e_{24} & e_{25} & e_{26} & e_{27} & e_{28} & e_{29} & e_{30} & e_{31} & e_{32} & e_{33} & e_{34} & e_{35} & e_{36} & e_{37} & e_{38} & e_{39} \end{pmatrix}. \tag{21.4}$$

The toric diagram of Model 18 is given by

$$G_t = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & s_{10} & s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} & s_{17} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} & e_{11} & e_{12} & e_{13} & e_{14} & e_{15} & e_{16} & e_{17} & e_{18} & e_{19} & e_{20} & e_{21} & e_{22} & e_{23} & e_{24} & e_{25} & e_{26} & e_{27} & e_{28} & e_{29} & e_{30} & e_{31} & e_{32} & e_{33} & e_{34} & e_{35} & e_{36} & e_{37} & e_{38} & e_{39} \end{pmatrix}, \tag{21.5}$$

where figure 45 shows the toric diagram with brick matching labels.

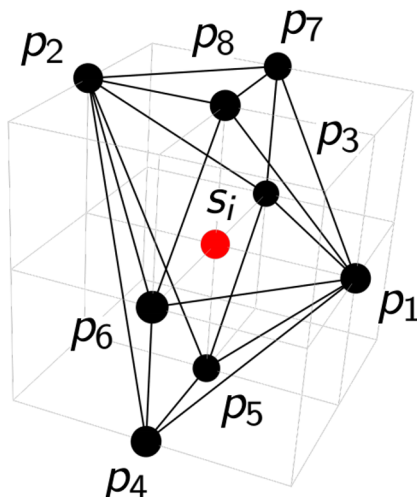


Figure 45. Toric diagram for Model 18.

The Hilbert series of the mesonic moduli space of Model 18 is

$$\begin{aligned}
 g_1(t_i, y_s, y_{o_1}, y_{o_2}, y_{o_3}, y_{o_4}, y_{o_5}, y_{o_6}, y_{o_7}; \mathcal{M}_{18}) &= \frac{P(t_i, y_s, y_{o_1}, y_{o_2}, y_{o_3}, y_{o_4}, y_{o_5}, y_{o_6}, y_{o_7}; \mathcal{M}_{18})}{(1 - y_s y_{o_1} y_{o_2}^3 y_{o_3}^2 y_{o_4}^2 y_{o_5} y_{o_6}^3 y_{o_7}^2 t_1 t_3 t_4^2 t_5^2 t_6)} \\
 &\times \frac{1}{(1 - y_s y_{o_1} y_{o_2}^3 y_{o_3}^2 y_{o_4}^2 y_{o_5} y_{o_6}^3 y_{o_7}^2 t_2 t_3 t_4^2 t_5^2 t_6)(1 - y_s y_{o_1}^2 y_{o_2}^2 y_{o_3}^3 y_{o_4} y_{o_5} y_{o_6}^3 y_{o_7}^2 t_1 t_3^2 t_4^2 t_5^2 t_7)} \\
 &\times \frac{1}{(1 - y_s y_{o_1}^2 y_{o_2}^2 y_{o_3}^3 y_{o_4} y_{o_5} y_{o_6}^3 y_{o_7}^2 t_2 t_3^2 t_4^2 t_5^2 t_7)(1 - y_s y_{o_1} y_{o_2}^3 y_{o_3}^3 y_{o_4}^2 y_{o_5} y_{o_6}^3 y_{o_7}^2 t_1^2 t_4^2 t_5^2 t_6 t_8)} \\
 &\times \frac{1}{(1 - y_s y_{o_1} y_{o_2}^3 y_{o_3}^3 y_{o_4}^2 y_{o_5} y_{o_6}^3 y_{o_7}^2 t_2^2 t_4^2 t_5^2 t_6 t_8)(1 - y_s y_{o_1}^3 y_{o_2} y_{o_3}^3 y_{o_4} y_{o_5} y_{o_6}^3 y_{o_7}^2 t_1^2 t_3^2 t_5^2 t_7 t_8)} \\
 &\times \frac{1}{(1 - y_s y_{o_1}^3 y_{o_2} y_{o_3}^3 y_{o_4}^2 y_{o_5} y_{o_6}^3 y_{o_7}^2 t_2^2 t_3^2 t_5^2 t_7 t_8)(1 - y_s y_{o_1}^2 y_{o_2}^2 y_{o_3}^3 y_{o_4}^2 y_{o_5} y_{o_6}^3 y_{o_7}^4 t_1^3 t_4^2 t_6^2 t_7 t_8)} \\
 &\times \frac{1}{(1 - y_s y_{o_1}^2 y_{o_2}^2 y_{o_3}^3 y_{o_4}^2 y_{o_5} y_{o_6}^3 y_{o_7}^4 t_2^3 t_4^2 t_6^2 t_7 t_8)(1 - y_s y_{o_1}^3 y_{o_2} y_{o_3}^2 y_{o_4}^2 y_{o_5} y_{o_6}^3 y_{o_7}^4 t_1^3 t_3 t_6^2 t_7^2 t_8)} \\
 &\times \frac{1}{(1 - y_s y_{o_1}^3 y_{o_2} y_{o_3}^2 y_{o_4}^2 y_{o_5} y_{o_6}^3 y_{o_7}^4 t_2^3 t_3 t_6^2 t_7^2 t_8)}, \tag{21.6}
 \end{aligned}$$

where t_i are the fugacities for the extremal brick matchings p_i . Furthermore, we have the fugacity y_s that counts the product brick matchings of the internal toric point $s_1 \dots s_{17}$. The fugacities y_{o_1}, \dots, y_{o_6} , count products of extra GLSM fields, which are $o_1 \dots o_5$, $o_6 \dots o_8$, $o_9 \dots o_{10}$, $o_{11} \dots o_{17}$, $o_{18} \dots o_{34}$ and $o_{35} \dots o_{39}$, respectively. We note that setting the fugacities $y_{o_1} = 1, \dots, y_{o_6} = 1$ does not change the overall characterization of the mesonic moduli space by the Hilbert series, indicating that the extra GLSM fields, as expected, correspond to an over-parameterization of the moduli space.

	SU(2) _x	U(1) _{b₁}	U(1) _{b₂}	U(1)	fugacity
p_1	+1	0	0	r_1	t_1
p_2	-1	0	0	r_2	t_2
p_3	0	+1	0	r_3	t_3
p_4	0	-1	0	r_4	t_4
p_5	0	0	+1	r_5	t_5
p_6	0	0	-1	r_6	t_6
p_7	0	0	0	r_7	t_7
p_8	0	0	0	r_8	t_8

Table 71. Global symmetry charges on the extremal brick matchings p_i of Model 18.

By setting $t_i = t$ for the fugacities of the extremal brick matchings, and all other fugacities to 1, the unrefined Hilbert series takes the following form

$$\begin{aligned}
 g_1(t, 1, 1, 1, 1, 1, 1, 1, 1; \mathcal{M}_{18}) &= \frac{(1-t)^4}{(1-t^7)^3(1-t^8)^2(1-t^9)^3} \times (1+4t+10t^2+20t^3 \\
 &+ 35t^4+56t^5+84t^6+121t^7+176t^8+263t^9+396t^{10}+589t^{11}+856t^{12}+1211t^{13} \\
 &+ 1668t^{14}+2228t^{15}+2873t^{16}+3572t^{17}+4294t^{18}+5008t^{19}+5683t^{20}+6288t^{21} \\
 &+ 6796t^{22}+7188t^{23}+7468t^{24}+7636t^{25}+7692t^{26}+7636t^{27}+7468t^{28}+7188t^{29} \\
 &+ 6796t^{30}+6288t^{31}+5683t^{32}+5008t^{33}+4294t^{34}+3572t^{35}+2873t^{36}+2228t^{37} \\
 &+ 1668t^{38}+1211t^{39}+856t^{40}+589t^{41}+396t^{42}+263t^{43}+176t^{44}+121t^{45}+84t^{46} \\
 &+ 56t^{47}+35t^{48}+20t^{49}+10t^{50}+4t^{51}+t^{52}), \tag{21.7}
 \end{aligned}$$

where the palindromic numerator indicates that the mesonic moduli space is Calabi-Yau.

The global symmetry of Model 18 and the charges on the extremal brick matchings under the global symmetry are summarized in table 71. We can use the following fugacity map,

$$t = t_7 = t_8, \quad x = \frac{t_8}{t_2}, \quad b_1 = \frac{t_8}{t_4}, \quad b_2 = \frac{t_5}{t_8}, \tag{21.8}$$

where $t_1 t_2 = t_3 t_4 = t_5 t_6 = t_8^2$, in order to rewrite the Hilbert series for Model 18 in terms of characters of irreducible representations of $SU(2) \times U(1) \times U(1)$.

The highest weight form of the Hilbert series of Model 18 is

$$\begin{aligned}
 h_1(t, \mu, b_1, b_2; \mathcal{M}_{18}) &= \frac{1 - \mu^2 t^8}{(1 - \mu^2 b_1^{-2} t^8)(1 - \mu^2 b_2^{-2} t^8)(1 - \mu^2 b_1^{-2} b_2^{-2} t^8)(1 - \mu^2 b_1^2 t^8)} \\
 &\times \frac{1}{(1 - \mu^2 b_2^2 t^8)(1 - \mu^2 b_1^2 b_2^2 t^8)} \times (1 + 2\mu^2 t^8 - \mu^4 b_1^{-2} t^{16} - \mu^4 b_1^2 t^{16} - \mu^4 b_2^{-2} t^{16} \\
 &- \mu^4 b_1^{-2} b_2^{-2} t^{16} - \mu^4 b_2^2 t^{16} - \mu^4 b_1^2 b_2^2 t^{16} + 2\mu^6 t^{24} + \mu^8 t^{32}), \tag{21.9}
 \end{aligned}$$

where $\mu^m \sim [m]_{SU(2)_x}$. Here in highest weight form, the fugacity μ counts the highest weight of irreducible representations of $SU(2)_x$. The fugacities b_1 and b_2 count charges under the two $U(1)$ factors of the mesonic flavor symmetry.

generator	$SU(2)_x$	$U(1)_{\tilde{b}_1}$	$U(1)_{\tilde{b}_2}$
$p_1 p_3 p_4^2 p_5^2 p_6$ $so_1 o_2^3 o_3^2 o_4^2 o_5 o_6^3 o_7^2 t^7$	1	-1	0
$p_2 p_3 p_4^2 p_5^2 p_6$ $so_1 o_2^3 o_3^2 o_4^2 o_5 o_6^3 o_7^2 t^7$	0	-1	0
$p_1 p_3^2 p_4 p_5^2 p_7$ $so_1^2 o_2^3 o_3^2 o_4 o_5 o_6^3 o_7^2 t^7$	1	0	1
$p_2 p_3^2 p_4 p_5^2 p_7$ $so_1^2 o_2^3 o_3^2 o_4 o_5 o_6^3 o_7^2 t^7$	0	0	1
$p_1^2 p_4^2 p_5 p_6^2 p_8$ $so_1 o_2^3 o_3 o_4^3 o_5^2 o_6^3 o_7^2 t^8$	1	-1	-1
$p_1 p_2 p_4^2 p_5 p_6^2 p_8$ $so_1 o_2^3 o_3 o_4^3 o_5^2 o_6^3 o_7^2 t^8$	0	-1	-1
$p_2^2 p_4^2 p_5 p_6^2 p_8$ $so_1 o_2^3 o_3 o_4^3 o_5^2 o_6^3 o_7^2 t^8$	-1	-1	-1
$p_1^2 p_3 p_4 p_5 p_6 p_7 p_8$ $so_1^2 o_2^3 o_3^2 o_4^2 o_5^2 o_6^3 o_7^2 t^8$	1	0	0
$p_1 p_2 p_3 p_4 p_5 p_6 p_7 p_8$ $so_1^2 o_2^3 o_3^2 o_4^2 o_5^2 o_6^3 o_7^2 t^8$	0	0	0
$p_2^2 p_3 p_4 p_5 p_6 p_7 p_8$ $so_1^2 o_2^3 o_3^2 o_4^2 o_5^2 o_6^3 o_7^2 t^8$	-1	0	0
$p_1^2 p_3^2 p_5 p_7 p_8$ $so_1^3 o_2 o_3^3 o_4 o_5^2 o_6^3 o_7^2 t^8$	1	1	1
$p_1 p_2 p_3^2 p_5 p_7 p_8$ $so_1^3 o_2 o_3^3 o_4 o_5^2 o_6^3 o_7^2 t^8$	0	1	1
$p_2^2 p_3^2 p_5 p_7 p_8$ $so_1^3 o_2 o_3^3 o_4 o_5^2 o_6^3 o_7^2 t^8$	-1	1	1
$p_1^3 p_4 p_6^2 p_7 p_8^2$ $so_1^2 o_2^3 o_3 o_4^3 o_5^3 o_6^4 o_7^2 t^9$	1	0	-1
$p_1^2 p_2 p_4 p_6^2 p_7 p_8^2$ $so_1^2 o_2^3 o_3 o_4^3 o_5^3 o_6^4 o_7^2 t^9$	0	0	-1
$p_1 p_2^2 p_4 p_6^2 p_7 p_8^2$ $so_1^2 o_2^3 o_3 o_4^3 o_5^3 o_6^4 o_7^2 t^9$	-1	0	-1
$p_2^3 p_4 p_6^2 p_7 p_8^2$ $so_1^2 o_2^3 o_3 o_4^3 o_5^3 o_6^4 o_7^2 t^9$	-2	0	-1
$p_1^3 p_3 p_6 p_7 p_8^2$ $so_1^3 o_2 o_3^3 o_4^2 o_5^3 o_6^4 o_7^2 t^9$	1	1	0
$p_1^2 p_2 p_3 p_6 p_7 p_8^2$ $so_1^3 o_2 o_3^3 o_4^2 o_5^3 o_6^4 o_7^2 t^9$	0	1	0
$p_1 p_2^2 p_3 p_6 p_7 p_8^2$ $so_1^3 o_2 o_3^3 o_4^2 o_5^3 o_6^4 o_7^2 t^9$	-1	1	0
$p_2^3 p_3 p_6 p_7 p_8^2$ $so_1^3 o_2 o_3^3 o_4^2 o_5^3 o_6^4 o_7^2 t^9$	-2	1	0

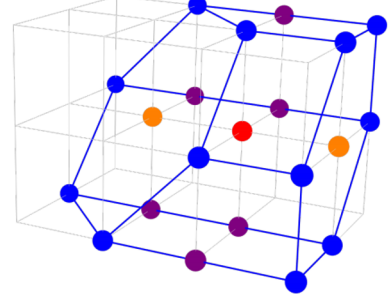


Table 72. The generators and lattice of generators of the mesonic moduli space of Model 18 in terms of brick matchings with the corresponding flavor charges.

The plethystic logarithm of the Hilbert series is

$$\begin{aligned}
 \text{PL}[g_1(t, x, b_1, b_2; \mathcal{M}_{18})] &= ([1]b_1^{-1}b_2 + [1]b_1b_2^2)t^7 + ([2] + [2]b_1^{-2}b_2^{-1} + [2]b_1^2b_2)t^8 \\
 &+ ([3]b_1^{-1}b_2^{-2} + [3]b_1b_2^{-1})t^9 - b_2^3t^{14} - ([1]b_1^{-3} + 2[1]b_1^{-1}b_2 + 2[1]b_1b_2^2 \\
 &+ [1]b_1^3b_2^3 + [3]b_1^{-1}b_2 + [3]b_1b_2^2)t^{15} - (2 + b_1^{-4}b_2^{-2} + b_1^{-2}b_2^{-1} + b_1^2b_2 + b_1^4b_2^2 \\
 &+ 3[2] + 2[2]b_1^{-2}b_2^{-1} + 2[2]b_1^2b_2 + 3[4] + [4]b_1^{-2}b_2^{-1} + [4]b_1^2b_2)t^{16} \\
 &- ([1]b_1^3 + [1]b_1^{-3}b_2^{-3} + 2[1]b_1^{-1}b_2^{-2} + 2[1]b_1b_2^{-1} + [3]b_1^3 + [3]b_1^{-3}b_2^{-3} \\
 &+ 2[3]b_1^{-1}b_2^{-2} + 2[3]b_1b_2^{-1} + [5]b_1^{-1}b_2^{-2} + [5]b_1b_2^{-1})t^{17} - (b_2^{-3} + [2]b_1^{-2}b_2^{-4} \\
 &+ [2]b_2^{-3} + [2]b_1^2b_2^{-2} + [4]b_2^{-3})t^{18} \dots, \tag{21.10}
 \end{aligned}$$

where $[m] = [m]_{SU(2)_x}$. From the plethystic logarithm, we see that the mesonic moduli space is a non-complete intersection.

By using the following fugacity map

$$\tilde{t} = t_7 = t_8, \quad \tilde{x} = \frac{t_8^2}{t_2^2}, \quad \tilde{b}_1 = \frac{t_6 t_8}{t_2 t_4}, \quad \tilde{b}_2 = \frac{t_2 t_8^2}{t_4 t_6^2}, \tag{21.11}$$

where $t_1 t_2 = t_3 t_4 = t_5 t_6 = t_8^2$, the mesonic flavor charges on the gauge invariant operators become \mathbb{Z} -valued. The generators in terms of brick matchings and their corresponding rescaled mesonic flavor charges are summarized in table 72. The generator lattice as shown

generator	$SU(2)_x$	$U(1)_{b_1}$	$U(1)_{b_2}$
$P_{17}X_{73}X_{32}X_{26}X_{61} = P_{28}X_{8,12}X_{12,7}X_{73}X_{32} = P_{39}X_{98}X_{8,12}X_{12,7}X_{73} = X_{26}P_{6,12}X_{12,7}X_{73}X_{32}$ $= P_{17}X_{7,10}X_{10,5}X_{52}X_{26}X_{61} = P_{28}X_{8,12}X_{12,7}X_{7,10}X_{10,5}X_{52} = X_{14}P_{4,10}X_{10,5}X_{52}X_{26}X_{61} = P_{5,11}X_{11,8}X_{8,12}X_{12,7}X_{7,10}X_{10,5}$ $= X_{26}P_{6,12}X_{12,7}X_{7,10}X_{10,5}X_{52}$	1	-1	0
$Q_{17}X_{73}X_{32}X_{26}X_{61} = Q_{28}X_{8,12}X_{12,7}X_{73}X_{32} = Q_{39}X_{98}X_{8,12}X_{12,7}X_{73} = X_{26}Q_{6,12}X_{12,7}X_{73}X_{32}$ $= Q_{17}X_{7,10}X_{10,5}X_{52}X_{26}X_{61} = Q_{28}X_{8,12}X_{12,7}X_{7,10}X_{10,5}X_{52} = X_{14}Q_{4,10}X_{10,5}X_{52}X_{26}X_{61} = Q_{5,11}X_{11,8}X_{8,12}X_{12,7}X_{7,10}X_{10,5}$ $= X_{26}Q_{6,12}X_{12,7}X_{7,10}X_{10,5}X_{52}$	0	-1	0
$P_{17}X_{7,10}X_{10,5}X_{56}X_{61} = X_{14}P_{4,10}X_{10,5}X_{56}X_{61} = P_{5,11}X_{11,12}X_{12,7}X_{7,10}X_{10,5} = X_{56}P_{6,12}X_{12,7}X_{7,10}X_{10,5}$ $= P_{17}X_{7,10}X_{10,3}X_{32}X_{26}X_{61} = P_{28}X_{8,12}X_{12,7}X_{7,10}X_{10,3}X_{32} = P_{39}X_{98}X_{8,12}X_{12,7}X_{7,10}X_{10,3} = X_{14}P_{4,10}X_{10,3}X_{32}X_{26}X_{61}$ $= X_{26}P_{6,12}X_{12,7}X_{7,10}X_{10,3}X_{32}$	1	0	1
$Q_{17}X_{7,10}X_{10,5}X_{56}X_{61} = X_{14}Q_{4,10}X_{10,5}X_{56}X_{61} = Q_{5,11}X_{11,12}X_{12,7}X_{7,10}X_{10,5} = X_{56}Q_{6,12}X_{12,7}X_{7,10}X_{10,5}$ $= Q_{17}X_{7,10}X_{10,3}X_{32}X_{26}X_{61} = Q_{28}X_{8,12}X_{12,7}X_{7,10}X_{10,3}X_{32} = Q_{39}X_{98}X_{8,12}X_{12,7}X_{7,10}X_{10,3} = X_{14}Q_{4,10}X_{10,3}X_{32}X_{26}X_{61}$ $= X_{26}Q_{6,12}X_{12,7}X_{7,10}X_{10,3}X_{32}$	0	0	1
$P_{28}X_{8,12}X_{12,4}P_{4,10}X_{10,5}X_{52} = P_{4,10}X_{10,5}P_{5,11}X_{11,8}X_{8,12}X_{12,4} = X_{26}P_{6,12}X_{12,4}P_{4,10}X_{10,5}X_{52}$ $= P_{17}X_{73}P_{39}X_{95}X_{52}X_{26}X_{61} = P_{28}X_{8,12}X_{12,7}X_{73}P_{39}X_{95}X_{52} = P_{39}X_{95}P_{5,11}X_{11,8}X_{8,12}X_{12,7}X_{73}$ $= X_{26}P_{6,12}X_{12,7}X_{73}P_{39}X_{95}X_{52}$	1	-1	-1
$P_{28}X_{8,12}X_{12,4}Q_{4,10}X_{10,5}X_{52} = Q_{28}X_{8,12}X_{12,4}P_{4,10}X_{10,5}X_{52} = P_{4,10}X_{10,5}Q_{5,11}X_{11,8}X_{8,12}X_{12,4}$ $= X_{26}Q_{6,12}X_{12,4}P_{4,10}X_{10,5}X_{52} = Q_{4,10}X_{10,5}P_{5,11}X_{11,8}X_{8,12}X_{12,4} = X_{26}P_{6,12}X_{12,4}Q_{4,10}X_{10,5}X_{52}$ $= P_{17}X_{73}Q_{39}X_{95}X_{52}X_{26}X_{61} = P_{28}X_{8,12}X_{12,7}X_{73}Q_{39}X_{95}X_{52} = Q_{17}X_{73}P_{39}X_{95}X_{52}X_{26}X_{61}$ $= Q_{28}X_{8,12}X_{12,7}X_{73}P_{39}X_{95}X_{52} = P_{39}X_{95}Q_{5,11}X_{11,8}X_{8,12}X_{12,7}X_{73} = X_{26}Q_{6,12}X_{12,7}X_{73}P_{39}X_{95}X_{52}$ $= Q_{39}X_{95}P_{5,11}X_{11,8}X_{8,12}X_{12,7}X_{73} = X_{26}P_{6,12}X_{12,7}X_{73}Q_{39}X_{95}X_{52}$	0	-1	-1
$Q_{28}X_{8,12}X_{12,4}Q_{4,10}X_{10,5}X_{52} = Q_{4,10}X_{10,5}Q_{5,11}X_{11,8}X_{8,12}X_{12,4} = X_{26}Q_{6,12}X_{12,4}Q_{4,10}X_{10,5}X_{52}$ $= Q_{17}X_{73}Q_{39}X_{95}X_{52}X_{26}X_{61} = Q_{28}X_{8,12}X_{12,7}X_{73}Q_{39}X_{95}X_{52} = Q_{39}X_{95}Q_{5,11}X_{11,8}X_{8,12}X_{12,7}X_{73}$ $= X_{26}Q_{6,12}X_{12,7}X_{73}Q_{39}X_{95}X_{52}$	-1	-1	-1

Table 73. The generators in terms of bifundamental chiral fields for Model 18 (*Part 1*).

in table 72 is a convex lattice polytope, which is reflexive. It is the dual of the toric diagram of Model 18 shown in figure 45. For completeness, table 73 and table 74 show the generators of Model 18 in terms of chiral fields with the corresponding mesonic flavor charges.

22 Conclusions

In this work, we have studied the 18 regular reflexive polytopes in dimension 3 as toric diagrams of Calabi-Yau 4-folds corresponding to $2d(0, 2)$ supersymmetric gauge theories. These $2d$ quiver gauge theories are realized by brane brick models. It is natural to search for brane brick models for each of the Calabi-Yau 4-folds associated to smooth Fano 3-folds and to investigate what detailed information about these gauge theories can be extracted from the underlying geometries. Guided by these questions, the following comprehensive results have been presented in this work:

- *Gauge theory identification.* For each of the 18 regular reflexive polytopes in 3 dimensions corresponding to toric Calabi-Yau 4-folds and smooth Fano 3-folds, we constructed a brane brick model realizing a $2d(0, 2)$ supersymmetric gauge theory. These $2d$ theories are worldvolume theories of D1-branes probing the Calabi-Yau singularities.
- *Moduli space computation.* The toric Calabi-Yau 4-folds are the mesonic moduli spaces of the $2d(0, 2)$ supersymmetric gauge theories realized by brane brick models. For each model, the generating function of mesonic gauge invariant operators, the Hilbert series, was calculated using the Molien integral formula. The fugacities of the Hilbert series can be chosen to count brick matchings of the brane brick model that correspond to GLSM fields as well as to points in the toric diagram of the Calabi-Yau

4-fold. Furthermore, the fugacities also refer to charges under the global symmetry of the $2d$ gauge theory.

- *Moduli space characterization.* For the 18 brane brick models under consideration, we expressed the generators of the mesonic moduli space both in terms of chiral fields of the $2d$ gauge theory as well as brick matchings. This is the first time the geometry of all toric Calabi-Yau 4-folds and smooth Fano 3-folds corresponding to the 18 regular reflexive polytopes has been associated to the moduli space generators of supersymmetric gauge theories.
- *Reflexive polytope duality.* The generators of the mesonic moduli space carry mesonic flavor symmetry charges. The mesonic charges carried by the generators can be represented as points on a \mathbb{Z}^3 lattice, similar to the points of the toric diagram Δ_3 of the Calabi-Yau 4-folds. The convex hull of all such points forms a convex polytope, which we call as the generator lattice of the mesonic moduli space. It turns out that for the brane brick models corresponding to the 18 toric Calabi-Yau 4-folds with regular reflexive polytopes as their toric diagrams, the generator lattice of the corresponding mesonic moduli spaces is another reflexive polytope. In fact, this work verified for all 18 brane brick models that the generator lattice of the corresponding mesonic moduli spaces is the polar reflexive dual of the toric diagram Δ_3 .

Overall, this work has considerably expanded our understanding of the correspondence between $2d$ $(0, 2)$ supersymmetric gauge theories, brane brick models and toric Calabi-Yau 4-folds. At the same time, it opens several avenues of future inquiry. We expect to be able to report on further related developments in the near future.

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A Numerators for the fully refined Hilbert series

A.1 Model 1

$$\begin{aligned}
 P(t_i, y_s; \mathcal{M}_1) = & 1 + y_s t_1^3 t_2 + y_s t_1^2 t_2^2 + y_s t_1 t_2^3 + y_s t_1^3 t_3 + y_s t_1^2 t_2 t_3 + y_s t_1 t_2^2 t_3 + y_s t_2^3 t_3 + y_s t_1^2 t_3^2 + y_s t_1 t_2 t_3^2 + \\
 & y_s t_2^2 t_3^2 + y_s^2 t_1^3 t_2^3 t_3 + y_s t_1 t_3^3 + y_s t_2 t_3^3 + y_s^2 t_1^3 t_2^3 t_3^2 + y_s^2 t_1^2 t_3^3 t_3^2 + y_s t_1^3 t_4 + y_s t_1^2 t_2 t_4 + y_s t_1 t_2^2 t_4 + y_s t_2^3 t_4 + \\
 & y_s t_1^2 t_3 t_4 + y_s t_1 t_2 t_3 t_4 + y_s t_2^2 t_3 t_4 + y_s^2 t_1^3 t_2^3 t_3 t_4 + y_s t_1 t_3^2 t_4 + y_s t_2 t_3^2 t_4 + y_s^2 t_1^3 t_2^2 t_3^2 t_4 + y_s^2 t_1^2 t_3^2 t_3^2 t_4 + y_s t_3^3 t_4 +
 \end{aligned}$$

$$\begin{aligned}
& y_s^6 y_{o_1}^{12} y_{o_2}^{12} t_1^7 t_2^6 t_3^6 t_4^6 t_5^6 + y_s^6 y_{o_1}^{12} y_{o_2}^{12} t_1^6 t_2^6 t_3^6 t_4^6 t_5^6 + y_s^6 y_{o_1}^{12} y_{o_2}^{12} t_1^5 t_2^6 t_3^6 t_4^6 t_5^6 + y_s^6 y_{o_1}^{12} y_{o_2}^{12} t_1^7 t_2^5 t_3^7 t_4^6 t_5^6 + \\
& y_s^6 y_{o_1}^{12} y_{o_2}^{12} t_1^6 t_2^5 t_3^7 t_4^6 t_5^6 + y_s^6 y_{o_1}^{12} y_{o_2}^{12} t_1^5 t_2^7 t_3^7 t_4^6 t_5^6 - y_s^5 y_{o_1}^8 y_{o_2}^{12} t_1^6 t_2^6 t_3^4 t_4^3 t_5^7 - y_s^5 y_{o_1}^8 y_{o_2}^{12} t_1^6 t_2^6 t_3^4 t_4^3 t_5^7 - \\
& y_s^5 y_{o_1}^8 y_{o_2}^{12} t_1^6 t_2^6 t_3^4 t_4^3 t_5^7 - y_s^5 y_{o_1}^8 y_{o_2}^{12} t_1^6 t_2^6 t_3^4 t_4^3 t_5^7 - y_s^5 y_{o_1}^8 y_{o_2}^{12} t_1^6 t_2^6 t_3^4 t_4^3 t_5^7 + y_s^6 y_{o_1}^{11} y_{o_2}^{13} t_1^7 t_2^6 t_3^5 t_4^5 t_5^7 + \\
& y_s^6 y_{o_1}^{11} y_{o_2}^{13} t_1^6 t_2^6 t_3^5 t_4^5 t_5^7 + y_s^6 y_{o_1}^{11} y_{o_2}^{13} t_1^6 t_2^6 t_3^5 t_4^5 t_5^7 + y_s^7 y_{o_1}^{14} y_{o_2}^{14} t_1^7 t_2^7 t_3^4 t_4^7 t_5^7,
\end{aligned}$$

where y_s counts $\prod_{a=1}^{11} s_a$, y_{o_1} counts $o_1 o_2$ and y_{o_2} counts $o_3 o_4$.

A.7 Model 7

$$\begin{aligned}
P(t_i, y_s, y_{o_1}, y_{o_2}, y_{o_3}; \mathcal{M}_7) = & 1 + y_s y_{o_1}^3 y_{o_2}^2 y_{o_3}^3 t_1^2 t_2^3 t_3^4 + y_s y_{o_1}^4 y_{o_2}^3 y_{o_3}^3 t_1 t_2^2 t_3^4 + y_s y_{o_1}^2 y_{o_2}^2 y_{o_3}^2 t_1^2 t_2 t_3 t_4 t_5 + \\
& y_s y_{o_1}^3 y_{o_2}^3 y_{o_3}^2 t_1 t_2^2 t_3 t_4 t_5 - y_s^2 y_{o_1}^4 y_{o_2}^3 y_{o_3}^5 t_1^5 t_2^3 t_3^4 t_5 + y_s y_{o_1}^4 y_{o_2}^4 y_{o_3}^2 t_2^3 t_3^4 t_5 - y_s^2 y_{o_1}^5 y_{o_2}^4 y_{o_3}^5 t_1^4 t_2^3 t_3^4 t_5 - \\
& y_s^2 y_{o_1}^6 y_{o_2}^5 y_{o_3}^3 t_1^3 t_2^3 t_3^4 t_5 + y_s y_{o_1}^2 y_{o_2}^2 y_{o_3}^3 t_1 t_2^2 t_4 t_5^2 - y_s^2 y_{o_1}^3 y_{o_2}^3 y_{o_3}^4 t_1^5 t_2^2 t_3^4 t_5^2 - 2 y_s^2 y_{o_1}^4 y_{o_2}^4 y_{o_3}^4 t_1^4 t_2^2 t_3^4 t_5^2 - \\
& 2 y_s^2 y_{o_1}^5 y_{o_2}^4 y_{o_3}^4 t_1^3 t_2^3 t_3^4 t_5^2 - y_s^2 y_{o_1}^6 y_{o_2}^4 y_{o_3}^4 t_1^2 t_2^4 t_3^4 t_5^2 - y_s^2 y_{o_1}^7 y_{o_2}^4 y_{o_3}^4 t_1 t_2^5 t_3^4 t_5^2 - y_s^2 y_{o_1}^3 y_{o_2}^4 y_{o_3}^3 t_1^4 t_2 t_3 t_4 t_5^3 - \\
& y_s^2 y_{o_1}^4 y_{o_2}^5 y_{o_3}^3 t_1^3 t_2^3 t_3^4 t_5^3 + y_s^3 y_{o_1}^5 y_{o_2}^6 y_{o_3}^3 t_1^7 t_2^3 t_3^4 t_5^3 - y_s^2 y_{o_1}^5 y_{o_2}^6 y_{o_3}^3 t_1^2 t_2^3 t_3^4 t_5^3 + y_s^3 y_{o_1}^6 y_{o_2}^6 y_{o_3}^3 t_1^6 t_2^3 t_3^4 t_5^3 + \\
& y_s^3 y_{o_1}^7 y_{o_2}^7 y_{o_3}^3 t_1^5 t_2^3 t_3^4 t_5^3 + y_s^3 y_{o_1}^6 y_{o_2}^7 y_{o_3}^3 t_1^5 t_2^3 t_3^4 t_5^3 + y_s^4 y_{o_1}^9 y_{o_2}^8 y_{o_3}^8 t_1^5 t_2^3 t_3^4 t_5^3 + y_s^4 y_{o_1}^{10} y_{o_2}^8 y_{o_3}^8 t_1^6 t_2^3 t_3^4 t_5^3 + \\
& y_s y_{o_1}^2 y_{o_2}^3 y_{o_3}^3 t_1^2 t_3 t_4 t_6 + y_s y_{o_1}^3 y_{o_2}^3 y_{o_3}^3 t_1 t_2 t_3^2 t_4 t_6 + y_s y_{o_1}^4 y_{o_2}^3 y_{o_3}^3 t_2^2 t_3 t_4 t_6 + y_s y_{o_1} y_{o_2} y_{o_3}^2 t_1^2 t_3 t_5 t_6 + \\
& y_s y_{o_1}^2 y_{o_2}^2 y_{o_3}^3 t_1 t_2 t_3 t_4 t_5 t_6 - y_s^2 y_{o_1}^3 y_{o_2}^2 y_{o_3}^3 t_1^3 t_3 t_4 t_5 t_6 + y_s y_{o_1}^3 y_{o_2}^3 y_{o_3}^2 t_2^2 t_3 t_4 t_5 t_6 - y_s^2 y_{o_1}^4 y_{o_2}^3 y_{o_3}^4 t_1^2 t_2^3 t_4 t_5 t_6 - \\
& y_s^2 y_{o_1}^5 y_{o_2}^4 y_{o_3}^3 t_1^2 t_2^3 t_3^4 t_5 t_6 + y_s y_{o_1} y_{o_2}^2 y_{o_3}^3 t_1 t_2 t_5^2 t_6 + y_s y_{o_1}^2 y_{o_2}^3 y_{o_3}^3 t_2 t_4 t_5 t_6 - 2 y_s^2 y_{o_1}^3 y_{o_2}^4 y_{o_3}^4 t_1^2 t_2^3 t_4 t_5 t_6 - \\
& 2 y_s^2 y_{o_1}^4 y_{o_2}^4 y_{o_3}^4 t_1^3 t_2^2 t_3^4 t_5 t_6 - y_s^2 y_{o_1}^5 y_{o_2}^4 y_{o_3}^4 t_1^2 t_2^3 t_3^4 t_5 t_6 - y_s^2 y_{o_1}^6 y_{o_2}^4 y_{o_3}^4 t_1 t_2^4 t_3^4 t_5 t_6 - \\
& y_s^3 y_{o_1}^7 y_{o_2}^7 y_{o_3}^3 t_1^5 t_2^3 t_3^4 t_5 t_6 - y_s^2 y_{o_1}^7 y_{o_2}^7 y_{o_3}^4 t_1^5 t_2^3 t_3^4 t_5 t_6 - y_s^2 y_{o_1}^2 y_{o_2}^3 y_{o_3}^3 t_1^2 t_2 t_3 t_4 t_5^3 t_6 + \\
& y_s^3 y_{o_1}^4 y_{o_2}^6 y_{o_3}^3 t_1^7 t_2^3 t_3^4 t_5 t_6 - y_s^2 y_{o_1}^4 y_{o_2}^6 y_{o_3}^3 t_1^2 t_2^3 t_3^4 t_5 t_6 + y_s^3 y_{o_1}^5 y_{o_2}^6 y_{o_3}^3 t_1^6 t_2^3 t_3^4 t_5 t_6 + y_s^3 y_{o_1}^6 y_{o_2}^6 y_{o_3}^3 t_1^5 t_2^3 t_3^4 t_5 t_6 + \\
& y_s^4 y_{o_1}^8 y_{o_2}^8 y_{o_3}^3 t_1^7 t_2^3 t_3^4 t_5 t_6 + y_s^4 y_{o_1}^9 y_{o_2}^8 y_{o_3}^3 t_1^6 t_2^3 t_3^4 t_5 t_6 + y_s^4 y_{o_1}^{10} y_{o_2}^8 y_{o_3}^3 t_1^5 t_2^3 t_3^4 t_5 t_6 + y_s y_{o_1}^2 y_{o_2}^3 y_{o_3}^3 t_1 t_2 t_4 t_6^2 + \\
& y_s y_{o_1}^3 y_{o_2}^3 y_{o_3}^3 t_2 t_3 t_4 t_6^2 + y_s^2 y_{o_1}^6 y_{o_2}^6 y_{o_3}^3 t_1^2 t_2^3 t_4 t_6^2 + y_s y_{o_1} y_{o_2} y_{o_3}^2 t_1 t_3 t_5 t_6^2 + y_s y_{o_1}^2 y_{o_2}^2 y_{o_3}^2 t_2 t_3 t_4 t_5 t_6^2 - \\
& y_s^2 y_{o_1}^3 y_{o_2}^3 y_{o_3}^4 t_1^3 t_3 t_4 t_5 t_6^2 - y_s^2 y_{o_1}^4 y_{o_2}^3 y_{o_3}^5 t_1^2 t_2^3 t_3^4 t_5 t_6^2 + y_s^2 y_{o_1}^5 y_{o_2}^3 y_{o_3}^5 t_1^2 t_2^3 t_3^4 t_5 t_6^2 - y_s^3 y_{o_1}^7 y_{o_2}^8 y_{o_3}^8 t_1^5 t_2^3 t_3^4 t_5 t_6^2 - \\
& 2 y_s^2 y_{o_1}^3 y_{o_2}^4 y_{o_3}^4 t_1^3 t_2^2 t_3^4 t_5 t_6^2 - y_s^2 y_{o_1}^4 y_{o_2}^4 y_{o_3}^4 t_1^2 t_2^2 t_3^4 t_5 t_6^2 - y_s^2 y_{o_1}^5 y_{o_2}^4 y_{o_3}^4 t_1 t_2^3 t_3^4 t_5 t_6^2 - y_s^3 y_{o_1}^6 y_{o_2}^5 y_{o_3}^5 t_1^5 t_2^3 t_3^4 t_5 t_6^2 - \\
& y_s^2 y_{o_1}^6 y_{o_2}^6 y_{o_3}^4 t_1^2 t_2^3 t_4 t_5 t_6^2 - 2 y_s^3 y_{o_1}^7 y_{o_2}^6 y_{o_3}^7 t_1^4 t_2^3 t_4 t_5 t_6^2 - y_s^3 y_{o_1}^8 y_{o_2}^7 t_1^5 t_2^3 t_4 t_5 t_6^2 - y_s^2 y_{o_1}^2 y_{o_2}^3 y_{o_3}^3 t_3^3 t_2 t_3 t_5 t_6^2 - \\
& y_s^2 y_{o_1}^3 y_{o_2}^4 y_{o_3}^3 t_1^2 t_2^3 t_3 t_4 t_5 t_6^2 + y_s^3 y_{o_1}^4 y_{o_2}^6 y_{o_3}^3 t_1^6 t_2^3 t_3 t_4 t_5 t_6^2 - y_s^2 y_{o_1}^4 y_{o_2}^6 y_{o_3}^3 t_1 t_2^3 t_3 t_4 t_5 t_6^2 + y_s^3 y_{o_1}^5 y_{o_2}^6 y_{o_3}^3 t_1^5 t_2^3 t_3 t_4 t_5 t_6^2 - \\
& y_s^2 y_{o_1}^5 y_{o_2}^6 y_{o_3}^3 t_1^2 t_2^3 t_3 t_4 t_5 t_6^2 + y_s^3 y_{o_1}^7 y_{o_2}^7 y_{o_3}^3 t_1^7 t_2^3 t_3 t_4 t_5 t_6^2 + y_s^4 y_{o_1}^8 y_{o_2}^8 y_{o_3}^3 t_1^6 t_2^3 t_3 t_4 t_5 t_6^2 + 2 y_s^4 y_{o_1}^{10} y_{o_2}^8 y_{o_3}^3 t_1^5 t_2^3 t_3 t_4 t_5 t_6^2 + \\
& y_s^3 y_{o_1}^7 y_{o_2}^8 y_{o_3}^3 t_1^2 t_2^3 t_4 t_5 t_6^2 + y_s^3 y_{o_1}^8 y_{o_2}^8 y_{o_3}^3 t_1^6 t_2^3 t_4 t_5 t_6^2 + y_s^4 y_{o_1}^9 y_{o_2}^8 y_{o_3}^3 t_1^5 t_2^3 t_4 t_5 t_6^2 + y_s^4 y_{o_1}^{10} y_{o_2}^8 y_{o_3}^3 t_1^4 t_2^3 t_4 t_5 t_6^2 + \\
& y_s^3 y_{o_1}^7 y_{o_2}^8 y_{o_3}^3 t_1^2 t_2^3 t_4 t_5 t_6^2 + y_s^3 y_{o_1}^8 y_{o_2}^8 y_{o_3}^3 t_1^6 t_2^3 t_4 t_5 t_6^2 - y_s^4 y_{o_1}^9 y_{o_2}^8 y_{o_3}^3 t_1^5 t_2^3 t_4 t_5 t_6^2 - y_s^5 y_{o_1}^{10} y_{o_2}^{11} y_{o_3}^9 t_1^6 t_2^3 t_4 t_5 t_6^2 - \\
& y_s^5 y_{o_1}^{11} y_{o_2}^{12} y_{o_3}^9 t_1^6 t_2^3 t_4 t_5 t_6^2 - y_s^2 y_{o_1}^3 y_{o_2}^2 y_{o_3}^3 t_1^3 t_3 t_4 t_5 t_6^3 - y_s^2 y_{o_1}^4 y_{o_2}^3 y_{o_3}^3 t_1^2 t_2^3 t_4 t_5 t_6^3 - y_s^2 y_{o_1}^5 y_{o_2}^4 y_{o_3}^3 t_1 t_2^3 t_4 t_5 t_6^3 - \\
& y_s^2 y_{o_1}^6 y_{o_2}^5 y_{o_3}^3 t_2^3 t_3 t_4 t_5 t_6^3 + y_s^3 y_{o_1}^8 y_{o_2}^8 y_{o_3}^3 t_3^3 t_3^5 t_5 t_6^3 - 2 y_s^2 y_{o_1}^3 y_{o_2}^3 y_{o_3}^4 t_1^2 t_2^2 t_4 t_5 t_6^3 - 2 y_s^2 y_{o_1}^4 y_{o_2}^4 y_{o_3}^4 t_1 t_2^2 t_2^2 t_5 t_6^3 + \\
& y_s^3 y_{o_1}^5 y_{o_2}^7 y_{o_3}^3 t_1^5 t_2 t_3 t_4 t_5 t_6^3 - 2 y_s^2 y_{o_1}^5 y_{o_2}^7 y_{o_3}^3 t_1^2 t_2^3 t_4 t_5 t_6^3 + y_s^3 y_{o_1}^6 y_{o_2}^7 y_{o_3}^3 t_1^6 t_2^3 t_4 t_5 t_6^3 + y_s^3 y_{o_1}^7 y_{o_2}^7 y_{o_3}^3 t_1^5 t_2^3 t_4 t_5 t_6^3 - \\
& y_s^2 y_{o_1}^2 y_{o_2}^3 y_{o_3}^3 t_1^2 t_2 t_3 t_5 t_6^3 - y_s^2 y_{o_1}^3 y_{o_2}^3 y_{o_3}^4 t_1 t_2 t_3 t_4 t_5 t_6^3 + 2 y_s^3 y_{o_1}^4 y_{o_2}^6 y_{o_3}^3 t_1^5 t_2 t_3 t_4 t_5 t_6^3 - y_s^2 y_{o_1}^4 y_{o_2}^5 y_{o_3}^3 t_3^2 t_3 t_4 t_5 t_6^3 + \\
& 2 y_s^3 y_{o_1}^5 y_{o_2}^6 y_{o_3}^3 t_1^4 t_2^3 t_3^2 t_4 t_5 t_6^3 + 3 y_s^3 y_{o_1}^6 y_{o_2}^6 y_{o_3}^3 t_1^3 t_3^3 t_3^3 t_4 t_5 t_6^3 + y_s^3 y_{o_1}^7 y_{o_2}^6 y_{o_3}^3 t_1^2 t_4 t_3^4 t_4 t_5 t_6^3 + y_s^4 y_{o_1}^9 y_{o_2}^8 y_{o_3}^3 t_1^4 t_5^3 t_4 t_5 t_6^3 + \\
& y_s^3 y_{o_1}^4 y_{o_2}^5 y_{o_3}^3 t_1^4 t_2^2 t_4 t_4 t_5 t_6^3 + y_s^3 y_{o_1}^5 y_{o_2}^5 y_{o_3}^3 t_3^3 t_3^2 t_2^2 t_4 t_5 t_6^3 + 2 y_s^3 y_{o_1}^6 y_{o_2}^7 y_{o_3}^5 t_1^2 t_4 t_2^3 t_4 t_5 t_6^3 + 2 y_s^4 y_{o_1}^9 y_{o_2}^8 y_{o_3}^8 t_1^4 t_5^3 t_4 t_5 t_6^3 + \\
& 2 y_s^4 y_{o_1}^{10} y_{o_2}^8 y_{o_3}^8 t_1^3 t_6 t_4 t_4 t_5 t_6^3 - y_s^5 y_{o_1}^{11} y_{o_2}^{10} y_{o_3}^{11} t_1^7 t_2^5 t_3^6 t_4 t_5 t_6^3 - y_s^5 y_{o_1}^{12} y_{o_2}^{11} y_{o_3}^{11} t_1^6 t_2^6 t_3^6 t_4 t_5 t_6^3 + y_s^3 y_{o_1}^5 y_{o_2}^7 y_{o_3}^4 t_1^2 t_2 t_3 t_4 t_5 t_6^3 - \\
& 2 y_s^4 y_{o_1}^7 y_{o_2}^8 y_{o_3}^7 t_1^5 t_2^3 t_3^4 t_5 t_6^3 + y_s^4 y_{o_1}^8 y_{o_2}^7 t_1^4 t_2^3 t_3^4 t_5 t_6^3 + y_s^4 y_{o_1}^9 y_{o_2}^7 t_1^3 t_6 t_3^5 t_4 t_5 t_6^3 - 2 y_s^5 y_{o_1}^{10} y_{o_2}^{10} y_{o_3}^{10} t_1^7 t_2^5 t_3^5 t_4 t_5 t_6^3 - \\
& 2 y_s^5 y_{o_1}^{11} y_{o_2}^{10} y_{o_3}^{10} t_1^6 t_2^6 t_3^4 t_5 t_6^3 - y_s^5 y_{o_1}^{10} y_{o_2}^9 y_{o_3}^9 t_1^6 t_2^6 t_3^4 t_5 t_6^3 - y_s^5 y_{o_1}^{11} y_{o_2}^9 y_{o_3}^9 t_1^5 t_2^6 t_3^4 t_5 t_6^3 - y_s^5 y_{o_1}^{12} y_{o_2}^9 y_{o_3}^9 t_1^4 t_2^6 t_3^4 t_5 t_6^3 - \\
& y_s^2 y_{o_1}^3 y_{o_2}^2 y_{o_3}^3 t_1^2 t_3 t_4 t_5 t_6^4 - y_s^2 y_{o_1}^4 y_{o_2}^3 y_{o_3}^3 t_1 t_2 t_3^2 t_4 t_5 t_6^4 - y_s^2 y_{o_1}^5 y_{o_2}^4 y_{o_3}^3 t_2^2 t_3^2 t_4 t_5 t_6^4 - 2 y_s^2 y_{o_1}^3 y_{o_2}^3 y_{o_3}^4 t_1 t_2 t_2^2 t_4 t_5 t_6^4 - \\
& 2 y_s^2 y_{o_1}^4 y_{o_2}^4 y_{o_3}^4 t_2^2 t_3^2 t_4 t_5 t_6^4 + y_s^3 y_{o_1}^5 y_{o_2}^4 y_{o_3}^4 t_1 t_2 t_3^2 t_4 t_5 t_6^4 + y_s^3 y_{o_1}^6 y_{o_2}^5 y_{o_3}^4 t_1^2 t_2 t_3^2 t_4 t_5 t_6^4 - 2 y_s^3 y_{o_1}^7 y_{o_2}^6 y_{o_3}^7 t_1^2 t_3^2 t_4 t_5 t_6^4 + \\
& y_s^4 y_{o_1}^9 y_{o_2}^8 y_{o_3}^8 t_1^5 t_2^3 t_5 t_4 t_5 t_6^4 - y_s^2 y_{o_1}^2 y_{o_2}^3 y_{o_3}^3 t_1 t_2 t_3 t_5 t_6^4 - y_s^2 y_{o_1}^3 y_{o_2}^3 y_{o_3}^4 t_2^2 t_3 t_4 t_5 t_6^4 + 2 y_s^3 y_{o_1}^4 y_{o_2}^6 y_{o_3}^3 t_1^4 t_2 t_3 t_4 t_5 t_6^4 +
\end{aligned}$$

$$\begin{aligned}
 &2y_s^3 y_{o_1}^5 y_{o_2}^5 y_{o_3}^6 t_1^3 t_2^3 t_3^4 t_4^5 t_6^4 + 2y_s^4 y_{o_1}^8 y_{o_2}^7 y_{o_3}^9 t_1^5 t_2^3 t_4^4 t_5^4 t_6^4 + y_s^4 y_{o_1}^9 y_{o_2}^8 y_{o_3}^9 t_1^4 t_2^4 t_3^5 t_4^5 t_6^4 + y_s^4 y_{o_1}^{10} y_{o_2}^9 y_{o_3}^9 t_1^3 t_2^5 t_3^4 t_4^5 t_6^4 + \\
 &y_s^3 y_{o_1}^4 y_{o_2}^5 y_{o_3}^5 t_1^3 t_2^4 t_3 t_4^4 t_5^4 + y_s^4 y_{o_1}^7 y_{o_2}^8 y_{o_3}^8 t_1^5 t_2^3 t_3^4 t_4^4 t_6^4 + 3y_s^4 y_{o_1}^8 y_{o_2}^8 y_{o_3}^8 t_1^4 t_2^4 t_3^4 t_4^4 t_5^4 + 2y_s^4 y_{o_1}^9 y_{o_2}^8 y_{o_3}^8 t_1^3 t_2^5 t_3^4 t_4^5 t_6^4 - \\
 &y_s^5 y_{o_1}^{10} y_{o_2}^9 y_{o_3}^9 t_1^4 t_2^5 t_3^4 t_4^5 t_6^4 + 2y_s^4 y_{o_1}^{10} y_{o_2}^{10} y_{o_3}^8 t_1^2 t_2^3 t_3^4 t_4^5 t_6^4 - y_s^5 y_{o_1}^{11} y_{o_2}^{10} y_{o_3}^{11} t_1^2 t_2^3 t_3^4 t_4^5 t_6^4 - y_s^5 y_{o_1}^{12} y_{o_2}^{11} y_{o_3}^{11} t_1^2 t_2^3 t_3^4 t_4^5 t_6^4 + \\
 &y_s^4 y_{o_1}^7 y_{o_2}^8 y_{o_3}^7 t_1^4 t_2^4 t_3^4 t_4^5 t_6^4 + y_s^4 y_{o_1}^8 y_{o_2}^9 y_{o_3}^7 t_1^5 t_2^3 t_3^4 t_4^5 t_6^4 - 2y_s^5 y_{o_1}^9 y_{o_2}^{10} y_{o_3}^{10} t_1^7 t_2^3 t_3^4 t_4^5 t_6^4 + y_s^4 y_{o_1}^9 y_{o_2}^{10} y_{o_3}^7 t_1^2 t_2^3 t_3^4 t_4^5 t_6^4 - \\
 &2y_s^5 y_{o_1}^{10} y_{o_2}^{10} y_{o_3}^{10} t_1^6 t_2^5 t_3^4 t_4^5 t_6^4 - 2y_s^5 y_{o_1}^{11} y_{o_2}^{10} t_1^5 t_2^3 t_3^4 t_4^5 t_6^4 + y_s^4 y_{o_1}^6 y_{o_2}^8 y_{o_3}^6 t_1^4 t_2^3 t_3^4 t_4^5 t_6^4 - \\
 &y_s^5 y_{o_1}^9 y_{o_2}^9 y_{o_3}^9 t_1^4 t_2^3 t_3^4 t_4^5 t_6^4 - y_s^5 y_{o_1}^{10} y_{o_2}^9 t_1^5 t_2^3 t_3^4 t_4^5 t_6^4 - y_s^5 y_{o_1}^{11} y_{o_2}^{12} y_{o_3}^9 t_1^4 t_2^3 t_3^4 t_4^5 t_6^4 - \\
 &y_s^2 y_{o_1}^3 y_{o_2}^2 y_{o_3}^2 t_1^3 t_2^3 t_4^5 t_6^4 - y_s^2 y_{o_1}^4 y_{o_2}^3 y_{o_3}^2 t_1^2 t_3^2 t_4^5 t_6^4 - y_s^3 y_{o_1}^7 y_{o_2}^8 y_{o_3}^8 t_1^2 t_2^3 t_4^5 t_6^4 - y_s^2 y_{o_1}^3 y_{o_2}^4 y_{o_3}^3 t_1^2 t_2 t_4^5 t_6^4 + \\
 &y_s^3 y_{o_1}^5 y_{o_2}^4 y_{o_3}^7 t_1^3 t_2^4 t_3^4 t_5^4 t_6^4 - y_s^3 y_{o_1}^6 y_{o_2}^5 y_{o_3}^7 t_1^2 t_2^3 t_4^5 t_6^4 - y_s^3 y_{o_1}^7 y_{o_2}^6 y_{o_3}^7 t_1^3 t_2^3 t_4^5 t_6^4 + y_s^4 y_{o_1}^8 y_{o_2}^6 y_{o_3}^{10} t_1^5 t_2^2 t_3^4 t_4^5 t_6^4 + \\
 &y_s^4 y_{o_1}^9 y_{o_2}^7 y_{o_3}^{10} t_1^4 t_2^3 t_4^5 t_6^4 + 2y_s^3 y_{o_1}^4 y_{o_2}^6 y_{o_3}^6 t_1^2 t_2^3 t_3^4 t_5^4 t_6^4 + y_s^3 y_{o_1}^5 y_{o_2}^5 y_{o_3}^6 t_1^2 t_2^3 t_3^4 t_5^4 t_6^4 + y_s^3 y_{o_1}^6 y_{o_2}^5 y_{o_3}^6 t_1^2 t_2^3 t_3^4 t_5^4 t_6^4 + \\
 &y_s^4 y_{o_1}^7 y_{o_2}^9 y_{o_3}^9 t_1^5 t_2^2 t_3^4 t_5^4 t_6^4 + y_s^3 y_{o_1}^7 y_{o_2}^8 y_{o_3}^8 t_1^4 t_2^3 t_4^5 t_6^4 + 2y_s^4 y_{o_1}^8 y_{o_2}^7 y_{o_3}^8 t_1^4 t_2^3 t_4^5 t_6^4 + y_s^4 y_{o_1}^{10} y_{o_2}^9 y_{o_3}^8 t_1^2 t_2^5 t_3^4 t_5^4 t_6^4 + \\
 &y_s^3 y_{o_1}^4 y_{o_2}^5 y_{o_3}^5 t_1^2 t_2 t_3^4 t_4^5 t_6^4 - y_s^4 y_{o_1}^6 y_{o_2}^6 y_{o_3}^8 t_1^5 t_2^2 t_3^4 t_4^5 t_6^4 + y_s^3 y_{o_1}^7 y_{o_2}^6 y_{o_3}^8 t_1^2 t_2^3 t_3^4 t_4^5 t_6^4 + y_s^4 y_{o_1}^7 y_{o_2}^7 y_{o_3}^8 t_1^2 t_2^3 t_3^4 t_4^5 t_6^4 - \\
 &y_s^5 y_{o_1}^9 y_{o_2}^8 y_{o_3}^{11} t_1^7 t_2^3 t_4^4 t_5^4 t_6^4 + y_s^4 y_{o_1}^9 y_{o_2}^9 y_{o_3}^8 t_1^2 t_2^3 t_4^5 t_6^4 - y_s^5 y_{o_1}^{10} y_{o_2}^9 y_{o_3}^{11} t_1^6 t_2^3 t_4^5 t_6^4 + y_s^4 y_{o_1}^{10} y_{o_2}^{10} y_{o_3}^8 t_1^2 t_2^3 t_4^4 t_5^4 t_6^4 - \\
 &y_s^5 y_{o_1}^{11} y_{o_2}^{10} y_{o_3}^{11} t_1^5 t_2^3 t_4^4 t_5^4 t_6^4 - y_s^5 y_{o_1}^{12} y_{o_2}^{11} y_{o_3}^{11} t_1^4 t_2^3 t_4^4 t_5^4 t_6^4 - y_s^4 y_{o_1}^5 y_{o_2}^6 y_{o_3}^7 t_1^2 t_2^3 t_4^5 t_6^4 - 2y_s^4 y_{o_1}^7 y_{o_2}^8 y_{o_3}^8 t_1^2 t_2^3 t_3^4 t_4^5 t_6^4 - \\
 &y_s^5 y_{o_1}^8 y_{o_2}^8 y_{o_3}^9 t_1^7 t_2^3 t_4^5 t_6^4 - y_s^4 y_{o_1}^8 y_{o_2}^9 y_{o_3}^7 t_1^2 t_2^3 t_4^5 t_6^4 - y_s^5 y_{o_1}^9 y_{o_2}^9 y_{o_3}^{10} t_1^6 t_2^3 t_4^5 t_6^4 - y_s^5 y_{o_1}^{10} y_{o_2}^{10} y_{o_3}^{10} t_1^5 t_2^3 t_4^5 t_6^4 - \\
 &2y_s^5 y_{o_1}^{11} y_{o_2}^{10} y_{o_3}^{11} t_1^4 t_2^3 t_4^5 t_6^4 - y_s^4 y_{o_1}^7 y_{o_2}^8 y_{o_3}^8 t_1^2 t_2^3 t_4^5 t_6^4 + y_s^5 y_{o_1}^9 y_{o_2}^{10} y_{o_3}^9 t_1^5 t_2^3 t_4^5 t_6^4 - y_s^5 y_{o_1}^{10} y_{o_2}^{11} y_{o_3}^{11} t_1^4 t_2^3 t_4^5 t_6^4 - \\
 &2y_s^5 y_{o_1}^{11} y_{o_2}^{11} y_{o_3}^{11} t_1^3 t_2^3 t_4^5 t_6^4 + y_s^6 y_{o_1}^{12} y_{o_2}^{12} y_{o_3}^{12} t_1^6 t_2^3 t_4^5 t_6^4 + y_s^6 y_{o_1}^{13} y_{o_2}^{13} y_{o_3}^{12} t_1^6 t_2^3 t_4^5 t_6^4 + y_s^5 y_{o_1}^8 y_{o_2}^{10} y_{o_3}^8 t_1^5 t_2^3 t_3^4 t_4^5 t_6^4 + \\
 &y_s^6 y_{o_1}^{11} y_{o_2}^{12} y_{o_3}^{11} t_1^7 t_2^3 t_4^5 t_6^4 + y_s^6 y_{o_1}^{12} y_{o_2}^{13} y_{o_3}^{11} t_1^6 t_2^3 t_4^5 t_6^4 + y_s^3 y_{o_1}^4 y_{o_2}^6 y_{o_3}^6 t_1^2 t_2^3 t_4^5 t_6^4 + y_s^3 y_{o_1}^5 y_{o_2}^5 y_{o_3}^6 t_1^2 t_2^3 t_4^5 t_6^4 + \\
 &y_s^3 y_{o_1}^6 y_{o_2}^6 y_{o_3}^6 t_1^2 t_2^3 t_4^5 t_6^4 + y_s^4 y_{o_1}^8 y_{o_2}^8 y_{o_3}^8 t_1^2 t_2^3 t_4^5 t_6^4 + y_s^4 y_{o_1}^9 y_{o_2}^8 y_{o_3}^8 t_1^2 t_2^3 t_4^5 t_6^4 - y_s^5 y_{o_1}^9 y_{o_2}^9 y_{o_3}^{11} t_1^5 t_2^3 t_4^5 t_6^4 + \\
 &y_s^4 y_{o_1}^{10} y_{o_2}^{10} y_{o_3}^8 t_1^2 t_3^4 t_4^5 t_6^4 - y_s^5 y_{o_1}^{11} y_{o_2}^{10} y_{o_3}^{11} t_1^5 t_2^3 t_4^5 t_6^4 - y_s^5 y_{o_1}^{12} y_{o_2}^{11} y_{o_3}^{11} t_1^4 t_2^3 t_4^5 t_6^4 - y_s^5 y_{o_1}^7 y_{o_2}^8 y_{o_3}^8 t_1^2 t_2^3 t_4^5 t_6^4 - \\
 &y_s^4 y_{o_1}^7 y_{o_2}^8 y_{o_3}^7 t_1^2 t_2^3 t_4^5 t_6^4 - y_s^5 y_{o_1}^8 y_{o_2}^8 y_{o_3}^9 t_1^6 t_2^3 t_4^5 t_6^4 - y_s^5 y_{o_1}^9 y_{o_2}^9 y_{o_3}^{10} t_1^5 t_2^3 t_4^5 t_6^4 - 2y_s^5 y_{o_1}^{10} y_{o_2}^{10} y_{o_3}^{10} t_1^4 t_2^3 t_4^5 t_6^4 - \\
 &2y_s^5 y_{o_1}^{11} y_{o_2}^{11} y_{o_3}^{11} t_1^3 t_2^3 t_4^5 t_6^4 + y_s^6 y_{o_1}^{12} y_{o_2}^{12} y_{o_3}^{12} t_1^6 t_2^3 t_4^5 t_6^4 + y_s^6 y_{o_1}^{13} y_{o_2}^{13} t_1^6 t_2^3 t_4^5 t_6^4 - y_s^5 y_{o_1}^9 y_{o_2}^{10} y_{o_3}^9 t_1^5 t_2^3 t_4^5 t_6^4 - \\
 &2y_s^5 y_{o_1}^{10} y_{o_2}^{10} y_{o_3}^{10} t_1^4 t_2^3 t_4^5 t_6^4 - y_s^5 y_{o_1}^{11} y_{o_2}^{11} y_{o_3}^{11} t_1^3 t_2^3 t_4^5 t_6^4 + y_s^6 y_{o_1}^{12} y_{o_2}^{12} y_{o_3}^{12} t_1^6 t_2^3 t_4^5 t_6^4 + y_s^6 y_{o_1}^{13} y_{o_2}^{13} t_1^6 t_2^3 t_4^5 t_6^4 + \\
 &y_s^6 y_{o_1}^{13} y_{o_2}^{13} y_{o_3}^{12} t_1^6 t_2^3 t_4^5 t_6^4 + y_s^6 y_{o_1}^{10} y_{o_2}^{11} y_{o_3}^{11} t_1^5 t_2^3 t_4^5 t_6^4 + y_s^6 y_{o_1}^{11} y_{o_2}^{12} y_{o_3}^{11} t_1^6 t_2^3 t_4^5 t_6^4 + y_s^6 y_{o_1}^{12} y_{o_2}^{13} y_{o_3}^{11} t_1^6 t_2^3 t_4^5 t_6^4 + y_s^6 y_{o_1}^{13} y_{o_2}^{13} y_{o_3}^{12} t_1^6 t_2^3 t_4^5 t_6^4 + \\
 &y_s^3 y_{o_1}^4 y_{o_2}^6 y_{o_3}^6 t_1 t_2 t_3 t_4^5 t_6^4 + y_s^3 y_{o_1}^5 y_{o_2}^5 y_{o_3}^6 t_1^2 t_2^3 t_4^5 t_6^4 + y_s^4 y_{o_1}^8 y_{o_2}^7 y_{o_3}^8 t_1^2 t_2^3 t_4^5 t_6^4 + y_s^4 y_{o_1}^7 y_{o_2}^8 y_{o_3}^8 t_1^2 t_2^3 t_3^4 t_4^5 t_6^4 + \\
 &y_s^4 y_{o_1}^8 y_{o_2}^8 y_{o_3}^8 t_1^2 t_3^4 t_4^5 t_6^4 - y_s^5 y_{o_1}^9 y_{o_2}^8 y_{o_3}^{11} t_1^5 t_2^3 t_4^5 t_6^4 + y_s^4 y_{o_1}^9 y_{o_2}^8 y_{o_3}^8 t_1^2 t_2^3 t_4^5 t_6^4 - y_s^5 y_{o_1}^{10} y_{o_2}^9 y_{o_3}^{11} t_1^4 t_2^3 t_4^5 t_6^4 + \\
 &y_s^5 y_{o_1}^{11} y_{o_2}^{10} y_{o_3}^{11} t_1^3 t_2^3 t_4^5 t_6^4 - y_s^5 y_{o_1}^7 y_{o_2}^8 y_{o_3}^{10} t_1^2 t_2^3 t_4^5 t_6^4 - y_s^5 y_{o_1}^8 y_{o_2}^8 y_{o_3}^{10} t_1^5 t_2^3 t_4^5 t_6^4 - 2y_s^5 y_{o_1}^9 y_{o_2}^9 y_{o_3}^{10} t_1^4 t_2^3 t_4^5 t_6^4 - \\
 &2y_s^5 y_{o_1}^{10} y_{o_2}^{10} y_{o_3}^{10} t_1^3 t_2^3 t_4^5 t_6^4 - y_s^5 y_{o_1}^{11} y_{o_2}^{11} y_{o_3}^{11} t_1^2 t_2^3 t_4^5 t_6^4 + y_s^6 y_{o_1}^{12} y_{o_2}^{12} y_{o_3}^{12} t_1^6 t_2^3 t_4^5 t_6^4 - y_s^5 y_{o_1}^9 y_{o_2}^9 y_{o_3}^9 t_1^4 t_2^3 t_4^5 t_6^4 - \\
 &y_s^5 y_{o_1}^9 y_{o_2}^9 y_{o_3}^9 t_1^4 t_2^3 t_4^5 t_6^4 + y_s^6 y_{o_1}^{10} y_{o_2}^{10} y_{o_3}^{10} t_1^6 t_2^3 t_4^5 t_6^4 + y_s^6 y_{o_1}^{11} y_{o_2}^{11} y_{o_3}^{11} t_1^6 t_2^3 t_4^5 t_6^4 + y_s^6 y_{o_1}^{12} y_{o_2}^{12} y_{o_3}^{12} t_1^6 t_2^3 t_4^5 t_6^4 + \\
 &y_s^6 y_{o_1}^{13} y_{o_2}^{13} y_{o_3}^{12} t_1^6 t_2^3 t_4^5 t_6^4 + y_s^7 y_{o_1}^{14} y_{o_2}^{14} t_1^7 t_2^3 t_4^5 t_6^4 + y_s^7 y_{o_1}^{14} y_{o_2}^{14} t_1^7 t_2^3 t_4^5 t_6^4,
 \end{aligned}$$

A.8 Model 8

$$\begin{aligned}
 P(t_i, y_s, y_{o_1}, y_{o_2}, y_{o_3}; \mathcal{M}_8) &= 1 + y_s y_{o_1} y_{o_2} y_{o_3}^2 t_1^2 t_2 t_4 t_6 + y_s y_{o_1} y_{o_2} y_{o_3}^2 t_1 t_2^2 t_4 t_6 + y_s y_{o_1}^2 y_{o_2}^2 y_{o_3}^2 t_1^2 t_3 t_4 t_5 t_6 + \\
 &y_s y_{o_1}^2 y_{o_2}^2 y_{o_3}^2 t_1 t_2 t_3 t_4 t_5 t_6 + y_s y_{o_1}^2 y_{o_2}^2 y_{o_3}^2 t_2^2 t_3 t_4 t_5 t_6 - y_s^2 y_{o_1}^3 y_{o_2}^3 y_{o_3}^3 t_1^2 t_3^4 t_5 t_6 - y_s^2 y_{o_1}^3 y_{o_2}^3 y_{o_3}^3 t_1^2 t_2^3 t_5 t_6 - \\
 &y_s^2 y_{o_1}^3 y_{o_2}^3 y_{o_3}^3 t_1^3 t_4^5 t_6 + y_s y_{o_1}^3 y_{o_2}^3 y_{o_3}^2 t_1^2 t_3 t_4^5 t_6 + y_s y_{o_1}^3 y_{o_2}^3 y_{o_3}^2 t_1^2 t_3 t_4^5 t_6 - y_s^2 y_{o_1}^4 y_{o_2}^4 y_{o_3}^4 t_1^3 t_3^4 t_5 t_6 - \\
 &2y_s^2 y_{o_1}^4 y_{o_2}^4 y_{o_3}^3 t_1^2 t_2 t_3 t_4^5 t_6 - 2y_s^2 y_{o_1}^4 y_{o_2}^4 y_{o_3}^3 t_1^2 t_2 t_3 t_4^5 t_6 - y_s^2 y_{o_1}^4 y_{o_2}^4 y_{o_3}^3 t_1^2 t_2 t_3 t_4^5 t_6 + y_s y_{o_1}^4 y_{o_2}^4 y_{o_3}^3 t_1^3 t_4^5 t_6 - \\
 &y_s^2 y_{o_1}^5 y_{o_2}^5 y_{o_3}^5 t_1^2 t_2^3 t_4^5 t_6 - 2y_s^2 y_{o_1}^5 y_{o_2}^5 y_{o_3}^5 t_1 t_2 t_3 t_4^5 t_6 - y_s^2 y_{o_1}^5 y_{o_2}^5 y_{o_3}^5 t_2^2 t_3 t_4^5 t_6 + y_s y_{o_1}^5 y_{o_2}^5 y_{o_3}^4 t_1^2 t_2 t_3 t_4^5 t_6 + \\
 &y_s^3 y_{o_1}^6 y_{o_2}^6 y_{o_3}^6 t_1^2 t_2^3 t_4^5 t_6 + y_s^3 y_{o_1}^6 y_{o_2}^6 y_{o_3}^6 t_1^2 t_3 t_4^5 t_6 - y_s^2 y_{o_1}^6 y_{o_2}^6 y_{o_3}^3 t_1^2 t_2 t_3 t_4^5 t_6 - y_s^2 y_{o_1}^6 y_{o_2}^6 y_{o_3}^3 t_2^3 t_3 t_4^5 t_6 + \\
 &y_s^3 y_{o_1}^7 y_{o_2}^7 y_{o_3}^7 t_1^2 t_2 t_3 t_4^5 t_6 + y_s^3 y_{o_1}^7 y_{o_2}^7 y_{o_3}^7 t_1^2 t_2 t_3 t_4^5 t_6 + y_s^3 y_{o_1}^8 y_{o_2}^8 y_{o_3}^8 t_1^2 t_2 t_3 t_4^5 t_6 + y_s y_{o_1} y_{o_2}^2 y_{o_3}^3 t_1^2 t_2 t_3 t_6 + \\
 &y_s y_{o_1} y_{o_2}^2 y_{o_3}^3 t_1^2 t_2 t_3 t_6 + y_s y_{o_1} y_{o_2}^2 y_{o_3}^3 t_1^2 t_3 t_2 t_6 + y_s y_{o_1}^2 y_{o_2}^3 y_{o_3}^3 t_1^3 t_2 t_5 t_6 + y_s y_{o_1}^2 y_{o_2}^3 y_{o_3}^3 t_1^2 t_2 t_3 t_5 t_6 + \\
 &y_s y_{o_1}^2 y_{o_2}^3 y_{o_3}^3 t_1^2 t_2^2 t_5 t_6 + y_s y_{o_1}^2 y_{o_2}^3 y_{o_3}^3 t_2^2 t_3 t_5 t_6 - y_s^2 y_{o_1}^3 y_{o_2}^3 y_{o_3}^3 t_1^3 t_2 t_3 t_5 t_6 - 2y_s^2 y_{o_1}^3 y_{o_2}^3 y_{o_3}^3 t_1^2 t_3 t_2 t_5 t_6 - \\
 &2y_s^2 y_{o_1}^3 y_{o_2}^3 y_{o_3}^3 t_1^2 t_2 t_3 t_5 t_6 - 2y_s^2 y_{o_1}^3 y_{o_2}^3 y_{o_3}^3 t_1^2 t_3 t_2 t_5 t_6 - 2y_s^2 y_{o_1}^3 y_{o_2}^3 y_{o_3}^3 t_1^2 t_3 t_2 t_5 t_6 - y_s^2 y_{o_1}^3 y_{o_2}^3 y_{o_3}^3 t_1^2 t_3 t_2 t_5 t_6 + \\
 &y_s y_{o_1}^3 y_{o_2}^4 y_{o_3}^4 t_1^2 t_3 t_2 t_6 + y_s y_{o_1}^3 y_{o_2}^4 y_{o_3}^4 t_1 t_2 t_3 t_2 t_6 + y_s y_{o_1}^3 y_{o_2}^4 y_{o_3}^4 t_2 t_3 t_2 t_6 - 2y_s^2 y_{o_1}^4 y_{o_2}^4 y_{o_3}^4 t_1^4 t_2^3 t_4^5 t_6 - \\
 &3y_s^2 y_{o_1}^4 y_{o_2}^4 y_{o_3}^4 t_1^2 t_2 t_3 t_4^5 t_6 - 2y_s^2 y_{o_1}^4 y_{o_2}^4 y_{o_3}^4 t_1^2 t_2 t_3 t_4^5 t_6 - 3y_s^2 y_{o_1}^4 y_{o_2}^4 y_{o_3}^4 t_1 t_2^2 t_3 t_4^5 t_6 - \\
 \end{aligned}$$

$$\begin{aligned}
 & y_s^6 y_{o_1}^{12} y_{o_2}^{18} t_1^5 t_2^5 t_3^6 t_4^6 t_5^6 + y_s^6 y_{o_1}^{12} y_{o_2}^{18} t_1^6 t_2^6 t_3^6 t_4^6 t_5^6 + y_s^6 y_{o_1}^{12} y_{o_2}^{18} t_1^5 t_2^7 t_3^6 t_4^6 t_5^6 + y_s^6 y_{o_1}^{12} y_{o_2}^{18} t_1^7 t_2^5 t_3^7 t_4^6 t_5^6 + \\
 & y_s^6 y_{o_1}^{12} y_{o_2}^{18} t_1^6 t_2^6 t_3^7 t_4^6 t_5^6 + y_s^6 y_{o_1}^{12} y_{o_2}^{18} t_1^5 t_2^7 t_3^7 t_4^6 t_5^6 - y_s^4 y_{o_1}^5 y_{o_2}^9 t_1^3 t_2^3 t_3^2 t_4^5 t_7 - y_s^4 y_{o_1}^5 y_{o_2}^9 t_1^2 t_3^3 t_3^2 t_4^5 t_7 - \\
 & y_s^4 y_{o_1}^5 y_{o_2}^9 t_1^3 t_2^3 t_3^2 t_4^5 t_7 - y_s^4 y_{o_1}^5 y_{o_2}^9 t_1^2 t_3^3 t_3^2 t_4^5 t_7 - y_s^5 y_{o_1}^8 y_{o_2}^{13} t_1^4 t_2^4 t_3^2 t_4^5 t_7 - y_s^5 y_{o_1}^8 y_{o_2}^{13} t_1^3 t_2^4 t_3^2 t_4^5 t_7 - \\
 & y_s^5 y_{o_1}^8 y_{o_2}^{13} t_1^2 t_3^4 t_4^5 t_7 - y_s^5 y_{o_1}^8 y_{o_2}^{13} t_1^4 t_2^4 t_3^4 t_4^5 t_7 - y_s^5 y_{o_1}^8 y_{o_2}^{13} t_1^3 t_2^4 t_3^4 t_4^5 t_7 - y_s^5 y_{o_1}^8 y_{o_2}^{13} t_1^2 t_3^4 t_4^5 t_7 - \\
 & y_s^5 y_{o_1}^8 y_{o_2}^{13} t_1^2 t_3^4 t_4^5 t_7 - y_s^5 y_{o_1}^8 y_{o_2}^{13} t_1^4 t_2^4 t_3^4 t_4^5 t_7 - y_s^5 y_{o_1}^8 y_{o_2}^{13} t_1^3 t_2^4 t_3^4 t_4^5 t_7 + y_s^6 y_{o_1}^{11} y_{o_2}^{17} t_1^6 t_2^5 t_3^4 t_4^5 t_7 + \\
 & y_s^6 y_{o_1}^{11} y_{o_2}^{17} t_1^5 t_2^6 t_3^4 t_4^5 t_7 + y_s^6 y_{o_1}^{11} y_{o_2}^{17} t_1^7 t_2^4 t_3^4 t_4^5 t_7 + y_s^6 y_{o_1}^{11} y_{o_2}^{17} t_1^6 t_2^5 t_3^4 t_4^5 t_7 + y_s^6 y_{o_1}^{11} y_{o_2}^{17} t_1^5 t_2^6 t_3^4 t_4^5 t_7 + \\
 & y_s^6 y_{o_1}^{11} y_{o_2}^{17} t_1^4 t_2^7 t_3^4 t_4^5 t_7 + y_s^6 y_{o_1}^{11} y_{o_2}^{17} t_1^7 t_2^4 t_3^4 t_4^5 t_7 + y_s^6 y_{o_1}^{11} y_{o_2}^{17} t_1^6 t_2^5 t_3^4 t_4^5 t_7 + y_s^6 y_{o_1}^{11} y_{o_2}^{17} t_1^5 t_2^6 t_3^4 t_4^5 t_7 + \\
 & y_s^6 y_{o_1}^{11} y_{o_2}^{17} t_1^4 t_2^7 t_3^4 t_4^5 t_7 + y_s^6 y_{o_1}^{11} y_{o_2}^{17} t_1^7 t_2^4 t_3^4 t_4^5 t_7 + y_s^7 y_{o_1}^{14} y_{o_2}^{21} t_1^7 t_2^3 t_3^7 t_4^5 t_7,
 \end{aligned}$$

A.10 Model 10

$$\begin{aligned}
 P(t_i, y_s, y_{o_1}, y_{o_2}; \mathcal{M}_{10}) = & 1 + y_s y_{o_1}^2 y_{o_2} t_1 t_2 t_3 t_4^2 + y_s y_{o_1}^3 y_{o_2}^2 t_1^2 t_3 t_4 t_5 + y_s y_{o_1}^3 y_{o_2}^2 t_1^2 t_2 t_3^2 t_4 t_5 + \\
 & y_s y_{o_1}^3 y_{o_2}^2 t_1^2 t_2^2 t_3 t_4 t_5 + y_s y_{o_1}^3 y_{o_2}^2 t_3^2 t_3 t_4 t_5 - y_s^2 y_{o_1}^5 y_{o_2}^3 t_1^2 t_2^3 t_3^2 t_5 - y_s^2 y_{o_1}^5 y_{o_2}^3 t_1^2 t_2^3 t_3^2 t_5 + y_s y_{o_1}^4 y_{o_2}^3 t_1 t_2 t_3^2 t_5^2 + \\
 & y_s y_{o_1}^4 y_{o_2}^3 t_1^2 t_2^2 t_3^2 t_5^2 + y_s y_{o_1}^4 y_{o_2}^3 t_1 t_2^3 t_3^2 t_5^2 - y_s^2 y_{o_1}^6 y_{o_2}^4 t_1^2 t_2 t_3^2 t_4^2 t_5^2 - y_s^2 y_{o_1}^6 y_{o_2}^4 t_1^2 t_2 t_3^2 t_4^2 t_5^2 - y_s^2 y_{o_1}^6 y_{o_2}^4 t_1^2 t_3^2 t_4^2 t_5^2 - \\
 & y_s^2 y_{o_1}^6 y_{o_2}^4 t_1^2 t_2^2 t_3^2 t_4^2 t_5^2 - y_s^2 y_{o_1}^6 y_{o_2}^4 t_1 t_2^5 t_3^2 t_4^2 t_5^2 - y_s^3 y_{o_1}^9 y_{o_2}^6 t_1^5 t_2^4 t_3^4 t_5^2 - y_s^3 y_{o_1}^9 y_{o_2}^6 t_1^4 t_2^5 t_3^4 t_5^2 - y_s^2 y_{o_1}^3 y_{o_2}^2 t_1^2 t_2 t_3 t_4^2 t_6 - \\
 & y_s^2 y_{o_1}^3 y_{o_2}^2 t_1^2 t_2 t_3 t_4^2 t_6 + y_s^2 y_{o_1}^3 y_{o_2}^2 t_1^2 t_3 t_4 t_5 t_6 + y_s^2 y_{o_1}^3 y_{o_2}^2 t_1 t_2 t_3 t_4 t_5 t_6 + y_s^2 y_{o_1}^3 y_{o_2}^2 t_2^2 t_3 t_4 t_5 t_6 - \\
 & y_s^2 y_{o_1}^4 y_{o_2}^3 t_1^3 t_2^3 t_3^2 t_5 t_6 - 2 y_s^2 y_{o_1}^4 y_{o_2}^3 t_1^3 t_2^2 t_3^2 t_3^2 t_5 t_6 - 3 y_s^2 y_{o_1}^4 y_{o_2}^3 t_1^2 t_2^2 t_3^2 t_3^2 t_5 t_6 - 2 y_s^2 y_{o_1}^4 y_{o_2}^3 t_1 t_2^3 t_3^2 t_5 t_6 - \\
 & y_s^2 y_{o_1}^4 y_{o_2}^3 t_1^2 t_2^3 t_3^2 t_5 t_6 + y_s^3 y_{o_1}^6 y_{o_2}^4 t_1^4 t_2^3 t_3^2 t_5 t_6 + 2 y_s^3 y_{o_1}^6 y_{o_2}^4 t_1^3 t_2^3 t_3^2 t_5 t_6 + y_s^3 y_{o_1}^6 y_{o_2}^4 t_1^2 t_2^3 t_3^2 t_5 t_6 + \\
 & y_s y_{o_1}^3 y_{o_2}^3 t_1^3 t_2^2 t_3^2 t_6 + y_s y_{o_1}^3 y_{o_2}^3 t_1^2 t_2^2 t_3^2 t_6 + y_s y_{o_1}^3 y_{o_2}^3 t_1 t_2^2 t_3^2 t_6 + y_s y_{o_1}^3 y_{o_2}^3 t_2^2 t_3^2 t_6 - y_s^2 y_{o_1}^5 y_{o_2}^4 t_1^5 t_2^3 t_4^2 t_5^2 t_6 - \\
 & 2 y_s^2 y_{o_1}^5 y_{o_2}^4 t_1^4 t_2^3 t_3^2 t_4^2 t_5^2 t_6 - 3 y_s^2 y_{o_1}^5 y_{o_2}^4 t_1^3 t_2^3 t_3^2 t_4^2 t_5^2 t_6 - 3 y_s^2 y_{o_1}^5 y_{o_2}^4 t_1^2 t_2^3 t_3^2 t_4^2 t_5^2 t_6 - 2 y_s^2 y_{o_1}^5 y_{o_2}^4 t_1 t_2^4 t_3^2 t_4^2 t_5^2 t_6 - \\
 & y_s^2 y_{o_1}^5 y_{o_2}^4 t_1^5 t_2^3 t_3^2 t_4^2 t_5^2 t_6 + y_s^3 y_{o_1}^7 y_{o_2}^5 t_1^6 t_2^2 t_3^4 t_4^2 t_5^2 t_6 + 2 y_s^3 y_{o_1}^7 y_{o_2}^5 t_1^5 t_2^2 t_3^4 t_4^2 t_5^2 t_6 + 2 y_s^3 y_{o_1}^7 y_{o_2}^5 t_1^4 t_2^3 t_3^4 t_4^2 t_5^2 t_6 + \\
 & 2 y_s^3 y_{o_1}^7 y_{o_2}^5 t_1^3 t_2^4 t_3^4 t_4^2 t_5^2 t_6 + 2 y_s^3 y_{o_1}^7 y_{o_2}^5 t_1^2 t_2^5 t_3^4 t_4^2 t_5^2 t_6 + y_s^3 y_{o_1}^7 y_{o_2}^5 t_1 t_2^6 t_3^4 t_4^2 t_5^2 t_6 + y_s^2 y_{o_1}^6 y_{o_2}^5 t_1^3 t_3^4 t_4^2 t_5^2 t_6 - \\
 & y_s^3 y_{o_1}^8 y_{o_2}^6 t_1^5 t_2^3 t_3^3 t_4^2 t_5^2 t_6 - y_s^3 y_{o_1}^8 y_{o_2}^6 t_1^4 t_2^4 t_3^3 t_4^2 t_5^2 t_6 - y_s^3 y_{o_1}^8 y_{o_2}^6 t_1^3 t_2^5 t_3^3 t_4^2 t_5^2 t_6 + y_s^4 y_{o_1}^{10} y_{o_2}^7 t_1^6 t_2^3 t_3^3 t_4^2 t_5^2 t_6 + \\
 & 2 y_s^4 y_{o_1}^{10} y_{o_2}^7 t_1^5 t_2^4 t_3^3 t_4^2 t_5^2 t_6 + y_s^4 y_{o_1}^{10} y_{o_2}^7 t_1^4 t_2^5 t_3^3 t_4^2 t_5^2 t_6 + y_s y_{o_1} y_{o_2}^2 t_1 t_4 t_5 t_6^2 + y_s y_{o_1} y_{o_2}^2 t_2 t_4 t_5 t_6^2 - y_s^2 y_{o_1}^3 y_{o_2}^3 t_1^3 t_3 t_4^3 t_5 t_6^2 - \\
 & 2 y_s^2 y_{o_1}^3 y_{o_2}^3 t_1^2 t_2 t_3 t_4^3 t_5 t_6^2 - 2 y_s^2 y_{o_1}^3 y_{o_2}^3 t_1 t_2^2 t_3 t_4^3 t_5 t_6^2 + y_s^3 y_{o_1}^5 y_{o_2}^4 t_1^3 t_2^3 t_3^2 t_4^2 t_5 t_6^2 + \\
 & 2 y_s^3 y_{o_1}^5 y_{o_2}^4 t_1^2 t_2^3 t_3^2 t_4^2 t_5 t_6^2 + 2 y_s^3 y_{o_1}^5 y_{o_2}^4 t_1 t_2^4 t_3^2 t_4^2 t_5 t_6^2 - y_s^4 y_{o_1}^7 y_{o_2}^5 t_1^4 t_2^3 t_3^2 t_4^2 t_5 t_6^2 - \\
 & y_s^4 y_{o_1}^7 y_{o_2}^5 t_1^3 t_2^4 t_3^2 t_4^2 t_5 t_6^2 + y_s y_{o_1}^2 y_{o_2}^3 t_1^2 t_3 t_5^2 t_6^2 + y_s y_{o_1}^2 y_{o_2}^3 t_1 t_2 t_3 t_5^2 t_6^2 + y_s y_{o_1}^2 y_{o_2}^3 t_2^2 t_3 t_5^2 t_6^2 - y_s^2 y_{o_1}^4 y_{o_2}^4 t_1^4 t_2^3 t_4^2 t_5^2 t_6^2 - \\
 & 2 y_s^2 y_{o_1}^4 y_{o_2}^4 t_1^3 t_2 t_3 t_4^2 t_5^2 t_6^2 - 3 y_s^2 y_{o_1}^4 y_{o_2}^4 t_1^2 t_2^2 t_3 t_4^2 t_5^2 t_6^2 - 2 y_s^2 y_{o_1}^4 y_{o_2}^4 t_1 t_2^3 t_4^2 t_5^2 t_6^2 - y_s^2 y_{o_1}^4 y_{o_2}^4 t_2^3 t_4^2 t_5^2 t_6^2 + \\
 & y_s^3 y_{o_1}^6 y_{o_2}^5 t_1^5 t_2^2 t_3^4 t_4^2 t_5^2 t_6^2 + 2 y_s^3 y_{o_1}^6 y_{o_2}^5 t_1^4 t_2^3 t_3^4 t_4^2 t_5^2 t_6^2 + 3 y_s^3 y_{o_1}^6 y_{o_2}^5 t_1^3 t_2^3 t_3^4 t_4^2 t_5^2 t_6^2 + 2 y_s^3 y_{o_1}^6 y_{o_2}^5 t_1^2 t_2^3 t_3^4 t_4^2 t_5^2 t_6^2 + \\
 & y_s^3 y_{o_1}^6 y_{o_2}^5 t_1 t_2^4 t_3^4 t_4^2 t_5^2 t_6^2 - y_s^4 y_{o_1}^8 y_{o_2}^6 t_1^5 t_2^3 t_4^2 t_5^2 t_6^2 - y_s^4 y_{o_1}^8 y_{o_2}^6 t_1^4 t_2^4 t_3^4 t_4^2 t_5^2 t_6^2 - y_s^4 y_{o_1}^8 y_{o_2}^6 t_1^3 t_2^5 t_3^4 t_4^2 t_5^2 t_6^2 + \\
 & y_s^2 y_{o_1}^5 y_{o_2}^5 t_1^2 t_2^3 t_3^4 t_4^2 t_5^2 t_6^2 + y_s^2 y_{o_1}^5 y_{o_2}^5 t_1 t_2^4 t_3^4 t_4^2 t_5^2 t_6^2 - y_s^3 y_{o_1}^7 y_{o_2}^6 t_1^5 t_2^3 t_4^2 t_5^2 t_6^2 - 2 y_s^3 y_{o_1}^7 y_{o_2}^6 t_1^4 t_2^4 t_3^4 t_4^2 t_5^2 t_6^2 - \\
 & 2 y_s^3 y_{o_1}^7 y_{o_2}^6 t_1^3 t_2^5 t_3^4 t_4^2 t_5^2 t_6^2 - y_s^3 y_{o_1}^7 y_{o_2}^6 t_1^2 t_2^6 t_3^4 t_4^2 t_5^2 t_6^2 + 2 y_s^4 y_{o_1}^9 y_{o_2}^7 t_1^6 t_2^3 t_3^5 t_4^2 t_5^2 t_6^2 + \\
 & 2 y_s^4 y_{o_1}^9 y_{o_2}^7 t_1^5 t_2^4 t_3^5 t_4^2 t_5^2 t_6^2 + y_s^4 y_{o_1}^9 y_{o_2}^7 t_1^4 t_2^5 t_3^5 t_4^2 t_5^2 t_6^2 - y_s^5 y_{o_1}^{11} y_{o_2}^8 t_1^6 t_2^3 t_4^2 t_5^2 t_6^2 - y_s^5 y_{o_1}^{11} y_{o_2}^8 t_1^5 t_2^4 t_3^4 t_4^2 t_5^2 t_6^2 + \\
 & y_s^2 y_{o_1}^6 y_{o_2}^6 t_1^3 t_2^4 t_4^2 t_5^2 t_6^2 - y_s^3 y_{o_1}^8 y_{o_2}^7 t_1^5 t_2^3 t_4^2 t_4^2 t_5^2 t_6^2 - y_s^3 y_{o_1}^8 y_{o_2}^7 t_1^4 t_2^4 t_5^2 t_4^2 t_5^2 t_6^2 - y_s^3 y_{o_1}^8 y_{o_2}^7 t_1^3 t_2^5 t_4^2 t_4^2 t_5^2 t_6^2 + \\
 & y_s^4 y_{o_1}^{10} y_{o_2}^8 t_1^6 t_4 t_5 t_6^2 + y_s^4 y_{o_1}^{10} y_{o_2}^8 t_1^5 t_2^5 t_6 t_4 t_5 t_6^2 + y_s^4 y_{o_1}^{10} y_{o_2}^8 t_1^4 t_2^6 t_6 t_4 t_5 t_6^2 - y_s^5 y_{o_1}^{12} y_{o_2}^9 t_1^6 t_2^3 t_4^2 t_5 t_6^2 - \\
 & y_s^2 y_{o_1}^2 y_{o_2}^3 t_1 t_2 t_3 t_5 t_6^3 + y_s^3 y_{o_1}^4 y_{o_2}^4 t_1^3 t_2 t_3 t_5^2 t_6^3 + y_s^3 y_{o_1}^4 y_{o_2}^4 t_1 t_2^3 t_3 t_5^2 t_6^3 - y_s^4 y_{o_1}^6 y_{o_2}^5 t_1^3 t_2^3 t_2^7 t_5 t_6^3 - \\
 & y_s^2 y_{o_1}^3 y_{o_2}^4 t_1^3 t_3 t_4^2 t_5^3 t_6^3 - 3 y_s^2 y_{o_1}^3 y_{o_2}^4 t_1^2 t_2 t_3 t_4^2 t_5^3 t_6^3 - 3 y_s^2 y_{o_1}^3 y_{o_2}^4 t_1 t_2^2 t_3 t_4^2 t_5^3 t_6^3 - y_s^2 y_{o_1}^3 y_{o_2}^4 t_2^3 t_3 t_2^2 t_5^3 t_6^3 + \\
 & y_s^3 y_{o_1}^5 y_{o_2}^5 t_1^4 t_2^3 t_4^2 t_5^3 t_6^3 + 2 y_s^3 y_{o_1}^5 y_{o_2}^5 t_1^3 t_2^2 t_3^2 t_4^2 t_5^3 t_6^3 + 2 y_s^3 y_{o_1}^5 y_{o_2}^5 t_1^2 t_2^3 t_4^2 t_5^3 t_6^3 + y_s^3 y_{o_1}^5 y_{o_2}^5 t_1 t_2^4 t_3^4 t_4^2 t_5^3 t_6^3 - \\
 & y_s^4 y_{o_1}^7 y_{o_2}^6 t_1^4 t_2^3 t_3^4 t_4^2 t_5^3 t_6^3 - y_s^4 y_{o_1}^7 y_{o_2}^6 t_1^3 t_2^4 t_3^4 t_4^2 t_5^3 t_6^3 - y_s^2 y_{o_1}^4 y_{o_2}^5 t_1^3 t_2 t_3 t_4^2 t_5^3 t_6^3 - y_s^2 y_{o_1}^4 y_{o_2}^5 t_1^2 t_2^3 t_4^2 t_5^3 t_6^3 - \\
 & y_s^2 y_{o_1}^4 y_{o_2}^5 t_1 t_2^4 t_3^4 t_4^2 t_5^3 t_6^3 - y_s^3 y_{o_1}^6 y_{o_2}^6 t_1^5 t_2^3 t_3^4 t_5^3 t_6^3 - y_s^3 y_{o_1}^6 y_{o_2}^6 t_1^4 t_2^4 t_3^4 t_5^3 t_6^3 - 2 y_s^3 y_{o_1}^6 y_{o_2}^6 t_1^3 t_2^5 t_3^4 t_5^3 t_6^3 - \\
 & y_s^3 y_{o_1}^6 y_{o_2}^6 t_1^2 t_3^3 t_4^2 t_5^3 t_6^3 - 2 y_s^3 y_{o_1}^6 y_{o_2}^6 t_1 t_2^4 t_3^3 t_4^2 t_5^3 t_6^3 - y_s^3 y_{o_1}^6 y_{o_2}^6 t_1^5 t_2^3 t_3^4 t_5^3 t_6^3 - y_s^3 y_{o_1}^6 y_{o_2}^6 t_2^3 t_3^4 t_5^3 t_6^3 + \\
 & 2 y_s^4 y_{o_1}^8 y_{o_2}^7 t_1^6 t_2^2 t_3^4 t_5^3 t_6^3 + 2 y_s^4 y_{o_1}^8 y_{o_2}^7 t_1^5 t_2^3 t_3^4 t_5^3 t_6^3 + 3 y_s^4 y_{o_1}^8 y_{o_2}^7 t_1^4 t_2^4 t_3^4 t_5^3 t_6^3 + 2 y_s^4 y_{o_1}^8 y_{o_2}^7 t_1^3 t_2^5 t_3^4 t_5^3 t_6^3 + \\
 & 2 y_s^4 y_{o_1}^8 y_{o_2}^7 t_1^2 t_2^6 t_3^4 t_5^3 t_6^3 - y_s^5 y_{o_1}^{10} y_{o_2}^8 t_1^6 t_2^3 t_3^4 t_5^3 t_6^3 - y_s^5 y_{o_1}^{10} y_{o_2}^8 t_1^5 t_2^4 t_3^4 t_5^3 t_6^3 - y_s^5 y_{o_1}^{10} y_{o_2}^8 t_1^4 t_2^5 t_3^4 t_5^3 t_6^3 - \\
 & y_s^2 y_{o_1}^5 y_{o_2}^6 t_1^4 t_2^3 t_3^4 t_5^3 t_6^3 - y_s^2 y_{o_1}^5 y_{o_2}^6 t_1^3 t_2^4 t_3^4 t_5^3 t_6^3 - y_s^2 y_{o_1}^5 y_{o_2}^6 t_1^2 t_2^5 t_3^4 t_5^3 t_6^3 - y_s^2 y_{o_1}^5 y_{o_2}^6 t_1 t_2^6 t_3^4 t_5^3 t_6^3 +
 \end{aligned}$$

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