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Non-unitary TQFTs from 3D $\mathcal{N}=4$ rank 0 SCFTs

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Abstract: We propose a novel procedure of assigning a pair of non-unitary topological quantum field theories (TQFTs), $\text{TFT}_\pm[\mathcal{T}_{\text{rank }0}]$, to a (2+1)D interacting $\mathcal{N}=4$ superconformal field theory (SCFT) $\mathcal{T}_{\text{rank }0}$ of rank 0, i.e. having no Coulomb and Higgs branches. The topological theories arise from particular degenerate limits of the SCFT. Modular data of the non-unitary TQFTs are extracted from the supersymmetric partition functions in the degenerate limits. As a non-trivial dictionary, we propose that $F = \max_\alpha \left(-\log |S^{(+)}_{0\alpha}|\right) = \max_\alpha \left(-\log |S^{(-)}_{0\alpha}|\right)$, where F is the round three-sphere free energy of $\mathcal{T}_{\text{rank }0}$ and $S^{(\pm)}_{0\alpha}$ is the first column in the modular S-matrix of $\text{TFT}_\pm$. From the dictionary, we derive the lower bound on F, $F \geq -\log\left(\sqrt{\frac{5-\sqrt{5}}{10}}\right) \simeq 0.642965$, which holds for any rank 0 SCFT. The bound is saturated by the minimal $\mathcal{N}=4$ SCFT proposed by Gang-Yamazaki, whose associated topological theories are both the Lee-Yang TQFT. We explicitly work out the (rank 0 SCFT)/(non-unitary TQFTs) correspondence for infinitely many examples.

Keywords: Conformal Field Theory, Supersymmetric Gauge Theory, Supersymmetry and Duality, Topological Field Theories

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1 Introduction

Supersymmetric quantum field theories with 8 supercharges (8 Qs) provide a fertile ground for many interesting research topics connecting various areas in theoretical and mathematical physics. For example, Seiberg-Witten's approach [1] to 4-dimensional (4D) $\mathcal{N} = 2$ supersymmetric gauge theories provides analytic understanding of non-perturbative phenomena, such as confinement, in strongly coupled gauge theories. 4D $\mathcal{N} = 2$ superconformal field theories can be geometrically constructed using wrapped M5-branes in M-theory, and the 4D-2D correspondence connects physics of 4D supersymmetric field theories with mathematical structures on 2D Riemann surfaces in an unexpected way [2–4]. Interestingly, there exist non-trivial superconformal field theories with 8 supercharges in higher dimensional space-time, 5D and 6D, as predicted by String/M-theory [5–7]. More recently, it is found that 2D chiral algebras (resp. 1D topological quantum mechanics) appear as protected subsectors of 4D $\mathcal{N} = 2$ (resp. 3D $\mathcal{N} = 4$) superconformal field theories [8–14].

Extended SUSY gauge theories have rich structures in their vacuum moduli space, and one natural invariant is the rank, i.e. the complex dimension of the Coulomb branch. There have been numerous efforts in classifying SCFTs with 8 Qs for a given low rank in various space-time dimensions [15–25]. For 3D $\mathcal{N} = 4$ SCFTs, however, the rank is in general not a duality-invariant concept since the Coulomb and Higgs branches are exchanged under the 3D mirror symmetry [26]. For this reason, we hereby modify the definition of rank as the maximum of the dimension of Coulomb branch and that of the Higgs branch. Another peculiar fact about 3D $\mathcal{N} = 4$ theories is that there exist non-trivial interacting SCFTs of rank 0, as studied by two of the authors of this present paper in [27]. This is in contrast with the case of $D \geq 4$, where it is often implicitly assumed that there is no non-trivial interacting rank 0 SCFTs with 8 Qs, so that the classification program starts with rank 1. (Recently, 4D/5D rank 0 SCFTs were found through a geometrical engineering but it is yet unclear if they are interacting SCFTs [28].) Note that most of the classification schemes in previous studies do not work for rank 0 cases since the existence of Coulomb or Higgs branch operators is an crucial assumption in the analysis.

In this paper, we initiate the classification of rank 0 3D $\mathcal{N} = 4$ SCFTs by establishing the following correspondence:

$$\begin{aligned}
&\text{3D } \mathcal{N} = 4 \text{ superconformal field theories of rank 0} \\
&\longleftrightarrow \text{A pair of 3D non-unitary topological quantum field theories (TQFTs)}\,.
\end{aligned} \tag{1.1}$$

The non-unitary TQFTs emerge at particular choices of non-superconformal R-symmetry, $\nu = \pm 1$ in (2.3), of rank 0 $\mathcal{N} = 4$ SCFTs. In the limits, due to huge Bose/Fermi cancellations the unrefined superconformal index gets contributions only from Coulomb-branch or Higgs-branch operators and their descendants. For rank 0 theories, the index becomes trivial (i.e. 1) since there are no non-trivial Coulomb/Higgs branch operators. Other partition functions on various rigid supersymmetric Euclidean backgrounds also drastically simplify in the degenerate limits for rank 0 theories. Our correspondence says that the simplified partition functions are actually equal to the partition functions of corresponding non-unitary TQFTs on the same 3D spacetime. (See (2.7) for a precise statement.) Concrete dictionaries of the correspondence are given in table 1. In the degenerate limits, as seen in the superconformal index case, contributions from local operators become unimportant for rank 0 theories and only non-local loop operators become relevant physical observables. Similarly, loop operators are the only physical observables in general TQFTs. Using the correspondence, one can map the problem of classifying rank 0 SCFTs to the classification of non-unitary TQFTs, which is much easier to handle. Mathematically, TQFTs are described by modular tensor categories (MTCs) and classification of MTCs has been studied intensively in the literature [29–34]. The most basic quantity characterizing a 3D CFT is the round three-sphere free energy, usually denoted as F. The F always monotonically decreases under the renormalization group (RG) flow and thus is regarded as a proper measure of the degrees of freedom of 3D CFT [35–37]. In one of the most interesting and surprising dictionaries of the correspondence, the F of a rank 0 CFT is related to the modular S-matrix of a non-unitary TQFT in a very simple way as given in table 2. Combining the dictionary and universal algebraic properties of the S-matrix, we derive following lower bound on F

$$F \geq -\log\left(\sqrt{\frac{5-\sqrt{5}}{10}}\right), \tag{1.2}$$

which should hold for any rank 0 SCFTs. Interestingly, the lower bound is saturated by the minimal theory studied in [27].

The correspondence is similar in spirit with the (4D $\mathcal{N}=2$ SCFT)/(2D chiral algebra) correspondence mentioned above. In both correspondences, non-unitary algebraic structures, chiral algebras on the one hand and modular tensor category on the other, appear as protected subsectors of unitary superconformal field theories. But there are several crucial differences. First, two theories in our correspondence are defined on the same 3D space-time while the 2D chiral algebra lives in the 2D subspace of 4D space-time of the SCFT. Secondly, basic physical objects are BPS local operators in the (4D SCFTs)/(2D chiral algebra) story while BPS non-local loop operators are basic objects in our correspondence. That non-local loop operators play crucial roles can be a great advantage of our classification approach over the conventional conformal bootstrap approaches, since the latter are based on correlation functions of local operators. We note that the 3D non-unitary TQFTs are sensitive to the global structure of the 3D rank 0 SCFTs and two theories in the correspondence share the same one-form symmetry as well as their 't Hooft anomalies. In 3D, the quantity F (unlike the stress-energy tensor central charge C_T) is sensitive to the global structure of CFT and the conformal bootstrap approach never give a constraint on F but only on C_T, which is not a proper measure of the degrees of freedom in a strict sense [38].

The remaining part of paper is organized as follows. In section 2, we present the precise statement of the correspondence with several concrete dictionaries. As an application of the correspondence, we derive interesting lower bounds on F for rank 0 SCFTs. In section 3, we explicitly work out the correspondence in detail with several classes of infinitely many rank 0 SCFTs. The results are summarized in table 2. In appendix A, we give brief reviews on supersymmetric partition functions of 3D $\mathcal{N} \geq 3$ gauge theories and modular data of 3D TQFTs which are basic ingredients of the dictionaries. In other appendices, we collect technical details and supplementary materials.

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2 (Rank 0 SCFT)/(Non-unitary TQFTs) correspondence

In this section, we establish a correspondence between

$$\begin{aligned} &\mathcal{T}_{\text{rank } 0}: \text{ a 3D } \mathcal{N}=4 \text{ interacting SCFT with empty Coulomb and Higgs branches} \\ &\longrightarrow \text{TFT}_{\pm}[\mathcal{T}_{\text{rank } 0}]: \text{ a pair of 3D non-unitary TQFTs}\,. \end{aligned} \tag{2.1}$$

The basic dictionaries for the correspondence are summarized in table 1.

2.1 Non-unitary TQFTs from $\mathcal{N}=4$ SCFTs of rank 0

3D Rank 0 $\mathcal{N}=4$ SCFTs. In this paper, 3D $\mathcal{N}=4$ rank 0 SCFT is defined as

$$(\text{Rank 0 theory}) := (\text{Theory with no Coulomb and Higgs branches})\,. \tag{2.2}$$

3D $\mathcal{N}=4$ SCFTs have $\text{SO}(4) \simeq \text{SU}(2)_L \times \text{SU}(2)_R$ R-symmetry. The Coulomb (Higgs) branch is parametrized by chiral primary operators charged under the $\text{SU}(2)_R$ ($\text{SU}(2)_L$) symmetry while neutral under the $\text{SU}(2)_L$ ($\text{SU}(2)_R$) symmetry. In our definition, the rank 0 theory can have mixed branches parametrized by chiral primaries charged under both of $\text{SU}(2)_L$ and $\text{SU}(2)_R$. Rank 0 SCFTs in section 3 with $\mathcal{N}=5$ supersymmetry actually have the mixed branches. Rank 0 theories cannot have a continuous flavor symmetry commuting with the SO(4) R-symmetry, since a flavor current multiplet contains Higgs- or Coulomb-branch operators. By the same reasoning, the rank 0 theories cannot have SUSY more than $\mathcal{N}=5$.

Axial U(1) symmetry and R-symmetry mixing. Let R_ν and A be the two Cartan generators of the SO(4) R-symmetry:

$$\begin{aligned} R_{\nu=0} &:= R+R'\,, \quad A := R-R'\,, \\ R_\nu &:= R_{\nu=0}+\nu A\,. \end{aligned} \tag{2.3}$$

Here R and R' are the Cartans of $\text{SU}(2)_L$ and $\text{SU}(2)_R$ respectively. In our convention, they are normalized as $R, R' \in \frac{1}{2}\mathbb{Z}$. In terms of an $\mathcal{N}=2$ subaglebra, R_ν is the R-charge while A is the charge of a U(1) flavor symmetry (commuting with the $\mathcal{N}=2$ supersymmetry) called the axial U(1) symmetry. The mixing between the U(1) R-symmetry and the axial flavor symmetry is parametrized by ν. The IR superconformal R-symmetry corresponds to $\nu=0$.

Supersymmetric partition functions. Generally, the partition function $\mathcal{Z}^{\mathbb{B}}_{\mathcal{T}}(b^2, m, \nu; s)$ of a 3d $\mathcal{N}=4$ SCFT $\mathcal{T}$ on a rigid supersymmetric background $\mathbb{B}$ depends on the followings:

$$\begin{aligned} \mathcal{M}&: \text{ topology of } \mathbb{B}\,, \\ b^2 \text{ (or } q)&: \text{ squashing (or } \Omega\text{-deformation) parameter,} \\ m \text{ (or } \eta=e^m)&: \text{ real mass (or fugacity variable) for axial U(1) symmetry,} \\ \nu&: \text{ R-symmetry mixing parameter in (2.3),} \\ s \in H^1(\mathcal{M}, \mathbb{Z}_2)&: \text{ SUSY-compatible spin-structure.} \end{aligned} \tag{2.4}$$

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We consider supersymmetric backgrounds $\mathbb{B}$ whose topologies are given as $\mathcal{M}_{g,p}$, a degree p bundle over a genus g Riemann surface Σ_g:

$$\begin{tikzcd} S^1 \arrow[r, "p"] & \mathcal{M}_{g,p} \arrow[d] \\ & \Sigma_g \end{tikzcd} \tag{2.5}$$

We refer readers to appendix A.1 for a brief review on localizations on supersymmetric backgrounds. We can turn on the Ω-deformation parameter (sometimes called squashing parameter) only for $g=0$. For even p, one can consider two supersymmetric backgrounds depending on the choice of the spin structure along the fiber $[S^1] \in H_1(\mathcal{M}, \mathbb{Z}_2)$. We denote $s=+1$ ($s=-1$) for the periodic (anti-periodic) boundary condition for fermionic fields along the S^1. For $p=0$, the partition function can be regarded as a version of the BPS index $\mathcal{I}(s)$ and its spin structure dependence can be interpreted as

$$\mathcal{I}(s) = \begin{cases} \mathrm{Tr}_{\mathcal{H}_{\mathrm{BPS}}}(-1)^{R_\nu}, & \text{for } s=-1\ , \\ \mathrm{Tr}_{\mathcal{H}_{\mathrm{BPS}}}(-1)^{2j_3}, & \text{for } s=1\ . \end{cases} \tag{2.6}$$

As BPS indices, they can be defined without overall phase factor ambiguities. For $p \neq 0$, on the other hand, local counterterms affect the phase factor of the partition function [39] and it is non-trivial to keep track of the local counterterms. Throughout this paper, we are for the most part interested in the absolute value of partition functions. For $g=1$ and $p=0$, the supersymmetric partition function is independent of all the parameters and is simply an integer number called the Witten index.

Emergence of non-unitary TQFT in the limits $\nu \to \pm 1$. As main result of the paper, we propose that for any rank 0 $\mathcal{N}=4$ SCFT $\mathcal{T}_{\text{rank } 0}$ we can associate a pair of non-unitary TQFTs denoted by $\mathrm{TFT}_{\pm}[\mathcal{T}_{\text{rank } 0}]$, satisfying the following relation

$$\textbf{Main proposal: } \mathcal{Z}^{\mathbb{B}}_{\mathcal{T}_{\text{rank } 0}}\left(b^2, m \text{ (or } \eta), \nu; s\right) \xrightarrow{m\to 0 \text{ (or } \eta \to 1),\ \nu \to \pm 1} \mathcal{Z}^{\mathcal{M}_{g,p}}_{\mathrm{TFT}_{\pm}[\mathcal{T}_{\text{rank } 0}]}(s)\ . \tag{2.7}$$

The partition function $\mathcal{Z}^{\mathcal{M}_{g,p}}_{\mathrm{TFT}}$ of the topological theory depends only on the topological structure $\mathcal{M}_{g,p}$ of $\mathbb{B}$ and (possibly) a choice of a spin-structure s on it.[1] We claim that in the degenerate limits i) the rigid supersymmetry partition function becomes independent on the squashing parameter b^2 (or Ω-deformation parameter q) and ii) it becomes the partition function of a non-unitary topological quantum field theory.

We call a topological quantum field theory a non-spin (or bosonic) TQFT when its partition function is independent on the choice of the spin structure, and a spin (or fermionic)

[1] The overall phase factor of the partition function depends also on the choice of the framing. As with supersymmetric partition functions, there is no canonical choice of the framing for non-zero p and we mostly focus on the absolute values of the partition functions.

TQFT otherwise. We propose that

$$\begin{aligned}&\text{The non-unitary } \mathrm{TFT}_\pm[\mathcal{T}_{\text{rank }0}] \text{ is a spin (fermionic) TQFT,}\\&\quad \text{if } R_{\nu=\pm 1}+2j_2 \in 2\mathbb{Z}+1 \text{ for some BPS local operators.}\\&\quad \text{or equivalently,}\\&\quad \text{if } \mathcal{I}^{\text{sci}}(q,\eta,\nu=\pm 1;s) \text{ contains } q^{\frac{1}{2}(\text{odd integer})} \text{ terms.}\end{aligned} \tag{2.8}$$

Here $\mathcal{I}^{\text{sci}}$ is the superconformal index defined in (A.3) and the index at $\nu=\pm 1$ is in general a power series in $q^{1/2}$ since $R_{\nu=\pm 1}+2j_3 \in \mathbb{Z}$. From (2.6), we expect that the supersymmetric partition on $\mathcal{M}_{g,p=0}$ depends on the spin-structure s if the above condition is met. This is because that the condition $R_{\nu=\pm 1}+2j_3 \in 2\mathbb{Z}+1$ implies that $(-1)^{R_{\nu=\pm 1}} \neq (-1)^{2j_3}$ for some BPS local operators which acts on the Hilbert-space $\mathcal{H}_{\text{BPS}}$ non-trivially. The above condition gives sufficient but not necessary condition for $\mathrm{TFT}_\pm$ to be fermionic. To see this, consider a rank 0 SCFT $\mathcal{T}_{\text{rank }0}$ not satisfying the above condition whose associated non-unitary topological field theories, $\mathrm{TFT}_\pm[\mathcal{T}_{\text{rank }0}]$, are bosonic. Then, $\mathcal{T}_{\text{rank }0}\otimes \mathcal{T}_{\text{spin top}}$ with an unitary fermionic topological field theory $\mathcal{T}_{\text{spin top}}$ is still a rank 0 SCFT not satisfying the above condition since the decoupled topological sector does not contribute to the superconformal index. But its associated non-unitary TQFTs $\mathrm{TFT}_\pm[\mathcal{T}_{\text{rank }0}\otimes \mathcal{T}_{\text{spin top}}] = \mathrm{TFT}_\pm[\mathcal{T}_{\text{rank }0}]\otimes \mathcal{T}_{\text{spin top}}$ are fermionic due to the decoupled unitary spin TQFT $\mathcal{T}_{\text{spin top}}$.

The proposal in (2.7) can be easily proven for the case when $\mathcal{Z}^{\mathbb{B}}$ is the superconformal index (A.3). In the degenerate limit, $\nu\to 1$ and $\eta=1$, the index becomes

$$\mathcal{I}^{\text{sci}}(q,\eta,\nu;s) \xrightarrow{\eta=1,\nu\to 1} \begin{cases}\mathrm{Tr}_{\mathcal{H}_{\text{rad}}(S^2)}(-1)^{2j_3}q^{R+j_3}\ , & s=+1\ ,\\ \mathrm{Tr}_{\mathcal{H}_{\text{rad}}(S^2)}(-1)^{R}q^{R+j_3}\ , & s=-1\ .\end{cases} \tag{2.9}$$

All unitary multiplets of 3D $\mathcal{N}=4$ superconformal algebra are listed in [40]. From the classification, it is not difficult to check that the index above gets contributions only from operators in a short multiplet denoted by $B_1[0]^{(2R,2R')}_{\Delta}$ with $R=0$ in [40]. The bottom state inside the multiplet corresponds to a Coulomb branch operator parametrizing the Coulomb branch. From the superconformal index in the degenerate limit, one can actually compute the Hilbert-series of the Coulomb branch [41]. Since we are considering a rank 0 theory $\mathcal{T}_{\text{rank }0}$ with empty Coulomb branch, the index gets contributions only from the identity operator and becomes simply (q-independent) 1. Similarly, one can also confirm that the index becomes 1 in the other degenerate limit, $\nu\to -1$ and $\eta=1$, since there is no Higgs branch. In summary, from the superconformal multiplet analysis, we conclude that

$$\mathcal{I}^{\text{sci}}_{\mathcal{T}_{\text{rank }0}}(q,\eta,\nu;s) \xrightarrow{\eta=1,\nu\to\pm 1} 1, \quad \text{for any rank 0 theory } \mathcal{T}_{\text{rank }0}\ . \tag{2.10}$$

This proves the proposal in (2.7) for the case when $\mathcal{Z}^{\mathbb{B}}=$ (superconformal index) since $\mathcal{Z}^{S^2\times S^1}=1$ for all topological theories.

We currently do not know the full proof of the proposal (2.7) for other supersymmetric partition functions $\mathcal{Z}^{\mathbb{B}}$. As noticed in [34], however, the triviality of superconformal index for a non-conformal choice of R-symmetry is a strong evidence for the appearance of non-unitary TQFT, while the triviality of the index at the superconformal R-symmetry implies an emergence of a unitary TQFT at the infra-red fixed point. We explicitly test the proposal with infinitely many rank 0 SCFTs and various supersymmetric partition functions in section 3.

2.2 Dictionaries

In table 1, we summarize basic dictionaries of the correspondence. The dictionary in the first line is the most basic one and other dictionaries (except for the last one for F) follow from it. On the one hand, partition functions (with insertion of loop operators along the fiber S^1) of a topological field theory on the geometries $\mathcal{M}_{g,p}$ are determined by the modular data, i.e. S and T matrices, of the topological theory. On the other hand, the supersymmetric partition function on $\mathbb{B}$ with topology $\mathcal{M}_{g,p}$ can be computed using localization technique as briefly summarized in appendix A.2.

For a TQFT its rank, i.e. the size of modular matrices, is equal to the ground state degeneracy on the two torus. For a supersymmetric field theory, the degeneracy is equal to the Witten index. The dictionaries for $S^2_{0\alpha}, T_{\alpha\beta}$ and $W_\beta(\alpha)$ simply follow from comparison between (A.57) and (A.44). The Bethe-vacuum corresponding to the trivial simple object, $\alpha = 0$, can be determined by requiring that

$$\text{Trivial object } \alpha=0\text{:}\ \frac{1}{\sqrt{|\mathcal{H}_{\alpha=0}(m=0,\nu\to\pm1)|}} = S^\pm_{00} = |\mathcal{Z}^{S^3_b}(m=0,\nu\to\pm1)|\ . \tag{2.11}$$

In topological field theories simple objects (labeled by α) are loop operators, while in supersymmetric field theories α labels the types (gauge and flavor charge) of loop operators. Therefore, we expect that for each Bethe-vacuum α there is a corresponding supersymmetric loop operator $\mathcal{O}_\alpha(\vec{z})$, see around (A.44):

$$\text{(Bethe vacua)-to-(loop operators) map: } \vec{z}_\alpha \to \mathcal{O}^\pm_\alpha\ . \tag{2.12}$$

The trivial object $\alpha = 0$ corresponds to the identity operator. The map will be determined by requiring the following consistency conditions

$$\begin{aligned} &W^\pm_0(\alpha) = \frac{S^\pm_{\alpha 0}}{S^\pm_{00}} = \sqrt{\frac{\mathcal{H}_{\alpha=0}(m=0,\nu\to\pm1)}{\mathcal{H}_\alpha(m=0,\nu\to\pm1)}}\ \text{ and } W^\pm_0(\alpha) = \mathcal{O}^\pm_\alpha(\vec{z}_{\alpha=0})\big|_{\nu\to\pm1,m=0}\ , \\ &\Rightarrow \mathcal{O}^\pm_\alpha(\vec{z}_{\alpha=0}) = \sqrt{\frac{\mathcal{H}_{\alpha=0}(m=0,\nu\to\pm1)}{\mathcal{H}_\alpha(m=0,\nu\to\pm1)}}\ . \end{aligned} \tag{2.13}$$

The dictionary for F in the last line is one of the most non-trivial and interesting statements in this paper. It says that

$$\begin{aligned} &F[\mathcal{T}_{\text{rank } 0}] \\ &= -\log|S_{0\alpha_*}| \text{ of } \text{TFT}_\pm[\mathcal{T}_{\text{rank } 0}] \quad (\alpha_* \text{ is chosen such that } |S_{0\alpha_*}| \le |S_{0\alpha}| \text{ for other } \alpha)\ . \end{aligned} \tag{2.14}$$

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$\mathrm{TFT}_{\pm}[\mathcal{T}_{\text{rank }0}]$	$\mathcal{T}_{\text{rank }0}$
$\mathcal{Z}^{\mathcal{M}_{g,p}}_{\mathrm{TFT}_{\pm}}(s)$	BPS partition function $\mathcal{Z}^{\mathbb{B}}_{\mathcal{T}_{\text{rank }0}}\big\|_{\nu\to\pm1,m=0}(s)$ with (topology of $\mathbb{B}$) $=\mathcal{M}_{g,p}$
Spin or non-spin	Equation (2.8)
Rank N	Witten index
Simple objects	Bethe vacua $\{\vec{z}_\alpha\}_{\alpha=0}^{N-1}$ or BPS loop operators $\{\mathcal{O}^{\pm}_{\alpha}(\vec{z})\}_{\alpha=0}^{N-1}$
$(S^{\pm}_{0\alpha})^{-2}$	$\mathcal{H}_\alpha(m=0,\nu\to\pm1;s=-1)$
$T^{\pm}_{\alpha\beta}$ (only for non-spin)	$\delta_{\alpha\beta}\left(\frac{\mathcal{F}_\alpha}{\mathcal{F}_{\alpha=0}}\right)\big\|_{\nu\to\pm1,m=0}$
$(T^2)^{\pm}_{\alpha\beta}$	$\delta_{\alpha\beta}\left(\frac{\mathcal{F}_\alpha}{\mathcal{F}_{\alpha=0}}\right)^2\big\|_{\nu\to\pm1,m=0,s=-1}$
$S^{\pm}_{00}$	$\big\|\mathcal{Z}^{S^3_b}_{\mathcal{T}_{\text{rank }0}}(m=0,\nu\to\pm1)\big\|$
$W^{\pm}_{\beta}(\alpha)$	$\mathcal{O}^{\pm}_{\alpha}(\vec{z}_\beta)\|_{\nu\to\pm1,m=0}$
$\max_\alpha(-\log\|S^{\pm}_{0\alpha}\|)$	F (three-sphere free energy)

Table 1. Basic dictionaries in (rank 0 SCFT)/(non-unitary TQFTs) correspondence. $S^{\pm}_{\alpha\beta}$ and $T^{\pm}_{\alpha\beta}$ are modular matrices of $\mathrm{TFT}_{\pm}[\mathcal{T}_{\text{rank }0}]$. We define $W_\beta(\alpha):=\frac{S_{\alpha\beta}}{S_{0\beta}}=\langle\beta|\mathcal{O}^A_\alpha|\beta\rangle$, see (A.52), from which one can compute S-matrix $S_{\alpha\beta}=W_\beta(\alpha)W_0(\beta)S_{00}$. $\mathcal{H}_\alpha$ and $\mathcal{F}_\alpha$ are handle gluing and fibering operator at the α-th Bethe-vacuum respectively, see (A.40) and (A.43). $\mathcal{O}_{\alpha=0}$ corresponds to the trivial loop and $W_\beta(0)=1$. The rank of a TFT (not to be confused with the rank of its associated 3D $\mathcal{N}=4$ SCFT) is defined as the dimension of $\mathcal{H}(\mathbb{T}^2)$ for non-spin case while is defined as the dimension of $\mathcal{H}_{--}(\mathbb{T}^2)$, i.e. Hilbert-space in NS-NS sector, for spin case. Similarly, the S and T matrices (resp. S and T^2) for non-spin TQFT case (resp. spin TQFT case) are modular matrices acting on $\mathcal{H}(\mathbb{T}^2)$ (resp. $\mathcal{H}_{--}(\mathbb{T}^2)$). Simple objects are in one-to-one with a basis of $\mathcal{H}(\mathbb{T}^2)$ (resp. $\mathcal{H}_{--}(\mathbb{T}^2)$) for non-spin TQFT case (resp. spin TQFT case). We refer readers to appendix A.2 for a general review on the modular data of non-spin and spin TQFTs.

F is the free energy on round three-sphere, which is the quantity appearing in the F-theorem and is a proper measure of degree of freedom. The relation is surprising since it relates the quantity (F) at the superconformal point, $\nu=0$, to the quantity ($S_{0\alpha_*}$) in the degenerate limits $\nu=\pm1$.

One possible explanation for the dictionary above is as follows. In general, $S_{0\alpha_*}$ in a topological field theory computes the three-sphere partition function with an insertion of loop operator $\mathcal{O}^{\Gamma=(\text{unknot})}_{\alpha_*}$ of type α_* along the unknot in S^3. In the rank 0 theory $\mathcal{T}_{\text{rank }0}$, on the other hand, there is a flavor vortex loop operator associated to the U(1) axial flavor symmetry. The loop operator is known to act on the three-sphere partition as a difference operator shifting the parameter ν [42–44]

$$(\text{flavor vortex loop})_{\pm} \text{ of charge } \pm 1 \quad \longleftrightarrow \quad \exp(\pm \partial_\nu)\,. \tag{2.15}$$

This means that the S^3 partition function at the conformal point can be identified with the S^3 partition function with an insertion of the vortex loop operators of charge $+1(-1)$ in TFT_- (TFT_+)[2]

$$\begin{aligned} &|\mathcal{Z}^{S^3}(\nu=0)| = \exp(\partial_\nu)\cdot|\mathcal{Z}^{S^3}(\nu=-1)| = |\mathcal{Z}^{S^3+\mathcal{O}_{\text{flavor vortex}}}(\nu=-1)| \\ \Rightarrow\ &|\mathcal{Z}^{S^3}(\nu=0)| = |S_{0\alpha=(\text{flavor vortex})_+} \text{ of TFT}_-|\,. \end{aligned} \tag{2.16}$$

Hence if one identify $\alpha = (\text{flavor vortex})_+$ ($\alpha = (\text{flavor vortex})_-$) of TFT_- (TFT_+) with $\alpha = \alpha_*$ in (2.14), then the dictionary follows. Actually, according to F-maximization (A.28), we expect that

$$\begin{aligned} |\mathcal{Z}^{S^3}(\nu=0)| &= |S_{0\alpha=(\text{flavor vortex})_+} \text{ of TFT}_-| < |S_{00} \text{ of TFT}_-| = |\mathcal{Z}^{S^3}(\nu=-1)|\,, \\ |\mathcal{Z}^{S^3}(\nu=0)| &= |S_{0\alpha=(\text{flavor vortex})_-} \text{ of TFT}_+| < |S_{00} \text{ of TFT}_+| = |\mathcal{Z}^{S^3}(\nu=1)|\,. \end{aligned} \tag{2.17}$$

It is compatible with the desired identification, (flavor vortex loop) = (α_*in (2.14)). The property above is also compatible with the fact that $\text{TFT}_\pm$ are non-unitary since they violate the unitarity condition (A.56).

The argument above gives circumstantial evidence but not a full proof for the dictionary on F. The dictionary will be confirmed explicitly in section 3 with infinitely many examples. We leave general proof or disproof of the dictionary for future work.

2.3 Application: lower bounds on F

Here we derive interesting lower bounds on F for rank 0 SCFTs using the correspondence introduced in the previous subsection.

One immediate and interesting consequence of the dictionaries in table 1 is

$$F > -\log \frac{1}{\sqrt{\text{Witten index}}}\,, \quad \text{for all interacting } \mathcal{N}\geq 4 \text{ SCFT } \mathcal{T}_{\text{rank }0} \text{ of rank } 0\,. \tag{2.18}$$

This follows from the following fact

$$\sum_{\alpha=0}^{N-1} (S_{0\alpha})^2 = 1 \ \Rightarrow\ \text{min}_\alpha |S_{0\alpha}| = |S_{0\alpha_*} \text{ in } (2.14)| < \frac{1}{\sqrt{N}}\,, \tag{2.19}$$

combined with the dictionaries for Witten index and F. More interestingly, using the dictionary, one can prove following

$$F \geq -\log\sqrt{\frac{5-\sqrt{5}}{10}} = 0.642965\,, \quad \text{for any interacting } \mathcal{N}\geq 4 \text{ SCFT } \mathcal{T}_{\text{rank }0} \text{ of rank } 0\,. \tag{2.20}$$

[2] If we turn on the squashing parameter b, the partition function at $\nu=-1$ with the flavor vortex loop is given as $\mathcal{Z}^{S^3_b+\mathcal{O}_{\text{flavor vortex}}}(m=0,\nu=-1) = \mathcal{Z}^{S^3_b}\left(m=i\pi(b-\frac{1}{b}),\nu=0\right)$. Interestingly, the partition function is actually independent b [45, 46] and $|\mathcal{Z}^{S^3_b}\left(m=i\pi(b-\frac{1}{b}),\nu=0\right)| = |\mathcal{Z}^{S^3_b}\left(m=i\pi(b-\frac{1}{b}),\nu=0\right)|_{b=1} = |\mathcal{Z}^{S^3_b}(m=0,\nu=0)| = e^{-F}$.

The bound is saturated by the minimal $\mathcal{T}_{\min}$ theory which will be introduced in section 3. The inequality above follows from the following fact combined with the dictionary for F

$$|S_{0\alpha_*}\text{in }(2.14)| \leq \sqrt{\frac{5-\sqrt{5}}{10}} = 0.525731 \quad \text{for all non-unitary TFT}\,. \tag{2.21}$$

Thanks to (2.19), we only need to check the inequality for non-unitary TQFTs up to rank 3 since and $\frac{1}{\sqrt{4}} = 0.5 < \sqrt{\frac{5-\sqrt{5}}{10}}$. Let us first consider rank 2 case. Let $x = S_{00}^2$ and $y = S_{01}^2$ where $S_{\alpha\beta}$ is the S-matrix of a non-unitary TQFT. Then the two positive real numbers x and y should satisfy followings

$$\text{GSD}_{g=0} = x + y = 1\,, \quad \text{GSD}_{g=2} = \frac{1}{x} + \frac{1}{y} = k \in \mathbb{Z}_{>0}\,. \tag{2.22}$$

Here GSD_g denotes the ground state degeneracy on genus g Riemann surface, see (A.58). One can solve the equations and we have two solutions

$$x = \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{k-4}{k}}\,, \quad y = \frac{1}{2} \mp \frac{1}{2}\sqrt{\frac{k-4}{k}}\,. \tag{2.23}$$

Imposing the conditions, $x, y \in \mathbb{R}_{>0}$ and $x > y$ (non-unitarity condition, see (A.56)), we have only one solution for each $k > 4$. As the natural number k increases, $y^{1/2} = |S_{0\alpha_*}|$ in the solution decreases. Thus, we have

$$\begin{aligned} &|S_{0\alpha_*}| = y^{1/2} \leq (y^{1/2} \text{ at } k = 5) = \sqrt{\frac{5-\sqrt{5}}{10}}\,, \\ &\text{for any rank 2 non-unitary TQFT}\,. \end{aligned} \tag{2.24}$$

Now let us move on to the rank 3 case. Let $x = S_{00}^2, y = S_{01}^2$ and $z = S_{02}^2 = S_{0\alpha_*}^2$. Then, the three positive real numbers should satisfy

$$\begin{aligned} &\text{GSD}_{g=0} = x + y + z = 1\,, \quad \text{GSD}_{g=2} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = k_1 \in \mathbb{Z}_{>0}\,, \\ &\text{GSD}_{g=3} = \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = k_2 \in \mathbb{Z}_{>0}\,. \end{aligned} \tag{2.25}$$

One can confirm that any solution to equations above satisfying the non-unitarity conditions, $\min(x, y) \geq z$ and $\max(x, y) > z$, have following property[3]

$$\begin{aligned} &|S_{0\alpha_*}| = z^{1/2} \leq (z^{1/2} \text{ at } k_1 = 10 \text{ and } k_2 = 36) = \frac{1}{2} < \sqrt{\frac{5-\sqrt{5}}{10}}\,, \\ &\text{for any rank 3 non-unitary TQFT}\,. \end{aligned} \tag{2.26}$$

We are not certain if there is a rank 3 non-unitary TQFT saturating the bound $|S_{0\alpha_*}| = \frac{1}{2}$. The results in (2.19), (2.24) and (2.26) imply the bound in (2.21), from which the conclusion in (2.20) follows.

[3]Since $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \leq \frac{3}{z}$, $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} \leq \frac{3}{z^2}$ and thus $z \leq \min(\frac{3}{k_1}, \sqrt{\frac{3}{k_2}})$, it is enough to check that $z^{1/2} \leq \frac{1}{2}$ for only finitely many cases ($0 < k_1 \leq 12$ and $0 < k_2 \leq 48$).

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$\mathcal{T}_{\text{rank } 0}$	$\text{TFT}_{\pm}[\mathcal{T}_{\text{rank } 0}]$	Set of $\{\lvert S^{\pm}_{0\alpha}\rvert\}$	$\exp(-F)$
$\mathcal{T}_{\text{min}}$	(Lee-Yang)	$\left\{\sqrt{\frac{5+\sqrt{5}}{10}}, \sqrt{\frac{5-\sqrt{5}}{10}}\right\}$	$\sqrt{\frac{5-\sqrt{5}}{10}}$
$(\mathrm{U}(1)_1 + H)$	$\mathrm{Gal}_d(\mathrm{SU}(2)_6)/\mathbb{Z}_2^f$ (with $d = \zeta_6^3$)	$\{2\zeta_6^1, 2\zeta_6^3\}$	$2\zeta_6^1$
$\mathrm{SU}(2)_k^{\frac{1}{2}\oplus\frac{1}{2}}$ $(\lvert k\rvert > 1)$	$\mathrm{Gal}_d(\mathrm{SU}(2)_{4\lvert k\rvert-2})/\mathbb{Z}_2^f$ (with $d = \zeta_{4\lvert k\rvert-2}^{2\lvert k\rvert-1}$)	$\{2\zeta_{4\lvert k\rvert-2}^{2n-1}\}_{n=1}^{\lvert k\rvert}$	$2\zeta^1_{4\lvert k\rvert-2}$
$T[\mathrm{SU}(2)]_{k_1,k_2}$	See the caption	$\left\{\left(\frac{1}{\sqrt{2}}\zeta^n_{\lvert k_1k_2-1\rvert-2}\right)^{\otimes 2}\right\}_{n=1}^{\lvert k_1k_2-1\rvert-1}$	$\frac{1}{\sqrt{2}}\zeta^1_{\lvert k_1k_2-1\rvert-2}$
$\frac{T[\mathrm{SU}(2)]}{\mathrm{SU}(2)^{\text{diag}}_{\lvert k\rvert=3}}$	$(\text{Lee-Yang})^{\otimes 2}\otimes \mathrm{U}(1)_2$	$\left\{\frac{1}{\sqrt{10}}^{\otimes 4}, \frac{5+\sqrt{5}}{10\sqrt{2}}^{\otimes 2}, \frac{5-\sqrt{5}}{10\sqrt{2}}^{\otimes 2}\right\}$	$\frac{5-\sqrt{5}}{10\sqrt{2}}$
$\frac{T[\mathrm{SU}(2)]}{\mathrm{SU}(2)^{\text{diag}}_{\lvert k\rvert=4}}$	$\frac{\mathrm{Gal}_{\zeta^7_{10}}(\mathrm{SU}(2)_{10})\times \mathrm{SU}(2)_2}{\mathbb{Z}_2^{\text{diag}}}$	$\left\{\frac{1}{2}, \frac{1}{2\sqrt{3}}^{\otimes 5}, \frac{3+\sqrt{3}}{12}^{\otimes 2}, \frac{3-\sqrt{3}}{12}^{\otimes 2}\right\}$	$\frac{3-\sqrt{3}}{12}$
$\frac{T[\mathrm{SU}(2)]}{\mathrm{SU}(2)^{\text{diag}}_{\lvert k\rvert=5}}$	$\mathrm{Gal}_d((G_2)_3)\otimes \mathrm{U}(1)_{-2}$ $\left(d = \sqrt{\frac{5}{84}+\frac{1}{4\sqrt{21}}}\right)$	$\left\{\frac{1}{\sqrt{6}}^{\otimes 2}, \frac{1}{\sqrt{14}}^{\otimes 6}, \sqrt{\frac{5}{84}\pm\frac{1}{4\sqrt{21}}}^{\otimes 2}\right\}$	$\sqrt{\frac{5}{84}-\frac{1}{4\sqrt{21}}}$
$\frac{T[\mathrm{SU}(2)]}{\mathrm{SU}(2)^{\text{diag}}_{\lvert k\rvert\geq 6}}$	**?**	$\left\{\frac{1}{\sqrt{2\lvert k\rvert-4}}^{\otimes(\lvert k\rvert-3)}, \frac{1}{\sqrt{2\lvert k\rvert+4}}^{\otimes(\lvert k\rvert+1)}, \left(\frac{1}{\sqrt{8\lvert k\rvert-16}}+\frac{1}{\sqrt{8\lvert k\rvert+16}}\right)^{\otimes 2}, \left(\frac{1}{\sqrt{8\lvert k\rvert-16}}-\frac{1}{\sqrt{8\lvert k\rvert+16}}\right)^{\otimes 2}\right\}$	$\frac{1}{\sqrt{8\lvert k\rvert-16}} - \frac{1}{\sqrt{8\lvert k\rvert+16}}$

Table 2. Non-unitary TQFTs from rank 0 $\mathcal{N}=4$ SCFTs. $\mathrm{Gal}_d(\text{TFT})$ denotes a Galois conjugate of an unitary topological field theory TFT with $S_{00}(\mathrm{Gal}_d[\text{TFT}]) = d$. We define $\zeta_k^n := \sqrt{\frac{2}{k+2}}\sin\frac{n\pi}{k+2}$. For the rank 0 SCFT $\mathcal{T}_{\text{rank } 0} = T[\mathrm{SU}(2)]_{k_1,k_2} := \frac{T[\mathrm{SU}(2)]}{\mathrm{SU}(2)^C_{k_1}\times \mathrm{SU}(2)^H_{k_2}}$ with $\lvert k_1k_2-1\rvert > 3$ and $\min(\lvert k_1\rvert,\lvert k_2\rvert) > 2$, the corresponding non-unitary TQFTs are $\text{TFT}_+ = \left[\mathrm{Gal}_{\zeta^{\lvert k_2\rvert}_{\lvert k_1k_2-1\rvert-2}}(\mathrm{SU}(2)_{\lvert k_1k_2-1\rvert-2})\right]\otimes \mathrm{U}(1)_2$ and $\text{TFT}_- = \left[\mathrm{Gal}_{\zeta^{\lvert k_1\rvert}_{\lvert k_1k_2-1\rvert-2}}(\mathrm{SU}(2)_{\lvert k_1k_2-1\rvert-2})\right]\otimes \mathrm{U}(1)_2$. For $\mathcal{T}_{\text{rank } 0} = \frac{T[\mathrm{SU}(2)]}{\mathrm{SU}(2)^{\text{diag}}_{\lvert k\rvert\geq 6}}$, we could not identify their associatec TFTs with previously known non-unitary TQFTs.

3 Examples

In this section, we introduce infinitely many examples of (2+1)D $\mathcal{N}=4$ interacting rank 0 superconformal field theories $\mathcal{T}_{\text{rank } 0}$. Using the dictionary in table 1, we compute the set $\{\lvert S^{\pm}_{0\alpha}\rvert\}$ for non-unitary TQFTs $\text{TFT}_{\pm}[\mathcal{T}_{\text{rank } 0}]$. We also independently compute the three-sphere free energy F by performing the localization integral for $\mathcal{Z}^{S^3_b}$ at $b=1, m=0, \nu=0$ and confirm the non-trivial dictionary for F. See table 2 for the summary.

3.1 The minimal $\mathcal{N}=4$ SCFT $\mathcal{T}_{\text{min}}$

3.1.1 SUSY enhancement

In [27], it was claimed that

$$\begin{aligned}&(\text{3D } \mathcal{N}=2 \text{ gauge theory, } \text{U}(1)_{k=-3/2} \text{ coupled to a chiral multiplet } \Phi \text{ of charge } +1) \\ &\xrightarrow{\text{at IR}} (\text{3D } \mathcal{N}=4 \text{ superconformal field theory } \mathcal{T}_{\text{min}})\,.\end{aligned} \tag{3.1}$$

As a quick evidence for the SUSY enhancement, there are following two gauge invariant monopole operators in the theory

$$\hat{\phi}\hat{\phi}|\mathfrak{m}=+1\rangle\,, \quad \hat{\psi}^*|\mathfrak{m}=-1\rangle\,, \tag{3.2}$$

with following quantum numbers [27, 47, 48]

$$\begin{aligned} &A=-\mathfrak{m}=-1, \quad R=\frac{1-R_\Phi}{2}|\mathfrak{m}|+2R_\Phi=1, \qquad j=1\,, \quad \Delta=2\,, \\ &A=-\mathfrak{m}=1, \quad\ \ R=\frac{1-R_\Phi}{2}|\mathfrak{m}|+(1-R_\Phi)=1, \quad j=1\,, \quad \Delta=2\,. \end{aligned} \tag{3.3}$$

Here A is the charge of $\text{U}(1)_{\text{top}}$ and R is the charge of the superconformal U(1) R-symmetry; $j\in\frac{\mathbb{Z}}{2}$ and Δ are the Lorentz spin and the conformal dimension respectively. We use the fact that [27]

$$R_\Phi := (\text{Superconformal U(1) R-symmetry charge of } \Phi) = \frac{1}{3}\,, \tag{3.4}$$

which can be determined by the F-maximization [49]. Here $|\mathfrak{m}\rangle\in\mathcal{H}_{\text{rad}}(S^2)$ (the radially quantized Hilbert-space on S^2) is a half BPS bare monopole operator. The bare monopole operator has a U(1) gauge charge $-\frac{3}{2}\mathfrak{m}+\frac{1}{2}|\mathfrak{m}|$ and should be dressed by excitations ($\hat{\phi},\hat{\psi}$ and their complex conjugations) of matter fields to be gauge-invariant. The dressed monopole operators are 1/4 BPS local operators. According to the classification in [40], the monopole operators above, if they survive at the IR superconformal point, must belong to an extra SUSY-current multiplet of the 3D $\mathcal{N}=2$ superconformal algebra. The multiplet consists of conformal primaries of the following quantum numbers as well as their conformal descendants,

$$\left[(R,j,\Delta)=\left(0,\frac{1}{2},\frac{3}{2}\right)\right] \xrightarrow{Q,\bar{Q}} [(R,j,\Delta)=(\pm1,1,2)] \xrightarrow{Q,\bar{Q}} \left[(R,j,\Delta)=\left(0,\frac{3}{2},\frac{5}{2}\right)\right]. \tag{3.5}$$

Here $Q:=Q_1+iQ_2$ and $\bar{Q}:=Q_1-iQ_2$ are the $\mathcal{N}=2$ supercharges. The local operators in the top component with $(R,j,\Delta)=\left(0,\frac{3}{2},\frac{5}{2}\right)$ correspond to the conserved current for the extra supersymmetry, whose existence guarantees the SUSY enhancement.

Further, it was claimed in [27] that the infra-red (IR) superconformal field theory (SCFT) $\mathcal{T}_{\text{min}}$ is the minimal 3D $\mathcal{N}=4$ SCFT having the smallest three-sphere free-energy

F and the smallest non-zero stress-energy tensor central charge C_T whose exact values are [27, 50]

$$
\begin{aligned}
F(\mathcal{T}_{\text{min}}) &= -\log \sqrt{\frac{5-\sqrt{5}}{10}} \\
&\simeq 0.642965\,, \\
\frac{C_T(\mathcal{T}_{\text{min}})}{C_T(\text{free theory with single }\Phi)} &= \frac{8}{26}\left(8-\frac{5\sqrt{5+2\sqrt{5}}}{\pi}\right) \\
&\simeq 0.992549\,.
\end{aligned} \tag{3.6}
$$

There is no vacuum moduli space in the minimal theory and thus the minimal SCFT is of rank 0.

Superconformal index. Applying the localization results in [51, 52] (see also appendix A of [53]), the superconformal index for $\mathcal{T}_{\text{min}}$ can be written as

$$
\begin{aligned}
&\mathcal{I}^{\text{sci}}_{\mathcal{T}_{\text{min}}}(q,\eta,\nu;s=1) = \sum_{\mathfrak{m}\in\mathbb{Z}} \oint_{|a|=1} \frac{da}{2\pi i a}\, q^{\frac{|\mathfrak{m}|}{6}} (a(-1)^{\mathfrak{m}})^{-\frac{3\mathfrak{m}}{2}-\frac{|\mathfrak{m}|}{2}} (\eta q^{\frac{\nu}{2}})^{-\mathfrak{m}} \text{P.E.}[f_{\text{single}}(q,a;\mathfrak{m})] \\
&\text{with } f_{\text{single}}(q,a;\mathfrak{m}) := \frac{q^{\frac{1}{6}+\frac{|\mathfrak{m}|}{2}} a}{1-q} - \frac{q^{\frac{5}{6}+\frac{|\mathfrak{m}|}{2}} a^{-1}}{1-q}\,.
\end{aligned} \tag{3.7}
$$

In the above, we use the superconformal R charge of Φ given in (3.4). At the conformal point, $\nu = 0$, the index becomes

$$
\mathcal{I}^{\text{sci}}_{\mathcal{T}_{\text{min}}}(q,u,\nu=0;s=1) = 1-q+\left(\eta+\frac{1}{\eta}\right)q^{3/2} - 2q^2 + \left(\eta+\frac{1}{\eta}\right)q^{5/2} - 2q^3 + \dots\,. \tag{3.8}
$$

The terms in $q^{3/2}$ come from the monopole operators in (3.3) and the index is compatible with the claimed SUSY enhancement [54]. On the other hand, the index at the non-conformal point $\nu = \pm 1$ is

$$
\begin{aligned}
&\mathcal{I}^{\text{sci}}_{\mathcal{T}_{\text{min}}}(q,\eta,\nu,s=1)\big|_{\nu\to\pm1} \\
&= 1 + \left(-1+\eta^{\mp1}\right)q + \left(-2+\eta+\frac{1}{\eta}\right)q^2 + \left(-2+\eta+\frac{1}{\eta}\right)q^3 + \dots\,.
\end{aligned} \tag{3.9}
$$

As anticipated from the superconformal multiplet analysis in (2.10), the index becomes (q-independent) 1 in the degenerate limits, $\nu \to \pm 1$ and $\eta \to 1$. This reconfirm that the $\mathcal{N} = 4$ theory is of rank 0 and gives a strong signal that a topological field theory emerges in the limits.

3.1.2 Lee-Yang TQFT in degenerate limits

Here we claim that

$$
\text{TFT}_{\pm}[\mathcal{T}_{\text{min}}] = (\text{Lee-Yang TQFT}) \tag{3.10}
$$

with explicit checks of the dictionaries in table 1. The non-unitary TQFT has following modular data

$$S = \begin{pmatrix} \sqrt{\frac{1}{10}\left(\sqrt{5}+5\right)} & -\sqrt{\frac{1}{10}\left(5-\sqrt{5}\right)} \\ -\sqrt{\frac{1}{10}\left(5-\sqrt{5}\right)} & -\sqrt{\frac{1}{10}\left(\sqrt{5}+5\right)} \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 \\ 0 & \exp\left(-\frac{2\pi i}{5}\right) \end{pmatrix}. \tag{3.11}$$

Squashed three-sphere partition function. The squashed three-sphere partition function for the minimal theory is ($\hbar = 2\pi i b^2, b \in \mathbb{R}$)

$$\mathcal{Z}^{S^3_b}_{\mathcal{T}_{\min}}(b,m,\nu) = \int \frac{dZ}{\sqrt{2\pi\hbar}}\, e^{-\frac{Z^2+2Z\left(m+\left(i\pi+\frac{\hbar}{2}\right)\nu\right)}{2\hbar}} \psi_\hbar(Z)\,. \tag{3.12}$$

Here $\psi_\hbar$ is a special function called the quantum dilogarithm, for which readers are referred to appendix D. The partition function in the limit $b \to 1$ was studied in [50] and one can check that

$$\begin{aligned} &\left|\mathcal{Z}^{S^3_b}_{\mathcal{T}_{\min}}(b,m=0,\nu\to\pm1)\right| \xrightarrow{b\to1} \sqrt{\frac{1}{10}\left(\sqrt{5}+5\right)} + \sum_{n=2}^{\infty} s_n(1-b)^n\,, \\ &\text{with } s_n = 0 \text{ up to arbitrary higher order } n\,. \end{aligned} \tag{3.13}$$

That the partition function becomes independent on the squashing parameter b in the degenerate limit is a strong signal that the theory becomes topological. Furthermore, the partition function is identical to the S^3 partition function (S_{00}, see (3.11)) of the Lee-Yang TQFT,

$$\left|\mathcal{Z}^{S^3_b}_{\mathcal{T}_{\min}}(b,m=0,\nu=\pm1)\right| = \left|\mathcal{Z}^{S^3}_{\text{Lee-Yang}}\right|. \tag{3.14}$$

The free energy $F = -\log\left|\mathcal{Z}^{S^3_b}_{\mathcal{T}_{\min}}(b=1,m=0,\nu=0)\right|$ of the minimal theory, given in (3.6), nicely matches with $\max_\alpha(-\log|S_{0\alpha}|)$, see (3.11) for the S-matrix,

$$(F \text{ of } \mathcal{T}_{\min}) = \left(\max_\alpha(-\log|S_{0\alpha}|) \text{ of Lee-Yang TQFT}\right), \tag{3.15}$$

which confirms the dictionary for F in table 1.

Perturbative invariants $\boldsymbol{S}^\alpha_n$. The integrand in (3.12) can be expanded as

$$\begin{aligned} &\log \mathcal{I}_\hbar(Z,m,\nu) = \log\left(e^{-\frac{Z^2+2Z\left(m+\left(i\pi+\frac{\hbar}{2}\right)\nu\right)}{2\hbar}}\psi_\hbar(Z)\right) \\ &\xrightarrow{\hbar\to0} \frac{1}{\hbar}\mathcal{W}_0(Z,m,\nu) + \mathcal{W}_1(Z,m,\nu) + \ldots \quad \text{with} \\ &\qquad \mathcal{W}_0 = \text{Li}_2(e^{-Z}) - \frac{Z^2}{2} - Z(m+i\pi\nu)\,, \quad \mathcal{W}_1 = -\frac{1}{2}\log(1-e^{-Z}) - \frac{Z\nu}{2}\,. \end{aligned} \tag{3.16}$$

There are two Bethe-vacua (A.31) determined by the following algebraic equation

$$\text{Bethe-vacua of } \mathcal{T}_{\min} : \left\{ z : \frac{(z-1)e^{-m-i\pi\nu}}{z^2} = 1 \right\}. \tag{3.17}$$

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In the degenerate limits, $m = 0$ and $\nu \to \pm 1$, the two Bethe-vacua approach following values

$$z_{\alpha=0} \to \frac{1}{2}\left(\sqrt{5}-1\right) , \quad z_{\alpha=1} \to \frac{1}{2}\left(-\sqrt{5}-1\right) . \tag{3.18}$$

Perturbative invariants (A.33) of two saddle points associated to the two Bethe-vacua in the degenerate limits are

$$\begin{aligned}
&\mathcal{S}_0^{\alpha=0} \to \frac{7\pi^2}{30} , && \mathcal{S}_1^{\alpha=0} \to -\frac{1}{2}\log\left(\frac{5-\sqrt{5}}{2}\right) , && \mathcal{S}_2^{\alpha=0} \to -\frac{7}{120} ,\\
&\mathcal{S}_0^{\alpha=1} \to -\frac{17\pi^2}{30} , && \mathcal{S}_1^{\alpha=1} \to -\frac{1}{2}\log\left(\frac{5+\sqrt{5}}{2}\right) , && \mathcal{S}_2^{\alpha=1} \to -\frac{7}{120} ,\\
&\mathcal{S}_{n\geq 3}^{\alpha} \to 0 .
\end{aligned} \tag{3.19}$$

That this is compatible with the expected properties in (A.35) is a highly non-trivial evidence for emergence of topological theory in the degenerate limits.

Fibering and Handle gluing. Using the formulae in (A.43) and (A.40) combined with the above computation of $\mathcal{S}_n^{\alpha}$, we have

$$\begin{aligned}
&\Big\{\mathcal{F}_\alpha(m=0,\nu\to\pm1,s=-1)\Big\}_{\alpha=0,1} \longrightarrow \left\{\exp\left(-\frac{7i\pi}{60}\right), \exp\left(\frac{17i\pi}{60}\right)\right\} ,\\
&\Big\{\mathcal{H}_\alpha(m=0,\nu\to\pm1,s=-1)\Big\}_{\alpha=0,1} \longrightarrow \left\{\frac{5-\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2}\right\} .
\end{aligned} \tag{3.20}$$

Since the $1/\sqrt{\mathcal{H}_{\alpha=0}}$ is equal to $|\mathcal{Z}^{S^3_b}|$ at $\nu = \pm 1$ in (3.13), the $z_{\alpha=0}$ is indeed the Bethe-vacuum corresponding to the trivial object according to the criterion in (2.11). The computations above also confirm the dictionary for $S_{0\alpha}^{-2}$ and $T_{\alpha\beta}$ in table 1, see (3.11) for modular matrices of Lee-Yang TQFT.

Supersymmetric loop operator. For a U(1) gauge theory, the supersymmetric dyonic loop operator $\mathcal{O}_{(p,q)}$ of (electric charge, magnetic charge)=(p,q) is

$$\mathcal{O}_{(p,q)} = z^p(1-z^{-1})^q . \tag{3.21}$$

The consistency condition in (2.13) is met when we choose the (Bethe vacua)-to-(loop operators) map as follow

$$\mathcal{O}_{\alpha=0} = (\text{identity operator}) , \quad \mathcal{O}_{\alpha=1} = \mathcal{O}_{(p,q)=(1,0)} . \tag{3.22}$$

Then, using the dictionary in table 1

$$W_{\beta=0,1}(0) = 1 , \quad W_{\beta=0}(1) = z_0 = \frac{1}{2}(\sqrt{5}-1) , \quad W_{\beta=1}(1) = z_1 = \frac{1}{2}(-\sqrt{5}-1) . \tag{3.23}$$

From $\mathcal{W}_\beta(\alpha)$, one can compute the S-matrix using the formula $S_{\alpha\beta} = S_{00}W_\beta(\alpha)W_0(\beta)$, and confirm that it is identical to the S-matrix of the Lee-Yang TQFT given in (3.11).

3.2 $\mathrm{U}(1)_1 + H$: SUSY enhancement $\mathcal{N}=3 \to \mathcal{N}=5$

3.2.1 SUSY enhancement

We define

$$\begin{aligned}(\mathrm{U}(1)_k + H) := (&\text{3D } \mathcal{N}=4 \text{ U(1) gauge theory with CS level } k \\ &\text{coupled to a hypermultiplet of charge } +1)\ .\end{aligned} \tag{3.24}$$

For non-zero k, the theory has $\mathcal{N}=3$ supersymmetry instead of $\mathcal{N}=4$ since the CS term breaks some of the $\mathcal{N}=4$ supersymmetry. The $\mathcal{N}=3$ theory has the $\mathrm{U}(1)_{\mathrm{top}}$ flavor symmetry associated to the dynamical U(1) gauge theory. As pointed out in [55, 56]

$$\begin{aligned}&\text{For } k=1, \text{ the } (\mathrm{U}(1)_k + H) \text{ has enhanced } \mathcal{N}=5 \text{ supersymmetry at IR} \\ &\text{and the resulting IR SCFT is of rank 0}\ .\end{aligned} \tag{3.25}$$

The $\mathrm{U}(1)_{\mathrm{top}} = \mathrm{SO}(2)_{\mathrm{top}}$ symmetry becomes the U(1) axial symmetry, which is a subgroup of $\mathrm{SO}(4) \subset \mathrm{SO}(5)$ R-symmetry, in the supersymmetry enhancement.

$$\mathrm{SO}(3)_R \times \mathrm{SO}(2)_{\mathrm{top}} \xrightarrow{\mathrm{RG}} \mathrm{SO}(5)_R\ . \tag{3.26}$$

For $|k|=1$, there are two BPS monopole operators whose quantum numbers are

$$\begin{aligned}&A=+1, \quad j=\frac{1}{2}, \quad R=1, \quad \Delta=\frac{3}{2} \quad \text{and} \\ &A=-1, \quad j=\frac{1}{2}, \quad R=1, \quad \Delta=\frac{3}{2}\ .\end{aligned} \tag{3.27}$$

Here A is the charge of $\mathrm{U}(1)_{\mathrm{top}}$ and $R \in \frac{\mathbb{Z}}{2}$ is the spin of SO(3) R-symmetry. j and Δ are the Lorentz spin and the conformal dimension respectively. The BPS operators belong to extra SUSY-current multiplet [40] of $\mathcal{N}=3$ superconformal algebra, which consists of the following conformal primaries and their descendants

$$\begin{aligned}&[(R,j,\Delta)=(0,0,1)] \xrightarrow{\mathbf{Q}} \left[(R,j,\Delta)=\left(1,\frac{1}{2},\frac{3}{2}\right)\right] \xrightarrow{\mathbf{Q}} \\ &[(R,j,\Delta)=(1,1,2)] \oplus [(R,j,\Delta)=(0,0,2)] \xrightarrow{\mathbf{Q}} \left[(R,j,\Delta)=\left(0,\frac{3}{2},\frac{5}{2}\right)\right]\ .\end{aligned} \tag{3.28}$$

Here $\mathbf{Q}=(Q_1,Q_2,Q_3)$ are the $\mathcal{N}=3$ supercharges. The local operators in the top component of the supermultiplet with $(R,j,\Delta)=\left(0,\frac{3}{2},\frac{5}{2}\right)$ correspond to the conserved current for extra supersymmetry, and thus the existence of the multiplet implies the supersymmetry enhancement [55, 56].

Superconformal index. Using the localization summarized in A.1, the index is given as

$$\begin{aligned}&\mathcal{I}^{\mathrm{sci}}_{\mathrm{U}(1)_k+H}(q,\eta,\nu;s=1) \\ &= \sum_{\mathfrak{m}} \oint_{|a|=1} \frac{da}{2\pi i a} q^{\frac{|\mathfrak{m}|}{4}} ((-1)^{\mathfrak{m}} a)^{k\mathfrak{m}} (\eta q^{\frac{1}{2}\nu})^{-\mathfrak{m}} \mathrm{P.E.}[f_{\mathrm{single}}(q,\mathbf{a},\eta;\mathfrak{m})]\ ,\end{aligned} \tag{3.29}$$

where η is a fugacity of $\mathrm{U}(1)_{\mathrm{top}}$ and

$$f_{\mathrm{single}}(q,\mathbf{a},\eta;\mathfrak{m}) := \frac{q^{\frac{1}{4}+\frac{1}{2}|\mathfrak{m}|}(a+\frac{1}{a})}{1-q} - \frac{q^{\frac{3}{4}+\frac{1}{2}|\mathfrak{m}|}(a+\frac{1}{a})}{1-q} \,. \tag{3.30}$$

Using the above expression, the index can be evaluated and we find[4]

$$\begin{aligned} &\mathcal{I}^{\mathrm{sci}}_{\mathrm{U}(1)_k+H}(q,\eta,\nu;s=1) \\ &= \begin{cases} 1+q^{1/2}+\left(-\eta-\frac{1}{\eta}-1\right)q+\left(\eta+\frac{1}{\eta}+2\right)q^{3/2}+\dots, & |k|=1 \\ 1+q^{1/2}-q+\dots\,, \quad |k|>1 \end{cases} \end{aligned} \tag{3.31}$$

The term $(-\eta-\frac{1}{\eta})q$ comes from the monopole operators (3.27) in extra SUSY-current multiplet and implies the SUSY enhancement. Note that the SUSY enhancement occurs only at $|k|=1$.

In the degenerate limit $\nu=\pm1$ and $\eta=1$, the index becomes

$$\begin{aligned} &\mathcal{I}^{\mathrm{sci}}_{\mathrm{U}(1)_k+H}(q,\eta=1,\nu=\pm1;s=1) \\ &= \begin{cases} 1+(1-\eta^{\mp1})\,q^{1/2}+(\eta^{\mp1}-1)\,q+\dots\big|_{\eta=1}=1\,, & (|k|=1) \\ \text{Non-trivial power series in } q^{1/2}\,, & (|k|>1) \end{cases}\,. \end{aligned} \tag{3.32}$$

It is compatible the expectation that the theory is a $\mathcal{N}=4$ (actually $\mathcal{N}=5$) SCFT of rank 0 when $|k|=1$. It also implies that there emerge non-unitary TQFTs in the degenerate limits only when $|k|=1$. The non-unitary TQFTs are expected to be fermionic according to (2.8).

3.2.2 Non-unitary TQFTs in degenerate limits

Here we extract modular data of $\mathrm{TFT}_\pm[\mathrm{U}(1)_{k=1}+H]$ by computing various supersymmetric partition functions.

Squashed three-sphere partition function. It can be written as (see appendix A.1)

$$\begin{aligned} &\mathcal{Z}^{S^3_b}_{\mathrm{U}(1)_k+H}(b,m,\nu) = \int \frac{dZ}{\sqrt{2\pi\hbar}}\,\mathcal{I}_\hbar(Z,m,\nu) \ \ \text{with} \\ &\quad \mathcal{I}_\hbar(Z,m,\nu) = \exp\left(\frac{kZ^2}{2\hbar}\right)\exp\left(-\frac{ZW}{\hbar}\right)\bigg|_{W=m+(\pi i+\frac{\hbar}{2})\nu} \prod_{\epsilon_1\in\{\pm1\}} \Psi_\hbar\left[\epsilon_1 Z+\frac{\pi i}{2}+\frac{\hbar}{4}\right]. \end{aligned} \tag{3.33}$$

Here $\Psi_\hbar(X) := \psi_\hbar(X)\exp\left(\frac{X^2}{4\hbar}\right)$ as defined in (A.25). In a round sphere limit ($b=1$) with $k=1$ and $m=0$, the integral reduces to (using appendix D)

$$\mathcal{Z}^{S^3}_{\mathrm{U}(1)_1+H}(b=1,m=0,\nu) = \frac{e^{i\delta}}{4\pi}\int dZ \frac{e^{\frac{kZ^2}{4\pi i}}e^{-Z\nu}}{\cosh(Z/2)}\,. \tag{3.34}$$

[4]The same computation was done in [55, 56].

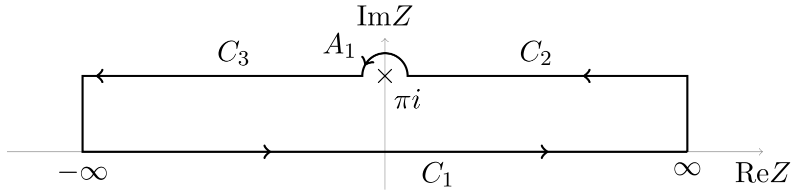

Figure 1. A contour for the evaluation of (3.34). There is a simple pole at $Z = \pi i$ inside the contour.

Here $e^{i\delta}$ is an unimportant phase factor sensitive to local counterterms. For $\nu = 0$, this integration can be exactly evaluated by applying the residue theorem to the integral along the contour depicted in figure 1:

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \mathrm{d}Z \frac{e^{\frac{Z^2}{4\pi i}}}{\cosh(Z/2)} + \int_{\infty}^{-\infty} \mathrm{d}Z \frac{e^{\frac{Z^2}{4\pi i}} e^{\frac{Z}{2}} e^{\frac{\pi i}{4}}}{i \sinh(Z/2)} + \pi i(-2ie^{\frac{\pi i}{4}}) = 2\pi i(-2ie^{\frac{\pi i}{4}})\,, \\
\Rightarrow &\int_{-\infty}^{\infty} \mathrm{d}Z \frac{e^{\frac{Z^2}{4\pi i}}}{\cosh(Z/2)} = 2\pi e^{\frac{\pi i}{4}} + e^{-\frac{\pi i}{4}} \int_{-\infty}^{\infty} \mathrm{d}Z \frac{e^{\frac{Z^2}{4\pi i}} e^{\frac{Z}{2}}}{\sinh(Z/2)} \\
&\qquad = 2\pi e^{\frac{\pi i}{4}} + \frac{e^{-\frac{\pi i}{4}}}{2} \left(\int_{-\infty}^{\infty} \mathrm{d}Z \frac{e^{\frac{Z^2}{4\pi i}} e^{\frac{Z}{2}}}{\sinh(Z/2)} - \int_{-\infty}^{\infty} \mathrm{d}Z \frac{e^{\frac{Z^2}{4\pi i}} e^{-\frac{Z}{2}}}{\sinh(Z/2)} \right) \\
&\qquad = 2\pi e^{\frac{\pi i}{4}} + e^{-\frac{\pi i}{4}} \int_{-\infty}^{\infty} \mathrm{d}Z e^{\frac{Z^2}{4\pi i}} \\
&\qquad = 2\pi(e^{\frac{\pi i}{4}} + e^{-\frac{\pi i}{2}}) = 4\pi e^{-\frac{\pi i}{8}} \sin\left(\frac{\pi}{8}\right)\,.
\end{aligned} \tag{3.35}
$$

Here, the residue at the simple pole $Z = \pi i$ is $-2ie^{\frac{\pi i}{4}}$. The first, second, and third term in the first line comes from the path C_1, $C_2 + C_3$, and an arc A_1 respectively. At the third equality, we have used the changing variable as $Z \to -Z$ from the last term in the second line.

Likewise, the integration for $\nu = \pm 1$ can be computed exactly in a similar way by using the same contour and we found

$$
\int_{-\infty}^{\infty} \mathrm{d}Z \frac{e^{\frac{Z^2}{4\pi i}}\, e^{-Z}}{\cosh(Z/2)} = 4\pi e^{-\frac{5\pi}{8}} \sin\left(\frac{3\pi}{8}\right)\,. \tag{3.36}
$$

Restoring the overall factor $1/(4\pi)$ in (3.34), we finally have

$$
\begin{aligned}
\exp(-F) &= \left|\mathcal{Z}^{S_b^3}_{\mathrm{U}(1)_1+H}(b=1, m=0, \nu=0)\right| = \sin\left(\frac{\pi}{8}\right) = (4+2\sqrt{2})^{-1/2}\,, \\
(S_{00} \text{ of TFT}_\pm) &= \left|\mathcal{Z}^{S_b^3}_{\mathrm{U}(1)_1+H}(b=1, m=0, \nu\to\pm 1)\right| = \sin\left(\frac{3\pi}{8}\right) = (4-2\sqrt{2})^{-1/2}\,.
\end{aligned} \tag{3.37}
$$

The partition functions in the degenerate limits $\nu \to \pm 1$ are actually independent on the squashing parameter b, as we will check it perturbatively in (3.41), and equality in the 2nd line holds for arbitrary $b \in \mathbb{R}$.

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Perturbative invariants $\mathcal{S}_n^\alpha$. The integrand in (3.33) at $k=1$, after a shift $Z \to Z + \frac{1}{2}\left(\pi i + \frac{\hbar}{2}\right)$ of dummy integral variable, can be expanded as

$$
\begin{aligned}
&\log \mathcal{I}_\hbar(Z,m,\nu)|_{Z\to Z+\frac{1}{2}(\pi i+\frac{\hbar}{2})} \xrightarrow{\hbar\to 0} \frac{1}{\hbar}\mathcal{W}_0(Z,m,\nu)+\mathcal{W}_1(Z,m,\nu)+\dots \quad \text{with} \\
&\mathcal{W}_0(m=0) = \text{Li}_2(e^{-Z+i\pi})+\text{Li}_2(e^Z)+Z^2+\pi i(1-\nu)Z+c_0\pi^2\,, \\
&\mathcal{W}_1(m=0) = \frac{1}{2}\left(-i\pi\nu-\nu Z+Z-\log\left(1-e^Z\right)\right)+ic_1\pi\,,
\end{aligned}
\tag{3.38}
$$

where c_0 and c_1 are Z-independent rational numbers. There are two Bethe-vacua (A.31) determined by a following algebraic equation

$$
\text{Bethe-vacua of } (\text{U}(1)_1+H) \text{ at } m=0\;:\; \left\{z\;:\;\frac{(-1)^\nu z(z+1)}{z-1}=1\right\}. \tag{3.39}
$$

In the degenerate limits, $m=0$ and $\nu\to\pm1$, the two Bethe-vacua approach following values

$$
z_{\alpha=0}\to\left(\sqrt{2}-1\right),\quad z_{\alpha=1}\to\left(-\sqrt{2}-1\right)\,. \tag{3.40}
$$

Perturbative invariants (A.33) of two saddle points associated to the two Bethe-vacua in the degenerate limits are

$$
\begin{aligned}
&\mathcal{S}_0^{\alpha=0}\to-\frac{11}{12}\pi^2\,, \qquad \mathcal{S}_1^{\alpha=0}\to-\frac{\pi i}{8}-\frac{1}{2}\log\left(4-2\sqrt{2}\right)\,, \\
&\mathcal{S}_0^{\alpha=1}\to\frac{1}{12}\pi^2\,, \qquad \mathcal{S}_1^{\alpha=1}\to-\frac{\pi i}{8}-\frac{1}{2}\log\left(4+2\sqrt{2}\right)\,, \\
&\text{Im}[\mathcal{S}^\alpha_{n=2}]\,,\quad \mathcal{S}^\alpha_{n\geq3}\to 0\,.
\end{aligned}
\tag{3.41}
$$

That this is compatible with the expected properties in (A.35) is a highly non-trivial evidence for emergence of topological theory in the degenerate limits.

Fibering and Handle gluing. Using the formulae in (A.43) and (A.40) combined with the above computation of $\mathcal{S}_n^\alpha$, we have

$$
\begin{aligned}
&\Big\{\mathcal{F}_\alpha(m=0,\nu\to\pm1,s=-1)\Big\}_{\alpha=0,1}\longrightarrow\left\{\exp\left(\frac{11i\pi}{24}\right),\exp\left(-\frac{i\pi}{24}\right)\right\}\,, \\
&\Big\{\mathcal{H}_\alpha(m=0,\nu\to\pm1,s=-1)\Big\}_{\alpha=0,1}\longrightarrow\left\{\left(4-2\sqrt{2}\right),\left(4+2\sqrt{2}\right)\right\}.
\end{aligned}
\tag{3.42}
$$

Since the $1/\sqrt{\mathcal{H}_{\alpha=0}}$ is equal to $|\mathcal{Z}^{S^3_b}|$ at $\nu=\pm1$ in (3.37), the $z_{\alpha=0}$ is indeed the Bethe-vacuum corresponding to the trivial object according to the criterion in (2.11).

Supersymmetric loop operators. The consistency condition in (2.13) is met when we choose the (Bethe vacua)-to-(loop operators) map as follows:

$$
\mathcal{O}_{\alpha=0}=(\text{identity operator})\,,\quad \mathcal{O}_{\alpha=1}=\mathcal{O}_{(p,q)=(1,0)}\,. \tag{3.43}
$$

Then, using the dictionary in table 1,

$$W_{\beta=0,1}(0) = 1\ , \quad W_{\beta=0}(1) = z_0 = \sqrt{2} - 1\ , \quad W_{\beta=1}(1) = z_1 = -\sqrt{2} - 1\ . \tag{3.44}$$

From $\mathcal{W}_\beta(\alpha)$, one can compute the S-matrix using the formula $S_{\alpha\beta} = S_{00} W_\beta(\alpha) W_0(\beta)$ and the result is

$$S = \begin{pmatrix} \sin\frac{3\pi}{8} & \sin\frac{\pi}{8} \\ \sin\frac{\pi}{8} & -\sin\frac{3\pi}{8} \end{pmatrix}\ . \tag{3.45}$$

Since the TQFTs, $\mathrm{TFT}_\pm[\mathrm{U}(1)_1 + H]$, are spin TQFTs, only the modular T^2 matrix is well-defined and according to the dictionary in table 1

$$T^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ . \tag{3.46}$$

The modular data (S and T^2) of the spin non-unitary TQFT, $\mathrm{TFT}_\pm[\mathrm{U}(1)_1 + H]$, is identical to that of $\mathrm{Gal}_d(\mathrm{SU}(2)_6)/\mathbb{Z}_2^f$ with $d = \frac{1}{2}\sin\left(\frac{3\pi}{8}\right)$.

3.3 $\mathrm{SU}(2)_k^{\frac{1}{2}\oplus\frac{1}{2}}$: $\mathcal{N}=5$ theory

The theory is defined as

$$\begin{aligned}\left(\mathrm{SU}(2)_k^{\frac{1}{2}\oplus\frac{1}{2}}\right) := &\ \mathrm{SU}(2) \text{ gauge theory coupled to a half hypermultiplet} \\ &\text{and a half twisted-hypermultiplet} \\ &\text{in fundamental representations with Chern-Simons level } k\ .\end{aligned} \tag{3.47}$$

The theory is can be regarded as a special case of $\mathrm{O}(M) \times \mathrm{Sp}(2N)$ type quiver theories (with $M = N = 1$) which have $\mathcal{N} = 5$ supersymmetry [57].

3.3.1 IR phases

Superconformal index. The superconformal index (A.3) of the $\mathcal{N} = 5$ theory is

$$\begin{aligned}&\mathcal{I}^{\mathrm{sci}}_{\mathrm{SU}(2)_k^{\frac{1}{2}\oplus\frac{1}{2}}}(q,\eta,\nu;s=1) \\ &= \sum_{\mathfrak{m}=0}^{\infty} \oint_{|a|=1} \frac{da}{2\pi i a} \Delta(\mathfrak{m},a) a^{2k\mathfrak{m}} q^{\frac{|\mathfrak{m}|}{2}} \mathrm{P.E.}\left[f_{\mathrm{single}}(q,\eta,a;\nu,\mathfrak{m})\right]\ .\end{aligned} \tag{3.48}$$

Here we define

$$\begin{aligned} f_{\mathrm{single}}(q,\eta,a;\nu,\mathfrak{m}) &:= \frac{q^{\frac{1}{4}+\frac{|\mathfrak{m}|}{2}}}{1+q^{\frac{1}{2}}}(a+a^{-1})\left((\eta q^{1/2\nu})^{\frac{1}{2}} + (\eta q^{1/2\nu})^{-\frac{1}{2}}\right)\ , \\ \Delta(\mathfrak{m},a) &:= \frac{1}{\mathrm{Sym}(\mathfrak{m})} q^{-|\mathfrak{m}|}(1-a^2 q^{|\mathfrak{m}|})(1-a^{-2}q^{|\mathfrak{m}|})\ , \\ \text{with } \mathrm{Sym}(\mathfrak{m}) &:= \begin{cases} 2 & \text{if } \mathfrak{m}=0\ , \\ 1 & \text{if } \mathfrak{m}>0\ . \end{cases} \end{aligned} \tag{3.49}$$

Using the formula, one can compute the superconformal index and check that

$$\begin{aligned}&\mathcal{I}^{\rm sci}_{\mathrm{SU}(2)_k^{\frac{1}{2}\oplus\frac{1}{2}}}(q,\eta,\nu=0;s=1)\\&=\begin{cases}\infty & \text{if } k=0\\ 1 & \text{if } |k|=1\\ 1+q^{\frac{1}{2}}+\left(-1-\frac{1}{\eta}-\eta\right)q+\left(2+\eta+\frac{1}{\eta}\right)q^{\frac{3}{2}}+\dots & \text{if } |k|\geq 2\,.\end{cases}\end{aligned} \tag{3.50}$$

The higher order terms (represented by $\dots$) depend on k for $|k|\geq 2$. From the computation, one can determine the basic property of theory appearing at the IR. The triviality of the index for $|k|=1$ implies that the theory has a mass gap and the IR physics is described by an unitary TQFT. The divergence of the index is a signal of emergence of a free chiral theory that decouples with the other part of the theory [58]. The non-triviality of the index implies that the theory flows to a superconformal field theory. In summary, from the index computation we conclude that

$$\mathrm{SU}(2)_k^{\frac{1}{2}\oplus\frac{1}{2}} \xrightarrow{\text{at IR}} \begin{cases}\text{Contains decoupled free chirals} & \text{if } |k|=0\,,\\ \text{Unitary TQFT} & \text{if } |k|=1\,,\\ \text{3D } \mathcal{N}=5 \text{ SCFT} & \text{if } |k|\geq 2\,.\end{cases} \tag{3.51}$$

In the degenerate limits, $\nu\to\pm 1$ and $\eta\to 1$, the index becomes (for $|k|\geq 2$)

$$\begin{aligned}&\mathcal{I}^{\rm sci}_{\mathrm{SU}(2)_k^{\frac{1}{2}\oplus\frac{1}{2}}}(q,\eta,\nu{=}\pm1;s{=}1)=1{+}\,(1{-}\eta^{\mp1})q^{1/2}-(\eta^{\mp1}{-}1)q+(2{-}\eta^{\pm1}{-}\eta^{\mp2})q^{3/2}+\dots\,,\\&\mathcal{I}^{\rm sci}_{\mathrm{SU}(2)_k^{\frac{1}{2}\oplus\frac{1}{2}}}(q,\eta{=}1,\nu{=}\pm1;s{=}1)=1\,.\end{aligned} \tag{3.52}$$

It implies that the theory is of rank 0 and non-unitary TQFTs, $\mathrm{TFT}_\pm[\mathrm{SU}(2)_k^{\frac{1}{2}\oplus\frac{1}{2}}]$, emerges when $|k|\geq 2$ in the limits. In addition, we expect that they are spin TQFTs according to (2.8). The $\mathrm{TFT}_\pm[\mathrm{SU}(2)_k^{\frac{1}{2}\oplus\frac{1}{2}}]$ in the table 2 is indeed a spin TQFT since it is given by a $\mathbb{Z}_2^f$ quotient (fermionic anyon condensation) of a bosonic TQFT.

3.3.2 Non-unitary TQFTs in degenerate limits

Squashed three-sphere partition function. The partition function of the $\mathcal{N}=5$ theory is

$$\begin{aligned}&\mathcal{Z}^{S^3_b}_{\mathrm{SU}(2)_k^{\frac{1}{2}\oplus\frac{1}{2}}}(b,m,\nu)=\int\frac{\mathrm{d}Z}{\sqrt{2\pi\hbar}}\mathcal{I}_\hbar(Z,m,\nu)\ \ \text{with}\\&\mathcal{I}_\hbar(Z,m,\nu)=\frac{1}{2}(2\sinh(Z))\left(2\sinh\left(\frac{2\pi iZ}{\hbar}\right)\right)\exp\left(\frac{kZ^2}{\hbar}\right)\\&\qquad\times\prod_{\epsilon_1,\epsilon_2\in\{\pm1\}}\Psi_\hbar\left[\epsilon_1 Z+\epsilon_2\frac{m+\nu\left(i\pi+\frac{\hbar}{2}\right)}{2}+\frac{\pi i}{2}+\frac{\hbar}{4}\right]\,.\end{aligned} \tag{3.53}$$

At $b = 1$ and $m = 0$, the localization integral is simplified as

$$\mathcal{Z}^{S_b^3}_{\mathrm{SU}(2)_k^{\frac{1}{2}\oplus\frac{1}{2}}}(b=1,m=0,\nu) = \frac{e^{\frac{\pi i}{12}}}{2\pi}\int dZ \frac{\sinh^2(Z)}{\cosh(Z)+\cosh(i\pi\nu)}\exp\left(\frac{kZ^2}{2\pi i}\right). \tag{3.54}$$

At $\nu = 0, \pm 1$, the partition function is exactly computable and we obtain

$$\begin{aligned}\exp(-F) &= \left|\mathcal{Z}^{S_b^3}_{\mathrm{SU}(2)_k^{\frac{1}{2}\oplus\frac{1}{2}}}(b=1,m=0,\nu=0)\right| = \sqrt{\frac{2}{|k|}}\sin\left(\frac{\pi}{4|k|}\right),\\ (S_{00} \text{ of TFT}_\pm) &= \left|\mathcal{Z}^{S_b^3}_{\mathrm{SU}(2)_k^{\frac{1}{2}\oplus\frac{1}{2}}}(b=1,m=0,\nu\to\pm1)\right| = \sqrt{\frac{2}{|k|}}\sin\left(\frac{\pi(2|k|-1)}{4|k|}\right).\end{aligned} \tag{3.55}$$

Bethe-vacua and Handle gluing operators in the degenerate limits. Since there is a $\mathbb{Z}_2$ symmetry $(m,\nu) \leftrightarrow (-m,-\nu)$, we only consider the limit $\nu \to 1$. In the limit $\nu \to 1$, the asymptotic expansion coefficients $\mathcal{W}_0$ and $\mathcal{W}_1$ of the integrand are

$$\begin{aligned}&\log\mathcal{I}_\hbar(Z,m,\nu) \xrightarrow{\hbar\to0} \frac{1}{\hbar}\mathcal{W}_0(Z,m,\nu)+\mathcal{W}_1(Z,m,\nu)+O(\hbar) \;\text{ with}\\ &\mathcal{W}_0(Z,m,\nu=1) = -\frac{\pi^2}{2}+\frac{\pi i m}{2}+\frac{m^2}{4}\pm2\pi i Z+(k+1)Z^2+\sum_{\epsilon_1,\epsilon_2\in\{\pm1\}}\mathrm{Li}_2(e^{-\epsilon_1 Z-\epsilon_2\frac{m}{2}}),\\ &\mathcal{W}_1(Z,m,\nu=1) = \frac{1}{2}\left(\pi i+\frac{m}{2}-\log(1-e^{\frac{m}{2}-Z})-\log(1-e^{\frac{m}{2}+Z})\right)+\log(\sinh(Z)).\end{aligned} \tag{3.56}$$

The Bethe vacua equation at $\nu = 1$ is

$$\exp(\partial_Z\mathcal{W}_0(Z,m,\nu=1))\Big|_{Z\to\log(z),m\to\log(\eta)} = \frac{(\sqrt{\eta}-z)(\sqrt{\eta}z+1)}{(\sqrt{\eta}+z)(\sqrt{\eta}z-1)}z^{2k} = 1. \tag{3.57}$$

In the degenerate limit $\eta \to 1$, the equation simplifies as $z^{2k} = -1$ and there are $|k|$ Bethe-vacua after taking into account of the Weyl $\mathbb{Z}_2$ quotient, $z \leftrightarrow 1/z$,

$$\text{Bethe-vacua: } z_\alpha = (-1)^{\frac{2k-2\alpha+1}{2|k|}}, \quad \alpha = 1,\cdots,|k|. \tag{3.58}$$

Now, the handle gluing in the degenerate limit, $m = 0, \nu = 1$, is given by

$$\begin{aligned}\mathcal{H}(z) &= \exp\left(-2\mathcal{S}_1^\alpha\right)|_{Z\to\log(z),\nu\to1,m\to0} = \frac{1}{4}e^{-2\mathcal{W}_1}\partial_Z\partial_Z\mathcal{W}_0\Big|_{Z\to\log(z),\nu\to1,m\to0}\\ &= \frac{2kz}{(z+1)^2}.\end{aligned} \tag{3.59}$$

The factor $1/4$ comes from $1/|\mathrm{Weyl}(\mathrm{SU}(2))|^2$, see (A.34). By plugging the eq. (3.58) in the eq. (3.59), we have

$$\left(\mathcal{H}_\alpha \text{ of } \mathrm{SU}(2)_k^{\frac{1}{2}\oplus\frac{1}{2}}\right) = \left(\sqrt{\frac{2}{|k|}}\sin\left(\frac{\pi(2\alpha-1)}{4|k|}\right)\right)^{-2}, \quad \alpha = 1,\ldots,|k|. \tag{3.60}$$

The set of $\{|S_{0\alpha}| = (\mathcal{H}_\alpha)^{-1/2}\}$ is identical to the set $\{|S_{0\alpha}|\}$ of the $\mathrm{SU}(2)_{4k-2}/\mathbb{Z}_2^f$ theory. It implies that the non-unitary TQFT $\mathrm{TFT}_\pm[\mathrm{SU}(2)_k^{\frac{1}{2}\oplus\frac{1}{2}}]$ is a Galois conjugate of $\mathrm{SU}(2)_{4|k|-2}/\mathbb{Z}_2^f$ with S_{00} in (3.55).

	q_1	q_2	q_3	q_4	ϕ_0
$\mathrm{U}(1)_{\text{gauge}}$	1	1	-1	-1	0
$\mathrm{U}(1)^H$	1	-1	1	-1	0
$\mathrm{U}(1)_{\text{axial}}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	-1
$\mathrm{U}(1)^C$	0	0	0	0	0

Table 3. Charge assignment in $T[\mathrm{SU}(2)]$ theory. $\mathrm{U}(1)^H, \mathrm{U}(1)^C$ and $\mathrm{U}(1)_{\text{axial}}$ denote the Cartans of $\mathrm{SU}(2)^H, \mathrm{SU}(2)^C$ and axial $\mathrm{U}(1) \subset \mathrm{SO}(4)_R$ symmetry respectively.

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3.4 $T[\mathrm{SU}(2)]_{k_1,k_2}$ and $T[\mathrm{SU}(2)]_{k_1,k_2}/\mathbb{Z}_2$

$T[\mathrm{SU}(2)]$ is the 3D theory living on the S duality domain wall in 4D $\mathcal{N}=4$ SYM [59]. The theory is the 3d $\mathcal{N}=4$ SQED with two fundamental hyper-multiplets. See table 3 for the matter contents of the theory. Let the four $\mathcal{N}=2$ chiral fields in the two $\mathcal{N}=4$ hyper-multiplets be q_1, q_2, q_3, q_4 and the adjoint $\mathcal{N}=2$ chiral field in the $\mathcal{N}=4$ vector multiplet be ϕ_0. The theory has $\mathrm{SU}(2)^H \times \mathrm{SU}(2)^C$ flavor symmetry at the IR as well as the SO(4) R-symmetry. The charge assignments for chiral fields under the Cartan subalgebra of the gauge and global symmetries are:

By gauging the two SU(2)s with non-zero Chern-Simons level k_1 and k_2, we obtain infinitely many rank 0 3D $\mathcal{N}=4$ SCFTs which will be denoted as

$$\begin{aligned} T[\mathrm{SU}(2)]_{k_1,k_2} &:= \frac{T[\mathrm{SU}(2)]}{\mathrm{SU}(2)^H_{k_1} \times \mathrm{SU}(2)^C_{k_2}} \\ &:= \Big(\text{Gauging } \mathrm{SU}(2)^H \times \mathrm{SU}(2)^C \text{ of } T[\mathrm{SU}(2)] \text{ with Chern-Simons levels } k_1 \text{ and } k_2\Big) . \end{aligned} \tag{3.61}$$

As argued in [27], the gauging does not break the supersymmetry down to $\mathcal{N}=3$ thanks to the nilpotent property of the moment map operators, $\vec{\mu}^H$ and $\vec{\mu}^C$, of the two SU(2)s.

The theory has $\mathbb{Z}_2^H \times \mathbb{Z}_2^C$ one-form symmetry originating from the center symmetry $\mathbb{Z}_2 \times \mathbb{Z}_2$ of the $\mathrm{SU}(2)^H \times \mathrm{SU}(2)^C$ gauge group. The discrete one-form symmetry has 't Hooft anomaly characterized by the following bulk action[5]

$$S_{\text{anom}} = \pi \int_{\mathcal{M}_4} \left(k_1 \frac{\mathcal{P}(w_2^H)}{2} + k_2 \frac{\mathcal{P}(w_2^H)}{2} + w_2^H \cup w_2^C \right) \quad (\text{mod } 2\pi) . \tag{3.62}$$

Here w_2^H (w_2^C) is the 2nd Stiefel-Whitney class, valued in $H^2(\mathcal{M}_4, \mathbb{Z}_2)$, of the $\mathrm{SO}(3)_H = (\mathrm{SU}(2)_H)/\mathbb{Z}_2$ and $\mathrm{SO}(3)_C = \mathrm{SU}(2)_C/\mathbb{Z}_2$ bundle respectively. $\mathcal{P}$ is the Pontryagin square operation,

$$\mathcal{P} \; : \; H^2(\mathcal{M}_4, \mathbb{Z}_2) \to H^4(\mathcal{M}_4, \mathbb{Z}_2) , \tag{3.63}$$

[5] The anomaly can be interpreted as a dependence of the partition function $\mathcal{Z}^{\mathcal{M}_3=\partial X_4}$ on the choice of a 4-manifold X_4 having $\mathcal{M}_3$ as a boundary. The difference between two partition functions, $\mathcal{Z}^{\mathcal{M}_3=\partial X_4}$ and $\mathcal{Z}^{\mathcal{M}_3=\partial Y_4}$, with two choices of 4-manifolds, X_4 and Y_4, is determined by the bulk action S_{anom} as $\frac{\mathcal{Z}^{\mathcal{M}_3=\partial X_4}}{\mathcal{Z}^{\mathcal{M}_3=\partial Y_4}} = e^{iS_{\text{anom}}[\mathcal{M}_4]}$. Here $\mathcal{M}_4 = X_4 \bigcup \overline{Y_4}$ is a closed 4-manifold obtained by gluing X_4 and orientation reversed Y_4 along the common boundary $\mathcal{M}_3$.

which satisfies $\mathcal{P}(w_2) = w_2^2 \pmod 2$. On spin manifold $\mathcal{M}_4$, the $\frac{1}{2}\mathcal{P}(\omega) \in \mathbb{Z}$. The first two terms in (3.62) come from the Chern-Simons action of two SU(2) gauge fields [60] while the last term is from the anomaly polynomial of $T[\mathrm{SU}(2)]$ theory [61]. From the anomaly polynomial, one can confirm that the following $\mathbb{Z}_2$ one-form symmetry is anomaly free

$$\text{Anomaly free one-form } \mathbb{Z}_2 \text{ symmetry}: \begin{cases} \mathbb{Z}_2^H\,, & k_1 \in 2\mathbb{Z} \text{ and } k_2 \in 2\mathbb{Z}+1\,, \\ \mathbb{Z}_2^C\,, & k_2 \in 2\mathbb{Z} \text{ and } k_1 \in 2\mathbb{Z}+1\,, \\ \mathbb{Z}_2^{\text{diag}} \subset \mathbb{Z}_2^C \times \mathbb{Z}_2^H\,, & \text{otherwise}\,. \end{cases} \tag{3.64}$$

The $\mathbb{Z}_2$ one-form symmetry can be gauged and we define

$$\begin{aligned} &T[\mathrm{SU}(2)]_{k_1,k_2}/\mathbb{Z}_2 \\ &:= (\text{Theory after gauging the anomaly free one-form } \mathbb{Z}_2 \text{ symmetry in } T[\mathrm{SU}(2)]_{k_1,k_2})\,. \end{aligned} \tag{3.65}$$

3.4.1 IR phases

Superconformal index. Index of the theory $T[\mathrm{SU}(2)]_{k_1,k_2}$ (or $T[\mathrm{SU}(2)]_{k_1,k_2}/\mathbb{Z}_2$) is

$$\begin{aligned} &\mathcal{I}^{\text{sci}}(q,\eta,\nu;s=1) \\ &= \sum_{\mathfrak{m}_1,\mathfrak{m}_2} \oint_{|a_1|=1,|a_2|=1} \left(\prod_{i=1}^{2} \frac{\Delta(\mathfrak{m}_i,a_i)da_i}{2\pi i a_i} (a_i(-1)^{\mathfrak{m}_i})^{2k_i\mathfrak{m}_i} \right) \mathcal{I}^{\text{sci}}_{T[\mathrm{SU}(2)]}(a_1,a_2,\eta,\nu;\mathfrak{m}_1,\mathfrak{m}_2)\,. \end{aligned} \tag{3.66}$$

Here the generalized superconformal index for $T[\mathrm{SU}(2)]$ theory is

$$\begin{aligned} &\mathcal{I}^{\text{sci}}_{T[\mathrm{SU}(2)]}(a_1,a_2,\eta,\nu;\mathfrak{m}_1,\mathfrak{m}_2) = \mathcal{I}^{\text{sci}}_{T[\mathrm{SU}(2)]}(a_1,a_2,\eta,\nu=0;\mathfrak{m}_1,\mathfrak{m}_2)\big|_{\eta\to\eta q^{\frac{\nu}{2}}} \text{ with} \\ &\mathcal{I}^{\text{sci}}_{T[\mathrm{SU}(2)]}(a_1,a_2,\eta,\nu=0;\mathfrak{m}_1,\mathfrak{m}_2) \\ &= \sum_{\mathfrak{n}} \oint_{|u|=1} \frac{du}{2\pi i u} ((-1)^{\mathfrak{m}_2}a_2)^{-2\mathfrak{n}} ((-1)^{\mathfrak{n}}u)^{-2\mathfrak{m}_2} (q^{\frac{1}{2}}\eta^{-1})^{\frac{1}{2}(|\mathfrak{m}_1+\mathfrak{n}|+|\mathfrak{m}_1-\mathfrak{n}|)} \mathrm{P.E.}[f_{\text{single}}]\,, \\ &\text{where} \\ &f_{\text{single}}(a_1,a_2,\eta,u;\mathfrak{m}_1,\mathfrak{n}) = \frac{q^{\frac{1}{4}}\sqrt{\eta}-q^{\frac{3}{4}}\sqrt{\eta^{-1}}}{1-q} q^{\frac{1}{2}|\mathfrak{m}_1+\mathfrak{n}|} \left(a_1 u + \frac{1}{a_1 u}\right) \\ &\qquad + \frac{q^{\frac{1}{4}}\sqrt{\eta}-q^{\frac{3}{4}}\sqrt{\eta^{-1}}}{1-q} q^{\frac{1}{2}|\mathfrak{m}_1-\mathfrak{n}|} \left(\frac{a_1}{u} + \frac{u}{a_1}\right) + \frac{q^{\frac{1}{2}}}{1-q}\left(\frac{1}{\eta}-\eta\right)\,. \end{aligned} \tag{3.67}$$

From Dirac quantization conditions, following monopole fluxes are allowed

$$\mathfrak{n},\mathfrak{m}_1,\mathfrak{m}_2 \in \frac{1}{2}\mathbb{Z} \text{ with } \mathfrak{n}\pm\mathfrak{m}_1 \in \mathbb{Z}\,. \tag{3.68}$$

In the above formula, however, we are only summing over following monopole fluxes

$$\text{for } T[\mathrm{SU}(2)]_{k_1,k_2} \text{ theory: } \mathfrak{n},\mathfrak{m}_1,\mathfrak{m}_2 \in \mathbb{Z}\,, \tag{3.69}$$

since we are summing over SU(2) bundles, i.e. SO(3) bundles with trivial w_2. The localization saddle point (A.8) with SU(2) monopole flux $\mathfrak{m}$ has non-trivial w_2 if and only if

$$\mathfrak{m} \in \mathbb{Z} + \frac{1}{2} \, . \tag{3.70}$$

For the theory (3.65) after gauging the $\mathbb{Z}_2$ one-form symmetry, we also need to sum over gauge bundle with non-trivial $w_2^{\mathbb{Z}_2}$. For the superconformal index for the $T[\mathrm{SU}(2)]_{k_1,k_2}/\mathbb{Z}_2$ theory (3.65), the summation range of monopole fluxes are

$$\text{for } T[\mathrm{SU}(2)]_{k_1,k_2}/\mathbb{Z}_2 \text{ theory: } \begin{cases} \mathfrak{n}+\mathfrak{m}_1, 2\mathfrak{m}_1, \mathfrak{m}_2 \in \mathbb{Z}\,, & k_1 \in 2\mathbb{Z} \text{ and } k_2 \in 2\mathbb{Z}+1\,, \\ \mathfrak{n}, \mathfrak{m}_1, 2\mathfrak{m}_2 \in \mathbb{Z}\,, & k_2 \in 2\mathbb{Z} \text{ and } k_1 \in 2\mathbb{Z}+1\,, \\ \mathfrak{n}+\mathfrak{m}_1, 2\mathfrak{m}_1, \mathfrak{m}_1-\mathfrak{m}_2 \in \mathbb{Z}\,, & \text{otherwise}\,. \end{cases} \tag{3.71}$$

From the formulae in (3.66), (3.67), (3.69), (3.71), one can compute the superconformal indices and check followings

$$\begin{aligned} &\mathcal{I}^{\text{sci}}(q,\eta,\nu=0;s=1) \text{ of } T[\mathrm{SU}(2)]_{k_1,k_2} \text{ (or } T[\mathrm{SU}(2)]_{k_1,k_2}/\mathbb{Z}_2) \\ &= \begin{cases} \text{Non-trivial power series in } q^{1/2} & \text{if } |k_1k_2-1|>3 \text{ and } \min(|k_1|,|k_2|)>1\,, \\ 0 & \text{if } |k_1k_2-1|=1\,, \\ 1 & \text{if } |k_1k_2-1|=3 \text{ or } (|k_1k_2-1|>3 \text{ and } \min(|k_1|,|k_2|)=1)\,, \\ 1 \text{ (or 2)} & \text{if } |k_1k_2-1|=2\,, \\ \infty & \text{if } |k_1k_2-1|=0\,. \end{cases} \end{aligned} \tag{3.72}$$

For the case when $|k_1k_2-1|=1$, the index vanishes and it implies that SUSY is spontaneously broken. For the case when $|k_1k_2-1|=2$, on the other hand, the index for $T[\mathrm{SU}(2)]_{k_1k_2}$ is just 1 while the index for $T[\mathrm{SU}(2)]_{k_1k_2}/\mathbb{Z}_2$ is surprisingly 2. It implies that theory $T[\mathrm{SU}(2)]_{k_1k_2}$ has a mass gap and flows to a topological theory and the UV $\mathbb{Z}_2$ one-form symmetry decouples at IR, i.e. the $\mathbb{Z}_2$ does not act faithfully on any IR observables. It means that the IR TQFT actually does not have the $\mathbb{Z}_2$ symmetry. The index for $T[\mathrm{SU}(2)]_{k_1k_2}/\mathbb{Z}_2$ becomes 2 just because we perform the gauging of the decoupled (so absent) one-form symmetry by hand. At the level of $S^2\times S^1$ partition function, the one-form gauging procedure is

$$\mathcal{Z}^{S^2\times S^1}_{\mathrm{TFT}/\mathbb{Z}_2} = \sum_{[\beta_2]\in H^2(S^2\times S^1,\mathbb{Z}_2)} \mathcal{Z}^{S^2\times S^1}_{\mathrm{TFT}}([\beta_2])\,. \tag{3.73}$$

Here the $[\beta_2]\in H^2(S^2\times S^1,\mathbb{Z}_2)=\mathbb{Z}_2$ is the background 2-form $\mathbb{Z}_2$ flat connections coupled to the one-form symmetry. Alternatively, the r.h.s. can be written as

$$\mathcal{Z}^{S^2\times S^1}_{\mathrm{TFT}} + \mathcal{Z}^{S^2\times S^1+\mathcal{O}^{[S^1]}_{\alpha_{\mathbb{Z}_2}}}_{\mathrm{TFT}}\,, \tag{3.74}$$

where first term is the $S^2\times S^1$ partition function (with trivial $[\beta_2]$) and the 2nd term is the partition function with insertion of loop operator along the $[S^1]$, the generator of

$H_1(S^2\times S^1,\mathbb{Z}_2) = H^2(S^2\times S^1,\mathbb{Z}_2)$, with the $\mathbb{Z}_2$ symmetry generating anyon $\alpha_{\mathbb{Z}_2}$. For general TFT with a faithful $\mathbb{Z}_2$ one-form symmetry, the first term is just 1 while the 2nd term vanishes and the total partition function $\mathcal{Z}^{S^2\times S^1}_{\mathrm{TFT}/\mathbb{Z}_2}$ becomes just 1 as expected. When the $\mathbb{Z}_2$ one-form symmetry is decoupled at IR, however, the 2nd term is also 1 since the anyon $\alpha_{\mathbb{Z}_2}$ actually becomes the trivial operator, $\mathcal{O}_{\alpha=0}$, at IR. So the result, $\mathcal{Z}^{S^2\times S^1}_{\mathrm{TFT}/\mathbb{Z}_2}=2$, is an artifact due to the "gauging" of the "absent" $\mathbb{Z}_2$ symmetry. As we will see below, one can actually confirm that the $\mathbb{Z}_2$ symmetry act trivially on Bethe-vacua of $T[\mathrm{SU}(2)]_{k_1,k_2}$ theory for the case when $|k_1k_2-1|=2$. In summary, from the index computation, we can conclude that

$$T[\mathrm{SU}(2)]_{k_1,k_2} \xrightarrow{\text{at IR}} \begin{cases} \text{Non-trivial } \mathcal{N}=4 \text{ SCFT} & \text{if } |k_1k_2-1|>3 \text{ and } \min(|k_1|,|k_2|)>1\,, \\ \text{SUSY broken} & \text{if } |k_1k_2-1|=1\,, \\ \text{Unitary TQFT} \quad \text{if } |k_1k_2-1|=3 & \text{or } (|k_1k_2-1|>3 \text{ and } \min(|k_1|,|k_2|)=1\,, \\ \text{Unitary TQFT with decoupled } \mathbb{Z}_2 & \text{if } |k_1k_2-1|=2\,, \\ \text{Decoupled free chirals} & \text{if } |k_1k_2-1|=0\,. \end{cases} \tag{3.75}$$

In the degenerate limits ($\nu\to\pm1$ and $\eta\to1$), on the other hand, the indices are (when $|k_1k_2-1|>3$ and $\min(|k_1|,|k_2|)>1$)

$$\begin{aligned} &\text{for } T[\mathrm{SU}(2)]_{k_1,k_2} \text{ theory}\,, \\ &\mathcal{I}^{\mathrm{sci}}(q,\eta,\nu=\pm1;s=1) = (\text{non-trivial power series in } q) \\ \text{and } \ &\mathcal{I}^{\mathrm{sci}}(q,\eta=1,\nu=\pm1;s=1)=1\,, \end{aligned} \tag{3.76}$$

while

$$\begin{aligned} &\text{for } T[\mathrm{SU}(2)]_{k_1,k_2}/\mathbb{Z}_2 \text{ theory}\,, \\ &\mathcal{I}^{\mathrm{sci}}(q,\eta,\nu=\pm1;s=1) = \begin{cases} \text{non-trivial power series in } q^{1/2} & \text{if } k_1k_2\in 4\mathbb{Z}+1\,. \\ \text{non-trivial power series in } q & \text{otherwise}\,. \end{cases} \\ &\text{and } \mathcal{I}^{\mathrm{sci}}(q,\eta=1,\nu=\pm1;s=1)=1\,. \end{aligned} \tag{3.77}$$

The computation implies that the IR theories are $\mathcal{N}=4$ SCFTs of rank 0 and non-unitary TQFTs

$$\mathrm{TFT}_\pm[T[\mathrm{SU}(2)_{k_1,k_2}]] \text{ or } \mathrm{TFT}_\pm[T[\mathrm{SU}(2)_{k_1,k_2}]/\mathbb{Z}_2] \tag{3.78}$$

emerge in the degenerate limits, $\eta\to1$ and $\nu\to\pm1$. According to the criterion in (2.8), we further expect that $\mathrm{TFT}_\pm[T[\mathrm{SU}(2)]_{k_1,k_2}/\mathbb{Z}_2]$ is a spin TQFT when $k_1k_2\in4\mathbb{Z}+1$. The non-unitary TQFTs associated to the $\mathcal{N}=4$ SCFT before the $\mathbb{Z}_2$ one-form symmetry gauging are given in table 2 and the TQFTs after the gauging are

$$\begin{aligned} \mathrm{TFT}_\pm[T[\mathrm{SU}(2)]_{k_1,k_2}/\mathbb{Z}_2] &= \frac{\mathrm{TFT}_\pm[T[\mathrm{SU}(2)]_{k_1,k_2}]}{\mathbb{Z}_2} = \frac{\mathrm{Gal}_{d_\pm}(\mathrm{SU}(2)_{|k_1k_2-1|-2})\otimes \mathrm{U}(1)_2}{\mathbb{Z}_2}\,, \\ \text{with } \ d_+ &= \zeta^{|k_2|}_{|k_1k_2-1|-2},\ d_- = \zeta^{|k_1|}_{|k_1k_2-1|-2}\,. \end{aligned} \tag{3.79}$$

Here $\mathbb{Z}_2$ is the anomaly free $\mathbb{Z}_2$ one-form symmetry in $\mathrm{Gal}_{d_\pm}(\mathrm{SU}(2)_{|k_1k_2-1|-2})\otimes \mathrm{U}(1)_2$. The topological theory has $\mathbb{Z}_2^{\mathrm{SU}(2)}\times\mathbb{Z}_2^{\mathrm{U}(1)}$ one-form symmetry and the following $\mathbb{Z}_2$ one-form symmetry is non-anomalous

$$\text{Anomaly free one-form } \mathbb{Z}_2 \text{ symmetry } : \begin{cases} \mathbb{Z}_2^{\mathrm{SU}(2)} & \text{if } k_1k_2 \in 2\mathbb{Z}+1 \ , \\ \mathbb{Z}_2^{\mathrm{diag}} & \text{otherwise} \ . \end{cases} \tag{3.80}$$

When $k_1k_2\in 4\mathbb{Z}_2+1$, the one-form $\mathbb{Z}_2$ symmetry is fermionic and the theory after the $\mathbb{Z}_2$ quotient becomes a spin TQFT,

$$\frac{\mathrm{Gal}_{d_\pm}(\mathrm{SU}(2)_{|k_1k_2-1|-2})\otimes \mathrm{U}(1)_2}{\mathbb{Z}_2} \text{ is a spin TQFT when } k_1k_2\in 4\mathbb{Z}+1 \ . \tag{3.81}$$

It confirms the criterion (2.8) combined with the superconformal index computation (3.77).

3.4.2 Non-unitary TQFTs in degenerate limits

Squashed three-sphere partition function. The partition function $\mathcal{Z}^{S_b^3}_{(k_1,k_2)}(b,m,\nu)$ of the $T[\mathrm{SU}(2)]_{k_1,k_2}$ theory is

$$\begin{aligned} &\mathcal{Z}^{S_b^3}_{(k_1,k_2)}(b,m,\nu) = \int \frac{dX_1 dX_2 dZ}{(2\pi\hbar)^{3/2}} \mathcal{I}_\hbar(X_1,X_2,Z,m,\nu) \ , \\ &\text{with } \mathcal{I}_\hbar(X_1,X_2,Z,m,\nu) = \mathcal{I}^{\mathrm{vec}}_\hbar(X_1,X_2)\times \mathcal{I}^{T[\mathrm{SU}(2)]}_\hbar(X_1,X_2,Z,m,\nu) \ . \end{aligned} \tag{3.82}$$

Here $\mathcal{I}^{\mathrm{vec}}_\hbar$ is the contribution from the vector multiplet for the $\mathrm{SU}(2)^H_{k_1}\times \mathrm{SU}(2)^C_{k_2}$ gauging:

$$\mathcal{I}^{\mathrm{vec}}_\hbar(Z_1,Z_2) = \exp\left(\frac{k_1X_1^2+k_2X_2^2}{\hbar}\right)\prod_{i=1}^{2}\frac{1}{2}\left(2\sinh X_i\right)\left(2\sinh\frac{2\pi i X_i}{\hbar}\right) \ . \tag{3.83}$$

$\mathcal{I}^{T[\mathrm{SU}(2)]}_\hbar$ is the contribution from the $T[\mathrm{SU}(2)]$ theory whose squashed three-sphere partition function is

$$\begin{aligned} &\mathcal{Z}^{S_b^3}_{T[\mathrm{SU}(2)]}(b,X_1,X_2,m,\nu) = \int\frac{dZ}{\sqrt{2\pi\hbar}}\,\mathcal{I}^{T[\mathrm{SU}(2)]}_\hbar(X_1,X_2,Z,m,\nu), \text{ where} \\ &\mathcal{I}^{T[\mathrm{SU}(2)]}_\hbar(X_1,X_2,Z,m,\nu) = \prod_{\epsilon_{1,2}=\pm1}\Psi_\hbar\left(\epsilon_1 Z+\epsilon_2X_1+\frac{m+\nu(i\pi+\frac{\hbar}{2})}{2}+\frac{\pi i}{2}+\frac{\hbar}{4}\right) \\ &\qquad\times\Psi_\hbar\left(-2\frac{m+\nu(i\pi+\frac{\hbar}{2})}{2}+\pi i+\frac{\hbar}{2}\right)\exp\left(-\frac{2ZX_2}{\hbar}-\frac{\left(\pi i+\frac{\hbar}{2}\right)}{\hbar}\frac{m+\nu\left(i\pi+\frac{\hbar}{2}\right)}{2}\right) \ . \end{aligned} \tag{3.84}$$

One of the non-trivial consistency checks is the mirror property

$$\mathcal{Z}^{S_b^3}_{T[\mathrm{SU}(2)]}(b,X_1,X_2,m=0,\nu) = \mathcal{Z}^{S_b^3}_{T[\mathrm{SU}(2)]}(b,X_2,X_1,m=0,-\nu) \ , \tag{3.85}$$

which we have checked numerically for various values of X_1, X_2, and ν. Particularly for $b=1, m=0, \nu=0$, we have [62–64] (see also appendix C.1)

$$\mathcal{Z}^{S_b^3}_{T[\mathrm{SU}(2)]}(b=1,X_1,X_2,m=0,\nu=0) = \frac{e^{\frac{2\pi i}{3}}}{2}\frac{\sin(\frac{X_1X_2}{\pi})}{\sinh(X_1)\sinh(X_2)} \ . \tag{3.86}$$

As in (3.54), this localization (3.82) also simplified at $b = 1$, $m = 0$, and is exactly computable at $\nu = 0, \pm 1$ (see appendix C.2).

$$
\begin{aligned}
\exp(-F) &= \left|\mathcal{Z}^{S^3_b}_{(k_1,k_2)}(b=1, m=0, \nu=0)\right| = \sqrt{\frac{1}{|k_1k_2-1|}}\sin\left(\frac{\pi}{|k_1k_2-1|}\right) , \\
(S_{00} \text{ of } \mathrm{TFT}_\pm) &= \left|\mathcal{Z}^{S^3_b}_{(k_1,k_2)}(b=1, m=0, \nu\to\pm1)\right| \\
&= \begin{cases} \sqrt{\frac{1}{|k_1k_2-1|}}\sin\left(\frac{\pi|k_2|}{|k_1k_2-1|}\right) , & \nu \to +1 , \\ \sqrt{\frac{1}{|k_1k_2-1|}}\sin\left(\frac{\pi|k_1|}{|k_1k_2-1|}\right) , & \nu \to -1 . \end{cases}
\end{aligned}
\tag{3.87}
$$

Bethe-vacua and Handle gluing operators in the degenerate limits. The asymptotic expansions, $\mathcal{W}_0$ and $\mathcal{W}_1$, of the localization integral are

$$
\log \mathcal{I}_\hbar(X_1, X_2, Z, m, \nu) \xrightarrow{\hbar\to0} \frac{1}{\hbar}\mathcal{W}_0(X_1, X_2, Z, m, \nu) + \mathcal{W}_1(X_1, X_2, Z, m, \nu) + O(\hbar) ,
$$

where

$$
\begin{aligned}
\mathcal{W}_0 &= (k_1+1)X_1^2 + k_2X_2^2 \pm 2\pi i X_1 \pm 2\pi i X_2 - 2X_2Z + Z^2 + \sum_{\epsilon_1,\epsilon_2=\pm1} \mathrm{Li}_2(e^{\epsilon_1X_1+\epsilon_2Z-\frac{m+i\pi\nu}{2}-\frac{\pi i}{2}}) , \\
\mathcal{W}_1 &= \frac{i\pi(\nu^2-\nu+1)}{2} + \frac{2m\nu-m}{4} + \frac{1}{4}\sum_{\epsilon_1,\epsilon_2=\pm1}(\nu-1)\log\left(1+e^{\epsilon_1X_1+\epsilon_2Z-\frac{m+i\pi\nu}{2}-\frac{\pi i}{2}}\right) \\
&\quad - \frac{\nu}{2}\log\left(1+e^{m+i\pi\nu}\right) + \log(\sinh X_1) + \log(\sinh X_2) .
\end{aligned}
\tag{3.88}
$$

In the $\mathcal{W}_0$ above, we have ignored terms which are independent on X_1, X_2 and Z. By extremizing the twisted superpotential

$$
\exp(\partial_{X_1}\mathcal{W}_0) = \exp(\partial_{X_2}\mathcal{W}_0) = \exp(\partial_Z\mathcal{W}_0) = 1 , \tag{3.89}
$$

we have following Bethe-vacua equations

$$
\begin{aligned}
&\frac{x_1^{2k_1}(wx_1+iz)(wx_1z+i)}{(wz+ix_1)(w+ix_1z)} = \frac{x_2^{2k_2}}{z^2} = \frac{(wz+ix_1)(wx_1z+i)}{x_2^2(wx_1+iz)(w+ix_1z)} = 1 , \\
&\qquad \text{where } \ x_1 = e^{X_1},\ x_2 = e^{X_2},\ z = e^Z,\ w = e^{\frac{m+i\pi\nu}{2}} .
\end{aligned}
\tag{3.90}
$$

At generic choice of w, there are $2 \times \left|(|k_1k_2-1|-1)\right|$ Bethe-vacua,

$$
\{(x_1, x_2, z) = ((x_1^*)_{n,\pm}, (x_2^*)_{n,\pm}, z^*_{n,\pm})\}_{n=1}^{\left||k_1k_2-1|-1\right|} , \tag{3.91}
$$

after removing the unphysical solutions, which are invariant under a non-trivial subgroup of the Weyl $\mathbb{Z}_2\times\mathbb{Z}_2$ acting as $x_i \to 1/x_i$, and quotienting by the Weyl group. The one-form symmetry $\mathbb{Z}_2^H \times \mathbb{Z}_2^C$, on the other hand, acts on the Bethe-vacua in the following way

$$
\mathbb{Z}_2^H \ : \ x_1 \to \pm x_1 \ , \quad \mathbb{Z}_2^C \ : \ x_2 \to \pm x_2 \ . \tag{3.92}
$$

When $|k_1 k_2 - 1| = 2$, the anomaly free $\mathbb{Z}_2 = \mathbb{Z}_2^{\text{diag}} \subset \mathbb{Z}_2^H \times \mathbb{Z}_2^C$ act trivially (modulo Weyl $\mathbb{Z}_2 \times \mathbb{Z}_2$) on the Bethe-vacua, i.e.

$$x_1^*, x_2^* \in \{i, -i\} \ . \tag{3.93}$$

This explains why we obtain 2 (instead of 1) in the index computation of $T[\mathrm{SU}(2)]_{k_1,k_2}/\mathbb{Z}_2$ theory when $|k_1 k_2 - 1| = 2$, see the paragraph below (3.72).

The handle gluing operator is

$$\mathcal{H}(x_1, x_2, z; m, \nu) = \frac{1}{16} \exp(-2\mathcal{W}_1) \det_{i,j} (\partial_i \partial_j \mathcal{W}_0) \ . \tag{3.94}$$

The set of the handle gluing operators evaluated at the Bethe-vacua in the degenerate limits are

$$\left\{ \mathcal{H}((x_1^*)_{n,\pm}, (x_2^*)_{n,\pm}, (z^*)_{n,\pm}) \right\}_{n=1}^{\big||k_1k_2-1|-1\big|} \xrightarrow{(m,\nu)\to(0,\pm 1)} \left\{ \frac{|k_1 k_2 - 1|}{\sin^2\left(\frac{n\pi}{|k_1k_2-1|}\right)}^{\otimes 2} \right\}_{n=1}^{\big||k_1k_2-1|-1\big|} . \tag{3.95}$$

3.5 $T[\mathrm{SU}(2)]/\mathrm{SU}(2)_k^{\text{diag}}$ and $T[\mathrm{SU}(2)]/$"$\mathrm{PSU}(2)_k^{\text{diag}}$"

Let us consider

$$\frac{T[\mathrm{SU}(2)]}{\mathrm{SU}(2)_k^{\text{diag}}} = \Big(\text{Gauging diagonal } \mathrm{SU}(2)^{\text{diag}} \text{ of } T[\mathrm{SU}(2)] \text{ with Chern-Simons level } k\Big) \ . \tag{3.96}$$

Thanks to the nilpotency of the moment map $\vec{\mu}^{\text{diag}}$ for the $\mathrm{SU}(2)^{\text{diag}}$ flavor symmetry, the theory remains $\mathcal{N} = 4$ theory even after the gauging with non-zero k [27, 55].

The theory has a $\mathbb{Z}_2$ one-form symmetry which corresponds to the center group of the gauged $\mathrm{SU}(2)^{\text{diag}}$ symmetry. The 't Hooft anomaly polynomial for the one-form symmetry is

$$S_{\text{anom}} = \pi \int_{\mathcal{M}_4} \left(k \frac{\mathcal{P}(w_2)}{2} \right) \quad (\text{mod } 2\pi) \ . \tag{3.97}$$

Thus, the one-form symmetry is anomalous for $k =$ (odd) while non-anomalous for $k =$ (even). For odd k, the theory can be tensored with a topological theory $\mathrm{U}(1)_2 = \mathrm{SU}(2)_1$, which also has anomalous $\mathbb{Z}_2$ one-form symmetry, and the diagonal $\mathbb{Z}_2$ one-form symmetry becomes non-anomalous. We define

$$\begin{aligned} &\frac{T[\mathrm{SU}(2)]}{\text{“}\mathrm{PSU}(2)_k^{\text{diag}}\text{”}} \\ &:= \begin{cases} \text{Gauging the } \mathbb{Z}_2 \text{ one-form symmetry of } \frac{T[\mathrm{SU}(2)]}{\mathrm{SU}(2)_k^{\text{diag}}} \ , & \text{even } k \ , \\ \text{Gauging the (diagonal) } \mathbb{Z}_2 \text{ one-form symmetry of } \left(\frac{T[\mathrm{SU}(2)]}{\mathrm{SU}(2)_k^{\text{diag}}} \otimes \mathrm{U}(1)_2 \right) \ , & \text{odd } k \ . \end{cases} \end{aligned} \tag{3.98}$$

3.5.1 IR phases

Superconformal index. The superconformal index of the $\frac{T[\mathrm{SU}(2)]}{\mathrm{SU}(2)^{\mathrm{diag}}_k}$ (or $\frac{T[\mathrm{SU}(2)]}{\text{“}\mathrm{PSU}(2)^{\mathrm{diag}}_k\text{”}}$) theory is

$$\begin{aligned}&\mathcal{I}^{\mathrm{sci}}(q,\eta,\nu;s=1)\\&=\sum_{\mathfrak{m}}\oint_{|a|=1}\frac{\Delta(\mathfrak{m},a)da}{2\pi i a}\,(a(-1)^{\mathfrak{m}})^{2k\mathfrak{m}}\,\mathcal{I}^{\mathrm{sci}}_{T[\mathrm{SU}(2)]}(a,a,\eta,\nu;\mathfrak{m},\mathfrak{m})\ ,\end{aligned} \tag{3.99}$$

where $\mathcal{I}^{\mathrm{sci}}_{T[\mathrm{SU}(2)]}$ is the index of $T[\mathrm{SU}(2)]$ theory given in (3.67). From the Dirac quantization conditions, the following monopole fluxes are allowed

$$\mathfrak{n},\mathfrak{m}\in\frac{\mathbb{Z}}{2}\ \text{ with }\mathfrak{n}\pm\mathfrak{m}\in\mathbb{Z}\ . \tag{3.100}$$

The summation range of monopole fluxes is

$$\begin{aligned}&\text{for }\frac{T[\mathrm{SU}(2)]}{\mathrm{SU}(2)^{\mathrm{diag}}_k}: &&\mathfrak{n},\mathfrak{m}\in\mathbb{Z}\ ,\\&\text{for }\frac{T[\mathrm{SU}(2)]}{\text{“}\mathrm{PSU}(2)^{\mathrm{diag}}_k\text{”}}: &&\mathfrak{n},\mathfrak{m}\in\frac{\mathbb{Z}}{2}\ \text{ with }\mathfrak{n}-\mathfrak{m}\in\mathbb{Z}\ .\end{aligned} \tag{3.101}$$

From superconformal index computation [55, 56],

$$\begin{aligned}&\mathcal{I}^{\mathrm{sci}}(q,\eta,\nu=0;s=1)\text{ for }\frac{T[\mathrm{SU}(2)]}{\mathrm{SU}(2)^{\mathrm{diag}}_k}\text{ or }\frac{T[\mathrm{SU}(2)]}{\text{“}\mathrm{PSU}(2)^{\mathrm{diag}}_k\text{”}}\\&=\begin{cases}1\ , & |k|<2\ ,\\ \infty\ , & |k|=2\ ,\\ \text{non-trivial power series in }q^{1/2}\ , & |k|>2\ .\end{cases}\end{aligned} \tag{3.102}$$

we expect that

$$\begin{aligned}&\frac{T[\mathrm{SU}(2)]}{\mathrm{SU}(2)^{\mathrm{diag}}_k}\text{ or }\frac{T[\mathrm{SU}(2)]}{\text{“}\mathrm{PSU}(2)^{\mathrm{diag}}_k\text{”}}\\&\xrightarrow{\text{at IR}}\begin{cases}\text{Unitary topological field theory} & \text{if }|k|<2\ ,\\ \text{Decoupled free chirals} & \text{if }|k|=2\ ,\\ \text{3D }\mathcal{N}=4\text{ SCFT} & \text{if }|k|>2\ .\end{cases}\end{aligned} \tag{3.103}$$

Superconformal indices in degenerate limits. In the degenerate limits ($\nu\to\pm1$ and $\eta\to1$), on the other hand, the indices are (when $|k|>2$)

$$\begin{aligned}&\text{for }\frac{T[\mathrm{SU}(2)]}{\mathrm{SU}(2)^{\mathrm{diag}}_k}\text{ theory}\ ,\\&\mathcal{I}^{\mathrm{sci}}\left(q,\eta,\nu=\pm1;s=1\right)=(\text{non-trivial power series in }q)\\&\text{and }\mathcal{I}^{\mathrm{sci}}\left(q,\eta=1,\nu=\pm1;s=1\right)=1\ ,\end{aligned} \tag{3.104}$$

while

$$\text{for } \frac{T[\mathrm{SU}(2)]}{\text{“PSU}(2)_k^{\text{diag}}\text{”}} \text{ theory },$$

$$\mathcal{I}^{\text{sci}}\left(q,\eta,\nu=\pm 1; s=1\right) = \begin{cases} \text{non-trivial power series in } q^{1/2} & \text{if } k\in 4\mathbb{Z}\,, \\ \text{non-trivial power series in } q & \text{otherwise}\,, \end{cases} \tag{3.105}$$

$$\text{and } \mathcal{I}^{\text{sci}}\left(q,\eta=1,\nu=\pm 1; s=1\right) = 1\,.$$

The computation implies that non-unitary TQFTs, $\mathrm{TFT}_{\pm}\Big[\frac{T[\mathrm{SU}(2)]}{\mathrm{SU}(2)_k^{\text{diag}}}\Big]$ and $\mathrm{TFT}_{\pm}\Big[\frac{T[\mathrm{SU}(2)]}{\text{“PSU}(2)_k^{\text{diag}}\text{”}}\Big]$, emerge in the degenerate limits and that $\mathrm{TFT}_{\pm}\Big[\frac{T[\mathrm{SU}(2)]}{\text{“PSU}(2)_k^{\text{diag}}\text{”}}\Big]$ are spin TQFTs when $k\in 4\mathbb{Z}$. From table 2, one can see that $\mathrm{TFT}_{\pm}\Big[\frac{T[\mathrm{SU}(2)]}{\text{“PSU}(2)_k^{\text{diag}}\text{”}}\Big]$ at $|k|=4$ are indeed spin TQFTs.

3.5.2 Non-unitary TQFTs in degenerate limits

Squashed three-sphere partition function. The squashed three-sphere partition function of the $T[\mathrm{SU}(2)]/\mathrm{SU}(2)_k^{\text{diag}}$ theory is realized as

$$\mathcal{Z}^{S_b^3}_{\text{diag}_k}(b,m,\nu) \equiv \frac{1}{2}\int \frac{dX}{\sqrt{2\pi\hbar}}\left(2\sinh(X)\right)\left(2\sinh\left(\frac{2\pi i X}{\hbar}\right)\right)e^{\frac{kX^2}{\hbar}}\mathcal{Z}^{S_b^3}_{T[\mathrm{SU}(2)]}(b,X,X,m,\nu)\,, \tag{3.106}$$

where $\mathcal{Z}^{S_b^3}_{T[\mathrm{SU}(2)]}$ is given in (3.84). As in (3.54), this localization (3.106) also simplified at $b=1$, $m=0$, and is exactly computable at $\nu=0,\pm1$ (see appendix C.3).

$$\begin{aligned} \exp(-F) &= \left|\mathcal{Z}^{S_b^3}_{\text{diag}_k}(b=1,m=0,\nu=0)\right| = \frac{1}{\sqrt{8(|k|-2)}} - \frac{1}{\sqrt{8(|k|+2)}}\,, \\ (S_{00} \text{ of } \mathrm{TFT}_{\pm}) &= \left|\mathcal{Z}^{S_b^3}_{\text{diag}_k}(b=1,m=0,\nu\to\pm1)\right| = \frac{1}{\sqrt{8(|k|-2)}} + \frac{1}{\sqrt{8(|k|+2)}}\,. \end{aligned} \tag{3.107}$$

Bethe-vacua and Handle gluing operators in the degenerate limits. Similar to (3.88), the asymptotic expansions $\mathcal{W}_0$ and $\mathcal{W}_1$ of the localization integral are

$$\begin{aligned} \mathcal{W}_0 &= (k+1)X^2 \pm 2\pi i X - 2XZ + Z^2 + \sum_{\epsilon_1,\epsilon_2=\pm1}\mathrm{Li}_2\big(e^{\epsilon_1 X+\epsilon_2 Z-\frac{m+i\pi\nu}{2}-\frac{\pi i}{2}}\big)\,, \\ \mathcal{W}_1 &= \frac{\pi i(\nu^2-\nu+1)}{2} + \frac{2m\nu-m}{4} + \frac{1}{4}\sum_{\epsilon_1,\epsilon_2=\pm1}(\nu-1)\log\left(e^{\epsilon_1 X+\epsilon_2 Z-\frac{m+i\pi\nu}{2}-\frac{\pi i}{2}}\right) \\ &\quad - \frac{\nu}{2}\log(1+e^{m+i\pi\nu}) + \log(\sinh(X))\,. \end{aligned} \tag{3.108}$$

We have ignored terms which are independent on X and Z in the expression for $\mathcal{W}_0$ given above. By extremizing the twisted superpotential

$$\exp(\partial_X \mathcal{W}_0) = \exp(\partial_Z \mathcal{W}_0) = 1\,, \tag{3.109}$$

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we have the Bethe-vacua equations

$$\frac{x^{2k}(wx+iz)(i+wxz)}{z^2(x-iwz)(iw-xz)} = \frac{(x-iwz)(i+wxz)}{x^2(z-iwx)(w+ixz)} = 1\,, \qquad \text{where} \quad x=e^X,\; z=e^Z \text{ and } w=e^{\frac{m+\pi i\nu}{2}}\,. \tag{3.110}$$

At generic choice of w, there are $2\times(2|k|+2)$ Bethe-vacua,

$$(x,z) = (x^*_{n,\pm}, z^*_{n,\pm})_{n=1}^{2|k|+2}\,, \tag{3.111}$$

after removing the unphysical solutions, which are invariant under $x_i \to 1/x_i$, and quotienting by the Weyl group. The handle gluing operator is

$$\mathcal{H}(x,z;m,\nu) = \frac{1}{4}\exp(-2\mathcal{W}_1)\det_{i,j}(\partial_i\partial_j\mathcal{W}_0)\,. \tag{3.112}$$

The set of the handle gluing operators evaluated at the Bethe-vacua in the degenerate limits are

$$\begin{aligned}&\left\{\mathcal{H}(x^*_{n,\pm}, z^*_{n,\pm})\right\}_{n=1}^{2|k|+2} \xrightarrow{(m,\nu)\to(0,\pm1)} \\ &\left\{\frac{1}{\sqrt{2A_k}}^{\otimes(|k|-3)}, \frac{1}{\sqrt{2B_k}}^{\otimes(|k|+1)}, \left(\frac{1}{\sqrt{8A_k}}+\frac{1}{\sqrt{8B_k}}\right)^{\otimes 2}, \left(\frac{1}{\sqrt{8A_k}}-\frac{1}{\sqrt{8B_k}}\right)^{\otimes 2}\right\}\,,\end{aligned} \tag{3.113}$$

where we define $A_k := |k|-2$, $B_k := |k|+2$.

For $|k|>2$, as concluded in (3.103), the theory $T[\mathrm{SU}(2)]/\mathrm{SU}(2)^{\mathrm{diag}}_k$ lands on 3D $\mathcal{N}=4$ SCFT of rank 0 at the end of RG. According to the dictionary in table 1, the above set should be equal to the set of $\{S^2_{0\alpha}\}$ for a non-unitary TQFT for the case when $|k|>2$.

$|k|=3$ case. The set of handle gluing (and S_{00} in (3.107)) is identical to the set of $\{|S^{-2}_{0\alpha}|\}$ (and S_{00}) for the (Lee-Yang)⊗(Lee-Yang)⊗$\mathrm{U}(1)_2$. From the computation, we arrive the conclusion in table 2.

$|k|=4$ case. The set is identical to the set of $\{|S^{-2}_{0\alpha}|\}$ for $\left(\mathrm{SU}(2)_{10}\times\mathrm{SU}(2)_2\right)/\mathbb{Z}_2^{\mathrm{diag}}$, see (A.99). Combined with the computation of S_{00} in (3.107), we arrive the conclusion in table 2.

$|k|=5$ case. The set of handle gluing is identical to the set of $\{|S^{-2}_{0\alpha}|\}$ for $(G_2)_3\times\mathrm{U}(1)_{-2}$, where G_2 is a exceptional group with dimension 14. Combined with the computation of S_{00} in (3.107), we arrive the conclusion in table 2.

$|k|>6$ case. For the cases, we could not identify $\mathrm{TFT}_\pm\left[\frac{T[\mathrm{SU}(2)]}{\mathrm{SU}(2)^{\mathrm{diag}}_k}\right]$ with previously known non-unitary TQFTs in the literature. It would be an interesting future work to better understand this novel series of non-unitary TQFTs.

3.6 Dualities among rank 0 theories

In this paper, we introduce a pair of non-unitary TQFTs, $\text{TFT}_\pm[\mathcal{T}_{\text{rank0}}]$, as an invariant of 3D rank 0 $\mathcal{N}=4$ SCFT $\mathcal{T}_{\text{rank0}}$. One natural question is

$$\begin{gathered}\text{Q: how powerful are } \text{TFT}_\pm[\mathcal{T}_{\text{rank0}}]\text{s in distinguishing } \mathcal{T}_{\text{rank0}}?\\ \text{i.e. are } \text{TFT}_\pm[\mathcal{T}_{\text{rank0}}]\text{s different for different } \mathcal{T}_{\text{rank0}}\text{s?}\end{gathered} \tag{3.114}$$

In this subsection, we provide evidences that the answer is affirmative. This is by examination of the duality between two $\mathcal{T}_{\text{rank 0}}$s whose associated non-unitary TQFTs are identical. Having common non-unitary TQFTs in the degenerate limits, $m=0$ and $\nu=\pm 1$, automatically implies that equivalence of all supersymmetric partition function $\mathcal{Z}^{\mathbb{B}}$ in the limits. Here we will confirm that the equivalence still holds at general values of m and ν.

Duality among $(\mathbf{U(1)}_{|k|=1}+H)$, $\mathbf{SU(2)}^{\frac{1}{2}\oplus\frac{1}{2}}_{|k|=2}$ and $\frac{T[\mathbf{SU(2)}]_{k_1=3,k_2=3}}{\mathbb{Z}_2}$. According to table 2, the non-unitary TQFTs associated to the 3 theories are all identical to the following theory

$$\text{TFT}_\pm = \text{Gal}_{d=\sin\left(\frac{3\pi}{8}\right)}(\text{SU}(2)_6)/\mathbb{Z}_2^f\,. \tag{3.115}$$

Using the explicit formulas in (3.29), (3.48), (3.49), (3.66), (3.67) and (3.71), one can confirm that the superconformal indices for the 3 theories are all equal to

$$\begin{aligned}&\mathcal{I}^{\text{sci}}(q,\eta,\nu=1;s=1)\\&=1+q^{\frac{1}{2}}-\left(1+\eta+\frac{1}{\eta}\right)q+\left(2+\eta+\frac{1}{\eta}\right)q^{\frac{3}{2}}-\left(2+\eta+\frac{1}{\eta}\right)q^2+\dots\,.\end{aligned} \tag{3.116}$$

One can check that computations of various other supersymmetric partition functions also support the duality.

Duality between $(\mathcal{T}_{\mathbf{min}})^{\otimes 2}\otimes\mathbf{U(1)_2}$ and $\frac{T[\mathbf{SU(2)}]}{\mathbf{SU(2)}^{\text{diag}}_{|k|=3}}$. One can check that [27, 55]

$$\begin{aligned}&\left(\mathcal{I}^{\text{sci}}(q,\eta,\nu=0;s=1)\text{ of }\frac{T[\text{SU}(2)]}{\text{SU}(2)^{\text{diag}}_{|k|=3}}\text{ given in (3.67), (3.99), (3.101)}\right)\\&=\left(\mathcal{I}^{\text{sci}}(q,\eta,\nu=0;s=1)\text{ of }\frac{T[\text{SU}(2)]}{\text{“PSU}(2)^{\text{diag}}_{|k|=3}\text{”}}\text{ given in (3.67), (3.99), (3.101)}\right)\\&=\left(\mathcal{I}^{\text{sci}}(q,\eta,\nu=0;s=1)\text{ of }\mathcal{T}_{\text{min}}\text{ given in (3.8)}\right)^2\\&=1-2q+2\left(\eta+\frac{1}{\eta}\right)q^{3/2}-3q^2+\left(2+\eta^2+\frac{1}{\eta^2}\right)q^3-4\left(\eta+\frac{1}{\eta}\right)q^{7/2}+\dots\,.\end{aligned} \tag{3.117}$$

From the index computation, it is tempting to identify both $\frac{T[\text{SU}(2)]}{\text{SU}(2)^{\text{diag}}_{|k|=3}}$ and $\frac{T[\text{SU}(2)]}{\text{“PSU}(2)^{\text{diag}}_{|k|=3}\text{”}}$ with $(\mathcal{T}_{\text{min}})^{\otimes 2}$. But this cannot be true since $\frac{T[\text{SU}(2)]}{\text{SU}(2)^{\text{diag}}_{|k|=3}}$ has the one-form $\mathbb{Z}_2$ symmetry while the other two theories do not. Further, the Witten indices of the three theories do not match:

$$\left(\text{Witten index of } (\mathcal{T}_{\rm min})^{\otimes 2}\right) = (\text{Witten index of } (\mathcal{T}_{\rm min}))^2 = 4\,,$$
$$\left(\text{Witten index of } \frac{T[\mathrm{SU}(2)]}{\text{``}\mathrm{PSU}(2)^{\rm diag}_{|k|=3}\text{''}}\right) = 4\,, \text{ but } \left(\text{Witten index of } \frac{T[\mathrm{SU}(2)]}{\mathrm{SU}(2)^{\rm diag}_{|k|=3}}\right) = 8\,.$$

From the computation of superconformal indices, Witten indices, and one-form $\mathbb{Z}_2$ symmetry matching, we propose that

$$\frac{T[\mathrm{SU}(2)]}{\mathrm{SU}(2)^{\rm diag}_{|k|=3}} = (\mathcal{T}_{\rm min})^{\otimes 2} \otimes \mathrm{U}(1)_2 \quad \text{at IR}. \tag{3.118}$$

The additional $\mathrm{U}(1)_2$ theory does not contribute to the superconformal index since there is no non-trivial local operator in the topological sector. The additional topological sector, however, doubles the Witten index since it has two ground states on two-torus and provide the one-form $\mathbb{Z}_2$ symmetry. The proposal is also compatible with following fact

$$\mathrm{TFT}_{\pm}\left[\frac{T[\mathrm{SU}(2)]}{\mathrm{SU}(2)^{\rm diag}_{|k|=3}}\right] = (\mathrm{TFT}_{\pm}[\mathcal{T}_{\rm min}])^{\otimes 2} \otimes \mathrm{U}(1)_2\,, \tag{3.119}$$

which is obvious from table 2. Using the duality, we can also confirm that

$$\begin{aligned}\frac{T[\mathrm{SU}(2)]}{\text{``}\mathrm{PSU}(2)^{\rm diag}_{|k|=3}\text{''}} &:= \left(\frac{T[\mathrm{SU}(2)]}{\mathrm{SU}(2)^{\rm diag}_{|k|=3}} \otimes \mathrm{U}(1)_2\right)/\mathbb{Z}_2^{\rm diag} = \left((\mathcal{T}_{\rm min})^{\otimes 2} \otimes \mathrm{U}(1)_2 \otimes \mathrm{U}(1)_2\right)/\mathbb{Z}_2^{\rm diag} \\ &= (\mathcal{T}_{\rm min})^{\otimes 2} \otimes \frac{\mathrm{U}(1)_2 \otimes \mathrm{U}(1)_2}{\mathbb{Z}_2^{\rm diag}} \simeq (\mathcal{T}_{\rm min})^{\otimes 2}\,.\end{aligned} \tag{3.120}$$

The theory $(\mathrm{U}(1)_2 \otimes \mathrm{U}(1)_2)/\mathbb{Z}_2^{\rm diag}$ is an almost-trivial theory, whose partition function on any closed 3-manifold is a pure phase factor. Throughout this paper, we have ignored the overall phase factor of the partition function and thus will also ignore such a decoupled almost trivial theory. The duality above has natural interpretation in terms of the 3D-3D correspondence for once-punctured torus bundles [65, 66], for which readers are referred to appendix B for details.

Duality between $\mathbf{SU(2)}^{\frac{1}{2}\oplus\frac{1}{2}}_{|k|=3}$ and $\frac{\boldsymbol{T}[\mathbf{SU(2)}]}{\text{``}\mathbf{PSU(2)}^{\rm diag}_{|k|=4}\text{''}}$. The superconformal index is

$$\begin{aligned}&\left(\mathcal{I}^{\rm sci}(q,\eta,\nu=0;s=1) \text{ of } \frac{T[\mathrm{SU}(2)]}{\mathrm{SU}(2)^{\rm diag}_{|k|=4}}\right) \\ &= 1 - q + \left(\eta+\frac{1}{\eta}\right)q^{3/2} - q^2 - 2\left(\eta+\frac{1}{\eta}\right)q^{5/2} + 2\left(3+\eta^2+\frac{1}{\eta^2}\right)q^3 + \dots\,, \\ &\left(\mathcal{I}^{\rm sci}(q,\eta,\nu=0;s=1) \text{ of } \frac{T[\mathrm{SU}(2)]}{\text{``}\mathrm{PSU}(2)^{\rm diag}_{|k|=4}\text{''}}\right) \\ &= 1 + q^{1/2} - \left(1+\eta+\frac{1}{\eta}\right)q + \left(2+\eta+\frac{1}{\eta}\right)q^{3/2} - q^2 - \left(\eta+\frac{1}{\eta}\right)\left(2+\eta+\frac{1}{\eta}\right)q^{5/2} + \dots\,.\end{aligned} \tag{3.121}$$

Two indices are different unlike in the $|k|=3$ case. Surprisingly, the index computation shows that the two theories actually have different amount of supersymmetries [54]:

$$\begin{aligned} \frac{T[\mathrm{SU}(2)]}{\mathrm{SU}(2)^{\mathrm{diag}}_{|k|=4}} &\quad \text{has } \mathcal{N}=4 \text{ SUSY },\\ \frac{T[\mathrm{SU}(2)]}{\text{“}\mathrm{PSU}(2)^{\mathrm{diag}}_{|k|=4}\text{”}} &\quad \text{has } \mathcal{N}=5 \text{ SUSY }. \end{aligned} \tag{3.122}$$

We can thus conclude that supersymmetry is enhanced under the $\mathbb{Z}_2$ one-form symmetry gauging in (3.98). We further claim that the $\mathcal{N}=5$ theory is actually dual to the following theory with manifest $\mathcal{N}=5$ supersymmetry:

$$\left(\frac{T[\mathrm{SU}(2)]}{\text{“}\mathrm{PSU}(2)^{\mathrm{diag}}_{|k|=4}\text{”}}\right) = \left(\mathrm{SU}(2)^{\frac{1}{2}\oplus\frac{1}{2}}_{|k|=3} \text{ in } (3.47)\right) \quad \text{at IR.} \tag{3.123}$$

The proposal can be checked using the superconformal index and various other supersymmetric partition functions. Neither theories in the duality has any one-form $\mathbb{Z}_2$ symmetry. The duality is also compatible with table 2 since

$$\begin{aligned} &\mathrm{TFT}_\pm\left[\frac{T[\mathrm{SU}(2)]}{\text{“}\mathrm{PSU}(2)^{\mathrm{diag}}_{|k|=4}\text{”}}\right] = \mathrm{TFT}_\pm\left[\frac{T[\mathrm{SU}(2)]}{\mathrm{SU}(2)^{\mathrm{diag}}_{|k|=4}}\right]/\mathbb{Z}_2 \\ &= \left(\frac{\mathrm{Gal}_{\zeta^7_{10}}(\mathrm{SU}(2)_{10})\otimes \mathrm{SU}(2)_2}{\mathbb{Z}_2^{\mathrm{diag}}}\right)/\mathbb{Z}_2 = \left(\frac{\mathrm{Gal}_{\zeta^7_{10}}(\mathrm{SU}(2)_{10})}{\mathbb{Z}_2}\right)\otimes\left(\frac{\mathrm{SU}(2)_2}{\mathbb{Z}_2}\right) \\ &\simeq \left(\frac{\mathrm{Gal}_{\zeta^7_{10}}(\mathrm{SU}(2)_{10})}{\mathbb{Z}_2}\right) = \mathrm{TFT}_\pm\left[\mathrm{SU}(2)^{\frac{1}{2}\oplus\frac{1}{2}}_{|k|=3}\right]. \end{aligned} \tag{3.124}$$

In the last line, we again ignore the almost trivial spin TQFT $\frac{\mathrm{SU}(2)_2}{\mathbb{Z}_2}$. In the second line, we use the following fact

$$\left(\frac{\mathrm{TFT}_1\otimes\mathrm{TFT}_2}{\mathbb{Z}_2^{\mathrm{diag}}}\right)/\mathbb{Z}_2 = \left(\frac{\mathrm{TFT}_1}{\mathbb{Z}_2}\right)\otimes\left(\frac{\mathrm{TFT}_2}{\mathbb{Z}_2}\right), \tag{3.125}$$

which holds for any two TQFTs, TFT_1 and TFT_2, which have non-anomalous $\mathbb{Z}_2$ one-form symmetries. The theory $T[\mathrm{SU}(2)]/\text{“}\mathrm{PSU}(2)^{\mathrm{diag}}_{|k|=4}\text{”}$ has yet another dual description with only manifest $\mathcal{N}=2$ supersymmetry which is expected from the geometrical aspects of the 3D-3D correspondence for a once-punctured torus bundle [65, 66]. The $\mathcal{N}=2$ dual is presented in appendix B.

4 Discussion

There are several interesting questions we want to address in future.

Relation with Rozansky-Witten theory. One well-known method of constructing a topological theory from a 3D $\mathcal{N}=4$ SCFT is using topological twisting as studied by Rozansky and Witten in [67]. An $\mathcal{N}=4$ SCFT $\mathcal{T}$ has $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$ R-symmetry and we can consider a pair of topological twisted theories, $\mathrm{RW}_+[\mathcal{T}]$ and $\mathrm{RW}_-[\mathcal{T}]$, using the $\mathrm{SU}(2)_L$ or $\mathrm{SU}(2)_R$ in the twisting respectively. It would be interesting to clarify the exact relation between the pair of topological twisted theories, $\mathrm{RW}_\pm[\mathcal{T}_{\text{rank }0}]$, and our $\mathrm{TFT}_\pm[\mathcal{T}_{\text{rank0}}]$ for rank 0 SCFT $\mathcal{T}_{\text{rank }0}$.[6]

Two theories, $\mathrm{RW}[\mathcal{T}_{\text{rank }0}]$ and $\mathrm{TFT}[\mathcal{T}_{\text{rank }0}]$, have the same ground state degeneracy GSD_g for all $g \geq 0$ since the partitial topological twisting on $\Sigma_g \times S^1$ using the $\mathrm{U}(1) \subset \mathrm{SU}(2)_R$ (or $\mathrm{U}(1) \subset \mathrm{SU}(2)_L$) symmetry is actually equivalent to the full topological twisting on the 3-manifold using the $\mathrm{SU}(2)_R$ ($\mathrm{SU}(2)_L$) symmetry [69, 70]. But two theories cannot be the same since one (RW) is unitary while the other (TFT) is non-unitary. From the comparison, we naturally conjecture that

$$\mathrm{TFT}_\pm[\mathcal{T}_{\text{rank }0}] \text{ is a Galois conjugate of } \mathrm{RW}_\pm[\mathcal{T}_{\text{rank }0}]\,. \tag{4.1}$$

3D non-unitary TQFTs from 4D $\mathcal{N}=2$ SCFTs. In [71], the authors constructed 3D non-unitary TQFTs from some 4D $\mathcal{N}=2$ Argyres-Douglas theories. The construction is somewhat similar to our construction of TFT, but the precise relation is not clear. In our construction, semi-simple non-unitary TQFTs appear in 3D SCFTs of rank 0, while their examples after dimension reduction to 3D are not of rank 0. It would be interesting to clarify for which classes of 4D $\mathcal{N}=2$ SCFTs their construction works, and to see if we can apply their construction in classification of 4D $\mathcal{N}=2$ SCFTs.

Are all non-unitary TQFTs correspond to rank 0 $\mathcal{N}=4$ SCFTs? In our paper, we assign a pair of non-unitary TQFTs to 3D rank 0 $\mathcal{N}=4$ SCFTs. But it is not clear if all non-unitary TQFTs can be constructed in this way. There are exotic non-unitary TQFTs which cannot be related to any unitary TQFT via Galois conjugation, as found at rank 6 in [30]. If the proposed relation in (4.1) is true, such exotic non-unitary TQFTs cannot be realized from rank 0 $\mathcal{N}=4$ SCFTs.

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[6]For a 3D $\mathcal{N}=4$ SCFT $\mathcal{T}$ of non-zero rank, the topological twisted theories $\mathrm{RW}_\pm[\mathcal{T}]$ are not genuine TQFTs since they do not obey the standard axioms of TQFT. Nevertheless, interesting 3-manifold/knot invariants can be studied using the $\mathrm{RW}_\pm[\mathcal{T}]$ [68].

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A Some reviews

In order to help better understand the correspondence in (2.7), we briefly review basic relevant aspects of (2+1)D non-unitary topological field theory and supersymmetric partition functions of 3D superconformal field theories.

A.1 Localization on 3D $\mathcal{N} \geq 3$ gauge theories

In this paper, we consider following 4 types of supersymmetric partition functions of 3D $\mathcal{N} = 4$ SCFT $\mathcal{T}$ of rank 0 theory

$$\begin{aligned}
\mathcal{I}^{\text{sci}}_{\mathcal{T}}(q,\eta,\nu;s) \;&:\quad \text{Superconformal index [51, 52]}\ ,\\
\mathcal{Z}^{S^3_b}_{\mathcal{T}}(b,m,\nu) \;&:\quad \text{Squashed three-sphere partition function [49, 72]}\ ,\\
\mathcal{I}^{\Sigma_g}_{\mathcal{T}}(\eta,\nu;s) \;&:\quad \text{(Topologically) twisted indices on } \Sigma_g \text{ [73–76]}\ ,\\
\mathcal{Z}^{\mathcal{M}_{g,p}}_{\mathcal{T}}(m,\nu;s) \;&:\quad \text{Twisted partition function on } \mathcal{M}_{g,p} \text{ [77–81]}\ .
\end{aligned} \tag{A.1}$$

These 4 types of partition functions are not totally exclusive.

$$\begin{aligned}
&\mathcal{Z}^{S^3_b}(b=1,m,\nu) = \mathcal{Z}^{\mathcal{M}_{g=0,p=1}}(m,\nu;s=1)\ ,\\
&\mathcal{I}^{\Sigma_g}(\eta,\nu;s)\big|_{\eta=e^m} = \mathcal{Z}^{\mathcal{M}_{g,p=0}}(m,\nu;s)\ .
\end{aligned} \tag{A.2}$$

In the partition function, m (resp. η) is the real mass parameter (resp. fugacity) associated to the axial U(1) symmetry while ν is the R-symmetry mixing parameter (2.3). Rank 0 $\mathcal{N} = 4$ SCFT cannot have any flavor symmetry commuting with $\mathcal{N} = 4$ supersymmetries and thus the BPS partition functions cannot be further refined.

3D $\mathcal{N} = 4$ SCFTs can appear as IR fixed points of 3D quantum field theories. Since there could be supersymmetry enhancement along the RG flow, we do not need to start from a UV theory with manifest $\mathcal{N} = 4$ symmetry. For the exact computation of supersymmetric partition using localization, however, the UV theory should have at least $\mathcal{N} = 2$ supersymmetry. In this paper, we study several examples of $\mathcal{N} = 4$ rank 0 SCFTs which appears as IR fixed points of $\mathcal{N} \geq 3$ supersymmetric theories. In the below, we summarize localization formulae for the BPS partition functions introduced above for $\mathcal{N} \geq 3$ gauge theories. In localization computations, $\mathcal{N} \geq 3$ gauge theories have several advantages over $\mathcal{N} = 2$ gauge theories. The local Lagrangian density of an $\mathcal{N} = 3$ gauge theory is uniquely determined by the choice of gauge group G, its Chern-Simons levels $\vec{k}$ and matter contents (hypermultiplets and twisted hypermultilplets in unitary representations of G). When the CS levels are all zero, i.e. $\vec{k} = \vec{0}$, the theory has $\mathcal{N} = 4$ supersymmetry. For $\mathcal{N} \geq 3$ gauge theories, the R-symmetry is non-abelian (SO(3)), and thus the IR R-symmetry is uniquely fixed (i.e. is not mixed with other abelian flavor symmetries) and we do not need to perform the F-maximization [49]. The localization for general $\mathcal{N} \geq 2$ theories can be done in a similar way but with some more complications.

Superconformal index. The superconformal index for a 3D $\mathcal{N} = 4$ SCFT is defined as

$$\mathcal{I}^{\text{sci}}(q,\eta,\nu;s) := \begin{cases} \text{Tr}_{\mathcal{H}_{\text{rad}}(S^2)}(-1)^{2j_3} q^{\frac{R_\nu}{2}+j_3}\eta^A\ , & s=1\ ,\\ \text{Tr}_{\mathcal{H}_{\text{rad}}(S^2)}(-1)^{R_\nu} q^{\frac{R_\nu}{2}+j_3}\eta^A\ , & s=-1\ . \end{cases} \tag{A.3}$$

Here the trace is taken over the radially quantized Hilbert-space $\mathcal{H}_{\rm rad}(S^2)$ on S^2 whose elements are local operators. $j_3 \in \frac{\mathbb{Z}}{2}$ is the Lorentz spin, the Cartan of SO(3) isometry on the S^2. The parameter q plays role as an Ω-deformation parameter. Only BPS operators satisfying following relation contribute to the index

$$\Delta = R + R' + j_3 \;, \tag{A.4}$$

where Δ is the conformal dimension. The index alternatively can be regarded as a partition function on $(S^2 \times S^1) = \mathcal{M}_{g=0,p=0}$ with a fixed metric, background electric fields coupled to $\mathrm{U}(1)_{R_\nu}$ and axial U(1) symmetry and spin-structure along the S^1. The indices at different ν are simply related as follows

$$\begin{aligned} \mathcal{I}^{\rm sci}(q,\eta,\nu;s=1) &= \mathcal{I}^{\rm sci}(q,\eta,\nu=0;s=1)\big|_{\eta\to\eta q^{\frac{\nu}{2}}} \;, \\ \mathcal{I}^{\rm sci}(q,\eta,\nu;s=-1) &= \mathcal{I}^{\rm sci}(q,\eta,\nu=0;s=-1)\big|_{\eta\to\eta(-q^{\frac{1}{2}})^\nu} \;. \end{aligned} \tag{A.5}$$

The two indices with different choices of the spin structure are related to each other in the following way

$$\mathcal{I}^{\rm sci}\left(q,\eta,\nu=\pm1;s=1\right) = \mathcal{I}^{\rm sci}\left(q,\eta,\nu=\pm1;s=-1\right)\big|_{q^{\frac{1}{2}}\to -q^{\frac{1}{2}}} \;. \tag{A.6}$$

Using localization, the superconformal index at $\nu=0$ is given as

$$\begin{aligned} &\mathcal{I}^{\rm sci}(q,\eta,\nu=0;s=1) \\ &= \sum_{\mathbf{m}} \oint_{|a_i|=1} \left(\prod_{i=1}^{\mathrm{rank}G} \frac{da_i}{2\pi i a_i}\right) \Delta_G(\mathbf{m},\mathbf{a};q) q^{\epsilon_0(\mathfrak{n})} \mathcal{I}_0^{cs}(\mathbf{m},\mathbf{a}) \mathrm{P.E.}[f_{\rm single}(q,\mathbf{a},\eta;\mathbf{m})] \;. \end{aligned} \tag{A.7}$$

In the localization, the saddle points are parametrized by $\{\mathfrak{m}_i, a_i\}_{i=1}^{\mathrm{rank}(G)}$,

$$\begin{aligned} &\frac{1}{2\pi}\int_{S^2} F = \mathbf{m}, \;\exp\left(i\int_{S^1} A\right) = \mathbf{a} \;\text{ and }\; \sigma = \frac{\mathbf{m}}{2} \\ &\text{with } \mathbf{m} := \sum_i \mathfrak{m}_i \mathbf{h}^i \text{ and } \mathbf{a} := \exp(\mathbf{A}) := \exp\left(\sum_i (\log a_i)\mathbf{h}^i\right) \;. \end{aligned} \tag{A.8}$$

Here σ is the adjoint real scalar in the $\mathcal{N}=2$ vector multiplet and $\{\mathbf{h}^i\}$ is a normalized basis of Cartan subalgebra of G. For $G = \mathrm{U}(N)$ or SU(2) case, the basis is chosen as

$$\begin{aligned} G &= \mathrm{U}(N), \quad \mathbf{h}^i := \mathrm{diag}\{0,\ldots,\overset{i\text{–th}}{1},\ldots,0\} \;, \\ G &= \mathrm{SU}(2), \quad\;\; \mathbf{h} = \mathbf{h}^{i=1} = \mathrm{diag}\{1,-1\} \;. \end{aligned} \tag{A.9}$$

Δ_G is the contribution from $\mathcal{N}=2$ vector multiplet

$$\Delta_G(\mathbf{m},\mathbf{a};q) := \frac{1}{\mathrm{Sym}(\mathbf{m})} \prod_{\lambda\in\Lambda^+_{\rm adj}} q^{-\frac{|\lambda(\mathbf{m})|}{2}} \left(1 - q^{\frac{1}{2}|\lambda(\mathbf{m})|} e^{\lambda(\mathbf{A})}\right)\left(1 - q^{\frac{1}{2}|\lambda(\mathbf{m})|} e^{-\lambda(\mathbf{A})}\right) \;. \tag{A.10}$$

The monopole flux $\mathbf{m}$ in (A.8) breaks the gauge group G to its subgroup $H(\mathbf{m})$

$$H(\mathbf{m}) := \{h \in G \;:\; [h,\mathbf{m}] = 0\} \;, \tag{A.11}$$

and Sym($\mathbf{m}$) is the order of the Weyl group of the subgroup,

$$\mathrm{Sym}(\mathbf{m}) := |\mathrm{Weyl}(H(\mathbf{m}))| \ . \tag{A.12}$$

Λ^+_{adj} is the set of positive roots of G.

$$\begin{aligned} &G = \mathrm{U}(N), \quad \{\lambda(\mathbf{m}) \ : \ \lambda \in \Lambda^+_{\text{adj}}\} = \{\mathfrak{m}_i - \mathfrak{m}_j \ : \ 0 < i < j \leq N\} \ , \\ &G = \mathrm{SU}(2), \quad \{\lambda(\mathbf{m}) \ : \ \lambda \in \Lambda^+_{\text{adj}}\} = \{\mathfrak{m}_1 - \mathfrak{m}_2\} \ . \end{aligned} \tag{A.13}$$

The single particle index is

$$f_{\text{single}}(q, \mathbf{a}, \eta; \mathbf{m}) = \sum_{\Phi} \sum_{\beta \in \rho_{R_\Phi}} \left(\frac{q^{\frac{1}{2}\Delta_\Phi + \frac{1}{2}|\beta(\mathbf{m})|} e^{\beta(\mathbf{A})} \eta^{q_A(\Phi)}}{1-q} - \frac{q^{\frac{1}{2}(2-\Delta_\Phi) + \frac{1}{2}|\beta(\mathbf{m})|} e^{-\beta(\mathbf{A})} \eta^{-q_A(\Phi)}}{1-q} \right). \tag{A.14}$$

Here the summation is over $\mathcal{N} = 2$ chiral multiplets Φ in the representation of R_Φ under the gauge group G. ρ_R is the set of weights of the representation R. $q_A(\Phi)$ and Δ_Φ are the axial U(1) symmetry and the conformal dimension of the chiral field Φ respectively. An $\mathcal{N} = 4$ hypermultiplet consists of two chiral multiplets with gauge charges R and $\overline{R}$, $q_A = \frac{1}{2}$ and $\Delta = \frac{1}{2}$. The adjoint chiral multiplet in $\mathcal{N} = 4$ vector multiplet has $q_A = -1$ and $\Delta = 1$. If one wants to introduce a Chern-Simons interaction in an $\mathcal{N} = 4$ gauge theory, it will break the $\mathcal{N} = 4$ supersymmetry down to $\mathcal{N} = 3$ symmetry. In the case, the R-symmetry is broken to SO(3) and thus we cannot introduce the fugacity η for the U(1) axial symmetry. The Casimir energy ϵ_0 is

$$\epsilon_0 = \frac{1}{2} (\partial_q f_{\text{single}})\big|_{q,\eta \to 1} = \sum_{\Phi} \sum_{\beta \in \rho_{R_\Phi}} \frac{(1-\Delta_\Phi)|\beta(\mathbf{m})|}{4} \ . \tag{A.15}$$

$\mathcal{I}^{cs}_0(\mathbf{m}, \mathbf{a})$ is the contribution from the classical CS term

$$\begin{aligned} \mathrm{U}(N)_k \ &: \ \prod_{i=1}^{N} (u_i(-1)^{\mathfrak{m}_i})^{k\mathfrak{m}_i} \ , \\ \mathrm{SU}(2)_k \ &: \ (u(-1)^{\mathfrak{m}})^{2k\mathfrak{m}} \ . \end{aligned} \tag{A.16}$$

For a U(N) dynamical gauge group, there is a U(1) topological symmetry whose Noether current is

$$j^\mu_{\text{top}} = -\epsilon^{\mu\nu\rho} \mathrm{Tr}(F_{\nu\rho}) \ . \tag{A.17}$$

The fugacity a and its background monpole flux $\mathfrak{m}_a$ for the topological symmetry can be introduced by including the following term to $\mathcal{I}^{cs}_0$

$$(a(-1)^{\mathfrak{m}_a})^{-\sum_{i=1}^{N} \mathfrak{m}_i} \left(\prod_{i=1}^{N} u_i(-1)^{\mathfrak{m}_i} \right)^{-\mathfrak{m}_a} \ . \tag{A.18}$$

The monopole flux $\mathbf{m}$ should satisfy the following Dirac quantization conditions

$$\begin{aligned} &\lambda(\mathbf{m}) \in \mathbb{Z}, \quad \forall \lambda \in \Lambda^+_{\rm adj} \text{ and} \\ &\beta(\mathbf{m}) \in \mathbb{Z}, \quad \forall \beta \in R_\Phi \ . \end{aligned} \tag{A.19}$$

There could be additional constraints on the monopole fluxes depending on the global structure of the $\mathcal{N} \geq 3$ gauge theories as we have seen in (3.69) and (3.71). In the localization summation, we need to sum over monopole flux $\mathbf{m}$ modulo the redundant Weyl symmetry of G.

Squashed three-sphere partition function $\mathcal{Z}^{S^3_b}(b, m, \nu)$. This is a partition function on $S^3 = \mathcal{M}_{g=0,p=1}$ with the following metric

$$ds^2(S^3_b) = |dz|^2 + |dw|^2 \ , \quad (z,w) \in \mathbb{C}^2 \text{ are subject to } b^{-2}|z|^2 + b^2|w|^2 = 1 \ . \tag{A.20}$$

To preserve some supercharges, a background field coupled to the $\mathrm{U}(1)_{R_\nu}$ symmetry is properly turned on. Using localization, the partition function can be given in the following integral form

$$\mathcal{Z}^{S^3_b}(b, m, \nu) = \int \left(\prod_{i=1}^{\mathrm{rank}(G)} \frac{dZ_i}{\sqrt{2\pi\hbar}} \right) \Delta_G(\mathbf{Z};\hbar) \mathcal{I}_\hbar(\mathbf{Z}, m, \nu) \ , \quad \hbar := 2\pi i b^2 \ . \tag{A.21}$$

Here $\{Z_i\}_{i=1}^{\mathrm{rank}(G)}$ parametrizes the Cartan subalgebra of G.

$$\mathbf{Z} = \sum_{i=1}^{\mathrm{rank}(G)} Z_i \mathbf{h}^i \in (\text{Cartan subalgebra of } G) \ , \tag{A.22}$$

and $\Delta_G(\mathbf{Z})$ is the contribution from the $\mathcal{N} = 2$ vector multiplet associated to the gauge group G

$$\Delta_G(\mathbf{Z};\hbar) := \frac{1}{|\mathrm{Weyl}(G)|} \prod_{\lambda \in \Lambda^+_{\rm adj}} \left[4 \sinh\left(\frac{1}{2}\lambda \cdot \mathbf{Z}\right) \sinh\left(\frac{\pi i}{\hbar}\lambda \cdot \mathbf{Z}\right) \right] . \tag{A.23}$$

$|\mathrm{Weyl}(G)|$ is the order of the Weyl group of G.

The integrand $\mathcal{I}_\hbar$ is determined by gauge group, matter contents and Chern-Simons levels of the $\mathcal{N} = 3$ gauge theory as follows:

- An $\mathcal{N} = 2$ chiral multiplet in a representation R under G with U(1) axial charge q_A and conformal dimension Δ contributes

$$\prod_\Phi \prod_{\beta \in \rho_{R_\Phi}} \Psi_\hbar \left(\beta \cdot \mathbf{Z} + q_A \left(m + \left(i\pi + \frac{\hbar}{2} \right) \nu \right) + \left(i\pi + \frac{\hbar}{2} \right) \Delta \right) . \tag{A.24}$$

  We define $\Psi_\hbar$ as

$$\Psi_\hbar(X) := \psi_\hbar(X) \exp\left(\frac{X^2}{4\hbar}\right) , \tag{A.25}$$

with $\psi_\hbar(x)$ being the non-compact quantum dilogarithm function. (We refer to D for details of the definition and basic properties of the function.) An $\mathcal{N}=4$ hypermultiplet consists of two $\mathcal{N}=2$ chiral multiplets with gauge charge R and $\overline{R}$, $q_A=\frac{1}{2}$ and $\Delta=\frac{1}{2}$. The adjoint $\mathcal{N}=2$ chiral multiplet in a $\mathcal{N}=4$ vector multiplet has $q_A=-1$ and $\Delta=1$.

- Chern-Simons term of gauge G of level k contributes the following term to the integrand

$$\exp\left(\frac{k}{2\hbar}\mathrm{Tr}(\mathbf{Z}^2)\right) . \tag{A.26}$$

The real mass m (FI parameter) and the R-symmetry mixing parameter ν of the U(1) topological symmetry for $G=\mathrm{U}(N)$ are introduced by adding the following term to the integrand

$$\exp\left(-\frac{W\mathrm{Tr}(\mathbf{Z})}{\hbar}\right)\bigg|_{W=m+(i\pi+\frac{\hbar}{2})\nu} . \tag{A.27}$$

The partition function at $b=1$, which corresponds to round three-sphere, enjoys interesting properties. Firstly, its free-energy is maximized at the superconformal R-charge choice, i.e.

$$F_{\nu=0}>F_{\nu\neq 0} , \quad \text{where } F_\nu := -\log|\mathcal{Z}^{S^3_b}(b=1,m=0,\nu)| . \tag{A.28}$$

Secondly, the round sphere free-energy F at conformal point

$$F=-\log|\mathcal{Z}^{S^3_b}(b=1,m=0,\nu=0)| , \tag{A.29}$$

always monotonically decreases under the RG flow. So the quantity F can be regarded as a proper measure of degrees of freedom.

Perturbative expansion of squashed three-sphere partition function integral. One can consider formal perturbative expansion of the localization integral in an asymptotic limit $\hbar\to 0$, to obtain infinitely many 3D SCFT invariants. In the limit, the integrand $\mathcal{I}_\hbar$ can be perturbatively expanded in the following form

$$\log\mathcal{I}_\hbar(\vec{Z},m,\nu)\xrightarrow{\hbar\to 0}\sum_{n=0}^{\infty}\hbar^{n-1}\mathcal{W}_n(\vec{Z},n,\nu) . \tag{A.30}$$

The leading part $\mathcal{W}_0$ corresponds to the twisted superpotential. By extremizing the twisted superpotential, we obtain Bethe-vacua

$$\text{Bethe-vacua:}\ \frac{\left\{\vec{z}:\ \left(\exp(\partial_{Z_i}\mathcal{W}_0)|_{\vec{Z}\to\log\vec{z}}\right)=1,\ w\cdot\vec{z}\neq\vec{z}\ \forall\text{ non-trivial } w\in\mathrm{Weyl}(G)\right\}_{i=1}^{\mathrm{rank}(G)}}{\mathrm{Weyl}(G)} . \tag{A.31}$$

Here Weyl(G) is the Weyl group of gauge group G. A Bethe-vacuum $\vec{z}_\alpha$ can be promoted to a saddle point $\vec{Z}_\alpha = \log \vec{z}_\alpha$ of the localization integral by properly shifting $\mathcal{W}_0$ as follows,

$$\mathcal{W}_0^{\vec{n}_\alpha} = \mathcal{W}_0 + 2\pi i \sum_i n_\alpha^i Z_i, \quad n_\alpha^i \in \mathbb{Z} \text{ is chosen such that } \partial_{Z_i} \mathcal{W}_0^{\vec{n}_\alpha}|_{\vec{Z}\to \log \vec{z}_\alpha} = 0 \,. \tag{A.32}$$

Then we can consider formal perturbative expansion of the localization integral around the saddle point

$$\begin{aligned}
&|\mathrm{Weyl}(G)| \times \int \prod_{i=1}^{\mathrm{rank}(G)} \frac{d(\delta Z_i)}{\sqrt{2\pi\hbar}} \exp\left(\frac{1}{\hbar}\mathcal{W}_0^{\vec{n}_\alpha}(\vec{Z}^\alpha + \delta\vec{Z}, m, \nu) + \sum_{n=1}^{\infty} \hbar^{n-1}\mathcal{W}_n(\vec{Z}+\delta\vec{Z}, , m, \nu)\right) \\
&\xrightarrow{\hbar\to 0} \exp\left(\sum_{n=0}^{\infty} \hbar^{n-1} \mathcal{S}_n^\alpha(m,\nu)\right) \,.
\end{aligned} \tag{A.33}$$

The factor $|\mathrm{Weyl}(G)|$ is multiplied since that many saddle points, which all give the same perturbative expansion, collapse into a single Bethe-vacuum after the Weyl quotient. The perturbative expansion can be formally computed by performing Gaussian integrals [50, 82]. For example,

$$\mathcal{S}_0^\alpha = \mathcal{W}_0^{\vec{n}_\alpha}(\vec{Z}_\alpha)\,, \quad \mathcal{S}_1^\alpha = -\frac{1}{2}\log\left(\det_{i,j} \frac{\partial^2 \mathcal{W}_0}{\partial Z_i \partial Z_j}\right)\bigg|_{\vec{Z}=\vec{Z}_\alpha} + \mathcal{W}_1(\vec{Z}_\alpha) + \log|\mathrm{Weyl}(G)|\,. \tag{A.34}$$

The proposal in (2.7) implies the following highly non-trivial constraints on the perturbative invariants for rank 0 SCFTs,

$$\begin{aligned}
&\mathrm{Im}[\mathcal{S}_0^\alpha(m=0,\nu)] \xrightarrow{\nu\to\pm 1} 0\,, \quad \mathrm{Im}[\mathcal{S}_2^\alpha(m=0,\nu)] \xrightarrow{\nu\to\pm 1} 0\,, \\
&\mathcal{S}_{n\geq 3}^\alpha(m=0,\nu) \xrightarrow{\nu\to\pm 1} 0\,.
\end{aligned} \tag{A.35}$$

This follows from the fact that the squashed three-sphere partition function becomes b-independent in the degenerate limits, $m=0$ and $\nu \to \pm 1$, modulo local counter terms which affect an overall factor of the following form

$$\exp\left(\pi i q_1\left(b^2 + \frac{1}{b^2}\right) + i\pi q_2\right)\bigg|_{q_1, q_2 \in \mathbb{Q}}\,. \tag{A.36}$$

Twisted indices and twisted partition functions. The twisted index is defined as

$$\mathcal{I}^{\Sigma_g}(\eta,\nu;s) = \begin{cases} \mathrm{Tr}_{\mathcal{H}(\Sigma_g;\nu)}(-1)^{2j_3}\eta^A\,, & s=1\,, \\ \mathrm{Tr}_{\mathcal{H}(\Sigma_g;\nu)}(-1)^{R_\nu}\eta^A\,, & s=-1\,. \end{cases} \tag{A.37}$$

Here $\mathcal{H}(\Sigma_g;\nu)$ is the Hilbert-space on Σ_g with topological twisting using the $\mathrm{U}(1)_{R_\nu}$ symmetry. Unlike the radially quantized Hilbert-space $\mathcal{H}_{\mathrm{rad}}(S^2)$, the Hilbert-space depends on the choice of the R-symmetry mixing parameter ν. Due to the topological twisting, the index is well-defined only when following Dirac quantization condition is satisfied

$$R_\nu \times (g-1) \in \mathbb{Z} \quad \text{for all local operators}\,. \tag{A.38}$$

Note that the condition is always satisfied in the degenerate limit $\nu = \pm 1$ since $R_{\nu=\pm 1} \in \mathbb{Z}$ which obvious from the fact that $R_{\nu=1} = 2R \in \mathbb{Z}$ and $R_{\nu=-1} = 2R' \in \mathbb{Z}$. For $g = 0$ case, the quantization condition is satisfied for all ν and the index is independent on the continuous parameter ν. Generally, the twisted indices can be written as follows

$$\mathcal{I}^{\Sigma_g}(\eta, \nu; s) = \sum_{\vec{z}_\alpha \,:\, \text{Bethe-vacua}} (\mathcal{H}_\alpha(\eta, \nu; s))^{g-1} . \tag{A.39}$$

Here $\mathcal{H}_\alpha$ is called the *handle gluing* operator at the α-th Bethe-vacuum. For $s = -1$ case, the operator is simply given as

$$\mathcal{H}_\alpha(\eta, \nu; s = -1) = e^{i\varphi} \exp\left(-2\mathcal{S}_1^\alpha(m, \nu)\right)\big|_{m = \log \eta} . \tag{A.40}$$

Here $e^{i\varphi}$ is a α-independent overall phase factor, affected by the local counter term (A.36), which can be fixed by requiring $\mathcal{I}^{\Sigma_g} \in \mathbb{Z}$ for all g up to a sign. For rank 0 SCFT, the phase factor is uniquely determined by requiring $\mathcal{I}^{\Sigma_{g=0}} = 1$ in the degenerate limits, $\eta \to 1$ and $\nu \to \pm 1$. Upon the proper choice of the phase factor, furthermore, the handle gluing operators become all positive real number in the degenerate limits,

$$\mathcal{H}_\alpha(\eta = 1, \nu \to \pm 1, s) > 0 , \quad \text{for all } \alpha . \tag{A.41}$$

This is compatible with the dictionary for the handle gluing operators in table 1. More generally, the twisted partition function is given in the following form

$$\mathcal{Z}^{\mathcal{M}_{g,p}}(m, \nu, s) = \sum_{\vec{z}_\alpha \,:\, \text{Bethe-vacua}} (\mathcal{H}_\alpha(\eta = e^m, \nu; s))^{g-1} (\mathcal{F}_\alpha(m, \nu; s))^p . \tag{A.42}$$

Here $\mathcal{F}_\alpha$ is called *fibering operator* at the α-th Bethe-vacuum. For $s = -1$ case, the operator is simply given as

$$\mathcal{F}_\alpha(\eta, \nu; s = -1) = \exp\left(\frac{\mathcal{S}_0^\alpha(m, \nu)}{2\pi i}\right) . \tag{A.43}$$

Supersymmetric loop operator $\mathcal{O}(\vec{z})$. In the twisted partition functions computation, an inclusion of a supersymmetric loop operator along the fiber S^1 in $\mathcal{M}_{g,p}$ corresponds to an inclusion of a (Weyl invariant) finite Laurent polynomial $\mathcal{O}(\vec{z})$ in $\{z_i\}_{i=1}^{\text{rank}(G)}$ with integer coefficients:

$$\mathcal{Z}^{\mathcal{M}_{g,p} + \mathcal{O}}(m, \nu, s) = \sum_{\vec{z}_\alpha \,:\, \text{Bethe-vacua}} (\mathcal{H}_\alpha(m, \nu; s))^{g-1} (\mathcal{F}_\alpha(m, \nu, s))^p \mathcal{O}(\vec{z}_\alpha) . \tag{A.44}$$

Here $\mathcal{Z}^{\mathcal{M}_{g,p}+\mathcal{O}}$ is the twisted partition function on $\Sigma_{g,p}$ with insertion of loop operator $\mathcal{O}$. For example, dyonic loop operator $\mathcal{O}_{(p,q)}$ of charge (electric charge, magnetic charge) $=$ (p, q) in a U(1) gauge theory is given as

$$\mathcal{O}_{(p,q)}(z) = z^p \left(1 - \frac{1}{z}\right)^q . \tag{A.45}$$

A.2 Modular data of 3D TQFT

In Bosonic (i.e. non-spin) TQFT. One basic characteristic quantity of 3D bosonic (i.e. non-spin) topological field theories is so-called *modular data*, which consists of S and T matrices. Let us denote components of the two matrices by

$$S_{\alpha\beta},\ T_{\alpha\beta}\ :\ \alpha,\beta=0,\dots N-1\ . \tag{A.46}$$

To understand the physical meaning of the matrices, let us consider the Hilbert space $\mathcal{H}(\mathbb{T}^2)$ on two-torus. In topological field theory, there is no local operator and the only physical observables are loop operators $\mathcal{O}^{\Gamma}_{\alpha=0,\dots,N-1}$. α labels types (gauge charge) of loop operators, sometimes called *anyons*, and the natural number N is called the rank of the topological field theory. Γ is the one-dimensional trajectory where the operator is supported. $\mathcal{O}_{\alpha=0}$ is the trivial loop operator, i.e. identity operator,

$$\mathcal{O}_{\alpha=0}=1\ . \tag{A.47}$$

One natural basis of the Hilbert-space $\mathcal{H}(\mathbb{T}^2)$ is

$$\text{Basis of } \mathcal{H}(\mathbb{T}^2)\ :\ \{|\alpha\rangle:=\mathcal{O}^{B}_{\alpha}|0\rangle\}_{\alpha=0}^{N-1}\ , \tag{A.48}$$

where B is a generator of $H_1(\mathbb{T}^2,\mathbb{Z})=\langle A,B\rangle$. A mapping class element $\varphi\in \mathrm{SL}(2,\mathbb{Z})$ acts on the Hibert-space as a unitary operator $\hat{\varphi}$. The operators $\{\hat{\varphi}\}_{\varphi\in\mathrm{SL}(2,\mathbb{Z})}$ form a unitary representation of $\mathrm{SL}(2,\mathbb{Z})$. The S and T matrices are nothing but[7]

$$S_{\alpha\beta}=\langle\alpha|\hat{\mathbb{S}}|\beta\rangle\ ,\quad T_{\alpha\beta}=\exp\left(\frac{2\pi i c_{2d}}{24}\right)\times\langle\alpha|\hat{\mathbb{T}}|\beta\rangle\ . \tag{A.49}$$

c_{2d} (mod 24) is the chiral central charge of boundary 2d chiral CFT. Here $\mathbb{S}$ and $\mathbb{T}$ are two canonical generators of $\mathrm{SL}(2,\mathbb{Z})$

$$\mathbb{S}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\ ,\quad \mathbb{T}=\begin{pmatrix}1&0\\1&1\end{pmatrix}\ . \tag{A.50}$$

The modular matrices contain a lot of information of the topological field theory. According to the Verlinde formula, the fusion coefficients $N_{\alpha\beta}^{\gamma}$ can be given as

$$N_{\alpha\beta}^{\gamma}=\sum_{\delta=0}^{N-1}\frac{S_{\delta\alpha}S_{\delta\beta}S^*_{\delta\gamma}}{S_{0\delta}}\ . \tag{A.51}$$

The S-matrix determines how the basic operators $\mathcal{O}^A_\alpha$ and $\mathcal{O}^B_\alpha$ act on the Hilbert-space

$$\mathcal{O}^A_\beta|\alpha\rangle=W_\beta(\alpha)|\alpha\rangle=\frac{S_{\alpha\beta}}{S_{\alpha0}}|\alpha\rangle\ ,\quad \mathcal{O}^B_\beta|\alpha\rangle=\sum_\gamma N^\gamma_{\alpha\beta}|\gamma\rangle\ . \tag{A.52}$$

[7]In our convention, T_{00} is fixed to be 1. Conventionally, $T_{\alpha\beta}$ is defined as $\exp\left(-\frac{2\pi i c_{2d}}{24}\right)\times T^{\text{ours}}_{\alpha\beta}$ such that $T_{\alpha\beta}$ is just $\langle\alpha|\hat{\mathbb{T}}|\beta\rangle$ without the phase factor.

T-matrix is a diagonal unitary matrix

$$\begin{aligned} T_{\alpha\beta} &= \delta_{\alpha\beta} e^{2\pi i h_\alpha} \ (h_{\alpha=0}=0) \ , \\ h_\alpha &= \text{topological spin of } \alpha\text{-th anyon} \ . \end{aligned} \tag{A.53}$$

The topological spin is defined only modulo 1. S and T matrices satisfies

$$S^2 = C \ (C^2=1) \ , \quad (ST)^3 = \exp\left(\frac{2\pi i c_{2d}}{8}\right) \times C \ . \tag{A.54}$$

The matrix $C_{\alpha\beta}$ is called charge conjugation. $S_{0\alpha}$ are real and they have following path-integral interpretation

$$\begin{aligned} S_{0\alpha} &= \mathcal{Z}^{S^3 + \mathcal{O}_\beta^{\Gamma=(\text{unknot})}} \\ &:= (\text{Partition function on } S^3 \text{ with a loop operator } \mathcal{O}_\alpha^\Gamma \text{ along the } \Gamma\text{=(unknot)}) \ . \end{aligned} \tag{A.55}$$

Especially, S_{00} is the partition function on S^3. In an unitary topological field theory, S matrix satisfies the following conditions

$$\text{Unitarity: } |S_{00}| \leq |S_{0\alpha}| \ . \tag{A.56}$$

Partition functions on $\mathcal{M}_{g,p}$ with insertion of a loop operator $\mathcal{O}_\beta^{[S^1]}$ along the fiber $[S^1]$ can be written as follows

$$\mathcal{Z}_{\text{TFT}}^{\mathcal{M}_{g,p}+\mathcal{O}_\beta^{[S^1]}} = \sum_{\alpha=0}^{N-1} (S_{0\alpha})^{2-2g} (T_{\alpha\alpha})^p W_\beta(\alpha) = \sum_{\alpha=0}^{N-1} (S_{0\alpha})^{2-2g-1} (T_{\alpha\alpha})^p S_{\alpha\beta} \ . \tag{A.57}$$

The partition function at $p=0$ without insertion of loop operator, i.e. $\beta=0$, counts ground state degeneracy GSD_g on a genus g Riemann surface,

$$\text{GSD}_g = \sum_{\alpha=0}^{N-1} (S_{0\alpha})^{2-2g} \ . \tag{A.58}$$

Since this counts actual numbers, the partition function at $p=0$ can be defined without any phase factor ambiguity. By contrast the partition function at non-zero p depends on the framing choice of the 3-manifold $\mathcal{M}_{g,p}$ as well as of the knot along the fiber $[S^1]$, and consequently there is no canonical framing choice. The formula above is only valid for a certain choice of the framing. The framing change affects the partition function by an overall phase factor of the form $\exp\left(\frac{2\pi i c_{2d}}{24}\mathbb{Z} + 2\pi i h_\beta \mathbb{Z}\right)$. For example, when $\mathcal{M}_{g=0,p=1} = S^3$

$$\mathcal{Z}_{\text{TFT}}^{\mathcal{M}_{g=0,p=1}+\mathcal{O}_\beta^{[S^1]}} = \sum_{\alpha=0}^{N-1} S_{0\alpha} T_{\alpha\alpha} S_{\alpha\beta} = (STS)_{0\beta} = \exp\left(\frac{2\pi i c_{2d}}{8} - 2\pi i h_\beta\right) \times S_{0\beta} \ ,$$

which is different from the $\mathcal{Z}^{S^3+\mathcal{O}_\beta^{\Gamma=(\text{unknot})=[S^1]}}$ in (A.55) by a phase factor.

In fermioinc (i.e. spin) TQFT. For this case the Hilbert space $\mathcal{H}(\mathbb{T}^2)$ depends on the choice of a spin-structure $H^1(\mathbb{T}^2, \mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2$. Let us consider following NS-NS sector

$$\mathcal{H}_{--}(\mathbb{T}^2) = (\text{Hilbert-space on } \mathbb{T}^2 \text{ with anti-periodic boundary conditions along both } S^1)$$

Similarly, one can consider four Hilbert-spaces $\mathcal{H}_{\pm\pm}$ depending on the choice of the spin-structure. On the $\mathcal{H}_{--}(\mathbb{T}^2)$, only a subgroup of $\mathrm{SL}(2,\mathbb{Z})$ generated by S and T^2 can act since T maps a state in $\mathcal{H}_{--}(\mathbb{T}^2)$ into a state in $\mathcal{H}_{-+}(\mathbb{T}^2)$. In other words, topological spins of anyons are defined only modulo 1/2 in spin TQFT.

Under anyon condensation, $/\mathbb{Z}_2$ and $/\mathbb{Z}_2^f$. In topological field theory, $\mathbb{Z}_2$ one-form symmetry is generated by an anyon $\mathcal{O}_{\alpha_{\mathbb{Z}_2}}$ satisfying the fusion rule $\mathcal{O}_{\alpha_{\mathbb{Z}_2}} \times \mathcal{O}_{\alpha_{\mathbb{Z}_2}} = 1$,

$$\mathcal{O}_{a_{\mathbb{Z}_2}} \; : \; \text{The anyon generating the one-form } \mathbb{Z}_2 \text{ symmetry} \; . \tag{A.59}$$

The topological spin for the anyon $\alpha_{\mathbb{Z}_2}$ can take only following values [83]

$$h_{\alpha_{\mathbb{Z}_2}} \in \left\{0, \frac{1}{2}, \pm\frac{1}{4}\right\} \; (\text{mod } 1) \; . \tag{A.60}$$

When $h_{\alpha_{\mathbb{Z}_2}} = \pm\frac{1}{4}$, the $\mathbb{Z}_2$ symmetry is anomalous and cannot be gauged. When $h_{\alpha_{\mathbb{Z}_2}} = 0$ (resp. $h_{\alpha_{\mathbb{Z}_2}} = 1/2$), on the other hand, $\mathbb{Z}_2$ is non-anomalous and is called a bosonic $\mathbb{Z}_2$ (resp. fermionic $\mathbb{Z}_2$) symmetry and is sometimes denoted as $\mathbb{Z}_2^b$ (resp. $\mathbb{Z}_2^f$).

$$\mathbb{Z}_2 \text{ is } \begin{cases} \text{anomalous} & \text{if } h_{\alpha_{\mathbb{Z}_2}} = \pm\frac{1}{4} \; (\text{mod } 1) \; , \\ \text{non-anomalous and bosonic} & \text{if } h_{\alpha_{\mathbb{Z}_2}} = 0 \; (\text{mod } 1) \; , \\ \text{non-anomalous and fermionic} & \text{if } h_{\alpha_{\mathbb{Z}_2}} = \frac{1}{2} \; (\text{mod } 1) \; . \end{cases} \tag{A.61}$$

Starting from a bosonic topological field theory TFT with bosonic or fermionic $\mathbb{Z}_2$ one-form symmetry, one can gauge the one-form symmetry to obtain another topological field theory, $\mathrm{TFT}/\mathbb{Z}_2$ or $\mathrm{TFT}/\mathbb{Z}_2^f$. The gauging procedure is sometimes called the *anyon condensation*. The resulting theory after gauging is a non-spin TQFT for bosonic $\mathbb{Z}_2$ case while the resulting theory is spin TQFT for fermionic $\mathbb{Z}_2^f$ case.

$$\begin{aligned} \mathrm{TFT}/\mathbb{Z}_2^b \; &: \; \text{Bosonic (non-spin) TQFT} \; , \\ \mathrm{TFT}/\mathbb{Z}_2^f \; &: \; \text{Fermionic (spin) TQFT} \; . \end{aligned} \tag{A.62}$$

The anomalous symmetry can be gauged only after tensoring with another TQFT, such as $\mathrm{U}(1)_{\pm 2}$, with anomalous $\mathbb{Z}_2$ one-form symmetry. At the level of modular data, the anyon condensation process can be summarized as follows. First consider the bosonic $\mathbb{Z}_2$ one-form symmetry gauging. After the gauging, the Hilbert-space on $\mathcal{H}(\mathbb{T}^2)$ is spanned by following basis

$$\begin{aligned} &\mathcal{H}^{\mathrm{TFT}/\mathbb{Z}_2}(\mathbb{T}^2) = \mathcal{H}^{\mathrm{TFT}/\mathbb{Z}_2}_{\text{untwisted}}(\mathbb{T}^2) \oplus \mathcal{H}^{\mathrm{TFT}/\mathbb{Z}_2}_{\text{twisted}}(\mathbb{T}^2) \; , \\ &\mathcal{H}^{\mathrm{TFT}/\mathbb{Z}_2}_{\text{untwisted}}(\mathbb{T}^2) = \mathrm{Span}\{|[\alpha]\rangle := \frac{1}{\sqrt{2}}(|\alpha\rangle + |\alpha_{\mathbb{Z}_2} \cdot \alpha\rangle) \; : \; \mathcal{O}^A_{\alpha_{\mathbb{Z}_2}}|\alpha\rangle = |\alpha\rangle, \;\; |\alpha_{\mathbb{Z}_2} \cdot \alpha\rangle \neq |\alpha\rangle\} \; , \\ &\mathcal{H}^{\mathrm{TFT}/\mathbb{Z}_2}_{\text{twisted}}(\mathbb{T}^2) = \mathrm{Span}\{|\alpha; \pm\rangle \; : \; \mathcal{O}^A_{\alpha_{\mathbb{Z}_2}}|\alpha\rangle = |\alpha\rangle, \;\; |\alpha_{\mathbb{Z}_2} \cdot \alpha\rangle = |\alpha\rangle\} \; . \end{aligned} \tag{A.63}$$

Here $|\alpha\rangle$ is the basis in (A.48) and see (A.52) for the action of $\mathcal{O}^A_\alpha$ and $\mathcal{O}^B_\alpha$ on the basis. On the basis given in (A.48), the $\mathcal{O}^A_{\alpha_{\mathbb{Z}_2}}$ acts as a diagonal matrix whose entries are all $+1$ or -1 while $\mathcal{O}^B_{\alpha_{\mathbb{Z}_2}}$ acts as a permutation matrix whose square is identity. In the above, we define $|\alpha_{\mathbb{Z}_2}\cdot\alpha\rangle := \mathcal{O}^B_{\alpha_{\mathbb{Z}_2}}|\alpha\rangle$. In the gauging procedure, we first discard basis elements which are odd (having eigenvalue -1) under the $\mathcal{O}^A_{\alpha_{\mathbb{Z}_2}}$. Then, we quotient the reduced Hilbert-space by the action of $\mathcal{O}^B_{\alpha_{\mathbb{Z}_2}}$. When a basis $|\alpha\rangle$ is invariant under both $\mathcal{O}^A_{\alpha_{\mathbb{Z}_2}}$ and $\mathcal{O}^B_{\alpha_{\mathbb{Z}_2}}$, the basis will be doubled to $\{|\alpha;\pm\rangle\}$. The modular data of the gauged theory is

$$\begin{aligned} &S^{\text{TFT}/\mathbb{Z}_2}_{[\alpha][\beta]} = 2S^{\text{TFT}}_{\alpha\beta}\,, \qquad &&S^{\text{TFT}/\mathbb{Z}_2}_{[\alpha=0](\beta;\pm)} = S^{\text{TFT}}_{0\beta}\,,\\ &h^{\text{TFT}/\mathbb{Z}_2}_{[\alpha]} = h^{\text{TFT}}_{\alpha}\,, \qquad &&h^{\text{TFT}/\mathbb{Z}_2}_{(\alpha;\pm)} = h^{\text{TFT}}_{\alpha} \end{aligned} \tag{A.64}$$

For other S-matrix elements, $S_{(\alpha;\pm)(\beta;\pm)}$ and $S_{[\alpha](\beta;\pm)}$, of TFT/$\mathbb{Z}_2$, we need to know more information on the mother TFT beyond modular data [84].

In the fermionic $\mathbb{Z}_2^f$ gauging, the Hilbert-space $\mathcal{H}_{--}(\mathbb{T}^2)$ of the resulting spin TQFT is

$$\mathcal{H}^{\text{TFT}/\mathbb{Z}_2^f}_{--}(\mathbb{T}^2) = \text{Span}\{|[\alpha]\rangle := \frac{1}{\sqrt{2}}(|\alpha\rangle + |\alpha_{\mathbb{Z}_2}\cdot\alpha\rangle)\ :\ \mathcal{O}^A_{\alpha_{\mathbb{Z}_2}}|\alpha\rangle = |\alpha\rangle\}\,. \tag{A.65}$$

Unlike the bosonic $\mathbb{Z}_2$ gauging, there is no twisted sector in the $\mathbb{Z}_2^f$ gauging since $|\alpha_{\mathbb{Z}_2}\cdot\alpha\rangle \neq |\alpha\rangle$. The modular data of the gauged theory is

$$\begin{aligned} &S^{\text{TFT}/\mathbb{Z}_2^f}_{[\alpha][\beta]} = 2S^{\text{TFT}}_{\alpha\beta}\,,\\ &h^{\text{TFT}/\mathbb{Z}_2^f}_{[\alpha]} = h^{\text{TFT}}_{\alpha}\ (\text{mod } 1/2)\,. \end{aligned} \tag{A.66}$$

Note that the topological spin of $|[\alpha]\rangle$ is only defined modulo 1/2 (instead of 1) after the quotient since $h^{\text{TFT}}_\alpha - h^{\text{TFT}}_{\alpha_{\mathbb{Z}_2}\cdot\alpha} = \pm 1/2$ for fermionic $\mathbb{Z}_2$. This is also compatible with the fact that anyon spins are defined only modulo 1/2 in spin-TQFT.

Under Galois conjugation. For a given unitary TQFT satisfying (A.56), there could be non-unitary TQFTs violating (A.56) called Galois conjugates. These non-unitary theories have several properties in common with the unitary TQFT. Galois conjugate pair has the same ground state degeneracy, $\text{GSD}_g = \mathcal{Z}^{\mathcal{M}_{g,p=0}}$, on any Riemann surface Σ_g. According to the formula in (A.57), it implies that

$$\text{Galois conjugate pair has the same set of } \{S^2_{0\alpha}\}_{\alpha=0}^{N-1}\,. \tag{A.67}$$

But the pair has different $S_{00} = |\mathcal{Z}^{S_3}|$ and the unitarity condition in (A.56) says that

$$S_{00}(\text{non-unitary Galois conjugate}) > S_{00}(\text{unitary TQFT})\,. \tag{A.68}$$

From the computation of $\mathcal{Z}^{\mathcal{M}_{g,p=0}}$ one can determines the set $\{S^2_{0\alpha}\}_{\alpha=0}^{N-1}$, while from the $|\mathcal{Z}^{S^3}|$ one can determine S_{00}. So, from the two computations, one can determine whether the TQFT is unitary satisfying (A.56) or not.

Example: $\mathrm{U}(1)_k$ theory. The action for the theory is given as

$$S = \frac{k}{4\pi}\int_{\mathcal{M}_3} A\wedge dA = \frac{k}{4\pi}\int_{X_4\,:\,\partial X_4=\mathcal{M}_3} F\wedge F\ . \tag{A.69}$$

The action depends on the choice of a 4-manifold X_4 whose boundary is $\mathcal{M}_3$. Two different choices of the 4-manifolds, say X_4 and Y_4, give the following difference in the action

$$\Delta S = \frac{k}{4\pi}\left(\int_{X_4} F\wedge F - \int_{Y_4} F\wedge F\right) = \frac{k}{4\pi}\int_{\mathcal{M}_4 := X_4\cup \overline{Y_4}} F\wedge F\ . \tag{A.70}$$

Here $\mathcal{M}_4$ is a closed orientable 4-manifold obtained by gluing X_4 and $\overline{Y}_4$, an orientation reversal of Y_4, along the common boundary $\mathcal{M}_3$. Since

$$\int_{\mathcal{M}_4} F\wedge F \in 4\pi^2\mathbb{Z} \quad (\text{for any closed orientiable } \mathcal{M}_4)\ , \tag{A.71}$$

the action is well-defined modulo πk and thus

$$e^{iS} \text{ depends only on } \mathcal{M}_3 \text{ (but not on } X_4\text{) when } k\in 2\mathbb{Z}\ . \tag{A.72}$$

On the other hand, if we restrict the case when $\mathcal{M}_4$ is a spin 4-manifold

$$\int_{\mathcal{M}_4} F\wedge F \in 8\pi^2\mathbb{Z} \quad (\text{for any closed spin } \mathcal{M}_4)\ . \tag{A.73}$$

It means that we choose a particular spin choice on $\mathcal{M}_3$ and the 4-manifold X_4 is chosen such that it has a spin structure which is compatible the spin structure of the boundary $\mathcal{M}_3$. Then the $\mathcal{M}_4 = X_4\cup\overline{Y}_4$ for two possible such extensions of $\mathcal{M}_3$ has a spin structure. Thus,

$$e^{iS} \text{ depends only on } \mathcal{M}_3 \text{ and its spin-structure (but not on } X_4\text{) when } k\in 2\mathbb{Z}+1\ . \tag{A.74}$$

Actually, the $\mathrm{U}(1)_k$ theory is a spin or non-spin TQFT depending on evenness/oddness of k

$$\mathrm{U}(1)_k \text{ is } \begin{cases} \text{non-spin (bosonic) TQFT} & \text{if } k\in 2\mathbb{Z}\ , \\ \text{spin (fermionic) TQFT} & \text{if } k\in 2\mathbb{Z}+1\ . \end{cases} \tag{A.75}$$

Modular data (S and T matrices) of $\mathrm{U}(1)_k$ with even k is

$$\begin{aligned} &k\in 2\mathbb{Z}_{>0}\ :\ S_{\alpha\beta} = \frac{1}{\sqrt{k}}e^{\frac{2\pi i\alpha\beta}{k}},\ \ T_{\alpha\beta} = \delta_{\alpha\beta}e^{2\pi i h_\alpha} \text{ with } h_\alpha := \frac{\alpha^2}{2k}\ (\text{mod } 1)\ , \\ &\qquad \text{where } \ \alpha,\beta = 0,1,\cdots,k-1\ . \end{aligned} \tag{A.76}$$

On the other hand, modular data (S and T^2 matrices) of $\mathrm{U}(1)_k$ with odd k is

$$\begin{aligned} &k\in 2\mathbb{Z}_{>0}-1\ :\ S_{\alpha\beta} = \frac{1}{\sqrt{k}}e^{\frac{2\pi i\alpha\beta}{k}},\ \ (T^2)_{\alpha\beta} = \delta_{\alpha\beta}e^{4\pi i h_\alpha} \text{ with } h_\alpha := \frac{\alpha^2}{2k}\ \left(\text{mod } \frac{1}{2}\right)\ , \\ &\qquad \text{where } \ \alpha,\beta = 0,1,\cdots,k-1\ . \end{aligned} \tag{A.77}$$

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The loop operator $\mathcal{O}_\alpha^\Gamma$ corresponds to the Wilson loop of U(1) gauge charge α, i.e.

$$\mathcal{O}_\alpha^\Gamma = \exp\left(i\alpha \oint_\Gamma A\right) . \tag{A.78}$$

The fusion coefficients of $\mathrm{U}(1)_k$ can be computed from (A.51) and (A.76) as

$$N_{\alpha\beta}^\gamma = \delta^\gamma_{\alpha+\beta\,(\mathrm{mod}\ k)} , \text{ i.e. } \quad \mathcal{O}_\alpha \times \mathcal{O}_\beta = \mathcal{O}_{\alpha+\beta(\mathrm{mod}\ k)} . \tag{A.79}$$

The $\mathrm{U}(1)_k$ theory has one-form $\mathbb{Z}_k$ symmetry generated by

$$\mathcal{O}_{\alpha_{\mathbb{Z}_k}} = \mathcal{O}_{\alpha=1} , \text{ which satisfies } (\mathcal{O}_{\alpha_{\mathbb{Z}_k}})^k = 1 . \tag{A.80}$$

For even k, the theory has $\mathbb{Z}_2 \subset \mathbb{Z}_k$ one-form symmetry generated by $\mathcal{O}_{\alpha_{\mathbb{Z}_2}} = \mathcal{O}_{\alpha=\frac{k}{2}}$. The topological spin of the symmetry generating anyon is

$$h_{\alpha_{\mathbb{Z}_2}} = \frac{k}{8}\ (\mathrm{mod}\ 1) = \begin{cases} \frac{1}{4}\ (\mathrm{mod}\ 1) & \text{if } k \in 4\mathbb{Z}+2 , \\ 0\ (\mathrm{mod}\ 1) & \text{if } k \in 8\mathbb{Z} , \\ \frac{1}{2}\ (\mathrm{mod}\ 1) & \text{if } k \in 8\mathbb{Z}+4 . \end{cases} \tag{A.81}$$

Thus, according to A.61

$$\mathbb{Z}_2 \text{ in } \mathrm{U}(1)_{k\in 2\mathbb{Z}} \text{ is } \begin{cases} \text{anomalous} & \text{if } k \in 4\mathbb{Z}+2 , \\ \text{non-anomalous and bosonic} & \text{if } k \in 8\mathbb{Z} , \\ \text{non-anomalous and fermionic} & \text{if } k \in 8\mathbb{Z}+4 . \end{cases} \tag{A.82}$$

Example: modular data of SU(2)$_k$. The action of the topological field theory is

$$S = \frac{k}{4\pi}\int_{\mathcal{M}_3} \mathrm{Tr}\left(A\wedge dA + \frac{2i}{3}A\wedge A\wedge A\right) = \frac{k}{4\pi}\int_{X_4\,:\,\partial X_4 = \mathcal{M}_3} \mathrm{Tr}\,(F\wedge F) . \tag{A.83}$$

The topological theory is a non-spin TQFT with a $\mathbb{Z}_2$ one-form symmetry. Modular data (S,T) of $\mathrm{SU}(2)_{k>0}$ TFT are $(\alpha,\beta = 0,1,\dots k)$

$$S_{\alpha\beta} = \sqrt{\frac{2}{k+2}}\sin\left(\frac{\pi(\alpha+1)(\beta+1)}{k+2}\right) , \quad T_{\alpha\beta} = \delta_{\alpha\beta}e^{2\pi i h_\alpha} \text{ with } h_\alpha = \frac{\alpha(\alpha+2)}{4(k+2)}\ (\mathrm{mod}\ 1) . \tag{A.84}$$

The loop operator $\mathcal{O}_\alpha$ corresponds to the Wilson loop operator in the representation $\mathrm{Sym}^{\otimes\alpha}\square$, α-th symmetric product of the fundamental representation, i.e.

$$\mathcal{O}_\alpha^\Gamma = \mathrm{Tr}_{R=\mathrm{Sym}^{\otimes\alpha}\square}\left(P\exp\left(i\oint_\Gamma A\right)\right) . \tag{A.85}$$

The $\mathbb{Z}_2$ one-form symmetry is generated by

$$\mathcal{O}_{\alpha_{\mathbb{Z}_2}} = \mathcal{O}_{\alpha=k} \text{ with } h_{\alpha_{\mathbb{Z}_2}} = \frac{k}{4} . \tag{A.86}$$

According to (A.61)

$$\text{The } \mathbb{Z}_2 \text{ is } \begin{cases} \text{anomalous} & \text{if } k \in 2\mathbb{Z}+1 , \\ \text{non-anomalous and bosonic} & \text{if } k \in 4\mathbb{Z} , \\ \text{non-anomalous and fermionic} & \text{if } k \in 4\mathbb{Z}+2 . \end{cases} \tag{A.87}$$

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Example: $\mathbf{U(1)_{4k}/\mathbb{Z}_2 = U(1)_k}$. The non-anomalous $\mathbb{Z}_2$ symmetry is generated by $\mathcal{O}_{\alpha=2k}$ which is bosonic (resp. fermionic) for even k (resp. odd k). After the one-form $\mathbb{Z}_2$ gauging, the Hilbert-space on the two-torus is

$$\begin{aligned}
k\in 2\mathbb{Z}_{>0}:\ \mathcal{H}^{\mathrm{U}(1)_{4k}/\mathbb{Z}_2}(\mathbb{T}^2) &= \mathrm{Span}\left\{|[\alpha]\rangle := \frac{1}{\sqrt{2}}(|2\alpha\rangle + |2\alpha+2k\rangle):\ \alpha=0,1,\dots,k-1\right\},\\
k\in 2\mathbb{Z}_{>0}-1:\ \mathcal{H}_{--}^{\mathrm{U}(1)_{4k}/\mathbb{Z}_2}(\mathbb{T}^2) &= \mathrm{Span}\left\{|[\alpha]\rangle := \frac{1}{\sqrt{2}}(|2\alpha\rangle + |2\alpha+2k\rangle):\ \alpha=0,1,\dots,k-1\right\}.
\end{aligned} \tag{A.88}$$

The modular data of the $\mathrm{U}(1)_{4k}/\mathbb{Z}_2$ theory is $(\alpha,\beta=0,1,\cdots,k-1)$

$$S^{\mathrm{U}(1)_{4k}/\mathbb{Z}_2}_{[\alpha][\beta]} = \frac{2}{\sqrt{4k}}e^{\frac{2\pi i(2\alpha)(2\beta)}{4k}} = S^{\mathrm{U}(1)_k}_{\alpha\beta}\ , \quad h^{\mathrm{U}(1)_{4k}/\mathbb{Z}_2}_{[\alpha]} = \begin{cases} \frac{(2\alpha)^2}{8k}\ (\mathrm{mod}\, 1) & \text{if } k\in 2\mathbb{Z}_{>0}\ ,\\ \frac{(2\alpha)^2}{8k}\ (\mathrm{mod}\, \frac{1}{2}) & \text{if } k\in 2\mathbb{Z}_{>0}-1\ . \end{cases} \tag{A.89}$$

It implies that $\mathrm{U}(1)_{4k}/\mathbb{Z}_2$ is actually the $\mathrm{U}(1)_k$ theory. From the $\mathbb{Z}_2$ gauging, one can also confirm that the $\mathrm{U}(1)_k$ is a non-spin (resp. spin) TQFT for even k (resp. odd k) since the $\mathbb{Z}_2$ theory is bosonic (resp. fermionic).

Example: $\mathbf{SU(2)_{2k}/\mathbb{Z}_2}$. For odd k, the $\mathbb{Z}_2$ one-form symmetry is fermionic and we have a spin topological theory after the $\mathbb{Z}_2$ gauging. The Hilbert-space on the two-torus in the NS-NS sector is

$$k\in 2\mathbb{Z}_{\geq 0}+1:\ \mathcal{H}_{--}^{\mathrm{SU}(2)_{2k}/\mathbb{Z}_2}(\mathbb{T}^2) = \mathrm{Span}\left\{|[\alpha]\rangle := \frac{1}{\sqrt{2}}(|2\alpha\rangle + |2k-\alpha\rangle):\ \alpha=0,\dots,\frac{(k-1)}{2}\right\}. \tag{A.90}$$

On the basis, the modular S, T^2 matrices are

$$\begin{aligned}
S^{\mathrm{SU}(2)_{2k}/\mathbb{Z}_2}_{[\alpha][\beta]} &= 2\sqrt{\frac{2}{2k+2}}\sin\left(\frac{\pi(2\alpha+1)(2\beta+1)}{2k+2}\right),\\
(T^2)^{\mathrm{SU}(2)_{2k}/\mathbb{Z}_2}_{[\alpha][\beta]} &= \delta_{\alpha\beta}e^{2\pi i h_{[\alpha]}}\ \text{ with } h_{[\alpha]} = \frac{\alpha(\alpha+1)}{2(k+1)}\ \left(\mathrm{mod}\ \frac{1}{2}\right)\ .
\end{aligned} \tag{A.91}$$

For even k, on the other hand, the $\mathbb{Z}_2$ one-form symmetry is bosonic and the resulting theory after the gauging is a bosonic topological field theory. The Hilbert space on the two-torus is

$$\begin{aligned}
k\in 2\mathbb{Z}_{>0}\ :\ \mathcal{H}^{\mathrm{SU}(2)_{2k}/\mathbb{Z}_2}(\mathbb{T}^2) &= \mathcal{H}^{\mathrm{SU}(2)_{2k}/\mathbb{Z}_2}_{\text{untwisted}}(\mathbb{T}^2)\oplus \mathcal{H}^{\mathrm{SU}(2)_{2k}/\mathbb{Z}_2}_{\text{twisted}}(\mathbb{T}^2)\ \text{ where}\\
\mathcal{H}^{\mathrm{SU}(2)_{2k}/\mathbb{Z}_2}_{\text{untwisted}} &= \mathrm{Span}\left\{|[\alpha]\rangle := \frac{1}{\sqrt{2}}(|2\alpha\rangle + |2k-\alpha\rangle)\ :\ \alpha=0,\dots,\frac{(k-2)}{2}\right\},\\
\mathcal{H}^{\mathrm{SU}(2)_{2k}/\mathbb{Z}_2}_{\text{twisted}} &= \mathrm{Span}\{|k;+\rangle,\ |k;-\rangle\}\ .
\end{aligned} \tag{A.92}$$

The modular data of the topological theory is

$$S^{\mathrm{SU}(2)_{2k}/\mathbb{Z}_2}_{[\alpha][\beta]} = 2\sqrt{\frac{2}{2k+2}}\sin\left(\frac{\pi(2\alpha+1)(2\beta+1)}{2k+2}\right)\,,\quad S^{\mathrm{SU}(2)_{2k}/\mathbb{Z}_2}_{[0](k;\pm)} = \sqrt{\frac{2}{2k+2}}\,,$$
and
$$h_{[\alpha]} = \frac{\alpha(\alpha+1)}{2(k+1)}\ (\mathrm{mod}\,1)\,,\quad h_{(k;\pm)} = \frac{k(k+2)}{8(k+1)}\ (\mathrm{mod}\ 1)\,. \tag{A.93}$$

Example: $\frac{\mathbf{SU(2)_{10}\times SU(2)_2}}{\mathbb{Z}_2^{\mathbf{diag}}}$. The Hilbert-space of $\mathrm{SU}(2)_{10}\times\mathrm{SU}(2)_2$ theory on the two torus is

$$\mathcal{H}^{\mathrm{SU}(2)_{10}\times\mathrm{SU}(2)_2}(\mathbb{T}^2) = \mathrm{Span}\{|\alpha_1,\alpha_2\rangle\ :\ 0\le\alpha_1\le 10,\ \ 0\le\alpha_2\le 2\}\,. \tag{A.94}$$

Modular data is

$$S^{\mathrm{SU}(2)_{10}\times\mathrm{SU}(2)_2}_{(\alpha_1,\alpha_2),(\beta_1,\beta_2)} = S^{\mathrm{SU}(2)_{10}}_{\alpha_1\beta_1}\times S^{\mathrm{SU}(2)_2}_{\alpha_2\beta_2},\quad T^{\mathrm{SU}(2)_{10}\times\mathrm{SU}(2)_2}_{(\alpha_1,\alpha_2),(\beta_1,\beta_2)} = T^{\mathrm{SU}(2)_{10}}_{\alpha_1\beta_1}\times T^{\mathrm{SU}(2)_2}_{\alpha_2\beta_2}\,. \tag{A.95}$$

The theory has $\mathbb{Z}_2^{(1)}\times\mathbb{Z}_2^{(2)}$ one-form symmetry generated by

$$\mathcal{O}_{\alpha_{\mathbb{Z}_2^{(1)}}} = \mathcal{O}_{(\alpha_1=10,\alpha_2=0)}\,,\quad \mathcal{O}_{\alpha_{\mathbb{Z}_2^{(2)}}} = \mathcal{O}_{(\alpha_1=0,\alpha_2=2)}\,. \tag{A.96}$$

Both $\mathbb{Z}_2^{(1)}$ and $\mathbb{Z}_2^{(2)}$ are fermionic. The diagonal $\mathbb{Z}_2^{\mathrm{diag}}$ one-form symmetry is generated by

$$\mathcal{O}_{\alpha_{\mathbb{Z}_2^{\mathrm{diag}}}} = \mathcal{O}_{(\alpha_1,\alpha_2)=(10,2)} \tag{A.97}$$

and is bosonic. After the $\mathbb{Z}_2^{\mathrm{diag}}$ gauging, the Hilbert-space on the two torus is

$$\begin{aligned}
&\mathcal{H}^{(\mathrm{SU}(2)_{10}\times\mathrm{SU}(2)_2)/\mathbb{Z}_2^{\mathrm{diag}}}(\mathbb{T}^2) = \mathcal{H}^{(\mathrm{SU}(2)_{10}\times\mathrm{SU}(2)_2)/\mathbb{Z}_2^{\mathrm{diag}}}_{\mathrm{untwisted}}(\mathbb{T}^2)\oplus\mathcal{H}^{(\mathrm{SU}(2)_{10}\times\mathrm{SU}(2)_2)/\mathbb{Z}_2^{\mathrm{diag}}}_{\mathrm{twisted}}(\mathbb{T}^2),\\
&\mathcal{H}^{(\mathrm{SU}(2)_{10}\times\mathrm{SU}(2)_2)/\mathbb{Z}_2^{\mathrm{diag}}}_{\mathrm{untwisted}}(\mathbb{T}^2) = \mathrm{Span}\Big\{|[\alpha]\rangle := \frac{1}{\sqrt{2}}\left(|2\alpha,0\rangle+|10-2\alpha,2\rangle\right):\ \alpha=0,\dots,5\Big\}\\
&\qquad\qquad\oplus\mathrm{Span}\Big\{|[\tilde\alpha]\rangle := \frac{1}{\sqrt{2}}\left(|2\tilde\alpha+1,1\rangle+|9-2\tilde\alpha,1\rangle\right):\ \tilde\alpha=0,\dots,1\Big\},\\
&\mathcal{H}^{(\mathrm{SU}(2)_{10}\times\mathrm{SU}(2)_2)/\mathbb{Z}_2^{\mathrm{diag}}}_{\mathrm{twisted}}(\mathbb{T}^2) = \mathrm{Span}\Big\{|5,1;+\rangle,\ |5,1;-\rangle\Big\}\,.
\end{aligned} \tag{A.98}$$

There are 10 simple objects and their $\{S_{0\alpha}\}$ are ($|0\rangle = |[\alpha=0]\rangle$)

$$\begin{aligned}
&S_{0[\alpha]} = \sqrt{\frac{1}{6}}\sin\left(\frac{\pi(2\alpha+1)}{12}\right)\,,\quad S_{0[\tilde\alpha]} = \sqrt{\frac{1}{3}}\sin\left(\frac{\pi(2\tilde\alpha+2)}{12}\right)\,,\\
&S_{0,(5,1;+)} = S_{0,(5,1;-)} = \frac{1}{2\sqrt{3}}\,.
\end{aligned} \tag{A.99}$$

Example: Lee-Yang TQFT as a Galois conjugation of Fibonacci TQFT. The Fibonacci topological field is

$$\text{Fibonacci TQFT: } \frac{\mathrm{SU}(2)_3 \otimes \mathrm{U}(1)_2}{\mathbb{Z}_2} \quad \text{or equivalently } (G_2)_1 \,. \tag{A.100}$$

The modular data of the bosonic topological field theory is

$$S = \begin{pmatrix} \sqrt{\frac{1}{10}\left(5-\sqrt{5}\right)} & \sqrt{\frac{1}{10}\left(\sqrt{5}+5\right)} \\ \sqrt{\frac{1}{10}\left(\sqrt{5}+5\right)} & -\sqrt{\frac{1}{10}\left(5-\sqrt{5}\right)} \end{pmatrix}\,, \quad T = \begin{pmatrix} 1 & 0 \\ 0 & \exp\left(\frac{4\pi i}{5}\right) \end{pmatrix}. \tag{A.101}$$

The non-unitary Lee-Yang TQFT, whose modular data is given in (3.11), is a Galois conjugate of the Fibonacci TQFT.

B Dual description for $\frac{T[\mathrm{SU}(2)]}{\text{“}\mathrm{PSU}(2)_k^{\mathrm{diag}}\text{”}}$

In term of the 3D-3D correspondence, the theory $\frac{T[\mathrm{SU}(2)]}{\text{“}\mathrm{PSU}(2)_k^{\mathrm{diag}}\text{”}}$ corresponds to a 3-manifold called the once-punctured torus bundle with monodromy matrix $\varphi = \mathbb{S}\mathbb{T}^k$ [65, 66, 85]

$$\mathcal{T}[(\Sigma_{1,1} \times S^1)_{\varphi = \mathbb{S}\mathbb{T}^k}; A = S^1_{\mathrm{punct}})] = \frac{T[\mathrm{SU}(2)]}{\text{“}\mathrm{PSU}(2)_k^{\mathrm{diag}}\text{”}}\,. \tag{B.1}$$

The once-punctured torus bundle $(\Sigma_{1,1} \times S^1)_\varphi$ with $\varphi \in \mathrm{SL}(2,\mathbb{Z})$ is defined as

$$\begin{aligned} &(\Sigma_{1,1} \times S^1)_\varphi = (\Sigma_{1,1} \times [0,1])/\sim\,, \quad \text{where} \\ &(x,0) \sim (\varphi(x),1)\,. \end{aligned} \tag{B.2}$$

Here $\Sigma_{g=1,h=1}$ is the once-punctured torus and $\varphi \in \mathrm{SL}(2,\mathbb{Z})$ is an element of mapping class group of the Riemann surface. The mapping torus actually depends only on the conjugacy class of φ in $\mathrm{SL}(2,\mathbb{Z})$, i.e.

$$(\Sigma_{1,1} \times S^1)_{\varphi_1} = (\Sigma_{1,1} \times S^1)_{\varphi_2} \;\text{ if and only if }\; \varphi_1 \sim \varphi_2\,. \tag{B.3}$$

Here $\varphi_1 \sim \varphi_2$ means that φ_1 are φ_2 are related to each other by conjugation in $\mathrm{SL}(2,\mathbb{Z})$. The mapping torus has a torus boundary. Generally, for 3-manifold N with a torus boundary, we need to choose primitive boundary 1-cycle $A \in H_1(\partial N, \mathbb{Z})$ to specify its associated 3D gauge theory $\mathcal{T}[N;A]$ [42, 61]. In the once-puncture torus bundle, there is a natural choice of the boundary 1-cycle $A = S^1_{\mathrm{punct}}$, which is the cycle encircling the puncture in $\Sigma_{1,1}$.

In the view point of 3D-3D correspondence, the conformal window in (3.103) can be geometrically understood from the following topological fact:

$$(\Sigma_{1,1} \times S^1)_{\varphi = \mathbb{S}\mathbb{T}^k} \text{ is } \begin{cases} \text{non-hyperbolic} & \text{if } |k| < 2\,, \\ \text{hyperbolic} & \text{if } |k| > 2\,. \end{cases} \tag{B.4}$$

We will focus on the case when the mapping torus $(\Sigma_{1,1} \times S^1)_\varphi$ is hyperbolic. For the case, the conjugacy class of φ can be decomposed into positive powers of $\mathbb{L}$ and $\mathbb{R}$ (up to sign)

$$\begin{aligned} &\varphi = \pm g \left(\mathbb{L}^{n_1}\mathbb{R}^{n_2}\mathbb{L}^{n_3}\ldots\mathbb{L}^{n_L}(\text{or } \mathbb{R}^{n_L})\right) g^{-1} \qquad n_i \in \mathbb{Z}_{>0}\,, \\ &\mathbb{L} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\,, \quad \mathbb{R} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\,. \end{aligned} \tag{B.5}$$

The mapping torus $(\Sigma_{1,1} \times S^1)_\varphi$ has an alternative topological description based on an ideal triangulation. Using an ideal triangulation of $(\Sigma_{1,1} \times S^1)_\varphi$, one can give an alternative description for $\mathcal{T}[(\Sigma_{1,1} \times S^1)_\varphi; A = S^1_{\text{punct}}]$ following the algorithm proposed in [42]. Interestingly, the 3D gauge theory based on an ideal triangulation has only manifest $\mathcal{N}=2$ supersymmetry. We expect that the $\mathcal{N}=2$ gauge theories have enhanced $\mathcal{N}=4$ supersymmetry at IR.

$k = 3$ ($\varphi = \mathbb{LR} \sim \mathbb{ST}^3$) case: $(\mathcal{T}_{\text{min}})^{\otimes 2}$. The corresponding mapping-torus can be decomposed into two ideal tetrahedrons [66]. The corresponding 3D $\mathcal{N}=2$ theory is [27]

$$\mathcal{T}[(\Sigma_{1,1} \times S^1)_{\varphi = \mathbb{LR}}; A = S^1_{\text{punct}}] = \left(\mathrm{U}(1)_{3/2} + \Phi\right) \otimes \left(\mathrm{U}(1)_{-3/2} + \Phi\right)\,. \tag{B.6}$$

The theory is nothing but $(\mathcal{T}_{\text{min}})^{\otimes 2}$ using the duality between $(\mathrm{U}(1)_{3/2} + \Phi)$ and $(\mathrm{U}(1)_{-3/2} + \Phi)$.

$k = 4$ ($\varphi = \mathbb{LLR} \sim \mathbb{ST}^4$) case: $\mathcal{N} = 2 \to \mathcal{N} = 5$. The corresponding mapping torus can be decomposed into three ideal tetrahedrons [66]. According to the algorithm in [42], the corresponding 3D $\mathcal{N}=2$ field theory is

$$\begin{aligned} &\mathcal{T}[(\Sigma_{1,1} \times S^1)_{\varphi = \mathbb{LLR}}; A = S^1_{\text{punct}}] \\ &= (\text{3D } \mathcal{N}=2 \text{ U}(1) \times \text{U}(1) \text{ gauge theory with mixed CS level } K \text{ coupled to} \\ &\quad\ \text{3 chiral multiplets } (\Phi_1, \Phi_2, \Phi_3) \text{ of charge } \mathbf{Q} \\ &\quad\ \text{with superpotential } W = (\Phi_1\Phi_2\Phi_3)^2 + V_{\mathbf{m}=(1,-1)})\,. \end{aligned} \tag{B.7}$$

The mixed CS level K for $\mathrm{U}(1) \times \mathrm{U}(1)$ gauge group is

$$K = \begin{pmatrix} -1 & -1/2 \\ -1/2 & -1 \end{pmatrix}\,. \tag{B.8}$$

Gauge charges $\mathbf{Q}$ for 3 chirals are assigned as follows

	U(1)	U(1)
Φ_1	1	0
Φ_2	−1	1
Φ_3	0	−1

$V_{\mathbf{m}=(\mathfrak{m}_1,\mathfrak{m}_2)}$ denotes the 1/2 BPS bare monopole operator with monopole fluxes $(\mathfrak{m}_1, \mathfrak{m}_2)$ coupled to the two gauge U(1)s. The bare monopole operator is a gauge-invariant 1/2 BPS chiral primary when $\mathfrak{m}_1+\mathfrak{m}_2=0$. The superpotential breaks the $\mathrm{U}(1)^3$ flavor symmetry to $\mathrm{U}(1)$. From the F-maximization, the IR superconformal R-charge ($\nu=0$) is determined as

$$R_{\nu=0}(\Phi_1)=R_{\nu=0}(\Phi_3)=1,\ R_{\nu=0}(\Phi_2)=-1\ . \tag{B.9}$$

The superconformal index at the IR conformal fixed point is

$$\begin{aligned}&\mathcal{I}^{\rm sci}_{\mathcal{T}[(\Sigma_{1,1}\times S^1)_{\varphi=\mathbb{LLR}},A=S^1_{\rm punct}]}(q,\eta,\nu=0;s=1)\\&=1+q^{1/2}-\left(\eta+\frac{1}{\eta}+1\right)q-(2+\eta+\eta^{-1})q^{3/2}+\dots\ .\end{aligned} \tag{B.10}$$

Surprisingly, the index show $\mathcal{N}=5$ supersymmetry instead of $\mathcal{N}=4$ [54]. Actually, from the superconformal index computation, one can confirm that the theory is dual to the following $\mathcal{N}=5$ gauge theory

$$\mathcal{T}[(\Sigma_{1,1}\times S^1)_{\varphi=\mathbb{LLR}};A=S^1_{\rm punct}]=\left(\mathrm{SU}(2)^{\frac{1}{2}\oplus\frac{1}{2}}_{|k|=3}\ \text{ in (3.47)}\right)\ . \tag{B.11}$$

C Contour integrals

We explicitly evaluate the contour integrals that appear in this paper.

C.1 $\mathcal{Z}^{S^3_b}_{T[\mathrm{SU}(2)]}(b=1,X_1,X_2,m=0,\nu)$

With the properties in appendix D, the partition function (3.84) for $b=1$, $m=0$, and $\nu=0$ is simplified as

$$\mathcal{Z}^{S^3_b}_{T[\mathrm{SU}(2)]}(b=1,X_1,X_2,m=0,\nu=0)=\frac{e^{\frac{2\pi i}{3}}}{4\pi}\int dZ\frac{e^{\frac{ZX_2}{\pi i}}}{\cosh(Z)+\cosh(X_1)}\ . \tag{C.1}$$

The contour integral of the integrand in (C.1) along the path in figure 2 is

$$\begin{aligned}&\int_{-\infty}^{\infty}dZ\frac{e^{\frac{iZX_2}{\pi}}}{\cosh(Z)+\sinh(Z)}+\int_{\infty}^{-\infty}dZ\frac{e^{\frac{iZX_2}{\pi}}e^{-2X_2}}{\cosh(Z)+\sinh(Z)}=2\pi i\left(\frac{e^{-\frac{iX_1X_2}{\pi}}e^{-X_2}}{\sinh(X_1)}-\frac{e^{\frac{iX_1X_2}{\pi}}e^{-X_2}}{\sinh(X_1)}\right)\\&\to\int_{-\infty}^{\infty}dZ\frac{e^{\frac{ZX_2}{\pi i}}}{\cosh(Z)+\sinh(Z)}=\frac{4\pi\sin\left(\frac{X_1X_2}{\pi}\right)}{\sinh(X_1)\left(e^{X_2}-e^{-X_2}\right)}=\frac{2\pi\sin\left(\frac{X_1X_2}{\pi}\right)}{\sinh(X_1)\sinh(X_2)}\ .\end{aligned} \tag{C.2}$$

Here, the first and second term of the first line of (C.2) are for the path C_1 and C_2 in the figure 2 respectively. Restoring the factor $\frac{e^{\frac{2\pi i}{3}}}{4\pi}$, we have

$$\mathcal{Z}^{S^3_b}_{T[\mathrm{SU}(2)]}(b=1,X_1,X_2,m=0,\nu=0)=\frac{e^{\frac{2\pi i}{3}}}{2}\frac{\sin\left(\frac{X_1X_2}{\pi}\right)}{\sinh(X_1)\sinh(X_2)}\ ,\ \text{ for }\ |\mathrm{Im}[X_1]|\le\pi\ . \tag{C.3}$$

The computations for gauging $\mathrm{SU}(2)^H$ or $\mathrm{SU}(2)^C$ of this $T[\mathrm{SU}(2)]$ theory at the conformal limit ($\nu=0$) are straightforward since they are always reduced to the simple Gaussian integration at the three-sphere partition function level.

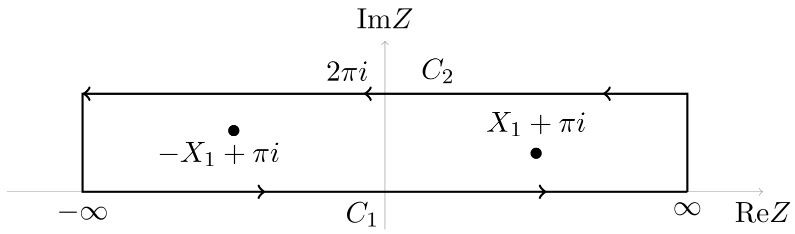

Figure 2. A contour for the evaluation of (C.1). Assuming $|\mathrm{Im}[X_1]| < \pi$, there are two simple poles at $Z = \pm X_1 + \pi i$ inside the path.

For the degenerate limit, say $\nu = 1$, the three-sphere partition function $\mathcal{Z}^{S_b^3}_{T[\mathrm{SU}(2)]}(b = 1, m = 0, \nu = 1)$ diverges due to the factor $\psi_{\hbar=2\pi i}\big(2\pi i(1-\nu)\big)$ from the adjoint matter. To handle it, we expand the partition function divided by this divergence around $\nu = 1$ as

$$\begin{aligned}\frac{\mathcal{Z}^{S_b^3}_{T[\mathrm{SU}(2)]}(b=1,m=0,\nu)}{\psi_{\hbar=2\pi i}\big(2\pi i(1-\nu)\big)} &= \left(\frac{e^{\frac{13\pi i}{12}}}{2\pi}\int \mathrm{d}Z\, e^{\frac{iZX_2}{\pi}}\right)\\ &+\left(\frac{e^{\frac{19\pi i}{12}}}{2}\int \mathrm{d}Z\, e^{\frac{iZX_2}{\pi}}\left(1-\frac{i}{\pi}\frac{Z\sinh(Z)-X_1\sinh(X_1)}{\cosh(Z)-\cosh(X_1)}\right)\right)(\nu-1)+\mathcal{O}((\nu-1)^2)\,.\end{aligned} \tag{C.4}$$

The divergence of $\psi_{\hbar=2\pi i}\big(2\pi i(1-\nu)\big)$ comes from the simple pole at $\nu = 1$

$$\psi_{\hbar=2\pi i}\big(2\pi i(1-\nu)\big) = -\frac{e^{\frac{\pi i}{12}}}{2\pi i}\frac{1}{(\nu-1)} + \frac{e^{\frac{\pi i}{12}}(\pi - i)}{2\pi} + \mathcal{O}((\nu-1)^1)\,. \tag{C.5}$$

The first term in (C.4) vanishes after $\mathrm{SU}(2)_k$ gauging since

$$\int \mathrm{d}X\mathrm{d}Z\, e^{\frac{iZX}{\pi}} e^{\frac{kX^2}{2\pi i}} \sinh^2(X) = 0\,. \tag{C.6}$$

This means that there is no divergence problem even for $\nu = 1$ if we are considering, say, the $\mathrm{SU}(2)_k$ gauged $T[\mathrm{SU}(2)]$ theory, since (C.4) always starts with linear $(\nu-1)$ term which cancels the diverging simple pole in (C.5) from the adjoint matter.

C.2 $\mathcal{Z}^{S_b^3}_{(k_1,k_2)}(b = 1, m = 0, \nu = \pm 1)$

With the help of appendix D, (C.5), and (C.6), the partition function (3.82) for $b = 1$, $\nu = 1$ is simplified as (with Gaussian integral of X_2)

$$\begin{aligned}\mathcal{Z}^{S_b^3}_{(k_1,k_2)}(b=1,m=0,\nu=1) &= \frac{e^{\frac{11\pi i}{12}}}{16\pi^3}\sqrt{\frac{2}{k_2}}\int \mathrm{d}Z\mathrm{d}X_1 \sinh^2(X_1)\left(\frac{Z\sinh(Z)-X_1\sinh(X_1)}{\cosh(Z)-\cosh(X_1)}\right)\\ &\times\left(e^{-\frac{2\pi i}{k_2}}e^{\frac{2Z}{k_2}}+e^{-\frac{2\pi i}{k_2}}e^{-\frac{2Z}{k_2}}-2\right)e^{-\frac{Z^2}{2\pi i k_2}}e^{\frac{k_1X_1^2}{2\pi i}}\,.\end{aligned} \tag{C.7}$$

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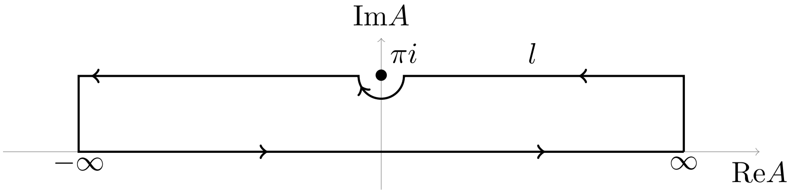

Figure 3. A path l for (C.10). There is a simple pole at $A = \pi i$.

Changing the variables as $X_1 \to A + B$, $Z \to A - B$, we have

$$\begin{aligned}\mathcal{Z}^{S_b^3}_{(k_1,k_2)}(b=1, m=0, \nu=1) &= \frac{e^{\frac{11\pi i}{12}}}{2\pi^3}\sqrt{\frac{2}{k_2}}\int \mathrm{d}A\mathrm{d}B \sinh^2(A+B)\frac{A\cosh(A)}{\sinh(A)} \\ &\times e^{\frac{(k_1k_2-1)}{2\pi i k_2}A^2} e^{\frac{(k_1k_2-1)}{2\pi i k_2}B^2} e^{\frac{(k_1k_2+1)}{\pi i k_2}AB}\left(e^{\frac{2A-2B-2\pi i}{k_2}}-1\right). \quad \text{(C.8)}\end{aligned}$$

where several even terms in the integrand under $A \to -A$, $B \to -B$ have been stacked up. With Gaussian integral of B, (C.8) is further evaluated as

$$\begin{aligned}\mathcal{Z}^{S_b^3}_{(k_1,k_2)}(b=1, m=0, \nu=1) &= \frac{e^{\frac{2\pi i}{3}}}{4\pi^2}\frac{1}{\sqrt{k_1k_2-1}}\int \mathrm{d}A\frac{A\cosh(A)}{\sinh(A)} \\ &\times\Bigg[e^{-\frac{2(A+\pi i)^2}{\pi i k_2}}\left(e^{-\frac{2(A+\pi i+k_2\pi i)^2}{\pi i k_2(k_1k_2-1)}}+e^{-\frac{2(A+\pi i-k_2\pi i)^2}{\pi i k_2(k_1k_2-1)}}-2e^{-\frac{2(A+\pi i)^2}{\pi i k_2(k_1k_2-1)}}\right) \\ &- e^{-\frac{2A^2}{\pi i k_2}}\left(e^{-\frac{2(A+k_2\pi i)^2}{\pi i k_2(k_1k_2-1)}}+e^{-\frac{2(A-k_2\pi i)^2}{\pi i k_2(k_1k_2-1)}}-2e^{-\frac{2A^2}{\pi i k_2(k_1k_2-1)}}\right)\Bigg]. \quad \text{(C.9)}\end{aligned}$$

The third and sixth terms of (C.9) can be evaluated from the following contour integrals, see figure 3:

$$\begin{aligned}&\oint_l \mathrm{d}A\frac{A\cosh(A)}{\sinh(A)}e^{-\frac{2A^2}{\pi i k_2}}e^{-\frac{2A^2}{\pi i k_2(k_1k_2-1)}} = 0 \\ &\to \int \mathrm{d}A\frac{A\cosh(A)}{\sinh(A)}\left(e^{-\frac{2A^2}{\pi i k_2}}e^{-\frac{2A^2}{\pi i k_2(k_1k_2-1)}} - e^{-\frac{2(A+\pi i)^2}{\pi i k_2}}e^{-\frac{2(A+\pi i)^2}{\pi i k_2(k_1k_2-1)}}\right) \\ &= \pi i a_+ + \frac{\pi i}{2}\int \mathrm{d}A\frac{\cosh(A)}{\sinh(A)}\left(e^{-\frac{2(A+\pi i)^2}{\pi i k_2}}e^{-\frac{2(A+\pi i)^2}{\pi i k_2(k_1k_2-1)}} - e^{-\frac{2(A-\pi i)^2}{\pi i k_2}}e^{-\frac{2(A-\pi i)^2}{\pi i k_2(k_1k_2-1)}}\right). \quad \text{(C.10)}\end{aligned}$$

where $a_+ = \pi i e^{-\frac{2\pi i k_1}{k_1k_2-1}}$ is the residue of $\frac{A\cosh(A)}{\sinh(A)}e^{-\frac{2A^2}{\pi i k_2}}e^{-\frac{2A^2}{\pi i k_2(k_1k_2-1)}}$ at $A = \pi i$. Again, the last integral in (C.10) can be evaluated from the below contour integral, see figure 4:

$$\begin{aligned}&\oint_{l'} \mathrm{d}A\,\frac{\cosh(A)}{\sinh(A)}e^{-\frac{2A^2}{\pi i k_2}}e^{-\frac{2A^2}{\pi i k_2(k_1k_2-1)}} = -2\pi i\tilde{a}_0 \\ &\to \int \mathrm{d}A\,\frac{\cosh(A)}{\sinh(A)}\left(e^{-\frac{2(A+\pi i)^2}{\pi i k_2}}e^{-\frac{2(A+\pi i)^2}{\pi i k_2(k_1k_2-1)}} - e^{-\frac{2(A-\pi i)^2}{\pi i k_2}}e^{-\frac{2(A-\pi i)^2}{\pi i k_2(k_1k_2-1)}}\right) \\ &= -\pi i(\tilde{a}_- + 2\tilde{a}_0 + \tilde{a}_+)\,. \quad \text{(C.11)}\end{aligned}$$

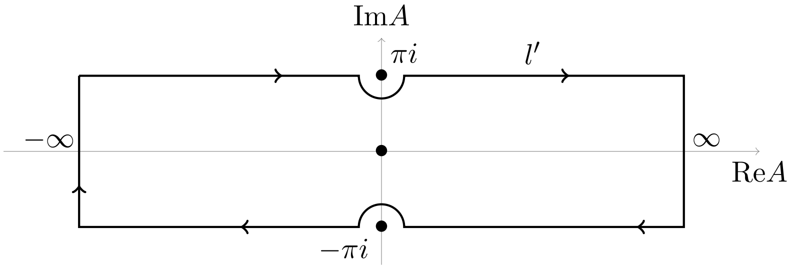

Figure 4. A path l' for (C.11). There are simple poles at $A = 0, \pm\pi i$.

where $\tilde{a}_0 = 1$, $\tilde{a}_\pm = e^{-\frac{2\pi i k_1}{k_1 k_2 - 1}}$ are the residues of $\frac{\cosh(A)}{\sinh(A)} e^{-\frac{2A^2}{\pi i k_2}} e^{-\frac{2A^2}{\pi i k_2 (k_1 k_2 - 1)}}$ at $A = 0, \pm\pi i$ respectively. Plugging this into (C.10), we have

$$\int \mathrm{d}A \, \frac{A\cosh(A)}{\sinh(A)} \left(e^{-\frac{2A^2}{\pi i k_2}} e^{-\frac{2A^2}{\pi i k_2 (k_1 k_2 - 1)}} - e^{-\frac{2(A+\pi i)^2}{\pi i k_2}} e^{-\frac{2(A+\pi i)^2}{\pi i k_2 (k_1 k_2 - 1)}} \right) = \pi^2 \,. \tag{C.12}$$

Likewise, the rest terms in (C.9) can also be evaluated in a similar way as

$$\begin{aligned} \int \mathrm{d}A \, \frac{A\cosh(A)}{\sinh(A)} \Bigg[& e^{-\frac{2(A+\pi i)^2}{\pi i k_2}} \left(e^{-\frac{2(A+\pi i + k_2 \pi i)^2}{\pi i k_2 (k_1 k_2 - 1)}} + e^{-\frac{2(A+\pi i - k_2 \pi i)^2}{\pi i k_2 (k_1 k_2 - 1)}} \right) \\ & - e^{-\frac{2A^2}{\pi i k_2}} \left(e^{-\frac{2(A+k_2 \pi i)^2}{\pi i k_2 (k_1 k_2 - 1)}} + e^{-\frac{2(A-k_2 \pi i)^2}{\pi i k_2 (k_1 k_2 - 1)}} \right) \Bigg] = -2\pi^2 e^{-\frac{2\pi i k_2}{k_1 k_2 - 1}} \,. \end{aligned} \tag{C.13}$$

Combining the two results (C.12) and (C.13), and restoring the overall factor in (C.9), we have

$$\begin{aligned} \mathcal{Z}^{S_b^3}_{(k_1,k_2)}(b=1, m=0, \nu=1) &= \frac{e^{\frac{2\pi i}{3}}}{4\pi^2} \frac{1}{\sqrt{k_1 k_2 - 1}} \left(2\pi^2 - 2\pi^2 e^{-\frac{2\pi i k_2}{k_1 k_2 - 1}} \right) \\ &= e^{\frac{7\pi i}{6} - \frac{k_2 \pi i}{k_1 k_2 - 1}} \frac{1}{\sqrt{k_1 k_2 - 1}} \sin\left(\frac{k_2 \pi}{k_1 k_2 - 1} \right) . \end{aligned} \tag{C.14}$$

For $\nu = -1$, thanks to (3.85), the only difference is nothing but an exchange of the role of k_1 and k_2:

$$\mathcal{Z}^{S_b^3}_{(k_1,k_2)}(b=1, m=0, \nu=-1) = e^{\frac{7\pi i}{6} - \frac{k_1 \pi i}{k_1 k_2 - 1}} \frac{1}{\sqrt{k_1 k_2 - 1}} \sin\left(\frac{k_1 \pi}{k_1 k_2 - 1} \right) . \tag{C.15}$$

C.3 $\mathcal{Z}^{S_b^3}_{\mathbf{diag}_k}(b=1, m=0, \nu=\pm 1)$

By the mirror-symmetry property (3.85) it is enough to consider the case of $\nu = 1$. With the help of appendix D, (C.5), and (C.6), the partition function (3.106) for $b = 1$, $\nu = 1$ is simplified as

$$\mathcal{Z}^{S_b^3}_{\mathrm{diag}_k}(b{=}1, m{=}0, \nu{=}1) = -\frac{e^{\frac{5\pi i}{12}}}{4\pi^3} \int \mathrm{d}Z \mathrm{d}X \sinh^2(X) e^{\frac{iZX}{\pi}} e^{\frac{kX^2}{2\pi i}} \left(\frac{X\sinh(X) - Z\sinh(Z)}{\cosh(X) - \cosh(Z)} \right) . \tag{C.16}$$

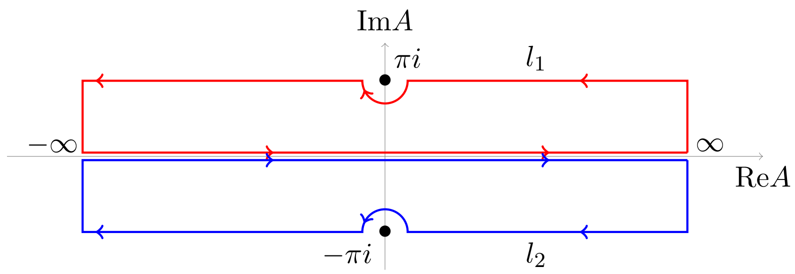

Figure 5. Two paths l_1 and l_2 for (C.19). There are simple poles at $A = \pm\pi i$.

Changing the variables as $X \to A+B$, $Z \to A-B$, we have

$$\mathcal{Z}^{S_b^3}_{\text{diag}_k}(b=1, m=0, \nu=1) = -\frac{e^{\frac{5\pi i}{12}}}{2\pi^3}\int \mathrm{d}A\mathrm{d}B \sinh^2(A+B) e^{\frac{kAB}{\pi i}} \\ \times e^{\frac{(k-2)A^2}{2\pi i}} e^{\frac{(k+2)B^2}{2\pi i}} \left(\frac{A\cosh(A)}{\sinh(A)} + \frac{B\cosh(B)}{\sinh(B)}\right). \tag{C.17}$$

We first consider the first term in the integrand which is an Gaussian integral of B.

$$\int \mathrm{d}A\mathrm{d}B \, \sinh^2(A+B) e^{\frac{kAB}{\pi i}} e^{\frac{(k-2)A^2}{2\pi i}} e^{\frac{(k+2)B^2}{2\pi i}} \frac{A\cosh(A)}{\sinh(A)} \\ \to \frac{\pi}{\sqrt{8i(k+2)}} \int \mathrm{d}A \, \frac{A\cosh(A)}{\sinh(A)} \left(e^{-\frac{2(A-\pi i)^2}{\pi i(k+2)}} + e^{-\frac{2(A+\pi i)^2}{\pi i(k+2)}} - 2e^{-\frac{2A^2}{\pi i(k+2)}}\right). \tag{C.18}$$

Now, consider contour integrals along the paths shown below, see figure 5:

$$\oint_{l_1} \mathrm{d}A \, \frac{A\cosh(A)}{\sinh(A)} e^{-\frac{2A^2}{\pi i(k+2)}} + \oint_{l_2} \mathrm{d}A \, \frac{A\cosh(A)}{\sinh(A)} e^{-\frac{2A^2}{\pi i(k+2)}} = 0 \\ \to \int \mathrm{d}A \, \frac{A\cosh(A)}{\sinh(A)} \left(e^{-\frac{2(A-\pi i)^2}{\pi i(k+2)}} + e^{-\frac{2(A+\pi i)^2}{\pi i(k+2)}} - 2e^{-\frac{2A^2}{\pi i(k+2)}}\right) \\ = \pi i(u_- - u_+) - \pi i \int \mathrm{d}A \, \frac{\cosh(A)}{\sinh(A)} \left(e^{-\frac{2(A+\pi i)^2}{\pi i(k+2)}} - e^{-\frac{2(A-\pi i)^2}{\pi i(k+2)}}\right), \tag{C.19}$$

where $u_\pm = \pm\pi i e^{-\frac{2\pi i}{k+2}}$ are the residues of $\frac{A\cosh(A)}{\sinh(A)} e^{-\frac{2A^2}{\pi i(k+2)}}$ at $A = \pm\pi i$. Again, the integral at the last term of (C.19) can be evaluated by considering the following path, see figure 6:

$$\oint_{l_3} \mathrm{d}A \, \frac{\cosh(A)}{\sinh(A)} e^{-\frac{2A^2}{\pi i(k+2)}} = -2\pi i v_0 \\ \to \int \mathrm{d}A \, \frac{\cosh(A)}{\sinh(A)} \left(e^{-\frac{2(A+\pi i)^2}{\pi i(k+2)}} - e^{-\frac{2(A-\pi i)^2}{\pi i(k+2)}}\right) = -\pi i(v_- + 2v_0 + v_+) = -2\pi i(1 + e^{-\frac{2\pi i}{k+2}}). \tag{C.20}$$

where $v_0 = 1$, $v_\pm = e^{-\frac{2\pi i}{k+2}}$ are the residues of $\frac{\cosh(A)}{\sinh(A)} e^{-\frac{2A^2}{\pi i(k+2)}}$ at $A = 0, \pm\pi i$ respectively. Combining the results (C.19) and (C.20), we have

$$\int \mathrm{d}A\mathrm{d}B \, \sinh^2(A+B) e^{\frac{kAB}{\pi i}} e^{\frac{(k-2)A^2}{2\pi i}} e^{\frac{(k+2)B^2}{2\pi i}} \frac{A\cosh(A)}{\sinh(A)} = -\frac{\pi^3}{\sqrt{2i(k+2)}}. \tag{C.21}$$

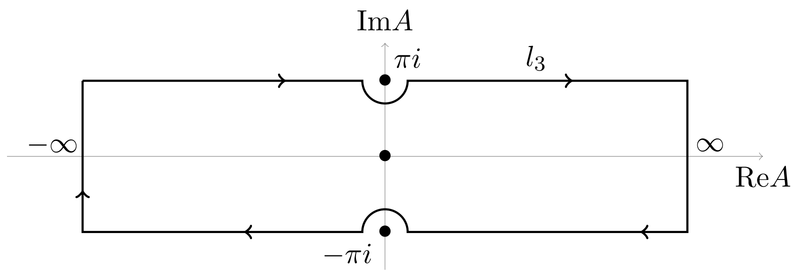

Figure 6. A path l_3 for (C.20). There are simple poles at $A = 0, \pm\pi i$.

Similarly, the second term in the integrand of (C.17) can be evaluated as

$$\int \mathrm{d}A\mathrm{d}B \sinh^2(A+B) e^{\frac{kAB}{\pi i}} e^{\frac{(k-2)A^2}{2\pi i}} e^{\frac{(k+2)B^2}{2\pi i}} \frac{B\cosh(B)}{\sinh(B)} = -\frac{\pi^3}{\sqrt{2i(k-2)}} \,. \tag{C.22}$$

Finally, with the results (C.21) and (C.22), and restoring the overall factor in (C.17), we have

$$\mathcal{Z}^{S_b^3}_{\mathrm{diag}_k}(b=1, m=0, \nu=1) = e^{\frac{\pi i}{6}} \left(\sqrt{\frac{1}{8(k-2)}} + \sqrt{\frac{1}{8(k+2)}} \right) . \tag{C.23}$$

D Quantum dilogarithm function

The quantum dilogarithm function (Q.D.L) $\psi_\hbar(Z)$ is defined by [86] ($\hbar = 2\pi i b^2$)

$$\psi_\hbar(Z) := \begin{cases} \prod_{r=1}^{\infty} \frac{1-q^r e^{-Z}}{1-\tilde{q}^{-r+1} e^{-\tilde{Z}}} & \text{if} \quad |q| < 1 \,, \\ \prod_{r=1}^{\infty} \frac{1-\tilde{q}^r e^{-\tilde{Z}}}{1-q^{-r+1} e^{-Z}} & \text{if} \quad |q| > 1 \,, \end{cases} \tag{D.1}$$

with

$$q := e^{2\pi i b^2} \,, \qquad \tilde{q} := e^{2\pi i b^{-2}} \,, \qquad \tilde{Z} := \frac{1}{b^2} Z \,, \tag{D.2}$$

where b is the squashing parameter. The function satisfies the following difference equations:

$$\psi_\hbar(Z + 2\pi i b^2) = (1 - e^{-Z})\psi_\hbar(Z) \,, \quad \psi_\hbar(Z + 2\pi i) = (1 - e^{-\frac{Z}{b^2}})\psi_\hbar(Z) \,. \tag{D.3}$$

At the special value $b = 1$, the Q.D.L simplifies as

$$\log \psi_{\hbar=2\pi i}(Z) = \frac{-(2\pi + iZ)\log(1 - e^{-Z}) + i\mathrm{Li}_2(e^{-Z})}{2\pi} \,, \tag{D.4}$$

and there is a special limit at $b = 1$

$$\lim_{p \to 0} p\, \psi_{\hbar=2\pi i}(p) = e^{\frac{\pi i}{12}} \,. \tag{D.5}$$

On the other hand, the asymptotic expansion when $\hbar = 2\pi i b^2 \to 0$ is given by

$$\log \psi_\hbar(Z) \xrightarrow{b^2\to 0} \sum_{n=0}^{\infty} \frac{B_n \hbar^{n-1}}{n!} \mathrm{Li}_{2-n}(e^{-Z}) \,. \tag{D.6}$$

Here B_n is the n-th Bernoulli number with $B_1 = 1/2$. For several computations in the main text, one needs to utilize the identity

$$\mathrm{Li}_2(u) + \mathrm{Li}_2(u^{-1}) = -\frac{\pi^2}{6} - \frac{1}{2}\big(\log(-u)\big)^2 \,. \tag{D.7}$$

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