(Phenomenology/Lattice-Compatible) SU(3) MχPT HD up to $\mathcal{O}(p^4)$ and the $\mathcal{O}(R^4)$-Large-N Connection

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ABSTRACT: Obtaining the values of the coupling constants of the low energy effective theory corresponding to QCD, compatible with experimental data, even in the (vector) mesonic sector from (the $\mathcal{M}$-theory uplift of) a UV-complete string theory dual, has thus far been missing in the literature. We take the first step in this direction by obtaining the values of the coupling constants of the $\mathcal{O}(p^4)$ χPT Lagrangian in the chiral limit involving the NGBs and ρ meson (and its flavor partners) from the $\mathcal{M}$-theory/type IIA dual of large-N thermal QCD, inclusive of the $\mathcal{O}(R^4)$ corrections. We observe that ensuring compatibility with phenomenological/lattice results (the values) as given in [1], requires a relationship relating the $\mathcal{O}(R^4)$ corrections and large-N suppression. In other words, QCD demands that the higher derivative corrections and the large-N suppressed corrections in its M/string theory dual, are related. As a bonus, we explicitly show that the $\mathcal{O}(R^4)$ corrections in the UV to the $\mathcal{M}$-theory uplift of the type IIB dual of large-N thermal QCD at low temperatures, can be consistently set to be vanishingly small.

Dedicated by one of the authors (AM) to the memory of his supervisor, the late Professor Daniel Koltun who introduced him to χPT 25 years ago — a subject the former has returned back to after 22 years — and instilled in the former the importance of uncompromising independence of thought and inquiry.

Contents

1 Introduction

Chiral Perturbation Theory (χPT) is an effective field theory of Quantum Chromodynamics (QCD) which describes the low energy regime (IR) of Quantum Chromodynamics (QCD). In χPT, the degrees of freedom are hadrons (which include mesons and baryons and hyperons but we will study only mesonic χPT). χPT Lagrangian consists of various terms which are invariant under chiral symmetry $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$, charge conjugation and parity symmetry of QCD. Chiral symmetry $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ is spontaneously broken to $\mathrm{SU}(N_f)_V$ yielding $(N_f^2 - 1)$ (pseudo-)Goldstone bosons where N_f is the number of flavors. As an effective field theory, it is renormalizable order-by-order in momentum.

The ρ vector meson can also be incorporated by augmenting the chiral symmetry with the inclusion of a HLS (Hidden Local Symmetry) whose gauge group is $G_{\text{global}} \times H_{local}$, where $G_{\text{global}} = \text{SU}(N_f)_L \times \text{SU}(N_f)_R$ and $H = \text{SU}(N_f)_V$ is the HLS [2]. The ρ meson and its flavor partners are identified with gauge boson of HLS. Chiral symmetry breaking scale is $\Lambda_\chi \sim 4\pi F_\pi \sim 1.1\,\text{GeV}$ [3] in the chiral limit [5]. Since Λ_χ is much larger than the ρ mass scale and $M_\rho < \Lambda_\chi$, therefore we can expand the generating functional of QCD in terms of p/Λ_χ or m/Λ_χ. At scale $\mu = \Lambda_\chi$, perturbative expansion in μ/Λ_χ breaks down. One can construct the most general form of the Lagrangian order by order in the derivative expansion consistently with the chiral symmetry. The "universal" leading order Lagrangian is constructed from the terms of $O(p^2)$. J. Gasser and H. Leutwyler worked out the SU(2) [4]/SU(3) [5] chiral perturbation theory lagrangian up to $O(p^4)$ and the renormalization of the coupling constants at scale $\mu = m_\eta$ [5]. At NLO, i.e. $O(p^4)$, there are 12 coupling constants "$(L_{i=1,2,\ldots 10}, H_1, H_2)$". Some of the low energy constants L_4, L_5, L_6, L_8 have been calculated from Lattice simulation from the MILC collaboration [6]. In [7], the ten low energy constants $L_{i=1,2,\ldots 10}$ were evaluated at scale $\mu = M_\rho$; [8] also contains the low energy constants at $\mu = M_\rho$. Updated values are given in [1].

QCD is a non-abelian gauge theory coupled to quarks. This theory can also be explored using gauge/gravity duality which is a duality between strongly coupled gauge theory and weakly coupled gravitational theory. Originally, it started with the AdS/CFT correspondance [9] which is a duality between strongly coupled $\mathcal{N} = 4$ SU(N) supersymmetric Yang Mills theory in large N limit and type IIB string theory on $AdS_5 \times S^5$ background. $\mathcal{N} = 4$ SU(N) supersymmetric Yang Mills theory is a conformal field theory but QCD is non-conformal. This correspondence has been generalised to study QCD. There are two approaches to study holographic theories. One is the top-down approach in which one starts with ten dimensional type IIB/IIA string theory or eleven dimensional $\mathcal{M}$-theory and then compactifies it to four dimensions with desired properties; this will be the approach followed in this work. The other is the bottom-up approach, in which one has to first know what properties we are looking for and then construct the gravitational dual, which usually involves an AdS_5.

As regards a top-down approach, Kruczenski et al. [10] considered an interesting probe $D6$-branes in a $D4$-brane type IIA supergravity background, which they used to explore various aspects of low energy phenomena in QCD. An important ingredient which was still missing from their model, however, were the massless pions as Nambu-Goldstone bosons associated with the spontaneous breaking of the $\text{U}(N_f)_L \times \text{U}(N_f)_R$ chiral symmetry in QCD. Following [10], Sakai-Sugimoto (SS) in [11, 12] considered a nice modification by looking at a $D4/D8/\overline{D8}$ system in type IIA supergravity background with anti-periodic boundary condition for fermions along a circle to break supersymmetry. This model exhibited chiral symmetry breaking with $D8 - \overline{D8}$ pairs merging into $D8$-branes. This model also yields massless pions which are identified with Nambu-Goldstone bosons associated with chiral symmetry breaking, and the lightest vector meson (ρ meson). The Sakai Sugimoto model is closely related to the HLS formalism which produces Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin-type relation among the couplings. Chern-Simons term on the probe brane leads to the Wess-Zumino-Witten term. In [12], Sakai Sugimoto obtained few low energy

constants (L_1, L_2, L_3) of SU(3) chiral perturbation theory at $\mathcal{O}(p^4)$ which were close to the values given in [7] for suitable choice of a parameter κ. Although Sakai Sugimoto model reproduces various physical quantities of low energy QCD but this model deviates from realistic QCD above the energy scale of the vector mesons because they obtain a four-dimensional theory by compactifying $D4$-branes on a circle of radius $\tilde{M}_{KK}^{-1}$ with an infinite tower of Kaluza-Klein modes of mass scale $\tilde{M}_{KK}$ arises. These Kaluza-Klein modes do not appear in realistic QCD. Further, the SS model caters to the IR and is not UV complete. This was taken care of by the (only) UV-complete $D3, D5/\overline{D5}, D7/\overline{D7}$ holographic dual of large-N thermal QCD of the McGill group [14], its type IIA SYZ mirror and the $\mathcal{M}$-theory uplift of the same (in particular in the 'MQGP limit' (2.4)) as constructed in [15] (with one of the co-authors [AM]).

In [2], the authors considered the Sakai Sugimoto model [11] as holographic QCD model and proposed a method to integrate out infinite number of higher KK modes appearing in the expansion of five dimensional gauge field which consists of infinite number of vector and axial vector fields including pion as Nambu Goldstone boson arising due to spontaneous chiral symmetry breaking. First, they truncated the spectrum at certain level so that number of fields "integrated in" becomes finite in the theory and then integrated out all the KK modes except pion and lowest lying vector mode (ρ meson and the flavor partners). Using this method they obtained the effective lagrangian up to $O(p^4)$. However this Lagrangian is not same as the SU(3) chiral perturbation theory Lagrangian [5]. The authors in [13] derived relations between the SU(3) low energy coupling constants of [5] and $O(p^4)$ couplings in [2].

The main takeaway from this paper is that QCD imposes a relationship between the higher derivative corrections and large-N suppression when comparing our $\mathcal{M}$-theory/type IIA holographic computational results for the low energy coupling constants of χPT Lagrangian up to $\mathcal{O}(p^4)$ and corresponding experimental values of these coupling constants.

The rest of the paper is organized as follows. In section 2, we give a quick review of the type IIB holographic dual of large-N thermal QCD as constructed in [14], its SYZ type IIA mirror (and its $\mathcal{M}$-theory uplift) as obtained in [15] and summarize results pertaining to the applications of the type IIA/ $\mathcal{M}$-theory dual of holographic QCD phenomenology as obtained by the group to which the authors of this paper belong. In section 3, we outline obtaining the chiral limit of the Chiral Perturbation Theory Lagrangian up to $\mathcal{O}(p^4)$ involving the π and ρ mesons as well as their flavor partners, using the HLS formalism/notation of [2]. In section 4 which forms the core of the paper with all the results, we obtain the values of the LECs of [5] up to $\mathcal{O}(p^4)$ as radial integrals using the type IIA SYZ mirror of [14] inclusive of the $\mathcal{O}(R^4)$ $\mathcal{M}$-theory corrections worked out in [16]. In the process of matching the holographic results with experimental values see that there is a deep connection between the large-N suppression and the aforementioned l_p^6 ($\mathcal{O}(R^4)$) corrections. Section 5 has a summary of the results and the lessons learnt. There are four supporting appendices. Appendix A is about showing that the $\mathcal{O}(R^4)$ corrections to the $\mathcal{M}$-theory uplift of large-N thermal QCD-like theories at low temperatures (i.e. below the deconfinement temperature) can consistently be made to be vanishingly small in the UV. Appendix B has some details pertaining to the evaluation of the coupling constants appearing in the "HLS"

chiral Lagrangian as radial integrals, which is relevant to the computation of section 4. Appendix C is a brief review of the HLS formalism and a derivation of the $\mathcal{O}(p^2)$ SU(3) χPT Lagrangian of [5] as well as contributions to the low energy coupling constants (LECs) of $\mathcal{O}(p^4)$ SU(3) χPT Lagrangian of [5] arising from integrating out the rho mesons from the theory. Appendix D provides details of obtaining the DBI action for the type IIA flavor $D6$-branes incorporating the $\mathcal{O}(R^4)$ corrections to the $\mathcal{M}$-theory uplift of large-N thermal QCD-like theories.

2 Brief review of the (UV complete) type IIB/IIA SYZ mirror (and $\mathcal{M}$-theory) holographic dual of large-N thermal QCD at intermediate coupling, and holographic QCD phenomenology

In the context of a UV-complete top-down holographic dual of large-N thermal QCD, the following summarizes the main features of the brane construct and the gravity dual of [14], its type IIA mirror and $\mathcal{M}$-theory uplift (up to $\mathcal{O}(R^4)$) as well as holographic phenomenological applications of the same.

- **Brane construct of [14]:** the type IIB string dual of [14] consists of N $D3$-branes placed at the tip of six-dimensional conifold, with M $D5$-branes wrapping the vanishing S^2, referred to as fractional $D3$-branes, and M $\overline{D5}$-branes distributed along the resolved S^2 placed at antipodal points relative to the M $D5$-branes. Denoting the average $D5/\overline{D5}$ separation by $\mathcal{R}_{D5/\overline{D5}}$, $r > \mathcal{R}_{D5/\overline{D5}}$, would be the UV. The N_f $D7$-branes, holomorphically embedded via Ouyang embedding [17] in the resolved conifold geometry, "smeared"/delocalized along the angular directions $\theta_{1,2}$, are present in the UV, the IR-UV interpolating region and dip into the (confining) IR (but do not touch the $D3$-branes; the shortest $D3 - D7$ string corresponding to the lightest quark). In addition, N_f $\overline{D7}$-branes are present in the UV and the UV-IR interpolating region for the reason given below. The following table summarizes the aforementioned brane construct wherein $S^2(\theta_1, \phi_1)$ denotes the vanishing two-sphere and (NP/SP of) $S_a^2(\theta_2, \phi_2)$ is the (North Pole/South Pole of the) resolved/blown-up two-sphere — a being the radius of the blown-up S^2 — and r_{UV} is the UV cut-off and $\frac{\epsilon}{\left(\mathcal{R}_{D5/\overline{D5}} - |\mu_{\text{Ouyang}}|^{\frac{2}{3}}\right)} < 1$ in table 1. Also, μ_{Ouyang} is the Ouyang embedding parameter that is defined as:

$$\left(r^6 + 9a^2 r^4\right)^{\frac{1}{4}} e^{\frac{i}{2}(\psi - \phi_1 - \phi_2)} \sin\left(\frac{\theta_1}{2}\right) \sin\left(\frac{\theta_2}{2}\right) = \mu_{\text{Ouyang}}, \tag{2.1}$$

effected by (3.4) for vanishingly small $|\mu_{\text{Ouyang}}|$, while describing the embedding of the flavor $D7$-branes in the resolved conifold geometry.

- In the UV, one has $\text{SU}(N+M) \times \text{SU}(N+M)$ color gauge group and $\text{SU}(N_f) \times \text{SU}(N_f)$ flavor gauge group. There occurs a partial Higgsing of $\text{SU}(N + M) \times \text{SU}(N + M)$ to $\text{SU}(N + M) \times \text{SU}(N)$ as one goes from $r > \mathcal{R}_{D5/\overline{D5}}$ to $r < \mathcal{R}_{D5/\overline{D5}}$. This happens because at energies less than $\mathcal{R}_{D5/\overline{D5}}$ (IR), the $\overline{D5}$-branes are integrated out resulting in the reduction of the rank of one of the product gauge groups (which is $\text{SU}(N + $

S. No.	Branes	World Volume
1.	$N\ D3$	$\mathbb{R}^{1,3}(t, x^{1,2,3}) \times \{r = 0\}$
2.	$M\ D5$	$\mathbb{R}^{1,3}(t, x^{1,2,3}) \times \{r = 0\} \times S^2(\theta_1, \phi_1) \times \mathrm{NP}_{S^2_a(\theta_2,\phi_2)}$
3.	$M\ \overline{D5}$	$\mathbb{R}^{1,3}(t, x^{1,2,3}) \times \{r = 0\} \times S^2(\theta_1, \phi_1) \times \mathrm{SP}_{S^2_a(\theta_2,\phi_2)}$
4.	$N_f\ D7$	$\mathbb{R}^{1,3}(t, x^{1,2,3}) \times \mathbb{R}_+(r \in [|\mu_{\mathrm{Ouyang}}|^{\frac{2}{3}}, r_{\mathrm{UV}}]) \times S^3(\theta_1, \phi_1, \psi) \times \mathrm{NP}_{S^2_a(\theta_2,\phi_2)}$
5.	$N_f\ \overline{D7}$	$\mathbb{R}^{1,3}(t, x^{1,2,3}) \times \mathbb{R}_+(r \in [\mathcal{R}_{D5/\overline{D5}} - \epsilon, r_{\mathrm{UV}}]) \times S^3(\theta_1, \phi_1, \psi) \times \mathrm{SP}_{S^2_a(\theta_2,\phi_2)}$

Table 1. The type IIB Brane Construct of [14].

number of $D5 -$ branes$) \times \mathrm{SU}(N +$ number of $\overline{D5} -$ branes$))$. By the same token, the $\overline{D5}$-branes are "integrated in" for energies more than $\mathcal{R}_{D5/\overline{D5}}$ (UV), resulting in the conformal Klebanov-Witten-like $\mathrm{SU}(M + N) \times \mathrm{SU}(M + N)$ product color gauge group.

- The pair of gauge couplings, $g_{\mathrm{SU}(N+M)}$ and $g_{\mathrm{SU}(N)}$ can be shown to flow oppositely. The flux of the NS-NS B through the vanishing S^2 is the reason for introduction of non-conformality which is why $M\ \overline{D5}$-branes were included in [14] to cancel the net $D5$-brane charge in the UV. Further, as the N_f flavor $D7$-branes enter the RG flow of the gauge couplings via the dilaton, their contribution therefore needs to be canceled by $N_f\ \overline{D7}$-branes which is the reason for their inclusion in the UV.

  Using UV-complete top-down type IIB holographic dual of lagre-N thermal QCD at finite gauge/'t Hooft coupling [14], delocalized Strominger-Yau-Zaslow (SYZ) type IIA mirror of [14] and its M theory uplift in the 'MQGP' limit as worked out in [15], our group has made the following contributions to the top-down holographic dual of large N thermal QCD phenomenology at intermediate gauge/'t Hooft coupling.

- In the IR, at the end of a Seiberg-like duality cascade, the number of colors N_c gets identified with M, which in the 'MQGP limit' to be discussed below, can be tuned to equal 3 (see [18]).

- **Gravity dual of the brane construct of** [**14**]: the finite temperature ($> T_c$) on the gauge/brane side is effected in the gravitational dual via a black hole in the latter. Turning on of the temperature (in addition to requiring a finite separation between the $M\ D5$-branes and $M\ \overline{D5}$-branes to provide a natural scale above which one is in the UV) corresponds in the gravitational dual to having a non-trivial resolution parameter of the conifold. IR confinement on the brane/gauge theory side corresponds to having a non-trivial deformation of the conifold geometry in the gravitational dual. The gravity dual is hence given by a resolved warped deformed conifold wherein the $D3$-branes and the $D5$-branes are replaced by fluxes in the IR, and the back-reactions are included in the warp factor and fluxes.

- **Color-Flavor Enhancement of Length Scale in the IR**: in the IR in the MQGP limit (2.4), with the inclusion of terms higher order in $g_s N_f$ in the RR and NS-NS three-form fluxes and the NLO terms in N in the metric, there occurs an IR color-flavor enhancement of the length scale as compared to a Planckian length scale in

Klebanov-Strassler (KS)'s model even for $\mathcal{O}(1)$ M, thereby ensuring that quantum corrections will be suppressed. This was discussed/summarized in [19]/[18]. Essentially, defining:

$$N_{\text{eff}}(r) = N \left[1 + \frac{3g_s M_{\text{eff}}^2}{2\pi N} \left(\log r + \frac{3g_s N_f^{\text{eff}}}{2\pi} \left(\log r \right)^2 \right) \right],$$

$$M_{\text{eff}}(r) = M + \frac{3g_s N_f M}{2\pi} \log r + \sum_{m \geq 1} \sum_{n \geq 1} N_f^m M^n f_{mn}(r),$$

$$N_f^{\text{eff}}(r) = N_f + \sum_{m \geq 1} \sum_{n \geq 0} N_f^m M^n g_{mn}(r), \tag{2.2}$$

wherein the type IIB axion $C_0 = N_f^{\text{eff}} \frac{(\psi - \phi_1 - \phi_2)}{4\pi}$, at the end of a Seiberg-like duality cascade, $N_{\text{eff}}(r_0 \in \text{IR}) = 0$ and writing the ten-dimensional warp factor $h \sim \frac{L^4}{r^4}$, the length scale L in the IR will be given by:

$$L \sim \sqrt[4]{M} N_f^{\frac{3}{4}} \sqrt{\left(\sum_{m \geq 0} \sum_{n \geq 0} N_f^m M^n f_{mn}(r_0) \right) \left(\sum_{l \geq 0} \sum_{p \geq 0} N_f^l M^p g_{lp}(r_0) \right)^{\frac{1}{4}}} L_{\text{KS}}, \tag{2.3}$$

$L_{KS} \sim \sqrt[4]{g_s M} \sqrt{\alpha'}$. The relation (2.3) implies that in the IR, relative to KS, there is a color-flavor enhancement of the length scale in the MQGP limit. Hence, in the IR, even for $N_c^{\text{IR}} = M = 3$ and $N_f = 2(u/d) + 1(s)$, upon inclusion of $n, m > 1$ terms in M_{eff} and N_f^{eff} in (2.2), $L \gg L_{\text{KS}}(\sim L_{\text{Planck}})$ in the MQGP limit (2.4), implying that *the stringy corrections are suppressed and one can trust supergravity calculations.*

- **Obtaining $N_c = 3$**: as explained in [18], in the IR, at the end of a Seiberg-like duality cascade, the number of colors N_c gets identified with M, which in the 'MQGP limit' (2.4) can be tuned to equal 3. This is briefly summarized now. One can identify N_c with the effective number N_{eff} of $D3$-branes and the effective number M_{eff} of $D5$-branes as: $N_C = N_{\text{eff}}(r) + M_{\text{eff}}(r)$. $N_{\text{eff}}(r)$ is defined via $\tilde{F}_5 \equiv dC_4 + B_2 \wedge F_3 = \mathcal{F}_5 + *\mathcal{F}_5$, wherein $\mathcal{F}_5 \equiv N_{\text{eff}} \times \text{Vol}(\text{Base of Resolved Warped Deformed Conifold})$. Similarly, M_{eff} is defined via $M_{\text{eff}} = \int_{S^3} \tilde{F}_3$ (the S^3 being dual to $e_\psi \wedge (\sin\theta_1 d\theta_1 \wedge d\phi_1 - B_1 \sin\theta_2 \wedge d\phi_2)$, B_1 being an 'asymmetry factor' defined in [14]; $e_\psi \equiv d\psi + \cos\theta_1 \, d\phi_1 + \cos\theta_2 \, d\phi_2$) and [20]: $\tilde{F}_3 (\equiv F_3 - \tau H_3) \propto M(r) \equiv M \frac{1}{1 + e^{\alpha(r - \mathcal{R}_{D5/\overline{D5}})}}, \alpha \gg 1$. As N_{eff} varies between $N \gg 1$ in the UV and 0 in the deep IR, and M_{eff} varies between 0 in the UV and M in the deep IR, N_c varies between M in the deep IR and a large value [in the MQGP limit of (2.4) for a large value of N] in the UV. Therefore, at very low energies, the number of colors N_c can be approximated by M, which in the MQGP limit is finite and can hence be taken to be equal to three. Additionally, one can set $N_f = 2(u/d) + 1(s)$. Hence, in the IR, this is somewhat like the Veneziano limit in which $\frac{N_f}{N_c}$ is fixed (but, unlike [14, 15], $N_c, N_f \to \infty$ in the Veneziano limit in, e.g., [21]) as (in the IR) $\frac{N_f}{N_c} \sim \frac{N_f}{M}$ in [14].

Thus, under the aforementioned Seiberg-like duality cascade, the N $D3$-branes are cascaded away and there is a finite M left at the end corresponding to a strongly

coupled IR-confining $SU(M)$ gauge theory; the finite temperature version of this $SU(M)$ gauge theory is what was considered in [14]. So, at the end of the Seiberg-like duality cascade in the IR, the number of colors N_c, identified with M, in the 'MQGP limit' can be tuned to equal 3.

- **The MQGP limit, type IIA Strominger-Yau-Zaslow (SYZ) mirror of [14] and its $\mathcal{M}$-theory uplift at intemediate gauge coupling**:

  – For constructing a holographic dual of thermal QCD-like theories, one would be required to consider intemediate gauge coupling (as well as finite number of colors) — dubbed as the 'MQGP limit' in [15]. From the perspective of gauge-gravity duality, this would hence require looking at the strong-coupling/non-perturbative limit of string theory — M theory. The MQGP limit in [15, 19] was defined as:

  $$g_s \sim \frac{1}{\mathcal{O}(1)}, M, N_f \equiv \mathcal{O}(1), \ N \gg 1, \ \frac{g_s M^2}{N} \ll 1. \tag{2.4}$$

  – The $\mathcal{M}$-theory uplift of the type IIB holographic dual of [14] was constructed in [15, 19] by working out the Strominger-Yau-Zaslow (SYZ) type IIA mirror of [14] effected via a triple T duality along a local special Lagrangian (sLag) T^3 — which could be identified with the T^2-invariant sLag of [22] with a large base $\mathcal{B}(r, \theta_1, \theta_2)$ (of a $T^3(\phi_1, \phi_2, \psi)$-fibration over $\mathcal{B}(r, \theta_1, \theta_2)$) [19, 23].[1] Let us briefly discuss the basic idea. Consider the aforementioned N D3-branes with its world-volume directions $x^{0,1,2,3}$ at the tip of conifold. Further, assuming the M D5-branes to be parallel to these $D3$-branes as well as wrapping the vanishing $S^2(\theta_1, \phi_1)$, a single T-dual along ψ yields N D4-branes wrapping the ψ circle and M D4-branes straddling a pair of orthogonal $NS5$-branes. The world volumes of these pair of $NS5$-branes correspond to the vanishing $S^2(\theta_1, \phi_1)$ and the blown-up $S^2(\theta_2, \phi_2)$ with a non-zero resolution parameter a (the radius of the blown-up $S^2(\theta_2, \phi_2)$). Two more T-dualities along ϕ_1 and ϕ_2, then convert the aforementioned pair of orthogonal $NS5$-branes into a pair of orthogonal Taub-NUT spaces, the N D4-branes into N color $D6$-branes and the M straddling $D4$-branes also to $D6$-branes. Also, in the presence of the aforementioned N_f flavor $D7$-branes (embedded holomorphically via the Ouyang embedding), oriented parallel to the $D3$-branes and "wrapping" a non-compact four-cycle $\Sigma^{(4)}(r, \psi, \theta_1, \phi_1)$, upon T-dualization yield N_f $D6$-branes "wrapping" a non-compact three-cycle $\Sigma^{(3)}(r, \theta_1, \phi_2)$. An uplift to $\mathcal{M}$-theory of the SYZ type IIA mirror so obtained, converts the $D6$-branes to KK monopoles that are variants

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[1]As explained in [18] also as a footnote, consider $D5$-branes wrapping the resolved S^2 of a resolved conifold geometry [24], which one knows, globally, breaks SUSY. As in [25], to begin with, SYZ is implemented wherein the pair of S^2s are replaced by a pair of T^2s in the delocalized limit, and the correct T-duality coordinates are identified. Then, when uplifting the mirror to M theory, it is found that a G_2-structure can be chosen which is free of the delocalization. For the SYZ mirror of the resolved warped deformed conifold which figures in the gravitational dual of large-N thermal QCD of [14], that gets uplifted to $\mathcal{M}$-theory with G_2 structure worked out in [15], the idea is exactly the same.

of Taub-NUT spaces. Therefore, all the branes are converted to geometry and fluxes and one ends up with $\mathcal{M}$-theory on a G_2-structure manifold. Similarly, one may perform identical three T-dualities on the gravity dual on the type IIB side, which is a resolved warped-deformed conifold with fluxes, to obtain another G_2 structure manifold, giving us the MQGP holographic dual of [15, 19].

Hence, the type IIB model of [14] make it an ideal holographic dual of thermal QCD because: (i) it is UV conformal (Landau poles are absent), (ii) it is IR confining, (iii) the quarks transform in the fundamental representation of flavor and color groups, and (iv) it is defined for the full range of temperature — both low and high.

- **Conceptual physics issues miscellanea**

    - *Regime of validity of the top-down holographic model and* Λ_{QCD}: a natural question that one would like to answer is the range of scales in which the top-down holographic model elaborated upon, is expected to match QCD. The basic idea, using the notations introduced earlier on in this section, that answers this question is that the range of variation of the radial coordinate in the supergravity dual corresponding to the energy scale in QCD-like theories is determined by: $\left\{ r \mid N_{\text{eff}}(r) = \int_{M_5(\theta_{1,2},\phi_{1,2},\psi)} (F_5 + B_2 \wedge C_3) \approx 0 \right\} \cap \{r \mid M_{\text{eff}} \equiv \mathcal{O}(1)\}$ — $M_5(\theta_{1,2},\phi_{1,2},\psi)$ being the base of the non-Kähler resolved warped deformed conifold — because this will ensure that the (effective) number of colors can be set to be $\mathcal{O}(1)$, and in fact 3. One can show that in the IR (wherein $|\log r| \gg 1$) in the MQGP limit, $N_{\text{eff}}(r) \sim N\left[1 + g_s \frac{g_s M^2(g_s N_f)}{N} \log^3 r\right]$ [28] and from (2.2), estimating: $M\left[1 + \frac{3g_s N_f M \log r}{2\pi}\right] \equiv \mathcal{O}(1)$, or $\log r = -\frac{2\pi(-\mathcal{O}(1)+M)}{3g_s M N_f}$ and substituting into N_{eff} yields, e.g., for $N \sim 10^2, N_f = 3, M \sim \mathcal{O}(1)$ as in the MQGP limit.

        The computations in this paper are in the low temperature limit, i.e., for temperatures below the deconfinement temperature whereat one has bound states of quarks such as mesons which is what we are interested in looking at. To therefore understand the upper limit in energy on the QCD side or r in the gravitational dual side, we therefore need to understand the non-perturbative QCD scale, Λ_{QCD}, in terms of the geometrical data of our top-down holographic model. Let us remind ourselves that (before the Seiberg-like dualities) the SU($M \times N$) and SU(N) gauge couplings $g_{\text{SU}(M+N)}$ and $g_{\text{SU}(N)}$ satisfy [14]: $4\pi^2 \left(\frac{1}{g^2_{\text{SU}(M+N)}} + \frac{1}{g^2_{\text{SU}(N)}}\right) e^{\phi^{\text{IIB}}(r,\theta_{10},\theta_{20})} = \pi$; $4\pi^2 \left(\frac{1}{g^2_{\text{SU}(M+N)}} - \frac{1}{g^2_{\text{SU}(N)}}\right) e^{\phi^{\text{IIB}}(r,\theta_{10},\theta_{20})} \sim \frac{1}{2\pi\alpha'} \int_{S^2(\theta_1,\phi_1)} B^{\text{IIB}} \sim g_s M_{\text{eff}}(r) N_f^{\text{eff}}(r) \log r \big|_{r \in \text{IR-UV interpolating region}}$. Note, everywhere r, a are in fact $\frac{r}{\mathcal{R}_{D5/\overline{D5}}}, \frac{a}{\mathcal{R}_{D5/\overline{D5}}}$ where $\mathcal{R}_{D5/\overline{D5}}$ is taken to be $\sqrt{3}a$ (a being the resolution parameter of the blown-up S^2) [28]. Hence, $\Lambda_{\text{QCD}} = \mathcal{R}_{D5/\overline{D5}} \sim \sqrt{3}a$ because near $r = \mathcal{R}_{D5/\overline{D5}}$, the aforementioned gauge couplings become very large indicating the onset of non-perturbative QCD. Recall, in the UV, i.e., $\forall r > \mathcal{R}_{D5/\overline{D5}}$, we obtain the conformal SU($N + M$) $\times$ SU($N + M$) Klebanov-Witten-like (asymptotically supersymmetric as the complexified three-form type

IIB flux G_3 becomes ISD for $r \gg \mathcal{R}_{D5/\overline{D5}}$) gauge theory, and in the IR, i.e., $\forall r < \mathcal{R}_{D5/\overline{D5}}$, we obtain the non-conformal non-supersymmetric $\mathrm{SU}(M+N) \times \mathrm{SU}(N)$ gauge theory, which after a Seiberg-like duality cascade, is expected to yield IR confining QCD-like gauge theory.

– *Hierarchy of scales in the gravitational dual*: there is an IR cut-off denoted by r_0 indicative of the boundary of the deep IR. The $D5 - \overline{D5}$ separation $\mathcal{R}_{D5/\overline{D5}}$ provides another scale so that at energies larger than $\mathcal{R}_{D5/\overline{D5}}$, the $D5 - \overline{D5}$ strings become massive and the gauge group, as already mentioned above, becomes the Klebanov-Witten-like UV-conformal $\mathrm{SU}(M + N) \times \mathrm{SU}(M + N)$, and at energies less than $\mathcal{R}_{D5/\overline{D5}}$ the gauge group is Higgsed down to the Klebanov-Strassler-like $\mathrm{SU}(M + N) \times \mathrm{SU}(N)$;[2] there is also a $\mathrm{U}(1)^M$ for massless strings starting and ending on the same $\overline{D5}$-brane. For all practical purposes, $r > \mathcal{R}_{D5/\overline{D5}}$ is in the UV. As one is always working in the "near-horizon" limit in which the ten-dimensional warp factor $h(r, \theta_{1,2}) \sim \frac{L^4}{r^4}\left(1 + f\left(r; N, M, N_f\right)\right)$, even in the UV (i.e., one drops the 1 that figures, e.g., in the warp factor in the Klebanov-Witten supergravity dual: $h = 1 + \frac{L^4}{r^4}$), where $L \equiv (4\pi g_s N \alpha')^{\frac{1}{4}}$, hence, the UV cut-off $r_{\mathrm{UV}} \stackrel{<}{\sim} L$. So, in our holographic dual, the gravitational analog of Λ_{QCD} and r_{UV} are separated.

- **Holographic QCD phenomenology**

  – In [26], two of the authors (VY and AM) along with K.Sil, studied the glueball spectra and evaluated the masses of $0^{++}, 0^{-+}, 0^{--}, 1^{++}, 2^{++}$ glueballs in type IIB/type IIA/M theory supergravity backgrounds using WKB quantization and by imposing Neumann/Dirichlet boundary conditions at the IR cut-off. It was found that WKB quantization produces masses of $0^{++}, 0^{-+}, 0^{--}, 1^{++}, 2^{++}$ glueballs very close to the lattice results.

  – In [27], two of the authors (VY and AM) along with K.Sil calculated (pseudo-)vector and (pseudo-)scalar meson spectra at finite coupling (part of the 'MQGP' limit), and compared their result with PDG data. It was found that masses of the (pseudo-)vector ($\rho[770], a1[1260], \rho[1450], a1[1640]$) and (pseudo-)scalar ($f0[980]/a0[980], f0[1370], f0[1450]$) mesons were closer to the PDG data than previously obtained in the literature.

  – In [29], two of the authors (VY and AM) studied (exotic) scalar glueball 0_E^{++} which correspond to metric fluctuations of the M theory uplift at finite coupling. Using the same (exotic) scalar glueball 0_E^{++}-meson interaction Lagrangian linear in (exotic) scalar glueball and quartic in meson fields was derived. Decay widths

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[2]In principle, there is another scale that exists in the type IIB holographic dual, which is provided by the embedding of the $D7$-branes via the Ouyang's embedding (2.1), where the modulus of the Ouyang embedding parameter $|\mu|^{\frac{2}{3}}$ gives a measure of the radial separation of the "deepest" embedded flavor $D7$-branes in the IR from the color $D3$-branes corresponding hence to the mass of the lightest quark. Hence, one could have a refinement of the hierarchy and assume the IR to be given by $r \in [r_0, |\mu|^{\frac{2}{3}}]$, the IR-UV interpolating region given by $r \in [|\mu|^{\frac{2}{3}}, \mathcal{R}_{D5/\overline{D5}}]$ and the UV corresponding to $r \in [\mathcal{R}_{D5/\overline{D5}}, r_{\mathrm{UV}}]$. However, for simplicity of calculations, we have merged the first two in this paper.

S. No.	Manifold	G-Structure	Non-Trivial Torsion Classes
1.	$M_6(r, \theta_1, \theta_2, \phi_1, \phi_2, \psi)$	SU(3)	$T_{SU(3)}^{IIA} = W_1 \oplus W_2 \oplus W_3 \oplus W_4 \oplus W_5 : W_4 \sim W_5$
2.	$M_7(r, \theta_1, \theta_2, \phi_1, \phi_2, \psi, x^{10})$	G_2	$T_{G_2}^{\mathcal{M}} = W_{14} \oplus W_{27}$
3.	$M_8(x^0, r, \theta_1, \theta_2, \phi_1, \phi_2, \psi, x^{10})$	SU(4)	$T_{SU(4)}^{\mathcal{M}} = W_2 \oplus W_3 \oplus W_5$
4.	$M_8(x^0, r, \theta_1, \theta_2, \phi_1, \phi_2, \psi, x^{10})$	Spin(7)	$T_{Spin(7)}^{\mathcal{M}} = W_1 \oplus W_2$

Table 2. Summary of IR G-Structure Torsion Classes of Six-/Seven-/Eight-Folds in the type IIA/$\mathcal{M}$-Theory Duals of Thermal QCD.

of the processess $G_E \to 2\pi, G_E \to 2\rho, \rho \to 2\pi, G_E \to 4\pi, G_E \to \rho + 2\pi$ as well as indirect four π decay associated with $G_E \to \rho + 2\pi \to 4\pi$ and $G_E \to 2\rho \to 4\pi$, were also obtained. By appropriate choice of combination of constants of integration appearing in the solutions to the EOMs of the profile functions of the π and ρ mesons and six metric perturbations these decay widths were shown to match exactly with PDG data.

- **$\mathcal{O}(R^4)$-corrected $\mathcal{M}$-theory uplift and G-structure torsion classes**
  - In [16], two of the authors (V. Yadav and A. Misra), worked out the $O(l_p^6)$ corrections to the aforementioned $\mathcal{M}$-theory metric arising from terms quartic in the eleven dimensional supergravity action.
  - The SU(3)/G_2/SU(4)/Spin(7)-structure torsion classes of the relevant six-, seven- and eight-folds associated with the aforementioned M theory uplift (near $\psi = 0/2\pi/4\pi$ coordinare patches) were worked out which can be summarized in table 2.

3 Obtaining $\mathcal{L}_{\chi PT}^{(4)}[\pi, \rho]$ from DBI on flavor $D6$ branes

In this section, similar to the discussion in [2] using the "HLS formalism", starting from the DBI action on flavor $D6$-branes (obtained from the SYZ type IIA mirror of the type IIB holographic dual of large-N thermal QCD as constructed in [14]) we review obtaining the χPT Lagrangian for π, ρ mesons and their flavor partners, up to $\mathcal{O}(p^4)$ wherein the coupling constants are obtained as appropriate radial integrals.

The type IIB dual corresponding to high temperatures, i.e., $T > T_c$, will involve a black hole with the metric given by [14]:

$$ds^2 = \frac{1}{\sqrt{h(r, \theta_{1,2})}} \left(-g(r)dt^2 + \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 \right) + \sqrt{h(r, \theta_{1,2})} \left(\frac{dr^2}{g(r)} + r^2 ds^2(\theta_{1,2}, \phi_{1,2}, \psi) \right),$$
(3.1)

where $g(r) = 1 - \frac{r_h^4}{r^4}$, and for low temperatures, i.e., $T < T_c$, is given by the thermal gravitational dual:

$$ds^2 = \frac{1}{\sqrt{h(r, \theta_{1,2})}} \left(-dt^2 + \left(dx^1\right)^2 + \left(dx^2\right)^2 + \tilde{g}(r) \left(dx^3\right)^2 \right) + \sqrt{h(r, \theta_{1,2})} \left(\frac{dr^2}{\tilde{g}(r)} + r^2 ds^2(\theta_{1,2}, \phi_{1,2}, \psi) \right)$$
(3.2)

where $\tilde{g}(r) = 1 - \frac{r_0^4}{r^4}$. One notes that $t \to x^3$, $x^3 \to t$ in (3.1) following by a Double Wick rotation in the new x^3, t coordinates obtains (3.2); $h(r, \theta_{1,2})$ is the ten-dimensional warp factor [14, 15]. This amounts to: $-g_{tt}^{\text{BH}}(r_h \to r_0) = g_{x^3 x^3}^{\text{Thermal}}(r_0)$, $g_{x^3 x^3}^{\text{BH}}(r_h \to r_0) = -g_{tt}^{\text{Themal}}(r_0)$ in the results of [16, 29] (See [31] in the context of Euclidean/black $D4$-branes in type IIA).

In (3.2), we will assume the spatial part of the solitonic $D3$ brane world volume to be given by $\mathbb{R}^2(x^{1,2}) \times S^1(x^3)$ where the period of $S^1(x^3)$ is given by a very large: $\frac{2\pi}{M_{\text{KK}}}$, where the very small $M_{\text{KK}} = \frac{2r_0}{L^2} \left[1 + \mathcal{O}\left(\frac{g_s M^2}{N} \right) \right]$, r_0 being the very small IR cut-off in the thermal background (see also [32]) and $L = (4\pi g_s N)^{\frac{1}{4}}$. So, $\lim_{M_{\text{KK}} \to 0} \mathbb{R}^2(x^{1,2}) \times S^1(x^3) = \mathbb{R}^3(x^{1,2,3})$, thereby recovering 4D Physics.

As explained in [33], the T^3-valued (x, y, z) (used for effecting SYZ mirror via a triple T-dual in [15, 19]) are defined via:

$$\phi_1 = \phi_{10} + \frac{x}{\sqrt{h_2} \left[h(r_0, \theta_{10,20}) \right]^{\frac{1}{4}} \sin \theta_{10} \ r_0},$$

$$\phi_2 = \phi_{20} + \frac{y}{\sqrt{h_4} \left[h(r_0, \theta_{10,20}) \right]^{\frac{1}{4}} \sin \theta_{20} \ r_0},$$

$$\psi = \psi_0 + \frac{z}{\sqrt{h_1} \left[h(r_0, \theta_{10,20}) \right]^{\frac{1}{4}} \ r_0}, \tag{3.3}$$

$h_{1,2,4}$ defined in [14], and one works up to linear order in (x, y, z). Up to linear order in r, i.e., in the IR, it can be shown [34] that $\theta_{10,20}$ can be promoted to global coordinates $\theta_{1,2}$ in all the results in the paper. The meson sector in the type IIA dual background of top-down holographic type IIB setup is given by the flavor $D6$-branes action. For

$$\theta_1 = \frac{\alpha_{\theta_1}}{N^{\frac{1}{5}}}, \ \theta_2 = \frac{\alpha_{\theta_2}}{N^{\frac{3}{10}}}, \tag{3.4}$$

i.e., restricting to the Ouyang embedding (2.1) for a vanishingly small $|\mu_{\text{Ouyang}}|$, one will assume that the embedding of the $D6$-brane will be given by $\iota : \Sigma^{1,6}\left(R^{1,3}, r, \theta_2 \sim \frac{\alpha_{\theta_2}}{N^{\frac{3}{10}}}, y \right) \hookrightarrow M^{1,9}$, effected by: $z = z(r)$. As obtained in [27] one sees that z=constant is a solution and by choosing $z = \pm \mathcal{C} \frac{\pi}{2}$, one can choose the $D6/\overline{D6}$-branes to be at "antipodal" points along the z coordinate. As in [27], we will be working with redefined (r, z) in terms of new variables (Y, Z):

$$r = r_0 e^{\sqrt{Y^2 + Z^2}},$$

$$z = \mathcal{C} \arctan \frac{Z}{Y}.$$

Vector mesons are obtained by considering gauge fluctuations of a background gauge field along the world volume of the embedded flavor $D6$-branes (with world volume $\Sigma_7(x^{0,1,2,3}, Z, \theta_2, \tilde{y}) = \Sigma_2(\theta_2, \tilde{y}) \times \Sigma_5(x^{0,1,2,3}, Z))$. Turning on a gauge field fluctuation $\tilde{F}$ about a small background gauge field F_0 and the backround $i^*(g + B)[i : \Sigma_7 \hookrightarrow M_{10},$

M_{10} being the ten-dimensional ambient space-time]. This implies:

$$\text{Str} \sqrt{-\det_{\Sigma_7(x^{0,1,2,3},Z,\theta_2,\tilde{y})}\left(i^*(g+B)+(F_0+\tilde{F})\right)}\bigg|_{Y=0} \delta\left(\theta_2 - \frac{\alpha_{\theta_2}}{N^{\frac{3}{10}}}\right)$$

$$=\sqrt{-\det_{\Sigma_2(\theta_2,\tilde{y})}\left(i^*(g+B)\right)} \, \text{Str}\sqrt{\det_{\Sigma_5(x^{0,1,2,3},Z)}\left(i^*(g+B)+(F_0+\tilde{F})\right)}\bigg|_{Y=0} \delta\left(\theta_2 - \frac{\alpha_{\theta_2}}{N^{\frac{3}{10}}}\right)$$

$$=\sqrt{-\det_{\Sigma_2(\theta_2,\tilde{y})}\left(i^*(g+B)\right)}\sqrt{\det_{\Sigma_5(x^{0,1,2,3},Z)}(i^*g)}$$

$$\cdot \text{Str}\left(\mathbf{1}_3 - \frac{1}{2}\left[(i^*g)^{-1}\left((F_0+\tilde{F})\right)\right]^2 + \dots\right)\bigg|_{Y=0} \delta\left(\theta_2 - \frac{\alpha_{\theta_2}}{N^{\frac{3}{10}}}\right), \tag{3.5}$$

where $Y=0$ is the SYZ mirror of the Ouyang embedding [27].

Picking up terms quadratic in $\tilde{F}$:

$$S_{D6}^{IIA} = \frac{T_{D_6}(2\pi\alpha')^2}{4}\left(\frac{\pi L^2}{r_0}\right)\text{Str}\int\prod_{i=0}^{3}dx^i dZ d\theta_2 dy\,\delta\left(\theta_2 - \frac{\alpha_{\theta_2}}{N^{\frac{3}{10}}}\right)e^{-\Phi}$$

$$\cdot \sqrt{-\det_{\theta_2 y}(\iota^*(g+B))}\sqrt{\det_{\mathbb{R}^{1,3},Z}(\iota^*g)}\,g^{\tilde{\mu}\tilde{\nu}}\tilde{F}_{\tilde{\nu}\tilde{\rho}}g^{\tilde{\rho}\tilde{\sigma}}\tilde{F}_{\tilde{\sigma}\tilde{\mu}}, \tag{3.6}$$

where $\tilde{\mu} = i(=0,1,2,3), Z$

To begin with, for simplicity let us assume the absence of any external (axial-)vector fields. Expanding $A_\mu(x^\nu, Z) = \sum_{n=1}^{\infty}\rho_\mu^{(n)}(x^\nu)\psi_n(Z)$, $A_Z(x^\nu, Z) = \sum_{n=0}^{\infty}\pi^{(n)}(x^\nu)\phi_n(Z)$, one obtains (as also earlier discussed in [11, 29, 30]):

$$-\frac{\mathcal{V}_{\Sigma_2}}{4}\int d^3x dZ \sum_{n,m}\text{tr}\left(\mathcal{V}_2(Z)\tilde{F}_{\mu\nu}^{(n)}\tilde{F}^{(m)\mu\nu}\psi_m(Z)\psi_n(Z) + \mathcal{V}_1(Z)\rho_\mu^{(m)}\rho^{(n)\mu}\dot{\psi}_m\dot{\psi}_n\right.$$

$$\left. + \mathcal{V}_1(Z)\partial_\mu\pi^n\partial^\mu\pi^m\phi_n\phi_m - \mathcal{V}_1(Z)\partial_\mu\pi^{(n)}\rho^{(m)\mu}\phi_n\dot{\psi}_m - \mathcal{V}_1(Z)\partial_\mu\pi^{(m)}\rho^{(n)\mu}\phi_m\dot{\psi}_n\right). \tag{3.7}$$

The terms quadratic in $\psi/\dot{\psi}(\,\cdot\,\equiv \frac{d}{dZ})$ in (3.7) are given as:

$$-\frac{\mathcal{V}_{\Sigma_2}}{4}\int d^3x dZ \sum_{n,m}\text{tr}\left(\mathcal{V}_2(Z)\tilde{F}_{\mu\nu}^{(n)}\tilde{F}^{(m)\mu\nu}\psi_m(Z)\psi_n(Z) + \mathcal{V}_1(Z)\rho_\mu^{(m)}\rho^{(n)\mu}\dot{\psi}_m\dot{\psi}_n\right), \tag{3.8}$$

where:

$$F_{\mu\nu}(x^\rho, Z) = \sum_n \tilde{F}_{\mu\nu}^{(n)}(x^\rho)\psi_n(Z),$$

$$\mathcal{V}_{\Sigma_2} = -\frac{T_{D_6}(2\pi\alpha')^2}{4}\int dy d\theta_2\,\delta\left(\theta_2 - \frac{\alpha_{\theta_2}}{N^{\frac{3}{10}}}\right),$$

and,

$$\mathcal{V}_1(z) = 2\sqrt{h}g^{zz}e^{-\phi}\sqrt{-\det_{\theta_2 y}(\iota^*(g+B))}\sqrt{\det_{\mathbb{R}^{1,3},Z}(\iota^*g)},$$

$$\mathcal{V}_2(z) = he^{-\phi}\sqrt{-\det_{\theta_2 y}(\iota^*(g+B))}\sqrt{\det_{\mathbb{R}^{1,3},Z}(\iota^*g)}. \tag{3.9}$$

The EOM satisfied by $\rho_\mu(x^\nu)^{(n)}$ is: $\partial_\mu \tilde{F}_{(n)}^{\mu\nu} + \partial_\mu \log \sqrt{g_{t,\mathbb{R}^{1,2},|Z|}} \tilde{F}_{(n)}^{\mu\nu} = \partial_\mu \tilde{F}_{(n)}^{\mu\nu} = \mathcal{M}_{(n)}^2 \rho_{(n)}^\nu$.
After integrating by parts once, and utilizing the EOM for $\rho_\mu^{(n)}$, one writes:

$$\int d^3x\, dZ \,\, \mathrm{tr}\left(-2\mathcal{V}_2(Z)\mathcal{M}_{(m)}^2 \psi_n^{\rho_\mu}\psi_m^{\rho_\mu} + \mathcal{V}_1(Z)\dot{\psi}_n^{\rho_\mu}\dot{\psi}_m^{\rho_\mu}\right)\rho^{\mu(n)}\rho_\mu^{(m)}, \qquad (3.10)$$

which yields the following equation of motion for $\psi_{(m)}^\mu$:

$$\psi_{(m)}^\mu : \frac{d}{dZ}\left(\mathcal{V}_1(Z)\dot{\psi}_{(m)}^\mu\right) + 2\mathcal{V}_2(Z)\mathcal{M}_{(m)}^2 \psi_m^\mu = 0. \qquad (3.11)$$

The normalization condition of $\psi_{(n)}$ are given as

$$\mathcal{V}_{\Sigma_2} \int dZ\, \mathcal{V}_2(Z)\, \psi_n \psi_{(m)} = \delta_{nm}$$

$$\frac{\mathcal{V}_{\Sigma_2}}{2} \int dZ\, \mathcal{V}_1(Z)\, \partial_Z \psi_n \partial_Z \psi_{(m)} = \mathcal{M}_{(n)}^2 \delta_{nm}. \qquad (3.12)$$

Thus the action for vector meson part for all $n \geq 1$ can be wriiten as

$$-\int d^3x \sum_n \mathrm{tr}\left(\frac{1}{4}\tilde{F}_{\mu\nu}^{(n)}\tilde{F}^{(n)\mu\nu} + \frac{\mathcal{M}_{(n)}^2}{2}\rho_\mu^{(n)}\rho^{(n)\mu}\right). \qquad (3.13)$$

To normalize the kinetic term for $\pi^{(n)}$, we impose the normalization condition for all n corresponding to $\pi^{(n)}$ which ranges from 0 to ∞

$$\frac{\mathcal{V}_{\Sigma_2}}{2} \int dZ\, \mathcal{V}_1(Z)\, \phi_n \phi_m = \delta_{nm}. \qquad (3.14)$$

From (3.12), it is seen that we can choose $\phi_n = \mathcal{M}_{(n)}^{-1}\dot{\psi}_n$ for all $n \geq 1$. For $n = 0$ corresponding to ϕ_0 we choose its form such as it is orthogonal to $\dot{\psi}_n$ for all $n \geq 1$. By writing $\phi_0 = \frac{C}{\mathcal{V}_1(Z)}$, we have

$$(\phi_0, \phi_n) \propto \int dZ\, C\partial_Z\psi = 0.$$

Thus the cross component in (3.7) vanishes for $n = 0$, and the remaining cross components can be absorbed in the $\rho_\mu^{(n)}$ by following a specific gauge transformation given as,

$$\rho_\mu^{(n)} \to \rho_\mu^{(n)} + \mathcal{M}_{(n)}^{-1}\partial_\mu \pi^{(n)}.$$

Then the action becomes:

$$-\int d^3x\,\, \mathrm{tr}\left[\frac{1}{2}\partial_\mu\pi^{(0)}\partial^\mu\pi^{(0)} + \sum_{n\geq 1}\left(\frac{1}{4}\tilde{F}_{\mu\nu}^{(n)}\tilde{F}^{(n)\mu\nu} + \frac{m_n^2}{2}\rho_\mu^{(n)}\rho^{(n)\mu}\right)\right]. \qquad (3.15)$$

Working in the $A_Z(x^\mu, Z) = 0$-gauge, integrating out all higher order vector and axial vector meson fields except keeping only the lowest vector meson field [2] $V_\mu^{(1)}(x^\mu) = g\rho_\mu(x^\mu) = \begin{pmatrix} \frac{1}{\sqrt{2}}(\rho_\mu^0 + \omega_\mu) & \rho_\mu^+ & K_\mu^{*+} \\ \rho_\mu^- & -\frac{1}{\sqrt{2}}(\rho_\mu^0 - \omega_\mu) & K_\mu^{*0} \\ K_\mu^{*-} & \bar{K}_\mu^{*0} & \phi_\mu \end{pmatrix}$ and lightest pseudo-scalar meson field i.e. $\pi =$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0+\frac{1}{\sqrt{6}}\eta_8+\frac{1}{\sqrt{3}}\eta_0 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0+\frac{1}{\sqrt{6}}\eta_8+\frac{1}{\sqrt{3}}\eta_0 & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta_8+\frac{1}{\sqrt{3}}\eta_0 \end{pmatrix}$$ meson, the gauge field $A_\mu(x^\nu,Z)$ up to $\mathcal{O}(\pi)$ is given by:

$$A_\mu(x^\nu,Z) = \frac{\partial_\mu\pi}{F_\pi}\psi_0(Z) - V_\mu(x^\nu)\psi_1(Z), \tag{3.16}$$

where $\psi_0(z) = \int_0^Z dZ'\phi_0(Z'), V_\mu^{(1)}(x^\nu) = \rho_\mu^{(1)} - \frac{1}{\mathcal{M}_{(1)}}\partial_\mu\pi^{(1)}$.

To introduce external vector $\mathcal{V}_\mu$ and axial vector fields $\mathcal{A}_\mu$, one could use the Hidden Local Symmetry (HLS) formalism of [2] and references therein, wherein $\frac{1}{F_\pi}\partial_\mu\pi \to \hat{\alpha}_{\mu\perp} = \frac{1}{F_\pi}\partial_\mu\pi + \mathcal{A}_\mu - \frac{i}{F_\pi}[\mathcal{V}_\mu,\pi] + \cdots$ (refer to (C.3)), and one also works with $\hat{\alpha}_{\mu||} \equiv -V_\mu + \mathcal{V}_\mu - \frac{i}{2F_\pi^2}[\partial_\mu\pi,\pi] + \cdots$ (refer to (C.4)). To obtain the low energy effective theory of QCD, again truncating the KK spectrum at certain level because mode expansion of the gauge field contains infinite number of vector meson fields $V_\mu^{(n)}(x^\mu)$ and axial vector meson fields $A_\mu^{(n)}(x^\mu)$ [2],

$$A_\mu(x^\mu,z) = \hat{\alpha}_{\mu\perp}(x^\mu)\psi_0(z) + (\hat{\alpha}_{\mu||}(x^\mu) + V_\mu^{(1)}(x^\mu)) + \hat{\alpha}_{\mu||}(x^\mu)\psi_1(z), \tag{3.17}$$

implying therefore

$$F_{\mu\nu} = -V_{\mu\nu}\psi_1 + v_{\mu\nu}(1+\psi_1) + a_{\mu\nu}\psi_0 - i[\hat{\alpha}_{\mu||},\hat{\alpha}_{\nu||}]\psi_1(1+\psi_1) + i[\hat{\alpha}_{\mu\perp},\hat{\alpha}_{\nu\perp}](1+\psi_1-\psi_0^2)$$
$$- i([\hat{\alpha}_{\mu\perp},\hat{\alpha}_{\nu||}] + [\hat{\alpha}_{\mu||},\hat{\alpha}_{\nu\perp}])\psi_1\psi_0. \tag{3.18}$$

From appendix C (based on [36]), as regards a chiral power counting, one notes that $M_\rho \equiv \mathcal{O}(p)$ implying $\hat{\alpha}_{\nu||} \equiv \frac{\mathcal{O}(p^3)}{M_\rho^2} \equiv \mathcal{O}(p), \hat{\alpha}_{\nu\perp} \equiv \mathcal{O}(p)$. Further, $V_{\mu\nu}, a_{\mu\nu}$ and $v_{\mu\nu}$ are of $\mathcal{O}(p^2)$. Hence, using (3.18), $(F_{\mu\nu}F^{\mu\nu})^m$ is of $\mathcal{O}(p^{4m}), m \in \mathbb{Z}^+$. Therefore, one considers the kinetic term ($m=1$) at $\mathcal{O}(p^4)$, which yields the following expansion:

$$F_{\mu\nu}F^{\mu\nu} = \psi_1^2 V_{\mu\nu}V^{\mu\nu} - \psi_1(1+\psi_1)V_{\mu\nu}v^{\mu\nu} - \psi_0\psi_1 V_{\mu\nu}a^{\mu\nu} + i\psi_1^2(1+\psi_1)V_{\mu\nu}[\hat{\alpha}_{||}^\mu,\hat{\alpha}_{||}^\nu]$$
$$- i\psi_1(1+\psi_1-\psi_0^2)V_{\mu\nu}[\hat{\alpha}_\perp^\mu,\hat{\alpha}_\perp^\nu] + i\psi_0\psi_1^2 V_{\mu\nu}([\hat{\alpha}_\perp^\mu,\hat{\alpha}_{||}^\nu] + [\hat{\alpha}_{||}^\mu,\hat{\alpha}_\perp^\nu]) - \psi_1(1+\psi_1)v_{\mu\nu}V^{\mu\nu}$$
$$+ (1+\psi_1)^2 v_{\mu\nu}v^{\mu\nu} + \psi_0(1+\psi_1)v_{\mu\nu}a^{\mu\nu} - i\psi_1(1+\psi_1)^2 v_{\mu\nu}[\hat{\alpha}_{||}^\mu,\hat{\alpha}_{||}^\nu]$$
$$+ i(1+\psi_1)(1+\psi_1-\psi_0^2)v_{\mu\nu}[\hat{\alpha}_\perp^\mu,\hat{\alpha}_\perp^\nu] - i\psi_0\psi_1(1+\psi_1)v_{\mu\nu}([\hat{\alpha}_\perp^\mu,\hat{\alpha}_{||}^\nu] + [\hat{\alpha}_{||}^\mu,\hat{\alpha}_\perp^\nu])$$
$$- \psi_0\psi_1 a_{\mu\nu}V^{\mu\nu} + \psi_0(1+\psi_1)a_{\mu\nu}v^{\mu\nu} + \psi_0^2 a_{\mu\nu}a^{\mu\nu} - i\psi_0\psi_1(1+\psi_1)a_{\mu\nu}[\hat{\alpha}_{||}^\mu,\hat{\alpha}_{||}^\nu]$$
$$+ i\psi_0(1+\psi_1-\psi_0^2)a_{\mu\nu}[\hat{\alpha}_\perp^\mu,\hat{\alpha}_\perp^\nu] - i\psi_0^2\psi_1 a_{\mu\nu}([\hat{\alpha}_\perp^\mu,\hat{\alpha}_{||}^\nu] + [\hat{\alpha}_{||}^\mu,\hat{\alpha}_\perp^\nu])$$
$$+ i\psi_1^2(1+\psi_1)[\hat{\alpha}_{\mu||},\hat{\alpha}_{\nu||}]V^{\mu\nu} - i\psi_1(1+\psi_1)^2[\hat{\alpha}_{\mu||},\hat{\alpha}_{\nu||}]v^{\mu\nu}$$
$$- i\psi_0\psi_1(1+\psi_1)[\hat{\alpha}_{\mu||},\hat{\alpha}_{\nu||}]a^{\mu\nu} - \psi_1^2(1+\psi_1)^2[\hat{\alpha}_{\mu||},\hat{\alpha}_{\nu||}][\hat{\alpha}_{||}^\mu,\hat{\alpha}_{||}^\nu]$$
$$+ \psi_1(1+\psi_1)(1+\psi_1-\psi_0^2)[\hat{\alpha}_{\mu||},\hat{\alpha}_{\nu||}][\hat{\alpha}_\perp^\mu,\hat{\alpha}_\perp^\nu] - \psi_0\psi_1^2(1+\psi_1)[\hat{\alpha}_{\mu||},\hat{\alpha}_{\nu||}]([\hat{\alpha}_\perp^\mu,\hat{\alpha}_{||}^\nu] + [\hat{\alpha}_{||}^\mu,\hat{\alpha}_\perp^\nu])$$
$$- i\psi_1(1+\psi_1-\psi_0^2)[\hat{\alpha}_{\mu\perp},\hat{\alpha}_{\nu\perp}]V^{\mu\nu} + i(1+\psi_1)(1+\psi_1-\psi_0^2)[\hat{\alpha}_{\mu\perp},\hat{\alpha}_{\nu\perp}]v^{\mu\nu}$$
$$+ \psi_1(1+\psi_1)(1+\psi_1-\psi_0^2)[\hat{\alpha}_{\mu\perp},\hat{\alpha}_{\nu\perp}][\hat{\alpha}_{||}^\mu,\hat{\alpha}_{||}^\nu] - (1+\psi_1-\psi_0^2)^2[\hat{\alpha}_{\mu\perp},\hat{\alpha}_{\nu\perp}][\hat{\alpha}_\perp^\mu,\hat{\alpha}_\perp^\nu]$$
$$+ \psi_0\psi_1(1+\psi_1-\psi_0^2)[\hat{\alpha}_{\mu\perp},\hat{\alpha}_{\nu\perp}]([\hat{\alpha}_\perp^\mu,\hat{\alpha}_{||}^\nu] + [\hat{\alpha}_{||}^\mu,\hat{\alpha}_\perp^\nu])$$
$$+ i\psi_0\psi_1^2([\hat{\alpha}_{\mu\perp},\hat{\alpha}_{\nu||}] + [\hat{\alpha}_{\mu||},\hat{\alpha}_{\nu\perp}])V^{\mu\nu} - i\psi_0\psi_1(1+\psi_1)([\hat{\alpha}_{\mu\perp},\hat{\alpha}_{\nu||}] + [\hat{\alpha}_{\mu||},\hat{\alpha}_{\nu\perp}])v^{\mu\nu}$$
$$- i\psi_0^2\psi_1([\hat{\alpha}_{\mu\perp},\hat{\alpha}_{\nu||}] + [\hat{\alpha}_{\mu||},\hat{\alpha}_{\nu\perp}])a^{\mu\nu} - \psi_0\psi_1^2(1+\psi_1)([\hat{\alpha}_{\mu\perp},\hat{\alpha}_{\nu||}] + [\hat{\alpha}_{\mu||},\hat{\alpha}_{\nu\perp}])[\hat{\alpha}_{||}^\mu,\hat{\alpha}_{||}^\nu]$$

$$+\psi_0\psi_1(1+\psi_1-\phi_0^2)([\hat{\alpha}_{\mu\perp},\hat{\alpha}_{\nu||}]+[\hat{\alpha}_{\mu||},\hat{\alpha}_{\nu\perp}])[\hat{\alpha}_\perp^\mu,\hat{\alpha}_\perp^\nu]$$
$$-\psi_0^2\psi_1^2([\hat{\alpha}_{\mu\perp},\hat{\alpha}_{\nu||}]+[\hat{\alpha}_{\mu||},\hat{\alpha}_{\nu\perp}])([\hat{\alpha}_\perp^\mu,\hat{\alpha}_{||}^\nu]+[\hat{\alpha}_{||}^\mu,\hat{\alpha}_\perp^\nu]). \tag{3.19}$$

Defining parity as $x^i \to -x^i$, i indexing the conformally Minkowskian spatial directions and $Z \to -Z$, given that $A_\mu(x,Z)$ will be odd, $\alpha_\perp$ will be even, $\alpha_{||}$ will be odd and V_μ will be odd implies $\psi_0(Z)$ will be odd and $\psi_1(Z)$ will be even. As coupling constants are assumed to be scalars and they are given by Z-integrals, the Z-dependent terms in the action must be separately of even-Z parity. As ψ_0 has odd Z-parity and ψ_1 has even Z-parity, therefore at $\mathcal{O}(p^4)$, terms with $(3\hat{\alpha}_{||}s,\ 1\hat{\alpha}_\perp$ or $3\hat{\alpha}_\perp s,\ 1\hat{\alpha}_{||})$, are dropped as they involve coefficients of the type $\psi_0^{2m+1}\psi_1^{2n}(Z)$ for appropriate postive integral values of n, m. Similarly, at $\mathcal{O}(p^2)$, $tr(\hat{\alpha}_{\mu\perp}\hat{\alpha}_{||}^\mu)$ accompanied by $\dot{\psi}_0\dot{\psi}_1(\ \cdot \equiv \frac{d}{dZ})$ of odd-Z parity, is dropped. At $\mathcal{O}(p^4)$, one hence obtains [2]:

$$\mathcal{L}_{(4)} \ni y_1 \operatorname{tr}[\hat{\alpha}_{\mu\perp}\hat{\alpha}_\perp^\mu\hat{\alpha}_{\nu\perp}\hat{\alpha}_\perp^\nu] + y_2 \operatorname{tr}[\hat{\alpha}_{\mu\perp}\hat{\alpha}_{\nu\perp}\hat{\alpha}_\perp^\mu\hat{\alpha}_\perp^\nu] + y_3 \operatorname{tr}[\hat{\alpha}_{\mu||}\hat{\alpha}_{||}^\mu\hat{\alpha}_{\nu||}\hat{\alpha}_{||}^\nu] + y_4 \operatorname{tr}[\hat{\alpha}_{\mu||}\hat{\alpha}_{\nu||}\hat{\alpha}_{||}^\mu\hat{\alpha}_{||}^\nu]$$
$$+ y_5 \operatorname{tr}[\hat{\alpha}_{\mu\perp}\hat{\alpha}_\perp^\mu\hat{\alpha}_{\nu||}\hat{\alpha}_{||}^\nu] + y_6 \operatorname{tr}[\hat{\alpha}_{\mu\perp}\hat{\alpha}_{\nu\perp}\hat{\alpha}_{||}^\mu\hat{\alpha}_{||}^\nu] + y_7 \operatorname{tr}[\hat{\alpha}_{\mu\perp}\hat{\alpha}_{\nu\perp}\hat{\alpha}_{||}^\nu\hat{\alpha}_{||}^\mu]$$
$$+ y_8 \left\{ \operatorname{tr}[\hat{\alpha}_{\mu\perp}\hat{\alpha}_{||}^\nu\hat{\alpha}_{\nu\perp}\hat{\alpha}_{||}^\mu] + \operatorname{tr}[\hat{\alpha}_{\mu\perp}\hat{\alpha}_{||}^\mu\hat{\alpha}_{\nu\perp}\hat{\alpha}_{||}^\nu] \right\} + y_9 \operatorname{tr}[\hat{\alpha}_{\mu\perp}\hat{\alpha}_{\nu||}\hat{\alpha}_\perp^\mu\hat{\alpha}_{||}^\nu]$$
$$+ z_1 \operatorname{tr}[v_{\mu\nu}v^{\mu\nu}] + z_2 \operatorname{tr}[a_{\mu\nu}a^{\mu\nu}] + z_3 \operatorname{tr}[v_{\mu\nu}V^{\mu\nu}] + iz_4 \operatorname{tr}[V_{\mu\nu}\hat{\alpha}_\perp^\mu\hat{\alpha}_\perp^\nu]$$
$$+ iz_5 \operatorname{tr}[V_{\mu\nu}\hat{\alpha}_{||}^\mu\hat{\alpha}_{||}^\nu] + iz_6 \operatorname{tr}[v_{\mu\nu}\hat{\alpha}_\perp^\mu\hat{\alpha}_\perp^\nu] + iz_7 \operatorname{tr}[v_{\mu\nu}\hat{\alpha}_{||}^\mu\hat{\alpha}_{||}^\nu] - iz_8 \operatorname{tr}\left[a_{\mu\nu}\left(\hat{\alpha}_\perp^\mu\hat{\alpha}_{||}^\nu + \hat{\alpha}_{||}^\mu\hat{\alpha}_\perp^\nu\right)\right] \tag{3.20}$$

where:

$$v_{\mu\nu} = \frac{1}{2}\left(\xi_R\mathcal{R}_{\mu\nu}\xi_R^\dagger + \xi_L\mathcal{L}_{\mu\nu}\xi_L^\dagger\right) \quad \text{and} \quad a_{\mu\nu} = \frac{1}{2}\left(\xi_R\mathcal{R}_{\mu\nu}\xi_R^\dagger - \xi_L\mathcal{L}_{\mu\nu}\xi_L^\dagger\right), \tag{3.21}$$

$\mathcal{L}_{\mu\nu} = \partial_{[\mu}\mathcal{L}_{\nu]} - i[\mathcal{L}_\mu,\mathcal{L}_\nu]$ and $\mathcal{R}_{\mu\nu} = \partial_{[\mu}\mathcal{R}_{\nu]} - i[\mathcal{R}_\mu,\mathcal{R}_\nu]$ and $\mathcal{L}_\mu = \mathcal{V}_\mu - \mathcal{A}_\mu$ where $\mathcal{R}_\mu = \mathcal{V}_\mu + \mathcal{A}_\mu$, and $\xi_L^\dagger(x^\mu) = \xi_R(x^\mu) = e^{\frac{i\pi(x^\mu)}{F_\pi}}$; also, $V_{\mu\nu} = \partial_{[\mu}V_{\nu]} - i[V_\mu,V_\nu]$.

The various couplings, using (3.19), are hence given by the following expressions [2]:

$$F_\pi^2 = -\frac{\mathcal{V}_{\Sigma_2}}{4}\langle\langle\dot{\psi}_0^2\rangle\rangle$$

$$aF_\pi^2 = -\frac{\mathcal{V}_{\Sigma_2}}{4}\langle\langle\dot{\psi}_1^2\rangle\rangle$$

$$\frac{1}{g^2} = \frac{\mathcal{V}_{\Sigma_2}}{2}\langle\psi_1^2\rangle$$

$$y_1 = -y_2 = -\frac{\mathcal{V}_{\Sigma_2}}{2}\langle(1+\psi_1-\psi_0)^2\rangle$$

$$y_3 = -y_4 = -\frac{\mathcal{V}_{\Sigma_2}}{2}\langle\psi_1^2(1+\psi_1)^2\rangle$$

$$y_5 = -\mathcal{V}_{\Sigma_2}\langle\psi_0^2\psi_1^2\rangle$$

$$y_6 = -y_7 = -\mathcal{V}_{\Sigma_2}\langle\psi_1(1+\psi_1)(1+\psi_1-\psi_0^2)\rangle$$

$$y_8 = -y_9 = -\mathcal{V}_{\Sigma_2}\langle\psi_0^2\psi_1^2\rangle$$

$$z_1 = -\frac{\mathcal{V}_{\Sigma_2}}{4}\langle(1+\psi_1)^2\rangle$$

$$z_2 = -\frac{\mathcal{V}_{\Sigma_2}}{4}\langle(\psi_0)^2\rangle$$

$$z_3 = \frac{\mathcal{V}_{\Sigma_2}}{2}\langle\psi_1(1+\psi_1)\rangle$$

$$z_4 = \mathcal{V}_{\Sigma_2}\langle\psi_1(1+\psi_1-\psi_0^2)\rangle$$

$$z_5 = -\mathcal{V}_{\Sigma_2}\langle\psi_1^2(1+\psi_1)\rangle$$

$$z_6 = -\mathcal{V}_{\Sigma_2}\langle(1+\psi_1)(1+\psi_1-\psi_0^2)\rangle$$

$$z_7 = \mathcal{V}_{\Sigma_2}\langle\psi_1(1+\psi_1)^2\rangle$$

$$z_8 = \mathcal{V}_{\Sigma_2}\langle\psi_0^2\psi_1\rangle \tag{3.22}$$

where:

$$\langle\langle A\rangle\rangle = \int_0^{+\infty}\mathcal{V}_1(z)AdZ \tag{3.23}$$

and

$$\langle A\rangle = \int_0^{+\infty}\mathcal{V}_2(z)AdZ. \tag{3.24}$$

4 The coupling constants in $\mathcal{L}_{\chi PT}^{[4]}(\pi,\rho)$ from S_{DBI}^{D6} incorporating $\mathcal{O}(R^4)$-corrections from M theory

This section has the core of the main results of this paper. Inclusive of the $\mathcal{O}(R^4)$ corrections to the $\mathcal{M}$-theory uplift of large-N thermal QCD as worked out in [16], we show how to obtain lattice-compatible values of the coupling constants up to $\mathcal{O}(p^4)$ of the χPT Lagrangian of [5] in the chiral limit. The $\mathcal{O}(R^4)$-corrections — indicated by a ˜ (e.g., the $\mathcal{M}$-theory metric: $\tilde{G}_{MN}^{\mathcal{M}} = G_{MN}^{\text{MQGP}}(1+f_{MN})$ [16]) in (D.1) — to the $D6$-brane DBI action is described in appendix D.

The $\mathcal{O}(\beta)$-corrected $\mathcal{M}$-theory metric of [15] in the MQGP limit adapted to the thermal background (3.2) near the $\psi = 2n\pi, n = 0, 1, 2$-branches up to $\mathcal{O}((r-r_0)^2)$ [and up to $\mathcal{O}((r-r_0)^3)$ for some of the off-diagonal components along the delocalized $T^3(x,y,z)$] — the components which do not receive an $\mathcal{O}(\beta)$ corrections, are not listed in (4.1) — is given below [16]:

$$\tilde{G}^{\mathcal{M}}_{x^3 x^3} = G^{\text{MQGP}}_{x^3 x^3}\left[1 + \frac{1}{4}\frac{4 b^8 (9b^2+1)^3 (4374 b^6 + 1035 b^4 + 9b^2 - 4)\,\beta M \left(\frac{1}{N}\right)^{9/4}\Sigma_1 (6a^2+r_0^2)\log(r_0)}{27\pi (18b^4 - 3b^2 - 1)^5 \log N^2 N_f r_0^2 \alpha_{\theta_2}^3 (9a^2+r_0^2)}(r-r_0)^2\right]$$

$$\tilde{G}^{\mathcal{M}}_{x^{0,1,2} x^{0,1,2}} = G^{\text{MQGP}}_{x^{0,1,2} x^{0,1,2}}\left[1 - \frac{1}{4}\frac{4 b^8 (9b^2+1)^4 (39 b^2 - 4) M \left(\frac{1}{N}\right)^{9/4}\beta (6a^2+r_0^2)\log(r_0)\Sigma_1}{9\pi (3b^2-1)^5 (6b^2+1)^4 \log N^2 N_f r_0^2 (9a^2+r_0^2)\alpha_{\theta_2}^3}(r-r_0)^2\right]$$

$$\tilde{G}^{\mathcal{M}}_{rr} = G^{\mathcal{M}}_{rr}\left[1 + \left(-\frac{2(9b^2+1)^4 b^{10} M (6a^2+r_0^2)\left((r-r_0)^2 + r_0^2\right)\Sigma_1}{3\pi(-18b^4+3b^2+1)^4 \log N N^{8/15} N_f (-27a^4 + 6a^2 r_0^2 + r_0^4)\alpha_{\theta_2}^3}\right.\right.$$
$$\left.\left. + C_{zz}^{(1)} - 2C_{\theta_1 z}^{(1)} + 2C_{\theta_1 x}^{(1)}\right)\beta\right]$$

$$\tilde{G}^{\mathcal{M}}_{\theta_1 x} = G^{\text{MQGP}}_{\theta_1 x}\left[1 + \left(-\frac{(9b^2+1)^4 b^{10} M (6a^2+r_0^2)\left((r-r_0)^2+r_0^2\right)\Sigma_1}{3\pi(-18b^4+3b^2+1)^4 \log N N^{8/15} N_f (-27a^4 + 6a^2 r_0^2 + r_0^4)\alpha_{\theta_2}^3} + C_{\theta_1 x}^{(1)}\right)\beta\right]$$

$$\tilde{G}^{\mathcal{M}}_{\theta_1 z} = G^{\text{MQGP}}_{\theta_1 z}\left[1 + \left(\frac{16(9b^2+1)^4 b^{12}\beta \left(\frac{(r-r_0)^3}{r_0^3}+1\right)\left(19683\sqrt{3}\alpha_{\theta_1}^6 + 3321\sqrt{2}\alpha_{\theta_2}^2 \alpha_{\theta_1}^3 - 40\sqrt{3}\alpha_{\theta_2}^4\right)}{243\pi^3(1-3b^2)^{10}(6b^2+1)^8 g_s^{9/4}\log N^4 N^{7/6} N_f^3 (-27a^4 r_0 + 6a^2 r_0^3 + r_0^5)\alpha_{\theta_1}^7 \alpha_{\theta_2}^6} + C_{\theta_1 z}^{(1)}\right)\right]$$

$$\tilde{G}^{\mathcal{M}}_{\theta_2 x} = G^{\text{MQGP}}_{\theta_2 x}\left[1 + \left(\frac{16(9b^2+1)^4 b^{12} \left(\frac{(r-r_0)^3}{r_0^3}+1\right)\left(19683\sqrt{3}\alpha_{\theta_1}^6 + 3321\sqrt{2}\alpha_{\theta_2}^2 \alpha_{\theta_1}^3 - 40\sqrt{3}\alpha_{\theta_2}^4\right)}{243\pi^3(1-3b^2)^{10}(6b^2+1)^8 g_s^{9/4}\log N^4 N^{7/6} N_f^3 (-27a^4 r_0 + 6a^2 r_0^3 + r_0^5)\alpha_{\theta_1}^7 \alpha_{\theta_2}^6} + C_{\theta_2 x}^{(1)}\right)\beta\right]$$

$$\tilde{G}^{\mathcal{M}}_{\theta_2 y} = G^{\text{MQGP}}_{\theta_2 y}\left[1 + \frac{3b^{10}(9b^2+1)^4 M\beta(6a^2+r_0^2)\left(1-\frac{(r-r_0)^2}{r_0^2}\right)\log(r_0)\Sigma_1}{\pi(3b^2-1)^5(6b^2+1)^4 \log N^2 N^{7/5} N_f (9a^2+r_0^2)\alpha_{\theta_2}^3}\right]$$

$$\tilde{G}^M_{\theta_2 z} = G^{\text{MQGP}}_{\theta_2 z}\left[1 + \left(\frac{3(9b^2+1)^4 b^{10} M (6a^2+r_0^2)\left(1-\frac{(r-r_0)^2}{r_0^2}\right)\log(r_0)\left(19683\sqrt{6}\alpha_{\theta_1}^6 + 6642\alpha_{\theta_2}^2\alpha_{\theta_1}^3 - 40\sqrt{6}\alpha_{\theta_2}^4\right)}{\pi(3b^2-1)^5(6b^2+1)^4\log N^2 N^{7/6} N_f (9a^2+r_0^2)\alpha_{\theta_2}^3} \right.\right.$$
$$\left.\left. + C^{(1)}_{\theta_2 z}\right)\beta\right]$$

$$\tilde{G}^M_{xy} = G^{\text{MQGP}}_{xy}\left[1 + \left(\frac{3(9b^2+1)^4 b^{10} M (6a^2+r_0^2)\left(\frac{(r-r_0)^2}{r_0^2}+1\right)\log(r_0)\alpha_{\theta_2}^3\Sigma_1}{\pi(3b^2-1)^5(6b^2+1)^4\log N^2 N^{21/20} N_f (9a^2+r_0^2)\alpha_{\theta_{2l}}^6} + C^{(1)}_{xy}\right)\beta\right]$$

$$\tilde{G}^M_{xz} = G^{\text{MQGP}}_{xz}\left[1 + \frac{18 b^{10}(9b^2+1)^4 M\beta(6a^2+r_0^2)\left(\frac{(r-r_0)^2}{r_0^2}+1\right)\log^3(r_0)\Sigma_1}{\pi(3b^2-1)^5(6b^2+1)^4\log N^4 N^{5/4} N_f (9a^2+r_0^2)\alpha_{\theta_2}^3}\right]$$

$$\tilde{G}^M_{yy} = G^{\text{MQGP}}_{yy}\left[1 - \frac{3b^{10}(9b^2+1)^4 M\left(\frac{1}{N}\right)^{7/4}\beta(6a^2+r_0^2)\log(r_0)\Sigma_1\left(\frac{(r-r_0)^2}{r_0^2}+1\right)}{\pi(3b^2-1)^5(6b^2+1)^4\log N^2 N_f r_0^2 (9a^2+r_0^2)\alpha_{\theta_2}^3}\right]$$

$$\tilde{G}^M_{yz} = G^{\text{MQGP}}_{yz}\left[1 + \left(\frac{64(9b^2+1)^8 b^{22} M\left(\frac{1}{N}\right)^{29/12}(6a^2+r_0^2)\left(\frac{(r-r_0)^3}{r_0^3}+1\right)\log(r_0)}{27\pi^4(3b^2-1)^{15}(6b^2+1)^{12} g_s^{9/4}\log N^6 N_f^4 r_0^3 (r_0^2-3a^2)(9a^2+r_0^2)^2\alpha_{\theta_1}^7\alpha_{\theta_2}^9}\right.\right.$$
$$\left.\left.\times\left(387420489\sqrt{2}\alpha_{\theta_1}^{12} + 87156324\sqrt{3}\alpha_{\theta_2}^2\alpha_{\theta_1}^9 + 5778054\sqrt{2}\alpha_{\theta_2}^4\alpha_{\theta_1}^6 - 177120\sqrt{3}\alpha_{\theta_2}^6\alpha_{\theta_1}^3 + 1600\sqrt{2}\alpha_{\theta_2}^8\right) + C^{(1)}_{yz}\right)\beta\right]$$

$$\tilde{G}^M_{zz} = G^{\text{MQGP}}_{zz}\left[1 + \left(C^{(1)}_{zz} - \frac{b^{10}(9b^2+1)^4 M\left(r_0^2-\frac{(r-r_0)^3}{r_0}\right)\log(r_0)\Sigma_1}{27\pi^{3/2}(3b^2-1)^5(6b^2+1)^4\sqrt{g_s}\log N^2 N^{23/20} N_f\alpha_{\theta_2}^5}\right)\beta\right]$$

$$\tilde{G}^M_{x^{10}x^{10}} = G^{\text{MQGP}}_{x^{10}x^{10}}\left[1 - \frac{27 b^{10}(9b^2+1)^4 M\left(\frac{1}{N}\right)^{5/4}\beta(6a^2+r_0^2)\left(1-\frac{(r-r_0)^2}{r_0^2}\right)\log^3(r_0)\Sigma_1}{\pi(3b^2-1)^5(6b^2+1)^4\log N^4 N_f r_0^2 (9a^2+r_0^2)\alpha_{\theta_2}^3}\right],$$

$$(4.1)$$

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where:

$$\Sigma_1 \equiv 19683\sqrt{6}\alpha_{\theta_1}^6 + 6642\alpha_{\theta_2}^2\alpha_{\theta_1}^3 - 40\sqrt{6}\alpha_{\theta_2}^4$$

$$\xrightarrow{\text{Global}} N^{\frac{6}{5}}\left(19683\sqrt{6}\sin^6\theta_1 + 6642\sin^2\theta_2\sin^3\theta_1 - 40\sqrt{6}\sin^4\theta_2\right), \qquad (4.2)$$

$\mathcal{C}_{MN}^{(1)}$ are constants of integration that figure in (4.1) after solving the EOMs for the $\mathcal{O}(\beta)$ metric perturbations f_{MN}, and G_{MN}^{MQGP} are the $\mathcal{M}$ theory metric components in the MQGP limit at $\mathcal{O}(\beta^0)$ [29]. The explicit dependence on $\theta_{10,20}$ of the $\mathcal{M}$-theory metric components up to $\mathcal{O}(\beta)$, using (3.4), is effected by the replacememnts: $\alpha_{\theta_1} \to N^{\frac{1}{5}}\sin\theta_{10}$, $\alpha_{\theta_2} \to N^{\frac{3}{10}}\sin\theta_{20}$ in (4.1). Also, see (3.3). The main Physics-related take-away is the following. From (4.1), one notes that in the IR: $r = \chi r_0, \chi \equiv \mathcal{O}(1)$, up to $\mathcal{O}(\beta)$:

$$f_{MN} \sim \beta\frac{(\log r_0)^m}{r_0^n N^{\beta_N}}, \ m \in \{0,1,3\}, \ n \in \{0,2,5,7\}, \ \beta_N > 0. \qquad (4.3)$$

As estimated in [28], $\left|\log\left(\frac{r_0}{\mathcal{R}_{D5/\overline{D5}}}\right)\right| \sim N^{\frac{1}{3}}$, implying there is a competition between Planckian and large-N suppression and infra-red enhancement arising from $m, n \neq 0$ in (4.3).

Now, using the standard Witten's prescription of reading off the type IIA metric (inclusive of the $\mathcal{O}(R^4)$ corrections):

$$ds_{11}^2 = e^{-\frac{2\phi^{\text{IIA}}}{3}}\left[\frac{1}{\sqrt{h(r,\theta_{1,2})}}\left(-dt^2 + \left(dx^1\right)^2 + \left(dx^2\right)^2 + \tilde{g}(r)\left(dx^3\right)^2\right)\right.$$

$$\left. + \sqrt{h(r,\theta_{1,2})}\left(\frac{dr^2}{\tilde{g}(r)} + ds_{\text{IIA}}^2(r,\theta_{1,2},\phi_{1,2},\psi)\right)\right] + e^{\frac{4\phi^{\text{IIA}}}{3}}\left(dx^{11} + A_{\text{IIA}}^{F_1^{\text{IIB}}+F_3^{\text{IIB}}+F_5^{\text{IIB}}}\right)^2,$$

$$(4.4)$$

where $A_{\text{IIA}}^{F_{i=1,3,5}^{\text{IIB}}}$ are the type IIA RR 1-forms obtained from the triple T/SYZ-dual of the type IIB $F_{1,3,5}^{\text{IIB}}$ fluxes in the type IIB holographic dual of [14].

Turning now to obtaining the EOM for the profile function of the vector mesons, we will use (3.11). Using (3.9) and (D.2), (D.3) and (D.5), one first obtains:

$$\mathcal{V}_1 = \mathcal{V}_1^{\text{LO}} + \mathcal{V}_1^{\mathcal{O}(R^4)} \text{ where :}$$

$$\mathcal{V}_1^{\mathcal{O}(R^4)} = \frac{\sqrt{h}e^{-2Z}(2e^{-\phi}G_{x^{10}x^{10}}^{\mathcal{M}}G_{rr}^{\mathcal{M}}TrC.F + e^{-\phi^{\text{IIA}}}\sqrt{g^{\text{IIA}}}(-\mathcal{F}_{x^{10}x^{10}}G_{rr}^{\mathcal{M}} - 2\mathcal{F}_{rr}G_{x^{10}x^{10}}^{\mathcal{M}} + 2G_{x^{10}x^{10}}^{\mathcal{M}}G_{rr}^{\mathcal{M}}))}{G_{x^{10}x^{10}}^{\mathcal{M}}{}^{3/2}G_{rr}^{\mathcal{M}}{}^2 r_0^2}$$

$$= -\frac{3g_s M\sqrt[5]{\frac{1}{N}}N_f^2 e^{-4Z}\left(e^{4Z}-1\right)(\mathcal{C}_{zz}^{(1)} - 2\mathcal{C}_{\theta_1 z}^{(1)} + 2\mathcal{C}_{\theta_1 x}^{(1)})\log\left(r_0 e^Z\right)\left(72a^2 r_0 e^Z\log\left(r_0 e^Z\right) + 3a^2 + 2r_0^2 e^{2Z}\right)}{8\pi^2\alpha_{\theta_1}\alpha_{\theta_2}^2};$$

$$\mathcal{V}_2 = \mathcal{V}_2^{\text{LO}} + \mathcal{V}_2^{\mathcal{O}(R^4)} \text{ where :}$$

$$\mathcal{V}_2^{\mathcal{O}(R^4)} = e^{-\phi^{\text{IIA}}}hTr(\mathcal{C}.\mathcal{F}) + \mathcal{E}_{x^{10}x^{10}}h\sqrt{g^{\text{IIA}}}$$

$$= \frac{3g_s^2 MN^{4/5}N_f^2 e^{-2Z}(\mathcal{C}_{zz}^{(1)} - 2\mathcal{C}_{\theta_1 z}^{(1)} + 2\mathcal{C}_{\theta_1 x}^{(1)})\log\left(r_0 e^Z\right)\left(72a^2 r_0 e^Z\log\left(r_0 e^Z\right) - 3a^2 + 2r_0^2 e^{2Z}\right)}{4\pi r_0^2\alpha_{\theta_1}\alpha_{\theta_2}^2},$$

$$(4.5)$$

where $\mathcal{V}_{1,2}^{\mathrm{LO}}$ are the LO terms as obtained in [29]. The equation of motion (3.11) satisfied by the profile function of the vector meson $\psi_1(z)$, using ideas similar to [29], can be rewritten as a Schödinger-like equation with a potential $\mathcal{V} = \mathcal{V}^{\mathrm{LO}} + \mathcal{V}^{\mathcal{O}(R^4)}$ where $\left(\mathcal{M}_{(1)} = m_0 \frac{r_0}{\sqrt{4\pi g_s N}}\right)$ and $\mathcal{V}^{\mathrm{LO}}$ is the LO potential as given in [29]. Further,

$$
\begin{aligned}
\mathcal{V}^{\mathcal{O}(R^4)} &= -2\mathcal{M}_{(1)}^2 \frac{\mathcal{V}_1^{\mathcal{O}(R^4)}\mathcal{V}_2}{\mathcal{V}_1^2} + 2\mathcal{M}_{(1)}^2 \frac{\mathcal{V}_2^{\mathcal{O}(R^4)}}{\mathcal{V}_1} - \frac{\mathcal{V}_1^{\mathcal{O}(R^4)}(\mathcal{V}_1')^{\,2}}{\mathcal{V}_1^3} + \frac{\mathcal{V}_1'\left(\mathcal{V}_1^{\mathcal{O}(R^4)}\right)'}{2\mathcal{V}_1^2} + \frac{\mathcal{V}_1^{\mathcal{O}(R^4)}\mathcal{V}_1''}{2\mathcal{V}_1^2} - \frac{\left(\mathcal{V}_1^{\mathcal{O}(R^4)}\right)''}{2\mathcal{V}_1} \\[2mm]
&= \beta N \Bigg(\frac{2\pi g_s^{7/2} \log N \mathcal{M}_{(1)}^2 N_f e^{-4Z}(\mathcal{C}_{zz}^{(1)} - 2\mathcal{C}_{\theta_1 z}^{(1)} + 2\mathcal{C}_{\theta_1 x}^{(1)})\left(72a^2 r_0 e^Z \log\left(r_0 e^Z\right) - 3a^2 + 2r_0{}^2 e^{2Z}\right)}{r_0{}^8 \left(e^{4Z}-1\right)\left(\frac{g_s e^{-4Z}}{r_0{}^4}\right)^{3/2} \tilde{\Omega}(Z)} \\[2mm]
&\quad + \frac{2\pi g_s{}^2 \log N \mathcal{M}_{(1)}^2 N_f e^{4Z}(\mathcal{C}_{zz}^{(1)} - 2\mathcal{C}_{\theta_1 z}^{(1)} + 2\mathcal{C}_{\theta_1 x}^{(1)})\left(72a^2 r_0 e^Z \log\left(r_0 e^Z\right) + 3a^2 + 2r_0{}^2 e^{2Z}\right)}{81 r_0{}^2 \left(e^{4Z}-1\right)\alpha_{\theta_1}^2 \tilde{\Omega}(Z)^{\,2}} \Bigg) \\[2mm]
&\quad \times \Bigg[243 a^2 e^{-2Z}\alpha_{\theta_1}^2 \Bigg(3\log\left(r_0 e^Z\right)\left(g_s N_f\left(8\log N r_0 e^Z + 1\right) + 32\pi r_0 e^Z\right) - g_s(\log N + 3)N_f \\[2mm]
&\quad - 72 g_s N_f r_0 e^Z \log^2\left(r_0 e^Z\right) - 4\pi\Bigg) + 162 r_0{}^2 \alpha_{\theta_1}^2\left(g_s \log N N_f - 3g_s N_f \log\left(r_0 e^Z\right) + 4\pi\right) \Bigg] \\[2mm]
&= -\beta \frac{(3b^2 - 2)\log N m_0{}^2 (\mathcal{C}_{zz}^{(1)} - 2\mathcal{C}_{\theta_1 z}^{(1)} + 2\mathcal{C}_{\theta_1 x}^{(1)})}{4\left(3b^2 + 2\right)\left(\log N - 3\log(r_0)\right)Z} + \mathcal{O}(Z^0),
\end{aligned}
\tag{4.6}
$$

where $a = \left(b + \gamma \frac{g_s M^2}{N}\left(1 + \log r_0\right)\right) r_0$, and,

$$
\begin{aligned}
\tilde{\Omega}(Z) &\equiv 3\log\left(r_0 e^Z\right)\left(3a^2\left(g_s N_f\left(8\log N r_0 e^Z - 1\right) + 32\pi r_0 e^Z\right) - 2g_s N_f r_0{}^2 e^{2Z}\right) \\[2mm]
&\quad + 3a^2(g_s(\log N - 3)N_f + 4\pi) - 216 a^2 g_s N_f r_0 e^Z \log^2\left(r_0 e^Z\right) + 2r_0{}^2 e^{2Z}(g_s \log N N_f + 4\pi).
\end{aligned}
\tag{4.7}
$$

As in [29], one defines $g(Z) \equiv \sqrt{\mathcal{V}_1(Z)}\psi_1(Z)$, where $g(Z)$ satisfies the following Schrödinger-like equation that, as mentioned above, is obtained from (3.11):

$$
g''(Z) + \left(\frac{\omega_1 + \beta \mathcal{C}^{zz}{}_{\theta_1 z \ \theta_1 x}}{Z} + \omega_2 + \frac{1}{4Z^2}\right) g(Z) = 0,
\tag{4.8}
$$

wherein:

$$
\begin{aligned}
\omega_1 &\equiv \frac{1}{4}\left(m_0{}^2 - 3b^2\left(m_0{}^2 - 2\right)\right) + 18b^2 r_h \log(r_h) \\[2mm]
&\quad - \frac{3b\gamma g_s M^2\left(m_0{}^2 - 2\right)\log(r_h)}{2N} + \frac{36 b\gamma g_s M^2 r_h \log^2(r_h)}{N}, \\[2mm]
\omega_2 &\equiv -\frac{4}{3} + \frac{3}{2}b^2\left(m_0{}^2 + 72 r_h - 4\right) - 36 b^2 r_h \log(r_h) \\[2mm]
&\quad + \frac{3b\gamma g_s M^2\left(m_0{}^2 - 4\right)\log(r_h)}{N} - \frac{72 b\gamma g_s M^2 r_h \log^2(r_h)}{N}, \\[2mm]
\mathcal{C}^{zz}{}_{\theta_1 z \ \theta_1 x} &= -\frac{(3b^2 - 2)\log N m_0{}^2 (\mathcal{C}_{zz}^{(1)} - 2\mathcal{C}_{\theta_1 z}^{(1)} + 2\mathcal{C}_{\theta_1 x}^{(1)})}{4\left(3b^2 + 2\right)\left(\log N - 3\log(r_0)\right)},
\end{aligned}
\tag{4.9}
$$

and whose solution (using arguments similar to the ones in [29]) is given in terms of Whittaker functions:

$$g(Z) = c_{\psi_1}^{(1)} M_{-\frac{i\left(\omega_1 + \beta \mathcal{C}^{zz}_{\theta_1 z\ \theta_1 x}\right)}{2\sqrt{\omega_2}},0}\left(2i\sqrt{\omega_2}Z\right) + c_{\psi_1}^{(2)} W_{-\frac{i\left(\omega_1 + \beta \mathcal{C}^{zz}_{\theta_1 z\ \theta_1 x}\right)}{2\sqrt{\omega_2}},0}\left(2i\sqrt{\omega_2}Z\right). \quad (4.10)$$

We pause here and note that the effect of the inclusion of the $\mathcal{O}(R^4)$ corrections into the EOM for the radial profile function $\psi_1(Z)$ for the ρ meson is a shift in the residue of the simple pole in the potential of the Schrödinger-like equation satisfied by the redefined ρ meson profile function $g(Z)$.

Using arguments similar to the ones in [29], implementing Neumann boundary condition ($\psi_1'(Z = 0) = 0$), one sees that in the IR (i.e., near $Z = 0$):

$$\psi_1(Z) = \sqrt{2}\mathcal{C}_{\psi_1}^{(1)\ \text{IR}}\sqrt{i\sqrt{\omega_2}}\left[1 - Z\left(\beta\mathcal{C}^{zz}_{\theta_1 z\ \theta_1 x} + \omega_1\right)\right]; \quad \mathcal{C}_{\psi_1}^{(1)\ \text{IR}} \equiv \mathcal{C}_{\psi_1}^{\text{IR}} = N^{-\Omega_{\psi_1}}, \ \Omega_{\psi_1} > 0. \quad (4.11)$$

Now, as explained in section 3,

$$\phi_0(Z) = \frac{\mathcal{C}_{\phi_0}^{\text{IR}}}{\mathcal{V}_1(Z)} = \frac{\phi_0^{(-1)}}{Z} + \phi_0^{(\text{constant})} + \phi_0^{(1)}Z + \phi_0^{(2)}Z^2 + \mathcal{O}(Z^3). \quad (4.12)$$

By requiring:

$$\alpha_{\theta_2} = \frac{9\sqrt{\log N - 3\log r_0}}{\sqrt{2}\sqrt{\log N + 3\log r_0}}N^{\frac{1}{10}}\alpha_{\theta_1}, \quad \log N > |\log r_0|,$$

$$b = \frac{1}{\sqrt{3}} + \epsilon, \quad (4.13)$$

for a very tiny ϵ to be ascertained later, one can set: $\phi_0^{(-1)} = \phi_0^{(1)} = 0$ and one obtains:

$$\phi_0(Z) = \frac{\pi^2 \mathcal{C}_{\phi_0}^{\text{IR}}\ N^{2/5}\alpha_{\theta_1}^3(\log N - 3\log r_0)\left(\frac{27}{8b^2 g_s \log N(\log N + 3\log r_0)} - \frac{81b^2\beta(\mathcal{C}_{zz}^{(1)} - 2\mathcal{C}_{\theta_1 z}^{(1)} + 2\mathcal{C}_{\theta_1 x}^{(1)})}{8\log(r_0)}\right)}{g_s M N_f^2 r_0^3(\log N + 3\log r_0)}$$
$$- \frac{\pi^2 \mathcal{C}_{\phi_0}^{IR}N^{2/5}\alpha_{\theta_1}^3(\log N - 3\log r_0)\left(\frac{9(3b^2+1)\beta(\mathcal{C}_{zz}^{(1)} - 2\mathcal{C}_{\theta_1 z}^{(1)} + 2\mathcal{C}_{\theta_1 x}^{(1)})}{4\log r_0} + \frac{1944b^4}{(3b^2+2)^4}\right)}{g_s \log r_0 M N_f^2 r_0^2(\log N + 3\log r_0)}Z^2 + \mathcal{O}(Z^3). \quad (4.14)$$

One can similarly show that one obtains the following profile functions in the UV:

$$\psi_1^{\text{UV}}(Z) = \mathcal{C}_{\psi_1}^{\text{UV}}\frac{e^{-2Z}}{Z^{\frac{3}{2}}},$$

$$\phi_0^{\text{UV}}(Z) = \mathcal{C}_{\phi_0}^{\text{UV}}\frac{e^{-2Z}}{Z^2}. \quad (4.15)$$

Let us discuss the normalization conditions on $\psi_1(Z)$ and $\phi_0(Z)$ and the consequent constraints on $\mathcal{C}_{\psi_1}^{\text{UV}}$ and $\mathcal{C}_{\phi_0}^{\text{UV}}(Z)$.

- The normalization condition on $\psi_1(Z)$ (3.12): $\left.\mathcal{V}_{\Sigma_2}\int_0^\infty dZ\mathcal{V}_2\left(\psi_1(Z)\right)^2\right|_{b=\frac{1}{\sqrt{3}}+\epsilon}=1$ obtains:

$$\mathcal{C}_{\psi_1}^{\mathrm{UV}}=\frac{\sqrt{\frac{\sqrt{7}(f_{r_0}-1)f_{r_0}g_s{}^2MN_f{}^2\mathcal{V}_{\Sigma_2}\mathcal{C}_{\psi_1}^{\mathrm{IR}}{}^2\log N\left(7\beta(\mathcal{C}_{zz}^{(1)}-2\mathcal{C}_{\theta_1 z}^{(1)}+2\mathcal{C}_{\theta_1 x}^{(1)})f_{r_0}{}^2\gamma^2 g_s{}^2M^4\log^2(N)+3456\epsilon^2(f_{r_0}+1)N^2\right)}{2\ 3^{3/4}\epsilon^{3/2}(f_{r_0}+1)N^{7/5}\alpha_{\theta_1}^3}-93312\pi}}{24\sqrt{-\frac{(f_{r_0}-1)g_s{}^2MN^{3/5}N_f{}^2\mathcal{V}_{\Sigma_2}}{\epsilon^2(f_{r_0}+1)\alpha_{\theta_1}^3\log N}}},$$

(4.16)

wherein, similar to [28], the IR cut-off r_0 is assumed to be given as $r_0=N^{-\frac{f_{r_0}}{3}}$. Let us impose $\mathcal{C}_{\psi_1}^{\mathrm{UV}}=0$ which is equivalent to:

$$\mathcal{V}_{\Sigma_2}=\frac{186624\ 3^{3/4}\pi\epsilon^{3/2}(f_{r_0}+1)N^{7/5}\alpha_{\theta_1}^3}{\sqrt{7}(f_{r_0}-1)f_{r_0}g_s{}^2\log NMN_f{}^2\mathcal{C}_{\psi_1}^{\mathrm{IR}}{}^2\left(7\beta(\mathcal{C}_{zz}^{(1)}-2\mathcal{C}_{\theta_1 z}^{(1)}+2\mathcal{C}_{\theta_1 x}^{(1)})f_{r_0}{}^2\gamma^2 g_s{}^2\log N^2M^4+3456\epsilon^2(f_{r_0}+1)N^2\right)}.$$

(4.17)

- The normalization condition on $\phi_0(Z)$ (3.14): $\frac{\mathcal{V}_{\Sigma_2}}{2}\int_0^\infty dZ\mathcal{V}_1\left(\phi_0(Z)\right)^2=1$ obtains:

$$\mathcal{C}_{\phi_0}^{\mathrm{UV}}=\frac{243\sqrt[4]{3}\pi^2\mathcal{C}_{\phi_0}^{\mathrm{IR}}\sqrt{\epsilon}N^{f_{r_0}+\frac{2}{5}}\sqrt{(f_{r_0}+1)(-\beta(\mathcal{C}_{zz}^{(1)}-2\mathcal{C}_{\theta_1 z}^{(1)}+2\mathcal{C}_{\theta_1 x}^{(1)})+2f_{r_0}+2)}}{32(f_{r_0}-1)^2g_s{}^2\left(\log N\right)^2MN_f{}^2}+\mathcal{O}(\epsilon^{\frac{3}{2}}).$$

(4.18)

Being proportional to $\sqrt{\epsilon}$ and assuming $\epsilon\ll 1$ (for black-hole gravity dual, $\epsilon<r_h^2\left(\log r_h\right)^{\frac{9}{2}}N^{-\frac{9}{10}}$ [16]), we will henceforth be approximating $\mathcal{C}_{\phi_0}^{\mathrm{UV}}\approx 0$, i.e., $\psi_1^{\mathrm{UV}}(Z)\approx 0,\phi_0^{\mathrm{UV}}(Z)\approx 0$.

To evaluate $y_{1,3,5,7}$ and $z_{1,\dots,8}$ using (3.22) along with (3.23) and (3.24), one will be splitting the radial integral into the IR and the UV, e.g., $\langle A\rangle[\tilde{G}_{MN}^{\mathrm{IR}}]+\langle A\rangle[G_{MN}^{\mathrm{UV}}]$, where using the results of appendix A, f_{MN}^{UV}'s are vanishingly small (impling $\tilde{G}_{MN}^{\mathrm{UV}}=G_{MN}^{\mathrm{UV}}$). Using (4.11) and (4.14), one arrives at the expressions for the coupling constants $y_{1,3,5,7}$ and $z_{1,\dots,8}$ as explained in appendix B:

$$y_{1,\dots,7},\ z_{1,\dots,8}\big|_{\sim\mathrm{IIB\ Ouyang}}$$
$$=\mathcal{V}_{\Sigma_2}\left(\mathcal{C}_{\psi_1}^{\mathrm{IR}}\right)^{n_{y_i/z_j}}\left(\mathcal{C}_{\phi_0}^{\mathrm{IR}}\right)^{m_{y_i/z_j}}$$
$$\times\left(\mathcal{F}_{y_i/z_j}(r_0;M,N,N_f)+\beta\left(\mathcal{C}_{zz}^{(1)}-2\mathcal{C}_{\theta_1 z}^{(1)}+2\mathcal{C}_{\theta_1 x}^{(1)}\right)\mathcal{H}_{y_i/z_j}(r_0;M,N,N_f)\right).$$

(4.19)

Further,

$$F_\pi^2=\mathcal{V}_{\Sigma_2}\left(\frac{243\pi^2\beta\left(\mathcal{C}_{zz}^{(1)}-2\mathcal{C}_{\theta_1 z}^{(1)}+2\mathcal{C}_{\theta_1 x}^{(1)}\right)\mathcal{C}_{\phi_0}^{\mathrm{IR}}{}^2 f_{r_0}(f_{r_0}+1)\log^2(3)\alpha_{\theta_1}^3 N^{\frac{4f_{r_0}}{3}+\frac{2}{5}}}{8192(f_{r_0}-1)^3g_s{}^3\left(\log N\right)^3MN_f^2}\right.$$
$$\left.+\frac{81\sqrt{3}\pi^2\mathcal{C}_{\phi_0}^{\mathrm{IR}}{}^2\epsilon f_{r_0}(f_{r_0}+1)^2\log(3)(\log(243)-6)\alpha_{\theta_1}^3 N^{\frac{4f_{r_0}}{3}+\frac{2}{5}}}{2048(f_{r_0}-1)^3g_s{}^3\left(\log N\right)^3MN_f^2}-\frac{243\pi^2\mathcal{C}_{\phi_0}^{\mathrm{IR}}{}^2 f_{r_0}(f_{r_0}+1)^2\log^2(3)\alpha_{\theta_1}^3 N^{\frac{4f_{r_0}}{3}+\frac{2}{5}}}{4096(f_{r_0}-1)^3g_s{}^3\left(\log N\right)^3MN_f^2}\right)$$

(4.20)

and

$$g_{\mathrm{YM}}^2=\frac{\log NN\left(7(\mathcal{C}_{zz}^{(1)}-2\mathcal{C}_{\theta_1 z}^{(1)}+2\mathcal{C}_{\theta_1 x}^{(1)})f_{r_0}{}^2\gamma^2 g_s{}^2M^4\log^2(N)+3456(f_{r_0}+1)\lambda_\epsilon^2\right)}{288\lambda_\epsilon^2\alpha_{\theta_1}^2\log N\left(\sqrt{3}\beta^{3/2}(\mathcal{C}_{zz}^{(1)}-2\mathcal{C}_{\theta_1 z}^{(1)}+2\mathcal{C}_{\theta_1 x}^{(1)})\lambda_\epsilon m_0^2-12(f_{r_0}+1)N\right)}.$$

(4.21)

In the chiral limit, the $\mathcal{O}(p^4)$ SU(3) χPT Lagrangian is given by [5]:

$$L_1\left(\text{Tr}(\nabla_\mu U^\dagger \nabla^\mu U)\right)^2 + L_2\left(\text{Tr}(\nabla_\mu U^\dagger \nabla_\nu U)\right)^2 + L_3\text{Tr}\left(\nabla_\mu U^\dagger \nabla^\mu U\right)^2$$
$$-iL_9 Tr\left(\mathcal{L}_{\mu\nu}\nabla^\mu U\nabla^\nu U^\dagger + \mathcal{R}_{\mu\nu}\nabla^\mu U\nabla^\nu U^\dagger\right) + L_{10} Tr\left(U^\dagger\mathcal{L}_{\mu\nu}U\mathcal{R}^{\mu\nu}\right) + H_1 Tr\left(\mathcal{L}_{\mu\nu}^2 + \mathcal{R}_{\mu\nu}^2\right),$$

$$(4.22)$$

where $\nabla_\mu U \equiv \partial_\mu U - i\mathcal{L}_\mu U + iU\mathcal{R}_\mu$, $U = e^{\frac{2i\pi}{F_\pi}}$. For completeness, we have reviewed the arguments of [36] to obtain (4.22)-like terms from the HLS Lagrangian by integrating out the ρ mesons in appendix C. Using hence (C.20) along with results of [38], one obtains relationships between the LECs y_i, z_i of (3.20) and the L_is of (4.22)–(4.29) for L_1, (4.37) for L_9 and (4.46) for L_{10}.

The parameters L_i and H_i are renormalized at one-loop level with all vertices in one-loop diagrams arising from the $\mathcal{O}(p^2)$ terms. Using dimensional regularization and performing renormalizations of the parameters via [5]:

$$L_i = L_i^r(\mu) + \Gamma_i\lambda(\mu), \qquad H_i = H_i^r(\mu) + \Delta_i\lambda(\mu), \qquad (4.23)$$

where μ is the renormalization point, and Γ_i and Δ_i are certain numbers given later; $\lambda(\mu)$ is the divergent part given by

$$\lambda(\mu) = -\frac{1}{2(4\pi)^2}\left[\frac{1}{\epsilon} - \ln\mu^2 + 1\right], \qquad (4.24)$$

where

$$\frac{1}{\epsilon} = \frac{2}{4-d} - \gamma_E + \ln 4\pi, \qquad (4.25)$$

d being the non-radial non-compact space-time dimensionality to be set to four. The constants Γ_i and Δ_i for SU(3) χPT theory were worked out in [5]:

$$\Gamma_1 = \frac{3}{32}, \quad \Gamma_2 = \frac{3}{16}, \quad \Gamma_3 = 0, \quad \Gamma_4 = \frac{1}{8}, \quad \Gamma_5 = \frac{3}{8},$$
$$\Gamma_6 = \frac{11}{144}, \quad \Gamma_7 = 0, \quad \Gamma_8 = \frac{5}{48}, \quad \Gamma_9 = \frac{1}{4}, \quad \Gamma_{10} = -\frac{1}{4}; \qquad (4.26)$$
$$\Delta_1 = -\frac{1}{8}, \quad \Delta_2 = \frac{5}{24}.$$

The analog of the 1-loop renormalization in χPT can be understood on the gravity dual side by noting that the latter requires holographic renormalization. This can be seen as follows. It can be shown [39] that the bulk on-shell $D = 11$ supergravity action inclusive of $\mathcal{O}(R^4)$-corrections is given by:

$$S_{D=11}^{\text{on-shell}} = \frac{1}{2}\left[-\frac{7}{2}S_{\text{EH}}^{(0)} + 2S_{\text{GHY}}^{(0)} + \beta\left(\frac{20}{11}S_{\text{EH}} - \frac{117}{22}\int_{M_{11}}\sqrt{-g^{(1)}}R^{(0)} + 2S_{\text{GHY}} - \frac{2}{11}\int_{M_{11}}\sqrt{-g^{(0)}}g_{(0)}^{MN}\frac{\delta J_0}{\delta g_{(0)}^{MN}}\right)\right].$$

$$(4.27)$$

The UV divergences of the various terms in (4.27) are summarized below:

$$\int_{M_{11}}\sqrt{-g}R\Big|_{\text{UV−divergent}}, \qquad \int_{\partial M_{11}}\sqrt{-h}K\Big|_{\text{UV−divergent}} \sim r_{\text{UV}}^4\log r_{\text{UV}},$$

$$\int_{M_{11}}\sqrt{-g}g^{MN}\frac{\delta J_0}{\delta g^{MN}}\Big|_{\text{UV−divergent}} \sim \frac{r_{\text{UV}}^4}{\log r_{\text{UV}}}. \qquad (4.28)$$

LECs	GL 1985 [5]	NLO 2014	NNLO free fit	NNLO BE14 [35]
$10^3\ L_1^r$	0.7(3)	1.0(1)	0.64)06	0.53(06)
$10^3\ L_2^r$	1.3(7)	1.6(2)	0.59(04)	0.81(04)
$10^3\ L_3^r$	−4.4(2.5)	−3.8(3)	−2.80(20)	−3.07(20)
$10^3\ L_4^r$	−0.3(5)	0.0(3)	0.76(18)	0.3
$10^3\ L_5^r$	1.4(5)	1.2(1)	0.50(07)	1.01(06)
$10^3\ L_6^r$	−0.2(3)	0.0(4)	0.49(25)	0.14(05)
$10^3\ L_7^r$	−0.4(2)	−0.3(2)	−0.19(08)	−0.34(09)
$10^3\ L_8^r$	0.9(3)	0.5(2)	0.17(11)	0.47(10)

Table 3. Various fits for NLO LECs $L_i^r (i = 1, 2, \ldots, 8)$.

i	$L_i^r(M_\rho)\ 10^3$	Source
1	0.4 ± 0.3	$K_{e4}, \pi\pi \to \pi\pi$
2	1.4 ± 0.3	$K_{e4}, \pi\pi \to \pi\pi$
3	-3.5 ± 1.1	$K_{e4}, \pi\pi \to \pi\pi$
4	-0.3 ± 0.5	Zweig rule
5	1.4 ± 0.5	$F_K : F_\pi$
6	-0.2 ± 0.3	Zweig rule
7	-0.4 ± 0.2	Gell-Mann-Okubo, L_5, L_8
8	0.9 ± 0.3	$M_{K0} - M_{K_+}, L_5, (m_s - \hat{m}) : (m_d - m_u)$
9	6.9 ± 0.7	$\langle r^2 \rangle_V^\pi$
10	-5.5 ± 0.7	$\pi \to e\nu\gamma$

Table 4. Phenomenological Values of the 1-loop renormalised couplings $L_i^r(M_\rho)$ of (4.22) [7]. Last column shows the source to extract this information.

It can be shown [39] that an appropriate linear combination of the boundary terms: $\int_{\partial M_{11}} \sqrt{-h} K \big|_{r=r_{\mathrm{UV}}}$ and $\int_{\partial M_{11}} \sqrt{-h} h^{mn} \frac{\partial J_0}{\partial h^{mn}} \big|_{r=r_{\mathrm{UV}}}$ serves as the appropriate counter terms to cancel the UV divergences (4.28).[3]

We will now discuss how to match our results with the experimental values of the 1-loop renormalized coupling constants $L_{1,2,3,9,10}^r$ in (4.22) and F_π^2 and g_{YM}^2; the experimental value of H_1 apparently is unavailable. Table 3 contains the values of the 1-loop renormalized values of the coupling constants in (4.22) [1].

Table 4 elaborates upon the column titled **GL 1985** [5].

We will now show how, in five steps, it is possible to match the phenomenological values of the $\mathcal{O}(p^4)$ SU(3) χPT Lagrangian [5] one-loop renormalized LECs $L_{1,9,10}^r$ as well as $F_\pi^2, g_{\mathrm{YM}}(\Lambda_{\mathrm{QCD}} = 0.4\,\mathrm{GeV}, \Lambda = 1.1\mathrm{Gev}, \mu = M_\rho)$ where Λ is the "HLS-QCD" matching scale [36] and μ is the renormalization scale, as well as the order of magnitude and signs of $L_{2,3}^r$.

---

[3]For consistency, one needs to impose the following relationship between the UV-valued effective number of flavor $D7$-branes of the parent type IIB dual, N_f^{UV} and $\log r_{\mathrm{UV}}$: $N_f^{\mathrm{UV}} = \frac{(\log r_{\mathrm{UV}})^{\frac{15}{2}}}{\log N}$.

- **Step 1: matching $L^r_{1,2,3}$.**

  Using (4.17), (4.21), (B.2), (B.3) and $y_2 = -y_1$ one obtains:

$$L^r_1 = \frac{L^r_2}{2} = -\frac{L^r_3}{6} = \frac{1}{g^2_{\rm YM}} - z_4 + y_2 \tag{4.29}$$

$$= \frac{1}{143360\sqrt{7}(f_{r_0}-1)f_{r_0}g_s{}^8\log N M^4 N_f^8 \alpha^3_{\theta_1}\mathcal{C}^{\rm IR}_{\psi_1}{}^2\Omega}\left\{3\pi(f_{r_0}+1)N^{7/5}\right.$$

$$\times\left(-\frac{4822335\sqrt{2}\sqrt[8]{3}\sqrt[4]{7}\pi^3\beta\left(\mathcal{C}^{(1)}_{zz}-2\mathcal{C}^{(1)}_{\theta_1 z}+2\mathcal{C}^{(1)}_{\theta_1 x}\right)\mathcal{C}^{\rm IR}_{\phi_0}{}^2\epsilon^{9/4}f_{r_0}(f_{r_0}+1)g_s{}^4 M^2 N_f^4\alpha^9_{\theta_1}\mathcal{C}^{\rm IR}_{\psi_1}{}^{N^{2f_{r_0}+\frac{7}{5}}}}{(f_{r_0}-1)^3\left(\log N\right)^3}\right.$$

$$+\frac{430080\sqrt{21}\beta\left(\mathcal{C}^{(1)}_{zz}-2\mathcal{C}^{(1)}_{\theta_1 z}+2\mathcal{C}^{(1)}_{\theta_1 x}\right)\epsilon^3(f_{r_0}-1)f_{r_0}g_s{}^8\log N M^4 m_0{}^2 N^{3/5}N_f^8\alpha^5_{\theta_1}\mathcal{C}^{\rm IR}_{\psi_1}{}^2}{\pi(f_{r_0}+1)}$$

$$+\frac{64\sqrt{\frac{7}{\pi}}(f_{r_0}-1)^2 g_s{}^9\log r_0 M^4 N_f^8\mathcal{C}^{\rm IR}_{\psi_1}{}^2 N^{\frac{2f_{r_0}}{3}-\frac{3}{5}}\Omega}{(f_{r_0}+1)^2} + \frac{26880\sqrt{7}(f_{r_0}-1)g_s{}^9\log r_0 M^4 N_f^8\mathcal{C}^{\rm IR}_{\psi_1}{}^2 N^{\frac{2f_{r_0}}{3}-\frac{2}{5}}\Omega}{(f_{r_0}+1)^2}$$

$$-\frac{8960\sqrt{7}(f_{r_0}-1)f_{r_0}g_s{}^9\log N M^4 N_f^8\alpha^2_{\theta_1}\mathcal{C}^{\rm IR}_{\psi_1}{}^2 N^{\frac{2f_{r_0}}{3}-\frac{2}{5}}\Omega}{f_{r_0}+1}$$

$$-\frac{132269760\sqrt{2}\sqrt[8]{3}\sqrt[4]{7}\pi^3\mathcal{C}^{\rm IR}_{\phi_0}{}^2\epsilon^{17/4}f_{r_0}(f_{r_0}+1)^2 g_s{}^4 M^2 N_f^4\alpha^9_{\theta_1}\mathcal{C}^{\rm IR}_{\psi_1}{}^{N^{2f_{r_0}+\frac{7}{5}}}}{(f_{r_0}-1)^3\left(\log N\right)^3}$$

$$-\frac{5160960\sqrt{7}\epsilon^2(f_{r_0}-1)f_{r_0}g_s{}^8\log N M^4 N^{3/5}N_f^8\alpha^5_{\theta_1}\mathcal{C}^{\rm IR}_{\psi_1}{}^2}{\pi}$$

$$\left.\left.-\frac{195259926456\sqrt[4]{3}\pi^7\mathcal{C}^{\rm IR}_{\phi_0}{}^4\epsilon^{13/2}f_{r_0}(f_{r_0}+1)^4\alpha^{15}_{\theta_1}N^{4f_{r_0}+\frac{11}{5}}}{(f_{r_0}-1)^7\left(\log N\right)^7}\right)\right\}, \tag{4.30}$$

where:

$$\Omega \equiv \left(7\beta\left(\mathcal{C}^{(1)}_{zz}-2\mathcal{C}^{(1)}_{\theta_1 z}+2\mathcal{C}^{(1)}_{\theta_1 x}\right)f_{r_0}{}^2\gamma^2 g_s{}^2\left(\log N\right)^2 M^4 + 3456\epsilon^2(f_{r_0}+1)N^2\right). \tag{4.31}$$

Now, as we are working up to $\mathcal{O}(\beta)$ and further due to the smallness of ϵ assuming working up to $\mathcal{O}(\epsilon^2)$, one sees that (4.29) gets simplified to read:

$$L^r_1 = \frac{1}{143360\sqrt{7}(f_{r_0}-1)f_{r_0}g_s{}^8\log N M^4 N_f^8\alpha^3_{\theta_1}}$$

$$\times\left\{3\pi(f_{r_0}+1)N^{7/5}\left(\frac{64\sqrt{\frac{7}{\pi}}(f_{r_0}-1)^2 g_s{}^9\log r_0 M^4 N_f^8 N^{\frac{2f_{r_0}}{3}-\frac{3}{5}}}{(f_{r_0}+1)^2}\right.\right.$$

$$+\frac{26880\sqrt{7}(f_{r_0}-1)g_s{}^9\log r_0 M^4 N_f^8 N^{\frac{2f_{r_0}}{3}-\frac{2}{5}}}{(f_{r_0}+1)^2} - \frac{8960\sqrt{7}(f_{r_0}-1)f_{r_0}g_s{}^9\log N M^4 N_f^8\alpha^2_{\theta_1}N^{\frac{2f_{r_0}}{3}-\frac{2}{5}}}{f_{r_0}+1}$$

$$\left.\left.-\frac{5160960\sqrt{7}(f_{r_0}-1)f_{r_0}g_s{}^8\log N M^4 N^{3/5}N_f^8\alpha^5_{\theta_1}}{\pi}\left(\frac{\epsilon^2}{\Omega}\right)\right)\right\} \tag{4.32}$$

As will be shown below (4.39) and (4.40), to match the experimental value of L^r_9, one

needs to set $f_{r_0} = 1 - \kappa, 0 < \kappa \ll 1$. Hence:

$$L_1^r = \frac{1}{143360\sqrt{7}f_{r_0}g_s{}^8 \log N M^4 N_f^8 \alpha_{\theta_1}^3} \left\{ 3\pi(f_{r_0}+1)N^{7/5} \left(\frac{26880\sqrt{7}g_s{}^9 \log r_0 M^4 N_f^8}{(f_{r_0}+1)^2} \right. \right.$$
$$\left. \left. - \frac{8960\sqrt{7}f_{r_0}g_s{}^9 \log N M^4 N_f^8 \alpha_{\theta_1}^2}{f_{r_0}+1} - \frac{5160960\sqrt{7}f_{r_0}g_s{}^8 \log N M^4 N^{\frac{1}{3}} N_f^8 \alpha_{\theta_1}^5}{\pi} \left(\frac{\epsilon^2}{\Omega} \right) \right) \right\}. \tag{4.33}$$

Further, as the $\log r_0$ in (4.33) is in fact $\log \left(\frac{r_0}{\mathcal{R}_{D5/\overline{D5}}} \right)$ — $\mathcal{R}_{D5/\overline{D5}} > r_0$ being the $D5$–$\overline{D5}$ separation — one sees from (4.33) that in order to obtain a positive value (as required from phenomenological value of L_1^r), $\Omega < 0$. Note, as shown below in (4.36), matching with the experimental value of the pion decay constant F_π requires an N-suppression in α_{θ_1}, implying the N enhancement in the last term in (4.33) is artificial. So, to ensure one does not pick up an $\mathcal{O}\left(\frac{1}{\beta}\right)$ contribution in L_1 from $\frac{\epsilon^2}{\Omega}$ in (4.33) and also to ensure that the third term in (4.33) required to partly compensate the first two negative terms in the same (as explained above) to produce a positive term, is not vanishingly small, from (4.31), one needs to set:

$$\epsilon = \lambda_\epsilon \frac{\sqrt{\beta}}{N}. \tag{4.34}$$

Finally, combining the above observations with the requirement to match the experimental value $L_1^{\exp} = 0.64 \times 10^{-3}$, one requires to implement the following constraint:

$$\left(\mathcal{C}_{zz}^{(1)} - 2\mathcal{C}_{\theta_1 z}^{(1)} + 2\mathcal{C}_{\theta_1 x}^{(1)} \right) = -\frac{493.7(\delta+1)(f_{r_0}+1)\lambda_\epsilon^2}{f_{r_0}{}^2 \gamma^2 g_s{}^2 (\log N)^2 M^4},$$
$$\delta = \frac{0.053\alpha_{\theta_1}^3 N^{-\frac{2f_{r_0}}{3}-1}}{g_s}. \tag{4.35}$$

We hence see from (4.34) that ϵ provides an expansion parameter connecting the $\frac{1}{N}$ and β expansions. Also, together with (4.35), this is the first connection between large-N and higher derivative corrections in the context of $\mathcal{M}$-theory dual of large-N thermal QCD-like theories.

From (4.29) upon comparison with experimental values of $L_{2,3}^r$, we see that one can obtain a match with their order of magnitude and sign, but not the exact numerical value.

- **Step 2: matching F_π^2.**
  Now, using (4.35) and (4.17), one can show that the difference of (4.20) and the experimental value of $F_\pi^2 = \frac{0.0037 N^{-\frac{2f_{r_0}}{3}-1}}{g_s}$,[4] vanishes for:

$$\alpha_{\theta_1} = \frac{0.03 \sqrt[24]{\beta}g_s{}^{11/3} \sqrt[12]{\lambda_\epsilon} (\log N)^{2/3} \sqrt[3]{M} N_f^{2/3} \sqrt[3]{\mathcal{C}_{\psi_1}^{\rm IR}} (1-f_{r_0})^{2/3} N^{-\frac{f_{r_0}}{3}-\frac{13}{60}}}{\sqrt[3]{\mathcal{C}_{\phi_0}^{\rm IR}} \sqrt[3]{f_{r_0}+1}}. \tag{4.36}$$

---

[4]F_π^2 can be made to match the experimental value of $92.3\,\mathrm{MeV}$ wherein from 0^{++}-glueball mass [18], one identifies: $\frac{1700}{4}MeV \equiv \frac{r_0}{\sqrt{4\pi g_s N}}$.

- **Step 3: matching L_9^r.**

  Using (C.20) and [38], one can show that:

  $$L_9^r = \frac{1}{8}\left(\frac{2}{g_{\text{YM}}^2} - 2z_3 - z_4 - z_6\right). \tag{4.37}$$

  By using (4.35), (4.17) and (B.3) one obtains:

  $$L_9^r = -\frac{0.0031(f_{r_0}-1)g_s N^{\frac{2f_{r_0}}{3}+\frac{4}{5}}}{(f_{r_0}+1)\alpha_{\theta_1}^3}. \tag{4.38}$$

  Now, assuming:

  $$f_{r_0} = 1 - \omega\alpha_{\theta_1}^3, \tag{4.39}$$

  where

  $$\omega = \frac{4.6 N^{-\frac{2f_{r_0}}{3}-\frac{4}{5}}}{g_s}, \tag{4.40}$$

  one sees one gets a match with the phenomenological/experimental value $L_9^{\text{exp}} = 6.9 \times 10^{-3}$. Substituting (4.39)–(4.40) into (4.35), one obtains:

  $$\left(\mathcal{C}_{zz}^{(1)} - 2\mathcal{C}_{\theta_1 z}^{(1)} + 2\mathcal{C}_{\theta_1 x}^{(1)}\right) \approx -\frac{987.4\lambda_\epsilon^2}{\gamma^2 g_s{}^2 \left(\log N\right)^2 M^4}. \tag{4.41}$$

  Consistency of (4.39), (4.40) and (4.36), setting $\sqrt[24]{\beta} \approx \beta^{0.04}$ to unity (because of the very small exponent of β), requires:

  $$\alpha_{\theta_1} = \frac{11.33\sqrt[3]{\mathcal{C}_{\phi_0}^{\text{IR}}}N^{55/36}}{g_s \sqrt[12]{\lambda_\epsilon}\sqrt[3]{M}N_f{}^{2/3}\sqrt[3]{\mathcal{C}_{\psi_1}^{\text{IR}}}\log^{\frac{2}{3}}(N)}. \tag{4.42}$$

- **Step 4: matching $g_{\text{YM}}^2(\Lambda_{\text{QCD}} = 0.4\,\text{GeV}, \Lambda = 1.1\,\text{GeV}, \mu = M_\rho)$.**

  Similarly, g_{YM}^2 can be chosen to match the experimental value 36 (at $\Lambda_{\text{QCD}} = 0.4\,\text{GeV}$ and the HLS-QCD matching scale "Λ" $= 1.1\,\text{GeV}$ [36]) and renormalization scale $\mu = M_\rho$ by imposing:

  $$\frac{12\delta N}{\alpha_{\theta_1}^2\left(\frac{855.11\beta^{3/2}(\delta+1)\lambda_\epsilon^3 m_0{}^2}{f_{r_0}{}^2\gamma^2 g_s{}^2(\log N)^2 M^4} + 12N\right)} = 36, \tag{4.43}$$

  which can be effected by:

  $$\gamma = \frac{10^6 \lambda_\epsilon^{3/2} m_0 \beta^{3/4}\alpha_{\theta_1}\sqrt{8.4(\delta+1)}}{f_{r_0}g_s\left(\log N\right)M\sqrt{N\left(3.27\times10^9\delta - 1.18\times10^{11}\alpha_{\theta_1}^2\right)}} \tag{4.44}$$

  By requiring the argument of the square root in the denominator of (4.44) to be positive, one hence obtains an upper bound on $\mathcal{C}_{\phi_0}^{\text{IR}}$ from (4.44):

  $$\mathcal{C}_{\phi_0}^{\text{IR}} < \frac{1.04\times10^{-13}\sqrt[8]{\beta}(f_{r_0}-1)^2 g_s{}^2\sqrt[4]{\lambda_\epsilon}MN_f^2\mathcal{C}_{\psi_1}^{\text{IR}}\ N^{-3f_{r_0}-\frac{73}{20}}\log^2(N)}{f_{r_0}+1}. \tag{4.45}$$

- **Step 5: matching L_{10}^r.**

  Now, using (4.17), (4.35), (B.3) and [37, 38] (which provides also the UV-finite part of the $\rho - \pi$ one-loop correction via an "$a(M_\rho)$" factor taken to be equal to 2) one can show that (for $N_f = 3$):

  $$
  \begin{aligned}
  L_{10}^r &= \frac{1}{4}\left(-\frac{1}{g_{\text{YM}}^2} + 2z_3 - 2z_2 + 2z_1\right) + \frac{11 N_f a(M_\rho)}{96(4\pi)^2} \\
  &= \frac{0.47 g_s N^{\frac{2f_{r_0}}{3}+1}}{\alpha_{\theta_1}} - \frac{8640. g_s N^{5/3}\alpha_{\theta_1}^2}{g_s N^{5/3} - 1831.63\alpha_{\theta_1}^3} + 4.4 \times 10^{-3}.
  \end{aligned}
  \tag{4.46}
  $$

  Now, using $L_{10}^r = -5.5 \times 10^{-3}$, one obtains the following value of the θ_1 delocalization parameter α_{θ_1}:

  $$
  \alpha_{\theta_1} = -2.8 \times 10^{-14} g_s N^{\frac{5}{3}} + \frac{\sqrt{4.2 \times 10^{-6} - 2.9 \times 10^{-11} g_s^2 N^{\frac{10}{3}}}}{2}.
  \tag{4.47}
  $$

  For $g_s = 0.1$, this implies $N < 140$. For numerical computation we set $N = 10^2, g_s = 0.1, M = N_f = 3$, and from (4.42) and (4.47) we obtain the following non-linear relation between $\mathcal{C}_{\psi_1}^{\text{IR}}, \lambda_\epsilon$ and $\mathcal{C}_{\phi_0}^{\text{IR}}$ (assumed to be also satisfying (4.45)):

  $$
  \frac{1.6\sqrt[3]{\mathcal{C}_{\phi_0}^{\text{IR}}}}{\lambda_\epsilon^{\frac{1}{12}}\sqrt[3]{\mathcal{C}_{\psi_1}^{\text{IR}}}} = 10^{-7}.
  \tag{4.48}
  $$

  The key results of this section are summarized below.

1. Fixing (eight non-zero) parameters:

   (a) a linear combination of constants of integration appearing in the solutions of the EOMs of the $\mathcal{O}(R^4)$ $\mathcal{M}$-theory uplift's metric components $G_{zz,\theta_1 z,\theta_1 x}^{\mathcal{M}}$;[5]

   (b) constants of integration $\mathcal{C}_{\psi_1,\phi_0}^{\text{IR}}$,[6] appearing in the solutions of the EOMs in the IR of respectively the ρ and π mesons;

   (c) ϵ (or equivalently λ_ϵ as in $\epsilon = \lambda_\epsilon \frac{\sqrt{\beta}}{N}$), γ as in $b = \frac{1}{\sqrt{3}} + \epsilon$ in the relationship between the resolution parameter a (i.e., the radius of the blown-up S^2) and the IR cut-off r_0: $a = \left(b + \gamma\mathcal{O}\left(\frac{g_s M^2}{N}\right)\right) r_0$;

   (d) θ_1 delocalization parameter α_{θ_1};

---

[5]In principle, there are other constants of integration appearing in other $\mathcal{O}(R^4)$ $\mathcal{M}$-theory metric components (4.1), but it turns out that there is a specific combination of only three that appears while matching χPT LECs up to $\mathcal{O}(p^4)$. Even though it is unclear why a specific combination, but it is intuitively evident that it involves $G_{zz,\theta_1 z,\theta_1 x}^{\mathcal{M}}$ as these essentially correspond to the S^3, part of the non-compact four-cycle wrapped by the flavor $D7$-branes in the type IIB dual of [14]; these $D7$-flavor branes are (triple) T dualized to the type IIA $D6$ flavor branes. This, in fact, is an extremely non-trivial signature of the four-cycle wrapped by the type IIB $D7$-branes, that manifests itself as $\mathcal{O}(R^4)$-corrections to the MQGP background [15]/$\mathcal{M}$-theory uplift of thermal QCD-like theories.

[6]$\mathcal{C}_{\psi_1,\phi_0}^{\text{UV}}$ can be self-consistently set to zero — see 2. and 3.

S. No.	Quantities whose experimental values are fitted to	Parameters of the holographic dual used for fitting	Equation numbers
1.	$L^r_{1,2,3}$	Specific linear combination of constants of integration appearing in solutions to $\mathcal{O}(R^4)$ corrections to $\mathcal{M}$-theory metric components $G^{\mathcal{M}}_{zz,\theta_1 z,\theta_1 x}$; f_{r_0}; γ; λ_ϵ;	(4.35) [using (4.34)]
		$\mathcal{O}(R^4) - \frac{1}{N}$ connection: $a - r_0$ relation must have an $\epsilon \sim \frac{l_p^3}{N}$ contribution at $\mathcal{O}\left(\frac{g_s M^2}{N}\right)^0$	(4.34)
2.	F^2_π; L^r_9	f_{r_0}; α_{θ_1}; $\mathcal{C}^{\mathrm{IR}}_{\phi_0}$; $\mathcal{C}^{\mathrm{IR}}_{\psi_1}$; λ_ϵ	(4.36), (4.38)–(4.42)[consistency check]
3.	$g_{\mathrm{SU}(3)}$	γ; upper bound on $\mathcal{C}^{\mathrm{IR}}_{\phi_0}$	(4.44), (4.45)
4.	L^r_{10}	α_{θ_1}; $\mathcal{C}^{\mathrm{IR}}_{\phi_0}$; $\mathcal{C}^{\mathrm{IR}}_{\psi_1}$	(4.47)–(4.48) [even though specific values of N, M, N_f, g_s chosen, but can find analog of (4.48) $\forall N < 140$ (4.47) respecting (2.4)]

Table 5. Summary of Matching of Parameters of the $\mathcal{M}$-theory uplift and Experimental values of 1-Loop renormalized χPT LECs, $F^2_\pi, g^2_{\mathrm{SU}(3)}$.

(e) $D6$-brane tension or equivalently α';

(f) f_{r_0} as in $r_0 = N^{-\frac{f_{r_0}}{3}}$

[all for given values of N, M, N_f in the MQGP limit (2.4)] of the top-down holographic dual by matching with experimental values of one-loop renormalized χPT Lagrangian's LECs $L^r_{1,2,3,9,10}, F^2_\pi, g_{\mathrm{SU}(3)}$: table 5 lists out the same.

Using the values of the parameters of our $\mathcal{M}$-theory dual of thermal QCD-like theories, one can also obtain the values of the LEC H_1 of (4.22), and in principle, the LECs of the χPT Lagrangian at $\mathcal{O}(p^6)$ [40]. We defer the latter in particular for later exploration.

2. Further, the normalization condition of the ρ-meson profile function $\psi_1(Z)$ ($n = 1$ mode) is used to determine the constant of integration $\mathcal{C}^{\mathrm{UV}}_{\psi_1}$ appearing in the solution to $\psi_1(Z)$ in the UV, in terms of $\mathcal{C}^{\mathrm{IR}}_{\psi_1}, f_{r_0}, \gamma, \epsilon, \lambda_\epsilon$ and T_{D6} or equivalently α' (via "$\mathcal{V}_{\Sigma_2}$"); it was shown that one could self-consistently set $\mathcal{C}^{\mathrm{UV}}_{\psi_1} = 0$.[7]

3. Substituting the expression for $T_{D6}/\alpha'/\mathcal{V}_{\Sigma_2}$ obtained above from the normalization condition of $\psi_1(Z)$ into the normalization condition for the profile function $\phi_0(Z)$, one could self-consistently set $\mathcal{C}^{\mathrm{UV}}_{\phi_0} = 0$.

---

[7]The ρ-meson mass parameter m_0 can be fixed by imposing Dirichlet boundary condition on $\psi_1(Z)$ at $Z = 0$ [29].

5 Summary

In this paper, we have shown that $\mathcal{O}\left(\frac{1}{N}\right)$-corrections of the $\mathcal{M}$-theory uplift of holographic thermal QCD of [14] as worked out in [15, 19] in conjunction with the $\mathcal{O}(R^4)$-corrections to the same as worked out in [16], can be used to match the experimental values of the coupling constants up to $\mathcal{O}(p^4)$ appearing in the SU(3) Chiral Perturbation theory Lagrangian for the pion and rho vector mesons as well as their flavor partners, in the chiral limit.

The following are the key lessons learnt in this paper:

- There is a particular combination of the constants of integration appearing in the solutions to the $\mathcal{O}(R^4)$ corrections to the $\mathcal{M}$-theory dual of thermal QCD that will appear in all the coupling constants of χPT at least up to $\mathcal{O}(p^4)$. As an example, in working in the $\psi = 0, 2\pi, 4\pi$ patches close to the type IIB Ouyang embedding effected by working near small $\theta_{1,2}$ the aforementioned combination is $\mathcal{C}_{zz}^{(1)} - 2\mathcal{C}_{\theta_1 z}^{(1)} + 2\mathcal{C}_{\theta_1 x}^{(1)}$, where $\mathcal{C}_{MN}^{(1)}$s are the constants of integration appearing in the solutions to the EOMs for the $\mathcal{O}(R^4)$-corrections $G_{MN}^{\mathcal{M},(1)} = G_{MN}^{\mathcal{M},(0)}(1 + f_{MN})$, being the MQGP metric of [14, 19]. This dependence is expected to change in the $\psi \neq 2n\pi, n = 0, 1, 2$-patches.

- Matching the result obtained from our $\mathcal{M}$-theory $\mathcal{O}(R^4)$-corrected holographic computation with the experimental values of one-loop renormalized $L_{1,2,3}^r$, one sees one that one is required to do two things. One, the $\mathcal{O}\left(\frac{1}{N}\right)$ correction to the leading order (in N) result in expressing the resolution parameter in terms of the IR cut-off, also must involve a term proportional to $\frac{l_p^3}{N}$, i.e., $a = \left(b + \gamma \frac{g_s M^2}{N}(1 + \log r_0)\right) r_0 \rightarrow \left(\tilde{b} + \lambda \frac{l_p^3}{N} + \gamma \frac{g_s M^2}{N}(1 + \log r_0)\right) r_0$. The second, the value of the aforementioned linear combination of integration constants figuring in the solutions to the $\mathcal{O}(R^4)$ corrections to the MQGP metric of [15, 19], gets fixed in terms of λ, γ, M, g_s and N. This is the first evidence of the relationship between the $\mathcal{O}\left(\frac{1}{N}\right)$ and $\mathcal{O}(\beta)$ corrections.

- Matching the experimental values of F_π^2 and the one-loop renormalized L_9^r and internal consistency, determine the angular delocalization in the polar angles $\theta_{1,2}$ (3.4) consistent with the type IIB Ouyang embedding of the flavor $D7$-branes in the type IIB holographic dual and its SYZ type IIA mirror [27]. Note, similar to as explained in [25], the SYZ type IIA mirror (and hence its M theory uplift) is independent of the angular delocalization. In the context of obtaining the values of the χPT Lagrangian's coupling constants this is encoded in the fact even though the aforementioned angular delocalization parameters $\alpha_{\theta_{1,2}}$ would change depending on the values of $\theta_{10,20}$ of (3.3), the corresponding values of $\alpha_{\theta_{1,2}}$ can always be found.

- Matching the experimental value of $g_{\rm YM}^2$ at $\Lambda_{\rm QCD} = 0.4\,{\rm GeV}$, the HLS-QCD matching scale $\Lambda = 1.1\,{\rm GeV}$ and renormalization scale $\mu = M_\rho$ with the value obtained from our setup, provide the following pair of results. One, the $\mathcal{O}\left(\frac{g_s M^2}{N}\right)$ correction appearing in the resolution parameter — IR cut-off relation turns out to also have an $l_p^{\frac{9}{2}}$ dependence apart from dependence on g_s, N, M, N_f, as well as the constants

of integration appearing in the solutions to the radial profile functions of the π and ρ mesons (and their flavor partners) in the IR. Second, one obtains an upper bound on the constant of integration appearing in the expression for the profile function of the π meson (and its flavor partners) in the IR.

- Upon matching with the experimental value of the one-loop renormalized L_{10}^r, one obtains a non-linear relation between the constants of integration appearing in the radial profile functions for the π and ρ mesons in the IR, as well as the coefficient of the $\frac{l_p^3}{N}$ term required to exist in the resolution-parameter-IR-cut-off relation (discussed in the second bullet above) upon matching with the experimental values of $L_{1,2,3}^r$. For numerical clarity, we explicitly wrote down the same for $N = 10^2$, $g_s = 0.1$, $M = N_f = 3$.

Acknowledgments

A Equations of motion for f_{MNs}, their large-N large-r (UV) limit and proof of f_{MNs} to be vanishingly small in the UV

In this appendix, we will show that f_{MN} is vanishingly small in the UV inclusive of the $\mathcal{O}(l_p^6 R^4)$ corrections to the $\mathcal{M}$-Theory uplift of large-N thermal QCD at low temperatures (i.e. below the deconfinement temperature) near the $\psi = 2n\pi, n = 0, 1, 2$-patches. In the following:

$$g_2(r) \equiv 1 - \frac{r_0^4}{r^4},$$

$$a_1(r) = \frac{3\left(-N_f^{\rm UV} \log\left(9a^2 r^4 + r^6\right) + \frac{8\pi}{g_s} + 2\log N N_f^{\rm UV} - 4N_f^{\rm UV} \log\left(\alpha_{\theta_1}\right) - 4N_f^{\rm UV} \log\left(\alpha_{\theta_2}\right) + 4N_f^{\rm UV} \log(4)\right)}{8\pi}$$

$$a_2(r) = \frac{12a^2 g_s M_{\rm UV}^2 N_f^{\rm UV}\left(c_1 + c_2 \log(r_0)\right)}{9a^2 + r^2}$$

$$a_3(r) = \frac{r^2 a_2(r)}{2N_f^{\rm UV}\left(6a^2 + r^2\right)}$$

$$a_4(r) = \frac{6a^2 + r^2}{9a^2 + r^2}$$

$$B(r) = \frac{3g_s M_{\rm UV}^2 \log(r)\left(-g_s \log N N_f^{\rm UV} + 2g_s N_f^{\rm UV} \log\left(\alpha_{\theta_1}\right) + 2g_s N_f^{\rm UV} \log\left(\alpha_{\theta_2}\right) + 12g_s N_f^{\rm UV} \log(r) + 6g_s N_f^{\rm UV} - 2g_s N_f^{\rm UV} \log(4) + 8\pi\right)}{32\pi^2}$$

$$b_1(r) = \frac{1}{6\sqrt{2}\pi^{5/4} r^2 \alpha_{\theta_1}\alpha_{\theta_2}^2}\left\{g_s^{3/4} M_{\rm UV} \times \left[-g_s N_f^{\rm UV}\left(r^2 - 3a^2\right) \log N(2\log(r) + 1) + \log(r)\left(4g_s N_f^{\rm UV}\left(r^2 - 3a^2\right) \log\left(\frac{1}{4}\alpha_{\theta_1}\alpha_{\theta_2}\right)\right.\right.\right.$$
$$\left.\left.\left. + 8\pi\left(r^2 - 3a^2\right) - 3g_s N_f^{\rm UV}\ r^2\right) + 2g_s N_f^{\rm UV}\left(r^2 - 3a^2\right) \log\left(\frac{1}{4}\alpha_{\theta_1}\alpha_{\theta_2}\right) + 18g_s N_f^{\rm UV}\left(r^2 - 3a^2(6r+1)\right)\log^2(r)\right]\right\}$$

$$b_2(r) = \frac{g_s^{7/4} M_{\rm UV} N_f^{\rm UV} \log(r)\left(36a^2 \log(r) + r\right)}{3\sqrt{2}\pi^{5/4} r\alpha_{\theta_2}^3},$$

$$\text{(A.1)}$$

wherein $M_{\mathrm{UV}} \equiv M_{\mathrm{eff}}(r > \mathcal{R}_{D5/\overline{D5}})$ and similarly $N_f^{\mathrm{UV}} \equiv N_f^{\mathrm{eff}}(r > \mathcal{R}_{D5/\overline{D5}})$. We should keep in mind that near the $\psi = \psi_0 \neq 2n\pi, n = 0, 1, 2$-patch, some $G_{rM}^{\mathcal{M}}, M \neq r$ and $G_{x^{10}N}^{\mathcal{M}}, N \neq x^{10}$ components are non-zero, making this exercise much more non-trivial. As shown in [16], the contributions from E_8 is sub-dominant as compared to the contributions from J_0 terms.

As the EOMs are too long, they have not been explicitly typed but their forms have been written out. The explicit forms of $\mathcal{F}_{MN}$ have been given.

Using (A.1), one obtains the following EOMs in the UV:

EOM_{MN} :

$$\sum_{\mathcal{M},\mathcal{N}=0}^{x^{10}} \sum_{p=0}^{2} \mathcal{H}_{MN}^{\mathcal{M}\mathcal{N}\ (p)} \left(r; r_0, a, N, M^{\mathrm{UV}}, N_f^{\mathrm{UV}}, g_s, \alpha_{\theta_{1,2}}\right) f_{\mathcal{M}\mathcal{N}}^{(p)}(r) + \beta \mathcal{F}_{MN} \left(r; r_0, a, N, M, N_f, g_s, \alpha_{\theta_{1,2}}\right) = 0,$$

(A.2)

where M, N run over the $D = 11$ coordinates, $f_{MN}^{(p)} \equiv \frac{d^p f_{MN}}{dr^p}, p = 0, 1, 2$.

In the UV, and in the large-N limit, one can show that EOMs are algebraic and are given as under:

$$EOM_{x^3x^3} : \quad -\frac{49r^2 g_2(r)\left(2f_{zz}(r) - 3f_{x^{10}x^{10}}(r) - 4f_{\theta_1 z}(r) - 5f_{\theta_2 y}(r)\right)}{3456\sqrt{\pi}\sqrt{g_s}\,\alpha_{\theta_1}^2 \alpha_{\theta_2}^4 b_1(r)^2} - \frac{7a^8\beta M_{\rm UV}\left(\frac{1}{N}\right)^{9/4}\Sigma_1 \log(r_0)}{216\pi^2 g_s \log N^2 N_f^{\rm UV} r^6 \alpha_{\theta_2}^3} = 0$$

$$EOM_{\mathbb{R}^{1,2}(t,x^{1,2})} : \quad -\frac{49r^2\left(2f_{zz}(r) - 3f_{x^{10}x^{10}}(r) - 4f_{\theta_1 z}(r) - 5f_{\theta_2 y}(r)\right)}{3456\sqrt{\pi}\sqrt{g_s}\,\alpha_{\theta_1}^2 \alpha_{\theta_2}^4 b_1(r)^2} - \frac{7a^8\beta M_{\rm UV}\left(\frac{1}{N}\right)^{9/4}\Sigma_1 \log(r)}{216\pi^2 g_s \log N^2 N_f^{\rm UV} r^6 \alpha_{\theta_2}^3} = 0$$

$$EOM_{rr} : \quad \frac{49\sqrt{\pi}\sqrt{g_s}N a_4(r)\left(2f_{zz}(r) - 3f_{x^{10}x^{10}}(r) - 4f_{\theta_1 z}(r) - 5f_{\theta_2 y}(r)\right)}{864 r^2 \alpha_{\theta_1}^2 \alpha_{\theta_2}^4 b_1(r)^2 g_2(r)} - \frac{2a^{12}\beta M_{\rm UV}\left(\frac{1}{N}\right)^{9/10}\Sigma_1}{3\sqrt{3}\pi^{5/4}\sqrt[4]{g_s}\log N^2 N_f^{\rm UV} r^{14}\alpha_{\theta_2}^5} = 0$$

$$EOM_{\theta_1\theta_2} : \quad -\frac{3969\sqrt[5]{N}\alpha_{\theta_1}^2\left(2f_{x^{10}x^{10}}(r) + f_{\theta_2 y}(r)\right)}{512\alpha_{\theta_2}^2} + \frac{a^{12}\beta g_s^{5/4}M_{\rm UV}^2\left(\frac{1}{N}\right)^{17/10}(-\Sigma_1)\log^2(r_0)}{26244\sqrt{2}\pi^{11/4}\log N r^{12}\alpha_{\theta_1}^6 \alpha_{\theta_2}^4} = 0$$

$$EOM_{\theta_1\theta_1} : \quad -\frac{3969 N^{3/10}\alpha_{\theta_1}^2 b_2(r)\left(2f_{x^{10}x^{10}}(r) + f_{\theta_2 y}(r)\right)}{512\alpha_{\theta_2}^2 b_1(r)} = 0$$

$$EOM_{\theta_1 x} : \quad \frac{49 N^{21/20}\left(2f_{zz}(r) - 3f_{x^{10}x^{10}}(r) - 4f_{\theta_1 z}(r) - 5f_{\theta_2 y}(r)\right)}{1728\alpha_{\theta_1}^2 \alpha_{\theta_2}^4 b_1(r)}$$
$$+ \frac{a^{12}\beta g_s M_{\rm UV}^2\left(\frac{1}{N}\right)^{17/20}\left(19683\sqrt{6}\alpha_{\theta_1}^6 + 6642\alpha_{\theta_2}^2\alpha_{\theta_1}^3 - 40\sqrt{6}\alpha_{\theta_2}^4\right)\log(r_0)}{18\sqrt{6}\pi^3 \log N r^{12}\alpha_{\theta_1}\alpha_{\theta_2}^7} = 0$$

$$EOM_{\theta_1 y} : \quad \frac{49 N^{7/20}\left(2f_{zz}(r) - 3f_{x^{10}x^{10}}(r) - 4f_{\theta_1 z}(r) - 5f_{\theta_2 y}(r)\right)}{128\sqrt{6}\alpha_{\theta_2}^3 b_1(r)} - \frac{11\sqrt{\frac{2}{3}}a^{12}\beta M_{\rm UV}\left(\frac{1}{N}\right)^{23/20}(-\Sigma_1)\log^2(r_0)}{3\pi^{3/2}\sqrt{g_s}\log N^2 N_f^{\rm UV} r^{12}\alpha_{\theta_1}^3 \alpha_{\theta_2}^3} = 0$$

$$EOM_{\theta_1 z} : \quad -\frac{49 N^{13/20}\left(2f_{zz}(r) - 3f_{x^{10}x^{10}}(r) - 4f_{\theta_1 z}(r) - 5f_{\theta_2 y}(r)\right)}{1152\alpha_{\theta_2}^4 b_1(r)} + \frac{a^{12}\beta g_s M_{\rm UV}^2\left(\frac{1}{N}\right)^{5/4}\alpha_{\theta_1}\Sigma_1 \log(r_0)}{12\sqrt{6}\pi^3 \log N r^{12}\alpha_{\theta_2}^7} = 0$$

$$EOM_{\theta_2\theta_2} : \quad f_{zz}(r) = f_{x^{10}x^{10}}(r)$$

$EOM_{\theta_2 x}$:
$$\frac{49 N^{23/20} b_2(r)\left(2 f_{zz}(r) - 3 f_{x^{10}x^{10}}(r) - 4 f_{\theta_1 z}(r) - 5 f_{\theta_2 y}(r)\right)}{1728 \alpha_{\theta_1}^2 \alpha_{\theta_2}^4 b_1(r)^2} + \frac{8 a^{14} \beta M_{\rm UV}\left(\frac{1}{N}\right)^{7/20} \Sigma_1 \log(r_0)}{243 \pi^{3/2} \sqrt{g_s} \log N^2 N_f^{\rm UV} r^{14} \alpha_{\theta_1}^4 \alpha_{\theta_2}^5} = 0$$

$EOM_{\theta_2 y}$:
$$-\frac{49 N^{17/20}\left(7\sqrt{2}\sqrt[4]{\pi}\sqrt[4]{g_s} f_{zz}(r) - 15\sqrt{2}\sqrt[4]{\pi}\sqrt[4]{g_s} f_{x^{10}x^{10}}(r) - 14\sqrt{2}\sqrt[4]{\pi}\sqrt[4]{g_s} f_{\theta_1 z}(r) + 5\sqrt{2}\sqrt[4]{\pi}\sqrt[4]{g_s} f_{\theta_2 y}(r)\right)}{15552 \alpha_{\theta_1}^4 \alpha_{\theta_2}^3 b_1(r)^2}$$
$$+\frac{8 a^{12} \beta\left(\frac{1}{N}\right)^{13/20}\left(19683\sqrt{6}\alpha_{\theta_1}^6 + 6642\alpha_{\theta_2}^2\alpha_{\theta_1}^3 - 40\sqrt{6}\alpha_{\theta_2}^4\right)}{243 g_s^2 \log N^2 N_f^{\rm UV} 2 r^{12} \alpha_{\theta_1}^6 \alpha_{\theta_2}} = 0$$

$EOM_{\theta_2 z}$:
$$-\frac{49 N^{3/4} b_2(r)\left(2 f_{zz}(r) - 3 f_{x^{10}x^{10}}(r) - 4 f_{\theta_1 z}(r) - 5 f_{\theta_2 y}(r)\right)}{1152 \alpha_{\theta_2}^4 b_1(r)^2} - \frac{4 a^{14} \beta M_{\rm UV}\left(\frac{1}{N}\right)^{3/4}(-\Sigma_1)\log(r)}{81 \pi^{3/2} \sqrt{g_s} \log N^2 N_f^{\rm UV} r^{14} \alpha_{\theta_1}^2 \alpha_{\theta_2}^5} = 0$$

EOM_{xx} :
$$\frac{144\sqrt{3}\sqrt[4]{\pi}a^{10}\beta\sqrt[4]{\frac{1}{N}}}{\log N^3 N_f^{\rm UV} 2 r^{10}} + \frac{g_s^{7/4} N^{9/10} \alpha_{\theta_2}^2 (\log N - 18\log r)^2\left(f_{zz}(r) - 2 f_{\theta_1 z}(r) + 2 f_{\theta_1 x}(r) - f_r(r)\right)}{\log r^2(\log N - 9\log r)^2} = 0$$

EOM_{xy} :
$$\frac{49 N^{6/5}\left(2 f_{zz}(r) - 3 f_{x^{10}x^{10}}(r) - 4 f_{\theta_1 z}(r) - 5 f_{\theta_2 y}(r)\right)}{3888\sqrt{6}\alpha_{\theta_1}^4 \alpha_{\theta_2}^5 b_1(r)^2} + \frac{8 a^{12} \beta\left(\frac{1}{N}\right)^{3/10}(-\Sigma_1)}{243\sqrt{3}\sqrt[4]{\pi}g_s^{9/4}\log N^3 N_f^{\rm UV} 2 r^{12} \alpha_{\theta_1}^6 \alpha_{\theta_2}^3} = 0$$

EOM_{xz} :
$$-\frac{49 N^{-3/2}\left(2 f_{zz}(r) - 3 f_{x^{10}x^{10}}(r) - 4 f_{\theta_1 z}(r) - 5 f_{\theta_2 y}(r)\right)}{34992 \alpha_{\theta_1}^4 \alpha_{\theta_2}^6 b_1(r)^2} = 0$$

EOM_{yy} :
$$\frac{16 a^{10} \beta\left(\frac{1}{N}\right)^{13/20}\left(-4 g_s N_f^{\rm UV}\log(\alpha_{\theta_1}) - 4 g_s N_f^{\rm UV}\log(\alpha_{\theta_2}) + 6 g_s N_f^{\rm UV}\log\left(\frac{1}{r_0}\right) + 4 g_s N_f^{\rm UV}\log(4) + 8\pi\right)}{27\sqrt{3}\sqrt[4]{\pi}g_s^{13/4}\log N^4 N_f^{\rm UV} 3 r^{10} \alpha_{\theta_1}^2 \alpha_{\theta_2}^4}$$
$$+\frac{49\sqrt{N}\left(16 f_{zz}(r) + 3 f_{x^{10}x^{10}}(r) + 4 f_{\theta_1 z}(r) + 5 f_{\theta_2 y}(r) - 36 f_{yz}(r) + 18 f_{yy}(r)\right)}{1728 \alpha_{\theta_1}^2 \alpha_{\theta_2}^4 b_1(r)^2}$$
$$+\frac{2\sqrt{2}a^{12}\beta(-\Sigma_1)}{27\sqrt[4]{\pi}g_s^{9/4}\log N^3 N N_f^{\rm UV} 2 r^{12} \alpha_{\theta_1}^4 \alpha_{\theta_2}^2} = 0$$

$$EOM_{yz} : -\frac{49N^{4/5}(2f_{zz}(r) - 3f_{x^{10}x^{10}}(r) - 4f_{\theta_1 z}(r) - 5f_{\theta_2 y}(r))}{2592\sqrt{6}\alpha_{\theta_1}^2 \alpha_{\theta_2}^5 b_1(r)^2} - \frac{8a^{12}\beta\left(\frac{1}{N}\right)^{21/20}\Sigma_1}{729g_s{}^2 \log N^3 N_f^{\mathrm{UV}}{}^2 r^{12}\alpha_{\theta_1}^4 \alpha_{\theta_2}} = 0$$

$$EOM_{zz} : \frac{49N^{11/10}(2f_{zz}(r) - 3f_{x^{10}x^{10}}(r) - 4f_{\theta_1 z}(r) - 5f_{\theta_2 y}(r))}{23328\alpha_{\theta_1}^2 \alpha_{\theta_2}^6 b_1(r)^2} - \frac{8\sqrt{2}a^{12}\beta\left(\frac{1}{N}\right)^{2/5}(-\Sigma_1)}{6561\sqrt[4]{\pi}g_s{}^{9/4} \log N^3 N_f^{\mathrm{UV}}{}^2 r^{12}\alpha_{\theta_1}^4 \alpha_{\theta_2}^4} = 0$$

$$EOM_{x^{10}x^{10}} : -\frac{49\sqrt{N}(16f_{zz}(r) + 3f_{x^{10}x^{10}}(r) - 32f_{\theta_1 z}(r) - 13f_{\theta_2 y}(r))}{1728\alpha_{\theta_1}^2 \alpha_{\theta_2}^4 a_1(r)^2 b_1(r)^2}$$
$$+ \frac{8\sqrt{\pi}a^{10}\beta(27\log N + 14)M_{\mathrm{UV}}\left(\frac{1}{N}\right)^{7/4}\Sigma_1 \log(r)}{81\sqrt{g_s} \log N^5 N_f^{\mathrm{UV}}{}^3 r^{10}\alpha_{\theta_2}^3} = 0$$

$$\tag{A.3}$$

where:

$$\Sigma_1 \equiv \left(-19683\sqrt{6}\alpha_{\theta_1}^6 - 6642\alpha_{\theta_2}^2 \alpha_{\theta_1}^3 + 40\sqrt{6}\alpha_{\theta_2}^4\right).$$

$$\tag{A.4}$$

From (A.3), one sees that the $\mathcal{O}(\beta)$-terms are further large-N-large-r-suppressed. Hence, in the UV one sees that one can consistently assume f_{MN} to be vanishingly small.

B Coupling constants $y_{1,3,5,7}$ and $z_{1,\ldots,8}$

In this appendix, we work out the IR and UV contributions to the coupling constants $y_{1,3,5,7}$ and $z_{1,2,\ldots,8}$. In the following, $\psi_0(Z) = \int_0^Z \phi_0(Z)dZ$. In the following, we will be splitting $\int dZ$ into $\int_{\mathrm{IR}} + \int_{\mathrm{UV}}$. Now, one can argue that $\int_{\mathrm{UV}} \sim \left(\mathcal{C}_{\phi_0}^{\mathrm{UV}}\right)^m \left(\mathcal{C}_{\psi_1}^{\mathrm{UV}}\right)^n, m, n \in \mathbb{Z}^+$, and as argued in section 4, we can self-consistently set $\mathcal{C}_{\phi_0}^{\mathrm{UV}} = \mathcal{C}_{\psi_1}^{\mathrm{UV}} = 0$, and one can hence argue that one can disregard $\int_{\mathrm{UV}}$. In the following calculations, one will need the value of $\mathcal{V}_2$ in the IR, which can be shown to be given by:

$$\mathcal{V}_2(Z \in \mathrm{IR}) = -\frac{3\left(3b^2 - 2\right) g_s{}^2 M N^{4/5} N_f{}^2 \log(r_0)(\log N - 3\log(r_0))}{2\pi \log N \alpha_{\theta_1} \alpha_{\theta_2}^2}$$

$$\frac{3\left(2 - 3b^2\right) \beta g_s{}^2 \log r_0 M N^{4/5} N_f{}^2 Z(\mathcal{C}_{zz}^{(1)} - 2\mathcal{C}_{\theta_1 z}^{(1)} + 2\mathcal{C}_{\theta_1 x}^{(1)})}{4\pi \alpha_{\theta_1} \alpha_{\theta_2}^2} \left(\frac{9b^2 g_s{}^2 M N^{4/5} N_f{}^2 \log(r_0)(\log N - 3\log(r_0))}{\pi \log N \alpha_{\theta_1} \alpha_{\theta_2}^2}\right).$$
(B.1)

Therefore, using (3.22) and using $b = \frac{1}{\sqrt{3}} + \epsilon$ [16] and setting $r_0 = N^{-\frac{f_{r_0}}{3}}$ [28], one obtains the following results for $y_{1,3,5,7}$:

- $y_1 = \int \mathcal{V}_2(Z)\left(1 + \psi_1(Z) - \psi_0^2(Z)\right)^2$

$$= \frac{177147\pi^6 \beta \mathcal{C}_{\phi_0}^{\mathrm{IR}}{}^4 \epsilon^6 f_{r_0}(f_{r_0} + 1)^3 \log r_0 \alpha_{\theta_1}^6 N^{4f_{r_0} + \frac{14}{5}} \left(\mathcal{C}_{zz}^{(1)} - 2\mathcal{C}_{\theta_1 z}^{(1)} + 2\mathcal{C}_{\theta_1 x}^{(1)}\right)}{8192(f_{r_0} - 1)^6 g_s{}^4 (\log N)^7 M^2 N_f^4}$$

$$- \frac{4782969\sqrt{3}\pi^7 \mathcal{C}_{\phi_0}^{\mathrm{IR}}{}^4 \epsilon^5 f_{r_0}(f_{r_0} + 1)^4 \alpha_{\theta_1}^9 N^{4f_{r_0} + \frac{11}{5}}}{20480(f_{r_0} - 1)^7 g_s{}^6 (\log N)^7 M^3 N_f^6}$$

- $y_3 = \int dZ \mathcal{V}_2 \psi_1^2(Z)\left(1 + \psi_1(Z)\right)^2$

$$= -\frac{21 g_s{}^2 \log r_0 M N^{4/5} N_f{}^2 \mathcal{C}_{\psi_1}^{\mathrm{IR}}{}^4 (\log N - 3\log r_0)}{8\pi \log N \alpha_{\theta_1} \alpha_{\theta_2}^2}$$

$$+ \frac{3^{9/8} 7^{3/4} \sqrt[4]{\epsilon} g_s{}^2 \log r_0 M N^{4/5} N_f{}^2 \mathcal{C}_{\psi_1}^{\mathrm{IR}}{}^3 (\log N - 3\log r_0)}{2\sqrt{2}\pi \log N \alpha_{\theta_1} \alpha_{\theta_2}^2}$$

$$+ 945\sqrt[3]{3}\sqrt[6]{\frac{\pi}{2}} \beta \epsilon g_s{}^{3/2} \left(\frac{1}{\log N}\right)^{2/3} \log r_0 M N^{3/10} N_f{}^{4/3} r_0{}^2 \mathcal{C}_{\psi_1}^{\mathrm{IR}}{}^4 (\mathcal{C}_{zz}^{(1)} - 2\mathcal{C}_{\theta_1 z}^{(1)} + 2\mathcal{C}_{\theta_1 x}^{(1)})$$

- $y_5 = \int dZ \mathcal{V}_2 \psi_0^2(Z)\left(\psi_1(Z)\right)^2 = \frac{6561\sqrt[4]{3}\sqrt{7}\pi^3 \mathcal{C}_{\phi_0}^{\mathrm{IR}}{}^2 \epsilon^{5/2} f_{r_0}(f_{r_0} + 1)^3 \alpha_{\theta_1}^5 \mathcal{C}_{\psi_1}^{\mathrm{IR}}{}^2 N^{2f_{r_0} + \frac{8}{5}}}{256(f_{r_0} - 1)^4 g_s{}^2 (\log N)^3 M N_f^2 \alpha_{\theta_2}^2}$

$$- \frac{63\sqrt[4]{3}\sqrt{7}\pi^3 \beta \mathcal{C}_{\phi_0}^{\mathrm{IR}}{}^2 \epsilon^{3/2} f_{r_0}(f_{r_0} + 1)\alpha_{\theta_1}^3 \mathcal{C}_{\psi_1}^{\mathrm{IR}}{}^2 N^{2f_{r_0} - \frac{3}{5}} \left(\mathcal{C}_{zz}^{(1)} - 2\mathcal{C}_{\theta_1 z}^{(1)} + 2\mathcal{C}_{\theta_1 x}^{(1)}\right)\left(6\epsilon N + f_{r_0}\gamma g_s \log N M^2\right)^2}{32768(f_{r_0} - 1)^3 g_s{}^2 (\log N)^3 M N_f^2}$$

- $y_7 = \int dZ \mathcal{V}_2 \psi_1(Z)(1 + \psi_1(Z))(1 + \psi_1(Z) - \psi_0(Z)^2)^2$

$$= \frac{4782969\sqrt[4]{3}\sqrt{7}\pi^7 \mathcal{C}_{\phi_0}^{\mathrm{IR}}{}^4 \epsilon^{9/2} f_{r_0}(f_{r_0} + 1)^4 \alpha_{\theta_1}^9 \mathcal{C}_{\psi_1}^{\mathrm{IR}}{}^2 N^{4f_{r_0} + \frac{11}{5}}}{40960(f_{r_0} - 1)^7 g_s{}^6 (\log N)^7 M^3 N_f^6}$$

$$- \frac{3720087\sqrt[4]{3}\sqrt{7}\pi^7 \beta \mathcal{C}_{\phi_0}^{\mathrm{IR}}{}^4 \epsilon^{9/2} f_{r_0}(f_{r_0} + 1)^3 \alpha_{\theta_1}^9 \mathcal{C}_{\psi_1}^{\mathrm{IR}}{}^2 N^{4f_{r_0} + \frac{11}{5}} \left(\mathcal{C}_{zz}^{(1)} - 2\mathcal{C}_{\theta_1 z}^{(1)} + 2\mathcal{C}_{\theta_1 x}^{(1)}\right)}{262144(f_{r_0} - 1)^7 g_s{}^6 (\log N)^7 M^3 N_f^6}.$$
(B.2)

Similarly, one obtains the following expressions for $z_{1,\ldots,8}$:

- $z_1 = \int dZ\,\mathcal{V}_2\,(1+\psi_1(Z))^2 = \dfrac{3\sqrt[4]{3}\sqrt{7}\sqrt{\epsilon}\,g_s^2 MN^{4/5}N_f^2 C_{\psi_1}^{\text{IR}\,2}\log(r_0)(\log N - 3\log(r_0))}{16\pi\alpha_{\theta_1}\alpha_{\theta_2}^2\log N}$

$\qquad +\dfrac{7\sqrt{7}\beta\sqrt{\epsilon}\,g_s^2\log r_0\,MN^{4/5}N_f^2 C_{\psi_1}^{\text{IR}\,2}\left(C_{zz}^{(1)}-2C_{\theta_1 z}^{(1)}+2C_{\theta_1 x}^{(1)}\right)\left(12\log N + \left(-36+6\log^2(3)+\log(9)\log(27)-\log(27)\log(81)\right)\log(r_0)\right)}{2048\cdot 3^{3/4}\pi\alpha_{\theta_1}\alpha_{\theta_2}^2(\log N - 3\log(r_0))}$

- $z_2 = \int dZ\,\mathcal{V}_2\,\psi_0^2(Z) = \dfrac{81\sqrt{3}\pi^3\beta C_{\phi_0}^{\text{IR}\,2}\epsilon^3(f_{r_0}+1)^2\alpha_{\theta_1}^3 N^{\frac{5f_{r_0}}{3}+\frac{7}{5}}\left(C_{zz}^{(1)}-2C_{\theta_1 z}^{(1)}+2C_{\theta_1 x}^{(1)}\right)}{128(f_{r_0}-1)^2 g_s(\log N)^2 MN_f^2}$

$\qquad -\dfrac{81\sqrt{3}\pi^3 C_{\phi_0}^{\text{IR}\,2}\epsilon^3 f_{r_0}(f_{r_0}+1)^2\alpha_{\theta_1}^3 N^{2f_{r_0}+\frac{7}{5}}}{256(f_{r_0}-1)^3 g_s^2(\log N)^3 MN_f^2}$

- $z_3 = \int dZ\,\mathcal{V}_2\,\psi_1(1+\psi_1)(Z) = \dfrac{3\sqrt[4]{3}\sqrt{7}\sqrt{\epsilon}\,g_s^2 MN^{4/5}N_f^2 C_{\psi_1}^{\text{IR}\,2}\log(r_0)(\log N - 3\log(r_0))}{8\pi\alpha_{\theta_1}\alpha_{\theta_2}^2\log N}$

$\qquad +\dfrac{7\sqrt[4]{3}\sqrt{7}\beta\sqrt{\epsilon}\,g_s^2 MN^{4/5}N_f^2 C_{\psi_1}^{\text{IR}\,2}\log(r_0)\left(C_{zz}^{(1)}-2C_{\theta_1 z}^{(1)}+2C_{\theta_1 x}^{(1)}\right)}{512\pi\alpha_{\theta_1}\alpha_{\theta_2}^2}$

- $z_4 = \int dZ\,\mathcal{V}_2\,\psi_1(1+\psi_1-\phi_0^2)(Z) = \dfrac{189\cdot 3^{3/8}\sqrt[4]{7}\pi^3\beta C_{\phi_0}^{\text{IR}\,2}\epsilon^{3/4}f_{r_0}(f_{r_0}+1)\alpha^3 C_{\psi_1}^{\text{IR}} N^{2f_{r_0}+\frac{7}{5}}\left(C_{zz}^{(1)}-2C_{\theta_1 z}^{(1)}+2C_{\theta_1 x}^{(1)}\right)}{16384\sqrt{2}(f_{r_0}-1)^3 g_s^2(\log N)^3 MN_f^2}$

$\qquad +\dfrac{81\cdot 3^{3/8}\sqrt[4]{7}\pi^3 C_{\phi_0}^{\text{IR}\,2}\epsilon^{11/4}f_{r_0}(f_{r_0}+1)^2\alpha_{\theta_1}^3 C_{\psi_1}^{\text{IR}} N^{2f_{r_0}+\frac{7}{5}}}{256\sqrt{2}(f_{r_0}-1)^3 g_s^2(\log N)^3 MN_f^2}$

- $z_5 = \int dZ\,\mathcal{V}_2\,\psi_1^2(1+\psi_1)(Z),\; = -\dfrac{3\sqrt[8]{3}7^{3/4}\sqrt[4]{\epsilon}\,g_s^2 MN^{4/5}N_f^2 C_{\psi_1}^{\text{IR}\,3}\log(r_0)\left(C_{zz}^{(1)}-2C_{\theta_1 z}^{(1)}+2C_{\theta_1 x}^{(1)}\right)}{4\sqrt{2}\pi\alpha_{\theta_1}\alpha_{\theta_2}^2\log N}$

$\qquad -\dfrac{21\sqrt[8]{3}7^{3/4}\beta\sqrt[4]{\epsilon}\,g_s^2 MN^{4/5}N_f^2 C_{\psi_1}^{\text{IR}\,3}\log(r_0)}{256\sqrt{2}\pi\alpha_{\theta_1}\alpha_{\theta_2}^2}$

- $z_6 = \int dZ\, \mathcal{V}_2 (1+\psi_1)(1+\psi_1-\psi_0^2)(Z) =$

$$\frac{567\ 3^{3/8}\sqrt[4]{7}\pi^3 \mathcal{C}_{\phi_0}^{\text{IR}}\ 2_\epsilon^{11/4} f_{r_0}(f_{r_0}+1)\alpha_{\theta_1}^3 \mathcal{C}_{\psi_1}^{\text{IR}}\ N^{2f_{r_0}+\frac{7}{5}}\left(\mathcal{C}_{zz}^{(1)}-2\mathcal{C}_{\theta_1 z}^{(1)}+2\mathcal{C}_{\theta_1 x}^{(1)}\right)}{8192\sqrt{2}(f_{r_0}-1)^3 g_s{}^2(\log N)^3 MN_f^2}$$

$$+\frac{81\ 3^{3/8}\sqrt[4]{7}\pi^3 \mathcal{C}_{\phi_0}^{\text{IR}}\ 2_\epsilon^{11/4} f_{r_0}(f_{r_0}+1)^2\alpha_{\theta_1}^3 \mathcal{C}_{\psi_1}^{\text{IR}}\ N^{2f_{r_0}+\frac{7}{5}}}{64\sqrt{2}(f_{r_0}-1)^3 g_s{}^2(\log N)^3 MN_f^2}$$

$$=\frac{567\ 3^{3/8}\sqrt[4]{7}\pi^3 \mathcal{C}_{\phi_0}^{\text{IR}}\ 2_\epsilon^{11/4} f_{r_0}(f_{r_0}+1)\alpha_{\theta_1}^3 \mathcal{C}_{\psi_1}^{\text{IR}}\ N^{2f_{r_0}+\frac{7}{5}}\left(\mathcal{C}_{zz}^{(1)}-2\mathcal{C}_{\theta_1 z}^{(1)}+2\mathcal{C}_{\theta_1 x}^{(1)}\right)}{8192\sqrt{2}(f_{r_0}-1)^3 g_s{}^2(\log N)^3 MN_f^2}$$

$$+\frac{81\ 3^{3/8}\sqrt[4]{7}\pi^3 \mathcal{C}_{\phi_0}^{\text{IR}}\ 2_\epsilon^{11/4} f_{r_0}(f_{r_0}+1)^2\alpha_{\theta_1}^3 \mathcal{C}_{\psi_1}^{\text{IR}}\ N^{2f_{r_0}+\frac{7}{5}}}{64\sqrt{2}(f_{r_0}-1)^3 g_s{}^2(\log N)^3 MN_f^2}$$

- $z_7 = \int dZ\, \mathcal{V}_2 \psi_1(1+\psi_1)^2(Z) =$

$$\frac{3\sqrt[8]{3}7^{3/4}\sqrt[4]{\epsilon}g_s^2 MN^{4/5}N_f\ \mathcal{C}_{\psi_1}^{\text{IR}}{}^3 \log(r_0)(\log N-3\log(r_0))}{4\sqrt{2}\pi\alpha_{\theta_1}\alpha_{\theta_2}^2 \log N}$$

$$-\frac{21\sqrt[8]{3}7^{3/4}\beta\sqrt[4]{\epsilon}g_s^2 MN^{4/5}N_f{}^2\mathcal{C}_{\psi_1}^{\text{IR}}{}^3 \log(r_0)\left(\mathcal{C}_{zz}^{(1)}-2\mathcal{C}_{\theta_1 z}^{(1)}+2\mathcal{C}_{\theta_1 x}^{(1)}\right)}{256\sqrt{2}\pi\alpha_{\theta_1}\alpha_{\theta_2}^2}$$

$$+\frac{48\ 3^{3/4}\sqrt{\frac{1}{\epsilon}}g_s^2 MN^{4/5}N_f{}^2 r_0 \log N \log^2(r_0)\mathcal{C}_{\psi_1}^{\text{UV}}}{\pi\log N\alpha_{\theta_1}\alpha_{\theta_2}^2}$$

- $z_8 = \int dZ\, \mathcal{V}_2 \psi_0^2\psi_1(Z) =$

$$\frac{567\ 3^{3/8}\sqrt[4]{7}\pi^3\beta\mathcal{C}_{\phi_0}^{\text{IR}}\ 2_\epsilon^{11/4} f_{r_0}(f_{r_0}+1)\alpha_{\theta_1}^3 \mathcal{C}_{\psi_1}^{\text{IR}}\ N^{2f_{r_0}+\frac{7}{5}}\left(\mathcal{C}_{zz}^{(1)}-2\mathcal{C}_{\theta_1 z}^{(1)}+2\mathcal{C}_{\theta_1 x}^{(1)}\right)}{8192\sqrt{2}(f_{r_0}-1)^3 g_s{}^2(\log N)^3 MN_f^2}$$

$$+\frac{81\ 3^{3/8}\sqrt[4]{7}\pi^3 \mathcal{C}_{\phi_0}^{\text{IR}}\ 2_\epsilon^{11/4} f_{r_0}(f_{r_0}+1)^2\alpha_{\theta_1}^3 \mathcal{C}_{\psi_1}^{\text{IR}}\ N^{2f_{r_0}+\frac{7}{5}}}{64\sqrt{2}(f_{r_0}-1)^3 g_s{}^2(\log N)^3 MN_f^2}.$$

$$\text{(B.3)}$$

C $G_{\text{global}} \times H_{\text{local}}$ HLS formalism and obtaining GL's mesonic χPT Lagrangian up to $\mathcal{O}(p^4)$ after integrating out the vector mesons from the HLS Lagrangian [36]

In this appendix, we summarize the HLS formalism and the arguments of how to obtain the SU(3) χPT Lagrangian of [5] up to (NLO in momentum below the chiral symmetry breaking scale) $\mathcal{O}(p^4)$ by integrating out the vector mesons from the HLS Lagrangian (both as discussed in detail in [36])

C.1 HLS formalism

The HLS formalism describes a model based on $G_{\text{global}} \times H_{\text{local}}$ symmetry, where $G = \text{SU}(N_f)_{\text{L}} \times \text{SU}(N_f)_{\text{R}}$ is the global chiral symmetry and $H = \text{SU}(N_f)_{\text{V}}$ is the H(idden) L(ocal) S(ymmetry). The building blocks of this model are $\text{SU}(N_f)$-matrix valued variables ξ_{L} and ξ_{R} which are introduced by splitting U in the ChPT as

$$U = \xi_{\text{L}}^\dagger \xi_{\text{R}} \,.$$

Now, under $G_{\text{global}} \times H_{\text{local}}$, $\xi_{\text{L,R}}(x)$ transform as follows:

$$\xi_{\text{L,R}}(x) \rightarrow \xi_{\text{L,R}}'(x) = h(x) \cdot \xi_{\text{L,R}}(x) \cdot g_{\text{L,R}}^\dagger \,,$$

where $h(x) \in H_{\text{local}}, g_{\text{L,R}} \in G_{\text{global}}$. These variables are parameterized as

$$\xi_{\text{L,R}} = e^{i\sigma/F_\sigma} e^{\mp i\pi/F_\pi} \,, \quad [\, \pi = \pi^a T_a \,, \ \sigma = \sigma^a T_a \,] \,,$$

where the matrix-valued π denotes the Nambu-Goldstone (NG) bosons associated with the spontaneous breaking of G chiral symmetry and σ denotes the NG bosons absorbed into the gauge bosons. Further, F_π and F_σ denote the relevant decay constants, and the parameter a is defined via: $a \equiv \frac{F_\sigma^2}{F_\pi^2}$. From ξ_{L} and ξ_{R} the following are constructed:

$$\alpha_{\perp\mu} = \left(\partial_\mu \xi_{\text{R}} \cdot \xi_{\text{R}}^\dagger - \partial_\mu \xi_{\text{L}} \cdot \xi_{\text{L}}^\dagger \right)/(2i) \,,$$

$$\alpha_{\parallel\mu} = \left(\partial_\mu \xi_{\text{R}} \cdot \xi_{\text{R}}^\dagger + \partial_\mu \xi_{\text{L}} \cdot \xi_{\text{L}}^\dagger \right)/(2i) \,,$$

which transform under H_{local} as

$$\alpha_{\perp\mu} \rightarrow h(x) \cdot \alpha_{\perp\mu} \cdot h^\dagger(x) \,,$$
$$\alpha_{\parallel\mu} \rightarrow h(x) \cdot \alpha_{\perp\mu} \cdot h^\dagger(x) - i\partial_\mu h(x) \cdot h^\dagger(x) \,.$$

The covariant derivatives of ξ_{L} and ξ_{R} can be obtained from the transformation properties of $\xi_{\text{L,R}}(x)$ as:

$$D_\mu \xi_{\text{L,R}} = \partial_\mu \xi_{\text{L,R}} - iV_\mu \xi_{\text{L,R}} \,,$$

where $V_\mu = V_\mu^a T_a$ are the gauge fields corresponding to H_{local}. Now, V_μ transforms under H_{local} as:

$$V_\mu \rightarrow h(x) \cdot V_\mu \cdot h^\dagger(x) - i\partial_\mu h(x) \cdot h^\dagger(x) \,.$$

The "covariantized 1-forms" defined as

$$\widehat{\alpha}_{\perp\mu} = \frac{1}{2i}\left(D_\mu\xi_R \cdot \xi_R^\dagger - D_\mu\xi_L \cdot \xi_L^\dagger\right) = \alpha_{\perp\mu}\,,$$

$$\widehat{\alpha}_{\|\mu} = \frac{1}{2i}\left(D_\mu\xi_R \cdot \xi_R^\dagger + D_\mu\xi_L \cdot \xi_L^\dagger\right) = \alpha_{\|\mu} - V_\mu\,,$$

transform homogeneously under H_{local}:

$$\widehat{\alpha}_{\perp,\|}^\mu \to h(x) \cdot \widehat{\alpha}_{\perp,\|}^\mu \cdot h^\dagger(x)\,.$$

Thus, one can construct the following two invariants:

$$\mathcal{L}_A \equiv F_\pi^2\,\text{tr}\left[\widehat{\alpha}_{\perp\mu}\widehat{\alpha}_\perp^\mu\right]\,,$$

$$a\mathcal{L}_V \equiv F_\sigma^2\,\text{tr}\left[\widehat{\alpha}_{\|\mu}\widehat{\alpha}_\|^\mu\right] = F_\sigma^2\,\text{tr}\left[\left(V_\mu - \alpha_{\|\mu}\right)^2\right]\,.$$

Therefore, the most general Lagrangian made out of $\xi_{L,R}$ and $D_\mu\xi_{L,R}$ with the lowest number of derivatives, i.e., at $\mathcal{O}(p^2)$ (see [4, 5, 36] for power counting arguments), is given by:

$$\mathcal{L} = \mathcal{L}_A + a\mathcal{L}_V\,.$$

Using the EOM for V_μ at $\mathcal{O}(p^2)$, one obtains:

$$V_\mu = \alpha_{\|\mu}\,.$$

Further, with the relation

$$\widehat{\alpha}_{\perp\mu} = \frac{1}{2i}\,\xi_L \cdot \partial_\mu U \cdot \xi_R^\dagger = \frac{i}{2}\,\xi_R \cdot \partial_\mu U^\dagger \cdot \xi_L^\dagger$$

substituted into (C.1), one obtains the following $\mathcal{O}(p^2)$ term in the χPT Lagrangian:

$$\mathcal{L} = \mathcal{L}_A = \frac{F_\pi^2}{4}\text{tr}\left[\partial_\mu U^\dagger \partial^\mu U\right]\,.$$

In the unitary gauge, $\sigma = 0$,[8] two $SU(N_f)$-matrix valued variables ξ_L and ξ_R are related via:

$$\xi_L^\dagger = \xi_R \equiv \xi = e^{i\pi/F_\pi}\,.$$

[8] This unitary gauge is not preserved though under the G_{global} transformation, which in general has the following form

$$\begin{aligned}
\xi \to \xi' &= \xi \cdot g_R^\dagger = g_L \cdot \xi \\
&= \exp\left[i\sigma'(\pi, g_R, g_L)/F_\sigma\right]\exp\left[i\pi'/F_\pi\right] \\
&= \exp\left[i\pi'/F_\pi\right]\exp\left[-i\sigma'(\pi, g_R, g_L)/F_\sigma\right]\,.
\end{aligned}$$

However, $\exp\left[i\sigma'(\pi, g_R, g_L)/F_\sigma\right]$ can be eliminated if we simultaneously perform the H_{local} gauge transformation:

$$h = \exp\left[i\sigma'(\pi, g_R, g_L)/F_\sigma\right] \equiv h\left(\pi, g_R, g_L\right)\,.$$

Therefore there is a global symmetry $G = SU(N_f)_L \times SU(N_f)_R$ under the following combined transformation:

$$G:\ \xi \to h\left(\pi, g_R, g_L\right) \cdot \xi \cdot g_R^\dagger = g_L \cdot \xi \cdot h^\dagger\left(\pi, g_R, g_L\right)\,.$$

C.2 Obtaining the $\mathcal{O}(p^2)$ SU(3) χPT Lagrangian by integrating out the vector mesons from the HLS Lagrangian

First, we introduce the external gauge fields $\mathcal{L}_\mu$ and $\mathcal{R}_\mu$ which include W boson, Z-boson and photon fields. This is done by gauging the G_{global} symmetry. The transformation properties of $\mathcal{L}_\mu$ and $\mathcal{R}_\mu$, $\mathcal{L}_\mu \to g_L \mathcal{L}_\mu g_L^\dagger - i\partial_\mu g_L . g_L^\dagger$ and $\mathcal{R}_\mu \to g_R \mathcal{R}_\mu g_R^\dagger - i\partial_\mu g_R . g_R^\dagger$ are used to define covariant derivatives of $\xi_{\text{L,R}}$:

$$
\begin{aligned}
D_\mu \xi_{\text{L}} &= \partial_\mu \xi_{\text{L}} - iV_\mu \xi_{\text{L}} + i\xi_{\text{L}} \mathcal{L}_\mu \,, \\
D_\mu \xi_{\text{R}} &= \partial_\mu \xi_{\text{R}} - iV_\mu \xi_{\text{R}} + i\xi_{\text{R}} \mathcal{R}_\mu \,.
\end{aligned}
\tag{C.1}
$$

It should be noticed that in the HLS these external gauge fields are included without assuming the vector dominance. It is outstanding feature of the HLS model that $\xi_{\text{L,R}}$ have two independent source charges and hence two independent gauge bosons are automatically introduced in the HLS model. Both the vector meson fields and external gauge fields are simultaneously incorporated into the Lagrangian fully consistent with the chiral symmetry. By using the above covariant derivatives two Maurer-Cartan 1-forms are constructed as

$$
\begin{aligned}
\widehat{\alpha}_{\perp\mu} &= \left(D_\mu \xi_{\text{R}} \cdot \xi_{\text{R}}^\dagger - D_\mu \xi_{\text{L}} \cdot \xi_{\text{L}}^\dagger \right) / (2i) \,, \\
\widehat{\alpha}_{\parallel\mu} &= \left(D_\mu \xi_{\text{R}} \cdot \xi_{\text{R}}^\dagger + D_\mu \xi_{\text{L}} \cdot \xi_{\text{L}}^\dagger \right) / (2i) \,.
\end{aligned}
\tag{C.2}
$$

These 1-forms upon expansion in a power series in π yield:

$$
\widehat{\alpha}_{\perp\mu} = \frac{1}{F_\pi} \partial_\mu \pi + \mathcal{A}_\mu - \frac{i}{F_\pi} [\mathcal{V}_\mu , \pi] - \frac{1}{6F_\pi^3} \Big[[\partial_\mu \pi , \pi] , \pi \Big] + \cdots \,,
\tag{C.3}
$$

$$
\widehat{\alpha}_{\parallel\mu} = \frac{1}{F_\sigma} \partial_\mu \sigma - V_\mu + \mathcal{V}_\mu - \frac{i}{2F_\pi^2} [\partial_\mu \pi , \pi] - \frac{i}{F_\pi} [\mathcal{A}_\mu , \pi] + \cdots \,,
\tag{C.4}
$$

where $\mathcal{V}_\mu = (\mathcal{R}_\mu + \mathcal{L}_\mu)/2$ and $\mathcal{A}_\mu = (\mathcal{R}_\mu - \mathcal{L}_\mu)/2$.

The covariantized 1-forms in eqs. (C.2) transform homogeneously:

$$
\widehat{\alpha}_{\parallel,\perp}^\mu \to h(x) \cdot \widehat{\alpha}_{\parallel,\perp}^\mu \cdot h^\dagger(x) \,.
\tag{C.5}
$$

Then we can construct two independent terms with lowest derivatives which are invariant under the full $G_{\text{global}} \times H_{\text{local}}$ symmetry as

$$
\mathcal{L}_{\text{A}} \equiv F_\pi^2 \operatorname{tr} [\hat{\alpha}_{\perp\mu} \hat{\alpha}_\perp^\mu] = \operatorname{tr} [\partial_\mu \pi \partial^\mu \pi] + \cdot \,,
\tag{C.6}
$$

$$
a\mathcal{L}_{\text{V}} \equiv F_\sigma^2 \operatorname{tr} \Big[\hat{\alpha}_{\parallel\mu} \hat{\alpha}_\parallel^\mu \Big] = \operatorname{tr} \Big[(\partial_\mu \sigma - F_\sigma V_\mu)(\partial^\mu \sigma - F_\sigma V^\mu) \Big] + \cdots \,,
\tag{C.7}
$$

where the expansions of the covariantized 1-forms in (C.3) and (C.4) were substituted to obtain the second expressions. These expansions imply that $\mathcal{L}_A$ generates the kinetic term of pseudoscalar meson, while $\mathcal{L}_V$ generates the kinetic term of the would-be NG boson σ in addition to the mass term of the vector meson.

The HLS gauge boson field strength defined by $V_{\mu\nu} \equiv \partial_\mu V_\nu - \partial_\nu V_\mu - i[V_\mu, V_\nu]$, which also transforms homogeneously:

$$
V_{\mu\nu} \to h(x) \cdot V_{\mu\nu} \cdot h^\dagger(x) \,,
\tag{C.8}
$$

is another building block of the χPT Lagrangian. Then the simplest term with $V_{\mu\nu}$ is the kinetic term of the gauge boson:

$$\mathcal{L}_{\text{kin}}(V_\mu) = -\frac{1}{2g^2}\,\text{tr}\left[V_{\mu\nu}V^{\mu\nu}\right], \tag{C.9}$$

where g is the HLS gauge coupling constant.

Now the Lagrangian with lowest derivatives is given by:

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_{\text{A}} + a\mathcal{L}_{\text{V}} + \mathcal{L}_{\text{kin}}(V_\mu) \\
&= F_\pi^2\,\text{tr}\left[\hat{\alpha}_{\perp\mu}\hat{\alpha}_\perp^\mu\right] + F_\sigma^2\,\text{tr}\left[\hat{\alpha}_{\parallel\mu}\hat{\alpha}_\parallel^\mu\right] - \frac{1}{2g^2}\,\text{tr}\left[V_{\mu\nu}V^{\mu\nu}\right].
\end{aligned} \tag{C.10}$$

C.3   Obtaining the $\mathcal{O}(p^4)$ SU(3) χPT Lagrangian by integrating out the vector mesons from the HLS Lagrangian

Integrating out the vector mesons in the Lagrangian of the HLS given in (C.10) we obtain the Lagrangian for pseudoscalar mesons. The resultant Lagrangian includes $\mathcal{O}(p^4)$ terms of the ChPT in addition to $\mathcal{O}(p^2)$ terms. To perform this it is convenient to introduce the following quantities:

$$\begin{aligned}
\alpha_{\perp\mu} &= \left(\mathcal{D}_\mu\xi_{\text{R}}\cdot\xi_{\text{R}}^\dagger - \mathcal{D}_\mu\xi_{\text{L}}\cdot\xi_{\text{L}}^\dagger\right)/(2i)\,, \\
\alpha_{\parallel\mu} &= \left(\mathcal{D}_\mu\xi_{\text{R}}\cdot\xi_{\text{R}}^\dagger + \mathcal{D}_\mu\xi_{\text{L}}\cdot\xi_{\text{L}}^\dagger\right)/(2i)\,,
\end{aligned} \tag{C.11}$$

where $\mathcal{D}_\mu\xi_{\text{L}}$ and $\mathcal{D}_\mu\xi_{\text{L}}$ are defined by

$$\begin{aligned}
\mathcal{D}_\mu\xi_{\text{L}} &= \partial_\mu\xi_{\text{L}} + i\xi_{\text{L}}\mathcal{L}_\mu\,, \\
\mathcal{D}_\mu\xi_{\text{R}} &= \partial_\mu\xi_{\text{R}} + i\xi_{\text{R}}\mathcal{R}_\mu\,.
\end{aligned} \tag{C.12}$$

The relations of these $\alpha_{\perp\mu}$ and $\alpha_{\parallel\mu}$ with $\hat{\alpha}_{\perp\mu}$ and $\hat{\alpha}_{\parallel\mu}$ in (C.2) are given by

$$\begin{aligned}
\hat{\alpha}_{\perp\mu} &= \alpha_{\perp\mu}\,, \\
\hat{\alpha}_{\parallel\mu} &= \alpha_{\parallel\mu} - V_\mu\,.
\end{aligned} \tag{C.13}$$

From the Lagrangian in (C.10) the equation of motion for the vector meson is given by

$$F_\sigma^2\left(V_\mu - \alpha_{\parallel\mu}\right) - \frac{1}{g^2}\left(\partial^\nu V_{\mu\nu} - i\left[V^\nu, V_{\mu\nu}\right]\right) = 0\,. \tag{C.14}$$

In the leading order of the derivative expansion the solution of (C.14) is given by

$$V_\mu = \alpha_{\parallel\mu} + \frac{1}{M_\rho^2}\mathcal{O}(p^3)\,, \tag{C.15}$$

consistent with (C.1). Substituting this into the field strength of the HLS gauge boson and performing the derivative expansion one can show that one obtains [36]:

$$V_{\mu\nu} = \xi_{\text{R}}\left(\mathcal{R}_{\mu\nu} + U^\dagger\mathcal{L}_{\mu\nu}U + \frac{i}{4}\nabla_\mu U^\dagger\cdot\nabla_\nu U - \frac{i}{4}\nabla_\nu U^\dagger\cdot\nabla_\mu U\right)\xi_{\text{R}}^\dagger + \frac{1}{M_\rho^2}\mathcal{O}(p^4)\,, \tag{C.16}$$

where,

$$\hat{\alpha}_{\perp\mu} = \frac{i}{2}\xi_{\mathrm{L}} \cdot \nabla_\mu U \cdot \xi_{\mathrm{R}}^\dagger = \frac{1}{2i}\xi_{\mathrm{R}} \cdot \nabla_\mu U^\dagger \cdot \xi_{\mathrm{L}}^\dagger, \tag{C.17}$$

has been used. By substituting (C.17) into the HLS Lagrangian, the first term in the HLS Lagrangian (C.10) becomes the first term in the leading order ChPT Lagrangian:

$$\mathcal{L}_{(2)}^{\mathrm{ChPT}}\Big|_{\chi=0} = \frac{F_\pi^2}{4}\mathrm{tr}\left[\nabla_\mu U^\dagger \nabla^\mu U\right]. \tag{C.18}$$

In addition, the second term in (C.10) with (C.14) substituted becomes of $\mathcal{O}(p^6)$ in the ChPT and the third term (the kinetic term of the HLS gauge boson) with (C.16) becomes of $\mathcal{O}(p^4)$ in the ChPT:

$$\begin{aligned}
\mathcal{L}_4^V = {}& \frac{1}{32g^2}\left(\mathrm{tr}\left[\nabla_\mu U \nabla^\mu U^\dagger\right]\right)^2 + \frac{1}{16g^2}\mathrm{tr}\left[\nabla_\mu U \nabla_\nu U^\dagger\right]\mathrm{tr}\left[\nabla^\mu U \nabla^\nu U^\dagger\right] \\
&- \frac{3}{16g^2}\mathrm{tr}\left[\nabla_\mu U \nabla^\mu U^\dagger \nabla_\nu U \nabla^\nu U^\dagger\right] - \frac{i}{4g^2}\mathrm{tr}\left[\mathcal{L}_{\mu\nu}\nabla^\mu U \nabla^\nu U^\dagger + \mathcal{R}_{\mu\nu}\nabla^\mu U^\dagger \nabla^\nu U\right] \\
&- \frac{1}{4g^2}\mathrm{tr}\left[\mathcal{L}_{\mu\nu}U\mathcal{R}^{\mu\nu}U^\dagger\right] - \frac{1}{8g^2}\left[\mathcal{L}_{\mu\nu}\mathcal{L}^{\mu\nu} + \mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}\right],
\end{aligned} \tag{C.19}$$

where we fixed $N_f = 3$. Comparing this with the $\mathcal{O}(p^4)$ terms of the ChPT Lagrangian given in (4.22), we obtain the contributions of vector mesons to the low-energy parameters of the ChPT:

$$\begin{aligned}
L_1^V &= \frac{1}{32g^2}, & L_2^V &= \frac{1}{16g^2}, & L_3^V &= -\frac{3}{16g^2}, \\
L_9^V &= \frac{1}{4g^2}, & L_{10}^V &= -\frac{1}{4g^2}.
\end{aligned} \tag{C.20}$$

D   Inclusion of $\mathcal{O}(R^4)$ corrections in the $D6$-brane DBI action

Inclusive of the $\mathcal{O}(R^4)$-corrections indicated by a ˜ (e.g., the $\mathcal{M}$-theory metric: $\tilde{G}_{MN}^{\mathcal{M}} = G_{MN}^{\mathrm{MQGP}}(1 + f_{MN})$ [16]) in (D.1), one sees that

$$\begin{aligned}
\mathcal{V}_1 &= 2\sqrt{h}e^{-\tilde{\phi}_{\mathrm{IIA}}}\sqrt{-\tilde{g}_{7\times7}^{\mathrm{IIA}}}\tilde{g}_{\mathrm{IIA}}^{ZZ}, \\
\mathcal{V}_2 &= he^{-\tilde{\phi}_{\mathrm{IIA}}}\sqrt{-\tilde{g}_{7\times7}^{\mathrm{IIA}}},
\end{aligned} \tag{D.1}$$

wherein using (4.4):

$$\begin{aligned}
e^{-\tilde{\phi}_{\mathrm{IIA}}} \mathcal{O}(R^4) &= \tilde{G}_{x^{10}x^{10}}^{\mathcal{M}}{}^{-\frac{3}{4}} = \left(G_{x^{10}x^{10}}^{\mathcal{M}} + \mathcal{F}_{x^{10}x^{10}}\right)^{-\frac{3}{4}} \equiv e^{-\phi_{\mathrm{IIA}}} + \mathcal{E}_{x^{10}x^{10}}\mathcal{F}_{x^{10}x^{10}}, \\
\mathcal{E}_{x^{10}x^{10}} &= -\frac{3}{4}\frac{1}{\left(G_{x^{10}x^{10}}^{\mathcal{M}}\right)^{\frac{7}{4}}} \\
\tilde{G}_{\mathcal{M}}^{ZZ} &= \frac{e^{-2Z}}{\sqrt{\tilde{G}_{x^{10}x^{10}}^{\mathcal{M}}}G_{\mathcal{M}}^{ZZ}r_0^2} - \frac{e^{-2Z}(\mathcal{F}_{x^{10}x^{10}}G_{\mathcal{M}}^{ZZ} + 2\mathcal{F}_{ZZ}\tilde{G}_{x^{10}x^{10}}^{\mathcal{M}})}{2\tilde{G}_{x^{10}x^{10}}^{\mathcal{M}}{}^{3/2}G_{\mathcal{M}}^{rr}{}^2 r_0^2}, \\
\sqrt{-\tilde{g}_{7\times7}} &= \sqrt{-g_{7\times7}} + \mathrm{Tr}(\mathcal{C}\mathcal{F}).
\end{aligned} \tag{D.2}$$

In (D.2),

$$\mathcal{F}_{MN} \equiv f_{MN} G_{MN}^{\mathcal{M}};$$

$$\mathcal{C}_{MN} \equiv \delta_M^t \delta_N^t \mathcal{C}_{tt} + \delta_M^{x^1} \delta_N^{x^1} \mathcal{C}_{x^1 x^1} + \delta_M^{x^2} \delta_N^{x^2} \mathcal{C}_{x^2 x^2} + \delta_M^{x^3} \delta_N^{x^3} \mathcal{C}_{x^3 x^3} + \delta_M^{r_0} \delta_N^{r_0} \mathcal{C}_{rr} + \delta_M^{\theta_2} \delta_N^{\theta_2} \mathcal{C}_{\theta_2 \theta_2} + \delta_M^{\theta_2} \delta_N^{\phi_2} \mathcal{C}_{\theta_2 y} + \delta_M^{\phi_2} \delta_N^{\phi_2} \mathcal{C}_{yy} + \delta_M^{x^{11}} \delta_N^{x^{11}} \mathcal{C}_{x^{10} x^{10}},$$

$$\mathcal{C}_{tt} = -\frac{\sqrt{-g} \left(-4 \mathcal{A}_{\theta_2 \theta_2}^{\mathcal{M}} G_{x^1 x^1}^{\mathcal{M}} G_{x^{10} x^{10}}^{\mathcal{M}} {}^3 G_{x^2 x^2}^{\mathcal{M}} G_{x^3 x^3}^{\mathcal{M}} G_{yy}^{\mathcal{M}} G_{rr}^{\mathcal{M}} - 4 B_{r\theta_2}^2 G_{x^1 x^1}^{\mathcal{M}} G_{x^{10} x^{10}}^{\mathcal{M}} {}^2 G_{x^2 x^2}^{\mathcal{M}} G_{x^3 x^3}^{\mathcal{M}} G_{rr}^{\mathcal{M}} {}^2 G_{x^2 x^2}^{\mathcal{M}} G_{x^3 x^3}^{\mathcal{M}} G_{rr}^{\mathcal{M}} G_{yy}^{\mathcal{M}} G_{tt}^{\mathcal{M}} + 4 G_{x^{10} x^{10}}^{\mathcal{M}} {}^2 G_{x^2 x^2}^{\mathcal{M}} G_{x^3 x^3}^{\mathcal{M}} G_{rr}^{\mathcal{M}} G_{\theta_2 y}^{\mathcal{M}} {}^2 G_{tt}^{\mathcal{M}} \right)}{\Delta}$$

$$= -\frac{2 \sqrt[6]{\pi} g_s {}^{3/2} M N N^{3/10} N_f \log(r_0 e^Z) \left(72 a^2 r_0 e^Z \log(r_0 e^Z) - 3 a^2 + 2 r_0{}^2 e^{2Z} \right)}{3^{2/3} \log N \left(1 - e^{-4Z} \right) \alpha_{\theta_1} \alpha_{\theta_2}^2 \mathcal{G}^{2/3}}$$

$$\mathcal{C}_{x^1 x^1} = -\frac{\sqrt{-g} \left(-4 \mathcal{A}_{\theta_2 \theta_2}^{\mathcal{M}} G_{x^{10} x^{10}}^{\mathcal{M}} {}^3 G_{x^2 x^2}^{\mathcal{M}} G_{x^3 x^3}^{\mathcal{M}} G_{rr}^{\mathcal{M}} G_{yy}^{\mathcal{M}} G_{tt}^{\mathcal{M}} - 4 B_{r\theta_2}^2 G_{x^{10} x^{10}}^{\mathcal{M}} {}^2 G_{x^2 x^2}^{\mathcal{M}} G_{x^3 x^3}^{\mathcal{M}} G_{rr}^{\mathcal{M}} G_{yy}^{\mathcal{M}} G_{tt}^{\mathcal{M}} + 4 G_{x^{10} x^{10}}^{\mathcal{M}} {}^2 G_{x^2 x^2}^{\mathcal{M}} G_{x^3 x^3}^{\mathcal{M}} G_{rr}^{\mathcal{M}} G_{\theta_2 y}^{\mathcal{M}} {}^2 G_{tt}^{\mathcal{M}} \right)}{\Delta}$$

$$= -\frac{2 \sqrt[6]{\pi} g_s {}^{3/2} M N N^{3/10} N_f \log(r_0 e^Z) \left(72 a^2 r_0 e^Z \log(r_0 e^Z) - 3 a^2 + 2 r_0{}^2 e^{2Z} \right)}{3^{2/3} \log N \alpha_{\theta_1} \alpha_{\theta_2}^2 \mathcal{G}^{2/3}},$$

$$\mathcal{C}_{x^2 x^2} = \mathcal{C}_{x^3 x^3} = -\frac{\sqrt{-g} \left(-4 \mathcal{A}_{\theta_2 \theta_2}^{\mathcal{M}} G_{x^1 x^1}^{\mathcal{M}} G_{x^{10} x^{10}}^{\mathcal{M}} {}^3 G_{x^3 x^3}^{\mathcal{M}} G_{rr}^{\mathcal{M}} G_{yy}^{\mathcal{M}} G_{tt}^{\mathcal{M}} - 4 B_{r\theta_2}^2 G_{x^1 x^1}^{\mathcal{M}} G_{x^{10} x^{10}}^{\mathcal{M}} {}^2 G_{x^3 x^3}^{\mathcal{M}} G_{rr}^{\mathcal{M}} G_{yy}^{\mathcal{M}} G_{tt}^{\mathcal{M}} + 4 G_{x^1 x^1}^{\mathcal{M}} G_{x^{10} x^{10}}^{\mathcal{M}} {}^2 G_{x^3 x^3}^{\mathcal{M}} G_{rr}^{\mathcal{M}} G_{\theta_2 y}^{\mathcal{M}} {}^2 G_{tt}^{\mathcal{M}} \right)}{\Delta}$$

$$= -\frac{2 \sqrt[6]{\pi} g_s {}^{3/2} M N^{3/10} N_f \log(r_0 e^Z) \left(72 a^2 r_0 e^Z \log(r_0 e^Z) - 3 a^2 + 2 r_0{}^2 e^{2Z} \right)}{3^{2/3} \log N \alpha_{\theta_1} \alpha_{\theta_2}^2 \mathcal{G}^{2/3}},$$

$$\mathcal{C}_{rr} = -\frac{\sqrt{-g} \left(-4 \mathcal{A}_{\theta_2 \theta_2}^{\mathcal{M}} G_{x^1 x^1}^{\mathcal{M}} G_{x^{10} x^{10}}^{\mathcal{M}} {}^3 G_{x^2 x^2}^{\mathcal{M}} G_{x^3 x^3}^{\mathcal{M}} G_{tt}^{\mathcal{M}} G_{yy}^{\mathcal{M}} - 4 B_{r\theta_2}^2 G_{x^1 x^1}^{\mathcal{M}} G_{x^{10} x^{10}}^{\mathcal{M}} {}^2 G_{x^2 x^2}^{\mathcal{M}} G_{x^3 x^3}^{\mathcal{M}} G_{tt}^{\mathcal{M}} G_{yy}^{\mathcal{M}} + 4 G_{x^1 x^1}^{\mathcal{M}} G_{x^{10} x^{10}}^{\mathcal{M}} {}^2 G_{x^2 x^2}^{\mathcal{M}} G_{x^3 x^3}^{\mathcal{M}} G_{\theta_2 y}^{\mathcal{M}} {}^2 G_{tt}^{\mathcal{M}} \right)}{\Delta}$$

$$= -\frac{\sqrt{g_s} M \left(\frac{1}{N} \right)^{7/10} N_f r_0{}^4 \left(e^{4Z} - 1 \right) \left(9 a^2 + r_0{}^2 e^{2Z} \right) \log(r_0 e^Z) \left(72 a^2 r_0 e^Z \log(r_0 e^Z) - 3 a^2 + 2 r_0{}^2 e^{2Z} \right)}{2 \cdot 3^{2/3} \pi^{5/6} \log N \alpha_{\theta_1} \alpha_{\theta_2}^2 \left(6 a^2 + r_0{}^2 e^{2Z} \right) \mathcal{G}^{2/3}},$$

$$\mathcal{C}_{\theta_2 \theta_2} = \frac{G_{x^{10} x^{10}}^{\mathcal{M}} G_{yy}^{\mathcal{M}} \sqrt{-g}}{2 \left(\mathcal{A}_{\theta_2 \theta_2}^{\mathcal{M}} G_{x^{10} x^{10}}^{\mathcal{M}} {}^2 G_{yy}^{\mathcal{M}} + B_{r\theta_2}^2 G_{yy}^{\mathcal{M}} - G_{x^{10} x^{10}}^{\mathcal{M}} G_{\theta_2 y}^{\mathcal{M}} {}^2 \right)} = \frac{\sqrt[6]{\pi} \left(\frac{1}{N} \right)^{13/10} r_0{}^4 e^{4Z} \alpha_{\theta_1} \alpha_{\theta_2}^2 \left(72 a^2 r_0 e^Z \log(r_0 e^Z) - 3 a^2 + 2 r_0{}^2 e^{2Z} \right) \mathcal{G}^{4/3}}{32 \cdot 3^{2/3} g_s {}^{5/2} \log N M N_f {}^3 \log(r_0 e^Z) \left(36 a^2 \log(r_0 e^Z) + r_0 e^Z \right)^2}$$

$$\mathcal{C}_{\theta_2 y} = -\frac{G_{x^{10} x^{10}}^{\mathcal{M}} G_{\theta_2 y}^{\mathcal{M}} \sqrt{-g}}{\mathcal{A}_{\theta_2 \theta_2}^{\mathcal{M}} G_{x^{10} x^{10}}^{\mathcal{M}} {}^2 G_{yy}^{\mathcal{M}} + B_{r\theta_2}^2 G_{yy}^{\mathcal{M}} - G_{x^{10} x^{10}}^{\mathcal{M}} G_{\theta_2 y}^{\mathcal{M}} {}^2} = -\frac{\pi^{5/12} \left(\frac{1}{N} \right)^{19/20} r_0 e^{4Z} \alpha_{\theta_2}^3 \left(72 a^2 r_0 e^Z \log(r_0 e^Z) - 3 a^2 + 2 r_0{}^2 e^{2Z} \right) \mathcal{G}^{4/3}}{72 \sqrt{2} 3^{2/3} g_s {}^{9/4} \log N M N_f {}^3 \alpha_{\theta_1} \log(r_0 e^Z) \left(36 a^2 \log(r_0 e^Z) + r_0 e^Z \right)^2}$$

$$\mathcal{C}_{yy} = \frac{\mathcal{A}^{\mathcal{M}}_{\theta_2\theta_2}\, G^{\mathcal{M}}_{x^{10}x^{10}}{}^2\,\sqrt{-g}}{2\left(\mathcal{A}^{\mathcal{M}}_{\theta_2\theta_2}\, G^{\mathcal{M}}_{x^{10}x^{10}}{}^2 G^{\mathcal{M}}_{yy} + B^2_{r\theta_2} - G^{\mathcal{M}}_{x^{10}x^{10}} G^{\mathcal{M}}_{\theta_2 y}{}^2\right)}$$

$$= \frac{g_s M\,\sqrt[5]{\frac{1}{N}}\, N_f r_0{}^2 e^{2Z}\log\left(r_0 e^Z\right)\left(72 a^2 r_0 e^Z \log\left(r_0 e^Z\right) - 3a^2 + 2 r_0{}^2 e^{2Z}\right)}{3^{2/3}\sqrt[3]{\pi}\log N\,\alpha_{\theta_1}^2 \alpha_{\theta_2}^2\,\mathcal{G}^{2/3}},$$

$$\mathcal{C}_{x^{10}x^{10}} = \frac{\sqrt{-g}\left(-18 \mathcal{A}^{\mathcal{M}}_{\theta_2\theta_2} G^{\mathcal{M}}_{x^1 x^1} G^{\mathcal{M}}_{x^{10}x^{10}}{}^2 G^{\mathcal{M}}_{x^2 x^2} G^{\mathcal{M}}_{x^3 x^3} G^{\mathcal{M}}_{rr} G^{\mathcal{M}}_{tt} G^{\mathcal{M}}_{yy} - 10 B^2_{r\theta_2} G^{\mathcal{M}}_{x^1 x^1} G^{\mathcal{M}}_{x^2 x^2} G^{\mathcal{M}}_{x^3 x^3} G^{\mathcal{M}}_{tt} + 14 G^{\mathcal{M}}_{x^1 x^1} G^{\mathcal{M}}_{x^{10}x^{10}} G^{\mathcal{M}}_{x^2 x^2} G^{\mathcal{M}}_{x^3 x^3} G^{\mathcal{M}}_{rr} G^{\mathcal{M}}_{\theta_2 y}{}^2 G^{\mathcal{M}}_{tt}\right)}{\Delta};$$

$$\Delta = 8 G^{\mathcal{M}}_{x^1 x^1} G^{\mathcal{M}}_{x^{10}x^{10}} G^{\mathcal{M}}_{x^2 x^2} G^{\mathcal{M}}_{x^3 x^3} G^{\mathcal{M}}_{rr} G^{\mathcal{M}}_{tt}\left(\mathcal{A}^{\mathcal{M}}_{\theta_2\theta_2} G^{\mathcal{M}}_{x^{10}x^{10}}{}^2 G^{\mathcal{M}}_{yy} + B^2_{r\theta_2} - G^{\mathcal{M}}_{x^{10}x^{10}} G^{\mathcal{M}}_{\theta_2 y}{}^2\right)$$

$$= \frac{27\sqrt[3]{3}\, g_s M\,\sqrt[5]{\frac{1}{N}}\, N_f r_0{}^2 e^{2Z}\log\left(r_0 e^Z\right)\left(72 a^2 r_0 e^Z \log\left(r_0 e^Z\right) - 3a^2 + 2 r_0{}^2 e^{2Z}\right)\mathcal{G}^{4/3}}{128\pi^{7/3}\log N\,\alpha_{\theta_1}\alpha_{\theta_2}^2};$$

$$\mathcal{A}^{\mathcal{M}}_{\theta_2\theta_2} = \left(x\left(\widetilde{\widetilde{F^{IIB}_3}}\right)_{x\theta_2} + y\left(\widetilde{\widetilde{F^{IIB}_3}}\right)_{y\theta_2} + z\left(\widetilde{\widetilde{F^{IIB}_3}}\right)_{z\theta_2} + r\left(\widetilde{\widetilde{*F^{IIB}_5}}\right)_{r\theta_2}\right)^2 \sim \left(z\left(\widetilde{\widetilde{F^{IIB}_3}}\right)_{z\theta_2} + r\left(\widetilde{\widetilde{*F^{IIB}_5}}\right)_{r\theta_2}\right)^2$$

$$= \frac{9 g_s^{7/2} M^2 N^{11/10} N_f{}^4 e^{-2Z}\log^2\left(r_0 e^Z\right)\left(36 a^2 \log\left(r_0 e^Z\right) + r_0 e^Z\right)^2}{2\pi^{5/2} r_0{}^2 \alpha_{\theta_1}^2 \alpha_{\theta_2}^4}, \tag{D.3}$$

where in the last line use has been made of that in the IR:

$$x\left(\widetilde{\widetilde{\widetilde{F_3^{IIB}}}}\right)_{x\theta_2} \sim N^{\frac{3}{20}}\log r, \qquad\qquad y\left(\widetilde{\widetilde{\widetilde{F_3^{IIB}}}}\right)_{y\theta_2} \sim N^{\frac{1}{4}}\log r,$$

$$z\left(\widetilde{\widetilde{\widetilde{F_3^{IIB}}}}\right)_{z\theta_2} \sim N^{\frac{11}{20}}\log r, \qquad\qquad \left(\widetilde{\widetilde{\widetilde{*F_5^{IIB}}}}\right)_{r\theta_2} \sim N^{\frac{1}{20}}\left(\frac{\log r}{r_0}\right)^2; \qquad \text{(D.4)}$$

the triple tildes $\widetilde{\widetilde{\widetilde{}}}$ imply a tripe T-dual of the type IIB background of [14]; x, y, z are the delocalized T^3 coordinates using for effecting SYZ mirror symmetry via a triple T dual in [15]. In (D.3), $\mathcal{G} \equiv 8\pi e^{-\phi^{IIB}}$.

One hence obtains:

$$\mathcal{E}_{x^{10}x^{10}}\mathcal{F}_{x^{10}x^{10}}$$

$$= \frac{243b^{10}\left(9b^2+1\right)^4\beta M\left(\frac{1}{N}\right)^{5/4}e^Z\left(e^Z-2\right)\left(-19683\sqrt{6}\alpha_{\theta_1}^6 - 6642\alpha_{\theta_2}^2\alpha_{\theta_1}^3 + 40\sqrt{6}\alpha_{\theta_2}^4\right)\left(r_0^2-3a^2\right)\log^3(r_0)}{16\pi^2\left(3b^2-1\right)^5\left(\log N\right)^3\alpha_{\theta_2}^3\left(6b^2r_0+r_0\right)^4},$$
$$\text{(D.5)}$$

relevant to obtaining the $\mathcal{O}(R^4)$-corrected type IIA dilaton via (D.2).

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