RECEIVED: April 16, 2021

REVISED: August 4, 2021

ACCEPTED: August 4, 2021

PUBLISHED: August 23, 2021

Virasoro algebras, kinematic space and the spectrum of modular Hamiltonians in CFT_2

Suchetan Das,^{a,b} Bobby Ezhuthachan,^b Somnath Porey^b and Baishali Roy^b

^a*Department of Physics, Indian Institute of Technology Kanpur, Kanpur 208016, India*

^b*Ramakrishna Mission Vivekananda Educational and Research Institute, Belur Math, Howrah-711202, West Bengal, India*

E-mail: suchetan@iitk.ac.in, bobby.ezhuthachan@rkmvu.ac.in, somnathhimu00@gm.rkmvu.ac.in, baishali.roy025@gm.rkmvu.ac.in

ABSTRACT: We construct an infinite class of eigenmodes with integer eigenvalues for the Vacuum Modular Hamiltonian of a single interval N in 2d CFT and study some of its interesting properties, which includes its action on OPE blocks as well as its bulk duals. Our analysis suggests that these eigenmodes, like the OPE blocks have a natural description on the so called kinematic space of CFT_2 and in particular realize the Virasoro algebra of the theory on this kinematic space. Taken together, our results hints at the possibility of an effective description of the CFT_2 in the kinematic space language.

KEYWORDS: AdS-CFT Correspondence, Conformal Field Theory, Field Theories in Lower Dimensions

ARXIV EPRINT: [2101.10211](https://arxiv.org/abs/2101.10211)

Contents

1	Introduction	1
2	Modular Hamiltonian in 2D CFT and its spectrum	4
2.1	OPE Blocks	4
2.2	A new class of modular eigenmodes and its properties	5
2.3	Action on the OPE blocks	6
2.4	MVA and the Kinematic space	8
2.4.1	Realizing $\mathbb{L}_n$ on k-space	8
3	The global subalgebra of the MVA	9
3.1	Symmetries of the CFT_2 causal diamonds	9
3.2	g-MVA and modular inclusions	11
4	Pulling the $\mathbb{L}_n$ into the bulk	12
5	Discussion	13
A	Modular inclusion in CFT_2 and finite dimensional system	15
A.1	Modular inclusion	15
A.2	Modular inclusion in vacuum CFT_2	16
A.3	Modular inclusion in finite dimensional Hilbert space	18
B	Commutation relation of modular generators and Virasoro algebra	20

1 Introduction

Research over the past several years has made it abundantly clear that Quantum information/entropy related ideas play a crucial role in developing a deeper understanding of Quantum Field Theory and Quantum Gravity. The algebraic formulation of QFT (AQFT) in terms of algebra of observables associated to causal domains of spatial subregions [1, 2], seems to be particularly well suited for such entropic studies. The many successes of this approach include formulating a precise version of various entropy bounds in QFT [3–5], developing a deeper understanding of RG flows in terms of relative entropy of states, [6–10], proofs of various null energy conditions in QFT [11–14], developing a more precise understanding of bulk Reconstruction [15–30] among others.

A key role in most of these studies is played by the (total) modular hamiltonian [31].¹ In the AQFT formulation, the modular hamiltonian operator K_Σ^ψ , for a particular state

¹Total modular hamiltonian is defined as the difference of modular hamiltonians of a given subregion and its complement. In the rest of the note, we refer modular hamiltonian as the total modular hamiltonian.

$|\psi\rangle$ generates an automorphism of the algebra of the operators localized in the causal diamond $\mathcal{D}(\Sigma)$ associated with the spatial subregion Σ . Under this automorphism, operators localized within $\mathcal{D}(\Sigma)$ transform into each other, thus generating a flow called the modular flow.² In applications to holography, the importance of the modular hamiltonian operator comes from its identification, at leading order in the inverse bulk newton's constant ($\frac{1}{G_N}$), with the corresponding bulk modular hamiltonian operator, where the corresponding bulk region is the bulk causal diamond associated with the region bounded by the RT surface and Σ [15]. These modular flows play an important role in the entanglement wedge reconstruction program.³ It has also been argued that the emergence of a semiclassical bulk spacetime might itself be understood from the algebra of modular hamiltonians of all subregions in the boundary QFT [30].

Given its relevance, particularly in the context of bulk reconstruction program alluded to above, it would be a useful endeavor to study the modular hamiltonian operator in detail both in general QFTs as well more specifically in simple but concrete examples. One way to characterize these operators would be through the spectrum of its eigenstates. It is reasonable to expect that this spectrum would encode the entanglement content of the QFT. The modular hamiltonian K_Σ^ψ for a state $|\psi\rangle$ and a spatial region Σ annihilates the state, ie $K_\Sigma^\psi|\psi\rangle = 0$. One may then construct its eigenstates by acting on $|\psi\rangle$ with a special class of operators ($\mathcal{O}_\omega$) which has the following commutation relation with the modular hamiltonian $[K, \mathcal{O}_\omega] = \omega \mathcal{O}_\omega$. These are referred to as the modular eigenmodes. It's easy to see that the fourier transform of the modular evolved operators (ie: $\int dse^{is\omega} \mathcal{O}_\omega$) are modular eigenmodes. These eigenmodes, in particular the zero modes play a crucial role in reconstruction of bulk fields inside the entanglement wedge [17, 19]. The zero modes, which commute with the modular hamiltonian may be thought of as local symmetries of the corresponding state, in the sense that correlation function of operators inside the region $\mathcal{D}(\Sigma)$, would be invariant under transformations (of the operators) generated by the zero modes. These are local because for the same given state, but for a different region, the modular hamiltonian and therefore the zero modes would be different. It has been argued that in the bulk these local symmetries generated by the zero modes corresponding to large diffeomorphisms which are not trivial on the RT surface [22]. Thus the modular eigenmodes seem to have a very important bearing on the emergence of bulk geometry itself.

Given the above motivations, a detailed study of the modular eigenmodes in these theories would be interesting. While for generic states and regions the modular hamiltonian as well its eigenmodes are nonlocal operators, there are a few examples, where they take a simple form as an integral of local fields. The simplest example of which is the case of the single interval in the vacuum state of a CFT₂. In [37], two of us had shown that OPE blocks of primary fields of different spins are nonzero modular eigenmodes of the modular hamiltonian of the single interval, in the vacuum of CFT₂, where the endpoints of the

²Under this flow, an operator $\mathcal{O} \rightarrow \mathcal{O}(s)$, where $\mathcal{O}(s) \equiv e^{isK} \mathcal{O} e^{-isK}$. Both $\mathcal{O}$ and $\mathcal{O}_s$ have support within $\mathcal{D}_\Sigma$.

³Recently, a different, but related, notion of the connes co-cycle flows also have been discussed in the context of extracting bulk information from the entanglement wedge region which is casually disconnected from the boundary [32, 33].

interval corresponds to the location of the two primary fields whose expansion define the OPE block. This generalizes known results in the literature that scalar OPE blocks are modular zero eigenmodes [17, 20].

In this note, we continue with the study initiated in [37], of the eigenmodes of the vacuum modular hamiltonian for a single interval (labelled as N) in 2D CFT. We find a new infinite class of modular eigenmodes with integer eigenvalues and discuss some of their interesting properties. The key point we want to make here is that like the OPE blocks, these new eigenmodes we construct here have a natural description on the so-called kinematic space(k-space) [34], which is essentially the space of causal diamonds in CFT₂. In particular, they realize the virasoro algebra of the CFT₂ on this k-space. As evidence of this fact we show that OPE blocks, which are local fields in the k-space description, transform as modes of a primary field under this ‘k-space virasoro algebra’, which we refer to as the modular virasoro algebra(MVA) in the bulk of the text. Moreover, as we show, a subset of the new modular eigenmodes, which generate the global subalgebra of the MVA representation can be directly identified with the modular hamiltonians of subregions of N . We believe that these observations, taken together, hint at the possibility of an equivalent effective description of the CFT₂ in the k-space language, a detailed study of which we leave for future work.

This draft is organized as follows. In the next section, after presenting a brief summary of the known examples of modular eigenmodes- the OPE blocks, and their bulk duals, we present the new class of eigenmodes and discuss its interesting properties. Specifically, we show that these eigenmodes together with the modular hamiltonian satisfy the virasoro algebra. The details of this calculation are presented in appendix B. We also compute the commutator of these new class of modular eigenmodes with the OPE blocks, and show that the result is same as that of the usual (local) virasoro generators with the modes of a primary field in CFT₂. This fact suggests that the OPE blocks transform as modes of a primary field under conformal transformations generated by the MVA. Since the OPE blocks can be described naturally as fields living on the so-called kinematic space(k-space), this also suggest that the new eigenmodes have a natural action on the k-space. We explore the kinematic aspects of this question in subsection 2.4.

In section 3, we focus on the global subalgebra of the MVA. Interestingly, we show that it is isomorphic to the algebra of the modular hamiltonians of N as well as its subregions N' and N'' and that they implement the so-called modular inclusion within the lightcone of N . For completeness we review the definition of the modular inclusion and, as an example, discuss the modular inclusion for finite dimensional Hilbert space in appendix A.

In section 4, we discuss the bulk dual of our construction. In particular we show the emergence of the RT geodesic very naturally from our constructions. We conclude with a summary of our results as well a discussion on some of the questions and directions opened up in the light of these results, in section 5.

Note added. While we were in the final stages of preparing this article, [58] was posted on the arxiv, in which the authors construct Virasoro algebra from generators constructed out of light-ray operators. Although the expression for those operators are similar to

what we refer here as ‘modular Virasoro generators’, the context and motivation of the approaches seem to us to be different.

2 Modular Hamiltonian in 2D CFT and its spectrum

The Modular Hamiltonian of a single interval with endpoints (z_2, z_3) on a constant time slice ($t = 0$) in the vacuum of 2D CFT is given by the following integral expression:

$$K = \int_{-\infty}^{\infty} d\zeta \frac{(z_2 - \zeta)(\zeta - z_3)}{z_2 - z_3} T_{\zeta\zeta}(\zeta) + \int_{-\infty}^{\infty} d\bar{\zeta} \frac{(z_2 - \bar{\zeta})(\bar{\zeta} - z_3)}{z_2 - z_3} T_{\bar{\zeta}\bar{\zeta}}(\bar{\zeta}) \quad (2.1)$$

Modular eigenmodes are operators which satisfy the following commutation relation with the modular hamiltonian $[K, \mathcal{O}] = \lambda \mathcal{O}$. In [17, 37], it was shown that global OPE blocks in CFT_2 , are eigen-modes of vacuum modular Hamiltonian. We therefore begin this section, with a brief review of the OPE blocks in CFT_2 .

2.1 OPE Blocks

In CFT, a global OPE block B_k^{ij} is defined as the contribution of a conformal family (ie a given primary field $\mathcal{O}_k$ of dimension $h_k, \bar{h}_k$ and all it’s global descendants) to the OPE of two primary operators $(\mathcal{O}_i, \mathcal{O}_j)$ of dimensions $(h_i, \bar{h}_i)$ and $(h_j, \bar{h}_j)$ respectively [34]. Mathematically,

$$\mathcal{O}_i(z_1, \bar{z}_1) \mathcal{O}_j(z_2, \bar{z}_2) = z_{12}^{-(h_i+h_j)} \bar{z}_{12}^{-(\bar{h}_i+\bar{h}_j)} \sum_k C_{ijk} B_k^{ij}(z_1, \bar{z}_1; z_2, \bar{z}_2) \quad (2.2)$$

Here, C_{ijk} is the OPE coefficient, which is dynamical input of the theory. The above equation tells us how the OPE block transforms under global conformal transformations and this is enough to fix the form of the OPE blocks. Indeed, B_k^{ij} has an integral expression which can be derived [34, 38] using the shadow operator formalism [39, 40] and takes the following form,

$$B_k^{ij}(z_1, \bar{z}_1; z_2, \bar{z}_2) = \int_{z_1}^{z_2} d\zeta \int_{\bar{z}_1}^{\bar{z}_2} d\bar{\zeta} \left(\frac{(\zeta - z_1)(z_2 - \zeta)}{z_2 - z_1} \right)^{h_k-1} \left(\frac{z_2 - \zeta}{\zeta - z_1} \right)^{h_{ij}} \\ \times \left(\frac{(\bar{\zeta} - \bar{z}_1)(\bar{z}_2 - \bar{\zeta})}{\bar{z}_2 - \bar{z}_1} \right)^{\bar{h}_k-1} \left(\frac{\bar{z}_2 - \bar{\zeta}}{\bar{\zeta} - \bar{z}_1} \right)^{\bar{h}_{ij}} \mathcal{O}_k(\zeta, \bar{\zeta}) \quad (2.3)$$

One can now show,⁴ using the OPE of $T, \bar{T}$ with the primary field $\mathcal{O}$, that these OPE blocks are indeed eigenmodes of K , with eigenvalue proportional to the spin difference (l_{ij}) of the two operators.

$$[K, B_k^{ij}] = 2\pi i l_{ij} B_k^{ij} \quad (2.4)$$

The scalar zero-modes have been shown to be dual to the so-called geodesic operators [34], which are essentially smeared geodesic integrals of the appropriate bulk dual field

⁴See appendix A of [37], for the details of the proof.

of $\mathcal{O}_k$. The geodesic endpoints being the location of the two primary fields, whose OPE defines the specific OPE block in question.

$$B_k^{ij} = \int_{\lambda} ds e^{-s\Delta_{ij}} \phi(x(s), z(s), t(s)) \quad (2.5)$$

Here ϕ is the dual scalar field to $\mathcal{O}_k$. (x, t) are the boundary coordinates while z is the bulk coordinate, and the integral is over a geodesic with end points on the boundary. The smearing function is $e^{-s\Delta_{ij}}$, with $\Delta_{ij} = \Delta_i - \Delta_j$ being the difference in scaling dimensions of the two operators. This can be generalized to non zero scalar modes [37], where now the integral is over a Lorentzian cylinder. The cylindrical surface is generated by K and P_D which generate boosts in the plane normal to the geodesic and translations along the geodesic respectively.

$$B_k^{ij} = c_k \int_{\text{cylinder}} d\tilde{t} ds e^{-s(\theta)\Delta_{ij}} e^{-\tilde{t}(\rho, \theta)l_{ij}} \phi(x(\rho, \theta), z(\rho, \theta), t(\rho, \theta)) \quad (2.6)$$

Here l_{ij} is the spin difference between the two operators, $\tilde{t}$ and s labels the coordinates on the cylinder and the c_k is a normalization constant which can be fixed by the appropriate boundary condition. See [37] for the details of the derivation. In the next section, we introduce the new class of eigenmodes which are smeared integrals of T and $\bar{T}$ and discuss their bulk duals.

2.2 A new class of modular eigenmodes and its properties

We now present a new class of integrated operators, which are all eigenmodes of the modular hamiltonian. Unlike the OPE blocks, these exist in any 2d CFT and are not theory dependent. As advertised earlier, these also satisfy a virasoro algebra. For this reason, we label them as $\mathbb{L}_n$ and $\bar{\mathbb{L}}_n$. The explicit expressions are given below.

$$\mathbb{L}_n = a_n \int_{-\infty}^{\infty} d\zeta \frac{(z_2 - \zeta)^{-n+1} (\zeta - z_3)^{n+1}}{z_2 - z_3} T(\zeta) \quad (2.7)$$

$$\bar{\mathbb{L}}_n = \bar{a}_n \int_{-\infty}^{\infty} d\bar{\zeta} \frac{(z_2 - \bar{\zeta})^{n+1} (\bar{\zeta} - z_3)^{-n+1}}{z_2 - z_3} \bar{T}(\bar{\zeta}) \quad (2.8)$$

Note that naively the integrand in the above formulae, blow up at z_2 and z_3 , however we can regulate the integral, by choosing a deformed contour such that it doesn't pass through z_2 and z_3 . Equivalently, we can give both z_2 and z_3 a small imaginary component. The a_n are arbitrary normalization constants⁵. In this notation, $\mathbb{L}_0 + \bar{\mathbb{L}}_0$, with $a_0 = \bar{a}_0 = 1$ is the modular hamiltonian of the single interval with endpoints z_2 and z_3 . It can be shown that they satisfies,

$$[\mathbb{L}_0, \mathbb{L}_n] = -n\mathbb{L}_n, \quad [\bar{\mathbb{L}}_0, \bar{\mathbb{L}}_n] = -n\bar{\mathbb{L}}_n \quad (2.9)$$

It follows that the $\mathbb{L}_n$ and $\bar{\mathbb{L}}_n$ for $(n \neq 0)$ are indeed modular eigenmodes. Furthermore, if we normalize the $\mathbb{L}_n$ suitably, which can be done without any loss of generality, such that

⁵If we choose the normalization constant to be independent of the endpoints z_2 and z_3 , then it is easy to see that the L_n and $\bar{L}_n$ are really only a function of $(z_2 - z_3)$, however if the normalization constants are non trivial functions of the endpoints, then the eigenmodes are bi-local (z_2 and z_3)

the $a_n = r^n$ and $\bar{a}_n = \bar{r}^n$ where r and $\bar{r}$ are two arbitrary constants, then in fact these eigenmodes satisfy the virasoro algebra, with the correct central charge term.

$$[\mathbb{L}_m, \mathbb{L}_n] = (m - n)\mathbb{L}_{m+n} + \frac{c}{12}n(n^2 - 1)\delta_{m+n,0} \quad (2.10)$$

For this reason, we refer to this, as the modular virasoro algebra (MVA). As we explain in detail in section 3, there is a nice geometric interpretation of the global $\text{SO}(2, 2)$ subalgebra of the MVA. In particular, the generators of this global subalgebra ie: $\mathbb{L}_{0,\pm}$ and $\bar{\mathbb{L}}_{0,\pm}$ are linear combinations of the holomorphic and antiholomorphic components of the modular hamiltonians corresponding to the subregions $N'(z_1, z_2)$ and $N''(z_3, z_1)$ of N . See figure 1. This is particularly transparent if one parameterizes the normalization constant as follows: $r = \frac{1}{\bar{r}} = \left(\frac{z_2 - z_1}{z_3 - z_1}\right)$. As is clear from figure 1, the z_1 in this parametrization is the point within the line segment N , which divides it into N' and N'' . For this reason, in the remainder of the note, we use this normalization for the $\mathbb{L}_n$ and $\bar{\mathbb{L}}_n$.

Finally we note that there is an interesting ‘duality’ between the standard generators of the CFT, which we denote as $\mathbf{L}_n$, and the $\mathbb{L}_n$ we construct here. In particular, there exists a conformal transformation which interchanges the two. The explicit map between the two conformal frames is given in equation (4.2). Under this transformation, $\mathbb{L}_n \iff \mathbf{L}_n$. In particular, this means that the modular hamiltonian gets interchanged with the usual CFT hamiltonian.

As we will argue in the rest of the note, it is natural to interpret the $\mathbb{L}_n$ as realizing the virasoro algebra on the space of causal diamonds in the CFT_2 , which is termed as the kinematic space (k-space). Evidence of this is provided in the following sections, where we analyze the action of $\mathbb{L}_n$ on the OPE blocks which are bilocal operators in the CFT_2 but have a simple local description in the k-space, and later when we understand the geometric meaning of the global subalgebra of the MVA.

2.3 Action on the OPE blocks

We can compute the commutator of the “modular” $\mathbb{L}_n$ operators with the OPE blocks, by using the commutator of T with primary operators, which can be obtained from the $T\mathcal{O}$ OPE.

$$[T(\omega), \mathcal{O}_k(\zeta, \bar{\zeta})] = 2\pi i(h\partial_\zeta\delta(\zeta - \omega) + \delta(\zeta - \omega)\partial_\zeta)\mathcal{O}_k(\zeta, \bar{\zeta}), \quad (2.11)$$

$$[\bar{T}(\bar{\omega}), \mathcal{O}_k(\zeta, \bar{\zeta})] = -2\pi i(h\partial_{\bar{\zeta}}\delta(\bar{\zeta} - \bar{\omega}) + \delta(\bar{\zeta} - \bar{\omega})\partial_{\bar{\zeta}})\mathcal{O}_k(\zeta, \bar{\zeta}) \quad (2.12)$$

One then needs to evaluate the action of $\mathbb{L}_n$ on primary field $\mathcal{O}_k(\zeta, \bar{\zeta})$. Using (2.7), (2.11) we get

$$\begin{aligned} [\mathbb{L}_n, \mathcal{O}_k(\zeta, \bar{\zeta})] &= \frac{2\pi i}{z_2 - z_3}(z_2 - \zeta)^{-n}(\zeta - z_3)^n \left(\frac{z_{21}}{z_{31}}\right)^n \\ &\quad \times \left[h_k(n(z_2 - z_3) + (z_2 + z_3 - 2\zeta)) + (z_2 - \zeta)(\zeta - z_3)\partial_\zeta\right]\mathcal{O}(\zeta, \bar{\zeta}) \end{aligned} \quad (2.13)$$

$$\begin{aligned} [\bar{\mathbb{L}}_n, \mathcal{O}_k(\zeta, \bar{\zeta})] &= \frac{2\pi i}{z_2 - z_3}(z_2 - \bar{\zeta})^n(\bar{\zeta} - z_3)^{-n} \left(\frac{z_{21}}{z_{31}}\right)^{-n} \\ &\quad \times \left[\bar{h}_k(-n(z_2 - z_3) + (z_2 + z_3 - 2\bar{\zeta})) + (z_2 - \bar{\zeta})(\bar{\zeta} - z_3)\partial_{\bar{\zeta}}\right]\mathcal{O}(\zeta, \bar{\zeta}) \end{aligned} \quad (2.14)$$

Using (2.3) and (2.13), we now present the commutator of the $\mathbb{L}_n$ and the B_k^{ij} .

$$\begin{aligned}
 & [\mathbb{L}_n, B_k^{ij}(z_2, z_3)] \\
 &= \int_{z_3}^{z_2} d\zeta \int_{\bar{z}_3}^{\bar{z}_2} d\bar{\zeta} \left(\frac{(\zeta - z_3)(z_2 - \zeta)}{z_2 - z_3} \right)^{h_k-1} \left(\frac{z_2 - \zeta}{\zeta - z_3} \right)^{h_{ij}} \\
 &\quad \times \left(\frac{(\bar{\zeta} - \bar{z}_3)(\bar{z}_2 - \bar{\zeta})}{\bar{z}_2 - \bar{z}_3} \right)^{\bar{h}_k-1} \left(\frac{\bar{z}_2 - \bar{\zeta}}{\bar{\zeta} - \bar{z}_3} \right)^{\bar{h}_{ij}} \frac{2\pi i}{z_2 - z_3} (z_2 - \zeta)^{-n} (\zeta - z_3)^n \left(\frac{z_{21}}{z_{31}} \right)^n \\
 &\quad \times \left[h_k (n(z_2 - z_3) + (z_2 + z_3 - 2\zeta)) + (z_2 - \zeta)(\zeta - z_3) \partial_\zeta \right] \mathcal{O}_k(\zeta, \bar{\zeta}) \\
 &= (\text{T.D}) + 2\pi i (nh_k - n + h_{ij}) \left(\frac{z_{21}}{z_{31}} \right)^n \int_{z_3}^{z_2} d\zeta \int_{\bar{z}_3}^{\bar{z}_2} d\bar{\zeta} \left(\frac{(\zeta - z_3)(z_2 - \zeta)}{z_2 - z_3} \right)^{h_k-1} \\
 &\quad \times \left(\frac{z_2 - \zeta}{\zeta - z_3} \right)^{h_{ij}} \left(\frac{(\bar{\zeta} - \bar{z}_3)(\bar{z}_2 - \bar{\zeta})}{\bar{z}_2 - \bar{z}_3} \right)^{\bar{h}_k-1} \left(\frac{\bar{z}_2 - \bar{\zeta}}{\bar{\zeta} - \bar{z}_3} \right)^{\bar{h}_{ij}} \left(\frac{z_2 - \zeta}{\zeta - z_3} \right)^{-n} \mathcal{O}_k(\zeta, \bar{\zeta}) \\
 &= 2\pi i \left(\frac{z_{21}}{z_{31}} \right)^n [n(h_k - 1) + h_{ij}] B_k^{ij-n} ; \text{ for } (h_{ij} - h_k \leq n \leq h_{ij} + h_k) \quad (2.15)
 \end{aligned}$$

Here (T.D) is the total derivative term which vanishes for $(h_{ij} - h_k \leq n \leq h_{ij} + h_k)$.

Equation (2.15) is identical to the action of usual Virasoro generator $\mathbf{L}_n$ in CFT_2 on the modes ϕ_m of a primary field ϕ ,

$$[\mathbf{L}_n, \phi_m] = [n(h - 1) - m] \phi_{n+m} \quad (2.16)$$

with the identification $\phi_m = \left(\frac{z_{21}}{z_{31}} \right)^{-h_{ij}} B_k^{ij}$. Thus we see that the OPE blocks play the role of modes of some highest weight primary field representation of the MVA.

To summarize, the key results of this section are:

- The integrated stress tensor operators $\mathbb{L}_n$ and $\bar{\mathbb{L}}_n$ form an infinite class of modular eigenmodes of the modular hamiltonian corresponding to an single interval N with endpoints (z_1, z_2) .
- These modular eigenmodes are bi-local, similar to the OPE blocks, in that they are a function of the end points (z_1, z_2) of the interval (N) .
- Their commutators satisfy the virasoro algebra. For obvious reasons we refer to this representation of the virasoro algebra as the modular virasoro algebra (MVA). Moreover under a conformal transformation given in equation (4.2), the $\mathbb{L}_n \iff \mathbf{L}_n$. In particular, the modular hamiltonian is interchanged with the usual CFT hamiltonian.
- Under the transformations generated by the $\mathbb{L}_n$ and $\bar{\mathbb{L}}_n$, the OPE blocks transforms as should the modes of a primary operator in CFT_2 .

Now, the OPE blocks which are bi-local fields in the CFT_2 have a natural description as local fields on the so-called kinematic space (k-space) [34]–[36]. In the light of [c] and [d], it is natural to wonder whether there exists an “effective CFT” description in the k-space itself, with the OPE blocks being the primary fields in this “k-space cft”. We explore the kinematical aspects of this question in the next section.

2.4 MVA and the Kinematic space

The kinematic space of CFT_2 is defined as the space of a ‘pair of space like points’ in the CFT. Thus its a four dimensional space, with coordinates given by the coordinates of the two points $(t_2, x_2; t_3, x_3)$ with signature $(+, -, +, -)$. One can fix the metric on this space by demanding its invariance under conformal transformations of both points. This leads to a unique metric.

$$ds_{\text{k-space}}^2 = 2 \left[\frac{dz_2 dz_3}{(z_2 - z_3)^2} + \frac{d\bar{z}_2 d\bar{z}_3}{(\bar{z}_2 - \bar{z}_3)^2} \right] \text{ with } z_i = t_i + x_i, \bar{z}_i = t_i - x_i \quad (2.17)$$

Thus the 4d space factorizes into two 2d conformally flat space times, spanned by the two sets of k-space light cone coordinates (z_2, z_3) and $(\bar{z}_2, \bar{z}_3)$ respectively.

The k-space formalism allows us to visualize the OPE blocks, which are bi-local fields in the CFT_2 as local fields living on this k-space. Moreover it geometrizes the conformal kinematic properties of these OPE blocks. In particular, this means that the conformal casimir equation which the OPE blocks satisfies, derived from its conformal transformation properties as obtained from its definition in (2.2), translates in the k-space terminology into an ‘equations of motion’ to be satisfied by the OPE blocks on the k-space [34, 35]. For scalar OPE blocks, this is just the Klein-Gordon (KG) equation, while the spinning OPE blocks satisfy a slightly modified KG equation [38]. The short distance behaviour of the OPE blocks⁶ is now interpreted as a boundary condition to be imposed along the k-space coordinates $(z_{23}, \bar{z}_{23} \rightarrow 0)$

2.4.1 Realizing $\mathbb{L}_n$ on k-space

Points [c] and [d] of the last section seem to hint at the possibility of an effective CFT description in the k-space with the OPE blocks being modes of the highest weight representation of the corresponding MVA, which in this interpretation would act locally on the k-space. Now from the explicit form of the $\mathbb{L}_n$, it is clear that they are functions of $(z_2 - z_3)$. Now in our case, we had chosen the interval to be on a constant time slice, so that $z_2 = \bar{z}_2$ and $z_3 = \bar{z}_3$. If, on the other hand, had we chosen an arbitrary spacelike interval, then indeed $\mathbb{L}_n, \bar{\mathbb{L}}_n$ are function of $(z_{23} = z_2 - z_3)$ and $(\bar{z}_{23} = \bar{z}_2 - \bar{z}_3)$ respectively. As is clear from the k-space metric (2.17), the z_{23} , and $\bar{z}_{23}$ are spatial coordinates on the two decoupled spaces and not light cone coordinates. Thus the $\mathbb{L}_n$ and $\bar{\mathbb{L}}_n$ and should be thought of as generating independent spatial diffeomorphisms along z_{23} and $\bar{z}_{23}$ directions rather than generating conformal transformations in a 2d space.

It is still possible that there is a useful effective k-space description in terms of product of two CFT_1 ’s, with a 1d stress tensor which we denote as $\mathbb{T}(\zeta)$ and $\bar{\mathbb{T}}(\bar{\zeta})$ given by

$$\mathbb{T}(\zeta) = \sum_n \frac{\mathbb{L}_n(z_{23})}{(\zeta - z_{23})^{n+2}}, \text{ and } \bar{\mathbb{T}}(\bar{\zeta}) = \sum_n \frac{\bar{\mathbb{L}}_n(\bar{z}_{23})}{(\bar{\zeta} - \bar{z}_{23})^{n+2}} \quad (2.18)$$

⁶The short distance limit of an ope implies the ope block behaves like a single primary, i.e:

$$\lim_{z_2, \bar{z}_2 \rightarrow z_3, \bar{z}_3} B_k^{ij}(z_2, \bar{z}_2; z_3, \bar{z}_3) \sim z_{23}^{h_k} \bar{z}_{23}^{\bar{h}_k} \mathcal{O}_{h_k, \bar{h}_k}(z_3, \bar{z}_3).$$

such that the OPE blocks are modes of a field $\Phi(\zeta, \bar{\zeta})$ with respect to both of the 1d cfts. Where the field Φ could be formally mode-expanded in terms of the OPE blocks as follows:

$$\Phi(\zeta, \bar{\zeta}) = \sum_{h_{ij}, \bar{h}_{ij}} \frac{\mathcal{B}_k^{ij}(z_2, z_3; \bar{z}_2, \bar{z}_3)}{(\zeta - z_{23})^{h_k - h_{ij}} (\bar{\zeta} - \bar{z}_{23})^{\bar{h}_k - \bar{h}_{ij}}} \quad (2.19)$$

However to establish whether a consistent description of this type can be constructed, would involve proving that it satisfies consistent crossing equations among other things. We do not attempt to answer this question here. The only point we want to make here, is that the algebra of the OPE blocks with the $\mathbb{L}_n$ is consistent with the existence of such an effective k-space description.

Of course, if such an effective description does exist in k-space, it would only be a reformulation of the original 2d cft in the k-space language, and the stress tensors defined via equation (2.18), would also be related to the CFT stress tensor components. Nevertheless, we know that the k-space description is a useful intermediary between the AdS and CFT descriptions, because the k-space has the advantage of being directly identified as the space of bulk geodesics which end on the boundary [36]. This fact has been used to derive a very simple proof of the identification (2.5), (2.6) of OPE blocks as geodesic operators in AdS [34]. For these reasons, a way to incorporate cft dynamics in k-space language would be interesting from the AdS/CFT perspective. We hope to come back to this issue in the near future.

3 The global subalgebra of the MVA

In this section, we focus on the global subalgebra of the MVA, which is spanned by $\mathbb{L}_{0,\pm 1}$ and $\bar{\mathbb{L}}_{0,\pm 1}$. We point out that this subset of $\mathbb{L}_n$ has a nice geometric interpretation as modular hamiltonians of N itself as well as its subparts, labelled as N' and N'' . This in turn realizes the action of ‘modular inclusion’ [50]–[53], within $\mathcal{D}_N$. For brevity, we refer to this subalgebra as the g-MVA in the rest of the note. We begin with a short discussion on the symmetries of causal diamonds in CFT_2 .

3.1 Symmetries of the CFT_2 causal diamonds

CFT_2 causal diamonds associated with intervals on a constant time slice are preserved by a $\text{SO}(1, 1) \times \text{SO}(1, 1)$ subset of global conformal symmetry group $\text{SO}(2, 2)$. Due to chiral structure of symmetry algebras, one can find a right moving and a left moving conformal killing vector (CKV) which stabilizes the diamond. For a causal diamond with upper and lower tips at $(v, \bar{v})$ and $(u, \bar{u})$ respectively (in the light cone coordinates), the CKVs take the following form,

$$K^\zeta \partial_\zeta = \frac{(v - \zeta)(\zeta - u)}{v - u} \partial_\zeta, \quad K^{\bar{\zeta}} \partial_{\bar{\zeta}} = \frac{(\bar{v} - \bar{\zeta})(\bar{\zeta} - \bar{u})}{\bar{v} - \bar{u}} \partial_{\bar{\zeta}} \quad (3.1)$$

Here $\zeta (= X + T)$, $\bar{\zeta} (= X - T)$ are the lightcone coordinates. One can similarly define the corresponding conserved charges as:

$$K^R = \int_{-\infty}^{\infty} d\zeta \frac{(v - \zeta)(\zeta - u)}{u - v} T_{\zeta\zeta}(\zeta), \quad K^L = \int_{-\infty}^{\infty} d\bar{\zeta} \frac{(\bar{v} - \bar{\zeta})(\bar{\zeta} - \bar{u})}{\bar{v} - \bar{u}} T_{\bar{\zeta}\bar{\zeta}}(\bar{\zeta}) \quad (3.2)$$

Figure 1. Causal diagram of different regions on a $T = 0$ slice.

If we take an interval on $T = 0$ slice with endpoints (z_2, z_3) , the upper and lower tips of the corresponding causal diamond (say N) are located at $y^\mu = (\frac{z_2 - z_3}{2}, \frac{z_2 + z_3}{2})$ and $x^\mu = (\frac{z_3 - z_2}{2}, \frac{z_2 + z_3}{2})$ respectively. In this case: $(u, \bar{u}) \equiv (x^1 - x^0, x^1 + x^0) = (z_2, z_3)$ and $(v, \bar{v}) \equiv (y^1 - y^0, y^1 + y^0) = (z_3, z_2)$. The total modular Hamiltonian K_N for the interval N is the sum of K_N^R and K_N^L , i.e

$$K = K_N^R + K_N^L = \int_{-\infty}^{\infty} d\zeta \frac{(z_2 - \zeta)(\zeta - z_3)}{z_2 - z_3} T_{\zeta\zeta}(\zeta) + \int_{-\infty}^{\infty} d\bar{\zeta} \frac{(z_2 - \bar{\zeta})(\bar{\zeta} - z_3)}{z_2 - z_3} T_{\bar{\zeta}\bar{\zeta}}(\bar{\zeta}) \quad (3.3)$$

This can be derived from the expression of modular Hamiltonian of Rindler half space by a conformal transformation from Rindler wedge to CFT₂ causal diamond. One can similarly define P_D as the antisymmetric combination of K^R and K^L , i.e. $P_D = K^R - K^L$ [20]. Together, K and P_D generates a geometrical flow which preserve the diamond. K generate flows from lower tip to the upper tip, while P_D generates flows from left to the right tip of the diamond.

Consider a CFT₂ interval $N(z_3, z_2)$ on a time slice $T = 0$.⁷ Divide this line segment into two parts $N'(z_1, z_2)$ and $N''(z_3, z_1)$ around a point z_1 . The corresponding causal diamonds of N' and N'' divides the causal diamond of N into four parts such that $(N \supset N', N'', U, L)$, where U and L are the upper and lower diamond as shown in the figure (1).⁸ Using the TT OPE, it can be shown that the $K_{(N', N, N'')}^R$ satisfy the following algebra:

$$[K_{N'}^R, K_N^R] = 2\pi i (K_N^R - K_{N'}^R) \quad (3.4)$$

$$[K_N^R, K_{N''}^R] = 2\pi i (K_N^R - K_{N''}^R) \quad (3.5)$$

$$[K_{N'}^R, K_{N''}^R] = -2\pi i (K_{N'}^R + K_{N''}^R) \quad (3.6)$$

This is isomorphic to the holomorphic $SO(2, 1)$ subsector of the full $SO(2, 2)$ conformal algebras, with the following identifications:⁹

$$K_{N''}^R - K_N^R = \mathbb{L}_1 ; K_N^R = \mathbb{L}_0 ; K_{N'}^R - K_N^R = \mathbb{L}_{-1} \quad (3.7)$$

⁷I.e. $z_{3,2} = \bar{z}_{3,2}$

⁸We will be using the same labels interchangeably for the line segments as well as the corresponding causal diamonds.

⁹We have absorbed the $2\pi i$ factor by redefining $K^{R,L}$ as $\frac{1}{2\pi i} K^{R,L}$.

Similarly, $K_{(N',N,N'')}^L$ can be shown to satisfy the following commutation relations:

$$[K_{N'}^L, K_N^L] = 2\pi i (K_{N'}^L - K_N^L), \quad (3.8)$$

$$[K_N^L, K_{N''}^L] = 2\pi i (K_{N''}^L - K_N^L), \quad (3.9)$$

$$[K_{N'}^L, K_{N''}^L] = 2\pi i (K_{N'}^L + K_{N''}^L) \quad (3.10)$$

This is again isomorphic to the anti-holomorphic $SO(2,1)$ sub-algebra, with the identification:

$$K_{N'}^L - K_N^L = \bar{\mathbb{L}}_1; \quad K_N^L = \bar{\mathbb{L}}_0; \quad K_{N''}^L - K_N^L = \bar{\mathbb{L}}_{-1} \quad (3.11)$$

The remaining right and left chiral generators of the diamond U and L can be expressed in terms of $K_{N,N',N''}^{R,L}$ as follows:

$$K_U^R = K_{N''}^R, \quad K_U^L = K_{N'}^L, \quad K_L^R = K_{N'}^R, \quad K_L^L = K_{N''}^L \quad (3.12)$$

Thus, the six modular generators of the three diamonds, $(K_{N,U,L}^{R(L)})$ also satisfy the $SO(2,1) \times SO(2,1)$ global conformal algebra.

We denote the CKV's associated with these generators $\bar{\mathbb{L}}_{0,\pm 1}$ as $L_{0,\pm}$. It's easy to see that these are simply linear combinations of the standard representations of the global conformal generators $l_{1,0,-1}$ defined earlier. The exact relation between them are given by:

$$l_1 = \frac{2z_2z_3}{z_3 - z_2}L_0 + \frac{z_3^2(z_1 - z_2)}{(z_1 - z_3)(z_3 - z_2)}L_{-1} + \frac{z_2^2(z_1 - z_3)}{(z_1 - z_2)(z_3 - z_2)}L_1 \quad (3.13)$$

$$l_0 = \frac{(z_2 + z_3)}{z_2 - z_3}L_0 + \frac{z_3(z_1 - z_2)}{(z_1 - z_3)(z_2 - z_3)}L_{-1} + \frac{z_2(z_1 - z_3)}{(z_1 - z_2)(z_2 - z_3)}L_1 \quad (3.14)$$

$$l_{-1} = \frac{2}{z_3 - z_2}L_0 + \frac{(z_1 - z_2)}{(z_1 - z_3)(z_3 - z_2)}L_{-1} + \frac{(z_1 - z_3)}{(z_1 - z_2)(z_3 - z_2)}L_1 \quad (3.15)$$

With a similar set of relations between $\bar{l}_{1,0,-1}$ and $\bar{L}_{1,0,-1}$. As is clear from the definitions, in our set-up, $K_N = \mathbb{L}_0 + \bar{\mathbb{L}}_0$ and $\mathbb{L}_{1,-1}$ and $\bar{\mathbb{L}}_{1,-1}$ are it's ± 1 eigenmodes. These eigenmodes are constructed by the K s and P_D s of the regions which reside inside the N itself. This set-up exhibits some other features like modular inclusion as we discuss in the next subsection.

3.2 g-MVA and modular inclusions

Out of the left and right moving CKVs, one can notice that K_N , K_U , K_L and $P_{D,N}$, $P_{D,N'}$, $P_{D,N''}$ are closed under $SO(2,1)$ subalgebra separately.¹⁰

$$[P_{D,N}, P_{D,N'}] = 2\pi i (P_{D,N'} - P_{D,N}) \quad (3.16)$$

$$[P_{D,N}, P_{D,N''}] = 2\pi i (P_{D,N} - P_{D,N''}) \quad (3.17)$$

$$[P_{D,N''}, P_{D,N'}] = 2\pi i (P_{D,N''} + P_{D,N'}) \quad (3.18)$$

Hence, $P_{D,N''} - P_{D,N}$, $P_{D,N}$ and $P_{D,N'} - P_{D,N}$ satisfy the $SO(2,1)$ sub-algebra.

¹⁰However these two $SO(2,1)$ don't commute with each other.

In a similar fashion, one could also obtain the following

$$[K_N, K_U] = K_N - K_U \quad (3.19)$$

$$[K_N, K_L] = K_L - K_N \quad (3.20)$$

$$[K_U, K_L] = K_U + K_L \quad (3.21)$$

Here $K_U - K_N$, K_N and $K_L - K_N$ construct the another $SO(2, 1)$ sub-algebra.

The above commutation relations (3.16) and (3.19) has the structure of modular inclusion as we discuss in detail in the appendix (A). In particular, (3.16) gives an unitary geometric operation using which one can find a map of algebra of observable between the nested diamonds N , N' and N'' . Similarly, using (3.19) we have a map of algebra of observables between N , U and L . See appendix (A), for further details. Using the inclusion properties and the fact that K and P_D of any diamond can be constructed in the basis of modular generators of N , N' , N'' as in (3.13), we could in principle construct algebra of observables of any region (diamond) or provide a map to any diamond in the spacetime from N . However, since here the modular generators in vacuum are constructed out of conformal symmetries, this inclusion property i.e.: the map from different causal domains is just an artefact of the global conformal symmetry.

4 Pulling the $\mathbb{L}_n$ into the bulk

Given the explicit form of the $\mathbb{L}_n$ and $\bar{\mathbb{L}}_n$, one can read off the corresponding CKV's (L_n) from equations (2.7) and (3.1). Their explicit forms are as follows.

$$\begin{aligned} L_n &= \frac{(z_2 - \zeta)^{-n+1}(\zeta - z_3)^{n+1}}{z_3 - z_2} \left(\frac{z_2 - z_1}{z_3 - z_1} \right)^n \partial_\zeta, \\ \bar{L}_n &= -\frac{(z_2 - \bar{\zeta})^{n+1}(\bar{\zeta} - z_3)^{-n+1}}{z_3 - z_2} \left(\frac{z_3 - z_1}{z_2 - z_1} \right)^n \partial_{\bar{\zeta}} \end{aligned} \quad (4.1)$$

Following [42], in this section we will extend the CKV's into the bulk where they would generate bulk diffeomorphisms. From our previous discussion of section 3, we already know what these are for the special case of $n = 0, \pm 1$. Since the $\mathbb{L}_{0,\pm 1}$ and $\bar{\mathbb{L}}_{0,\pm 1}$ generate isometries of the causal diamond of the subregions N' and N'' , their duals would generate boosts around and translations along the respective RT geodesics for N' and N'' .

For the generic n case, we proceed as follows. We first use the following transformation $(\zeta, \bar{\zeta}) \rightarrow (\zeta', \bar{\zeta}')$ to transform $(L_n, \bar{L}_n) \rightarrow (l_n, \bar{l}_n)$.

$$\zeta' = \frac{1}{\beta} \left(\frac{\zeta - z_3}{z_2 - \zeta} \right), \quad \bar{\zeta}' = \beta \frac{z_2 - \bar{\zeta}}{\bar{\zeta} - z_3}, \quad \left(\text{with } \beta = \frac{z_{13}}{z_{21}} \right) \quad (4.2)$$

One can then extend these transformations into the bulk. $(y, \zeta, \bar{\zeta}) \rightarrow (y', \zeta', \bar{\zeta}')$. Working in the Fefferman Graham gauge [41], the corresponding dual bulk transformations are given by:

$$\begin{aligned} \zeta' &= \frac{1}{\beta} \left[\frac{\zeta - z_3}{z_2 - \zeta} - \frac{z_{23}}{z_2 - \zeta} \frac{y^2}{y^2 - (z_2 - \zeta)(\bar{\zeta} - z_3)} \right], \\ \bar{\zeta}' &= \beta \left[\frac{z_2 - \bar{\zeta}}{\bar{\zeta} - z_3} - \frac{z_{23}}{\bar{\zeta} - z_3} \frac{y^2}{y^2 - (z_2 - \zeta)(\bar{\zeta} - z_3)} \right], \quad y' = \frac{y}{y^2 - (z_2 - \zeta)(\bar{\zeta} - z_3)} \end{aligned} \quad (4.3)$$

In the $y \rightarrow 0$ limit, this equation reduces to equation (4.2), as it should. From the above transformations, we can now obtain the expression for the bulk counterpart of the L_n 's. In the primed coordinates, the action of the bulk l_n , is given by [42]

$$l_n^{(b)} = \delta_n \zeta' \partial_{\zeta'} + \delta_n \bar{\zeta}' \partial_{\bar{\zeta}'} + \delta_n y' \partial_{y'} \quad (4.4)$$

where:

$$\delta_n \zeta' = (-\zeta')^{n+1}, \quad \delta_n \bar{\zeta}' = -n(n+1)y'^2(-\zeta')^{n-1}, \quad \delta_n y' = \frac{1}{2}(n+1)y'(-\zeta')^n \quad (4.5)$$

Similar expression may be obtained for the $\bar{l}_n^b$. By using eq. (4.3) and eq. (4.5) in eq. (4.4), we can obtain the explicit expression for L_n^b

$$L_n^b = \frac{(-\zeta')^{n+1}}{z_{21} \cdot A \cdot RT} [U \partial_{\zeta} + V \partial_{\bar{\zeta}} + W \partial_y] \quad (4.6)$$

Here, the expression for ζ' is given by the first of the equations (4.3) and the explicit expressions of U , V , W , A and RT are given below.

$$\begin{aligned} U &= y^4 n(n+1) - A(A + z_{21}(n+1)y^2), \\ V &= -A(A + z_{21}(n+1)(z_2 - \zeta)(\bar{\zeta} - z_3)) + n(n+1)(\bar{\zeta} - z_3)^2(z_2 - \zeta)^2, \\ W &= A(2A + z_{21}((\bar{\zeta} - z_3)(z_2 - \zeta) + y^2)) + 2ny^2(\bar{\zeta} - z_3)(n - z_2 + \zeta) \\ &\quad + nz_{21}[(z_2 - \zeta)^2(\bar{\zeta} - z_3)^2(\zeta - z_3) + y^2(y^2(z_2 + z_3 - 2\zeta) + (z_2 - \zeta)^2(\bar{\zeta} - z_3))], \\ A &= (\bar{\zeta} - z_3)(\zeta - z_3)(z_2 - \zeta) + y^2(z_2 + z_3 - 2\zeta), \\ RT &= (y^2 - (z_2 - \zeta)(\bar{\zeta} - z_3)) \end{aligned} \quad (4.7)$$

An interesting feature of the above formulae is the emergence of the RT geodesic. For instance, notice the appearance of the RT geodesic expression $(y^2 - (z_2 - \zeta)(\bar{\zeta} - z_3))$ in the r.h.s. of eq. (4.3). Thus these equations blow up on the RT surface. This means that the bulk coordinates $(\zeta, \bar{\zeta}, y)$ cover only the region between the geodesic and the boundary. Thus they provide a natural set of coordinates for the entanglement wedge associated to N . The RT geodesic also appears in the expressions for the bulk counterparts of L_n 's given in equation (4.4). The fact that the bulk coordinates and the bulk extensions of the L_n 'know' about the RT geodesic is not surprising. It is simply a reflection of the fact that the boundary L_n are modular eigenmodes by construction and thus has information about the boundary causal diamond of N .

5 Discussion

In summary, we have constructed an infinite class of modular eigenmodes ($\mathbb{L}_n$) for the single interval in the vacuum of CFT_2 . These are expressed as smeared integrals of the stress tensor components and thus exist in any CFT_2 .¹¹ Our construction of these eigenmodes

¹¹Such smeared integrals of the stress tensor has appeared in several different contexts recently, for instance in the study of the light ray operators [54]–[58] as well in the context of the so-called dipolar quantization of CFT_2 as discussed in [59, 60]. We thank Bartek Czek for bringing these works on the dipolar quantization to our attention.

are intimately tied to the causal diamond of N . This fact manifests itself in many of its interesting features. For instance, one way in which this connection to the causal diamond manifests itself is in the way $\mathbb{L}_n$ acts on OPE blocks. We showed that this action is identical to the action of conformal generators on local primary fields in CFT_2 . Coupled with the fact that the OPE blocks have a local description as fields living on the k -space, which is the space of causal diamonds of the CFT_2 , this hints at the possibility of finding an equivalent effective description of the CFT on k -space. We argued that on this k -space, the $\mathbb{L}_n$ seem to generate 1d diffeomorphisms along two independent directions. Unfortunately our discussions are only at a kinematic level, and it would be nice if these ideas can be made more concrete.

The connection to the causal diamonds is even more transparent, in the subclass of the eigenmodes corresponding to $n = 0, \pm 1$. In fact, as we showed, these generators are essentially linear combinations of the modular hamiltonians of the causal diamonds for the subregions N' and N'' of N . We further showed how this structure of the g -MVA realizes modular inclusions within in this setup. The half sided modular inclusion has been studied previously in some examples like certain regions on null plane in higher dimensions [47] and it has been used to show that in certain special situation, black hole interior could be reconstructed from the algebra of exterior region [48]. In our example, the inclusion structure emerges quite naturally due to the rich symmetric structure of the vacuum.¹²

Finally we also discussed the action of the bulk counterparts of the $\mathbb{L}_n$ on the bulk spacetime. We saw that these dual descriptions already ‘know’ about the bulk RT geodesic, which is again a reflection of the close connection of our construction with the causal diamond.

A natural question that arises is whether one can extend this construction of algebra and its representation beyond the vacuum in CFT for at least some class of excited states.¹³ Perhaps a more tractable direction to pursue would be to find the extension of such algebras for disconnected multi-interval cases where analytic expression of modular Hamiltonian are known [43, 44].¹⁴

We hope to return to some of these questions in the near future.

Acknowledgments

SD would like to acknowledge the support provided by the Max Planck Partner Group grant MAXPLA/PHY/2018577. The work of SP and BR was supported by a Junior Research Fellowship(JRF) from UGC.

¹²As a testing ground of such algebraic structure or modular properties it is always very useful to study them in quantum mechanical system having finite dimensional Hilbert space [2]. With this motivation in mind, in appendix A we study inclusion properties in an example of finite dimensional Hilbert space where inclusion algebras are satisfied trivially.

¹³For locally excited states in CFT_2 which are connected to vacuum by local conformal transformation, we do have local expression for modular Hamiltonian in single interval [19, 45]. However, we expect this case to be almost identical to the vacuum case.

¹⁴Recently analytic expression of modular Hamiltonian for intervals in BMS invariant field theories has been discussed where we could study similar construction to study algebra [46].

A Modular inclusion in CFT₂ and finite dimensional system

A.1 Modular inclusion

The Reeh-Schlieder theorem states¹⁵ that an algebra $\mathcal{A}_V$, made out of bounded operators restricted to an arbitrary small open set V in spacetime (flat), is enough to generate (by acting on vacuum) the full vacuum sector of the Hilbert space. Due to this property the vacuum state is said to be ‘cyclic’ w.r.t. the algebra of operators $\mathcal{A}_V$ in that small open region V . Incorporating microcausality, an obvious conclusion can be drawn that such a state is also separating w.r.t. $\mathcal{A}_V$, which means that, there exists no operator in V which annihilate the vacuum. In algebraic QFT, some useful quantum information quantities like Relative entropy, total modular Hamiltonian can be rigorously constructed for such cyclic and separating states of the QFT Hilbert space. In particular, a self-adjoint ‘modular operator’ Δ ($= e^{-K}$, K is the total modular Hamiltonian) and an antiunitary operator ‘modular conjugation’ J are the central objects of ‘Tomita-Takesaki theory’, which lies at the foundation of modular theory or modular algebra. The main result of the Tomita-Takesaki theory is that Δ defines an automorphism which maps an algebra of a region to itself while J defines an isomorphism from the algebra to its commutant $\mathcal{A}'_V$.

$$\Delta^{is} A \Delta^{-is} = \tilde{A}; \quad J A J = A', \quad (A, \tilde{A}) \in \mathcal{A}_V, \quad A' \in \mathcal{A}'_V, \quad \forall s \in \mathbb{R} \quad (\text{A.1})$$

The Δ generates a modular flow w.r.t. total modular Hamiltonian K . Here, the algebra $\mathcal{A}_V$ is considered to be a type of Von-Neumann algebra such that, $\mathcal{A}_V = \hat{\mathcal{A}}_V$. Where, $\hat{\mathcal{A}}_V$ is the algebra of the causal domain of the region V .

Within the context of the Tomita-Takesaki theory, a notion of inclusion of algebras has been discussed -the so-called ‘half-sided modular inclusion’ (hsmi) [49]–[51]. Take two Von-Neumann algebra of observables M, M' , such that the vacuum Ω is a common cyclic and separating state for both of them. We can define $\tilde{M} \subset M$ as the +hsmi if it satisfies the condition that M' is preserved under the modular flow of M , i.e

$$\Delta_M^{-it} \tilde{M} \Delta_M^{it} \subset \tilde{M}, \quad \forall t \geq 0 \quad (\text{A.2})$$

Here $\Delta_M, \Delta_{\tilde{M}}$ are modular operator of $M, \tilde{M}$.¹⁶ Once the above condition is satisfied, one can construct an one-parameter unitary group $U(a)$ on the Hilbert space such that,

$$U(a) = e^{iap}; \quad p \equiv \frac{1}{2\pi} (\ln \Delta_{\tilde{M}} - \ln \Delta_M) \geq 0; \quad \forall a \in \mathbb{R} \quad (\text{A.3})$$

The generator p is a positive operator. In such settings, the following properties hold:

$$\Delta_M^{it} U(a) \Delta_M^{-it} = \Delta_{\tilde{M}}^{it} U(a) \Delta_{\tilde{M}}^{-it} = U(e^{-2\pi t} a); \quad \forall a, t \in \mathbb{R} \quad (\text{A.4})$$

$$\Delta_{\tilde{M}}^{it} = U(1) \Delta_M^{it} U(-1); \quad \forall t \in \mathbb{R} \quad (\text{A.5})$$

¹⁵The readers may look at [2] for a recent review on algebraic QFT and modular theory

¹⁶The corresponding modular conjugation operators are $J_M, J_{\tilde{M}}$. However, in the present context, we won’t need the properties of J s and we only focus on modular flows generated by Δ s. For further details, we refer the readers to the following references [50, 51].

$$\tilde{M} = U(1)MU(-1) \quad (\text{A.6})$$

$$\Delta_M^{it} \Delta_{\tilde{M}}^{-it} = e^{i(-1+e^{-2\pi t})p} \quad (\text{A.7})$$

One can see that the first two relations are solved by

$$[K_M, K_{\tilde{M}}] = 2\pi ip; \quad K_{M, \tilde{M}} = -\ln \Delta_{M, \tilde{M}} \quad (\text{A.8})$$

We get the last two relations from the first two. Hence, if $\tilde{M} \subset M$ is a modular inclusion, then (A.8) must be satisfied. When the condition of inclusion (A.2) is satisfied for $t \leq 0$, it is called -hsmi. For that case, the commutation relation of modular Hamiltonian is given by $[K_M, K_{\tilde{M}}] = -2\pi ip$. Using this $\pm$ hsmi, a representation of the $\text{SL}(2, \mathbb{R})$ could be constructed in the following way [52, 53]

Theorem. *Let M, M_1, M_2 be Von-Neumann algebras on a Hilbert space $\mathcal{H}$ and Ω is a cyclic and separating state $\Omega \in \mathcal{H}$. Assume:*

1. $M_1 \subset M$ is a -hsmi
2. $M_2 \subset M$ is a +hsmi
3. $M_2 \subset M'_1$ is a -hsmi

(where M'_1 is the commutant of M_1 .) Then $\Delta_M^{it}, \Delta_{M_1}^{ir}, \Delta_{M_2}^{is}$, $t, r, s \in \mathbb{R}$ generate a representation of $\text{SL}(2, \mathbb{R})$ where,

$$P \equiv \frac{1}{2\pi} (\ln \Delta_{M_1} - \ln \Delta_M); \quad K \equiv \frac{1}{2\pi} (\ln \Delta_{M_2} - \ln \Delta_M); \quad D \equiv \frac{1}{2\pi} \ln \Delta_M \quad (\text{A.9})$$

In this way, the algebraic structure of modular inclusion provides an interesting way to construct chiral part of 2D conformal algebra.

A.2 Modular inclusion in vacuum CFT_2

Within the set up of section 3, we can explicitly see (3.16) and (3.19) exhibits both $\pm$ hsmi structure. However, (3.16) is constructed out of P_D which is not the modular Hamiltonian. However, we can see P_D s of N, N', N'' satisfies all the criterion of hsmi. Hence, in CFT_2 vacuum, we define two types of inclusion structure which we call ‘ K -inclusion’ and ‘ P_D -inclusion’.

P_D -inclusion. Let us first consider the two nested diamonds N' and N where $N' \subset N$. Since P_D generates a geometric flow from the left tip to the right tip of a diamond, the algebra of smaller nested diamond N' remain invariant under the flow of P_D of the larger diamond N i.e. $e^{-iP_D, Nt} N' e^{iP_D, Nt} \subset N'$. In such case, we call such inclusion $N' \subset N$ as the ‘ P_D -inclusion’ which satisfies (A.2). From the algebra of (3.16) we have seen that $P_{D, N'}$ and $P_{D, N}$ indeed satisfy half sided modular inclusion algebra which is $[P_{D, N}, P_{D, N'}] = 2\pi i(P_{D, N'} - P_{D, N})$. Since P_D is self adjoint, we could construct a self-adjoint $p \equiv P_{D, N} - P_{D, N'}$. Using $U(a)$, we can check the following inclusion property $N' = U(1)NU(-1)$, where $U(a) = e^{iap}$.

Here in the spacetime representation,

$$p_{(N,N')} = \frac{z_{31}(z_2 - \zeta)^2}{z_{12}z_{32}}\partial_\zeta + \frac{z_{31}(z_2 - \bar{\zeta})^2}{z_{12}z_{32}}\partial_{\bar{\zeta}} \quad (\text{A.10})$$

Hence the action of $U(1)$ on spacetime point $(\zeta, \bar{\zeta})$ gives

$$e^{p_{(N,N')}}(\zeta, \bar{\zeta}) = \left(\frac{\alpha z_2(\zeta - z_2) + \zeta}{\alpha(\zeta - z_2) + 1}, \frac{\alpha z_2(\bar{\zeta} - z_2) + \bar{\zeta}}{\alpha(\bar{\zeta} - z_2) + 1} \right); \quad \alpha = \frac{z_{31}}{z_{12}z_{32}} \quad (\text{A.11})$$

Here this particular $\text{SL}(2, \mathbb{R})$ transformation $\zeta \rightarrow \frac{(\alpha z_2 + 1)\zeta - \alpha z_2^2}{\alpha\zeta + 1 - \alpha z_2}$ gives the map from the larger diamond N to smaller diamond N' . For instance, the left tip (z_3, z_3) maps to (z_1, z_1) , upper tip (z_2, z_3) maps to that of N' i.e (z_2, z_1) and so on. Using the reverse transformation $U(-1)$ one could construct N from N' . Similarly we could treat $N'' \subset N$ as a -half sided P_D inclusion as the commutators gives an overall minus sign. In the same way, one can define

$$p_{(N,N'')} = \frac{z_{12}(z_3 - \zeta)^2}{z_{32}z_{31}}\partial_\zeta + \frac{z_{12}(z_3 - \bar{\zeta})^2}{z_{32}z_{31}}\partial_{\bar{\zeta}} \quad (\text{A.12})$$

Here the action of $U(-1)$ is given by

$$e^{-p_{(N,N'')}}(\zeta, \bar{\zeta}) = \left(\frac{(\beta z_3 - 1)\zeta - \beta z_3^2}{\beta\zeta - \beta z_3 - 1}, \frac{(\beta z_3 - 1)\bar{\zeta} - \beta z_3^2}{\beta\bar{\zeta} - \beta z_3 - 1} \right); \quad \beta = \frac{z_{12}}{z_{32}z_{31}} \quad (\text{A.13})$$

In this map $z_2 \rightarrow z_1$ and z_3 remains unchanged and thus it transforms N to N'' . Hence using $p_{N,N'}$, $p_{N,N''}$ consecutively we can map N' to N'' and vice versa. In this way, P_D inclusion gives a natural way to map between diamonds with the structures like N, N', N'' .

K-inclusion. Let us look at the another set of algebra described in (3.19) which provides another notion of modular inclusion which we call ‘ K -modular inclusion’. It consists of three diamonds N, U, L such that $U, L \subset N$. Since the total modular Hamiltonian K generates a flow from lower to upper tip, the algebra of U and L left unchanged under the flow of K^N , i.e $e^{-iK^N t}(U, L)e^{iK^N t} \subset (U, L)$. In a similar way of P_D -inclusion, here we can define a self adjoint $p \equiv K^N - K^{U,L}$. For instance, considering the inclusion $U \subset N$, we have

$$\frac{z_{12}(z_3 - \zeta)^2}{z_{32}z_{31}}\partial_\zeta + \frac{z_{13}(z_2 - \bar{\zeta})^2}{z_{12}z_{32}}\partial_{\bar{\zeta}} \quad (\text{A.14})$$

Hence the action of $U(1)$ gives,

$$e^{p_{(N,U)}}(\zeta, \bar{\zeta}) = \left(\frac{(\beta z_3 - 1)\zeta - \beta z_3^2}{\beta\zeta - \beta z_3 - 1}, \frac{\alpha z_2(\bar{\zeta} - z_2) + \bar{\zeta}}{\alpha(\bar{\zeta} - z_2) + 1} \right) \quad (\text{A.15})$$

In this map, one could see the left tip of N i.e (z_3, z_3) maps to left tip of U i.e (z_3, z_1) , the right tip (z_2, z_2) of N maps to that of U i.e (z_1, z_2) , the lower tip (z_2, z_3) maps to the same (z_1, z_1) and the upper tip remains unchanged for both diamonds. In the similar fashion, we could obtain the map from N to L using -half-sided K -inclusion of $L \subset N$.

Also both K -inclusion and P_D -inclusion of the form (3.19) and (3.16), satisfy $\text{SL}(2, \mathbb{R})$ algebra which we describe above as a theorem.

Using the fact that any modular generators of any diamond can be constructed from the modular algebra of N' , N , N'' and using the above mentioned K and P_D inclusion, we can now reproduce all causal diamonds and the fields of them from the modular conformal generators.

A.3 Modular inclusion in finite dimensional Hilbert space

Let us consider a finite dimensional quantum system and divide it into four subsystems A, A', B and B' , such that the dimensions of subsystems are related in the following way:

$$H_{\text{tot}} = H_A \otimes H_{A'} \otimes H_B \otimes H_{B'}; \quad d_A = d_{A'} = N, d_B = d_{B'} = N'. \quad (\text{A.16})$$

Without any loss of generality, we also assume that the total Hilbert space can be factorized as $H_{\text{tot}} = H_{AA'} \otimes H_{BB'}$, such that, there exists the state vectors $|\psi\rangle \in H_{\text{tot}}$, $|\phi\rangle \in H_{AA'}$ and $|\chi\rangle \in H_{BB'}$ which satisfy

$$|\psi\rangle = |\phi\rangle_{AA'} \otimes |\chi\rangle_{BB'}$$

We will first show that for such construction of the state $|\psi\rangle$, the modular inclusion criterion (A.2) will be automatically satisfied. Here we take M to be the system AB and $\tilde{M}$ to be A . To show this, we first define the corresponding density matrices and reduced density matrices as follows:

$$\rho_\psi = |\phi\rangle \langle \phi| \otimes |\chi\rangle \langle \chi| \quad (\text{A.17})$$

$$\rho_{AB} = \text{tr}_{A'B'} \rho = \text{tr}_{A'} |\phi\rangle \langle \phi| \otimes \text{tr}_{B'} |\chi\rangle \langle \chi| \quad (\text{A.18})$$

$$\rho_{A'B'} = \text{tr}_{AB} \rho = \text{tr}_A |\phi\rangle \langle \phi| \otimes \text{tr}_B |\chi\rangle \langle \chi| \quad (\text{A.19})$$

$$\rho_A = \text{tr}_{A'BB'} \rho = \text{tr}_{A'} |\phi\rangle \langle \phi| \quad (\text{A.20})$$

$$\rho_{A\mathbb{C}} = \text{tr}_B \rho = \text{tr}_A |\phi\rangle \langle \phi| \otimes |\chi\rangle \langle \chi| \quad (\text{A.21})$$

The total modular Hamiltonians $K_{M,\tilde{M}}$ or the modular operator $\Delta_{M,\tilde{M}}$ for the regions M and $\tilde{M}$ are defined as:

$$\Delta_M \equiv \Delta_{AB} = \rho_{AB} \otimes \rho_{A'B'}^{-1} \quad (\text{A.22})$$

$$\Delta_N \equiv \Delta_A = \rho_A \otimes \rho_{A\mathbb{C}}^{-1} \quad (\text{A.23})$$

To begin with, we use Schmidt decomposition of $|\phi\rangle_{AA'}$ and $|\chi\rangle_{BB'}$ as following:

$$|\phi\rangle_{AA'} = \sum_{i=1}^N C_i |i\rangle_A \otimes |i\rangle_{A'} \quad (\text{A.24})$$

And

$$|\chi\rangle_{BB'} = \sum_{k=1}^{N'} D_k |k\rangle_B \otimes |k\rangle_{B'} \quad (\text{A.25})$$

Hence in this basis, we get

$$\begin{aligned} \rho_{AB} &= \sum_{i,k=1}^{N,N'} |C_i|^2 |D_k|^2 |i\rangle_A |k\rangle_B \langle i|_A \langle k|_B \\ \rho_{A'B'} &= \sum_{i,k} |C_i|^2 |D_k|^2 |i\rangle_{A'} |k\rangle_{B'} \langle i|_{A'} \langle k|_{B'} \end{aligned}$$

Using the definition of (A.22), we get:

$$\Delta_{AB}^{it} = \left(\sum_{i,j,k,l} \frac{|C_i|^2 |D_j|^2}{|C_k|^2 |D_l|^2} \right)^{it} (|i\rangle_A |j\rangle_B \langle i|_A \langle j|_B) (|k\rangle_{A'} |l\rangle_{B'} \langle k|_{A'} \langle l|_{B'}). \quad (\text{A.26})$$

To show the inclusion condition (A.2), we need to define an operator which has support only in the region A, as:

$$\sum_{m,n} \mathcal{O}_{m,n} |m\rangle_A \langle n|_A \otimes \mathbb{I}_{A'} \otimes \mathbb{I}_B \otimes \mathbb{I}_{B'} \quad (\text{A.27})$$

Using the definition of Δ , it is straightforward to show that,

$$\begin{aligned} \Delta_{AB}^{-it} \left(\sum_{m,n} \mathcal{O}_{m,n} |m\rangle_A \langle n|_A \otimes \mathbb{I}_{A'} \otimes \mathbb{I}_B \otimes \mathbb{I}_{B'} \right) \Delta_{AB}^{it} \\ = \sum_{m,n} \left(\frac{|C_m|^2}{|C_n|^2} \right)^{it} \mathcal{O}_{m,n} |m\rangle_A \langle n|_A \otimes \mathbb{I}_{A'} \otimes \mathbb{I}_B \otimes \mathbb{I}_{B'} \end{aligned} \quad (\text{A.28})$$

From the above equation, it is clear that the state ψ satisfy the equation (A.2). With this, we want to check explicitly if it satisfies the condition of (A.8). To do so, we need to evaluate $\rho_{A\mathfrak{C}}$. However, $\rho_{A\mathfrak{C}}^{-1}$ may not be defined. Since we are calculating $\ln \Delta_A$, this won't matter. We can write,

$$\ln \Delta_{AB} = \ln \rho_{AB} \otimes \mathbb{I}_{A'B'} - \mathbb{I}_{AB} \otimes \ln \rho_{A'B'} \quad (\text{A.29})$$

$$\ln \Delta_A = \ln \rho_A \otimes \mathbb{I}_{A\mathfrak{C}} - \mathbb{I}_A \otimes \ln \rho_{A\mathfrak{C}} \quad (\text{A.30})$$

To calculate the commutation, we act $\ln \Delta_{AB}$ and $\ln \Delta_A$ consecutively on a basis state $|i\rangle_A |j\rangle_{A'} |i\rangle_B |j\rangle_{B'}$. One can see that, $\ln \Delta_{AB} |i\rangle_A |j\rangle_{A'} |i\rangle_B |j\rangle_{B'} = 0$ So,

$$\langle j'|_{B'} \langle i'|_B \langle j'|_{A'} \langle i'|_A (\ln \Delta_A \ln \Delta_{AB}) |i\rangle_A |j\rangle_{A'} |i\rangle_B |j\rangle_{B'} = 0 \quad (\text{A.31})$$

In the similar manner we can also check that,

$$\ln \Delta_A |i\rangle_A |j\rangle_{A'} |i\rangle_B |j\rangle_{B'} = (\ln |C_i|^2 - \ln |C_j|^2) |i\rangle_A |j\rangle_{A'} |i\rangle_B |j\rangle_{B'}$$

Since $\ln \Delta_{AB} |i\rangle_A |j\rangle_{A'} |i\rangle_B |j\rangle_{B'} = 0$, it follows from above that,

$$\langle j'|_{B'} \langle i'|_B \langle j'|_{A'} \langle i'|_A \ln \Delta_{AB} \ln \Delta_A |i\rangle_A |j\rangle_{A'} |i\rangle_B |j\rangle_{B'} = 0 \quad (\text{A.32})$$

So from the above equations, we finally get

$$\langle j'|_{B'} \langle i'|_B \langle j'|_{A'} \langle i'|_A [\ln \Delta_{AB}, \ln \Delta_A] |i\rangle_A |j\rangle_{A'} |i\rangle_B |j\rangle_{B'} = 0 \quad (\text{A.33})$$

Similarly, one can easily check that

$$\langle j'|_{B'} \langle i'|_B \langle j'|_{A'} \langle i'|_A (\ln \Delta_{AB} - \ln \Delta_A) |i\rangle_A |j\rangle_{A'} |i\rangle_B |j\rangle_{B'} = 0 \quad (\text{A.34})$$

Therefore in such example we get the desired inclusion properties (since it is true for any basis state)

$$[\ln \Delta_{AB}, \ln \Delta_A] = \ln \Delta_{AB} - \ln \Delta_A$$

Thus for such finite dimensional quantum system modular inclusion still holds.

B Commutation relation of modular generators and Virasoro algebra

Here we will reproduce the Virasoro algebra (2.10) from the expression of L_n s which is of the form (2.7), using the commutation relations of stress energy tensors. In CFT₂, TT OPE takes the following form,

$$T(z)T(\omega) = \frac{c/2}{(z-\omega)^4} + \frac{2T(\omega)}{(z-\omega)^2} + \frac{\partial T(\omega)}{z-\omega} + \text{regular terms} \quad (\text{B.1})$$

Using the Sokhotski-Plemelj formula, after analytically continuing to lightcone coordinate by $i\epsilon$ prescription, we get the following stress tensor commutators which we need to evaluate the L_n commutators.

$$[T(\zeta), T(\omega)] = 2\pi i \left[-\frac{c}{12} \partial_\omega^3 \delta(\omega - \zeta) + \delta(\omega - \zeta) \partial_\omega T(\omega) + 2\partial_\omega \delta(\omega - \zeta) T(\omega) \right] \quad (\text{B.2})$$

$$[\bar{T}(\bar{\zeta}), \bar{T}(\bar{\omega})] = -2\pi i \left[-\frac{c}{12} \partial_{\bar{\omega}}^3 \delta(\bar{\omega} - \bar{\zeta}) + \delta(\bar{\omega} - \bar{\zeta}) \partial_{\bar{\omega}} \bar{T}(\bar{\omega}) + 2\partial_{\bar{\omega}} \delta(\bar{\omega} - \bar{\zeta}) \bar{T}(\bar{\omega}) \right] \quad (\text{B.3})$$

Inserting this into the commutator of $\mathbb{L}_m$, we have

$$\begin{aligned} [\mathbb{L}_m, \mathbb{L}_n] &= \int_{-\infty}^{\infty} d\zeta \int_{-\infty}^{\infty} d\omega \frac{(z_2 - \zeta)^{-m+1} (\zeta - z_3)^{m+1}}{(z_2 - z_3)^2} \left(\frac{z_{21}}{z_{31}} \right)^{m+n} (z_2 - \omega)^{-n+1} (\omega - z_3)^{n+1} \\ &\quad \times 2\pi i \left[-\frac{c}{12} \partial_\omega^3 \delta(\omega - \zeta) + \delta(\omega - \zeta) \partial_\omega T(\omega) + 2\partial_\omega \delta(\omega - \zeta) T(\omega) \right] \end{aligned} \quad (\text{B.4})$$

First let us consider the last two term (ignoring the c term) of the $[T, T]$, we have

$$\begin{aligned} \frac{1}{2\pi i} [\mathbb{L}_m, \mathbb{L}_n]^{(1)} &= - \int_{-\infty}^{\infty} d\zeta \frac{(z_2 - \zeta)^{-m-n+2} (\zeta - z_3)^{m+n+2}}{(z_2 - z_3)^2} \left(\frac{z_{21}}{z_{31}} \right)^{m+n} \partial_\zeta T(\zeta) + 2(\text{T.D})_1 \\ &\quad - 2 \int_{-\infty}^{\infty} d\zeta \frac{(z_2 - \zeta)^{-m+1} (\zeta - z_3)^{m+1}}{(z_2 - z_3)^2} \left(\frac{z_{21}}{z_{31}} \right)^{m+n} T(\zeta) \\ &\quad \times \left[(n-1)(z_2 - \zeta)^{-n} (\zeta - z_3)^{n+1} + (n+1)(z_2 - \zeta)^{-n+1} (\zeta - z_3)^n \right] \\ &= 2(\text{T.D})_1 - (\text{T.D})_2 \\ &\quad + \int_{-\infty}^{\infty} d\zeta \frac{(z_2 - \zeta)^{-m-n+1} (\zeta - z_3)^{m+n+1}}{(z_2 - z_3)^2} \\ &\quad \times \left(\frac{z_{21}}{z_{31}} \right)^{m+n} [(m+n-2)(\zeta - z_3) + (m+n+2)(z_2 - \zeta)] T(\zeta) \\ &\quad - \int_{-\infty}^{\infty} d\zeta \frac{(z_2 - \zeta)^{-m-n+1} (\zeta - z_3)^{m+n+1}}{(z_2 - z_3)^2} \\ &\quad \times \left(\frac{z_{21}}{z_{31}} \right)^{m+n} [(2n-2)(\zeta - z_3) + (2n+2)(z_2 - \zeta)] T(\zeta) \\ &= 2(\text{T.D})_1 - (\text{T.D})_2 \\ &\quad + (m-n) \int_{-\infty}^{\infty} d\zeta \frac{(z_2 - \zeta)^{-m-n+1} (\zeta - z_3)^{m+n+1}}{z_2 - z_3} \left(\frac{z_{21}}{z_{31}} \right)^{m+n} T(\zeta) \zeta \end{aligned} \quad (\text{B.5})$$

We can identify the last term as the $(m-n)\mathbb{L}_{m+n}$. Here $(T.D)_{1,2}$ are two total derivative terms coming from the intermediate steps of the partial integration. Here,

$$(T.D)_1 = \int_{-\infty}^{\infty} d\zeta \frac{(z_2 - \zeta)^{-m+1}(\zeta - z_3)^{m+1}}{(z_2 - z_3)^2} \times \left(\frac{z_{21}}{z_{31}}\right)^{m+n} \left[(z_2 - \omega)^{-n+1}(\omega - z_3)^{n+1} \delta(\omega - \zeta) T(\omega) \right]_{\omega=-\infty}^{\omega=\infty} \quad (B.6)$$

Due to the Dirac delta function, the total derivative term inside the bracket, vanishes. Hence this term vanishes. The other term is,

$$(T.D)_2 = \left(\frac{z_{21}}{z_{31}}\right)^{m+n} \left[\frac{(z_2 - \zeta)^{-m-n+2}(\zeta - z_3)^{m+n+2}}{(z_2 - z_3)^2} T(\zeta) \right]_{\zeta=-\infty}^{\zeta=\infty} \quad (B.7)$$

To analyze this term, we need to look at the behavior of the stress tensor near spacetime infinity. From the transformation property of stress tensor we know, $T'(\zeta') = \left(\frac{\partial \zeta'}{\partial \zeta}\right)^{-2} + \text{Schwarzian derivative term}$. We choose a global transformation¹⁷ $\zeta' = \frac{a\zeta+b}{c\zeta+d}$ at $\zeta = \zeta_0 = -\frac{d}{c} + \epsilon$, such that $\zeta'_0 \sim \frac{1}{\epsilon}$. For such choice, we get the transformation of stress tensor in the following way,

$$T'(\zeta'_0) = (c\zeta_0 + d)^4 T(\zeta_0) \sim \frac{1}{\epsilon^4} T(\zeta_0) \quad (B.8)$$

Hence, for $\epsilon \rightarrow 0$, we get the behavior of the stress tensor near infinity as $T(\zeta)|_{\zeta \rightarrow \infty} \sim \frac{1}{\zeta^4}$. Using this, if we look at the term $(T.D)_2$, we get,

$$(T.D)_2 = \left(\frac{z_{21}}{z_{31}}\right)^{m+n} \lim_{\Lambda \rightarrow \infty} \left[\frac{(z_2 - \Lambda)^{-m-n+2}(\Lambda - z_3)^{m+n+2}}{(z_2 - z_3)^2} \frac{1}{\Lambda^4} - (\Lambda \rightarrow -\Lambda) \right] = 0 \quad (B.9)$$

Hence, both the total derivative terms vanishes. Let us now see the contribution coming from the central charge(c) part of the stress tensor commutator.

$$\begin{aligned} \frac{1}{2\pi i} [\mathbb{L}_m, \mathbb{L}_n]^{(2)} = & -\frac{c}{12} \left(\frac{z_{21}}{z_{31}}\right)^{m+n} \int_{-\infty}^{\infty} d\zeta \int_{-\infty}^{\infty} d\omega \frac{(z_2 - \zeta)^{-m+1}(\zeta - z_3)^{m+1}}{(z_2 - z_3)^2} \\ & \times (z_2 - \omega)^{-n+1}(\omega - z_3)^{n+1} \partial_{\omega}^3 \delta(\omega - \zeta) \end{aligned} \quad (B.10)$$

Let us denote the constant term $\frac{c}{12} \left(\frac{z_{21}}{z_{31}}\right)^{m+n} \frac{1}{(z_2 - z_3)^2} \equiv A$. In a similar fashion of the previous calculation, after some simple algebraic steps the final integration is of the following form

$$\begin{aligned} \frac{1}{2\pi i} [\mathbb{L}_m, \mathbb{L}_n]^{(2)} = & (T.D)_3 + (T.D)_4 + (T.D)_5 \\ & + n(n^2 - 1)A(z_2 - z_3)^3 \int_{-\infty}^{\infty} d\zeta (z_2 - \zeta)^{-m-n-1}(\zeta - z_3)^{m+n-1} \end{aligned} \quad (B.11)$$

¹⁷Hence we can get rid of the Schwarzian derivative term.

Here, the total derivative terms (T.D)_{3,4,5} are getting vanished due to the presence of dirac delta function and it's derivatives as we argued before. After carefully choosing a contour, we get the final result of the complex integration as (we choose $\text{Re}[z_2] > 0, \text{Re}[z_3] < 0$)

$$\frac{1}{2\pi i} [\mathbb{L}_m, \mathbb{L}_n]^{(2)} = \frac{c}{12} n(n^2 - 1) \left(\frac{z_{21}}{z_{31}} \right)^{m+n} \frac{\left(\frac{-z_2}{z_2} \right)^{m+n} - \left(\frac{-z_3}{z_3} \right)^{m+n}}{m+n}; \quad (\text{B.12})$$

This term vanishes for any $m+n \neq 0, \in \mathbb{Z}$. To extract the contribution for $m+n=0$, we can perform an analytic continuation by choosing $m+n=\epsilon$, taking $\epsilon \rightarrow 0$. This gives,

$$\frac{1}{2\pi i} [\mathbb{L}_m, \mathbb{L}_n]^{(2)} = \frac{c}{12} n(n^2 - 1) \lim_{\epsilon \rightarrow 0} \left(\frac{z_{21}}{z_{31}} \right)^\epsilon \frac{(-1)^\epsilon - (-1)^{-\epsilon}}{\epsilon} = \frac{c}{12} n(n^2 - 1) 2\pi i \quad (\text{B.13})$$

Hence, combining $[\mathbb{L}_m, \mathbb{L}_n]^{(1)}$ and $[\mathbb{L}_m, \mathbb{L}_n]^{(2)}$, we finally have

$$[\mathbb{L}_m, \mathbb{L}_n] = (m-n) \mathbb{L}_{m+n} + \frac{c}{12} n(n^2 - 1) \delta_{m+n,0} \quad (\text{B.14})$$

Here, $\mathbb{L}_{m,n}$ s are redefined as $\mathbb{L}_{m,n} \rightarrow \frac{1}{2\pi i} \mathbb{L}_{m,n}$.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] R. Haag, *Local quantum physics: fields, particles, algebras*, Springer, Germany (1992).
- [2] E. Witten, *APS medal for exceptional achievement in research: invited article on entanglement properties of quantum field theory*, *Rev. Mod. Phys.* **90** (2018) 045003.
- [3] H. Casini, *Relative entropy and the Bekenstein bound*, *Class. Quant. Grav.* **25** (2008) 205021 [[arXiv:0804.2182](https://arxiv.org/abs/0804.2182)] [[INSPIRE](#)].
- [4] R. Bousso, H. Casini, Z. Fisher and J. Maldacena, *Proof of a quantum bousso bound*, *Phys. Rev. D* **90** (2014) 044002 [[arXiv:1404.5635](https://arxiv.org/abs/1404.5635)] [[INSPIRE](#)].
- [5] R. Bousso, H. Casini, Z. Fisher and J. Maldacena, *Entropy on a null surface for interacting quantum field theories and the Bousso bound*, *Phys. Rev. D* **91** (2015) 084030 [[arXiv:1406.4545](https://arxiv.org/abs/1406.4545)] [[INSPIRE](#)].
- [6] H. Casini and M. Huerta, *A finite entanglement entropy and the c-theorem*, *Phys. Lett. B* **600** (2004) 142 [[hep-th/0405111](https://arxiv.org/abs/hep-th/0405111)] [[INSPIRE](#)].
- [7] H. Casini and M. Huerta, *On the RG running of the entanglement entropy of a circle*, *Phys. Rev. D* **85** (2012) 125016 [[arXiv:1202.5650](https://arxiv.org/abs/1202.5650)] [[INSPIRE](#)].
- [8] H. Casini, E. Teste and G. Torroba, *Relative entropy and the RG flow*, *JHEP* **03** (2017) 089 [[arXiv:1611.00016](https://arxiv.org/abs/1611.00016)] [[INSPIRE](#)].
- [9] H. Casini, E. Testé and G. Torroba, *Markov property of the conformal field theory vacuum and the a theorem*, *Phys. Rev. Lett.* **118** (2017) 261602 [[arXiv:1704.01870](https://arxiv.org/abs/1704.01870)] [[INSPIRE](#)].
- [10] H. Casini, I. Salazar Landea and G. Torroba, *Irreversibility in quantum field theories with boundaries*, *JHEP* **04** (2019) 166 [[arXiv:1812.08183](https://arxiv.org/abs/1812.08183)] [[INSPIRE](#)].

- [11] T. Faulkner, R.G. Leigh, O. Parrikar and H. Wang, *Modular Hamiltonians for deformed half-spaces and the averaged null energy condition*, *JHEP* **09** (2016) 038 [[arXiv:1605.08072](#)] [[INSPIRE](#)].
- [12] J. Koeller, S. Leichenauer, A. Levine and A. Shahbazi-Moghaddam, *Local modular Hamiltonians from the quantum null energy condition*, *Phys. Rev. D* **97** (2018) 065011 [[arXiv:1702.00412](#)] [[INSPIRE](#)].
- [13] S. Balakrishnan, T. Faulkner, Z.U. Khandker and H. Wang, *A general proof of the quantum null energy condition*, *JHEP* **09** (2019) 020 [[arXiv:1706.09432](#)] [[INSPIRE](#)].
- [14] F. Ceyhan and T. Faulkner, *Recovering the QNEC from the ANEC*, *Commun. Math. Phys.* **377** (2020) 999 [[arXiv:1812.04683](#)] [[INSPIRE](#)].
- [15] D.L. Jafferis, A. Lewkowycz, J. Maldacena and S.J. Suh, *Relative entropy equals bulk relative entropy*, *JHEP* **06** (2016) 004 [[arXiv:1512.06431](#)] [[INSPIRE](#)].
- [16] T. Faulkner and A. Lewkowycz, *Bulk locality from modular flow*, *JHEP* **07** (2017) 151 [[arXiv:1704.05464](#)] [[INSPIRE](#)].
- [17] D. Kabat and G. Lifschytz, *Local bulk physics from intersecting modular Hamiltonians*, *JHEP* **06** (2017) 120 [[arXiv:1703.06523](#)] [[INSPIRE](#)].
- [18] G. Sárosi and T. Ugajin, *Modular Hamiltonians of excited states, OPE blocks and emergent bulk fields*, *JHEP* **01** (2018) 012 [[arXiv:1705.01486](#)] [[INSPIRE](#)].
- [19] S. Das and B. Ezhuthachan, *Modular Hamiltonians and large diffeomorphisms in AdS_3* , *JHEP* **12** (2018) 096 [[arXiv:1808.09136](#)] [[INSPIRE](#)].
- [20] B. Czech, L. Lamprou, S. Mccandlish and J. Sully, *Modular Berry connection for entangled subregions in AdS/CFT* , *Phys. Rev. Lett.* **120** (2018) 091601 [[arXiv:1712.07123](#)] [[INSPIRE](#)].
- [21] T. Faulkner, M. Li and H. Wang, *A modular toolkit for bulk reconstruction*, *JHEP* **04** (2019) 119 [[arXiv:1806.10560](#)] [[INSPIRE](#)].
- [22] B. Czech, J. De Boer, D. Ge and L. Lamprou, *A modular sewing kit for entanglement wedges*, *JHEP* **11** (2019) 094 [[arXiv:1903.04493](#)] [[INSPIRE](#)].
- [23] J. De Boer and L. Lamprou, *Holographic order from modular chaos*, *JHEP* **06** (2020) 024 [[arXiv:1912.02810](#)] [[INSPIRE](#)].
- [24] D.D. Blanco, H. Casini, L.-Y. Hung and R.C. Myers, *Relative entropy and holography*, *JHEP* **08** (2013) 060 [[arXiv:1305.3182](#)] [[INSPIRE](#)].
- [25] N. Lashkari, J. Lin, H. Ooguri, B. Stoica and M. Van Raamsdonk, *Gravitational positive energy theorems from information inequalities*, *JHEP* **12** (2016) 109.
- [26] D. Blanco, H. Casini, M. Leston and F. Rosso, *Modular energy inequalities from relative entropy*, *JHEP* **01** (2018) 154 [[arXiv:1711.04816](#)] [[INSPIRE](#)].
- [27] T. Faulkner, M. Guica, T. Hartman, R.C. Myers and M. Van Raamsdonk, *Gravitation from entanglement in Holographic CFTs*, *JHEP* **03** (2014) 051 [[arXiv:1312.7856](#)] [[INSPIRE](#)].
- [28] T. Faulkner, F.M. Haehl, E. Hijano, O. Parrikar, C. Rabideau and M. Van Raamsdonk, *Nonlinear gravity from entanglement in conformal field theories*, *JHEP* **08** (2017) 057 [[arXiv:1705.03026](#)] [[INSPIRE](#)].
- [29] S.R. Roy and D. Sarkar, *Bulk metric reconstruction from boundary entanglement*, *Phys. Rev. D* **98** (2018) 066017 [[arXiv:1801.07280](#)] [[INSPIRE](#)].

- [30] D. Kabat and G. Lifschytz, *Emergence of spacetime from the algebra of total modular Hamiltonians*, *JHEP* **05** (2019) 017 [[arXiv:1812.02915](#)] [[INSPIRE](#)].
- [31] H. Casini, M. Huerta and R.C. Myers, *Towards a derivation of holographic entanglement entropy*, *JHEP* **05** (2011) 036 [[arXiv:1102.0440](#)] [[INSPIRE](#)].
- [32] R. Bousso, V. Chandrasekaran, P. Rath and A. Shahbazi-Moghaddam, *Gravity dual of Connes cocycle flow*, *Phys. Rev. D* **102** (2020) 066008 [[arXiv:2007.00230](#)] [[INSPIRE](#)].
- [33] A. Levine, A. Shahbazi-Moghaddam and R.M. Soni, *Seeing the entanglement wedge*, *JHEP* **06** (2021) 134 [[arXiv:2009.11305](#)] [[INSPIRE](#)].
- [34] B. Czech, L. Lamprou, S. McCandlish, B. Mosk and J. Sully, *A stereoscopic look into the bulk*, *JHEP* **07** (2016) 129 [[arXiv:1604.03110](#)] [[INSPIRE](#)].
- [35] J. de Boer, F.M. Haehl, M.P. Heller and R.C. Myers, *Entanglement, holography and causal diamonds*, *JHEP* **08** (2016) 162 [[arXiv:1606.03307](#)] [[INSPIRE](#)].
- [36] B. Czech, L. Lamprou, S. McCandlish and J. Sully, *Integral geometry and holography*, *JHEP* **10** (2015) 175 [[arXiv:1505.05515](#)] [[INSPIRE](#)].
- [37] S. Das and B. Ezhuthachan, *Spectrum of modular Hamiltonian in the vacuum and excited states*, *JHEP* **10** (2019) 009 [[arXiv:1906.00726](#)] [[INSPIRE](#)].
- [38] S. Das, *Comments on spinning OPE blocks in AdS_3/CFT_2* , *Phys. Lett. B* **792** (2019) 397 [[arXiv:1811.09375](#)] [[INSPIRE](#)].
- [39] S. Ferrara, A.F. Grillo, G. Parisi and R. Gatto, *Covariant expansion of the conformal four-point function*, *Nucl. Phys. B* **49** (1972) 77 [Erratum *ibid.* **53** (1973) 643] [[INSPIRE](#)].
- [40] D. Simmons-Duffin, *Projectors, shadows, and conformal blocks*, *JHEP* **04** (2014) 146 [[arXiv:1204.3894](#)] [[INSPIRE](#)].
- [41] M. Bañados, *Three-dimensional quantum geometry and black holes*, *AIP Conf. Proc.* **484** (1999) 147 [[hep-th/9901148](#)] [[INSPIRE](#)].
- [42] N. Anand, H. Chen, A.L. Fitzpatrick, J. Kaplan and D. Li, *An exact operator that knows its location*, *JHEP* **02** (2018) 012 [[arXiv:1708.04246](#)] [[INSPIRE](#)].
- [43] H. Casini and M. Huerta, *Reduced density matrix and internal dynamics for multicomponent regions*, *Class. Quant. Grav.* **26** (2009) 185005 [[arXiv:0903.5284](#)] [[INSPIRE](#)].
- [44] J. Erdmenger, P. Fries, I.A. Reyes and C.P. Simon, *Resolving modular flow: a toolkit for free fermions*, *JHEP* **12** (2020) 126 [[arXiv:2008.07532](#)] [[INSPIRE](#)].
- [45] J. Cardy and E. Tonni, *Entanglement hamiltonians in two-dimensional conformal field theory*, *J. Stat. Mech.* **1612** (2016) 123103 [[arXiv:1608.01283](#)] [[INSPIRE](#)].
- [46] L. Apolo, H. Jiang, W. Song and Y. Zhong, *Modular Hamiltonians in flat holography and $(W)AdS/WCFT$* , *JHEP* **09** (2020) 033 [[arXiv:2006.10741](#)] [[INSPIRE](#)].
- [47] H. Casini, E. Teste and G. Torroba, *Modular Hamiltonians on the null plane and the Markov property of the vacuum state*, *J. Phys. A* **50** (2017) 364001 [[arXiv:1703.10656](#)] [[INSPIRE](#)].
- [48] R. Jefferson, *Comments on black hole interiors and modular inclusions*, *SciPost Phys.* **6** (2019) 042 [[arXiv:1811.08900](#)] [[INSPIRE](#)].
- [49] H.J. Borchers, *The CPT theorem in two-dimensional theories of local observables*, *Commun. Math. Phys.* **143** (1992) 315 [[INSPIRE](#)].

- [50] H.W. Wiesbrock, *Half sided modular inclusions of von Neumann algebras*, *Commun. Math. Phys.* **157** (1993) 83 [Erratum *ibid.* **184** (1997) 683] [[INSPIRE](#)].
- [51] H.J. Borchers, *On revolutionizing quantum field theory with Tomita's modular theory*, *J. Math. Phys.* **41** (2000) 3604 [[INSPIRE](#)].
- [52] H.-W. Wiesbrock, *Symmetries and modular intersections of von Neumann algebras*, *Lett. Math. Phys.* **39** (1997) 203 [[INSPIRE](#)].
- [53] H.W. Wiesbrock, *Modular intersections of von Neumann algebras in quantum field theory*, *Commun. Math. Phys.* **193** (1998) 269 [[INSPIRE](#)].
- [54] P. Kravchuk and D. Simmons-Duffin, *Light-ray operators in conformal field theory*, *JHEP* **11** (2018) 102 [[arXiv:1805.00098](#)] [[INSPIRE](#)].
- [55] K.-W. Huang, *Lightcone commutator and stress-tensor exchange in $d > 2$ CFTs*, *Phys. Rev. D* **102** (2020) 021701 [[arXiv:2002.00110](#)] [[INSPIRE](#)].
- [56] A. Belin, D.M. Hofman, G. Mathys and M.T. Walters, *On the stress tensor light-ray operator algebra*, *JHEP* **05** (2021) 033 [[arXiv:2011.13862](#)].
- [57] K.W. Huang, *$d > 2$ stress-tensor OPE near a line*, *Phys. Rev. D* **103** (2021) 121702 [[arXiv:2103.09930](#)].
- [58] M. Beşken, J. De Boer and G. Mathys, *On local and integrated stress-tensor commutators*, [arXiv:2012.15724](#) [[INSPIRE](#)].
- [59] N. Ishibashi and T. Tada, *Infinite circumference limit of conformal field theory*, *J. Phys. A* **48** (2015) 315402 [[arXiv:1504.00138](#)] [[INSPIRE](#)].
- [60] N. Ishibashi and T. Tada, *Dipolar quantization and the infinite circumference limit of two-dimensional conformal field theories*, *Int. J. Mod. Phys. A* **31** (2016) 1650170 [[arXiv:1602.01190](#)] [[INSPIRE](#)].