Observables in inhomogeneous ground states at large global charge

Simeon Hellerman,^a Nozomu Kobayashi,^{a,b} Shunsuke Maeda^{a,b}
and Masataka Watanabe^{a,b}

^a*Kavli Institute for the Physics and Mathematics of the Universe (WPI),
The University of Tokyo Institutes for Advanced Study, The University of Tokyo,
Kashiwa, Chiba 277-8583, Japan*

^b*Department of Physics, Faculty of Science, The University of Tokyo,
Bunkyo-ku, Tokyo 133-0022, Japan*

E-mail: simeon.hellerman.3@gmail.com, nozomu.kobayashi@ipmu.jp,
shunsuke.maeda@ipmu.jp, masataka.watanabe@weizmann.ac.il

ABSTRACT: As a sequel to previous work, we extend the study of the ground state configuration of the $D = 3$, Wilson-Fisher conformal $O(4)$ model. In this work, we prove that for generic ratios of two charge densities, ρ_1/ρ_2 , the ground-state configuration is inhomogeneous and that the inhomogeneity expresses itself towards longer spatial periods. This is the direct extension of the similar statements we previously made for $\rho_1/\rho_2 \ll 1$. We also compute, at fixed set of charges, ρ_1, ρ_2 , the ground state energy and the two-point function(s) associated with this inhomogeneous configuration on the torus. The ground state energy was found to scale $(\rho_1 + \rho_2)^{3/2}$, as dictated by dimensional analysis and similarly to the case of the $O(2)$ model. Unlike the case of the $O(2)$ model, the ground also strongly violates cluster decomposition in the large-volume, fixed-density limit, with a two-point function that is negative definite at antipodal points of the torus at leading order at large charge.

KEYWORDS: Conformal Field Theory, Effective Field Theories, Renormalization Group

ARXIV EPRINT: [1804.06495](https://arxiv.org/abs/1804.06495)

Contents

1	Introduction	1
2	Goldstone counting and the (in)homogeneity of large-charge ground states	4
3	The $O(4)$ model at arbitrary charge densities	7
3.1	Parametrizing the charge density	9
3.2	Conformal sigma model from linear sigma model	9
3.3	Restriction to fixed average charge densities $\rho_{1,2}$	10
3.4	Equation of motion for the conformal sigma model	10
4	Observables	17
4.1	Ground state energy	17
4.2	Two-point functions	18
4.3	Cluster decomposition and the infinite volume limit	19
5	Results and discussions	20

1 Introduction

Global symmetries can give conformal field theories interesting and useful simplifications. Asymptotic expansions for dimensions and OPE coefficients of charged local operators in terms of large global charge are sometimes possible [1–4], to any given order, using a small number of unknown coefficients which come from terms in the effective Lagrangian in the large charge sector. These are strikingly parallel to the large-spin expansion using the light-cone bootstrap¹ [7, 8], in spite of the fact that the method of large global charge includes the use of Lagrangian methods as opposed to the abstract conformal bootstrap. These methods, of course, work best in the regime of large charge, and complement the regime of $O(1)$ charges and operator dimensions [9–13], which is effectively accessed using the method of linear programming [14–16] to solve the bootstrap equations.

In [1–3], the operator dimensions of charged local operators are calculated by radially quantizing the large-charge effective Lagrangian on a spherical spatial slice. The idea of these papers are as follows: if we consider a system with large global charge J and charge density ρ , on a spatial slice of the scale R_{geometry} , the large-charge effective Lagrangian has its UV scale at $E_{\text{UV}} \equiv \rho^{\frac{1}{D-1}}$ and IR at $E_{\text{IR}} = 1/R_{\text{geometry}}$, and this large hierarchy, $E_{\text{IR}}/E_{\text{UV}} \propto J^{-\frac{1}{D-1}}$, renders the theory weakly coupled. In other words, we can compute

¹See also more recent work [5, 6].

various physical quantities perturbatively in terms of $1/J$, with all the quantum corrections and higher-derivative terms suppressed.

Before studying the operator dimensions and various physical quantities, the first thing necessary here is to know the structure of the large-charge effective Lagrangian and the nature of the ground state with a fixed set of charges. In the limit where the charge goes to infinity $J \rightarrow \infty$, one may also take the size of the sphere to infinity $\mathcal{V} \rightarrow \infty$ and consider the system on an infinite flat space with fixed average charge density, $\rho \equiv J/\mathcal{V}$. In this limit, many possibilities may be realized:

- (a) The ground state may become a homogeneous configuration, related directly to the thermodynamic limit in infinite volume, at finite chemical potential and zero temperature. In some cases, this is indeed so, and various interesting new phases of matter with spontaneously broken conformal and Lorentz symmetries have been derived, which describe such breaking patterns [1–3].
- (b) The ground state may be homogeneous but the thermodynamic limit in infinite volume may not exist. This possibility is realized by some well-known and perfectly well-behaved CFT, including free complex scalar fields, and superconformal theories with moduli spaces of supersymmetric vacua in flat space. These CFT have discrete spectrum and perfectly well-defined thermodynamics on the sphere, but not in flat space; in the presence of curvature, the ground state at large R -charge is homogeneous and satisfies $T_{00} \propto (\text{Ricci scalar})^{+\frac{1}{2}} \rho$, and so the space of ground states collapses at zero curvature, even in finite volume. Examples include superconformal theories with vacuum manifolds, such as the $3D \mathcal{N} = 2$ superconformal XYZ model at large R -charge and [17], $4D \mathcal{N} = 2$ rank 1 SCFTs at large R -charge [18, 19].
- (c-IR) The ground state may be inhomogeneous in finite volume, with the scale of the inhomogeneity set by the scale of the geometry itself. For instance, on a torus, the ground state may break the translational symmetry to $\mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2}$ or $\mathbb{Z}_k \times U(1)$, depending on the geometry of the torus and the details of the CFT. This case is partly studied in [20] and will be expanded in this paper later.
- (c-UV) The ground state may be inhomogeneous on the scale set by the charge density itself; this possibility would be realized, for instance, by a striped phase, with the periodicity set by $\rho^{-\frac{1}{2}}$ times some fixed constant determined by the dynamics. While we know no examples of a relativistic CFT with such a ground state at large global charge, it is a logical possibility that may be realized in some not-yet-discovered theories. (However see recent work [21] in which the ground state with large charge combined with large angular momentum can be shown to break translational invariance spontaneously at the scale of the charge density itself, in certain limits.)

There is actually one other possibility that the ground state is not semiclassical at all, i.e., the fluctuations are not suppressed by powers of E_{IR}/μ . This by no means happens when all the degrees of freedom are coupled in a generic way. You can nonetheless come up with several examples where this happens — One is two completely decoupled CFTs, one of

which the symmetry acts trivially, and another is just two-dimensional CFT with a global symmetry, in which the current degree of freedom decouples because of Sugawara decomposition. Note that even in 2D CFTs with currents, when the theory has a continuous, rather than discrete, spectrum, the Sugawara decomposition doesn't apply and the theory does not realise this possibility. To the contrary, this kind of theories generically shows a simplification at large charge limit, and has been studied in the context of relativistic strings in the Regge limit [22–28]. We will not further comment on this non-semiclassical possibility which is irrelevant to the topic of this paper.

Studying and classifying the above possibilities are quite important in checking the validity of the effective field theories at large global charge. Especially, when the large-charge ground state realizes the case (c-UV), the suppression of quantum fluctuations in powers of E_{IR}/μ is never there in the first place. Reassuringly, though, there are several facts against (c-UV) both in the cases of $D = 3$, $O(2)$ and $O(4)$ Wilson-fisher fixed points at large charge/set of charges. We will collect evidences against (c-UV) in the first two parts of this paper. In the first part, we are going to, prove that the $O(2)$ model and the $O(4)$ model with vanishing ρ_1 or ρ_2 (charge densities), at large charge realise possibility (a). Also, we prove that for generic charge ratios in the $O(4)$ model on the torus, it is possible to eliminate the possibility that the ground state configuration is inhomogeneous in both cycles.

In the second part of this paper we will particularly concentrate on the Wilson-Fisher $O(4)$ fixed point in $D = 3$ for any set of total charges, $J_{1,2} = \int d^2 \rho_{1,2}$, which are taken to be large in arbitrary fixed ratio. (We partly answered the same question in the limit $J_1/J_2 \ll 1$ in [20]; in this regime the ground state is nearly homogeneous and the inhomogeneity can be treated as a perturbation.) This case is the simplest nontrivial example with an inhomogeneous ground state at large global charge. (Very interesting cases of inhomogeneous ground states combining large charge and angular momentum have also recently been discovered [21].) Extending the result of [2], we find that the ground state solution for large charges in generic charge ratio is inhomogeneous with a family of classical solutions periodic in one spatial direction; but, that the family of solutions has an energetic preference for longer spatial periods, which is eventually bounded by the longest scale of the geometry of the spatial slice. This means that for any set of large charge densities the ground state configuration varies very slowly compared to the scale of the charge density, so that the large-charge effective Lagrangian is parametrically reliable, and the possibility (c-IR) is internally consistent in this case. Although this analysis supports the consistency of the possibility (c-IR) to be realised, it still does not rule out the possibility (c-UV) as we work in the regime of the EFT throughout.

In the last part of this note where the issue with inhomogeneity has been settled, we will compute the classical energy and the two-point functions associated with the ground state configuration on a torus spatial geometry, $\mathbb{R}_t \times T^2$ using the above result. We show that the classical energy scales like $(\rho_1 + \rho_2)^{3/2}$ as expected from previous studies, with subleading correction which goes as $(\rho_1 + \rho_2)^{1/2}$. We also compute the two-point function on the torus from the inhomogeneous ground state, and see the resulting spatial dependence in term of operator insertions. Interestingly, even at leading order, it exhibits a dramatic

difference from that of the $O(2)$ model: the two-point function with insertions at two antipodal points is negative definite and nonzero at leading order in the charge, which can only occur together with a breakdown of cluster decomposition and spontaneous strong spatial inhomogeneity at the infrared scale. We hope these results can be checked against Monte-Carlo simulations as in [31].²

2 Goldstone counting and the (in)homogeneity of large-charge ground states

As promised, we are going to derive several facts about (in)homogeneity of the ground state configuration at large charge. on the torus. This can be done in a simple way by counting the number of Goldstone bosons and matching with the number of available light modes.

The following results can and should be generalised from torus spatial slice to the sphere spatial slice. However, because we only deal with the torus time slice in this paper, we are not going to mention the sphere case. Note that the sphere spatial slice adds a little more complexity to the problem because the symmetry group on it is not Abelian. But this is not in any way an obstacle in expanding these results, as you can use the method of [32] even in the case of non-Abelian symmetries.

A comment on helical symmetries and chemical potentials. Many discussions of finite-density ground states in the condensed matter literature, as well as some recent work on large quantum-number expansions in CFT [2], make use of chemical potentials in order to describe the large charge ground state. This is natural in the thermodynamic limit though slightly less so in finite volume, where the legendre transform between chemical potentials and densities must be replaced by a Fourier transform of the quantum partition function [4].

For this and other reasons, in the present work, we describe the ground state in terms of a classical solution with a helical symmetry, i.e., a symmetry under a combined time translation and global symmetry translation [1, 17, 25, 26]. In classical mechanics the two notions are precisely equivalent after a change of variables: the overall lowest-energy classical solution of the system with chemical potential is always static on general grounds of Hamiltonian mechanics; therefore after a time-dependent global symmetry transformation that removes the chemical potential term from the Hamiltonian, the lowest-energy ground state with a given charge must have a helical symmetry. Quantum corrections to the classical picture of the ground state can be added systematically in an asymptotic expansion in the inverse charge [4, 17].

The equivalence of these two descriptions of the ground state also emphasizes an important point that is sometimes ignored: a chemical potential always preserves the same symmetries — Lorentz and global symmetries — of the system that are preserved by the Hamiltonian without chemical potential: a constant chemical potential can be removed by a change of variables. The change in the ground state of the system induced by a chemical potential (at zero temperature) should always be viewed as spontaneous rather

²For example, see [36].

than explicit breaking. The description in terms of a helical solution merely emphasizes this fact which is otherwise obscured by the nontrivial transformation law after the change of variables.

While the helical frequency is a spontaneous rather than explicit breaking, the only light goldstone modes are those corresponding to symmetries commuting with the generator $\hat{g}$ defining the helical time-dependence $\exp\{i\hat{g}\omega t\}$, since the helical ground state is only time-independent up to a symmetry transformation by $\hat{g}$. The symmetries commuting with the helical frequency are precisely those which would commute with the chemical potential after the change of variables, and it is only these that generate light goldstone modes. The generators not commuting with the helical frequency are “massive goldstone bosons” [32] whose masses are above the cutoff but still precisely fixed by the symmetry algebra (See the supersymmetric version in the $W = \Phi^3$ model for [1], and the non-supersymmetric, $O(2N)$ version [2] for examples of massive goldstone fermions and bosons, respectively, in the sense of [32] in the context of the large charge expansion.). For purposes of counting light goldstone modes we can ignore these and count only symmetry generators that are spontaneously broken by the solution at fixed time, and commute with the helical frequency.³

We will not refer further to the chemical potential in the present paper; we have included these comments only to allow the reader to translate without difficulty between the two points of view.

The case of the $O(2)$ model. The discussion so far has led us to a simple rule for the counting of light modes: the light modes that can be understood as goldstone bosons, correspond one to one with symmetry generators that are spontaneously broken by the configuration at fixed time $t = 0$, and which commute with the generator $\hat{g}$ describing the helical frequency. As an example we will now apply this rule to the case of the large-charge ground state of the $O(2)$ model. We will see the rule gives a simple explanation for the spatial homogeneity of the ground state, that is independent of the details of the Wilsonian action at the conformal fixed point.

In the case of the $O(2)$ model the nature of the large-charge ground state is by now well-understood: in addition to the helical symmetry, the ground state is also spatially homogeneous, and the goldstone-counting rule shows this must be the case, because the $O(2)$ model does not have enough massless fields to realize more than one goldstone boson.

The proof of the homogeneity goes as follows. Assume otherwise, then the inhomogeneity is at some particular scale set by the charge density itself. Then in the IR, the effective action should contain one or more translational Goldstone bosons and one axion from the spontaneous broken $O(2)$ (combined with broken time translation symmetry). Now, remember we started out from a theory of one complex scalar, with two real degrees of freedom. The renormalization group flow takes this UV theory to an IR theory which inevitably includes fewer than two real light degrees of freedom, and hence contradicts with

³This formulation of the goldstone-counting rule, while not manifestly equivalent to the way of counting in [32], is more convenient for our purposes and does work out to the same answer, as illustrated in [2].

the above statement. So by contradiction, we know that the ground state configuration must be homogeneous for the $O(2)$ WF fixed point at large charge.

One could have considered a logical possibility that the charged ground state may be inhomogeneous, and indeed $O(2)$ or $U(1)$ symmetric CFT with a larger number of degrees of freedom, may spontaneously break translational symmetries, because they have enough degrees of freedom from the start, to do so. In the $O(2)$ case, the homogeneity of the ground state is related to the small number of light fields available, and does not follow automatically from the symmetries of the conformal fixed point.

The case of the $O(4)$ model with only one nonzero charge. Now we consider the next-simplest case, already analyzed in [2], and apply the goldstone-counting argument to reproduce some results of that paper regarding the spatial homogeneity of the ground state for various choices of ratios of charge density. In [2] it was shown that a particular choice of (conjugacy classes of) large-charge limit, have a spatially homogeneous ground state, namely those in which the charge density of the $SO(4) \simeq SU(2) \times SU(2)$ symmetry of the CFT is chosen to lie entirely inside one of the two $SU(2)$ factors. That is, letting ρ_1, ρ_2 be the Cartan eigenvalues of the total charge matrix (divided by a factor of the spatial volume), one can show that a classical ground state of the system with those charges, is homogeneous.

Here without loss of generality we set $\rho_2 = 0$, and let us use the notation q as in (3.1). In terms of q , this condition translates to, by looking at (3.5) and (3.6), $q_2 \partial_t q_2 = 0$. Just assume $q_2 = 0$ for the moment, and then the subgroup of $O(4)$ which preserves this condition is $SU(2) \times U(1)$ (overall phase rotation and $SU(2)$ rotation of q_1).

Because of how q is defined, q takes the following form,

$$q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} e^{i\omega_1 t} \sin(p(x)) \\ e^{i\omega_2 t} \cos(p(x)) \end{pmatrix}. \quad (2.1)$$

So in order to have $q_2 = 0$, you inevitably have $p(x) = \pi/2$, so that the configuration is homogeneous.

Now consider the case where $\partial_t q_2 = 0$. The subgroup of $O(4)$ which preserves this condition is the same $SU(2) \times U(1)$ as before. The solution to the equation of motion spontaneously breaks this $SU(2) \times U(1)$ into a smaller subgroup. If we assume the ground state configuration is homogeneous, one of such solutions is

$$\partial_t q = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.2)$$

but the subgroup of $SU(2) \times U(1)$ which respects this particular ground state is only $U(1)$ phase rotation of $\partial_t q_2$.

So, schematically, we get the following breaking pattern;

$$O(4) \xrightarrow{\text{helical frequency}} U(2) = U(1) \times SU(2) \xrightarrow{\text{spontaneous breaking}} U(1) \quad (2.3)$$

The dimension of the coset is $\dim(U(2)/U(1)) = 3$, so we have as many as 3 Goldstone bosons. Note that this counting is precisely what is given in [2]. If you were to break the

ground state homogeneity, you add one or more translational Goldstone bosons to these, making the total number of them 4 or more. But you only have three light real degrees of freedom in the IR; thus by contradiction, the ground state configuration must always be homogeneous in the one-charge case.

The case of the $O(4)$ model with two nonzero charges. In this case, we know from [2] that the configuration is inhomogeneous, so let us first assume the configuration is inhomogeneous only in one direction of the torus. As you still have the freedom to phase rotate q_1 and q_2 , respectively the pattern of breaking by the helical frequency becomes

$$O(4) \xrightarrow{\text{helical frequency}} U(1) \times U(1) \quad (2.4)$$

The inhomogeneity will not let us choose a ground state configuration which is special, like $\partial_t q = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Rather, the ground state configuration can only lie at a very generic point, hence preserves no subgroup of $U(1) \times U(1)$. The spontaneous breaking pattern of it, along with translational symmetries, is therefore,

$$U(1) \times U(1) \times \{\text{translation}\}^2 \xrightarrow{\text{spontaneous breaking}} \{\text{translation}\} \quad (2.5)$$

so that you have three Goldstone boson in the system by looking at the coset dimension.

Now, if you were to break one more translational symmetry, there would be four light Goldstone bosons in the system; however the theory has only had three light real fields with which to realize such excitations, which would be a contradiction, regardless of the form of the Wilsonian action for the light fields. Therefore the translational symmetry of the torus can be only broken in one direction for the case of the $O(4)$ model at generic set of large charges.

3 The $O(4)$ model at arbitrary charge densities

We move on to prove that the possibility (c-IR) is self-consistent by extending the result in [20] to the case of any sets of charges, $J_{1,2}$. In this paper, we denote by $J_{1,2}$ the two independent positive real eigenvalues of the matrix defined by the $SO(4)$ Noether charge. Now, rather than requiring the limit where one of the global charge is much less than the other as was done in [20], we study the ground state of the critical $O(4)$ model, fixing the spatially averaged global charge densities $\rho_{1,2} \equiv J_{1,2}/\mathcal{V}$ in an arbitrary ratio J_1/J_2 . As in [20] we integrate out the mode which becomes heavy at large charge and work within a weakly-coupled conformal sigma model with target space S^3 . The Lagrangian density for this conformal effective theory is singular in the vacuum but is not meant to be used there; it is only meant to be used to study states of energy $O(1)$ or less, above the ground state of large charge density.

In [20], we reproduced the no-go result of [2] in the context of the conformal sigma model: for generic charge densities or chemical potentials for the two independent $O(4)$ charges, there is no spatially homogeneous classical solution. The only homogeneous ground

states occur when the two chemical potentials are equal, or equivalently when the real antisymmetric 4×4 matrix defining the $O(4)$ charge of the state has a vanishing determinant, which in our conventions and those of [2], means $\rho_1 = 0$ or $\rho_2 = 0$.

For any classical theory, the lowest energy solution carrying given global charges always leaves unbroken a helical symmetry, by which we mean a symmetry under combined time translation and global symmetry rotation. Regarding the spatial configurations are necessarily inhomogeneous in the $O(4)$ model with generic $O(4)$ charges, by virtue of [2], as we have stated earlier. Hence in this note, we search for the lowest-energy inhomogeneous helical solution.

For inhomogeneous ground states, the most important qualitative question is whether the inhomogeneity is on the ultraviolet scale, set by the charge density itself, or on the infrared scale, set by the boundary conditions or overall geometry and topology of the spatial slice. It is only in the case where the inhomogeneity is on the infrared scale, that the large-charge effective theory can be used in a straightforward manner.

In the space of CFT with global symmetries, this question does not have a simplistic universal answer: the answer appears to depend on the theory and on the choice of global symmetry quantum numbers. For angular momentum in a single plane, for example, the lowest state is generally a small number of quanta each carrying a large angular momentum [7, 8] and thus the inhomogeneity is on the UV rather than IR scale. On the other hand, for theories in $D \geq 4$ the ground state with angular momentum in multiple planes, taken large in fixed ratio, may have a smooth, semiclassical ground state solution generically, which is inhomogeneous on the infrared scale. Both these behaviours are visible even in free theories.

For a given set of global quantum numbers, then, the behavior of the ground state is a question to be settled by direct calculation. For the case of the $O(4)$ model with generic $O(4)$ charges $\rho_{1,2}$ taken large in fixed ratio, it was shown in [20] that the lowest state has inhomogeneity on the infrared scale, for ρ_1/ρ_2 (or ρ_2/ρ_1) sufficiently small. In the present paper, we show that this result holds for any value of the ratio ρ_1/ρ_2 , and thus the ground state properties at large charge can be analyzed consistently in the effective conformal sigma model for any charge ratio.

The simplest candidate helical solutions leave the translational symmetry unbroken in one spatial direction and break it in the other direction down to a discrete subgroup whose period is ℓ . Two natural questions, which are closely related arise because of this fact. One is which value of ℓ has the lowest energy. The other one is the range of ℓ where the effective field theory is reliable. We will now try to answer this question using effective field theory, and are going to show that where $\ell \gg \sqrt{\rho^{-1}}$, the lowest energy is achieved where ℓ is largest, that is the size of the underlying geometry itself.

Note that it does *not* logically exclude the possibility of a striped phase, where the scale is set by $\sqrt{\rho^{-1}}$ itself so that the EFT breaks down. Although this possibility could not be realized in the case of the $O(2)$ model due to Goldstone counting, in the case of the $O(4)$ model we could not rule out this possibility on the basis of Goldstone-counting alone, due to the fact that any value of ℓ would realize the same symmetry-breaking pattern, and so the Hamiltonian could a priori favor either long or short distance scales for ℓ . The

EFT analysis shows only that a solution with period ℓ that is long on the infrared scale, energetically prefers as long a spatial period ℓ as possible, so that there is no internal instability in the EFT towards inhomogeneity on short scales.

3.1 Parametrizing the charge density

A convenient parametrization for the $O(4)$ charge is as follows. The charge densities take value in the adjoint of $O(4)$, the group of antisymmetric 4×4 complex matrices, which has real eigenvalues that occur in pairs with equal magnitude and opposite sign. We denote two (out of four) positive eigenvalues of the charge density matrix to be J_1 and J_2 ; in the infinite volume limit $\mathcal{V} \rightarrow \infty$, we may take these to infinity, fixing the spatially averaged charge densities $\rho_{1,2} \equiv J_{1,2}/\mathcal{V}$.

For helical solutions, this specific choice of parametrization is equivalent to choosing a complex basis for the fundamental of $U(2) \subset SO(4)$ and parametrizing the charge generator by the two matrix elements on the diagonal. Note that the off-diagonal matrix elements always vanish because the lowest-energy classical solution is always helical. This is the same convention used in the earlier work, [2].

We are now going to put the system on $\mathbb{R} \times \mathbb{R}^2$ and seek for the ground state configuration. This should be regarded as the infinite volume limit of the geometry $\mathbb{R} \times S^2$ or $\mathbb{R} \times T^2$. We will comment on the ground state configurations in finite volume case later on.

3.2 Conformal sigma model from linear sigma model

Now that the convenient parametrization of the charge densities has been made, we describe the $O(4)$ model by four real scalars $X^{1,2,3,4}$, which is then organised into $Q \equiv \begin{pmatrix} X_1 + iX_2 \\ X_3 + iX_4 \end{pmatrix}$, a complex $SU(2)$ doublet. The interacting IR fixed point of the model is given by starting from the UV Lagrangian with the kinetic term for Q plus a quartic potential proportional to $|Q|^4$, with a fine-tuned mass term so that it actually flows to a non-trivial fixed point. We parametrise Q as amplitudes and angles, which is given by

$$Q = A \times q, \quad q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \quad (3.1)$$

where $|q_1|^2 + |q_2|^2 = 1$. We give a large VEV to the A -field, and the resulting leading action in the IR includes a term that is proportional to A^6 , as explained in [1]. The IR Lagrangian we get, henceforth, is

$$\mathcal{L}_{\text{IR}} = \frac{1}{2}(\partial A)^2 + \frac{\gamma}{2}A^2\partial q^\dagger\partial q - \frac{h^2}{6}A^6. \quad (3.2)$$

This is under a RG normalization condition that the two-derivative kinetic term of A is canonical.

Note that we have omitted other terms like Ricci coupling and higher derivative terms, because we are only using the leading large-charge-density term. We already know that there are no homogeneous ground state configurations, so the suppression of these terms, however, should be proven *a posteriori*. Those terms are only suppressed when the scale

of the inhomogeneity, L , is much larger than the UV scale, $(\rho_1 + \rho_2)^{-\frac{1}{2}}$ — otherwise the large-charge EFT is not within its range of validity, and the higher-derivative operators and quantum corrections are out of control, and there is no simplification of the dynamics at large charge densities. In the following we first assume L to be much smaller than the UV scale and derive the ground state configuration, and then justify this assumption later, *a posteriori*.

Under this assumption, we first integrate the A field out, which has a mass scale defined by the charge density itself, which is above the Wilsonian scale we are talking about. By virtue of the EOM for A , we have, as an equilibrium value of A ,

$$\frac{\delta \mathcal{L}_{\text{IR}}}{\delta A} = 0 \iff A^2 = \sqrt{\frac{\partial q^\dagger \partial q}{\gamma^{-1} h^2}} \quad (3.3)$$

Now by using this and the original IR Lagrangian, (3.2), we get

$$\mathcal{L} = b_q \mathcal{L}_0^{3/2} = b_q (\partial q^\dagger \partial q)^{3/2}, \quad (3.4)$$

which is the conformal sigma model whose target space is S^3 and where $|q| = 1$ and $b_q = \sqrt{\gamma^3 h^{-2}}/3$ is an undetermined coefficient from the original large-charge effective action [1].

3.3 Restriction to fixed average charge densities $\rho_{1,2}$

We now put the theory on $\mathbb{R}^2$, so we inevitably have to use the concept of the “fixed average charge density” instead of that of total charge, which is ill-defined. We therefore impose the following conditions onto Noether currents,

$$-\frac{2ib_q}{3} \int dx^i \sqrt{\mathcal{L}_0} [q^\dagger \partial_t q - \text{c.c.}] / \mathcal{V} = \rho_1 + \rho_2 \quad (3.5)$$

$$-\frac{2ib_q}{3} \int dx^i \sqrt{\mathcal{L}_0} [q^\dagger \sigma^3 \partial_t q - \text{c.c.}] / \mathcal{V} = \rho_1 - \rho_2, \quad (3.6)$$

where $\mathcal{V}$ indicates the total volume of the space. Also, let us set $\rho_1 < \rho_2$ for simplicity, but we will comment on the $\rho_2 < \rho_1$ case later on.

The energy density can also be derived from the Lagrangian, which is

$$\mathcal{H} = b_q \sqrt{\dot{q}^\dagger \dot{q} - \partial_i q^\dagger \partial^i q} \times (2\dot{q}^\dagger \dot{q} + \partial_i q^\dagger \partial^i q). \quad (3.7)$$

We will therefore look for the minimizer of above under constraints (3.5) and (3.6).

3.4 Equation of motion for the conformal sigma model

To achieve the ground state solution of (3.4), we set an ansatz that the solution is at least homogeneous in one of the spatial directions, y , and only varies spatially in the x direction.

We also use the helical nature of the ground state solution and the invariance under the combination of $t \rightarrow -t$ and the complex conjugation. Basically these ansatz sets the solution of the form

$$q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} e^{i\omega_1 t} \sin(p(x)) \\ e^{i\omega_2 t} \cos(p(x)) \end{pmatrix}, \quad (3.8)$$

where we are free to set $\omega_1 > \omega_2$ and $p(x)$ takes value in $\mathbb{R}$. Under this parametrization, (3.4) becomes

$$\mathcal{L} = b_q \left[-p'(x)^2 + V(p) \right]^{3/2}, \quad (3.9)$$

where

$$V(p) = \omega_2^2 + (\omega_1^2 - \omega_2^2) \sin^2(p). \quad (3.10)$$

For general helical solutions depending on no more than one spatial direction, we can simplify the equations by reducing them to first order. Local conservation of momentum in the x -direction implies $0 = \partial_\mu T_x^\mu$. For a helical solution independent of the y -direction, the stress tensor is independent altogether of y and t , so the pressure T_{xx} in the x -direction is simply a constant:

$$\partial_\mu T_{xx} = 0.$$

Define a constant κ , of mass dimension $+1$ as the cube root of the pressure, with the coefficient of proportionality $b_q^{-\frac{1}{3}}$ to simplify the formulae:

$$\kappa \equiv b_q^{-\frac{1}{3}} T_{xx}^{\frac{1}{3}}.$$

Now we will use the general formula for the stress tensor in a theory with a Lagrangian that is first-order in derivatives acting on fields of vanishing conformal weight:

$$T_\mu{}^\nu \equiv \delta_\mu{}^\nu \mathcal{L} - \sum_A \Theta_{,\mu}^A \frac{\delta \mathcal{L}}{\delta \Theta_{,\nu}^A} \quad (3.11)$$

where Θ^A runs over all the fields in the system, in this case the three Goldstones parametrizing the S^3 target space. For a helical solution, with $\dot{\chi}_{1,2} = \omega_{1,2}$, the Lagrangian density is

$$\begin{aligned} \mathcal{L} &= b_q \mathcal{L}_0^{\frac{3}{2}}, \\ \mathcal{L}_0 &\equiv V - p'^2, \\ V &\equiv \omega_2^2 + (\omega_1^2 - \omega_2^2) \sin^2(p). \end{aligned}$$

modulo terms of second order in $\chi'_{1,2}$, which do not contribute to the stress tensor in a helical solution because $\chi'_{1,2}$ vanishes in the helical solution itself.

So the stress tensor is

$$T_{xx} = \mathcal{L} - p'(x) \frac{\delta \mathcal{L}}{\delta p'} = b_q \sqrt{-p'(x)^2 + V(p(x))} \left(2p'(x)^2 + V(p(x)) \right), \quad (3.12)$$

which we know is a constant and have already set to be equal to $b_q \kappa^3$. We now have the EOM, which is

$$-\frac{\kappa^6}{4} = -\frac{b_q^{-2} T_{xx}^2}{4} = \left(p'(x)^2 - V(p(x)) \right) \left(p'(x)^2 + \frac{V(p(x))}{2} \right)^2, \quad (3.13)$$

where $\kappa > 0$ because of the positivity of T_{xx} . The meat of this is that the equation of motion has now been reduced to a first-order equation (3.13) with one undetermined constant of motion, κ . In principle we could invert (3.13) algebraically to solve for $(p')^2$ in terms of

$V(p)$ for a given κ , using Cardano's formula for the general solution to a cubic equation, and then solve the first-order autonomous ODE for p as a function of x . However most of the complication involved in such a solution is unnecessary, because we are working only within the regime where the fields are varying on scales L long compared to the ultraviolet scale set by $\omega_{1,2}$, so we need only solve the EOM under the condition

$$p' \ll \omega_{1,2} . \quad (3.14)$$

Indeed, there are other terms in the effective Lagrangian that we have omitted, which would become important if we were to work outside this regime. But now we will now organize the first-order EOM in such a way as to exploit the condition (3.14) in order to solve it.

Remark. Before proceeding, let us make a few comments about how to choose the right solution when solving (3.13) for $(p')^2$. By imposing $(p')^2 > 0$ and $\kappa > 0$, we can have multiple solutions depending on the value of κ . Here, however, we only restrict attention to the case where $0 \leq (p')^2 \leq V(p)/2$. This is equivalent to imposing a condition that $p(x)$ must have a point where its derivative is vanishing. This is natural when we eventually want to put the system on S^2 and compute the dimensions of operators using the Neumann boundary condition on the poles. When we put the system on a torus, T^2 , there can possibly be solutions on different “winding numbers”, that are characterised by $p(x/\ell) = p(0) + n\pi$. The winding number here is not a topological charge, because the map from T^2 to S^3 can only be trivial homotopically. This means that on the torus we will also have to compute their energies separately, to know the true ground state configuration. We will, however, just assume the lowest solution is achieved when $n = 0$ even on the torus — at least we know that this is the case in the homogeneous case, and the continuity requires the statement is also true in a certain subset of J_1/J_2 near zero. Also, this is physically related to the existence of soft modes discussed in [2], and adding soft modes intuitively should increase the energy of the system, not otherwise. Again for these reasons, we will hereafter only consider solutions which has a point at which its derivative vanishes.

Scales in the equation of motion. The dimensionful quantities in the EOM are p', ω_1 and ω_2 , and we are going to consider the regime (3.14). The two frequencies $\omega_{1,2}$ are independent a priori, but their relationship will be fixed in terms of the spatial period of the solution.

First thing to notice is that the spatial period of the solution goes to infinity at $\omega_1 = \omega_2$, because the EOM just gives $p'(x) = 0$. This tells us straight away that the difference in frequencies, $\omega_1 - \omega_2$ must scale differently than either individual frequency: while we can hold ω_1 or ω_2 fixed while taking $p' \rightarrow 0$, we see that $\omega_1 - \omega_2$ must vanish in the limit $p' \rightarrow 0$. Let us now rewrite the first order EOM (3.13) to emphasize the distinction in scalings. We define $\omega_- \equiv \omega_1 - \omega_2$ and expand κ as $\kappa_{[0]} + \Delta\kappa$, where $\kappa_{[0]}$ is the value of κ for a homogeneous solution with a given ω_2 , and $\Delta\kappa \equiv \kappa - \kappa_{[0]}$ is the difference, which must scale as a positive power of ω_- . We could further expand $\Delta\kappa$ as a series $\kappa_{[1]} + \kappa_{[2]} + \dots$ where $\kappa_{[p]}$ is the term of order ω_-^p ; this will not be necessary, however, as we will only be

interested in first-order quantities. So using the formula (3.10), which we recap here,

$$V(p) = \omega_2^2 + (\omega_1^2 - \omega_2^2) \sin^2(p), \quad (3.15)$$

we see that $V = O(\omega_2^2) + O(\omega_2\omega_-)$. So V^3 is of order $O(\omega_2^6) + O(\omega_2^5\omega_-)$. So now, the l.h.s. of (3.13) is identically independent of spacetime; therefore the x -dependent parts of the r.h.s. will have to cancel order by order in ω_-/ω_2 .

The two types of x -dependent terms on the r.h.s. are $p'^2\omega_2^4$ and $\omega_-\omega_2^5 \sin^2(p_0)$. In order for them to cancel, if one is treating p_0 as $O(1)$, one needs to scale ω_- as p'^2/ω_2 .

To make this more concrete, we want to define the length scale ℓ as the inverse of the maximum x -derivative of the p -field. This isn't quite the right thing, because we want to get the right order of magnitude for the length scale not only when the amplitude of the oscillations is of $O(1)$ but also when it is small, whereas assigning ℓ to be $\frac{1}{p'_{\max}}$ would go to infinity in the limit when the amplitude of the oscillations is small but the period is fixed. To repair this deficiency, we multiply by $\sin p_0$, and define:

$$\ell \equiv \frac{\sin(p_0)}{p'_{\max}}. \quad (3.16)$$

We will express ℓ in terms of the actual period of the solution later.

This defines the general length scale characterizing the solution, and we expect the actual spatial period of the solution to be of order ℓ ; we will confirm this expectation when we find the ground-state classical solution exactly.

In terms of ℓ , then, we see ω_- must scale as

$$\omega_- = O\left(\frac{1}{\ell^2\omega_2}\right).$$

We require that there be a point at which p' vanishes, and at that point we have to be able to satisfy the EOM anyway. So this fixes $\Delta\kappa$ at first order completely in terms of ω_2, ω_- and p_0 :

$$\begin{aligned} \Delta\kappa &= \kappa_{[1]} + O(\ell^{-4}\omega_2^{-3}), \\ \kappa_{[1]} &= \omega_- \sin^2(p_0), \end{aligned}$$

which we can also write as

$$\kappa_{[1]} = \omega_2 \eta, \quad \eta \equiv \frac{\omega_-}{\omega_2} \sin^2(p_0).$$

Now, at first order in ω_- , the EOM reads:

$$p'^2 = 2\omega_2\omega_- [\sin^2(p_0) - \sin^2(p)], \quad (3.17)$$

which we can also write as

$$p'^2 = 2\omega_2^2 [\eta - \epsilon \sin^2(p)], \quad \epsilon \equiv \frac{\omega_-}{\omega_2}. \quad (3.18)$$

The quantities ϵ and η are related simply by

$$\eta = \sin^2(p_0)\epsilon,$$

so the ratio η/ϵ is always less than 1, and goes to zero in the linearized limit, where the amplitude of oscillation is small.

The maximum value of p' occurs when $p = 0$, where it takes the value

$$(p')_{\max} = \sqrt{2\omega_2\omega_-} |\sin(p_0)| = \sqrt{2\epsilon}\omega_2 |\sin(p_0)| = \sqrt{2\eta}\omega_2, \quad (3.19)$$

and so the general scale ℓ of the solution, which we defined in (3.16), is

$$\ell = \frac{1}{\sqrt{2\omega_2\omega_-}} = \frac{1}{\sqrt{2\epsilon}\omega_2}. \quad (3.20)$$

This low-energy EOM (3.17) for the helical solution is exactly the equation of motion for the angle θ of the pendulum with length $\mathbf{L}$ in a gravitational field $\mathbf{g}$ under the identification of $p = \theta/2$ and $2\omega_2\omega_- = \mathbf{g}/\mathbf{L}$. We know the exact solution to this type of differential equations so now we can simply solve the equation (3.17) and using it, we can compute the energy of the large charge ground state.

Now we have the analytic solution to (3.17), which is

$$\frac{\sin(p)}{\sin(p_0)} = \operatorname{sn}\left(\frac{x}{\ell}; \sin(p_0)\right) \quad (3.21)$$

where ℓ was already given in (3.20), and $\operatorname{sn}(x; k)$ is the Jacobian elliptic function with modulus k . The quarter period of the solution, L , is given by

$$L = \ell F\left(\frac{\pi}{2}; \sin(p_0)\right) \quad (3.22)$$

where

$$F(p; k) \equiv \int_0^p \frac{d\hat{p}}{\sqrt{1 - k^2 \sin^2(\hat{p})}} = \operatorname{sn}^{-1}(\sin(p); k) \quad (3.23)$$

Fixing the charge densities. We have now solved for the spatial dependence $p(x)$ in terms of the amplitude p_0 and the frequencies $\omega_{1,2}$. However we do not really want ℓ to be an output and we do not really want the spatial frequencies to be inputs. We would like to invert the relationship between the frequencies $\omega_{1,2}$ and the densities $\rho_{1,2}$ so that the latter are the independent variable.

The expressions for the average charge densities given by (3.5) and (3.6), are

$$\rho_1 = \frac{8b_q}{3\mathcal{V}} \int dx^i \omega_1 \sqrt{-p'(x)^2 + V(p(x))} \sin^2(p(x)) \quad (3.24)$$

$$\rho_2 = \frac{8b_q}{3\mathcal{V}} \int dx^i \omega_2 \sqrt{-p'(x)^2 + V(p(x))} \cos^2(p(x)), \quad (3.25)$$

We can compute these integrals analytically at leading-order in the low-energy expansion $\ell \gg 1/\omega_2$. Using the equation of motion (3.20), we can substitute

$$\sqrt{-p'(x)^2 + V(p(x))} = \frac{1}{\sqrt{2\eta}} \frac{p'(x)(1+\eta)}{\sqrt{1 - \frac{\epsilon}{\eta} \sin^2(p(x))}} + O(\epsilon^2). \quad (3.26)$$

Now as the previous subsection essentially states that the low-energy expansion we are interested in is in terms of $\epsilon, \eta \ll 1$ where $\eta/\epsilon = \sin^2(p_0) = O(1)$, we only have to keep track of the leading order contribution in terms of this expansion rule. (3.26), therefore means that at first order in ϵ and η , we can now change the variable in the integrand from x to p itself. Doing this, and using $\eta/\epsilon = \sin^2(p_0)$, we have

$$\rho_1 = \frac{8b_q\omega_2(1+\epsilon)(1+\eta)}{3L\sqrt{2\eta}} \int_0^{p_0} dp \frac{\sin^2(p)}{\sqrt{1 - \frac{\sin^2(p)}{\sin^2(p_0)}}} \quad (3.27)$$

$$= \frac{8b_q\omega_2^2(1+\epsilon+\eta)}{3} \times \Delta\left(\frac{\pi}{2}; \sin(p_0)\right) \quad (3.28)$$

$$\rho_2 = \frac{8b_q\omega_2^2(1+\eta)}{3} \left[1 - \Delta\left(\frac{\pi}{2}; \sin(p_0)\right) \right] \quad (3.29)$$

where

$$\begin{aligned} \Delta(p; k) &\equiv \frac{F(p; k) - E(p; k)}{F(p; k)} \\ F(p; k) &\equiv \int_0^p \frac{d\hat{p}}{\sqrt{1 - k^2 \sin^2(\hat{p})}} \\ E(p; k) &\equiv \int_0^p d\hat{p} \sqrt{1 - k^2 \sin^2(\hat{p})} \end{aligned}$$

As the first argument of the elliptic integral is always $\pi/2$ in this paper, we will omit it hereafter. Incidentally, these object is called the complete elliptic integrals (of the first/second kind).

$$\begin{aligned} \Delta(k) &\equiv \frac{F(k) - E(k)}{F(k)} \\ K(k) &\equiv F(k) \equiv F(\pi/2; k) \equiv \int_0^{\pi/2} \frac{d\hat{p}}{\sqrt{1 - k^2 \sin^2(\hat{p})}} \\ E(k) &\equiv E(p; k) \equiv \int_0^{\pi/2} d\hat{p} \sqrt{1 - k^2 \sin^2(\hat{p})} \end{aligned}$$

These expressions are a bit complicated, and it is instructive to see how they behave in the linearized limit $p_0 \ll 1$, using the fact $\Delta(\frac{\pi}{2}; \sin(p_0)) \sim \frac{\sin^2(p_0)}{2} \sim \frac{p_0^2}{2}$ at leading order in p_0 . We can also immediately reproduce the statement that limit $p_0 \ll 1$ and the limit $J_1 \ll J_2$ is the same thing.

Average energy density. The average energy density for the ground state configuration can be calculated from (3.7) as

$$\mathcal{E} = \frac{b_q}{\mathcal{V}} \int d^2x \sqrt{-p'(x)^2 + V(p(x))} (p'(x)^2 + 2V(p(x))) \quad (3.30)$$

$$= \frac{b_q \kappa^3}{2} + \frac{9}{16} (\rho_1 \omega_1 + \rho_2 \omega_2). \quad (3.31)$$

Now by using

$$(\omega_2)^2 = \frac{3(\rho_1 + \rho_2)}{8b_q} \left(1 - \eta - \frac{\epsilon \rho_1}{\rho_1 + \rho_2} \right), \quad (3.32)$$

we get

$$\mathcal{E} = \frac{3\sqrt{3}}{8\sqrt{2b_q}} (\rho_1 + \rho_2)^{3/2} \times \left(1 + \frac{\eta}{2} + \frac{\epsilon \rho_1}{2(\rho_1 + \rho_2)} \right) \quad (3.33)$$

or you could also write this way using $\eta = \epsilon \sin^2(p_0)$,

$$\mathcal{E} = \frac{3\sqrt{3}}{8\sqrt{2b_q}} (\rho_1 + \rho_2)^{3/2} \times \left[1 + \frac{\eta}{2} \left(1 + \frac{\rho_1}{\sin^2(p_0)(\rho_1 + \rho_2)} \right) \right] \quad (3.34)$$

Now we also have the constraint due to fixed charges, which is

$$\frac{\rho_1}{\rho_1 + \rho_2} = \Delta \left(\frac{\pi}{2}; \sin(p_0) \right) \left(1 + \epsilon \times \Delta \left(\frac{\pi}{2}; \sin(p_0) \right) \right) \quad (3.35)$$

or equivalently,

$$\Delta \left(\frac{\pi}{2}; \sin(p_0) \right) = \frac{\rho_1}{\rho_1 + \rho_2} \left(1 - \frac{\epsilon \rho_2}{\rho_1 + \rho_2} \right), \quad (3.36)$$

which determines $\sin(p_0) = \sqrt{\eta/\epsilon}$. We also have, from the constraint on the charges,

$$\rho_2 = \frac{4b_q \sin^2(p_0)}{3\ell^2} \left[1 - \Delta \left(\frac{\pi}{2}; \sin(p_0) \right) \right] \times \left(1 + \frac{1}{\eta} \right) \quad (3.37)$$

hence

$$\eta = \left[\frac{3(\rho_1 + \rho_2)\ell^2}{4b_q \sin^2(p_0)} - 1 \right]^{-1} = \frac{4b_q \sin^2(p_0)}{3(\rho_1 + \rho_2)\ell^2} \left(1 + O(1/\ell^2) \right) \quad (3.38)$$

The energy density, in terms of the length scale ℓ is, therefore,

$$\mathcal{E} = \frac{3\sqrt{3}}{8\sqrt{2b_q}} (\rho_1 + \rho_2)^{3/2} \times \left(1 + \frac{A}{\ell^2} \right), \quad (3.39)$$

where

$$A \equiv \frac{2b_q}{3(\rho_1 + \rho_2)} \left(\sin^2(p_0) + \frac{\rho_1}{\rho_1 + \rho_2} \right) > 0. \quad (3.40)$$

(This reproduces the result in [20] for $\rho_1/(\rho_1 + \rho_2) \rightarrow 0$.) Therefore, the ground state configuration is achieved when $L = \ell F(\frac{\pi}{2}; \sin(p_0)) \rightarrow \infty$, which is the same result we have already got for $p_0 \ll 1$ in [20].

Two branches of the solution. We have worked in the regime where $\rho_1 < \rho_2$, but what happens if we make $\rho_1 > \rho_2$? Does $\sin(p_0)$ go very close to 1 and as a consequence? The answer is no: one immediate reason is that the expression for the energy must be symmetric under the exchange of ρ_1 and ρ_2 . We could also say that the exchanging the role of them when $\rho_1 > \rho_2$ is energetically favourable. Hence, when ρ_1 exceeds the value of ρ_2 , there is a first order phase transition (strictly only a crossover but gets infinitely sharp at large- J limit) and $\sin(p_0)$ decrease as ρ_1/ρ_2 gets bigger, eventually reaching the same homogeneous solution, $p_0 = 0$.

4 Observables

We compute various quantities for the $D = 3$ Wilson-Fisher conformal $O(4)$ model on $T^2 \times \mathbb{R}$ in this section. Here we set $\ell_2 < \ell_1$ to be the spatial period of the torus, the spatial slice. Imposing the boundary condition on the toric geometry, then, we have $4\ell F(\frac{\pi}{2}; \sin(p_0)) = 4L = \ell_1$ or equivalently, $\ell = \frac{\ell_1}{4F(\frac{\pi}{2}; \sin(p_0))}$. Note that hereafter we only focus ourselves on the case where $\rho_1 < \rho_2$. The result of the other region can be obtained by swapping the two charges, as explained above.

Since we only use the leading order effective Lagrangian and there is no term of order $O(\sqrt{J})$ on the torus, the results for the ground state energy is true up to order $O(\sqrt{J})$. Even though this statement is true, note that the full classification of operators for this model has never been done to this day, and we need this if we wish to use this Lagrangian to estimate the CFT data in a controlled fashion.

4.1 Ground state energy

By using the result of the previous section, we have

$$\mathcal{E} = \frac{3\sqrt{3}}{8\sqrt{2}b_q} \rho^{3/2} \times \left(1 + \frac{16A}{\ell_1^2} \times \left(F \left(\sqrt{\Delta^{-1} \left(\frac{\rho_1}{\rho_1 + \rho_2} \right)} \right) \right)^2 \right), \quad (4.1)$$

where

$$A = \frac{b_q}{3\rho} \left(\Delta^{-1} \left(\frac{\rho_1}{\rho_1 + \rho_2} \right) + \frac{\rho_1}{\rho_1 + \rho_2} \right) \quad (4.2)$$

where Δ^{-1} is the inverse function of $\Delta(k) \equiv \Delta(\pi/2, k)$.

The total energy of the system on the torus is therefore

$$E = \frac{3\sqrt{3}}{8\sqrt{2}b_q\ell_1\ell_2} J^{3/2} \times \left[1 + \frac{16b_q\ell_2}{3J\ell_1} \mathcal{C} \left(\frac{J_1}{J_1 + J_2} \right) \right], \quad (4.3)$$

where

$$\mathcal{C}(x) \equiv \left(\Delta^{-1}(x) + x \right) \times \left(F \left(\sqrt{\Delta^{-1}(x)} \right) \right)^2. \quad (4.4)$$

The complicated function $\mathcal{C}(x)$ can be computed numerically, whose result we plot in figure 1.

Figure 1. Plot of $\mathcal{C}(x)$ for $0 < x < 1/2$. Since $x \equiv \frac{J_1}{J_1+J_2}$ and we take $J_1 < J_2$, this is the region of interest.

In particular, when $J_1 = J_2$, we have

$$\mathcal{C}(1/2) = 7.144, \quad (4.5)$$

so

$$E(J_1 = J_2) = \frac{3\sqrt{3}}{8\sqrt{2b_q\ell_1\ell_2}} J^{3/2} \times \left[1 + \frac{38.1b_q\ell_2}{J\ell_1} \right]. \quad (4.6)$$

We can also see that there is no such corrections in the case where $J_1 = 0$, as expected since this will lead to the homogeneous configuration.

4.2 Two-point functions

Let us calculate the two point function $\langle q_1^*(0,0)q_1(x_1,x_2) \rangle$. At leading order there are just classical contributions, which amount to

$$\langle q_1^*(0)q_1(x) \rangle \propto \int dy_1 dy_2 q_1^*(y_1, y_2) q_1(x_1 + y_1, x_2 + y_2) \quad (4.7)$$

because of the translational symmetry. Then we have

$$\langle q_1^*(0)q_1(x) \rangle = \int dy_1 dy_2 q_1^*(y_1, y_2) q_1(x_1 + y_1, x_2 + y_2) \quad (4.8)$$

$$\propto \int dy_1 \sin(p(y_1)) \sin(p(x_1 + y_1)) \quad (4.9)$$

$$= \sigma^2 \int_0^{4L} dy_1 \operatorname{sn} \left(\frac{y_1}{\ell}; \sigma \right) \operatorname{sn} \left(\frac{x_1 + y_1}{\ell}; \sigma \right) \quad (4.10)$$

Figure 2. $2\pi \langle q_1^*(0) q_1(x_1) \rangle$ and $2\pi \langle q_2^*(0) q_2(x_1) \rangle$ in terms of x_1 when $J_1 = J_2$. Note that they coincide with each other at $x_1 = 0$.

where $\Delta(\pi/2; \sigma) = \rho_1/(\rho_1 + \rho_2)$. Likewise, we have

$$\langle q_2^*(0) q_2(x) \rangle \propto \int_0^{4L} dy_1 \operatorname{dn} \left(\frac{y_1}{\ell}; \sigma \right) \operatorname{dn} \left(\frac{x_1 + y_1}{\ell}; \sigma \right) \quad (4.11)$$

Note that we have kept all the σ dependence to check the result in the homogeneous limit, $\sigma \rightarrow 0$. Also incidentally, because of the charge conservation, $\langle q_1^*(0) q_2(x) \rangle$ must vanish.

These integrals can be performed analytically too, but this gives rather involved expressions involving compositions of elliptic integrals and their inverses, and derivatives of those. It is simpler for practical purposes to express the observables in terms of numerically evaluated integrals; as an illustration, we compute these integrals numerically when $\rho_1 = \rho_2$. We also set $\ell = 1$ with no loss of generality, since the theory is conformal. Note that the constraint $\Delta(\pi/2; \sigma) = \rho_1/(\rho_1 + \rho_2) = 1/2$ is equivalent to $\sigma = 0.91 \dots$. Now by using this, we have the graph of the two-point function below (figure 2). Meanwhile in the limit where $\rho_1 \rightarrow 0$ the solution is just $p(x) = 0$ and we just get $\langle q_1^*(0) q_1(x_1) \rangle = 0$ and $\langle q_2^*(0) q_2(x_1) \rangle = 1$.

4.3 Cluster decomposition and the infinite volume limit

We have seen that the large-charge ground state of the system with a generic charge ratio J_2/J_1 , is inhomogeneous, and that the inhomogeneity is energetically favored at the longest possible distance scales. In finite volume, with a toroidal spatial slice, this pattern of global and translational symmetry breaking, produces a distinctive signature in the correlation functions in the large-charge ground state. Although our focus is on finite-volume observables, it is worth trying to understand the meaning of the infrared-enhanced inhomogeneity in the infinite-volume limit.

Since the inhomogeneity is clearly relevant at the longest distance scales, these IR-inhomogeneous solutions, may be understood as realizing a disordered phase of some kind,

in which certain local operators fail to cluster. This is distinct from a striped phase, in which the inhomogeneity would have a characteristic scale, which is inevitably fixed by the average charge density itself, in the case of a conformal field theory such as the critical $O(4)$ model.

The phenomenon of violation of cluster decomposition is quite common under renormalization group flow. What is unfamiliar here is the direct visibility of cluster-nondecomposition in large-charge perturbation theory, despite the strong coupling of the underlying model, where the perturbation parameter is $1/(\mu|x|)$, with $|x|$ being the distance between operators and μ being the chemical potential. The non-clustering of the two-point function comes from averaging over classical solutions that break the translational invariance spontaneously at the IR scale. Other examples of perturbatively calculable breakdown of cluster decomposition under renormalization group flow are known, and are related by dualities to cases where defect operators fail to cluster due to strong coupling effects: for instance, see [33–35].

5 Results and discussions

In this paper we began with a general argument, by counting the number of Goldstones and matching with the number of available light modes, that for the $O(2)$ model at large global charge, the ground state configuration can only be homogeneous regardless of the detailed form of the Wilsonian action at the fixed point. We went on to extend the argument to give similar constraints for the ground state of the $O(4)$ model, especially that there can be inhomogeneity in only one direction in the case of generic charge densities.

We have also followed up on the argument due to [2] that for the $O(4)$ model with a generic set of fixed large charges, the lowest energy classical solution is inhomogeneous: making the argument more concrete, we have constructed the inhomogeneous ground state solution explicitly. We also see that there are two branches of the configuration, namely when $J_1 < J_2$ and when vice versa; there is a continuous but nondifferentiable dependence of observables on the charge ratio, at the point $J_2/J_1 = 1$. This solution confirms the analysis of [20] at charge densities close to the homogeneous case, $J_2/J_1 \ll 1$.

We have computed the leading order energy on the torus and saw that it scales as $(J_1 + J_2)^{3/2}$ at leading order just like the $O(2)$ model. We computed the two-point functions on the torus too. These results can be checked with Monte-Carlo simulations, and it would be interesting if one could compute those quantities numerically and verify our results. However be warned that the result can only be checked by running a simulation at quite a low temperature, due to the presence of the soft modes of [2].

The appearance of perturbatively calculable disorder is interesting, and should make it possible to be far more explicit about the long-distance behavior of the $O(4)$ model (and other $O(2N)$ models) at finite chemical potential. In principle one could calculate explicitly which sets of operators obey cluster decomposition and which do not. However such a calculation would take a more thorough study of the classical solutions than we have performed so far. For instance, one would have to be more careful about the possibility of ground states preserving nontrivial combinations of translational and internal global sym-

metries while breaking each separately; we have ignored those possible breaking patterns in this paper.

In future work we hope to study the large-charge EFT on the sphere, in order to compute ground-state conformal dimensions via radial quantization. There the symmetry breaking pattern will be much more interesting because of the non-Abelian nature of the symmetry group on the sphere spatial slice. Also, the accuracy of the large- J expansion can be improved by computing subleading corrections from higher-derivative operators in the EFT and quantum effects.

Acknowledgments

The authors are grateful to Domenico Orlando and Susanne Reffert for valuable discussions. SM and MW acknowledge the support by JSPS Research Fellowship for Young Scientists. The work of SH is supported by the World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan; by the JSPS Program for Advancing Strategic International Networks to Accelerate the Circulation of Talented Researchers; and also supported in part by JSPS KAKENHI Grant Numbers JP22740153, JP26400242. SH is also grateful to the Physics Department at Harvard University, the Walter Burke Institute for Theoretical Physics at Caltech, and the Galileo Galilei Institute for generous hospitality while this work was in progress.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] S. Hellerman, D. Orlando, S. Reffert and M. Watanabe, *On the CFT Operator Spectrum at Large Global Charge*, *JHEP* **12** (2015) 071 [[arXiv:1505.01537](https://arxiv.org/abs/1505.01537)] [[INSPIRE](#)].
- [2] L. Álvarez-Gaumé, O. Loukas, D. Orlando and S. Reffert, *Compensating strong coupling with large charge*, *JHEP* **04** (2017) 059 [[arXiv:1610.04495](https://arxiv.org/abs/1610.04495)] [[INSPIRE](#)].
- [3] A. Monin, D. Pirtskhalava, R. Rattazzi and F.K. Seibold, *Semiclassics, Goldstone Bosons and CFT data*, *JHEP* **06** (2017) 011 [[arXiv:1611.02912](https://arxiv.org/abs/1611.02912)] [[INSPIRE](#)].
- [4] O. Loukas, *Abelian scalar theory at large global charge*, *Fortsch. Phys.* **65** (2017) 1700028 [[arXiv:1612.08985](https://arxiv.org/abs/1612.08985)] [[INSPIRE](#)].
- [5] L.F. Alday, *Large Spin Perturbation Theory for Conformal Field Theories*, *Phys. Rev. Lett.* **119** (2017) 111601 [[arXiv:1611.01500](https://arxiv.org/abs/1611.01500)] [[INSPIRE](#)].
- [6] L.F. Alday, *Solving CFTs with Weakly Broken Higher Spin Symmetry*, *JHEP* **10** (2017) 161 [[arXiv:1612.00696](https://arxiv.org/abs/1612.00696)] [[INSPIRE](#)].
- [7] Z. Komargodski and A. Zhiboedov, *Convexity and Liberation at Large Spin*, *JHEP* **11** (2013) 140 [[arXiv:1212.4103](https://arxiv.org/abs/1212.4103)] [[INSPIRE](#)].
- [8] A.L. Fitzpatrick, J. Kaplan, D. Poland and D. Simmons-Duffin, *The Analytic Bootstrap and AdS Superhorizon Locality*, *JHEP* **12** (2013) 004 [[arXiv:1212.3616](https://arxiv.org/abs/1212.3616)] [[INSPIRE](#)].

- [9] F. Kos, D. Poland, D. Simmons-Duffin and A. Vichi, *Precision Islands in the Ising and $O(N)$ Models*, *JHEP* **08** (2016) 036 [[arXiv:1603.04436](#)] [[INSPIRE](#)].
- [10] F. Kos, D. Poland, D. Simmons-Duffin and A. Vichi, *Bootstrapping the $O(N)$ Archipelago*, *JHEP* **11** (2015) 106 [[arXiv:1504.07997](#)] [[INSPIRE](#)].
- [11] F. Kos, D. Poland and D. Simmons-Duffin, *Bootstrapping the $O(N)$ vector models*, *JHEP* **06** (2014) 091 [[arXiv:1307.6856](#)] [[INSPIRE](#)].
- [12] P. Dey, A. Kaviraj and K. Sen, *More on analytic bootstrap for $O(N)$ models*, *JHEP* **06** (2016) 136 [[arXiv:1602.04928](#)] [[INSPIRE](#)].
- [13] K. Diab, L. Fei, S. Giombi, I.R. Klebanov and G. Tarnopolsky, *On C_J and C_T in the Gross-Neveu and $O(N)$ models*, *J. Phys. A* **49** (2016) 405402 [[arXiv:1601.07198](#)] [[INSPIRE](#)].
- [14] R. Rattazzi, V.S. Rychkov, E. Tonni and A. Vichi, *Bounding scalar operator dimensions in 4D CFT*, *JHEP* **12** (2008) 031 [[arXiv:0807.0004](#)] [[INSPIRE](#)].
- [15] S. El-Showk, M.F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin and A. Vichi, *Solving the 3D Ising Model with the Conformal Bootstrap*, *Phys. Rev. D* **86** (2012) 025022 [[arXiv:1203.6064](#)] [[INSPIRE](#)].
- [16] S. El-Showk, M.F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin and A. Vichi, *Solving the 3d Ising Model with the Conformal Bootstrap II. c -Minimization and Precise Critical Exponents*, *J. Stat. Phys.* **157** (2014) 869 [[arXiv:1403.4545](#)] [[INSPIRE](#)].
- [17] S. Hellerman, S. Maeda and M. Watanabe, *Operator Dimensions from Moduli*, *JHEP* **10** (2017) 089 [[arXiv:1706.05743](#)] [[INSPIRE](#)].
- [18] S. Hellerman and S. Maeda, *On the Large R -charge Expansion in $\mathcal{N} = 2$ Superconformal Field Theories*, *JHEP* **12** (2017) 135 [[arXiv:1710.07336](#)] [[INSPIRE](#)].
- [19] S. Hellerman, S. Maeda, D. Orlando, S. Reffert and M. Watanabe, *Universal correlation functions in rank 1 SCFTs*, *JHEP* **12** (2019) 047 [[arXiv:1804.01535](#)] [[INSPIRE](#)].
- [20] S. Hellerman, N. Kobayashi, S. Maeda and M. Watanabe, *A Note on Inhomogeneous Ground States at Large Global Charge*, *JHEP* **10** (2019) 038 [[arXiv:1705.05825](#)] [[INSPIRE](#)].
- [21] G. Cuomo, A. de la Fuente, A. Monin, D. Pirtskhalava and R. Rattazzi, *Rotating superfluids and spinning charged operators in conformal field theory*, *Phys. Rev. D* **97** (2018) 045012 [[arXiv:1711.02108](#)] [[INSPIRE](#)].
- [22] D.J. Gross and P.F. Mende, *String Theory Beyond the Planck Scale*, *Nucl. Phys. B* **303** (1988) 407 [[INSPIRE](#)].
- [23] D.J. Gross and P.F. Mende, *The High-Energy Behavior of String Scattering Amplitudes*, *Phys. Lett. B* **197** (1987) 129 [[INSPIRE](#)].
- [24] J. Polchinski and A. Strominger, *Effective string theory*, *Phys. Rev. Lett.* **67** (1991) 1681 [[INSPIRE](#)].
- [25] M. Baker and R. Steinke, *An Effective string theory of Abrikosov-Nielsen-Olesen vortices*, *Phys. Lett. B* **474** (2000) 67 [[hep-ph/9905375](#)] [[INSPIRE](#)].
- [26] J. Sonnenschein and D. Weissman, *Quantizing the rotating string with massive endpoints*, *JHEP* **06** (2018) 148 [[arXiv:1801.00798](#)] [[INSPIRE](#)].
- [27] S. Hellerman and S. Maeda, *On Vertex Operators in Effective String Theory*, [arXiv:1701.06406](#) [[INSPIRE](#)].

- [28] S. Hellerman, S. Maeda, J. Maltz and I. Swanson, *Effective String Theory Simplified*, *JHEP* **09** (2014) 183 [[arXiv:1405.6197](#)] [[INSPIRE](#)].
- [29] S. Hellerman and I. Swanson, *String Theory of the Regge Intercept*, *Phys. Rev. Lett.* **114** (2015) 111601 [[arXiv:1312.0999](#)] [[INSPIRE](#)].
- [30] K.G. Wilson and M.E. Fisher, *Critical exponents in 3.99 dimensions*, *Phys. Rev. Lett.* **28** (1972) 240 [[INSPIRE](#)].
- [31] D. Banerjee, S. Chandrasekharan and D. Orlando, *Conformal dimensions via large charge expansion*, *Phys. Rev. Lett.* **120** (2018) 061603 [[arXiv:1707.00711](#)] [[INSPIRE](#)].
- [32] H. Watanabe, T. Brauner and H. Murayama, *Massive Nambu-Goldstone Bosons*, *Phys. Rev. Lett.* **111** (2013) 021601 [[arXiv:1303.1527](#)] [[INSPIRE](#)].
- [33] S. Hellerman, A. Henriques, T. Pantev, E. Sharpe and M. Ando, *Cluster decomposition, T-duality, and gerby CFT's*, *Adv. Theor. Math. Phys.* **11** (2007) 751 [[hep-th/0606034](#)] [[INSPIRE](#)].
- [34] A. Caldararu, J. Distler, S. Hellerman, T. Pantev and E. Sharpe, *Non-birational twisted derived equivalences in abelian GLSMs*, *Commun. Math. Phys.* **294** (2010) 605 [[arXiv:0709.3855](#)] [[INSPIRE](#)].
- [35] S. Hellerman and E. Sharpe, *Sums over topological sectors and quantization of Fayet-Iliopoulos parameters*, *Adv. Theor. Math. Phys.* **15** (2011) 1141 [[arXiv:1012.5999](#)] [[INSPIRE](#)].
- [36] D. Banerjee, S. Chandrasekharan, D. Orlando and S. Reffert, *Conformal dimensions in the large charge sectors at the $O(4)$ Wilson-Fisher fixed point*, *Phys. Rev. Lett.* **123** (2019) 051603 [[arXiv:1902.09542](#)] [[INSPIRE](#)].