RECEIVED: May 29, 2021

REVISED: July 21, 2021

ACCEPTED: July 21, 2021

PUBLISHED: August 9, 2021

Celestial superamplitude in $\mathcal{N} = 4$ SYM theory

Hongliang Jiang

*Centre for Research in String Theory, School of Physics and Astronomy,
Queen Mary University of London,
Mile End Road, E1 4NS, U.K.*

E-mail: h.jiang@qmul.ac.uk

ABSTRACT: Celestial amplitude is a new reformulation of momentum space scattering amplitudes and offers a promising way for flat holography. In this paper, we study the celestial amplitudes in $\mathcal{N} = 4$ Super-Yang-Mills (SYM) theory aiming at understanding the role of superconformal symmetry in celestial holography. We first construct the superconformal generators acting on the celestial superfield which assembles all the on-shell fields in the multiplet together in terms of celestial variables and Grassmann parameters. These generators satisfy the superconformal algebra of $\mathcal{N} = 4$ SYM theory. We also compute the three-point and four-point celestial super-amplitudes explicitly. They can be identified as the conformal correlation functions of the celestial superfields living at the celestial sphere. We further study the soft and collinear limits which give rise to the super-Ward identity and super-OPE on the celestial sphere, respectively. Our results initiate a new perspective of understanding the well-studied $\mathcal{N} = 4$ SYM amplitudes via 2D celestial conformal field theory.

KEYWORDS: Conformal Field Theory, Scattering Amplitudes, Gauge-gravity correspondence, Superspaces

ARXIV EPRINT: [2105.10269](https://arxiv.org/abs/2105.10269)

Contents

1	Introduction	2
2	Superconformal symmetry on the celestial sphere	4
2.1	Kinematics on the celestial sphere	4
2.2	Celestial amplitude	6
2.3	Poincaré symmetry	7
2.4	Bulk conformal symmetry	8
2.5	Bulk superconformal symmetry and $\mathcal{N} = 4$ SYM	10
2.5.1	Superconformal symmetry in spinor-helicity basis	10
2.5.2	Quasi-supersymmetry generators in the celestial basis	11
2.5.3	Celestial on-shell superfield and celestial superamplitude	12
3	Celestial superamplitudes in $\mathcal{N} = 4$ SYM theory	17
3.1	Celestial amplitudes in YM theory	17
3.2	Four-point celestial superamplitudes in SYM	18
3.3	Structure at higher points	20
3.4	Three-point celestial superamplitudes in SYM	21
4	Soft theorem and Ward identity	24
4.1	Conformally soft gluon theorem in YM	24
4.2	Conformally supersoft theorem in SYM	26
4.3	Super-Ward identity from conformally supersoft theorem	29
5	Collinear limit and OPE	29
5.1	OPE in YM	30
5.2	OPE in SYM	30
5.3	OPE of soft mode	32
5.3.1	$\Delta \rightarrow 1$	32
5.3.2	$\Delta \rightarrow \frac{1}{2}$	33
5.3.3	$\Delta \rightarrow 0$	34
6	Conclusion and outlook	36
A	Soft and collinear limit in YM and SYM	37
A.1	Soft theorem in YM and SYM	37
A.2	Collinear limit in YM	38
A.3	Collinear limit in SYM	39

1 Introduction

In recent years, a new formalism of scattering amplitude was developed in the study of flat holography [1]. In contrast to the conventional momentum basis manifesting the translational invariance, the new formalism makes full use of the Lorentz symmetry of spacetime and expands the wave functions of particles in terms of the boost eigenstates. Mathematically, by performing the Mellin transformation on the momentum space scattering amplitude [1–3], one obtains the so-called *celestial amplitude* which behaves just like a correlation function in 2D CFT under Lorentz transformation [1, 4, 5]. As such, this reformulation offers a promising approach to establish flat holography by relating the quantum gravity scattering process in the 4D bulk Minkowski spacetime with the observables of the 2D *celestial conformal field theory* (CCFT) living on the boundary celestial sphere.¹ This goes beyond the paradigm of AdS/CFT [6] which is well-understood now after two decades of efforts. Guided by the general holographic principle [7, 8] and considering the great success of AdS/CFT, it is natural to ask how to concretely realize the holographic duality for quantum gravity in the absence of cosmological constant? Recent fruitful results are showing that celestial holography in terms of CCFT opens up an interesting avenue to this question!

Besides the connection to flat holography, the CCFT also offers a very different perspective to understand the scattering amplitude itself. For example, the soft limits and collinear limits feature some universal properties of quantum fields and are vital for the self-consistency of S-matrix. It turns out that they just correspond to the standard Ward identity and OPE in CCFT [9–13]. The CCFT also enables us to succinctly characterize the infinite number of non-trivial symmetries in asymptotically flat spacetime [14]. Furthermore, the CCFT provides an advantageous language in describing the UV and IR behavior of scattering amplitudes [15].

Despite its compelling role, the CCFT itself remains poorly understood. On the one hand, the celestial conformal field theory behaves in many aspects like an ordinary two dimensional conformal field theory. Many techniques in the ordinary CFT can be borrowed and applied to the CCFT. For example, one can construct the stress tensor [3, 16] and it is meaningful to discuss the operator production expansion (OPE) of various operators [9, 13]. On the other hand, the CCFT features lots of peculiarities. In particular, the spectra of operators and the notion of inner product, conjugation seem to be drastically different from the conventional CFTs [4]. See [17, 18] for recent discussions. Nevertheless, the rich symmetries and various self-consistency conditions already impose stringent constraints on CCFT.

In this paper, we add one more ingredient to the CCFT by considering the quantum field theories in 4D with superconformal symmetry.² In particular, we will be interested in the $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory. The $\mathcal{N} = 4$ SYM theory enjoys a huge

¹Although such a reformation can be studied in arbitrary dimensions and for arbitrary fields with arbitrary masses, here we will be mainly focusing on the massless fields in four dimensions.

²It is worth emphasizing that the supersymmetry discussed in this paper refers to the supersymmetry in the 4D bulk instead of the conventional 2D supersymmetry. Also, the conformal symmetry here refers to the 4D bulk conformal symmetry, which is different from the conformal symmetry on the 2D celestial sphere induced by the Lorentz transformation in 4D.

amount of symmetries which make it tractable or even exactly solvable in many situations. Especially, the scattering amplitudes in $\mathcal{N} = 4$ SYM have been intensively studied. Furthermore, the $\mathcal{N} = 4$ SYM theory provides the first and most successful example of AdS/CFT correspondence. Given its unique role, we are naturally led to study the celestial amplitudes of $\mathcal{N} = 4$ SYM theory and its role in flat holography.

We begin our studies with symmetry, which is arguably one of the most important guiding principles in physics. For celestial amplitude, a very natural question is how the various symmetries act on the celestial amplitudes and how to constrain the structure of celestial amplitudes by imposing symmetries. For Poincaré symmetry and conformal symmetry, the symmetry generators acting on celestial amplitudes are found in [19]. While for supersymmetry, it was also discussed in [20, 21]. Here we generalize the discussions to superconformal symmetry and especially the maximal superconformal symmetry enjoyed by the $\mathcal{N} = 4$ super-Yang-Mills theory. To this aim, we make full use of the on-shell superfield formalism by introducing Grassmann variables. We construct the superconformal generators in celestial superspace and checked that they indeed satisfy the superconformal algebra. These superconformal generators act on the on-shell celestial superfield which we constructed explicitly and fulfills a representation of the superconformal algebra. These superconformal generators can also act on the celestial super-amplitude, which is the Mellin transformation of momentum space superamplitude, and can be regarded as the super-correlator of on-shell celestial superfields. The superconformal symmetry thus imposes non-trivial constraints on the super-correlators: they must be invariant under the action superconformal generators.

Using Mellin transformation, we also explicitly compute the three-point and four-point celestial superamplitudes. In particular, we are able to recast the celestial superamplitude of $\mathcal{N} = 4$ SYM theory in a form with not only manifest Lorentz symmetry but also obvious dilatation symmetry and supersymmetry, in contrast to the manifest translational invariance and supersymmetry of momentum space superamplitude. By expanding the celestial superamplitudes in Grassmann variables, one can obtain all the component amplitudes including the gluon amplitudes.

We will also study the soft limit and collinear limit of scattering amplitudes. In these limits, the amplitudes diverge, featuring some universal behavior of quantum fields. For pure Yang-Mills (YM) theory, these two limits enable us to establish the Ward identity and extract the OPE of celestial operators, respectively [9–13]. We generalize these discussions to the $\mathcal{N} = 4$ SYM theory. In particular, the leading supersoft theorem in momentum space turns into the conformally supersoft theorem on the celestial sphere. Taking into account the color factors, this further leads to the Ward identity relating celestial super-correlators with and without soft “super”-current insertion. Physically, this arises from the asymptotic large gauge transformation of gluons, and the soft “super”-current is essentially the soft gluons with both helicities, generating the “super”-Kac-Moody symmetry on the celestial sphere. Again, the “super” here refers to the bulk supersymmetry and differs from the standard super-Kac-Moody in 2D.

While in the collinear limit, the two particles pass through the nearby points on the celestial sphere, thus corresponding to the coincident limit of operators. So the collinear

limit enables us to derive the OPE of operators in celestial CFT [9, 13]. Indeed, using the super-split factor for the collinear limit of SYM theory, we are able to compute the OPE between two super-operators in $\mathcal{N} = 4$ SYM theory. The super-OPE encodes all the component OPEs which can be obtained easily by expanding in Grassmann variables.

We further consider the super-OPE in the limit $\Delta \rightarrow 1, 1/2, 0, \dots$. The resulting super-OPEs manifest themselves with a pole in conformal dimension Δ . The operators with these special values of conformal dimension are thus the soft “super”-current. In particular, when $\Delta \rightarrow 1$, the resulting OPE agrees with the one derived directly from the soft gluon theorem, thus providing a consistent check of CCFT. While for $\Delta = 1/2, 0$, they correspond to soft gluinos and soft scalars, which are related to soft gluons via supersymmetry.

This paper is organized as follows. In section 2, we start with the symmetry aspect of celestial amplitudes by constructing explicitly all the generators of superconformal symmetry as well as the corresponding superfield. In section 3, we compute the three and four-point celestial super-amplitudes explicitly and comment on the general structure at higher points. In section 4, we study the soft limit of $\mathcal{N} = 4$ SYM theory which results in the conformally soft theorem and Ward identity for celestial operators. In section 5, we compute the OPE of super-operators from the collinear limit of $\mathcal{N} = 4$ SYM theory. We summarize the results of this paper in section 6 and outlook possible future directions. We also include appendix A, where we review the soft and collinear limits of YM and $\mathcal{N} = 4$ SYM theory.

Note added. After the completion of this work, we learned that Andreas Brandhuber, Graham R. Brown, Joshua Gowdy, Bill Spence and Gabriele Travaglini, who are in the same group at QMUL, had been working independently on a very similar problem [22], which overlaps with our section 2 and 3. The results in these two papers are consistent although using different approaches. We were not aware of the work of each other until the final stage due to Covid restrictions.

2 Superconformal symmetry on the celestial sphere

In this section, we will study the symmetry aspect of CCFT. We will construct the superconformal generators (2.78)–(2.89) in terms of the celestial sphere coordinates and Grassmann variables. They act on the celestial on-shell superfield (2.66) which can be obtained as the Mellin transformation of the on-shell superfield in momentum space. These generators are shown to satisfy the superconformal algebra and impose Ward identity constraints on the celestial superamplitudes.

In our discussion, we make full use of the on-shell superfield formalism which is especially powerful for $\mathcal{N} = 4$ SYM theory and brings a lot of simplifications. Note that the role of supersymmetry was studied before in [20] where the discussions were mainly based on the component form of each multiplet.

2.1 Kinematics on the celestial sphere

Let us start the celestial story with kinematics. A massless particle travels along light-like direction in spacetime and is described by null momentum p^μ satisfying $p^\mu p_\mu = 0$. We can

parametrize the null momentum in terms of complex coordinates $(z, \bar{z})$ on the so-called celestial sphere

$$p^\mu = (E, p^x, p^y, p^z) = \omega q^\mu(z, \bar{z}), \quad (2.1)$$

where $\omega > 0$ and the null vector q^μ is

$$q^\mu(z, \bar{z}) = \left(1 + z\bar{z}, \quad z + \bar{z}, \quad -i(z - \bar{z}), \quad 1 - z\bar{z}\right). \quad (2.2)$$

The point $(z, \bar{z})$ can be regarded as the point on the celestial sphere where the massless particle crosses, while ω is the energy along this null direction.

Under $\text{SL}(2, \mathbb{C})$, the $(z, \bar{z})$ coordinates on the celestial sphere transform as

$$z \rightarrow \frac{az + b}{cz + d}, \quad \bar{z} \rightarrow \frac{\bar{a}\bar{z} + \bar{b}}{\bar{c}\bar{z} + \bar{d}}, \quad a, b, c, d \in \mathbb{C}, \quad ad - bc = 1. \quad (2.3)$$

As a consequence, we have

$$p^\mu \rightarrow \Lambda^\mu{}_\nu p^\nu, \quad q^\mu \rightarrow |cz + d|^{-2} \Lambda^\mu{}_\nu q^\nu, \quad \omega \rightarrow |cz + d|^2 \omega, \quad (2.4)$$

where $\Lambda^\mu{}_\nu$ is the associated Lorentz transformation satisfying $\Lambda^T \eta \Lambda = \eta$. This is not surprising as $\text{SO}(3, 1) \simeq \text{SL}(2, \mathbb{C})$. Therefore the Lorentz transformation in 4D Minkowski spacetime induces the conformal transformation on the 2D celestial sphere, suggesting the existence of a conformal field theory living there.

A nice property of null vector q^μ is

$$q^\mu(z, \bar{z}) q_\mu(z', \bar{z}') = -2|z - z'|^2. \quad (2.5)$$

A spinning massless particle also carries internal degrees of freedom which can be described in terms of polarization vectors. For gauge boson, the two polarizations can be naturally chosen as

$$\sqrt{2}\epsilon_+^\mu(q) = \sqrt{2}\epsilon_z^\mu = \partial_z q^\mu = (\bar{z}, 1, -i, -\bar{z}), \quad \sqrt{2}\epsilon_-^\mu(q) = \sqrt{2}\epsilon_{\bar{z}}^\mu = \partial_{\bar{z}} q^\mu = (z, 1, i, -z), \quad (2.6)$$

satisfying

$$\epsilon_z \cdot q = \epsilon_{\bar{z}} \cdot q = 0, \quad \epsilon_z \cdot \epsilon_z = \epsilon_{\bar{z}} \cdot \epsilon_{\bar{z}} = 0, \quad \epsilon_z \cdot \epsilon_{\bar{z}} = 1. \quad (2.7)$$

We are interested in the particles involved in a scattering process where crucially, we need to distinguish whether the particle is incoming or outgoing. So the precise way to parametrize a null momentum p^μ in scattering amplitudes is

$$p^\mu = \varepsilon \omega q^\mu(z, \bar{z}) = \varepsilon \omega \left(1 + z\bar{z}, \quad z + \bar{z}, \quad -i(z - \bar{z}), \quad 1 - z\bar{z}\right), \quad \omega > 0, \quad z, \bar{z} \in \mathbb{C}, \quad (2.8)$$

where the extra factor $\varepsilon = \text{sgn}(p^0)$ further takes into account whether the particle is incoming ($\varepsilon = -1$) or outgoing ($\varepsilon = +1$).

In the spinor-helicity formalism,

$$p^{\alpha\dot{\alpha}} = \sigma_\mu^{\alpha\dot{\alpha}} p^\mu = \begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^3 \end{pmatrix} = \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}}, \quad (2.9)$$

where we choose $\sigma_\mu^{\alpha\dot{\alpha}} = (1, \boldsymbol{\sigma})$. For real physical momentum, the two spinors are related by complex conjugation $\tilde{\lambda}^{\dot{\alpha}} = \varepsilon(\lambda^\alpha)^*$.

For the null vector introduced in (2.2), we have

$$q^{\alpha\dot{\alpha}} = \sigma_\mu^{\alpha\dot{\alpha}} q^\mu = \begin{pmatrix} 2 & 2\bar{z} \\ 2z & 2z\bar{z} \end{pmatrix} = 2 \begin{pmatrix} 1 \\ z \end{pmatrix} \begin{pmatrix} 1 & \bar{z} \end{pmatrix}. \quad (2.10)$$

This suggests that we can take

$$\lambda^\alpha \equiv \langle p |^\alpha = \sqrt{2\omega} \begin{pmatrix} 1 \\ z \end{pmatrix} = \sqrt{2\omega} \langle q |^\alpha, \quad \tilde{\lambda}^{\dot{\alpha}} \equiv |p]^{\dot{\alpha}} = \varepsilon \sqrt{2\omega} \begin{pmatrix} 1 \\ \bar{z} \end{pmatrix} = \varepsilon \sqrt{2\omega} |q]^{\dot{\alpha}}, \quad (2.11)$$

where we introduce the following notation for later convenience

$$\langle q |^\alpha = \begin{pmatrix} 1 \\ z \end{pmatrix}, \quad |q]^{\dot{\alpha}} = \begin{pmatrix} 1 \\ \bar{z} \end{pmatrix}. \quad (2.12)$$

The angle and square brackets of spinors are

$$\langle ij \rangle = -2\sqrt{\omega_i \omega_j} z_{ij}, \quad z_{ij} = z_i - z_j, \quad (2.13)$$

$$[ij] = 2\varepsilon_i \varepsilon_j \sqrt{\omega_i \omega_j} \bar{z}_{ij}, \quad \bar{z}_{ij} = \bar{z}_i - \bar{z}_j. \quad (2.14)$$

2.2 Celestial amplitude

The scattering amplitude in momentum space is given by

$$\mathcal{A}_n(J_i, p_i^\mu) = A_n(J_i, p_i^\mu) \delta^4\left(\sum_i p_i^\mu\right), \quad p_i^\mu = \varepsilon_i \omega_i q_i^\mu, \quad (2.15)$$

where ε_i labels the ingoing (−) and outgoing (+) particle, J_i is the helicity of massless particle, and δ -function is to enforce the momentum conservation. The scattering amplitudes should respect the symmetry of the theory, especially the Poincaré symmetries which consist of translations and Lorentz transformations. In momentum space, this means

$$\sum_j \varepsilon_j P_j^\mu \mathcal{A}_n(J_i, p_i^\mu) = 0, \quad \sum_j M_j^{\mu\nu} \mathcal{A}_n(J_i, p_i^\mu) = 0, \quad (2.16)$$

where the generators of translations and Lorentz transformations act in momentum space as

$$P^\mu = \varepsilon p^\mu = \omega q^\mu, \quad M^{\mu\nu} = p^\mu \frac{\partial}{\partial p_\nu} - p^\nu \frac{\partial}{\partial p_\mu}. \quad (2.17)$$

The amplitude in the form of (2.15) obviously satisfies the first equation in (2.16), while the second equation in (2.16) is not manifest anymore. So, the momentum space scattering amplitudes make the translation symmetry manifest but leave the Lorentz symmetry obscure.

By contrast, the celestial amplitude makes the Lorentz symmetry evident, while sacrificing the manifestation of translational invariance. The celestial amplitude is defined as the Mellin transformation of momentum space amplitude

$$\mathcal{M}_n(\Delta_i, J_i, z_i, \bar{z}_i) = \left(\prod_{j=1}^n \int_0^\infty d\omega_j \omega_j^{\Delta_j-1} \right) \mathcal{A}_n(J_i, p_i^\mu), \quad (2.18)$$

where we parametrize the momentum of each particle as (2.8). As such, the celestial amplitude can be regarded as a conformal correlator on the celestial sphere

$$\mathcal{M}_n(\Delta_i, J_i, z_i, \bar{z}_i) = \langle O_{\Delta_1, J_1}^{\varepsilon_1}(z_1, \bar{z}_1) \cdots O_{\Delta_n, J_n}^{\varepsilon_n}(z_n, \bar{z}_n) \rangle, \quad (2.19)$$

and transforms under $\text{SL}(2, \mathbb{C})$ as

$$\mathcal{M}_n \left(\Delta_i, J_i, \frac{az_i + b}{cz_i + d}, \frac{\bar{a}\bar{z}_i + \bar{b}}{\bar{c}\bar{z}_i + \bar{d}} \right) = \left(\prod_{j=1}^n (cz_j + d)^{\Delta_j + J_j} (\bar{c}\bar{z}_j + \bar{d})^{\Delta_j - J_j} \right) \mathcal{M}_n(\Delta_i, J_i, z_i, \bar{z}_i), \quad (2.20)$$

where $a, b, c, d \in \mathbb{C}$, $ad - bc = 1$. This $\text{SL}(2, \mathbb{C})$ symmetry on the celestial sphere is just the Lorentz symmetry of the spacetime. So the Lorentz symmetry is manifest, while the translation symmetry is not obvious anymore. To show that Poincaré symmetries are indeed preserved, we first need find the generators of those symmetries in the celestial basis.

Note that the conformal dimension and spin/helicity are related to the holomorphic and anti-holomorphic weights as

$$\Delta = h + \bar{h}, \quad J = h - \bar{h}, \quad (2.21)$$

where J in 2d is the spin of the operator, while in 4d it is the helicity of the particle.

In order to have a complete and delta-function-normalizable conformal primary wavefunctions in conformal basis, the conformal dimension should reside in the principal continuous series of the unitary representations of $\text{SL}(2, \mathbb{C})$ [4]

$$\Delta = 1 + i\lambda, \quad \lambda \in \mathbb{R}. \quad (2.22)$$

We will also use the following relations later:

$$\partial_\Delta \equiv \partial_{i\lambda} \equiv -i\partial_\lambda \equiv \frac{1}{2}(\partial_h + \partial_{\bar{h}}), \quad \partial_J \equiv \frac{1}{2}(\partial_h - \partial_{\bar{h}}). \quad (2.23)$$

2.3 Poincaré symmetry

As in the ordinary 2d CFT case, the $\text{SL}(2, \mathbb{C})$ is generated by $L_k, \bar{L}_k$, $k = 0, \pm 1$ whose actions on primary operator $O_{h, \bar{h}}$ with conformal weights $h, \bar{h}$ are [23]

$$\mathcal{L}_k \cdot O_{h, \bar{h}}(z, \bar{z}) \equiv [\mathcal{L}_k, O_{h, \bar{h}}(z, \bar{z})] = L_k O_{h, \bar{h}}(z, \bar{z}), \quad (2.24)$$

$$\bar{\mathcal{L}}_k \cdot O_{h, \bar{h}}(z, \bar{z}) \equiv [\bar{\mathcal{L}}_k, O_{h, \bar{h}}(z, \bar{z})] = \bar{L}_k O_{h, \bar{h}}(z, \bar{z}), \quad (2.25)$$

where the differential operators $L_k, \bar{L}_k$ are given by

$$L_k = h(k+1)z^k + z^{k+1}\partial_z, \quad \bar{L}_k = \bar{h}(k+1)\bar{z}^k + \bar{z}^{k+1}\partial_{\bar{z}}, \quad (2.26)$$

while $\mathcal{L}_k, \bar{\mathcal{L}}_k$ should be regarded as charges in the field theory.³ They satisfy the commutation relations

$$[\mathcal{L}_k, \mathcal{L}_l] = (k-l)\mathcal{L}_{k+l}, \quad [\bar{\mathcal{L}}_k, \bar{\mathcal{L}}_l] = (k-l)\bar{\mathcal{L}}_{k+l}, \quad (2.27)$$

³More precisely, they are the mode expansion of stress tensor in the case of Virasoro symmetry.

and

$$[L_k, L_l] = -(k-l)L_{k+l}, \quad [\bar{L}_k, \bar{L}_l] = -(k-l)\bar{L}_{k+l}, \quad (2.28)$$

Note that there is an extra minus sign in (2.28). The difference of the extra minus comes from the fact that transformations compose in the opposite order when acting on the coordinates.⁴ We will mostly work with $L_k, \bar{L}_k$ when discussing the conformal symmetry. The generators of Lorentz transformations are the linear combinations of these $\text{SL}(2, \mathbb{C})$ conformal generators $L_k, \bar{L}_k$.

Let us now switch to translation symmetry. In momentum space, the action of translation just multiplies the amplitudes by the momentum (2.16). In celestial basis, the insertion of momentum $P_j^\mu = \omega_j q_j^\mu$ in the integrand (2.18) implements a multiplication of q_j^μ and a shift of conformal dimension $\Delta_j \rightarrow \Delta_j + 1$ (without changing the helicity). As a result, $h_j \rightarrow h_j + \frac{1}{2}, \bar{h}_j \rightarrow \bar{h}_j + \frac{1}{2}$. Therefore the translation generators in the celestial basis are given by [19]

$$P^\mu = q^\mu e^{\partial_\Delta} = q^\mu e^{(\partial_h + \partial_{\bar{h}})/2}. \quad (2.29)$$

Using (2.9), we can rewrite it as⁵

$$P^{\alpha\dot{\alpha}} = \frac{1}{2} \sigma_\mu^{\alpha\dot{\alpha}} P^\mu = \frac{1}{2} \sigma_\mu^{\alpha\dot{\alpha}} q^\mu e^{(\partial_h + \partial_{\bar{h}})/2} = |q|^{\dot{\alpha}} \langle q |^\alpha e^{(\partial_h + \partial_{\bar{h}})/2}, \quad (2.30)$$

whose form further suggests us to write in the product form:

$$\mathbf{p}^\alpha = \langle q |^\alpha e^{\partial_h/2} = \left(\frac{1}{z} \right) e^{\partial_h/2}, \quad \tilde{\mathbf{p}}^{\dot{\alpha}} = |q|^{\dot{\alpha}} e^{\partial_{\bar{h}}/2} = \left(\frac{1}{\bar{z}} \right) e^{\partial_{\bar{h}}/2}, \quad P^{\alpha\dot{\alpha}} = \frac{1}{2} \sigma_\mu^{\alpha\dot{\alpha}} P^\mu = \mathbf{p}^\alpha \tilde{\mathbf{p}}^{\dot{\alpha}}. \quad (2.31)$$

2.4 Bulk conformal symmetry

Besides the Poincaré symmetry considered above, we can further consider the 4D conformal symmetry acting on bulk spacetime. Especially, the special conformal generator in the celestial basis has been worked out in [19]. As translation generators, we find special conformal generators can also be compactly rewritten as

$$\mathbf{k}^\alpha = \left(\frac{\partial_z}{z\partial_z + 2h - 1} \right) e^{-\partial_h/2}, \quad \tilde{\mathbf{k}}^{\dot{\alpha}} = \left(\frac{\partial_{\bar{z}}}{\bar{z}\partial_{\bar{z}} + 2\bar{h} - 1} \right) e^{-\partial_{\bar{h}}/2}, \quad K^{\alpha\dot{\alpha}} = \frac{1}{2} \sigma_\mu^{\alpha\dot{\alpha}} K^\mu = \mathbf{k}^\alpha \tilde{\mathbf{k}}^{\dot{\alpha}}. \quad (2.32)$$

Finally, the dilatation is given by

$$D = 1 - (h + \bar{h}) = 1 - \Delta. \quad (2.33)$$

We can then work out the full conformal algebra in this basis. Let us first show two useful relations: ($\varepsilon^{12} = -\varepsilon^{21} = 1$)

$$[\mathbf{p}^\alpha, \mathbf{k}^\beta] = \mathbf{p}^\alpha \mathbf{k}^\beta - \mathbf{k}^\beta \mathbf{p}^\alpha = \varepsilon^{\alpha\beta}, \quad \mathbf{k}^\alpha \mathbf{p}^\beta - \mathbf{p}^\beta \mathbf{k}^\alpha = -2(h-1)\varepsilon^{\alpha\beta}, \quad (2.34)$$

⁴More specifically, $\mathcal{L}_k \cdot (\mathcal{L}_l \cdot O) = L_l L_k O$. See the appendix of [24] for more detailed discussions on this point.

⁵Here we introduce the factor 1/2 for simplicity, otherwise there will be many factors of $\sqrt{2}$ flowing around.

and its conjugate

$$[\tilde{\mathbf{p}}^{\dot{\alpha}}, \tilde{\mathbf{k}}^{\dot{\beta}}] = \tilde{\mathbf{p}}^{\dot{\alpha}} \tilde{\mathbf{k}}^{\dot{\beta}} - \tilde{\mathbf{k}}^{\dot{\beta}} \tilde{\mathbf{p}}^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}}, \quad \tilde{\mathbf{k}}^{\dot{\alpha}} \tilde{\mathbf{p}}^{\dot{\beta}} - \tilde{\mathbf{k}}^{\dot{\beta}} \tilde{\mathbf{p}}^{\dot{\alpha}} = -2(\bar{h} - 1)\varepsilon^{\dot{\alpha}\dot{\beta}}. \quad (2.35)$$

We can then construct

$$L^{\alpha\beta} = \frac{1}{2}(\mathbf{k}^{\alpha}\mathbf{p}^{\beta} + \mathbf{k}^{\beta}\mathbf{p}^{\alpha}), \quad L^{\dot{\alpha}\dot{\beta}} = \frac{1}{2}(\tilde{\mathbf{k}}^{\dot{\alpha}}\tilde{\mathbf{p}}^{\dot{\beta}} + \tilde{\mathbf{k}}^{\dot{\beta}}\tilde{\mathbf{p}}^{\dot{\alpha}}), \quad (2.36)$$

which enables us to write

$$\mathbf{k}^{\alpha}\mathbf{p}^{\beta} = L^{\alpha\beta} - (h - 1)\varepsilon^{\alpha\beta}, \quad \mathbf{p}^{\alpha}\mathbf{k}^{\beta} = L^{\alpha\beta} + h\varepsilon^{\alpha\beta}, \quad (2.37)$$

and likewise for $L^{\dot{\alpha}\dot{\beta}}$.

It turns out that these are just the $L_k, \bar{L}_k$ in (2.26). More precisely, we have

$$L^{11} = L_{-1}, \quad L^{12} = L^{21} = L_0, \quad L^{22} = L_1, \quad L^{\dot{1}\dot{1}} = \bar{L}_{-1}, \quad L^{\dot{1}\dot{2}} = L^{\dot{2}\dot{1}} = \bar{L}_0, \quad L^{\dot{2}\dot{2}} = \bar{L}_1. \quad (2.38)$$

So $L^{\alpha\beta}, L^{\dot{\alpha}\dot{\beta}}$ are just the generators of $\text{SL}(2, \mathbb{C})$.⁶

With these generators, one can show that they indeed satisfy the conformal algebra $\mathfrak{so}(4, 2)$:

$$[K^{\alpha\dot{\alpha}}, P^{\beta\dot{\beta}}] = \varepsilon^{\alpha\beta} L^{\dot{\alpha}\dot{\beta}} + \varepsilon^{\dot{\alpha}\dot{\beta}} L^{\alpha\beta} + D\varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}}, \quad (2.39)$$

$$[D, K^{\alpha\dot{\alpha}}] = -K^{\alpha\dot{\alpha}}, \quad (2.40)$$

$$[D, P^{\alpha\dot{\alpha}}] = P^{\alpha\dot{\alpha}}, \quad (2.41)$$

$$[L^{\alpha\beta}, L^{\gamma\dot{\delta}}] = \varepsilon^{\alpha\delta} L^{\gamma\dot{\beta}} - \varepsilon^{\gamma\beta} L^{\alpha\dot{\delta}}, \quad [L^{\dot{\alpha}\dot{\beta}}, L^{\gamma\dot{\delta}}] = \varepsilon^{\dot{\alpha}\dot{\delta}} L^{\gamma\dot{\beta}} - \varepsilon^{\gamma\dot{\beta}} L^{\dot{\alpha}\dot{\delta}}, \quad (2.42)$$

$$[L^{\alpha\beta}, P^{\gamma\dot{\gamma}}] = -\varepsilon^{\gamma\beta} P^{\alpha\dot{\gamma}} + \frac{1}{2}\varepsilon^{\alpha\beta} P^{\gamma\dot{\gamma}}, \quad [L^{\dot{\alpha}\dot{\beta}}, P^{\gamma\dot{\gamma}}] = \varepsilon^{\dot{\alpha}\dot{\gamma}} P^{\gamma\dot{\beta}} - \frac{1}{2}\varepsilon^{\dot{\alpha}\dot{\beta}} P^{\gamma\dot{\gamma}}, \quad (2.43)$$

$$[L^{\alpha\beta}, K^{\gamma\dot{\gamma}}] = -\varepsilon^{\gamma\beta} K^{\alpha\dot{\gamma}} + \frac{1}{2}\varepsilon^{\alpha\beta} K^{\gamma\dot{\gamma}}, \quad [L^{\dot{\alpha}\dot{\beta}}, K^{\gamma\dot{\gamma}}] = \varepsilon^{\dot{\alpha}\dot{\gamma}} K^{\gamma\dot{\beta}} - \frac{1}{2}\varepsilon^{\dot{\alpha}\dot{\beta}} K^{\gamma\dot{\gamma}}. \quad (2.44)$$

For any field theory with conformal symmetry, the amplitude should respect these symmetries. So we have

$$\begin{aligned} \sum_j \varepsilon_j P_j^{\alpha\dot{\beta}} \mathcal{M}_n(h, \bar{h}_i, z_i, \bar{z}_i) &= \sum_j L_j^{\alpha\beta} \mathcal{M}_n(h, \bar{h}_i, z_i, \bar{z}_i) = \sum_j L_j^{\dot{\alpha}\dot{\beta}} \mathcal{M}_n(h, \bar{h}_i, z_i, \bar{z}_i) \\ &= \sum_j \varepsilon_j K_j^{\alpha\dot{\beta}} \mathcal{M}_n(h, \bar{h}_i, z_i, \bar{z}_i) = \sum_j D_j \mathcal{M}_n(h, \bar{h}_i, z_i, \bar{z}_i) = 0, \end{aligned} \quad (2.45)$$

where the subscript j means the action of the generators on the j -th particle.

For comparison, we also write down the conformal generators in the spinor-helicity basis [25]:

$$\mathbf{P}^{\alpha\dot{\alpha}} = \lambda^{\alpha} \tilde{\lambda}^{\dot{\alpha}}, \quad \mathbf{K}^{\alpha\dot{\alpha}} = \frac{\partial}{\partial \lambda_{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_{\dot{\alpha}}}, \quad \mathbf{D} = \frac{1}{2} \lambda^{\alpha} \frac{\partial}{\partial \lambda^{\alpha}} + \frac{1}{2} \tilde{\lambda}^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}^{\dot{\alpha}}} + 1, \quad (2.46)$$

$$\mathbf{L}^{\alpha\beta} = \frac{1}{2} \left(\lambda^{\alpha} \frac{\partial}{\partial \lambda_{\beta}} + \lambda^{\beta} \frac{\partial}{\partial \lambda_{\alpha}} \right), \quad \mathbf{L}^{\dot{\alpha}\dot{\beta}} = \frac{1}{2} \left(\tilde{\lambda}^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_{\dot{\beta}}} + \tilde{\lambda}^{\dot{\beta}} \frac{\partial}{\partial \tilde{\lambda}_{\dot{\alpha}}} \right). \quad (2.47)$$

⁶The Lorentz generators of 4D spacetime can be written as $M^{\mu\nu} = (\sigma^{\mu\nu})_{\alpha\beta} L^{\alpha\beta} + (\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}} L^{\dot{\alpha}\dot{\beta}}$ where $\sigma^{\mu\nu} \propto \sigma^{[\mu} \sigma^{\nu]}$, $\bar{\sigma}^{\mu\nu} \propto \bar{\sigma}^{[\mu} \bar{\sigma}^{\nu]}$.

particle	G^+	G^-	$\Phi_{AB} = -\Phi_{BA}$	Γ_A	$\bar{\Gamma}^A$
helicity J	+1	-1	0	$+\frac{1}{2}$	$-\frac{1}{2}$
on-shell d.o.f.	1	1	6	4	4
$SU(4)_R$	singlet (1)	singlet (1)	anti-symmetric (6)	fundamental (4)	anti-fundamental ($\bar{4}$)

Table 1. On-shell degrees of freedom of $\mathcal{N} = 4$ SYM. Note that the color index for gauge group is suppressed.

2.5 Bulk superconformal symmetry and $\mathcal{N} = 4$ SYM

Now we turn to supersymmetry. Our main interest in this paper is $\mathcal{N} = 4$ SYM theory which enjoys the maximally superconformal symmetry $\mathfrak{psu}(2,2|4)$ in 4D. So to study celestial amplitudes in $\mathcal{N} = 4$ SYM theory, we also would like to find the generators of the superconformal symmetry. We will first review super-amplitudes in the spinor-helicity basis and then generalize to the celestial basis.

Let us first recall the field content of $\mathcal{N} = 4$ SYM theory which includes one spin-1 gluon, four spin-1/2 gluinos and six real scalars. All of them transform in the adjoint representation of the gauge group. For scattering amplitude, we only need to consider the on-shell degrees of freedom (d.o.f). There are $8_B + 8_F$ on-shell degrees of freedom and we list them in table 1.

2.5.1 Superconformal symmetry in spinor-helicity basis

It is convenient to introduce the anti-commuting Grassmann variables η^A with $A=1, \dots, 4$. Then the on-shell degrees of freedom of $\mathcal{N} = 4$ SYM theory can be packaged into an on-shell superfield as follows [25]:

$$\begin{aligned} \Psi(p, \eta) = & G^+(p) + \eta^A \Gamma_A(p) + \frac{1}{2!} \eta^A \eta^B \Phi_{AB}(p) \\ & + \frac{1}{3!} \epsilon_{ABCD} \eta^A \eta^B \eta^C \bar{\Gamma}^D(p) + \frac{1}{4!} \epsilon_{ABCD} \eta^A \eta^B \eta^C \eta^D G^-(p). \end{aligned} \quad (2.48)$$

Note each component field in this superfield has different helicities. We can formally assign η^A helicity $\frac{1}{2}$, then each term in the on-shell superfield (2.48) carries the same helicity. Mathematically, we can introduce the following generators

$$\Omega = -\frac{1}{2} \lambda^\alpha \frac{\partial}{\partial \lambda^\alpha} + \frac{1}{2} \tilde{\lambda}^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}^{\dot{\alpha}}} + \frac{1}{2} \eta^A \partial_A, \quad \partial_A = \frac{\partial}{\partial \eta^A}, \quad (2.49)$$

whose action on the superfield is then $\Omega \Psi = \Psi$.

The Poincaré and conformal supersymmetry generators acting on the superfield (2.48) are given by [25]:

$$Q^{\alpha A} = \lambda^\alpha \eta^A, \quad \tilde{Q}_A^{\dot{\alpha}} = \tilde{\lambda}^{\dot{\alpha}} \frac{\partial}{\partial \eta^A}, \quad (2.50)$$

$$S_A^\alpha = \lambda^\alpha \frac{\partial}{\partial \eta^A}, \quad \tilde{S}^{\dot{\alpha} A} = \tilde{\lambda}^{\dot{\alpha}} \eta^A. \quad (2.51)$$

Besides, the R -symmetry generators are given by [25]:

$$R^A{}_B = \eta^A \partial_B - \frac{1}{4} \delta_B^A \eta^C \partial_C. \quad (2.52)$$

Together with those in (2.46) and (2.47), they generate the full superconformal symmetry of $\mathcal{N} = 4$ SYM.

The superamplitude provides a compact way to write down the amplitudes of SYM theory:

$$\mathcal{A}_n(p_i, \eta_i) = \langle \Psi(p_1, \eta_1) \cdots \Psi(p_n, \eta_n) \rangle. \quad (2.53)$$

Performing the η expansion and comparing both sides, one can get the scattering amplitudes of all the component fields. The superamplitude is severely constrained by the superconformal symmetry of SYM theory. Especially this implies $\sum_j O_j \mathcal{A}_n(\lambda_i, \tilde{\lambda}_i, \eta_i) = 0$ where $O \in \{L, P, D, K, Q, S, R\}$. We will not spell out all the details here which can be found in [25].

We can then read off the Q -supersymmetry variation of the component field defined by⁷

$$\delta_\epsilon \Psi(p, \eta) = \epsilon_{\alpha A} Q^{\alpha A} \Psi(p, \eta). \quad (2.54)$$

Expanding both sides in components, we have

$$\delta_\epsilon \Psi(p, \eta) = \delta_\epsilon G^+(p) + \eta^A \delta_\epsilon \Gamma_A(p) + \frac{1}{2!} \eta^A \eta^B \delta_\epsilon \Phi_{AB}(p) + \cdots, \quad (2.55)$$

$$\epsilon_{\alpha A} Q^{\alpha A} \Psi(p, \eta) = \epsilon_{\alpha A} \lambda^\alpha \left(\eta^A G^+(p) + \eta^A \eta^B \Gamma_B(p) + \frac{1}{2!} \eta^A \eta^B \eta^C \Phi_{BC}(p) + \cdots \right), \quad (2.56)$$

which gives the supersymmetry variations of various component fields:

$$\delta_\epsilon G^+ = 0, \quad \delta_\epsilon \Gamma_A = \epsilon_{\alpha A} \lambda^\alpha G^+ = -\langle \epsilon_A \lambda \rangle G^+, \quad \delta_\epsilon \Phi_{AB} = -\langle \epsilon_A \lambda \rangle \Gamma_B + \langle \epsilon_B \lambda \rangle \Gamma_A, \quad \cdots. \quad (2.57)$$

2.5.2 Quasi-supersymmetry generators in the celestial basis

Now we would like to perform a similar construction in the celestial basis. First we need to find the supersymmetry generators in terms of the celestial coordinates. Motivated by the construction in the spinor-helicity basis in (2.50) and (2.51), it is natural to guess that the Poincaré and conformal supersymmetry generators take the following form:

$$Q^{\alpha A} = p^\alpha \eta^A, \quad \tilde{Q}^{\dot{\alpha}}_A = \tilde{p}^{\dot{\alpha}} \frac{\partial}{\partial \eta^A}, \quad (2.58)$$

$$S^\alpha_A = k^\alpha \frac{\partial}{\partial \eta^A}, \quad \tilde{S}^{\dot{\alpha} A} = \tilde{k}^{\dot{\alpha}} \eta^A, \quad (2.59)$$

where $p, \tilde{p}, k, \tilde{k}$ are given in (2.31) and (2.32). In particular, it is easy to check that they satisfy the supersymmetry algebra as desired:

$$\{Q^{\alpha A}, \tilde{Q}^{\dot{\alpha}}_B\} = \delta_B^A P^{\alpha \dot{\alpha}}, \quad \{S^\alpha_A, \tilde{S}^{\dot{\alpha} B}\} = \delta_A^B K^{\alpha \dot{\alpha}}. \quad (2.60)$$

⁷One can also consider the $\bar{Q}$ -supersymmetry variation: $\delta_{\bar{\epsilon}} \Phi(p, \eta) = \bar{\epsilon}^{\dot{\alpha}}_A \bar{Q}^{\dot{\alpha} A} \Phi(p, \eta)$ as well as the $S, \tilde{S}$ -supersymmetry variation in a similar way.

Besides, we also have the R-symmetry generators which take the same form as (2.52):

$$R^A{}_B = \eta^A \partial_B - \frac{1}{4} \delta_B^A \eta^C \partial_C, \quad (2.61)$$

as well as

$$\Omega = h - \bar{h} + \frac{1}{2} \eta^A \partial_A. \quad (2.62)$$

We can further compute

$$\{Q^{\alpha A}, S_B^\beta\} = \varepsilon^{\alpha\beta} R^A{}_B + \delta_B^A L^{\alpha\beta} - \frac{1}{2} \varepsilon^{\alpha\beta} \delta_B^A (D - \Omega + 1), \quad (2.63)$$

$$\{\bar{Q}_B^{\dot{\alpha}}, \bar{S}^{\dot{\beta} A}\} = -\varepsilon^{\dot{\alpha}\dot{\beta}} R^A{}_B + \delta_B^A L^{\dot{\alpha}\dot{\beta}} - \frac{1}{2} \varepsilon^{\dot{\alpha}\dot{\beta}} \delta_B^A (D + \Omega - 1), \quad (2.64)$$

where we used the identities (2.34) and (2.37). Note that when acting on the on-shell superfield or superamplitudes, we have $\Omega = 1$.

One can also check all the rest of the commutators. Indeed they generate the $\mathfrak{psu}(2, 2|4)$ superconformal algebra, which is the symmetry of $\mathcal{N} = \text{SYM}$ theory. However, it turns out that the generators constructed in this way are a bit subtle as they don't act on the physical on-shell superfield as we expected. For this reason, the generators constructed above will be called quasi-superconformal generators. We will explain this point in detail in the following.

2.5.3 Celestial on-shell superfield and celestial superamplitude

Now we also would like to have the analog of the superamplitude (2.53) in the celestial basis. For this purpose, we need to generalize the notion of superfield in (2.48). The superconformal generators can then act on the celestial on-shell superfield and realize a representation of the superconformal algebra. It turns out that this brings a subtlety which will be resolved at the end of this section.

Celestial on-shell celestial superfield from Mellin transformation. A natural way to obtain the superfield in the celestial basis is to perform the Mellin transformation of (2.48):

$$\hat{\Psi}_\Delta(z, \bar{z}, \eta) = \int_0^\infty d\omega \omega^{\Delta-1} \Psi(\varepsilon \omega q^\mu(z, \bar{z}), \eta). \quad (2.65)$$

Since the η -expansion and Mellin integral commute, we can perform Mellin transformation component by component. Each component has the same conformal dimension, but different helicity. The results can thus be compactly written as

$$\begin{aligned} \hat{\Psi}_{h, \bar{h}}(z, \bar{z}, \eta) = & \mathbf{G}^+(z, \bar{z}, h, \bar{h}) + \eta^A \Gamma_A \left(z, \bar{z}, h - \frac{1}{4}, \bar{h} + \frac{1}{4} \right) + \frac{1}{2!} \eta^A \eta^B \Phi_{AB} \left(z, \bar{z}, h - \frac{1}{2}, \bar{h} + \frac{1}{2} \right) \\ & + \frac{1}{3!} \epsilon_{ABCD} \eta^A \eta^B \eta^C \bar{\Gamma}^D \left(z, \bar{z}, h - \frac{3}{4}, \bar{h} + \frac{3}{4} \right) \\ & + \frac{1}{4!} \epsilon_{ABCD} \eta^A \eta^B \eta^C \eta^D \mathbf{G}^- \left(z, \bar{z}, h - 1, \bar{h} + 1 \right) \end{aligned} \quad (2.66)$$

with the extra condition $h - \bar{h} = 1$. Then obviously every component indeed has the same conformal dimension $\Delta = h + \bar{h}$ and its own correct helicity (note that the Grassmann variable η has $\Delta = 0$ and $J = 1/2$).

The superamplitude is given by

$$\widehat{\mathcal{M}}_n(h_i, \bar{h}_i, z_i, \bar{z}_i, \eta_i) = \langle \widehat{\Psi}_{h_1, \bar{h}_1}(z_1, \bar{z}_1, \eta_1) \cdots \widehat{\Psi}_{h_n, \bar{h}_n}(z_n, \bar{z}_n, \eta_n) \rangle. \quad (2.67)$$

However, this on-shell superfield is not appropriate for the supercharge $Q, \bar{Q}$ constructed in (2.50). From the supersymmetry variation in (2.57), we obtain

$$\delta_\epsilon \Gamma_A(\Delta) = \int d\omega \omega^{\Delta-1} \delta_\epsilon \Gamma_A(\omega) = \int d\omega \omega^{\Delta-1} (z - \zeta_A) \sqrt{\omega} G^+(\omega) = (z - \zeta_A) G^+ \left(\Delta + \frac{1}{2} \right), \quad (2.68)$$

where we used $\epsilon_A^\alpha = \frac{1}{\sqrt{2}}(1, \zeta_A)^T$.⁸ As a result, we have

$$\delta_\epsilon \Gamma_A(h, \bar{h}) = (z - \zeta_A) G^+ \left(h + \frac{1}{2}, \bar{h} \right), \quad \bar{h} = h - \frac{1}{2}. \quad (2.69)$$

However, applying the supercharge (2.58) to the (2.66), the supersymmetry variation rule $\delta_\epsilon \widehat{\Phi} = \epsilon_{\alpha A} Q^{\alpha A} \widehat{\Phi}$ leads to $\delta_\epsilon \Gamma_A(h, \bar{h}) = (z - \zeta_A) G^+ \left(h + \frac{3}{4}, \bar{h} - \frac{1}{4} \right)$ which is not consistent with the above result (2.69). To reproduce the expected supersymmetry variations, one needs to modify the rule as follows:

$$\delta_\epsilon \widehat{\Psi} = \epsilon_{\alpha A} \widehat{Q}^{\alpha A} \widehat{\Psi}, \quad \delta_{\bar{\epsilon}} \widehat{\Psi} = \bar{\epsilon}_{\dot{\alpha}}^A \widehat{Q}_A^{\dot{\alpha}} \widehat{\Psi}. \quad (2.70)$$

where

$$\widehat{Q}^{\alpha A} = e^{\frac{\partial_{\bar{h}} - \partial_h}{4}} Q^{\alpha A} = \left(\frac{1}{z} \right) e^{\frac{\partial_{\bar{h}} + \partial_h}{4}} \eta^A, \quad \widehat{Q}_A^{\dot{\alpha}} = e^{\frac{\partial_h - \partial_{\bar{h}}}{4}} Q_A^{\dot{\alpha}} = \left(\frac{1}{\bar{z}} \right) e^{\frac{\partial_{\bar{h}} + \partial_h}{4}} \partial_A. \quad (2.71)$$

These supercharges are just those used in [20, 21]. However, at this moment, it is not obvious how to similarly modify the rest of generators in order to fulfill a representation acting on (2.66). This will be achieved at the end of this subsection. Before that, let us stick to the quasi-superconformal generators that we constructed in the previous subsection and try to find a way to fulfill its representation.

Quasi-on-shell superfield. It turns out that we can introduce the following superfield

$$\begin{aligned} \Psi_{h, \bar{h}}(z, \bar{z}, \eta) &= G^+(z, \bar{z}, h, \bar{h}) + \eta^A \Gamma_A(z, \bar{z}, h, \bar{h}) + \frac{1}{2!} \eta^A \eta^B \Phi_{AB}(z, \bar{z}, h, \bar{h}) \\ &+ \frac{1}{3!} \epsilon_{ABCD} \eta^A \eta^B \eta^C \bar{\Gamma}^D(z, \bar{z}, h, \bar{h}) + \frac{1}{4!} \epsilon_{ABCD} \eta^A \eta^B \eta^C \eta^D G^-(z, \bar{z}, h, \bar{h}). \end{aligned} \quad (2.72)$$

Applying the Poincaré supersymmetry generators (2.58) to the above superfield, the corresponding supersymmetry transformation rules

$$\delta_\epsilon \Psi = \epsilon_{\alpha A} Q^{\alpha A} \Psi, \quad \delta_{\bar{\epsilon}} \Psi = \bar{\epsilon}_{\dot{\alpha}}^A Q_A^{\dot{\alpha}} \Psi \quad (2.73)$$

indeed give the right variation (2.69).

⁸Note that $\epsilon_{\alpha A} \in \mathbb{CP}^1 \simeq S^2$. So ζ_A transforms under the $SL(2, \mathbb{C})$ transformation in the same way as z .

However, there is a subtle point related to the superfield (2.72). Different components there can not become on-shell simultaneously due to their different helicities but the same value of $h - \bar{h}$. For this reason, we will call (2.72) *quasi-on-shell superfield*. Nevertheless, this subtlety does not bring any physical pathologies because we always need to pick exactly one component in the multiplet for each external leg to get the amplitudes in a physical scattering process. So we can regard (2.72) as a formal sum where the conformal weights $h, \bar{h}$ are arbitrary a priori. The on-shell condition for $h - \bar{h}$ is imposed appropriately only after the component field is specified.

The corresponding superamplitude is given then by

$$\mathcal{M}_n(p_i, \eta_i) = \langle \Psi_{h_1, \bar{h}_1}(z_1, \bar{z}_1, \eta_1) \cdots \Psi_{h_n, \bar{h}_n}(z_n, \bar{z}_n, \eta_n) \rangle. \quad (2.74)$$

To obtain the component field amplitude, we first perform the η -expansion, choose a specific component for each external leg, and finally impose the helicity constraint for each $h_i - \bar{h}_i$.

Celestial on-shell superfield, celestial superamplitude and superconformal generators. In the above discussions, we have constructed the superconformal generators (2.58), (2.59) whose action on the quasi-on-shell superfield (2.72) fulfills the representation of the superconformal algebra. Although we have argued that the quasi-on-shell superfield (2.72) does not bring physical pathologies, its interpretation is still obscure conceptually. In the following, we will construct a new set of superconformal generators. They act on the celestial on-shell-superfield (2.66) and also realize a representation of the superconformal algebra.

By comparing the component expansion of two superfields (2.66) and (2.72), it is not difficult to find that they can be related as follows:

$$\widehat{\Psi} = e^{-\frac{1}{2}\eta^A \partial_A \partial_J} \Psi = e^{-\frac{1}{4}\eta^A \partial_A (\partial_h - \partial_{\bar{h}})} \Psi, \quad \Psi = e^{\frac{1}{2}\eta^A \partial_A \partial_J} \widehat{\Psi}, \quad (2.75)$$

where $e^{-\frac{1}{2}\eta^A \partial_A \partial_J}$ implements the shift of h and $\bar{h}$ according to the number of η 's.

We have shown that Ψ fulfills the representation of the superconformal algebra with generators $\{P, L, K, D, Q, S, R\}$. To find the generators acting on the on-shell superfield $\widehat{\Psi}$, we can naturally modify generators in the following way

$$\widehat{O} = e^{-\frac{1}{2}\eta^A \partial_A \partial_J} O e^{\frac{1}{2}\eta^A \partial_A \partial_J} = e^{-\frac{1}{4}\eta^A \partial_A (\partial_h - \partial_{\bar{h}})} O e^{\frac{1}{4}\eta^A \partial_A (\partial_h - \partial_{\bar{h}})}, \quad (2.76)$$

where $O \in \{P, L, K, D, Q, S, R\}$ is any superconformal generators we constructed before. Now $\widehat{O}$ acts on the on-shell superfield $\widehat{\Psi}$ in (2.66). By construction, this should also fulfill the superconformal representation and in particular the generators should satisfy the superconformal algebra. In the followings, we will write down the generators $\{\widehat{P}, \widehat{L}, \widehat{K}, \widehat{D}, \widehat{Q}, \widehat{S}, \widehat{R}\}$ explicitly and verify these statements.

Let us start by observing the following useful relation:

$$e^{-\eta^A \partial_A \partial_x} \eta^{A_1} \cdots \eta^{A_k} \partial_{B_1} \cdots \partial_{B_l} f(x) e^{\eta^A \partial_A \partial_x} = f(x - \eta^A \partial_A) \eta^{A_1} \cdots \eta^{A_k} \partial_{B_1} \cdots \partial_{B_l} e^{(l-k)\partial_x}, \quad (2.77)$$

which can be proved by inserting $1 = e^{\eta^A \partial_A \partial_x} e^{-\eta^A \partial_A \partial_x}$ between ∂_{B_l} and $f(x)$.⁹ This immediately implies that $e^{-\eta^A \partial_A \partial_x} \eta^B \partial_C e^{\eta^A \partial_A \partial_x} = \eta^B \partial_C$. So we have

$$\hat{R}^A{}_B = R^A{}_B = \eta^A \partial_B - \frac{1}{4} \delta_B^A \eta^C \partial_C, \quad (2.78)$$

and

$$\begin{aligned} \hat{\Omega} &= e^{-\frac{1}{4} \eta^A \partial_A (\partial_h - \partial_{\bar{h}})} \left(h - \bar{h} + \frac{1}{2} \eta^A \partial_A \right) e^{\frac{1}{4} \eta^A \partial_A (\partial_h - \partial_{\bar{h}})} = \left(h - \frac{1}{4} \eta^A \partial_A \right) - \left(\bar{h} + \frac{1}{4} \eta^A \partial_A \right) + \frac{1}{2} \eta^A \partial_A \\ &= h - \bar{h} = J. \end{aligned} \quad (2.79)$$

So we identically have $\hat{\Omega} \hat{\Phi} = \hat{\Phi}$ as $h - \bar{h} = 1$ for G^+ gluon.

It is also easy to find that

$$\hat{D} = D = 1 - h - \bar{h}. \quad (2.80)$$

The special conformal generators are now:

$$\widehat{K}^{\dot{\alpha}\alpha} = e^{-\frac{1}{4} \eta^A \partial_A (\partial_h - \partial_{\bar{h}})} \tilde{k}^{\dot{\alpha}} k^{\alpha} e^{\frac{1}{4} \eta^A \partial_A (\partial_h - \partial_{\bar{h}})} = \hat{k}^{\dot{\alpha}} \hat{k}^{\alpha}, \quad (2.81)$$

where

$$\hat{k}^{\alpha} = e^{-\frac{1}{4} \eta^A \partial_A \partial_h} k^{\alpha} e^{\frac{1}{4} \eta^A \partial_A \partial_h} = \left(z \partial_z + 2h - \frac{1}{2} \eta^A \partial_A - 1 \right) e^{-\partial_h/2}, \quad (2.82)$$

$$\hat{k}^{\dot{\alpha}} = e^{\frac{1}{4} \eta^A \partial_A \partial_{\bar{h}}} \tilde{k}^{\dot{\alpha}} e^{-\frac{1}{4} \eta^A \partial_A \partial_{\bar{h}}} = \left(\bar{z} \partial_{\bar{z}} + 2\bar{h} + \frac{1}{2} \eta^A \partial_A - 1 \right) e^{-\partial_{\bar{h}}/2}. \quad (2.83)$$

It is worth comparing the special conformal generators here with that in (2.32): the only difference is the shift of $h, \bar{h}$ by $\mp \frac{1}{4} \eta^A \partial_A$. Looking at the superfield (2.66), the shift is indeed reasonable because it ensures that the special conformal generators act on each component in a correct way with its right value of $h, \bar{h}$.

The Lorentz generators become:

$$\hat{L}_{\alpha\beta} = \frac{1}{2} \left(\hat{k}^{\alpha} \mathbf{p}^{\beta} + \hat{k}^{\beta} \mathbf{p}^{\alpha} \right), \quad \hat{L}_{\dot{\alpha}\dot{\beta}} = \frac{1}{2} \left(\hat{k}^{\dot{\alpha}} \tilde{\mathbf{p}}^{\dot{\beta}} + \hat{k}^{\dot{\beta}} \tilde{\mathbf{p}}^{\dot{\alpha}} \right). \quad (2.84)$$

The translation generators remain the same:

$$\hat{P}^{\dot{\alpha}\alpha} = P^{\dot{\alpha}\alpha} = \tilde{\mathbf{p}}^{\dot{\alpha}} \mathbf{p}^{\alpha}. \quad (2.85)$$

The Q -supersymmetry generators are

$$\hat{Q}^{\alpha A} = e^{-\frac{1}{4} \eta^A \partial_A (\partial_h - \partial_{\bar{h}})} \mathbf{p}^{\alpha} \eta^A e^{\frac{1}{4} \eta^A \partial_A (\partial_h - \partial_{\bar{h}})} = \mathbf{p}^{\alpha} \eta^A e^{-\frac{1}{4} (\partial_h - \partial_{\bar{h}})} = \langle q |^{\alpha} e^{\frac{1}{2} \partial_{\Delta}} \eta^A, \quad (2.86)$$

$$\hat{\bar{Q}}_{\dot{A}} = e^{-\frac{1}{4} \eta^A \partial_A (\partial_h - \partial_{\bar{h}})} \tilde{\mathbf{p}}^{\dot{A}} \partial_A e^{\frac{1}{4} \eta^A \partial_A (\partial_h - \partial_{\bar{h}})} = \tilde{\mathbf{p}}^{\dot{A}} \partial_A e^{\frac{1}{4} (\partial_h - \partial_{\bar{h}})} = [q]_{\dot{A}} e^{\frac{1}{2} \partial_{\Delta}} \partial_A, \quad (2.87)$$

where we used the relation (2.77). The Q -supercharges constructed now agree with (2.71).

⁹Note the relation $[\eta^A \partial_A, \eta^B] = \eta^B$, $[\eta^A \partial_A, \partial_B] = -\partial_B$, $[\eta^A \partial_A, \eta^B \partial_B] = 0$.

Making full use of (2.77), one also finds the S -supersymmetry generators:

$$\hat{S}_A^\alpha = \hat{k}^\alpha \partial_A e^{\frac{1}{4}(\partial_h - \partial_{\bar{h}})} = \left(z \partial_z + 2h - \frac{1}{2} \eta^B \partial_B - 1 \right) \partial_A e^{-\frac{1}{4}(\partial_h + \partial_{\bar{h}})}, \quad (2.88)$$

$$\hat{S}^{\dot{\alpha}} A = \hat{k}^{\dot{\alpha}} \eta^A e^{\frac{1}{4}(-\partial_h + \partial_{\bar{h}})} = \left(\bar{z} \partial_{\bar{z}} + 2\bar{h} + \frac{1}{2} \eta^B \partial_B - 1 \right) \eta^A e^{-\frac{1}{4}(\partial_h + \partial_{\bar{h}})}. \quad (2.89)$$

These are all the superconformal generators acting on the on-shell superfield $\hat{\Phi}$. As we said, by construction they satisfy the superconformal algebra. Now we can compute some commutators of these generators and verify this claim.

First of all, it is easy to see that

$$\left\{ \hat{Q}^{\alpha A}, \hat{Q}_B^{\dot{\alpha}} \right\} = \delta_B^A \hat{P}^{\alpha \dot{\alpha}}, \quad \left\{ \hat{S}_A^\alpha, \hat{S}^{\dot{\alpha}} B \right\} = \hat{k}^{\dot{\alpha}} k^\alpha \left\{ \partial_A e^{\frac{1}{4}(\partial_h - \partial_{\bar{h}})}, \eta^B e^{\frac{1}{4}(-\partial_h + \partial_{\bar{h}})} \right\} = \delta_A^B \hat{K}^{\alpha \dot{\alpha}}, \quad (2.90)$$

where we used the identity

$$[AB, CD]_\pm = [A, C]BD + CA[B, D]_\pm \quad \text{if} \quad [B, C] = [A, D] = 0, \quad (2.91)$$

and noticed that

$$\left[\hat{k}^\alpha, \eta^A e^{-\frac{1}{4}(\partial_h - \partial_{\bar{h}})} \right] = 0, \quad \left[\hat{k}^{\dot{\alpha}}, \partial_A e^{-\frac{1}{4}(\partial_{\bar{h}} - \partial_h)} \right] = 0, \quad \left[\hat{k}^\alpha, \hat{k}^{\dot{\alpha}} \right] = 0. \quad (2.92)$$

The identities (2.34) and (2.37) are generalized to:

$$[\mathbf{p}^\alpha, \hat{k}^\beta] = \varepsilon^{\alpha\beta}, \quad \hat{k}^\alpha \mathbf{p}^\beta - \hat{k}^\beta \mathbf{p}^\alpha = -2 \left(h - \frac{1}{4} \eta^A \partial_A - 1 \right) \varepsilon^{\alpha\beta}, \quad (2.93)$$

$$\hat{k}^\alpha \mathbf{p}^\beta = \hat{L}^{\alpha\beta} - \left(h - \frac{1}{4} \eta^A \partial_A - 1 \right) \varepsilon^{\alpha\beta}, \quad \mathbf{p}^\alpha \hat{k}^\beta = \hat{L}^{\alpha\beta} + \left(h - \frac{1}{4} \eta^A \partial_A \right) \varepsilon^{\alpha\beta}. \quad (2.94)$$

They enable us to check

$$\begin{aligned} \{ \hat{Q}^{\alpha A}, \hat{S}_B^{\dot{\beta}} \} &= [\mathbf{p}^\alpha, \hat{k}^\beta] \eta^A \partial_B + \hat{k}^\beta \mathbf{p}^\alpha \delta_B^A = \varepsilon^{\alpha\beta} \eta^A \partial_B + \left(\hat{L}^{\alpha\beta} + \left(h - \frac{1}{4} \eta^A \partial_A - 1 \right) \varepsilon^{\alpha\beta} \right) \delta_B^A \\ &= \hat{L}^{\alpha\beta} \delta_B^A + \varepsilon^{\alpha\beta} \hat{R}_B^A - \frac{1}{2} \varepsilon^{\alpha\beta} (\hat{D} - \hat{\Omega} + 1). \end{aligned} \quad (2.95)$$

All the rest of the commutators can also be checked similarly and indeed they generate the superconformal algebra $\mathfrak{psu}(2, 2|4)$.

The superamplitudes should be invariant under the action of these symmetries, so it must satisfy the Ward identities for all the superconformal generators:

$$\begin{aligned} \sum_j \left\{ \varepsilon_j \hat{P}_j, \hat{L}_j^{\alpha\beta}, \hat{L}_j^{\dot{\alpha}\dot{\beta}}, \varepsilon_j \hat{K}_j^{\alpha\dot{\beta}}, \hat{D}_j, (\hat{Q}_A^\alpha)_j, \varepsilon_j (\hat{Q}^{\dot{\alpha}A})_j, (\hat{R}_B^A)_j, (\hat{S}_A^\alpha)_j, \varepsilon_j (\hat{S}^{\dot{\alpha}A})_j \right\} \\ \times \widehat{\mathcal{M}}_n(h_i, \bar{h}_i, z_i, \bar{z}_i, \eta_i) = 0, \end{aligned} \quad (2.96)$$

in addition to the helicity constraint $\hat{\Omega}_j \widehat{\mathcal{M}}_n(h_i, \bar{h}_i, z_i, \bar{z}_i, \eta_i) = \widehat{\mathcal{M}}_n(h_i, \bar{h}_i, z_i, \bar{z}_i, \eta_i)$ for each j .

Especially, the Lorentz invariance is manifest. By generalizing (2.20), the celestial superamplitudes transforms under $SL(2, \mathbb{C})$ as

$$\begin{aligned} & \widehat{\mathcal{M}}_n(h_i, \bar{h}_i, z_i, \bar{z}_i, \eta_i) \\ &= \left(\prod_{j=1}^n (cz_j + d)^{-2h_j + \frac{1}{2}\eta_j^A \partial_{jA}} (\bar{c}\bar{z}_j + \bar{d})^{-2\bar{h}_j - \frac{1}{2}\eta_j^A \partial_{jA}} \right) \widehat{\mathcal{M}}_n \left(h_i, \bar{h}_i, \frac{az_i + b}{cz_i + d}, \frac{\bar{a}\bar{z}_i + \bar{b}}{\bar{c}\bar{z}_i + \bar{d}}, \eta_i \right) \\ &= \left(\prod_{j=1}^n (cz_j + d)^{-2h_j} (\bar{c}\bar{z}_j + \bar{d})^{-2\bar{h}_j} \right) \widehat{\mathcal{M}}_n \left(h_i, \bar{h}_i, \frac{az_i + b}{cz_i + d}, \frac{\bar{a}\bar{z}_i + \bar{b}}{\bar{c}\bar{z}_i + \bar{d}}, \left(\frac{cz_i + d}{\bar{c}\bar{z}_i + \bar{d}} \right)^{\frac{1}{2}} \eta_i \right). \end{aligned} \quad (2.97)$$

The second equality follows because the celestial superamplitude is the sum of different component terms where each term contains specific number of η 's; shifting the power in $cz_j + d, \bar{c}\bar{z}_j + \bar{d}$ according to the number of η_j just implements the transformation of η_j itself under $SL(2, \mathbb{C})$, which is also consistent with its conformal weight $(h_\eta, \bar{h}_\eta) = (\frac{1}{4}, -\frac{1}{4})$. One can also check explicitly that all the superconformal generators leave the conformal weights invariant as desired, otherwise different terms in the sum in (2.96) carry different weights due to the action of generators on different particles and thus break the $SL(2, \mathbb{C})$ covariance.

In the following, we will be mainly using the on-shell superfield $\widehat{\Psi}$ as its physical meaning is clearer and has manifest $SL(2, \mathbb{C})$ covariance. To ease the notation, we will ignore the hat on the superfield (2.66), the superamplitude (2.67) and generators (2.78)–(2.89).

3 Celestial superamplitudes in $\mathcal{N} = 4$ SYM theory

In this section, we will compute the (color-ordered) celestial superamplitudes in $\mathcal{N} = 4$ SYM theory, focusing especially on the three-point and four-point case. This is done by performing a Mellin transformation on the momentum space superamplitudes. As we will see explicitly, the resulting celestial superamplitudes indeed can be regarded as the correlators of celestial superoperators and it has also manifest conformal invariance and supersymmetry. The computation technique here closely follows that in pure YM case [5]. We will first discuss the four-point amplitude, and then the three-point case.

3.1 Celestial amplitudes in YM theory

Before considering the superamplitudes in SYM, let us first review the YM case. Due to the conformal symmetry, the YM amplitudes have the following scaling property

$$\mathcal{A}_n(L\omega_i, z_i, \bar{z}_i) = L^{-n} \mathcal{A}_n(\omega_i, z_i, \bar{z}_i), \quad \mathcal{A}_n(\omega_i, z_i, \bar{z}_i) = A_n(p_i) \delta^4(P). \quad (3.1)$$

Then the celestial amplitudes are given by the Mellin transform (2.18). For four-particle scattering $1, 2 \rightarrow 3, 4$, actually the scaling property of YM amplitude ensures that the celestial amplitude takes the following form [5]:

$$\mathcal{M}_4 = \prod_{i=1}^4 \int_0^\infty d\omega_i \omega_i^{\Delta_i - 1} A_4(\omega_i, z_i, \bar{z}_i) \delta \left(\sum_i \varepsilon_i \omega_i q_i \right) \quad (3.2)$$

$$= \frac{\pi}{2} \delta \left(\sum_i \lambda_i \right) \Theta(\chi - 1) \delta(\chi - \bar{\chi}) \left(\prod_{i=1}^4 s_{*i}^{\lambda_i} \right) \frac{1}{|z_{13} z_{24}|^2} A_4(s_{*j}, z_j, \bar{z}_j), \quad (3.3)$$

where Θ equals to 0 (1) for negative (positive) argument and

$$s_{*1} = s_{*4} \frac{z_{24} \bar{z}_{34}}{z_{12} \bar{z}_{13}}, \quad s_{*2} = -s_{*4} \frac{z_{34} \bar{z}_{14}}{z_{23} \bar{z}_{12}}, \quad s_{*3} = -s_{*4} \frac{z_{24} \bar{z}_{14}}{z_{23} \bar{z}_{13}}, \quad (3.4)$$

The cross-ratio is defined as

$$\chi = \frac{z_{12} z_{34}}{z_{13} z_{24}}, \quad \bar{\chi} = \frac{\bar{z}_{12} \bar{z}_{34}}{\bar{z}_{13} \bar{z}_{24}}. \quad (3.5)$$

The different factors in (3.3) can be explained as follows. The factor $\delta(\sum_i \lambda_i)$ accounts for the scaling property (3.1) as well as the following integral

$$\int_0^\infty d\omega \omega^{i\lambda-1} = 2\pi \delta(\lambda). \quad (3.6)$$

Physically this is just the manifestation of the conformal symmetry: it ensures that the Ward identity (2.96) for dilatation (2.33) is identically satisfied. The factor $\delta(\chi - \bar{\chi})$ just comes from momentum conservation which ensures that in the four particle scattering process the momenta of for particles should lie on the same plane. The intersection of such a plane with celestial sphere forces the cross-ratio to be real.

For concreteness, let us consider the four-gluon scattering with $1^-, 2^- \rightarrow 3^+, 4^+$, namely $\varepsilon_{1,2} = -1, \varepsilon_{3,4} = 1$ and $J_{1,2} = -1, J_{3,4} = +1$. The color-ordered amplitude in momentum space is given by

$$\begin{aligned} \mathcal{A}_4(\omega_j, z_i, \bar{z}_i) &= A_4(p_i) \delta^4 \left(\sum_i p_i \right) \\ &= \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \delta^4 \left(\sum_i \varepsilon_i \omega_i q_i \right) = \frac{\omega_1 \omega_2}{\omega_3 \omega_4} \frac{z_{12}^3}{z_{23} z_{34} z_{41}} \delta^4 \left(\sum_i \varepsilon_i \omega_i q_i \right). \end{aligned} \quad (3.7)$$

Plugging into (3.3), we get the corresponding celestial amplitude¹⁰

$$\mathcal{M}_4 = -\frac{\pi}{2} \Theta(\chi-1) \delta(\chi-\bar{\chi}) \delta\left(\sum_j \lambda_j\right) K\left(\frac{i\lambda_1}{2}, \frac{i\lambda_2}{2}, 1+\frac{i\lambda_3}{2}, 1+\frac{i\lambda_4}{2}; 1+\frac{i\lambda_1}{2}, 1+\frac{i\lambda_2}{2}, \frac{i\lambda_3}{2}, \frac{i\lambda_4}{2}\right) \frac{\chi^{\frac{5}{3}}}{(1-\chi)^{\frac{1}{3}}} \quad (3.8)$$

where we introduce the kinematic factor K

$$K(h_1, h_2, h_3, h_4; \bar{h}_1, \bar{h}_2, \bar{h}_3, \bar{h}_4) = \prod_{i < j} z_{ij}^{\frac{h}{3} - h_i - h_j} \bar{z}_{ij}^{\frac{\bar{h}}{3} - \bar{h}_i - \bar{h}_j}, \quad h = \sum_i h_i, \quad \bar{h} = \sum_i \bar{h}_i. \quad (3.9)$$

So the four-gluon celestial amplitude indeed takes the form of the four-point function in CFT with weights $h_i, \bar{h}_i$ indicated above.

3.2 Four-point celestial superamplitudes in SYM

Now we move to the amplitudes in SYM theory. The four-point color-ordered superamplitude in momentum basis is given by [25]

$$\mathcal{A}_4(\omega_j, z_j, \bar{z}_j) = \frac{\delta^4(P) \delta^8(Q)}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} = \frac{\delta^4(P) \delta^8(Q)}{(-2)^4 \omega_1 \omega_2 \omega_3 \omega_4} \frac{1}{z_{12} z_{23} z_{34} z_{41}} \quad (3.10)$$

¹⁰Actually, there is also an overall factor $(-1)^{i\lambda_2 + i\lambda_3}$ which will be suppressed here and below.

where

$$\delta^8(\mathbf{Q}) = \delta^8 \left(\sum_{i=1}^4 \lambda_i^\alpha \eta_i^A \right) = \prod_{A=1}^4 \sum_{\substack{i,j=1 \\ i < j}}^4 \langle ij \rangle \eta_i^A \eta_j^A = \prod_{A=1}^4 \sum_{\substack{i,j=1 \\ i < j}}^4 -2\sqrt{\omega_i \omega_j} z_{ij} \eta_i^A \eta_j^A \quad (3.11)$$

The delta function $\delta^4(P), \delta^8(\mathbf{Q})$ guarantees that the amplitude conserves the momenta and is supersymmetric invariant. The celestial amplitude can be again obtained through Mellin transformation. In general for $\mathcal{A}_4(\omega_i, z_i, \bar{z}_i) = \mathbf{A}_4(\omega_i, z_i, \bar{z}_i) \delta^4(P) \delta^8(\mathbf{Q})$, we have

$$\mathcal{M}_4 = \prod_{k=1}^4 \int_0^\infty d\omega_k \omega_k^{\Delta_k-1} \mathcal{A}_4(\omega_i, z_i, \bar{z}_i) \quad (3.12)$$

$$= \prod_{k=1}^4 \int_0^\infty d\omega_k \omega_k^{\Delta_k-1} \mathbf{A}_4(\omega_i, z_i, \bar{z}_i) \delta \left(\sum_i \varepsilon_i \omega_i q_i \right) \prod_{A=1}^4 \sum_{\substack{i,j=1 \\ i < j}}^4 -2\sqrt{\omega_i \omega_j} z_{ij} \eta_i^A \eta_j^A \quad (3.13)$$

$$= \prod_{A=1}^4 \sum_{\substack{i,j=1 \\ i < j}}^4 -2z_{ij} \eta_i^A \eta_j^A e^{\frac{\partial_{\Delta_i} + \partial_{\Delta_j}}{2}} \prod_{k=1}^4 \int_0^\infty d\omega_k \omega_k^{\Delta_k-1} \mathbf{A}_4(\omega_i, z_i, \bar{z}_i) \delta \left(\sum_i \varepsilon_i \omega_i q_i \right) \quad (3.14)$$

$$= \prod_{A=1}^4 \sum_{\substack{i,j=1 \\ i < j}}^4 z_{ij} \eta_i^A \eta_j^A e^{\frac{\partial_{\Delta_i} + \partial_{\Delta_j}}{2}} \times \left[\frac{\pi}{2} \delta \left(\sum_k \lambda_k + 4i \right) \Theta(\chi - 1) \delta(\chi - \bar{\chi}) \left(\prod_{k=1}^4 s_{*k}^{i\lambda_k} \right) \frac{(-2)^4 \mathbf{A}_4(s_{*j}, z_j, \bar{z}_j)}{|z_{13} z_{24}|^2} \right]. \quad (3.15)$$

For the superamplitude in (3.10), we can easily compute

$$\begin{aligned} & \left(\prod_{k=1}^4 s_{*k}^{i\lambda_k} \right) \frac{\mathbf{A}_4(s_{*j}, z_j, \bar{z}_j)}{|z_{13} z_{24}|^2} \delta \left(\sum_k \lambda_k + 4i \right) \\ &= -K(\mathbf{h}_i; \bar{\mathbf{h}}_i) \chi^{\frac{1}{3}} (1 - \chi)^{\frac{1}{3}} \delta \left(\sum_k \lambda_k + 4i \right), \quad \mathbf{h}_i \equiv 1 + \frac{i\lambda_i}{2}, \quad \bar{\mathbf{h}}_i \equiv \frac{i\lambda_i}{2}, \end{aligned} \quad (3.16)$$

where we use the fact that the cross-ratio is real $\chi = \bar{\chi}$.

Therefore the four-point celestial super-amplitude (2.67) can be written as:

$$\begin{aligned} \mathcal{M}_4 &= \langle \Psi_{h_1, \bar{h}_1}(z_1, \bar{z}_1, \eta_1) \cdots \Psi_{h_4, \bar{h}_4}(z_4, \bar{z}_4, \eta_4) \rangle \\ &= -\frac{\pi}{2} \delta \left(\sum_k \lambda_k \right) \Theta(\chi - 1) \delta(\chi - \bar{\chi}) \prod_{A=1}^4 \sum_{\substack{i,j=1 \\ i < j}}^4 z_{ij} \eta_i^A \eta_j^A e^{\frac{\partial_{\Delta_i} + \partial_{\Delta_j}}{2}} \left[\chi^{\frac{1}{3}} (1 - \chi)^{\frac{1}{3}} K(\mathbf{h}_i; \bar{\mathbf{h}}_i) \right]. \end{aligned} \quad (3.18)$$

The celestial super-amplitudes can be regarded as the conformal correlation function of superoperators on the celestial sphere.¹¹ The shift operation $e^{\partial_{\Delta_i}/2}$ in the fermionic delta

¹¹Note that the fermionic delta function (3.28) has vanishing weights under $\text{SL}(2, \mathbb{C})$. Indeed, obviously $e^{\partial_{\Delta_i}/2}$ has weight $(h, \bar{h}) = (\frac{1}{4}, \frac{1}{4})$, η_i has weight $(h, \bar{h}) = (\frac{1}{4}, -\frac{1}{4})$, and $\langle q_i |^\alpha$ has weight $(h, \bar{h}) = (-\frac{1}{2}, 0)$ as can be seen from (2.4).

function seems to be unconventional; but actually it can be replaced with the multiplication factor $K^{-1}e^{\partial\Delta_i/2}K = \prod_{k \neq i}(z_{ik}\bar{z}_{ik})^{-\frac{1}{4}}$ once acting on the square bracket.

Performing the η expansion, we get all the component amplitudes. For example, consider the $\prod_{A=1}^4 \eta_1^A \eta_2^A$ term in the expansion. Using

$$\begin{aligned} & \left(z_{12} e^{\frac{\partial_i \lambda_1 + \partial_i \lambda_2}{2}} \right)^4 K \left(1 + \frac{i\lambda_i}{2}; \frac{i\lambda_i}{2} \right) \chi^{\frac{1}{3}} (1 - \chi)^{\frac{1}{3}} \\ &= K \left(\frac{i\lambda_1}{2}, \frac{i\lambda_2}{2}, 1 + \frac{i\lambda_3}{2}, 1 + \frac{i\lambda_4}{2}; \frac{1+i\lambda_1}{2}, 1 + \frac{i\lambda_2}{2}, \frac{i\lambda_3}{2}, \frac{i\lambda_4}{2} \right) \frac{\chi^{\frac{5}{3}}}{(1-\chi)^{\frac{1}{3}}}, \end{aligned} \quad (3.19)$$

one obtains

$$\begin{aligned} & \langle \Phi_{h_1, \bar{h}_1}(z_1, \bar{z}_1, \eta_1) \cdots \Phi_{h_n, \bar{h}_n}(z_n, \bar{z}_n, \eta_n) \rangle \Big|_{\eta_1^4 \eta_2^4} \\ &= -\frac{\pi}{2} \delta \left(\sum_k \lambda_k \right) \Theta(\chi-1) \delta(\chi-\bar{\chi}) K \left(\frac{i\lambda_1}{2}, \frac{i\lambda_2}{2}, 1 + \frac{i\lambda_3}{2}, 1 + \frac{i\lambda_4}{2}; \frac{1+i\lambda_1}{2}, 1 + \frac{i\lambda_2}{2}, \frac{i\lambda_3}{2}, \frac{i\lambda_4}{2} \right) \frac{\chi^{\frac{5}{3}}}{(1-\chi)^{\frac{1}{3}}}. \end{aligned} \quad (3.20)$$

This agrees with the four-gluon celestial superamplitude in (3.8).

Similarly, we can consider the four gluino amplitude by picking up the term $\eta_1^1 \eta_3^1 \prod_{A=2}^4 \eta_2^A \eta_4^A$:

$$\begin{aligned} & \langle \Phi_{h_1, \bar{h}_1}(z_1, \bar{z}_1, \eta_1) \cdots \Phi_{h_n, \bar{h}_n}(z_n, \bar{z}_n, \eta_n) \rangle \Big|_{\eta_1^1 \eta_3^1 \prod_{A=2}^4 \eta_2^A \eta_4^A} \\ &= -\frac{\pi}{2} \delta \left(\sum_k \lambda_k \right) \Theta(\chi-1) \delta(\chi-\bar{\chi}) K(h_i, \bar{h}_i) \frac{1}{\chi(1-\chi)^{\frac{1}{3}}}, \end{aligned} \quad (3.21)$$

where the conformal weights are

$$h_1 = \frac{3}{4} + \frac{i\lambda_1}{2}, \quad \bar{h}_1 = \frac{1}{4} + \frac{i\lambda_1}{2}, \quad (3.22)$$

$$h_2 = \frac{1}{4} + \frac{i\lambda_2}{2}, \quad \bar{h}_2 = \frac{3}{4} + \frac{i\lambda_2}{2}, \quad (3.23)$$

$$h_3 = \frac{3}{4} + \frac{i\lambda_3}{2}, \quad \bar{h}_3 = \frac{1}{4} + \frac{i\lambda_3}{2}, \quad (3.24)$$

$$h_4 = \frac{1}{4} + \frac{i\lambda_4}{2}, \quad \bar{h}_4 = \frac{3}{4} + \frac{i\lambda_4}{2}. \quad (3.25)$$

This corresponds to the four-gluino scattering $1^{+\frac{1}{2}}, 2^{-\frac{1}{2}} \rightarrow 3^{+\frac{1}{2}}, 4^{-\frac{1}{2}}$. The result can be checked to agree with the one obtained from Mellin transformation directly. All the rest of component amplitudes can be obtained in a similar way.

3.3 Structure at higher points

Just like the super-amplitude (3.10) written in a form with manifest translation symmetry and supersymmetry, the celestial superamplitude (3.18) can also be written as

$$\mathcal{M}_4 = \delta(D) \delta^8(Q) \left[-\frac{\pi}{2} \Theta(\chi-1) \delta(\chi-\bar{\chi}) \chi^{\frac{1}{3}} (1-\chi)^{\frac{1}{3}} K(\mathbf{h}_i; \bar{\mathbf{h}}_i) \right], \quad (3.26)$$

where the dilatation and supercharges are

$$\delta(D) = \delta \left(\sum_i D_i \right) = \delta \left(\sum_i \lambda_i \right), \quad (3.27)$$

$$\delta^8(Q) = \delta^8 \left(\sum_i \langle q_i |^\alpha \eta_i^A e^{\frac{1}{2} \partial_{\Delta_i}} \right) = \prod_{A=1}^4 \sum_{i < j} z_{ij} \eta_i^A \eta_j^A e^{\frac{\partial_{\Delta_i} + \partial_{\Delta_j}}{2}}. \quad (3.28)$$

They follow from our superconformal generators defined in (2.80) and (2.86). The celestial amplitude written in this form has thus manifest dilatation symmetry and supersymmetry. This structure can be generalized to higher points.

The n -point superamplitudes in momentum space take the following form [25]

$$\mathcal{A}_n = \frac{\delta^4(P) \delta^8(Q)}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle} H_n(\lambda_i, \tilde{\lambda}_i, \eta_i), \quad n \geq 4, \quad (3.29)$$

where

$$H_n(\lambda_i, \tilde{\lambda}_i, \eta_i) = H_n^{(0)} + H_n^{(4)} + H_n^{(8)} + \cdots + H_n^{(4n-16)}, \quad H_n^{(l)} \sim \mathcal{O}(\eta^l). \quad (3.30)$$

Each term in the expansion corresponds to MHV, NMHV, $\dots$, $\overline{\text{MHV}}$, respectively. For MHV, $H_n^{(0)} = 1$.

Similarly, the corresponding celestial superamplitudes are supposed to have the following general structure:

$$\mathcal{M}_n = \delta(D) \delta^8(Q) \mathcal{F}_n(h_i, \bar{h}_i, z_i, \bar{z}_i, \eta_i), \quad (3.31)$$

where $\delta(D), \delta^8(Q)$ are given in (3.27) and (3.28). The function $\mathcal{F}_n$ can be regarded as an n -point function in CCFT and transforms covariantly under $\text{SL}(2, \mathbb{C})$.¹² It also admits an expansion as (3.30) and each term can be written as the product of the kinematic factor (namely the n -point generalization of (3.9)), the function of cross-ratios, and Grassmann variables. The explicit form of $\mathcal{F}_n$ can be obtained from Mellin transformation as in the YM case.¹³ The expression (3.31) reduces to (3.18) when $n = 4$ as expected. So the general structure in (3.31) has not only the manifest Lorentz invariance, but also the manifest dilatation symmetry and supersymmetry.

In principle, we can compute all the higher-point celestial superamplitudes explicitly. For pure YM theory, the higher point was considered in [26]. Instead of doing tedious higher point computation, we will consider the three-point celestial amplitudes in the rest of this section.

3.4 Three-point celestial superamplitudes in SYM

The three-point amplitudes vanish on-shell identically in Minkowski signature due to the kinematic constraints. But we can define the three-point celestial in the (2,2) split signature. As such, the z_{ij} and $\bar{z}_{ij}$ become real and independent variables. And there are two helicity configurations for three-point amplitude, MHV and $\overline{\text{MHV}}$.

¹²More precisely, both $\mathcal{M}_n$ and $\mathcal{F}_n$ obey (2.97). Note $\delta(D), \delta^8(Q)$ are left invariant under $\text{SL}(2, \mathbb{C})$.

¹³Considering the rich symmetry of $\mathcal{N} = 4$ SYM, it would be interesting to see whether it is possible to bootstrap $\mathcal{F}_n$ without performing Mellin transformation.

For three-point gluon amplitude $\mathcal{A}_3(\omega_j, z_j, \bar{z}_j) = A_3(\omega_j, z_j, \bar{z}_j)\delta(\sum_i \varepsilon_i \omega_i q_i)$, one can obtain the celestial amplitude by performing the Mellin transformation and using again its scaling property. The momentum conservation leads to two branches. In particular, on the branch where $z_{ij} \neq 0$, the celestial amplitude reads¹⁴

$$\mathcal{M}_3 = \frac{\pi}{2} \delta\left(\sum \lambda_i\right) \prod_{i=1}^3 s_{*i}^{i\lambda_i} A_3(s_{*j}, z_j, \bar{z}_j) \frac{\delta(\bar{z}_{13})\delta(\bar{z}_{23})\text{sgn}(z_{23}z_{31})}{z_{23}z_{31} s_{*3}} \Theta\left(\frac{\varepsilon_1}{\varepsilon_3} \frac{z_{23}}{z_{12}}\right) \Theta\left(\frac{\varepsilon_2}{\varepsilon_3} \frac{z_{31}}{z_{12}}\right), \quad (3.32)$$

where

$$s_{*1} = \varepsilon_1 \frac{z_{23}}{D_3}, \quad s_{*2} = \varepsilon_2 \frac{z_{31}}{D_3}, \quad s_{*3} = \varepsilon_3 \frac{z_{12}}{D_3}, \quad D_3 = \varepsilon_1 z_{23} + \varepsilon_2 z_{31} + \varepsilon_3 z_{12}. \quad (3.33)$$

The celestial amplitude on the other branch $\bar{z}_{ij} \neq 0$ is similarly obtained by exchanging $z_{ij} \leftrightarrow \bar{z}_{ij}$.

For MHV amplitude $1^-2^-3^+$ in YM theory, we have

$$\mathcal{A}_{--+}(\omega_j, z_j, \bar{z}_j) = \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle} \delta^4(P) = -2 \frac{\omega_1 \omega_2}{\omega_3} \frac{z_{12}^3}{z_{23} z_{31}} \delta^4\left(\sum_i \varepsilon_i \omega_i q_i^\mu\right). \quad (3.34)$$

The corresponding celestial amplitude is thus given by

$$\begin{aligned} \mathcal{M}_{--+} = & -\pi \delta\left(\sum_i \lambda_i\right) \frac{z_{12}^3}{z_{23}^2 z_{13}^2} \left(\frac{\varepsilon_1}{\varepsilon_3} \frac{z_{23}}{z_{12}}\right)^{1+i\lambda_1} \left(\frac{\varepsilon_2}{\varepsilon_3} \frac{z_{31}}{z_{12}}\right)^{1+i\lambda_2} \\ & \times \text{sgn}(z_{23}z_{31})\delta(\bar{z}_{13})\delta(\bar{z}_{23})\Theta\left(\frac{\varepsilon_1}{\varepsilon_3} \frac{z_{23}}{z_{12}}\right)\Theta\left(\frac{\varepsilon_2}{\varepsilon_3} \frac{z_{31}}{z_{12}}\right). \end{aligned} \quad (3.35)$$

The three-point MHV superamplitude in SYM theory is

$$\mathcal{A}_3^{\text{MHV}}(\omega_j, z_j, \bar{z}_j) = \frac{\delta^4(P)\delta^8(Q)}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} = \frac{\delta^4(P)\delta^8(Q)}{(-2)^3 \omega_1 \omega_2 \omega_3} \frac{1}{z_{12} z_{23} z_{31}}, \quad (3.36)$$

where

$$\delta^8(Q) = \delta^8\left(\sum_{i=1}^3 \lambda_i^\alpha \eta_i^A\right) = \langle 12 \rangle^4 \delta^4\left(\eta_1 - \frac{\langle 23 \rangle}{\langle 12 \rangle} \eta_3\right) \delta^4\left(\eta_2 - \frac{\langle 31 \rangle}{\langle 12 \rangle} \eta_3\right) = \prod_{A=1}^4 \sum_{i,j=1}^4 \langle ij \rangle \eta_i^A \eta_j^A, \quad (3.37)$$

where we present two equivalent representations.

Written in terms of the celestial variables and replacing $\omega_i \rightarrow e^{\partial_{\Delta_i}}$ in the Mellin integral, $\delta^8(Q)$ becomes

$$(-2)^4 z_{12}^4 \delta^4\left(\eta_1 e^{\frac{1}{2}\partial_{\Delta_1}} - \frac{z_{23}}{z_{12}} \eta_3 e^{\frac{1}{2}\partial_{\Delta_3}}\right) \delta^4\left(\eta_2 e^{\frac{1}{2}\partial_{\Delta_2}} - \frac{z_{31}}{z_{12}} \eta_3 e^{\frac{1}{2}\partial_{\Delta_3}}\right) = \prod_{A=1}^4 \sum_{i,j=1}^4 -2 z_{ij} e^{\frac{1}{2}(\partial_{\Delta_i} + \partial_{\Delta_j})} \eta_i^A \eta_j^A. \quad (3.38)$$

¹⁴Here we used the equation:

$$\delta\left(\sum_i \varepsilon_i s_i q_i\right) \delta\left(\sum_i s_i - 1\right) \Big|_{z_{ij} \neq 0} = \frac{\delta(\bar{z}_{13})\delta(\bar{z}_{23})}{4s_1 s_2 s_3 D_3^2} \prod_{i=1}^3 \delta(s_i - s_{*i}) = \frac{\delta(\bar{z}_{13})\delta(\bar{z}_{23})\text{sgn}(z_{23}z_{31})}{4z_{23}z_{31}s_3} \prod_{i=1}^3 \delta(s_i - s_{*i}).$$

Then one can derive the three-point celestial superamplitude which generally takes the form

$$\begin{aligned} \mathcal{M}_3 = 2\pi\delta\left(\sum_i \lambda_i\right) \frac{\delta(\bar{z}_{13})\delta(\bar{z}_{23})\text{sgn}(z_{23}z_{31})}{4z_{23}z_{31}s_{*3}} \Theta\left(\frac{\epsilon_1}{\epsilon_3}\frac{z_{23}}{z_{12}}\right) \Theta\left(\frac{\epsilon_2}{\epsilon_3}\frac{z_{31}}{z_{12}}\right) z_{12}^4(-2)^4 A_3(s_{*j}, z_j, \bar{z}_j) \\ \times \delta^4\left(\eta_1 e^{\frac{1}{2}\partial_{\Delta_1}} - \frac{z_{23}}{z_{12}}\eta_3 e^{\frac{1}{2}\partial_{\Delta_3}}\right) \delta^4\left(\eta_2 e^{\frac{1}{2}\partial_{\Delta_2}} - \frac{z_{31}}{z_{12}}\eta_3 e^{\frac{1}{2}\partial_{\Delta_3}}\right) \prod_{i=1}^3 s_{*i}^{i\lambda_i}. \end{aligned} \quad (3.39)$$

Applying to MHV in (3.35), we get the celestial super-amplitude

$$\begin{aligned} \mathcal{M}_3^{\text{MHV}} = -\pi\delta\left(\sum_i \lambda_i\right) \frac{\delta(\bar{z}_{13})\delta(\bar{z}_{23})\text{sgn}(z_{23}z_{31})}{z_{12}z_{23}^2z_{31}^2} \Theta\left(\frac{\epsilon_1}{\epsilon_3}\frac{z_{23}}{z_{12}}\right) \Theta\left(\frac{\epsilon_2}{\epsilon_3}\frac{z_{31}}{z_{12}}\right) \frac{z_{12}^4}{s_{*1}s_{*2}s_{*3}^2} \\ \times \delta^4\left(\eta_1 e^{\frac{1}{2}\partial_{\Delta_1}} - \frac{z_{23}}{z_{12}}\eta_3 e^{\frac{1}{2}\partial_{\Delta_3}}\right) \delta^4\left(\eta_2 e^{\frac{1}{2}\partial_{\Delta_2}} - \frac{z_{31}}{z_{12}}\eta_3 e^{\frac{1}{2}\partial_{\Delta_3}}\right) \prod_{i=1}^3 s_{*i}^{i\lambda_i} \end{aligned} \quad (3.40)$$

$$\begin{aligned} = -\pi\delta\left(\sum_i \lambda_i\right) \frac{\delta(\bar{z}_{13})\delta(\bar{z}_{23})\text{sgn}(z_{23}z_{31})}{z_{12}z_{23}^2z_{31}^2} \Theta\left(\frac{\epsilon_1}{\epsilon_3}\frac{z_{23}}{z_{12}}\right) \Theta\left(\frac{\epsilon_2}{\epsilon_3}\frac{z_{31}}{z_{12}}\right) \\ \times \prod_{A=1}^4 \sum_{\substack{i,j=1 \\ i < j}}^3 z_{ij} \sqrt{\frac{s_{*i}}{s_{*3}} \frac{s_{*j}}{s_{*3}}} \eta_i^A \eta_j^A \times \left(\frac{s_{*1}}{s_{*3}}\right)^{i\lambda_1-1} \left(\frac{s_{*2}}{s_{*3}}\right)^{i\lambda_2-1}. \end{aligned} \quad (3.41)$$

where we also present an equivalent expression in the second equality for later discussions. Picking up the term $\eta_1^4\eta_2^4$, one gets

$$\begin{aligned} \mathcal{M}_3^{\text{MHV}} \Big|_{\eta_1^4\eta_2^4} \\ = -\pi\delta\left(\sum \lambda_i\right) \frac{\delta(\bar{z}_{13})\delta(\bar{z}_{23})\text{sgn}(z_{23}z_{31})}{z_{12}z_{23}^2z_{31}^2} \Theta\left(\frac{\epsilon_1}{\epsilon_3}\frac{z_{23}}{z_{12}}\right) \Theta\left(\frac{\epsilon_2}{\epsilon_3}\frac{z_{31}}{z_{12}}\right) \frac{z_{12}^4}{s_{*1}s_{*2}s_{*3}^2} s_{*1}^{i\lambda_1+2} s_{*2}^{i\lambda_2+2} s_{*3}^{i\lambda_3} \\ = -\pi\delta\left(\sum \lambda_i\right) \frac{z_{12}^3}{z_{23}^2z_{31}^2} \text{sgn}(z_{23}z_{31}) \delta(\bar{z}_{13})\delta(\bar{z}_{23}) \Theta\left(\frac{\epsilon_1}{\epsilon_3}\frac{z_{23}}{z_{12}}\right) \\ \times \Theta\left(\frac{\epsilon_2}{\epsilon_3}\frac{z_{31}}{z_{12}}\right) \left(\frac{\epsilon_1}{\epsilon_3}\frac{z_{23}}{z_{12}}\right)^{1+i\lambda_1} \left(\frac{\epsilon_2}{\epsilon_3}\frac{z_{31}}{z_{12}}\right)^{1+i\lambda_2}. \end{aligned} \quad (3.42)$$

This agrees with (3.35).

One can also consider the $\overline{\text{MHV}}$ superamplitude:

$$\mathcal{A}_3^{\overline{\text{MHV}}}(\omega_j, z_j, \bar{z}_j) = \frac{\delta^4(P)\delta^4([12]\eta_3 + [23]\eta_1 + [31]\eta_2)}{[12][23][31]} \quad (3.43)$$

$$= 2 \frac{\omega_1\omega_2\omega_3}{\bar{z}_{12}\bar{z}_{23}\bar{z}_{31}} \delta^4(P) \prod_{A=1}^4 \left(\epsilon_1\epsilon_2 \frac{\bar{z}_{12}}{\sqrt{\omega_3}} \eta_3^A + \epsilon_2\epsilon_3 \frac{\bar{z}_{23}}{\sqrt{\omega_1}} \eta_1^A + \epsilon_1\epsilon_3 \frac{\bar{z}_{31}}{\sqrt{\omega_2}} \eta_2^A \right), \quad (3.44)$$

whose celestial counterpart can be similarly derived

$$\begin{aligned} \mathcal{M}_3^{\overline{\text{MHV}}} = \pi\delta\left(\sum \lambda_i\right) \frac{\delta(z_{13})\delta(z_{23})\text{sgn}(\bar{z}_{23}\bar{z}_{31})}{\bar{z}_{12}\bar{z}_{23}^2\bar{z}_{31}^2} \Theta\left(\frac{\epsilon_1}{\epsilon_3}\frac{\bar{z}_{23}}{\bar{z}_{12}}\right) \Theta\left(\frac{\epsilon_2}{\epsilon_3}\frac{\bar{z}_{31}}{\bar{z}_{12}}\right) \bar{s}_{*1}\bar{s}_{*2} \\ \times \delta^4\left(\epsilon_1\epsilon_2\bar{z}_{12}\eta_3 e^{-\frac{1}{2}\partial_{\Delta_3}} + \epsilon_2\epsilon_3\bar{z}_{23}\eta_1 e^{-\frac{1}{2}\partial_{\Delta_1}} + \epsilon_1\epsilon_3\bar{z}_{31}\eta_2 e^{-\frac{1}{2}\partial_{\Delta_2}}\right) \prod_{i=1}^3 \bar{s}_{*i}^{i\lambda_i}, \end{aligned} \quad (3.45)$$

where $\bar{s}_{*i}$ is given in (3.33) by replacing $z \rightarrow \bar{z}$. Picking up the term η_3^4 , one recovers the $\overline{\text{MHV}}$ for gluon three-point amplitude

$$\mathcal{M}_3^{\overline{\text{MHV}}}|_{\eta_3^4} = \pi \delta(\sum \lambda_i) \frac{\bar{z}_{12}^3}{\bar{z}_{23}^2 \bar{z}_{31}^2} \delta(z_{13}) \delta(z_{23}) \text{sgn}(\bar{z}_{23} \bar{z}_{31}) \Theta\left(\frac{\varepsilon_1}{\varepsilon_3} \frac{\bar{z}_{23}}{\bar{z}_{12}}\right) \Theta\left(\frac{\varepsilon_2}{\varepsilon_3} \frac{\bar{z}_{31}}{\bar{z}_{12}}\right) \left(\frac{\varepsilon_2}{\varepsilon_3} \frac{\bar{z}_{31}}{\bar{z}_{12}}\right)^{i\lambda_1+1} \left(\frac{\varepsilon_1}{\varepsilon_3} \frac{\bar{z}_{23}}{\bar{z}_{12}}\right)^{i\lambda_2+1}. \quad (3.46)$$

The full three-point celestial super-amplitude is the sum of two helicity configurations: $\mathcal{M}_3 = \mathcal{M}_3^{\text{MHV}} + \mathcal{M}_3^{\overline{\text{MHV}}}$.

4 Soft theorem and Ward identity

In this section we study the soft limit of $\mathcal{N} = 4$ SYM theory. The soft limit captures some universal properties of quantum field theory. Indeed it has been understood in recent years that the soft theorems can be regarded as the Ward identity of various asymptotic symmetries. See [27] for a review. For graviton, the soft theorems arise from the BMS symmetries which include super-translation and super-rotation [28]. For gluon, the underlying symmetry for the leading soft theorem is the large gauge transformation [29]. However, the relations between symmetry and soft theorem are obscure in the conventional momentum space where the soft limit corresponds to taking one particle soft with vanishing energy.

Such a connection becomes much more obvious once we go to the celestial sphere and study the celestial amplitude. It turns out in the celestial basis, the energetically soft theorem becomes the conformally soft theorem [9–12]. Such a conformally soft theorem can be further translated into the Ward identity in CCFT. The Ward identity is just the standard Ward identity in CFT which relates the CFT correlators with and without the insertion of the soft symmetry current. For the leading soft gluon theorem, the corresponding soft symmetry current is just the Kac-Moody current.

We will generalize all these discussions to $\mathcal{N} = 4$ SYM theory. We will show the conformally supersoft theorem where one operator has dimension $\Delta \rightarrow 1$. Translating into the language of CCFT, this corresponds to the super-Ward identity.

4.1 Conformally soft gluon theorem in YM

Let us review the conformally soft gluon theorem in YM theory first [10]. In the definition of celestial amplitude (2.18), one can choose the j -th particle and take the limit $\lambda_j \rightarrow 0$ which leads to

$$\lim_{\lambda_j \rightarrow 0} i\lambda_j \mathcal{M}_n(\Delta_i, J_i, z_i, \bar{z}_i) = 2 \left(\prod_{\substack{j=1 \\ k \neq j}}^n \int_0^\infty d\omega_k \omega_k^{\Delta_k-1} \right) \int_0^\infty d\omega_j \delta(\omega_j) \omega_j \mathcal{A}_n(\omega_i, z_i, \bar{z}_i). \quad (4.1)$$

It is easy to recognize that the right hand side just picks the residue of momentum space amplitude $\mathcal{A}_n(\omega_i, z_i, \bar{z}_i)$ at $\omega_j = 0 = p_j$. The (energetically) soft theorem for color-ordered partial amplitude is given by in (A.1), (A.2). In terms of the celestial coordinates, the leading soft (positive helicity gluon) theorem takes the form:

$$\lim_{\omega_j \rightarrow 0} \omega_j \mathcal{A}_n(\omega_i, z_i, \bar{z}_i) = -\frac{1}{2} \frac{z_{j-1,j+1}}{z_{j-1,j} z_{j,j+1}} \mathcal{A}_{n-1}(1, \dots, j-1, j+1, \dots, n), \quad (4.2)$$

where the j -th particle is omitted on the right hand side. Plugging this into (4.1), we get the tree-level conformally soft theorem [10]

$$\lim_{\lambda_j \rightarrow 0} i\lambda_j \mathcal{M}_n(\Delta_i, J_i, z_i, \bar{z}_i) = -\frac{1}{2} \frac{z_{j-1,j+1}}{z_{j-1,j} z_{j,j+1}} \mathcal{M}_{n-1}(1, \dots, j-1, j+1, \dots, n), \quad (4.3)$$

where the factor can also be written as $1/z_{j-1,j} + 1/z_{j,j+1}$.

The conformal correlator is related to the full amplitude which includes the color factors and takes the following form at tree level:

$$\begin{aligned} & \langle O_{\Delta_1, J_1}^{a_1}(z_1, \bar{z}_1) O_{\Delta_2, J_2}^{a_2}(z_2, \bar{z}_2) \cdots O_{\Delta_n, J_n}^{a_n}(z_n, \bar{z}_n) \rangle \\ &= g_{\text{YM}}^{n-2} \sum_{\sigma \in S_{n-1}} \mathcal{M}_n(1, \sigma_2, \dots, \sigma_n) \text{Tr}(T^{a_1} T^{a_{\sigma_2}} \cdots T^{a_{\sigma_n}}), \end{aligned} \quad (4.4)$$

where σ acts by permuting the label $\{2, 3, \dots, n\}$.

If we define

$$J^a(z) = \lim_{\Delta \rightarrow 1} (\Delta - 1) O_{\Delta, +}^a, \quad (4.5)$$

which has conformal weights $(h, \bar{h}) = (1, 0)$, then the conformally soft theorem (4.12) becomes the Ward identity of a current algebra (after choosing normalization properly):

$$\begin{aligned} & \langle J^a(z) O_{\Delta_1, J_1}^{b_1}(z_1, \bar{z}_1) \cdots O_{\Delta_n, J_n}^{b_n}(z_n, \bar{z}_n) \rangle \\ &= \sum_{j=1}^n \frac{f^{abc}}{z - z_j} \langle O_{\Delta_1, J_1}^{b_1}(z_1, \bar{z}_1) \cdots O_{\Delta_j, J_j}^c(z_j, \bar{z}_j) \cdots O_{\Delta_n, J_n}^{b_n}(z_n, \bar{z}_n) \rangle, \end{aligned} \quad (4.6)$$

where f^{abc} is the structure constant of the Lie algebra.

In terms of OPE, we have

$$J^a(z) O_{\Delta, J}^b(w, \bar{w}) \sim \frac{f^{abc}}{z - w} O_{\Delta, J}^c(w, \bar{w}). \quad (4.7)$$

We can also consider the soft theorem for negative helicity gluon. Especially we have an anti-holomorphic current defined as

$$\bar{J}^a(\bar{z}) = \lim_{\Delta \rightarrow 1} (\Delta - 1) O_{\Delta, -}^a, \quad (4.8)$$

which obeys similar Ward identity and OPE as that in (4.6) (4.7).

So far, we only considered the leading soft theorem, namely the first term in (A.2). The second term in (A.2) corresponds to the subleading soft theorem. After performing Mellin transformation, one obtains [13, 20, 30]

$$\begin{aligned} & \langle K^a(z, \bar{z}) O_{\Delta_1, J_1}^{b_1}(z_1, \bar{z}_1) \cdots O_{\Delta_n, J_n}^{b_n}(z_n, \bar{z}_n) \rangle \\ &= \sum_{j=1}^n \frac{\varepsilon_j f^{ab_j c}}{z - z_j} \left(-2\bar{h}_j + 1 + (\bar{z} - \bar{z}_j) \frac{\partial}{\partial \bar{z}_j} \right) \langle O_{\Delta_1, J_1}^{b_1}(z_1, \bar{z}_1) \cdots O_{\Delta_{j-1}, J_{j-1}}^c(z_{j-1}, \bar{z}_{j-1}) \cdots O_{\Delta_n, J_n}^{b_n}(z_n, \bar{z}_n) \rangle \end{aligned} \quad (4.9)$$

where

$$K^a(z, \bar{z}) = \lim_{\Delta \rightarrow 0} \Delta O_{\Delta, +}^a. \quad (4.10)$$

4.2 Conformally supersoft theorem in SYM

The (leading) super-soft theorem of SYM is given by (A.3) and (A.6). In terms of celestial coordinates, it reads

$$\omega_j \mathcal{A}_n(\cdots, j-1, j, j+1, \cdots) \xrightarrow{\omega_j \rightarrow 0} -\frac{1}{2} \left(\frac{z_{j-1, j+1}}{z_{j-1, j} z_{j, j+1}} + \frac{\bar{z}_{j-1, j+1}}{\bar{z}_{j-1, j} \bar{z}_{j, j+1}} \delta^4(\eta_j) \right) \mathcal{A}_n(\cdots, j-1, j+1, \cdots). \quad (4.11)$$

The formula (4.1) can also be used for super-amplitude above and gives rise to

$$\begin{aligned} & \lim_{\lambda_j \rightarrow 0} i \lambda_j \mathcal{M}_n(\cdots, j-1, j, j+1, \cdots) \\ &= -\frac{1}{2} \left(\frac{z_{j-1, j+1}}{z_{j-1, j} z_{j, j+1}} + \frac{\bar{z}_{j-1, j+1}}{\bar{z}_{j-1, j} \bar{z}_{j, j+1}} \delta^4(\eta_j) \right) \mathcal{M}_{n-1}(\cdots, j-1, j+1, \cdots). \end{aligned} \quad (4.12)$$

This is just the leading conformally super-soft theorem in SYM. In the following we will verify that this relation holds for the four-point celestial superamplitudes.

The four-point celestial superamplitude is computed in (3.18) and written in terms of kinematic factors and cross-ratio. For the soft theorem, it is more convenient to use the following equivalent expression:

$$\begin{aligned} \mathcal{M}_4 &= \frac{\pi}{2} \left(\sum_{i=1}^4 \lambda_i \right) \delta(z_{12} z_{34} \bar{z}_{13} \bar{z}_{24} - z_{13} z_{24} \bar{z}_{12} \bar{z}_{34}) \frac{1}{z_{12} z_{23} z_{34} z_{41}} \prod_{A=1}^4 \left(\sum_{\substack{i,j=1 \\ i < j}}^4 z_{ij} \sqrt{\frac{s_{*i}}{s_{*3}} \frac{s_{*j}}{s_{*3}}} \eta_i^A \eta_j^A \right) \\ &\times \left(\frac{s_{*1}}{s_{*3}} \right)^{-1+i\lambda_1} \left(\frac{s_{*2}}{s_{*3}} \right)^{-1+i\lambda_2} \left(\frac{s_{*4}}{s_{*3}} \right)^{-1+i\lambda_4} \Theta \left(\frac{s_{*1}}{s_{*3}} \right) \Theta \left(\frac{s_{*2}}{s_{*3}} \right) \Theta \left(\frac{s_{*4}}{s_{*3}} \right), \end{aligned} \quad (4.13)$$

where

$$\frac{s_{*1}}{s_{*3}} = \frac{\varepsilon_3}{\varepsilon_1} \frac{z_{23} \bar{z}_{34}}{z_{12} \bar{z}_{14}} = \frac{\varepsilon_3}{\varepsilon_1} \frac{\bar{z}_{23} z_{34}}{\bar{z}_{12} z_{14}}, \quad \frac{s_{*2}}{s_{*3}} = \frac{\varepsilon_3}{\varepsilon_2} \frac{z_{13} \bar{z}_{34}}{z_{12} \bar{z}_{42}} = \frac{\varepsilon_3}{\varepsilon_2} \frac{\bar{z}_{13} z_{34}}{\bar{z}_{12} z_{42}}, \quad \frac{s_{*4}}{s_{*3}} = \frac{\varepsilon_3}{\varepsilon_4} \frac{z_{23} \bar{z}_{13}}{z_{42} \bar{z}_{14}} = \frac{\varepsilon_3}{\varepsilon_4} \frac{\bar{z}_{23} z_{13}}{\bar{z}_{42} z_{14}}. \quad (4.14)$$

For each term, the equalities of the two different expressions arise from the delta-function $\delta(z_{12} z_{34} \bar{z}_{13} \bar{z}_{24} - z_{13} z_{24} \bar{z}_{12} \bar{z}_{34})$.¹⁵

Using the following identity [10]

$$\lim_{\epsilon \rightarrow 0} \frac{\epsilon}{2} |x|^{\epsilon-1} = \delta(x), \quad (4.15)$$

we get

$$\begin{aligned} & \lim_{\lambda_4 \rightarrow 0} i \lambda_4 \mathcal{M}_4 \\ &= \frac{\pi}{2} \left(\sum_{i=1}^4 \lambda_i \right) \delta(z_{12} z_{34} \bar{z}_{13} \bar{z}_{24} - z_{13} z_{24} \bar{z}_{12} \bar{z}_{34}) \frac{1}{z_{12} z_{23} z_{34} z_{41}} \prod_{A=1}^4 \left(\sum_{\substack{i,j=1 \\ i < j}}^4 z_{ij} \sqrt{\frac{s_{*i}}{s_{*3}} \frac{s_{*j}}{s_{*3}}} \eta_i^A \eta_j^A \right) \\ &\times \left(\frac{s_{*1}}{s_{*3}} \right)^{-1+i\lambda_1} \left(\frac{s_{*2}}{s_{*3}} \right)^{-1+i\lambda_2} \delta \left(\frac{s_{*4}}{s_{*3}} \right) \Theta \left(\frac{s_{*1}}{s_{*3}} \right) \Theta \left(\frac{s_{*2}}{s_{*3}} \right), \end{aligned} \quad (4.16)$$

where we used $\Theta(0) = 1/2$.

¹⁵Note that since we want to relate this four-point amplitude to three-point on-shell amplitude which only makes sense in split signature, we will thus also consider here the four-point amplitudes in the split signature, namely here z_{ij} and $\bar{z}_{ij}$ are all real and independent.

Examining the two expressions for $\frac{s_{*4}}{s_{*3}}$ in (4.14), we see that $\frac{s_{*4}}{s_{*3}} = 0$ has two branches $z_{13} = 0$ or $\bar{z}_{13} = 0$.¹⁶

In the first branch $\bar{z}_{13} = 0$, the delta functions can be simplified

$$\delta\left(\frac{\varepsilon_3}{\varepsilon_4} \frac{\bar{z}_{13} z_{23}}{\bar{z}_{14} z_{42}}\right) = \text{sgn}(\bar{z}_{14} z_{42} z_{23}) \frac{\bar{z}_{14} z_{42}}{z_{23}} \delta(\bar{z}_{13}), \quad (4.17)$$

$$\delta(z_{12} z_{34} \bar{z}_{13} \bar{z}_{24} - z_{13} z_{24} \bar{z}_{12} \bar{z}_{34}) = \delta(z_{13} z_{24} \bar{z}_{12} \bar{z}_{34}) = \frac{\text{sgn}(z_{13} z_{24} \bar{z}_{34})}{z_{13} z_{24} \bar{z}_{34}} \delta(\bar{z}_{12}), \quad (4.18)$$

$$\rightarrow \delta(z_{12} z_{34} \bar{z}_{13} \bar{z}_{24} - z_{13} z_{24} \bar{z}_{12} \bar{z}_{34}) \delta\left(\frac{\varepsilon_3}{\varepsilon_4} \frac{\bar{z}_{13} z_{23}}{\bar{z}_{14} z_{42}}\right) = \text{sgn}(z_{23} z_{31}) \frac{1}{z_{23} z_{31}} \delta(\bar{z}_{12}) \delta(\bar{z}_{13}), \quad (4.19)$$

so in this branch we have $\bar{z}_1 = \bar{z}_2 = \bar{z}_3$ and

$$\frac{s_{*1}}{s_{*3}} = \frac{\varepsilon_3}{\varepsilon_1} \frac{z_{23}}{z_{12}}, \quad \frac{s_{*2}}{s_{*3}} = \frac{\varepsilon_3}{\varepsilon_2} \frac{z_{31}}{z_{12}}. \quad (4.20)$$

Substituting them back into (4.16), one finds

$$\begin{aligned} \lim_{\lambda_4 \rightarrow 0} i\lambda_4 \mathcal{M}_4 \Big|_{\bar{z}_{13}=0} &= \frac{\pi}{2} \left(\sum_{i=1}^4 \lambda_i \right) \frac{1}{z_{12} z_{23} z_{34} z_{41}} \\ &\times \prod_{A=1}^4 \left(\sum_{\substack{i,j=1 \\ i < j}}^4 z_{ij} \sqrt{\frac{s_{*i}}{s_{*3}} \frac{s_{*j}}{s_{*3}}} \eta_i^A \eta_j^A \right) \left(\frac{s_{*1}}{s_{*3}} \right)^{-1+i\lambda_1} \left(\frac{s_{*2}}{s_{*3}} \right)^{-1+i\lambda_2} \\ &\times \text{sgn}(z_{23} z_{31}) \frac{1}{z_{23} z_{31}} \delta(\bar{z}_{12}) \delta(\bar{z}_{13}) \Theta\left(\frac{s_{*1}}{s_{*3}}\right) \Theta\left(\frac{s_{*2}}{s_{*3}}\right) \end{aligned} \quad (4.21)$$

$$\begin{aligned} &= \frac{\pi}{2} \left(\sum_{i=1}^4 \lambda_i \right) \frac{z_{31}}{z_{34} z_{41}} \frac{\text{sgn}(z_{23} z_{31}) \delta(\bar{z}_{12}) \delta(\bar{z}_{13})}{z_{12} z_{23}^2 z_{31}^2} \Theta\left(\frac{\varepsilon_3}{\varepsilon_1} \frac{z_{23}}{z_{12}}\right) \Theta\left(\frac{\varepsilon_3}{\varepsilon_2} \frac{z_{31}}{z_{12}}\right) \\ &\times \left(\frac{\varepsilon_3}{\varepsilon_1} \frac{z_{23}}{z_{12}} \right)^{-1+i\lambda_1} \left(\frac{\varepsilon_3}{\varepsilon_2} \frac{z_{31}}{z_{12}} \right)^{-1+i\lambda_2} \prod_{A=1}^4 \left(\sum_{\substack{i,j=1 \\ i < j}}^3 z_{ij} \sqrt{\frac{s_{*i}}{s_{*3}} \frac{s_{*j}}{s_{*3}}} \eta_i^A \eta_j^A \right) \end{aligned} \quad (4.22)$$

where in the parentheses i, j are summed over 1 to 3 because $s_{*4} = 0$ and has no contribution.

Compared to (3.41), we find

$$\lim_{\lambda_4 \rightarrow 0} i\lambda_4 \mathcal{M}_4 \Big|_{\bar{z}_{13}=0} = -\frac{1}{2} \frac{z_{31}}{z_{34} z_{41}} \mathcal{M}_3^{\text{MHV}}. \quad (4.23)$$

In the other branch $z_{13} = 0$, similarly we have

$$\begin{aligned} \lim_{\lambda_4 \rightarrow 0} i\lambda_4 \mathcal{M}_4 \Big|_{z_{13}=0} &= \frac{\pi}{2} \left(\sum_{i=1}^4 \lambda_i \right) \prod_{A=1}^4 \left(\sum_{\substack{i,j=1 \\ i < j}}^4 z_{ij} \sqrt{\frac{s_{*i}}{s_{*3}} \frac{s_{*j}}{s_{*3}}} \eta_i^A \eta_j^A \right) \left(\frac{s_{*1}}{s_{*3}} \right)^{-1+i\lambda_1} \left(\frac{s_{*2}}{s_{*3}} \right)^{-1+i\lambda_2} \\ &\times \text{sgn}(\bar{z}_{23} \bar{z}_{31}) \frac{1}{\bar{z}_{23} \bar{z}_{31}} \frac{1}{z_{12} z_{23} z_{34} z_{41}} \delta(z_{12}) \delta(z_{13}) \Theta\left(\frac{s_{*1}}{s_{*3}}\right) \Theta\left(\frac{s_{*2}}{s_{*3}}\right). \end{aligned} \quad (4.24)$$

¹⁶Note that we are now in the split signature, so z_{ij} and $\bar{z}_{ij}$ are independent real variables. Other choices, say $z_{23} = 0$, give vanishing or infinite s_{*1}/s_{*3} or s_{*2}/s_{*3} .

The result seems to be divergent due to the factor $1/(z_{12}z_{23})$ and $\delta(z_{13})\delta(z_{12})$. However the coefficient $z_{ij}\sqrt{s_{*i}s_{*j}}$ in the Grassmann term also vanishes because $s_{*4} = 0$ and $z_{ij} = 0$ for $i, j \in \{1, 2, 3\}$. These two factors may cancel and lead to a finite result. Indeed, we have $z_{ij} \propto z_{12}$ and $\sqrt{s_{*4}} \propto \sqrt{z_{12}}$. So in order to cancel the factor z_{12}^2 , the only possibility is to take $z_{i4}\sqrt{s_{*i}s_{*4}}$ for each A . Other choices would lead to higher order zeros and thus vanishing contribution.

Especially, we find

$$\frac{z_{i4}\sqrt{\frac{s_{*i}s_{*4}}{s_{*3}s_{*3}}}}{\varepsilon_j\varepsilon_k\bar{z}_{jk}/\sqrt{s_{*i}}} = \varepsilon_1\varepsilon_2\sqrt{\varepsilon_3\varepsilon_4}\sqrt{s_{*3}}\sqrt{\frac{z_{12}z_{14}\bar{z}_{13}\bar{z}_{32}}{\bar{z}_{12}^3\bar{z}_{34}}}, \quad (i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2), \quad (4.25)$$

which enables us to rewrite

$$\prod_{A=1}^4 \left(\sum_i^3 z_{i4} \sqrt{\frac{s_{*i}s_{*4}}{s_{*3}s_{*3}}} \eta_i^A \eta_4^A \right) = \delta^4 \left(\sum_{i=1}^3 \varepsilon_j \varepsilon_k \bar{z}_{jk} / \sqrt{s_{*i}} \eta_i \right) \delta^4(\eta_4) s_{*3}^2 \left(\frac{z_{12}z_{14}\bar{z}_{13}\bar{z}_{32}}{\bar{z}_{12}^3\bar{z}_{34}} \right)^2. \quad (4.26)$$

With this identity, (4.24) can be rewritten as

$$\begin{aligned} & \lim_{\lambda_4 \rightarrow 0} i\lambda_4 \mathcal{M}_4 \Big|_{z_{13}=0} \\ &= \frac{\pi}{2} \left(\sum_{i=1}^4 \lambda_i \right) \delta^4 \left(\frac{\varepsilon_2\varepsilon_3\bar{z}_{23}}{\sqrt{s_{*1}}} \eta_1 + \frac{\varepsilon_1\varepsilon_3\bar{z}_{31}}{\sqrt{s_{*2}}} \eta_2 + \frac{\varepsilon_1\varepsilon_2\bar{z}_{12}}{\sqrt{s_{*3}}} \eta_3 \right) \delta^4(\eta_4) \operatorname{sgn}(\bar{z}_{23}\bar{z}_{31}) \delta(z_{12}) \delta(z_{13}) \\ & \quad \times s_{*3}^2 \left(\frac{z_{12}z_{14}\bar{z}_{13}\bar{z}_{32}}{\bar{z}_{12}^3\bar{z}_{34}} \right)^2 \frac{1}{\bar{z}_{23}\bar{z}_{31}} \frac{1}{z_{12}z_{23}z_{34}z_{41}} \left(\frac{s_{*1}}{s_{*3}} \right)^{-1+i\lambda_1} \left(\frac{s_{*2}}{s_{*3}} \right)^{-1+i\lambda_2} \Theta \left(\frac{s_{*1}}{s_{*3}} \right) \Theta \left(\frac{s_{*2}}{s_{*3}} \right) \\ &= -\frac{\pi}{2} \left(\sum_{i=1}^4 \lambda_i \right) \delta^4 \left(\frac{\varepsilon_2\varepsilon_3\bar{z}_{23}}{\sqrt{s_{*1}}} \eta_1 + \frac{\varepsilon_1\varepsilon_3\bar{z}_{31}}{\sqrt{s_{*2}}} \eta_2 + \frac{\varepsilon_1\varepsilon_2\bar{z}_{12}}{\sqrt{s_{*3}}} \eta_3 \right) \delta^4(\eta_4) \operatorname{sgn}(\bar{z}_{23}\bar{z}_{31}) \delta(z_{12}) \delta(z_{13}) \\ & \quad \times s_{*3}^2 \frac{\bar{z}_{31}}{\bar{z}_{34}\bar{z}_{41}} \frac{1}{\bar{z}_{12}\bar{z}_{23}^2\bar{z}_{31}^2} \left(\frac{s_{*1}}{s_{*3}} \right)^{1+i\lambda_1} \left(\frac{s_{*2}}{s_{*3}} \right)^{1+i\lambda_2} \Theta \left(\frac{s_{*1}}{s_{*3}} \right) \Theta \left(\frac{s_{*2}}{s_{*3}} \right), \end{aligned} \quad (4.27)$$

where we used the following relation (see (4.14))

$$\frac{s_{*1}}{s_{*3}} = \frac{\varepsilon_3}{\varepsilon_1} \frac{\bar{z}_{23}}{\bar{z}_{12}}, \quad \frac{s_{*2}}{s_{*3}} = \frac{\varepsilon_3}{\varepsilon_2} \frac{\bar{z}_{31}}{\bar{z}_{12}}, \quad \frac{z_{23}}{z_{12}} = \frac{\bar{z}_{23}\bar{z}_{14}}{\bar{z}_{12}\bar{z}_{34}}. \quad (4.28)$$

Compared to (3.45), we find

$$\lim_{\lambda_4 \rightarrow 0} i\lambda_4 \mathcal{M}_4 \Big|_{z_{13}=0} = -\frac{1}{2} \delta^4(\eta_4) \frac{\bar{z}_{31}}{\bar{z}_{34}\bar{z}_{41}} \mathcal{M}_3^{\overline{\text{MHV}}}. \quad (4.29)$$

Combining (4.23) and (4.29) together, we obtain the conformally supersoft theorem

$$\lim_{\lambda_4 \rightarrow 0} i\lambda_4 \mathcal{M}_4 = -\frac{1}{2} \left(\frac{z_{31}}{z_{34}z_{41}} \mathcal{M}_3^{\text{MHV}} + \delta^4(\eta_4) \frac{\bar{z}_{31}}{\bar{z}_{34}\bar{z}_{41}} \mathcal{M}_3^{\overline{\text{MHV}}} \right) = -\frac{1}{2} \left(\frac{z_{31}}{z_{34}z_{41}} + \delta^4(\eta_4) \frac{\bar{z}_{31}}{\bar{z}_{34}\bar{z}_{41}} \right) \mathcal{M}_3, \quad (4.30)$$

where in the second equality the cross-terms disappear as $z_{13}\delta(z_{13}) = \bar{z}_{13}\delta(\bar{z}_{13}) = 0$. This thus gives the first non-trivial explicit check of the conformally supersoft theorem.

4.3 Super-Ward identity from conformally supersoft theorem

Following the discussion for pure YM theory, we can recast the conformally supersoft theorem (4.12) in terms of the conformal correlators, leading to the super-Ward identity in CCFT

$$\begin{aligned} & \langle \mathcal{J}^a(z, \bar{z}, \eta) \mathcal{O}_{\Delta_1}^{h_1 b_1}(z_1, \bar{z}_1) \cdots \mathcal{O}_{\Delta_n, J_n}^{b_n}(z_n, \bar{z}_n) \rangle \\ &= \sum_{j=1}^n \left(\frac{f^{ab_j c}}{z - z_j} + \frac{f^{ab_j c}}{\bar{z} - \bar{z}_j} \delta^4(\eta) \right) \langle \mathcal{O}_{\Delta_1, J_1}^{b_1}(z_1, \bar{z}_1) \cdots \mathcal{O}_{\Delta_j, J_j}^c(z_j, \bar{z}_j) \cdots \mathcal{O}_{\Delta_n, J_n}^{b_n}(z_n, \bar{z}_n) \rangle \end{aligned} \quad (4.31)$$

Then we can read the following OPE

$$\mathcal{J}^a(z_1, \bar{z}_1, \eta_1) \mathcal{O}_{\Delta}^b(z_2, \bar{z}_2, \eta_2) \sim \frac{f^{abc}}{z_{12}} \mathcal{O}_{\Delta}^c(z_2, \bar{z}_2, \eta_2) + \frac{f^{abc}}{\bar{z}_{12}} \delta^4(\eta_1) \mathcal{O}_{\Delta}^c(z_2, \bar{z}_2, \eta_2) \quad (4.32)$$

where we introduced

$$\mathcal{J}^a(z, \bar{z}, \eta) = \lim_{\Delta \rightarrow 1} (\Delta - 1) \mathcal{O}_{\Delta}(z, \bar{z}, \eta), \quad (4.33)$$

and $\mathcal{O}_{\Delta}$ is just the superoperator (essentially the celestial on-shell superfield (2.66))

$$\begin{aligned} \mathcal{O}_{\Delta}(z, \bar{z}, \eta) &= \mathbf{G}_{\Delta}^{+}(z, \bar{z}) + \eta^A \Gamma_{A \Delta}(z, \bar{z}) + \frac{1}{2!} \eta^A \eta^B \Phi_{AB \Delta}(z, \bar{z}) \\ &+ \frac{1}{3!} \epsilon_{ABCD} \eta^A \eta^B \eta^C \bar{\Gamma}_{\Delta}^D(z, \bar{z}) + \frac{1}{4!} \epsilon_{ABCD} \eta^A \eta^B \eta^C \eta^D \mathbf{G}_{\Delta}^{-}(z, \bar{z}). \end{aligned} \quad (4.34)$$

The discussions so far are at leading order. In principle, we can also generalize to subleading order. We leave the generalization to the future.

5 Collinear limit and OPE

Another useful property in CFT is the OPE: when two local operators in CFT are inserted at nearby points, this is equivalent to inserting a set of operators and superposing them at the same point. The spectra of operators and OPE coefficients are the defining datum of CFT. So to understand CCFT, we need to determine the OPE coefficients among all the celestial operators.

It turns out that the OPE in CCFT can be naturally obtained from the collinear limit in the scattering amplitudes. Indeed, in the collinear limit, the two particles travel along approximately the same direction and pass the nearby points in the celestial sphere, which is just the OPE limit. The translation from collinear limit to OPE is achieved through Mellin transformation. For gluon and graviton, this has been done in [9]. Actually one can also use the consistency of CCFT to bootstrap the OPE [13].

We will generalize these results to the $\mathcal{N} = 4$ SYM theory and obtain the super-OPE between super-operators via the Mellin transformation. We will also consider the OPE when the conformal dimension of one of the operators takes special value $\Delta = 1, 1/2, 0, \dots$, which correspond to conformally soft modes. At these values, the OPE has a pole and the operator involved is the soft “super”-current of the CCFT.

5.1 OPE in YM

The YM amplitudes in the collinear limit are given in (A.7)–(A.12). Written in terms of celestial variables, they read ($\omega = \omega_1 + \omega_2$)¹⁷

$$\mathcal{A}_n(1^+, 2^+, \dots) \xrightarrow{p_1//p_2} \frac{\omega}{\omega_1 \omega_2} \frac{1}{z_{12}} \mathcal{A}_{n-1}(P_{12}^+, \dots), \quad (5.1)$$

$$\mathcal{A}_n(1^-, 2^-, \dots) \xrightarrow{p_1//p_2} \frac{\omega}{\omega_1 \omega_2} \frac{1}{\bar{z}_{12}} \mathcal{A}_{n-1}(P_{12}^-, \dots), \quad (5.2)$$

$$\mathcal{A}_n(1^-, 2^+, \dots) \xrightarrow{p_1//p_2} \frac{\omega_1}{\omega \omega_2} \frac{1}{z_{12}} \mathcal{A}_{n-1}(P_{12}^-, \dots) + \frac{\omega_2}{\omega \omega_1} \frac{1}{\bar{z}_{12}} \mathcal{A}_{n-1}(P_{12}^+, \dots). \quad (5.3)$$

We would like to perform the Mellin transformation in order to obtain an equation in terms of conformal correlators. For this purpose, let us first consider the Mellin transformation of $\omega_1^\alpha \omega_2^\beta (\omega_1 + \omega_2)^\gamma$ in the split factor:¹⁸

$$\begin{aligned} & \int_0^\infty d\omega_2 \omega_2^{\Delta_2-1} \int_0^\infty d\omega_1 \omega_1^{\Delta_1-1} \omega_1^\alpha \omega_2^\beta (\omega_1 + \omega_2)^\gamma f(\omega_1 + \omega_2) \\ &= B(\Delta_1 + \alpha, \Delta_2 + \beta) \int_0^\infty d\omega \omega^{\Delta_P-1} f(\omega), \end{aligned} \quad (5.4)$$

where $\Delta_P = \Delta_1 + \Delta_2 + \alpha + \beta + \gamma$.

Using this formula and bearing in mind the identification of correlator with full amplitude (4.4), we can rewrite (5.1), (5.2), (5.3) in terms of correlators and then extract the following OPEs [9, 13]:

$$O_{\Delta_1,+}^a(z_1, \bar{z}_1) O_{\Delta_2,+}^b(z_2, \bar{z}_2) \sim \frac{f^{abc}}{z_{12}} B(\Delta_1 - 1, \Delta_2 - 1) O_{\Delta_1+\Delta_2-1,+}^c(z_2, \bar{z}_2), \quad (5.5)$$

$$O_{\Delta_1,-}^a(z_1, \bar{z}_1) O_{\Delta_2,-}^b(z_2, \bar{z}_2) \sim \frac{f^{abc}}{\bar{z}_{12}} B(\Delta_1 - 1, \Delta_2 - 1) O_{\Delta_1+\Delta_2-1,-}^c(z_2, \bar{z}_2), \quad (5.6)$$

$$\begin{aligned} O_{\Delta_1,+}^a(z_1, \bar{z}_1) O_{\Delta_2,-}^b(z_2, \bar{z}_2) &\sim \frac{f^{abc}}{z_{12}} B(\Delta_1 - 1, \Delta_2 + 1) O_{\Delta_1+\Delta_2-1,-}^c(z_2, \bar{z}_2) \\ &+ \frac{f^{abc}}{\bar{z}_{12}} B(\Delta_1 + 1, \Delta_2 - 1) O_{\Delta_1+\Delta_2-1,+}^c(z_2, \bar{z}_2). \end{aligned} \quad (5.7)$$

When $\Delta = 1, 0$, the operator O_Δ becomes soft currents, and we recover the OPE involving soft current, namely (4.7) and the singular term in (4.9).

5.2 OPE in SYM

Now we switch to the OPE of super-operators in SYM theory by considering the Mellin transformation of the collinear limit in SYM theory given in (A.15), (A.16), (A.17). We note that the super-split factor (A.17) is the same as the split factor in (A.10), up to the dressing factors involving η 's. So we expect that (A.17) will give rise to OPE similar to (5.6), up to the η -dressing factors. The dressing factors contain $\sqrt{z} = \sqrt{p_1/(p_1 + p_2)}$

¹⁷For concreteness, we assume that the collinear particles are all out-going. Generalizations to case with in-coming particles are straightforward.

¹⁸We used $B(x, y) = \int_0^1 dt t^{x-1} (1-t)^{y-1} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.

and $\sqrt{1-\bar{z}} = \sqrt{p_2/(p_1+p_2)}$. For collinear p_1, p_2 , they can also be written as $\sqrt{z} = \sqrt{\omega_1/(\omega_1+\omega_2)}$ and $\sqrt{1-z} = \sqrt{\omega_2/(\omega_1+\omega_2)}$. So an insertion of $\sqrt{z}$ corresponds to shifting $\alpha \rightarrow \alpha + \frac{1}{2}, \gamma \rightarrow \gamma - \frac{1}{2}$ in (5.4). This amounts to the action of $e^{\frac{1}{2}\partial_{\Delta_1}}$ on $B(\Delta_1 + \alpha, \Delta_2 + \beta)$ in (5.4). Similarly $\sqrt{1-\bar{z}}$ amounts to the action of $e^{\frac{1}{2}\partial_{\Delta_2}}$ on $B(\Delta_1 + \alpha, \Delta_2 + \beta)$ in (5.4).

As a consequence, we obtain the following OPE

$$\begin{aligned} & \mathcal{O}_{\Delta_1}^a(z_1, \bar{z}_1, \eta_1) \mathcal{O}_{\Delta_2}^b(z_2, \bar{z}_2, \eta_2) \\ & \sim \frac{f^{abc}}{z_{12}} \int d^4\eta_3 \left[\prod_{A=1}^4 \left(\eta_3^A - e^{\frac{1}{2}\partial_{\Delta_1}} \eta_1^A - e^{\frac{1}{2}\partial_{\Delta_2}} \eta_2^A \right) B(\Delta_1 - 1, \Delta_2 - 1) \right] \mathcal{O}_{\Delta_1+\Delta_2-1}^c(z_1, \bar{z}_1, \eta_3) \\ & \quad + \frac{f^{abc}}{\bar{z}_{12}} \int d^4\eta_3 \left[\prod_{A=1}^4 \left(\eta_1^A \eta_2^A - e^{\frac{1}{2}\partial_{\Delta_2}} \eta_1^A \eta_3^A + e^{\frac{1}{2}\partial_{\Delta_1}} \eta_2^A \eta_3^A \right) B(\Delta_1 - 1, \Delta_2 - 1) \right] \\ & \quad \times \mathcal{O}_{\Delta_1+\Delta_2-1}^c(z_1, \bar{z}_1, \eta_3). \end{aligned} \quad (5.8)$$

This expression can be simplified. First of all, we notice that

$$\int d\eta (\eta - \theta) f(\eta) = f(\theta). \quad (5.9)$$

This can be further generalized to

$$\int d^4\eta_3 \prod_{A=1}^4 (\eta_3^A - \eta^A) f(\eta_3) = f(\eta). \quad (5.10)$$

Secondly, one can also show

$$\int d\eta_3 (\eta_1 \eta_2 - \eta_1 \eta_3 + \eta_2 \eta_3) f(\eta_3) = (\eta_1 - \eta_2) f\left(\frac{\eta_1 + \eta_2}{2}\right) = (\eta_1 - \eta_2) f(\eta_2), \quad (5.11)$$

as well as its generalization

$$\int d\eta_3^A \left(\eta_1^A \eta_2^A - e^{\frac{1}{2}\partial_{\Delta_2}} \eta_1^A \eta_3^A + e^{\frac{1}{2}\partial_{\Delta_1}} \eta_2^A \eta_3^A \right) f(\eta_3^A) = \left(\eta_1^A e^{\frac{1}{2}\partial_{\Delta_2}} - \eta_2^A e^{\frac{1}{2}\partial_{\Delta_1}} \right) f\left(\eta_2^A e^{-\frac{1}{2}\partial_{\Delta_2}}\right). \quad (5.12)$$

With these identities, the OPE in (5.8) can be rewritten as

$$\begin{aligned} & \mathcal{O}_{\Delta_1}^a(z_1, \bar{z}_1, \eta_1) \mathcal{O}_{\Delta_2}^b(z_2, \bar{z}_2, \eta_2) \\ & \sim \frac{f^{abc}}{z_{12}} \mathcal{O}_{\Delta_1+\Delta_2-1}^c\left(z_2, \bar{z}_2, \eta_1 e^{\frac{1}{2}\partial_{\Delta_1}} + \eta_2 e^{\frac{1}{2}\partial_{\Delta_2}}\right) B(\Delta_1 - 1, \Delta_2 - 1) \\ & \quad + \frac{f^{abc}}{\bar{z}_{12}} \mathcal{O}_{\Delta_1+\Delta_2-1}^c\left(z_2, \bar{z}_2, \eta_2 e^{-\frac{1}{2}\partial_{\Delta_2}}\right) \delta^4\left(\eta_1 e^{\frac{1}{2}\partial_{\Delta_2}} - \eta_2 e^{\frac{1}{2}\partial_{\Delta_1}}\right) B(\Delta_1 - 1, \Delta_2 - 1). \end{aligned} \quad (5.13)$$

Note that the dimension shifting operator $e^{\pm\frac{1}{2}\partial_{\Delta_{1,2}}}$ only acts on the function $B(\Delta_1 - 1, \Delta_2 - 1)$. Also the holomorphic OPE singularity in the first term only appears when the total number of η 's are less than or equal to 4, while the anti-holomorphic OPE singularity in the second term appears when the total number of η 's are greater than or equal to 4.

One can then perform the η expansion on both sides and get the OPE of the component operators. For example, when considering the lowest component without η_1, η_2 , one gets the OPE of two gluon operators with positive helicity (5.5):

$$G_{\Delta_1}^{+a}(z_1, \bar{z}_1) G_{\Delta_2}^{+b}(z_2, \bar{z}_2) \sim \frac{f^{abc}}{z_{12}} B(\Delta_1 - 1, \Delta_2 - 1) G_{\Delta_1 + \Delta_2 - 1}^{+c}(z_2, \bar{z}_2). \quad (5.14)$$

And the OPE of two gluon operator with opposite helicity can be extracted from the η_2^4 term:

$$\begin{aligned} G_{\Delta_1}^{+a}(z_1, \bar{z}_1) G_{\Delta_2}^{-b}(z_2, \bar{z}_2) &\sim \frac{f^{abc}}{z_{12}} B(\Delta_1 - 1, \Delta_2 + 1) G_{\Delta_1 + \Delta_2 - 1}^{-c}(z_2, \bar{z}_2) \\ &+ \frac{f^{abc}}{\bar{z}_{12}} B(\Delta_1 + 1, \Delta_2 - 1) G_{\Delta_1 + \Delta_2 - 1}^{+c}(z_2, \bar{z}_2) \end{aligned} \quad (5.15)$$

which agrees with (5.7).

More OPEs are as follows:

$$\Gamma_{A \Delta_1}^a(z_1, \bar{z}_1) \Gamma_{B \Delta_2}^b(z_2, \bar{z}_2) \sim \frac{f^{abc}}{z_{12}} B\left(\Delta_1 - \frac{1}{2}, \Delta_2 - \frac{1}{2}\right) \Phi_{AB \Delta_1 + \Delta_2 - 1}^c(z_2, \bar{z}_2), \quad (5.16)$$

$$\begin{aligned} \bar{\Gamma}_{\Delta_1}^{A a}(z_1, \bar{z}_1) \Gamma_{B \Delta_2}^b(z_2, \bar{z}_2) &\sim \delta_B^A \frac{f^{abc}}{z_{12}} B\left(\Delta_1 + \frac{1}{2}, \Delta_2 - \frac{1}{2}\right) G_{\Delta_1 + \Delta_2 - 1}^{-c}(z_2, \bar{z}_2) \\ &+ \delta_B^A \frac{f^{abc}}{\bar{z}_{12}} B\left(\Delta_1 - \frac{1}{2}, \Delta_2 + \frac{1}{2}\right) G_{\Delta_1 + \Delta_2 - 1}^{+c}(z_2, \bar{z}_2), \end{aligned} \quad (5.17)$$

$$\Gamma_{A \Delta_1}^a(z_1, \bar{z}_1) G_{\Delta_2}^{+b}(z_2, \bar{z}_2) \sim \frac{f^{abc}}{z_{12}} B\left(\Delta_1 - \frac{1}{2}, \Delta_2 - 1\right) \Gamma_{A \Delta_1 + \Delta_2 - 1}^c(z_2, \bar{z}_2), \quad (5.18)$$

$$\Gamma_{A \Delta_1}^a(z_1, \bar{z}_1) G_{\Delta_2}^{-b}(z_2, \bar{z}_2) \sim \frac{f^{abc}}{\bar{z}_{12}} B\left(\Delta_1 + \frac{1}{2}, \Delta_2 - 1\right) \Gamma_{A \Delta_1 + \Delta_2 - 1}^c(z_2, \bar{z}_2), \quad (5.19)$$

$$\begin{aligned} \Phi_{AB \Delta_1}^a(z_1, \bar{z}_1) \Phi_{CD \Delta_2}^b(z_2, \bar{z}_2) &\sim \frac{f^{abc}}{z_{12}} \epsilon_{ABCD} B(\Delta_1, \Delta_2) G_{\Delta_1 + \Delta_2 - 1}^{-c}(z_2, \bar{z}_2) \\ &+ \frac{f^{abc}}{\bar{z}_{12}} \epsilon_{ABCD} B(\Delta_1, \Delta_2) G_{\Delta_1 + \Delta_2 - 1}^{+c}(z_2, \bar{z}_2). \end{aligned} \quad (5.20)$$

These OPEs agree with [20] up to the R-symmetry index A and the scalar Φ which are absent in $\mathcal{N} = 1$ super-Yang-Mills theory. All the rest OPEs can also be obtained but will not be written down explicitly here. So the super-OPE (5.13) compactly includes all the component OPE. This shows the power of superspace.

5.3 OPE of soft mode

With super-OPE (5.13), we can now discuss further the OPEs involving the conformally soft modes by setting the conformal dimension to special values.

5.3.1 $\Delta \rightarrow 1$

We can take the limit $\Delta \rightarrow 1$ and define

$$\mathcal{J}(z, \bar{z}, \eta) = \lim_{\Delta \rightarrow 1} (\Delta - 1) \mathcal{O}_\Delta(z, \bar{z}, \eta). \quad (5.21)$$

As we will see, this corresponds to the conformally soft gluons of both helicities. The OPE of $\mathcal{J}$ with $\mathcal{O}_\Delta$ can be obtained from (5.13). Especially, we only need to focus on the singular term in the limit $\Delta_1 \rightarrow 1$ on the right hand side of (5.13). This happens only when there is no $e^{\frac{1}{2}\partial_{\Delta_1}}$ acting on $B(\Delta_1 - 1, \Delta_2 - 1)$. Moreover, one has¹⁹

$$\lim_{\Delta_1 \rightarrow 1} B(\Delta_1 - 1, \Delta_2 - 1) = \frac{1}{\Delta_1 - 1}. \quad (5.22)$$

Since this is independent of the value of Δ_2 , we can also drop the shifting $e^{\pm \frac{1}{2}\partial_{\Delta_2}}$ in (5.13).

As a consequence, one has

$$\mathcal{J}^a(z_1, \bar{z}_1, \eta_1) \mathcal{O}_\Delta^b(z_2, \bar{z}_2, \eta_2) \sim \frac{f^{abc}}{z_{12}} \mathcal{O}_\Delta^c(z_2, \bar{z}_2, \eta_2) + \frac{f^{abc}}{\bar{z}_{12}} \delta^4(\eta_1) \mathcal{O}_\Delta^c(z_2, \bar{z}_2, \eta_2). \quad (5.23)$$

This is the same as (4.32) which was obtained based on the soft theorem.

Since we only have η_1^0 and η_1^4 terms on the right-hand side, $\mathcal{J}$ just corresponds to the soft gluons. And we can indeed rewrite $\mathcal{J}$ in (5.21) as

$$\mathcal{J}^a(z, \bar{z}, \eta) = J^a(z) + \bar{J}^a(\bar{z}) \delta^4(\eta), \quad (5.24)$$

where

$$J^a(z) = \lim_{\Delta \rightarrow 1} (\Delta - 1) \mathbf{G}_\Delta^{+a}(z, \bar{z}, \eta), \quad \bar{J}^a(\bar{z}) = \lim_{\Delta \rightarrow 1} (\Delta - 1) \mathbf{G}_\Delta^{-a}(z, \bar{z}, \eta). \quad (5.25)$$

The conformal weights $(h, \bar{h})$ of J and $\bar{J}$ are $(1, 0)$ and $(0, 1)$, respectively.

In (5.23), one can further take the limit $\Delta \rightarrow 1$. This leads to

$$\mathcal{J}^a(z_1, \bar{z}_1, \eta_1) \mathcal{J}^b(z_2, \bar{z}_2, \eta_2) \sim \frac{f^{abc}}{z_{12}} \mathcal{J}^c(z_2, \bar{z}_2, \eta_2) + \frac{f^{abc}}{\bar{z}_{12}} \mathcal{J}^c(z_2, \bar{z}_2, \eta_2) \delta^4(\eta_1). \quad (5.26)$$

Note that the right-hand side is not symmetric under the exchange of 1,2. This just reflects a subtlety here: the OPE of two gluons with opposite helicity depends on the order of soft limits. Mathematically the subtlety arises because (5.22) breaks down if $\Delta_2 = 1$.

5.3.2 $\Delta \rightarrow \frac{1}{2}$

We can also take the limit $\Delta \rightarrow 1/2$ and define

$$\mathcal{K}^a(z, \bar{z}, \eta) = \lim_{\Delta \rightarrow \frac{1}{2}} \left(\Delta - \frac{1}{2} \right) \mathcal{O}_\Delta^a(z, \bar{z}, \eta). \quad (5.27)$$

Following similar arguments above, here we allow exactly one $e^{\frac{1}{2}\partial_{\Delta_1}}$ acting on $B(\Delta_1 - 1, \Delta_2 - 1)$ in (5.13). This leads to the coefficient $B(\Delta_1 - 1/2, \Delta_2 - 1) = 1/(\Delta_1 - 1/2)$ in the limit $\Delta_1 \rightarrow 1/2$.

The OPE (5.13) then becomes

$$\mathcal{K}^a(z_1, \bar{z}_1, \eta_1) \mathcal{O}_\Delta^b(z_2, \bar{z}_2, \eta_2) \sim \frac{f^{abc}}{z_{12}} \mathcal{O}_{\Delta - \frac{1}{2}}^c(z_2, \bar{z}_2, \eta_1 + \eta_2) \Big|_{\eta_1} + \frac{f^{abc}}{\bar{z}_{12}} \delta^4(\eta_1 - \eta_2) \Big|_{\eta_1^3} \mathcal{O}_{\Delta - \frac{1}{2}}^c(z_2, \bar{z}_2, \eta_2), \quad (5.28)$$

¹⁹This is generically true unless $\Delta_2 = 1, 0, -1, -2, \dots$.

where $|\eta_1$ and $|\eta_1^3$ means the projection to terms with exactly one or three η_1 's. So $\mathcal{K}^a$ corresponds to the soft gluinos and we can rewrite

$$\mathcal{K}^a(z, \bar{z}, \eta) = \eta^A K_A^a(z, \bar{z}) + \frac{1}{3!} \epsilon_{ABCD} \eta^A \eta^B \eta^C \bar{K}^{D a}(z, \bar{z}), \quad (5.29)$$

where

$$K_A^a(z, \bar{z}) = \lim_{\Delta \rightarrow \frac{1}{2}} \left(\Delta - \frac{1}{2} \right) \Gamma_{A \Delta}^a(z, \bar{z}), \quad \bar{K}^{A a}(z, \bar{z}) = \lim_{\Delta \rightarrow \frac{1}{2}} \left(\Delta - \frac{1}{2} \right) \bar{\Gamma}_{\Delta}^{A a}(z, \bar{z}). \quad (5.30)$$

These soft gluinos are related to the soft gluons through supersymmetry. The conformal weights $(h, \bar{h})$ of K and $\bar{K}$ are $(1/2, 0)$ and $(0, 1/2)$, respectively. And we have

$$K^a(z_1, \bar{z}_1) O_{\Delta, J}^b(z_2, \bar{z}_2) \sim \frac{f^{abc}}{z_{12}} O_{\Delta - \frac{1}{2}, J - \frac{1}{2}}^c(z_2, \bar{z}_2), \quad (5.31)$$

$$\bar{K}^a(z_1, \bar{z}_1) O_{\Delta, J}^b(z_2, \bar{z}_2) \sim \frac{f^{abc}}{\bar{z}_{12}} O_{\Delta - \frac{1}{2}, J + \frac{1}{2}}^c(z_2, \bar{z}_2), \quad (5.32)$$

where we omit the R -symmetry indices for simplicity.

5.3.3 $\Delta \rightarrow 0$

We finally consider the limit $\Delta \rightarrow 0$ and define

$$\mathcal{L}^a(z, \bar{z}, \eta) = \lim_{\Delta \rightarrow 0} \Delta \mathcal{O}_{\Delta}^a(z, \bar{z}, \eta). \quad (5.33)$$

There are more complications in this case. In (5.13), there are two sources of singularities in the limit $\Delta \rightarrow 0$. In the first case, we allow exactly two $e^{\frac{1}{2}\partial_{\Delta_1}}$ acting on $B(\Delta_1 - 1, \Delta_2 - 1)$. This gives rise to the coefficient $B(\Delta_1, \Delta_2 - 1) = 1/\Delta_1$ in the limit $\Delta_1 \rightarrow 0$. In the second case, there is no $e^{\frac{1}{2}\partial_{\Delta_1}}$ acting on $B(\Delta_1 - 1, \Delta_2 - 1)$:

$$\lim_{\Delta_1 \rightarrow 0} B(\Delta_1 - 1, \Delta_2 - 1) = \frac{2 - \Delta_2}{\Delta_1}. \quad (5.34)$$

The OPE (5.13) then becomes

$$\begin{aligned} & \mathcal{L}^a(z_1, \bar{z}_1, \eta_1) \mathcal{O}_{\Delta_2}^b(z_2, \bar{z}_2, \eta_2) \\ & \sim \frac{f^{abc}}{z_{12}} \mathcal{O}_{\Delta_2 - 1}^c(z_2, \bar{z}_2, \eta_1 + \eta_2) \Big|_{\eta_1^2} + \frac{f^{abc}}{\bar{z}_{12}} \delta^4(\eta_1 - \eta_2) \Big|_{\eta_1^2} \mathcal{O}_{\Delta_2 - 1}^c(z_2, \bar{z}_2, \eta_2) \\ & \quad + \frac{f^{abc}}{z_{12}} \mathcal{O}_{\Delta_2 - 1}^c(z_2, \bar{z}_2, \eta_2 e^{\frac{1}{2}\partial_{\Delta_2}}) (2 - \Delta_2) \\ & \quad + \frac{f^{abc}}{\bar{z}_{12}} \mathcal{O}_{\Delta_2 - 1}^c(z_2, \bar{z}_2, \eta_2 e^{-\frac{1}{2}\partial_{\Delta_2}}) \delta^4(\eta_1) (-\Delta_2). \end{aligned} \quad (5.35)$$

In the above OPE, there are 2, 2, 0 and 4 η_1 's on the right hand side, respectively. So we can expand

$$\mathcal{L}^a(z, \bar{z}, \eta) = \frac{1}{2} \eta^A \eta^B L_{AB}^a(z, \bar{z}) + L^a(z, \bar{z}) + \bar{L}^a(z, \bar{z}) \delta^4(\eta), \quad (5.36)$$

where

$$L_{AB}^a(z, \bar{z}) = \lim_{\Delta \rightarrow 0} \Delta \Phi_{AB\Delta}^a(z, \bar{z}), \quad L^a(z, \bar{z}) = \lim_{\Delta \rightarrow 0} \Delta G_{\Delta}^{+a}(z, \bar{z}), \quad \bar{L}^a(z, \bar{z}) = \lim_{\Delta \rightarrow 0} \Delta G_{\Delta}^{-a}(z, \bar{z}). \quad (5.37)$$

They have weights $(h, \bar{h}) = (0, 0), (1/2, -1/2), (-1/2, 1/2)$, respectively. Then the OPE (5.35) decomposes into

$$\begin{aligned} \frac{1}{2} \eta^A \eta^B L_{AB}^a(z_1, \bar{z}_1) \mathcal{O}_{\Delta_2}^b(z_2, \bar{z}_2, \eta_2) &\sim \frac{f^{abc}}{z_{12}} \mathcal{O}_{\Delta_2-1}^c(z_2, \bar{z}_2, \eta_1 + \eta_2) \Big|_{\eta_1^2} \\ &\quad + \frac{f^{abc}}{\bar{z}_{12}} \delta^4(\eta_1 - \eta_2) \Big|_{\eta_1^2} \mathcal{O}_{\Delta_2-1}^c(z_2, \bar{z}_2, \eta_2), \end{aligned} \quad (5.38)$$

$$L^a(z_1, \bar{z}_1) \mathcal{O}_{\Delta_2}^b(z_2, \bar{z}_2, \eta_2) \sim \frac{f^{abc}}{z_{12}} \mathcal{O}_{\Delta_2-1}^c(z_2, \bar{z}_2, \eta_2 e^{\frac{1}{2} \partial_{\Delta_2}}) (2 - \Delta_2), \quad (5.39)$$

$$\bar{L}^a(z_1, \bar{z}_1) \mathcal{O}_{\Delta_2}^b(z_2, \bar{z}_2, \eta_2) \sim \frac{f^{abc}}{\bar{z}_{12}} \mathcal{O}_{\Delta_2-1}^c(z_2, \bar{z}_2, \eta_2 e^{-\frac{1}{2} \partial_{\Delta_2}}) (-\Delta_2). \quad (5.40)$$

To see closer, we also write down the component OPE after doing η expansion. The OPE in (5.38) becomes

$$L_{\bullet\bullet}^a(z, \bar{z})(z_1, \bar{z}_1) \mathcal{O}_{\Delta, J}^b(z_2, \bar{z}_2) \sim \frac{f^{abc}}{z_{12}} \mathcal{O}_{\Delta-1, J-1}^c(z_2, \bar{z}_2) + \frac{f^{abc}}{\bar{z}_{12}} \mathcal{O}_{\Delta-1, J+1}^c(z_2, \bar{z}_2), \quad (5.41)$$

where we again ignore the R -symmetry index. This corresponds to the soft scalar.

For (5.40) and (5.39), the component OPEs involving only gluons are:

$$L^a(z_1, \bar{z}_1) G_{\Delta}^{+b}(z_2, \bar{z}_2) \sim \frac{f^{abc}}{z_{12}} (2 - \Delta) G_{\Delta-1}^{+c}(z_2, \bar{z}_2), \quad (5.42)$$

$$L^a(z_1, \bar{z}_1) G_{\Delta}^{-b}(z_2, \bar{z}_2) \sim \frac{f^{abc}}{z_{12}} (-\Delta) G_{\Delta-1}^{-c}(z_2, \bar{z}_2), \quad (5.43)$$

$$\bar{L}^a(z_1, \bar{z}_1) G_{\Delta}^{+b}(z_2, \bar{z}_2) \sim \frac{f^{abc}}{\bar{z}_{12}} (-\Delta) G_{\Delta-1}^{+c}(z_2, \bar{z}_2), \quad (5.44)$$

$$\bar{L}^a(z_1, \bar{z}_1) G_{\Delta}^{-b}(z_2, \bar{z}_2) \sim \frac{f^{abc}}{\bar{z}_{12}} (2 - \Delta) G_{\Delta-1}^{-c}(z_2, \bar{z}_2). \quad (5.45)$$

These OPEs just arise from the sub-leading soft gluon theorem. Note that here we only derive the singular terms in the OPE. But for the sub-leading soft gluon theorem, there are actually also important phase factor terms $z_{12}/\bar{z}_{12}$ or $\bar{z}_{12}/z_{12}$ which appear in the Ward identity (4.9). The OPE above agrees with the singular term in (4.9). So it would be important to also find the non-singular sub-leading terms in the OPE by studying sub-leading collinear limit in SYM theory. We leave it to the future.

One can in principle continue the discussion to all $\Delta = -1/2, -1, -3/2, \dots$. All correspond to the conformally soft modes, giving rise to various Ward identities. For gluon and graviton, the symmetry algebra for the infinite set of symmetries is recently determined in [14].

6 Conclusion and outlook

To summarize, in this paper, we have initiated the study of the celestial amplitudes in $\mathcal{N} = 4$ SYM theory using the language of superfield. We constructed the superconformal generators and the corresponding celestial on-shell superfield living on the celestial sphere. We then computed the three- and four-point celestial superamplitudes explicitly in $\mathcal{N} = 4$ SYM theory. We also studied the $\mathcal{N} = 4$ SYM amplitudes in the soft and collinear limit and their consequences in celestial conformal field theory. The energetically super-soft theorem in momentum space becomes the conformally super-soft theorem which further leads to Ward identity of symmetry current in CCFT. And the collinear limit enables us to extract the super-OPEs of the super-operators.

In our discussion of $\mathcal{N} = 4$ SYM theory, we made full use of the superspace and on-shell superfield. The momentum space on-shell superspace techniques for theories with $\mathcal{N} < 4$ supersymmetry were studied in [31]. It is straightforward to generalize our celestial superspace to those theories with less supersymmetry. Therefore, our formalism provides a natural language to account for bulk supersymmetry and superconformal symmetry in CCFT.

In particular, it would also be interesting to generalize our celestial superfield techniques to the supergravity theories, especially the $\mathcal{N} = 8$ supergravity.²⁰ The asymptotic symmetries of those theories are given by the supersymmetric extension of BMS symmetry. The interplay between supertranslation, superrotation, and maximal supersymmetry is supposed to lead to a rich story on the celestial sphere. One could furthermore consider the celestial double copy relation between $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ supergravity by generalizing the non-supersymmetric celestial double copy in [32].

One reason for us to focus on $\mathcal{N} = 4$ SYM theory in this paper is because of its maximal superconformal symmetry in 4D. However, a very remarkable feature of $\mathcal{N} = 4$ SYM theory is that the scattering amplitudes (in the planar limit) has been shown to enjoy much larger symmetries, the Yangian symmetry, which can be regarded as the combination of superconformal and dual superconformal symmetry [33]. As superconformal symmetry, it is then natural to ask how to realize superconformal and Yangian symmetry on the celestial sphere. Once all various types of symmetries are understood, a bootstrap program for the CCFT of $\mathcal{N} = 4$ SYM theory may become feasible. A closely related question is how to relate the celestial sphere with twistor space where the $\mathcal{N} = 4$ SYM theory was interpreted as B-model topological string theory [34].

Another virtue of $\mathcal{N} = 4$ SYM theory is that the quantum corrections have been much more well-understood compared to pure YM theory.²¹ So it would be interesting to understand the fate of loop corrections in celestial amplitude. This is important for further understanding the full-fledged quantum gravity via celestial holography. Even at the tree

²⁰Note that the case with $\mathcal{N} = 1$ supersymmetry has already been considered in component form without using superfield [20].

²¹The infrared divergent loop amplitudes in planar $\mathcal{N} = 4$ super Yang-Mills theory has been considered in [35].

level, in the discussion of soft and collinear limits, we were mainly discussing the leading term. It would be nice to work out the other sub-leading corrections.

Finally, a useful operation for celestial operator is shadow transformation [4]. Therefore, it would be interesting to see how to perform the shadow transformation for a super-operator. The shadow transformation flips the spin of the operator and in particular, should be compatible with supersymmetry. However, the supersymmetries considered here are 4D supersymmetry in the bulk instead of the ordinary 2D supersymmetry on the celestial sphere. So it seems to be not sensible to combine $(z_1, \bar{z}_1, \eta_1^A)$ and $(z_2, \bar{z}_2, \eta_2^A)$ to form a distance that could be used to construct the supershadow. Nevertheless, it is worth studying whether one can assemble the shadow of each component operator together in a supersymmetric way.

Acknowledgments

We would like to thank Rodolfo Russo and Congkao Wen for discussions and comments on the manuscript. We also thank the authors of [22] for sharing their draft before the publication, and Congkao Wen for informing us the two parallel independent works on similar topics. This work was supported by the Royal Society grant, “*Relations, Transformations, and Emergence in Quantum Field Theory*”, and by the Science and Technology Facilities Council (STFC) Consolidated Grant ST/T000686/1 “*Amplitudes, strings & duality*”.

A Soft and collinear limit in YM and SYM

A.1 Soft theorem in YM and SYM

The soft theorem for (color-ordered) YM gluon amplitudes is

$$\mathcal{A}_n(\cdots, a, s^\pm, b, \cdots) \xrightarrow{p_s \rightarrow 0} \text{Soft}^{\text{YM}}(a, s^\pm, b) \mathcal{A}_n(\cdots, a, b, \cdots), \quad (\text{A.1})$$

where the soft factor for positive helicity soft gluon at tree level is given by [37]

$$\text{Soft}^{\text{YM}}(a, s^+, b) = \frac{\langle ab \rangle}{\langle as \rangle \langle sb \rangle} + \left(\frac{1}{\langle sb \rangle} \tilde{\lambda}_s^\alpha \frac{\partial}{\partial \tilde{\lambda}_b^\alpha} + \frac{1}{\langle as \rangle} \tilde{\lambda}_s^\alpha \frac{\partial}{\partial \tilde{\lambda}_a^\alpha} \right). \quad (\text{A.2})$$

The first term is the leading soft factor of order $\mathcal{O}(1/p_s)$, while the second term in parentheses is the subleading soft factor of order $\mathcal{O}(1/p_s^0)$. For negative helicity soft gluon, the soft factor is given by the conjugate $\lambda_i \leftrightarrow \tilde{\lambda}_i$.

The discussion can be generalized to the superamplitudes in $\mathcal{N} = 4$ SYM theory:

$$\mathcal{A}_n(\cdots, a, s, b, \cdots) \xrightarrow{p_s \rightarrow 0} \text{Soft}^{\text{SYM}}(a, s, b) \mathcal{A}_{n-1}(\cdots, a, b, \cdots), \quad (\text{A.3})$$

where each particle is labeled by $(\lambda_i, \tilde{\lambda}_i, \eta_i)$. The physical soft limit $p_s \rightarrow 0$ corresponds to setting $(\lambda_s, \tilde{\lambda}_s, \eta_s) \rightarrow (\epsilon \Lambda_s, \epsilon \tilde{\Lambda}_s, \eta_s)$ and then taking $\epsilon \rightarrow 0$. However, this is a little complicated. In the literature, one instead considers the holomorphic soft limit $(\lambda_s, \tilde{\lambda}_s, \eta_s) \rightarrow (\epsilon \Lambda_s, \tilde{\lambda}_s, \eta_s)$ and anti-holomorphic soft limit $(\lambda_s, \tilde{\lambda}_s, \eta_s) \rightarrow (\lambda_s, \epsilon \tilde{\Lambda}_s, \eta_s)$.

In the holomorphic soft limit $(\lambda_s, \tilde{\lambda}_s, \eta_s) \rightarrow (\epsilon \Lambda_s, \tilde{\lambda}_s, \eta_s)$, namely $\lambda_s \rightarrow 0$, the supersoft factor is given by [37]

$$\text{Soft}_{\text{hol}}^{\text{SYM}}(a, s, b) = \frac{\langle ab \rangle}{\langle as \rangle \langle sb \rangle} + \left[\frac{1}{\langle sb \rangle} \left(\tilde{\lambda}_s^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_b^{\dot{\alpha}}} + \eta_s^A \frac{\partial}{\partial \eta_b^A} \right) + \frac{1}{\langle as \rangle} \left(\tilde{\lambda}_s^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_a^{\dot{\alpha}}} + \eta_s^A \frac{\partial}{\partial \eta_a^A} \right) \right], \quad (\text{A.4})$$

while in the anti-holomorphic soft limit $(\lambda_s, \tilde{\lambda}_s, \eta_s) \rightarrow (\lambda_s, \epsilon \tilde{\Lambda}_s, \eta_s)$, namely $\tilde{\lambda}_s \rightarrow 0$, the supersoft factor is given by [37]

$$\text{Soft}_{\text{anti-hol}}^{\text{SYM}}(a, s, b) = \frac{[ab]}{\langle as \rangle [sb]} \delta^4 \left(\eta_s + \frac{[as]}{[ab]} \eta_b + \frac{[sb]}{[ab]} \eta_a \right) \left[1 + \left(\frac{[sb]}{[ab]} \eta_s^A \frac{\partial}{\partial \eta_b^A} + \frac{[as]}{[ab]} \eta_s^A \frac{\partial}{\partial \eta_a^A} \right) \right]. \quad (\text{A.5})$$

In these two limits above, we give both the leading and sub-leading soft factors.

In the physical soft limit $p_s \rightarrow 0$ or equivalently $\lambda_s, \tilde{\lambda}_s \rightarrow 0$, the leading soft factor is given by the sum of leading soft factors in holomorphic and anti-holomorphic soft limit:

$$\text{Soft}_{\text{leading}}^{\text{SYM}}(a, s, b) = \frac{\langle ab \rangle}{\langle as \rangle \langle sb \rangle} + \frac{[ab]}{[as][sb]} \delta^4(\eta_s). \quad (\text{A.6})$$

The two terms obviously correspond to the soft gluons of positive and negative helicity. Other components are absent here because their corresponding soft factors are subdominant (for example, the leading soft factor for gluinos scales as $1/\sqrt{p_s}$ [20]).

A.2 Collinear limit in YM

When two particles point to the same direction, the amplitude also becomes singular, known as the collinear singularity. More specifically, consider two particles with momenta p_1 and p_2 . The two particles can fuse into one particle with momentum $P_{12} \equiv p_1 + p_2$ and one can parametrize the collinear momenta as $p_1 = z P_{12}, p_2 = (1 - z) P_{12}$. In the collinear limit, the gluon amplitudes in YM theory satisfy

$$\mathcal{A}_n(1^{h_1}, 2^{h_2}, \dots) \xrightarrow{p_1/p_2} \sum_h \mathcal{A}_{n-1}(P_{12}^h, \dots) \text{Split}_{-h}(1^{h_1}, 2^{h_2}), \quad (\text{A.7})$$

where the split factors $\text{Split}_{-h}(1^{h_1}, 2^{h_2})$ are given by [25, 38]

$$\text{Split}_+(z; 1^+, 2^-) = \frac{(1-z)^2}{\sqrt{z(1-z)}} \frac{1}{\langle 12 \rangle}, \quad (\text{A.8})$$

$$\text{Split}_+(z; 1^-, 2^+) = \frac{z^2}{\sqrt{z(1-z)}} \frac{1}{\langle 12 \rangle}, \quad (\text{A.9})$$

$$\text{Split}_+(z; 1^-, 2^-) = \frac{1}{\sqrt{z(1-z)}} \frac{1}{[12]}, \quad (\text{A.10})$$

$$\text{Split}_+(z; 1^+, 2^+) = 0, \quad (\text{A.11})$$

and

$$\text{Split}_-(z; 1^{-h_1}, 2^{-h_2}) = \text{Split}_+(z; 1^{h_1}, 2^{h_2})|_{\langle 12 \rangle \leftrightarrow [12]}. \quad (\text{A.12})$$

One can actually derive the soft theorem by taking the consecutive collinear limits of the same type [38]. The soft factor derived in this way is in agreement with the leading term in (A.2) as well as its conjugate.

A.3 Collinear limit in SYM

We can also consider the collinear limit of SYM theory using the language of superamplitude. Here we follow the discussion in [38]. The SYM superamplitudes can be written as

$$\mathcal{A}_n = A_{n,2} + A_{n,4} + \cdots + A_{n,n-2}, \quad (\text{A.13})$$

which corresponds to MHV, NMHV, ..., $\overline{\text{MHV}}$. In the collinear limit, the SYM amplitude satisfies

$$A_{n,k}(1, 2, 3, \dots, n) \xrightarrow{P_{12}^2 \rightarrow 0} \sum_{l=1}^2 \int d^4 \eta_{P_{12}} \text{Split}_{1-l}(1, 2, P_{12}) A_{n-1,k-l+1}(P_{12}, 3, \dots, n), \quad (\text{A.14})$$

where $l = 1, 2$ corresponds to helicity preserving and helicity decreasing process. Summing over k on both sides gives

$$\mathcal{A}_n(1, 2, 3, \dots, n) \xrightarrow{P_{12}^2 \rightarrow 0} \sum_{l=1}^2 \int d^4 \eta_{P_{12}} \text{Split}_{1-l}(1, 2, P_{12}) \mathcal{A}_{n-1}(P_{12}, 3, \dots, n). \quad (\text{A.15})$$

For helicity preserving case, the super-split factor is given by [38]

$$\text{Split}_0(z; \eta_1, \eta_2, \eta_3) = \frac{1}{\sqrt{z(1-z)}} \frac{1}{\langle 12 \rangle} \prod_{A=1}^4 \left(\eta_3^A - \sqrt{z} \eta_1^A - \sqrt{1-z} \eta_2^A \right). \quad (\text{A.16})$$

While for helicity decreasing case, the super-split factor is given by [38]

$$\text{Split}_{-1}(z; \eta_1, \eta_2, \eta_3) = \frac{1}{\sqrt{z(1-z)}} \frac{1}{[12]} \prod_{A=1}^4 \left(\eta_1^A \eta_2^A - \sqrt{1-z} \eta_1^A \eta_3^A + \sqrt{z} \eta_2^A \eta_3^A \right). \quad (\text{A.17})$$

As in the pure Yang-Mills theory, one can also produce the soft limit by taking consecutive collinear limits. Taking $p_2 \rightarrow 0$ and thus $z \rightarrow 1$, the split factors reduce to

$$\text{Split}_0(z; \eta_1, \eta_2, \eta_{P_{12}}) \rightarrow \frac{\langle 13 \rangle}{\langle 12 \rangle \langle 23 \rangle} \prod_{A=1}^4 (\eta_{P_{12}}^A - \eta_1^A), \quad (\text{A.18})$$

$$\text{Split}_{-1}(z; \eta_1, \eta_2, \eta_{P_{12}}) \rightarrow \frac{[13]}{[12][23]} \prod_{A=1}^4 \eta_2^A (\eta_{P_{12}}^A - \eta_1^A). \quad (\text{A.19})$$

Inserting into (A.14), one obtains

$$A_{n,k}(1, 2, 3, \dots, n) \xrightarrow{p_2 \rightarrow 0} \frac{\langle 13 \rangle}{\langle 12 \rangle \langle 23 \rangle} A_{n-1,k}(1, 3, \dots, n), \quad (\text{A.20})$$

$$A_{n,k}(1, 2, 3, \dots, n) \xrightarrow{p_2 \rightarrow 0} \delta^4(\eta_2) \frac{[13]}{[12][23]} A_{n-1,k-1}(1, 3, \dots, n). \quad (\text{A.21})$$

Combining together, we have

$$\mathcal{A}_n(1, 2, 3, \dots, n) \xrightarrow{p_2 \rightarrow 0} \left(\frac{\langle 13 \rangle}{\langle 12 \rangle \langle 23 \rangle} + \frac{[13]}{[12][23]} \delta^4(\eta_2) \right) \mathcal{A}_{n-1}(1, 3, \dots, n). \quad (\text{A.22})$$

This is in agreement with the leading supersoft factor in (A.6).

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] S. Pasterski, S.-H. Shao and A. Strominger, *Flat Space Amplitudes and Conformal Symmetry of the Celestial Sphere*, *Phys. Rev. D* **96** (2017) 065026 [[arXiv:1701.00049](https://arxiv.org/abs/1701.00049)] [[INSPIRE](#)].
- [2] J. de Boer and S.N. Solodukhin, *A Holographic reduction of Minkowski space-time*, *Nucl. Phys. B* **665** (2003) 545 [[hep-th/0303006](https://arxiv.org/abs/hep-th/0303006)] [[INSPIRE](#)].
- [3] C. Cheung, A. de la Fuente and R. Sundrum, *4D scattering amplitudes and asymptotic symmetries from 2D CFT*, *JHEP* **01** (2017) 112 [[arXiv:1609.00732](https://arxiv.org/abs/1609.00732)] [[INSPIRE](#)].
- [4] S. Pasterski and S.-H. Shao, *Conformal basis for flat space amplitudes*, *Phys. Rev. D* **96** (2017) 065022 [[arXiv:1705.01027](https://arxiv.org/abs/1705.01027)] [[INSPIRE](#)].
- [5] S. Pasterski, S.-H. Shao and A. Strominger, *Gluon Amplitudes as 2d Conformal Correlators*, *Phys. Rev. D* **96** (2017) 085006 [[arXiv:1706.03917](https://arxiv.org/abs/1706.03917)] [[INSPIRE](#)].
- [6] J.M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [[hep-th/9711200](https://arxiv.org/abs/hep-th/9711200)] [[INSPIRE](#)].
- [7] G. 't Hooft, *Dimensional reduction in quantum gravity*, *Conf. Proc. C* **930308** (1993) 284 [[gr-qc/9310026](https://arxiv.org/abs/gr-qc/9310026)] [[INSPIRE](#)].
- [8] L. Susskind, *The World as a hologram*, *J. Math. Phys.* **36** (1995) 6377 [[hep-th/9409089](https://arxiv.org/abs/hep-th/9409089)] [[INSPIRE](#)].
- [9] W. Fan, A. Fotopoulos and T.R. Taylor, *Soft Limits of Yang-Mills Amplitudes and Conformal Correlators*, *JHEP* **05** (2019) 121 [[arXiv:1903.01676](https://arxiv.org/abs/1903.01676)] [[INSPIRE](#)].
- [10] M. Pate, A.-M. Raclariu and A. Strominger, *Conformally Soft Theorem in Gauge Theory*, *Phys. Rev. D* **100** (2019) 085017 [[arXiv:1904.10831](https://arxiv.org/abs/1904.10831)] [[INSPIRE](#)].
- [11] D. Nandan, A. Schreiber, A. Volovich and M. Zlotnikov, *Celestial Amplitudes: Conformal Partial Waves and Soft Limits*, *JHEP* **10** (2019) 018 [[arXiv:1904.10940](https://arxiv.org/abs/1904.10940)] [[INSPIRE](#)].
- [12] T. Adamo, L. Mason and A. Sharma, *Celestial amplitudes and conformal soft theorems*, *Class. Quant. Grav.* **36** (2019) 205018 [[arXiv:1905.09224](https://arxiv.org/abs/1905.09224)] [[INSPIRE](#)].
- [13] M. Pate, A.-M. Raclariu, A. Strominger and E.Y. Yuan, *Celestial Operator Products of Gluons and Gravitons*, [arXiv:1910.07424](https://arxiv.org/abs/1910.07424) [[INSPIRE](#)].
- [14] A. Guevara, E. Himwich, M. Pate and A. Strominger, *Holographic Symmetry Algebras for Gauge Theory and Gravity*, [arXiv:2103.03961](https://arxiv.org/abs/2103.03961) [[INSPIRE](#)].
- [15] N. Arkani-Hamed, M. Pate, A.-M. Raclariu and A. Strominger, *Celestial Amplitudes from UV to IR*, [arXiv:2012.04208](https://arxiv.org/abs/2012.04208) [[INSPIRE](#)].
- [16] D. Kapec, P. Mitra, A.-M. Raclariu and A. Strominger, *2D Stress Tensor for 4D Gravity*, *Phys. Rev. Lett.* **119** (2017) 121601 [[arXiv:1609.00282](https://arxiv.org/abs/1609.00282)] [[INSPIRE](#)].
- [17] E. Crawley, N. Miller, S.A. Narayanan and A. Strominger, *State-Operator Correspondence in Celestial Conformal Field Theory*, [arXiv:2105.00331](https://arxiv.org/abs/2105.00331) [[INSPIRE](#)].
- [18] S. Pasterski, A. Puhm and E. Trevisani, *Celestial Diamonds: Conformal Multiplets in Celestial CFT*, [arXiv:2105.03516](https://arxiv.org/abs/2105.03516) [[INSPIRE](#)].

- [19] S. Stieberger and T.R. Taylor, *Symmetries of Celestial Amplitudes*, *Phys. Lett. B* **793** (2019) 141 [[arXiv:1812.01080](#)] [[INSPIRE](#)].
- [20] A. Fotopoulos, S. Stieberger, T.R. Taylor and B. Zhu, *Extended Super BMS Algebra of Celestial CFT*, *JHEP* **09** (2020) 198 [[arXiv:2007.03785](#)] [[INSPIRE](#)].
- [21] S. Pasterski and A. Puhm, *Shifting Spin on the Celestial Sphere*, [arXiv:2012.15694](#) [[INSPIRE](#)].
- [22] A. Brandhuber, G.R. Brown, J. Gowdy, B. Spence and G. Travaglini, *Celestial Superamplitudes*, [arXiv:2105.10263](#) [[INSPIRE](#)].
- [23] P. Di Francesco, P. Mathieu and D. Senechal, *Conformal Field Theory*, Graduate Texts in Contemporary Physics, Springer-Verlag, New York (1997), [[DOI](#)] [[INSPIRE](#)].
- [24] J.-F. Fortin, K. Intriligator and A. Stergiou, *Current OPEs in Superconformal Theories*, *JHEP* **09** (2011) 071 [[arXiv:1107.1721](#)] [[INSPIRE](#)].
- [25] J.M. Henn and J.C. Plefka, *Scattering Amplitudes in Gauge Theories*, Springer, Berlin Germany (2014) [[DOI](#)] [[INSPIRE](#)].
- [26] A. Schreiber, A. Volovich and M. Zlotnikov, *Tree-level gluon amplitudes on the celestial sphere*, *Phys. Lett. B* **781** (2018) 349 [[arXiv:1711.08435](#)] [[INSPIRE](#)].
- [27] A. Strominger, *Lectures on the Infrared Structure of Gravity and Gauge Theory*, [arXiv:1703.05448](#) [[INSPIRE](#)].
- [28] A. Strominger, *On BMS Invariance of Gravitational Scattering*, *JHEP* **07** (2014) 152 [[arXiv:1312.2229](#)] [[INSPIRE](#)].
- [29] A. Strominger, *Asymptotic Symmetries of Yang-Mills Theory*, *JHEP* **07** (2014) 151 [[arXiv:1308.0589](#)] [[INSPIRE](#)].
- [30] S. Banerjee and S. Ghosh, *MHV Gluon Scattering Amplitudes from Celestial Current Algebras*, [arXiv:2011.00017](#) [[INSPIRE](#)].
- [31] H. Elvang, Y.-t. Huang and C. Peng, *On-shell superamplitudes in $N < 4$ SYM*, *JHEP* **09** (2011) 031 [[arXiv:1102.4843](#)] [[INSPIRE](#)].
- [32] E. Casali and A. Puhm, *Double Copy for Celestial Amplitudes*, *Phys. Rev. Lett.* **126** (2021) 101602 [[arXiv:2007.15027](#)] [[INSPIRE](#)].
- [33] J.M. Drummond, J.M. Henn and J. Plefka, *Yangian symmetry of scattering amplitudes in $N = 4$ super Yang-Mills theory*, *JHEP* **05** (2009) 046 [[arXiv:0902.2987](#)] [[INSPIRE](#)].
- [34] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, *Commun. Math. Phys.* **252** (2004) 189 [[hep-th/0312171](#)] [[INSPIRE](#)].
- [35] H.A. González, A. Puhm and F. Rojas, *Loop corrections to celestial amplitudes*, *Phys. Rev. D* **102** (2020) 126027 [[arXiv:2009.07290](#)] [[INSPIRE](#)].
- [36] A. Brandhuber, G.R. Brown, J. Gowdy, B. Spence and G. Travaglini, *Celestial Superamplitudes*, to appear.
- [37] S. He, Y.-t. Huang and C. Wen, *Loop Corrections to Soft Theorems in Gauge Theories and Gravity*, *JHEP* **12** (2014) 115 [[arXiv:1405.1410](#)] [[INSPIRE](#)].
- [38] L. Ferro, T. Łukowski and R. Moerman, *From momentum amplituhedron boundaries to amplitude singularities and back*, *JHEP* **07** (2020) 201 [[arXiv:2003.13704](#)] [[INSPIRE](#)].