

The planar limit of $\mathcal{N} = 2$ superconformal quiver theories

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ABSTRACT: We compute the planar limit of both the free energy and the expectation value of the 1/2 BPS Wilson loop for four dimensional $\mathcal{N} = 2$ superconformal quiver theories, with a product of $SU(N)$ s as gauge group and bi-fundamental matter. Supersymmetric localization reduces the problem to a multi-matrix model, that we rewrite in the zero-instanton sector as an effective action involving an infinite number of double-trace terms, determined by the relevant extended Cartan matrix. We find that the results, as in the case of $\mathcal{N} = 2$ SCFTs with a simple gauge group, can be written as sums over tree graphs. For the \widehat{A}_1 case, we find that the contribution of each tree can be interpreted as the partition function of a generalized Ising model defined on the tree; we conjecture that the partition functions of these models defined on trees satisfy the Lee-Yang property, i.e. all their zeros lie on the unit circle.

KEYWORDS: Matrix Models, Supersymmetric Gauge Theory, Wilson, 't Hooft and Polyakov loops

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1 Introduction

The emergence of quantum gravity from a gauge theory is one of the most fascinating issues that can be addressed with the AdS/CFT correspondence. Since the work of [1] it has been clear that not every conformal field theory (CFT) in the large N limit can be dual to a gravitational theory described by a two derivative Einstein-Hilbert action. For instance, for four dimensional CFTs a necessary condition is that the two central charges coincide in the large N limit, $a = c$ [1]. For instance, this property is satisfied by $\mathcal{N} = 4$ super Yang-Mills, but it is not satisfied by $\mathcal{N} = 2$ $SU(N)$ with $n_F = 2N$ hypermultiplets in the fundamental representation, thus ruling out that the large N limit of this CFT has a holographic dual well described by gravity.

Since the early days of the holographic correspondence, it has been important to find further examples of CFTs with holographic duals, beyond the original example of $\mathcal{N} = 4$ SYM. Four dimensional quiver gauge theories with $\mathcal{N} = 2$ superconformal symmetry satisfy an ADE classification [2], and for certain values of the marginal couplings, they are orbifolds of $\mathcal{N} = 4$ SYM and have a gravity dual [3, 4]. These quiver gauge CFTs constitute thus an interesting laboratory, as variation of their marginal couplings allows to connect CFTs with and without gravity duals in the large N limit [5–12].

In this work we will consider $\mathcal{N} = 2$ SCFTs with gauge group a product of $SU(N)$ s, paying special attention to the simplest case, the \widehat{A}_1 theory, with gauge group $SU(N) \times SU(N)$. This theory has two marginal couplings (g_1, g_2) and varying them one can reach an orbifold of $\mathcal{N} = 4$ SYM and $\mathcal{N} = 2$ $SU(N)$ SQCD. Our main technical tool will be supersymmetric localization [13]. Thanks to this tool, the planar free energy and expectation value of the $1/2$ BPS circular Wilson loop are known to all orders in the 't Hooft coupling for the limiting theories mentioned above $\mathcal{N} = 4$ SYM and $\mathcal{N} = 2$ $SU(N)$ SQCD [14–19].

Four dimensional $\mathcal{N} = 2$ quiver CFTs have already been studied using localization [7, 10–12, 20]. The novelty of this work is that we evaluate various quantities of these theories in the planar limit, to all orders in the 't Hooft couplings λ_i . We do so by applying the same strategy developed for CFTs with simple gauge groups in [19]. For these quiver CFTs, supersymmetric localization [13] reduces the evaluation of various quantities to matrix integrals. Compared to the case of $\mathcal{N} = 2$ SCFTs with a simple gauge group, the main novelty is that the resulting matrix models are multi-matrix models. In the simplest case, the model to solve is a two-matrix model. As in our recent work [19], we rewrite the 1-loop factor as an effective action involving an infinite number of double-trace terms, in the fundamental representation of the respective gauge groups. We then show that this double-trace form of the potential implies that the perturbative series considered admit a combinatorial formulation, as sums over tree graphs.

While we will present results valid for all $\mathcal{N} = 2$ quiver CFTs, we will pay special attention to the simplest theory, \widehat{A}_1 . This theory has a \mathbb{Z}_2 symmetry exchanging the two nodes of the quiver. Since the ranks of the gauge groups are equal, this \mathbb{Z}_2 symmetry amounts to exchanging $g_1 \leftrightarrow g_2$. We will be particularly interested in observables that transform nicely under this symmetry: the free energy and particular linear combinations of the usual 1/2 BPS circular Wilson loop defined for each node [7].

In section 2, after introducing the theories we will consider, we derive the perturbative series of the planar free energy, to all orders in the 't Hooft couplings λ_i . Let's present here the answer for the \widehat{A}_1 theory. It is convenient to define $\mathcal{F}_0(\lambda_1, \lambda_2) = F_0(\lambda_1, \lambda_2) - F_0(\lambda_1)^{\mathcal{N}=4} - F_0(\lambda_2)^{\mathcal{N}=4}$. The perturbative series is given by a sum over tree graphs,

$$\begin{aligned} \mathcal{F}_0(\lambda_1, \lambda_2) = & \sum_{m=1}^{\infty} (-2)^m \sum_{n_1, \dots, n_m=2}^{\infty} \frac{\zeta(2n_1 - 1) \dots \zeta(2n_m - 1)}{n_1 \dots n_m} (-1)^{n_1 + \dots + n_m} \\ & \times \sum_{k_1=1}^{n_1-1} \binom{2n_1}{2k_1} \dots \sum_{k_m=1}^{n_m-1} \binom{2n_m}{2k_m} \sum_{\substack{\text{unlabeled trees} \\ \text{with } m \text{ edges}}} \frac{1}{|\text{Aut}(T)|} \prod_{i=1}^{m+1} \bar{\mathcal{V}}_i, \end{aligned} \tag{1.1}$$

where the product at the end of the last line runs over the vertices of a tree, and $\bar{\mathcal{V}}_i$ are factors to be defined below. This expression is formally identical to the one found for $\mathcal{N} = 2$ SQCD in [19], except for the fact that now the factors $\bar{\mathcal{V}}_i$ depend on two 't Hooft couplings, λ_1 and λ_2 . The terms in (1.1) with a single value of the ζ function have already appeared in [20]. In the perturbative expansion of $\mathcal{F}_0(\lambda_1, \lambda_2)$ above, each product of values of the ζ function is accompanied by a polynomial in λ_1 and λ_2 , that can be rewritten as a palindromic polynomial in λ_2/λ_1 . Intriguingly, up to the order we have checked explicitly, all such polynomials have all roots on the unit circle of the complex λ_2/λ_1 plane. This is of course reminiscent of the seminal work by Lee and Yang [21] for the zeros of the partition function of the ferromagnetic Ising model on a graph. We are able to prove this property for all the terms in (1.1) with a single value of ζ , and formulate two conjectures for general trees.

In section 3, we compute the planar limit of the expectation value of the 1/2 BPS circular Wilson loop defined for the gauge group in one of the two nodes of the \widehat{A}_1 theory, and in the fundamental representation. The answer is now given as a sum over rooted

trees. This Wilson loop is defined for one of the two nodes of the quiver, so it does not transform nicely under the \mathbb{Z}_2 symmetry of the theory. For this reason we consider $\langle W \rangle_{\pm} = \langle W_1 \rangle \pm \langle W_2 \rangle$ (with the $\mathcal{N} = 4$ results subtracted). For $\langle W \rangle_{\pm}$ we find again that, up to the orders we have checked explicitly, all the polynomials in λ_2/λ_1 that appear have all roots on the unit circle.

In the appendices, we write the first terms in the explicit expansion of the planar free energy and expectation value of various Wilson loop operators.

This work leaves open a number of interesting problems. First, there are general arguments that the perturbative series of the planar limit of quantum field theories have finite radius of convergence [22]. We have been able to determine the domain of convergence of just a small subset of the perturbative series found in this paper - see also [20] - but rigorously determining the full domain of convergence of the full perturbative series seems like a much harder problem. Second, in the main text we formulate two conjectures on the zeros of the polynomials that appear in the perturbative series of the planar free energy and expectation values of Wilson loops. It would be interesting to prove these conjectures, and further investigate if this property is related to the integrability of these theories, that has been encountered both in the planar limit [6, 9, 23, 24] and in the full theory [25, 26].

2 The partition function of $\mathcal{N} = 2$ quiver CFT

In this section we introduce the theories we are going to study, and recall how supersymmetric localization reduces the evaluation of selected quantities to matrix integrals. In particular, we will study first the planar free energy of the theory. Following [27–29], the integrals are performed over the full Lie algebra instead of restricting to a Cartan subalgebra, and the 1-loop factor is rewritten as an effective action. We will focus on the planar limit and in this limit, as in [19], we will unravel the underlying graph structure of the perturbative expansion.

Let us start by briefly reviewing the classification and field content of $\mathcal{N} = 2$ superconformal quiver gauge theories with $SU(N)$ gauge groups. They are in one-to-one correspondence with simply-laced affine Lie algebras \widehat{ADE} , and thus follow an ADE classification [2]. The gauge sector and matter content are encoded in the extended Cartan matrix of the affine Lie algebra. The gauge group is

$$\prod_i SU(n_i N), \tag{2.1}$$

where n_i is the Dynkin index of the i -th node of the affine Dynkin diagram. The hypermultiplets transform in the representations

$$\oplus a_{ij} (n_i N, \overline{n_j N}), \tag{2.2}$$

where a_{ij} is the adjacency matrix of the Dynkin diagram.

These theories have a marginal coupling for each gauge group and, in the particular case where the complexified couplings satisfy

$$\tau_i = n_i \tau, \tag{2.3}$$

the quiver theory can be obtained as an orbifold of $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills by the discrete subgroup Γ of $SU(2)$ [2], which also follow an ADE classification. These theories can be engineered in string theory via a suitable brane configuration and even more, in a suitable limit, they admit a weakly curved gravity dual in terms of the $AdS_5 \times S^5/\Gamma$ geometry [3, 4]. On the other hand, when all the couplings are set to zero except one, say g_1 , the quiver theory reduces to $\mathcal{N} = 2$ SQCD.

After having reviewed $\mathcal{N} = 2$ superconformal quiver theories, let's discuss supersymmetric localization for them. Following [13] it is possible to localize the \widehat{ADE} theories on S^4 . It is also possible to localize the theory on a squashed sphere of parameter b for which in the limit $b \rightarrow 1$ we recover the sphere, in such configuration the exact partition function is given by

$$Z = \int da_I \mathcal{Z}_{1\text{-loop}}(a_I, b) |\mathcal{Z}_{\text{inst}}(a_I, b)|^2 e^{-\sum_{I=1}^n \frac{8\pi^2}{g_I^2} \text{Tra}_I^2}, \tag{2.4}$$

where a_I denotes the eigenvalues of the vector-multiplet scalars Φ_I restricted to the constant mode on S^4 . In what follows we will be mostly interested in quantities that are relevant in the $b \simeq 1$ limit, such as the Wilson loop operator, or even more just observables defined on the sphere. As usual we will restrict our analysis to the zero-instanton sector, thus neglecting $|\mathcal{Z}_{\text{inst}}|^2$, and expanding (2.4) in b we obtain

$$Z = \int da_I \mathcal{Z}_{1\text{-loop}}(a_I) e^{-\sum_{I=1}^n \frac{8\pi^2}{g_I^2} \text{Tra}_I^2} + \mathcal{O}((b-1)^2), \tag{2.5}$$

higher order terms in b were studied before in [10] and we refer the reader there for more details. The factor $\widehat{\mathcal{Z}}_{1\text{-loop}}$ is the 1-loop contribution determined by the matter content. For instance for the \widehat{A}_{n-1} theory it is given by

$$\mathcal{Z}_{1\text{-loop}} = \prod_{I=1}^n \frac{\prod_{i<j} H^2(ia_i^I - ia_j^I)}{\prod_{i,j} H(ia_i^I - ia_j^{I+1})}, \tag{2.6}$$

where we identify the node $n+1$ with the first one and $H(x)$ is the Barnes function whose expansion is given by

$$\log H(x) = -(1+\gamma)x^2 - \sum_{n=2}^{\infty} \frac{\zeta(2n-1)}{n} x^{2n}. \tag{2.7}$$

Following the previous works [27–29] the strategy will be once again to interpret the matter content as an effective action

$$S_{\text{int}} = -\log \mathcal{Z}_{1\text{-loop}}. \tag{2.8}$$

Given that the theory is conformal for arbitrary values of the couplings, the quadratic terms in (2.7) will exactly cancel and the effective action will start at order g_i^4 .

Let us first illustrate the process with the \widehat{A}_1 quiver since the extension to the general case is straightforward. In this case the field content of the \widehat{A}_1 quiver consists of

two vector multiplets in the adjoint: $(A_\mu^I, \Phi^I, \Phi'^I)$, $I = 1, 2$, and bi-fundamental matter: $(X, Y, X^\dagger, Y^\dagger)$: $D_\mu X = \partial_\mu X + A_\mu^1 X - X A_\mu^2$. The 1-loop factor reduces to

$$\mathcal{Z}_{1\text{-loop}} = \frac{\prod_{i < j} H^2(ia_i^1 - ia_j^1) H^2(ia_i^2 - ia_j^2)}{\prod_{i,j} H^2(ia_i^1 - ia_j^2)}. \tag{2.9}$$

Following the procedure presented in [19] and using (2.7) it is possible to arrive to the effective action, obtaining

$$\begin{aligned} S_{\text{int}} = & \sum_{n=2}^{\infty} \frac{\zeta(2n-1)(-1)^n}{n} \\ & \times \left[\sum_{k=1}^{n-1} \binom{2n}{2k} \left(\text{Tr } a_1^{2(n-k)} \text{Tr } a_1^{2k} + \text{Tr } a_2^{2(n-k)} \text{Tr } a_2^{2k} - 2 \text{Tr } a_1^{2(n-k)} \text{Tr } a_2^{2k} \right) \right. \\ & - \sum_{k=1}^{n-2} \binom{2n}{2k+1} \left(\text{Tr } a_1^{2(n-k)-1} \text{Tr } a_1^{2k+1} + \text{Tr } a_2^{2(n-k)-1} \text{Tr } a_2^{2k+1} \right. \\ & \left. \left. - 2 \text{Tr } a_1^{2(n-k)-1} \text{Tr } a_2^{2k+1} \right) \right], \tag{2.10} \end{aligned}$$

where all traces are in the fundamental representation of the respective gauge group. Let's comment upon a couple of features of this result: first, as we already encountered in our previous work for theories with simple gauge groups [19], the effective action involves infinite sums of double-trace terms, that split into even and odd powers. By the same large N counting arguments as in [19], the odd powers will not contribute to the planar computations, so we discard such terms in what follows. Second, the pattern of double-trace terms in (2.10) is dictated by the Cartan matrix of \widehat{A}_1 ,

$$\frac{1}{2} C = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{2.11}$$

This shouldn't be a surprise, since for $\mathcal{N} = 2$ quiver superconformal field theories, the matter content is fixed by the 1-loop β functions, which are captured by the generalized Cartan matrix [2]. This last observation allows us to generalize (2.10) to arbitrary $\mathcal{N} = 2$ superconformal quiver theory. The effective action, keeping just the terms with even powers, is

$$S_{\text{int}} = \frac{1}{2} \sum_{I,J} C_{IJ} \sum_{n=2}^{\infty} \frac{\zeta(2n-1)(-1)^n}{n} \sum_{k=1}^{n-1} \binom{2n}{2k} \text{Tr } a_I^{2(n-k)} \text{Tr } a_J^{2k}, \tag{2.12}$$

where C_{IJ} is the Cartan matrix of the corresponding affine Lie algebra.

2.1 Planar free energy

We turn now to the large N limit of the free energy on S^4 , $F(\lambda_i, N) = \log Z_{S^4}$. In fact, as usual, we will compute the difference of free energy with the Gaussian model,

$\mathcal{F}(\lambda_i, N) \equiv F(\lambda_i, N) - \sum_i F(\lambda_i) N^4$. Our goal is to determine the leading term in the large N expansion, i.e. $F(\lambda_i, N) = F_0(\lambda_i) N^2 + \dots$. In general we have

$$F(\lambda_i, N) = \log Z_{S^4} = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \left(\sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \langle S_{\text{int}}^k \rangle \right)^m, \quad (2.13)$$

the free energy scales like N^2 in the planar limit, so there are many cancellations in (2.13) and we need to fully identify the N^2 terms from (2.13) that survive these cancellations. The argument to extract those terms is exactly the same as in our recent work [19]: for a disconnected $2m$ -point function, the pieces that scale like N^2 are products of $m + 1$ connected correlators. These connected correlators in the planar limit are given by [30] (see also [31] for a more recent derivation)

$$\langle \text{Tr } a^{2k_1} \text{Tr } a^{2k_2} \dots \text{Tr } a^{2k_n} \rangle_c = \mathcal{V}(k_1, \dots, k_n) \tilde{\lambda}^d N^{2-n}, \quad \tilde{\lambda} = \frac{\lambda}{16\pi^2}, \quad (2.14)$$

with

$$\mathcal{V}(k_1, \dots, k_n) = \frac{(d-1)!}{(d-n+2)!} \prod_{i=1}^n \frac{(2k_i)!}{(k_i-1)! k_i!}, \quad d = \sum_{i=1}^n k_i. \quad (2.15)$$

The products of $m + 1$ connected correlators that contribute to the planar free energy are those where the $2m$ traces are distributed in a way that can be characterized by a tree graph [19]: for each correlator introduce a vertex, and join them by an edge if they have operators from the same double-trace. The contributions to $\mathcal{F}_0(\lambda)$ at fixed order in the number of values of ζ function are then obtained following a similar procedure as in our recent work [19], but with a couple of modifications. Terms with m values of the ζ function have m pairs of traces, coming from m double-trace terms, which are of the form $C_{IJ} \text{Tr } a_I^{2(n-k)} \text{Tr } a_J^{2k}$.

To find the contribution to the planar free energy at this order, first draw all the trees with m edges. For every tree, assign each of the m double-traces to one of the m edges; this labels the m edges of the tree, turning it into an edge-labeled tree. Next, add an arrow to each of the m edges, turning the tree into a directed edge-labeled tree. Assign $\text{Tr } a_I^{2(n-k)}$ to the vertex at the start (i.e. origin of the arrow) of the i -th edge. Assign $\text{Tr } a_J^{2k}$ to the vertex at the end (i.e. end of the arrow) of the i -th edge. This procedure assigns to each of the $m + 1$ vertices a number of traces equal to its degree α_j , i.e. the number of edges connected to that vertex. For each vertex, consider now the connected correlator of all its trace operators and assign it its numerical factor \mathcal{V}_j , eq. (2.15), times $\tilde{\lambda}_j^{d_j}$, with $j = 1, \dots, m + 1$. For the connected correlator to be nonzero, all traces at a given vertex must be of the same matrix, and this enforces that they have the same index. Finally, multiply the contribution of this tree graph by a product of m components of the Cartan matrix, one per edge, with the indices fixed by those at the vertices of each edge. Summing

over all the possible choices, we arrive at

$$\begin{aligned} \mathcal{F}_0(\tilde{\lambda}_1, \dots, \tilde{\lambda}_n) &= \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \sum_{n_1, \dots, n_m=2}^{\infty} \frac{\zeta(2n_1 - 1) \dots \zeta(2n_m - 1)}{n_1 \dots n_m} (-1)^{n_1 + \dots + n_m} \\ &\times \sum_{k_1=1}^{n_1-1} \binom{2n_1}{2k_1} \dots \sum_{k_m=1}^{n_m-1} \binom{2n_m}{2k_m} \frac{1}{2^m} \sum_{\substack{\text{directed trees} \\ \text{with } m \text{ labeled edges}}} \sum_{I, J} C_{I_1 J_1} \dots C_{I_m J_m} \prod_{i=1}^{m+1} \tilde{\lambda}_{I_i}^{d_i} \mathcal{V}_i. \end{aligned} \quad (2.16)$$

This expression is the perturbative series for the planar free energy of any $\mathcal{N} = 2$ superconformal quiver theory, with quiver determined by the affine Lie algebra with Cartan matrix C . In what follows, we will discuss mostly the simplest quiver theory, \widehat{A}_1 , that has gauge group $SU(N) \times SU(N)$, and Cartan matrix (2.11). This means that double-traces where both operators belong to the same gauge group, e.g. $\text{Tr} a_1^{2(n-k)} \text{Tr} a_1^{2k}$ are weighted with a +1, while mixed double-traces, e.g. $\text{Tr} a_1^{2(n-k)} \text{Tr} a_2^{2k}$ are weighted with a -1. The overall sign of a given product of correlators is then -1 raised to the number of mixed double-traces. These signs can be transferred from the edges to the vertices: just assign an extra factor $(-1)^{\alpha_j}$ to all vertices of the tree corresponding to correlators of, say, the second gauge group (this choice is arbitrary and the final result is independent of it). To convince oneself that these two rules are the same, write every sign on top of the edges of the tree: if it is a -1 assign it to the vertex with operators of the second gauge group. If it is a +1, and it is joining two vertices with operators of the second gauge group, just write $+1 = (-1)(-1)$ and again assign one -1 to each vertex. Then each vertex contributes a factor

$$\bar{\mathcal{V}}(x_1, \dots, x_\alpha) = \mathcal{V}(x_1, \dots, x_\alpha) \left(\tilde{\lambda}_1^{\sum_i x_i} + (-1)^\alpha \tilde{\lambda}_2^{\sum_i x_i} \right), \quad (2.17)$$

and the generic expression (2.16) simplifies to

$$\begin{aligned} \mathcal{F}_0(\tilde{\lambda}_1, \tilde{\lambda}_2) &= \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \sum_{n_1, \dots, n_m=2}^{\infty} \frac{\zeta(2n_1 - 1) \dots \zeta(2n_m - 1)}{n_1 \dots n_m} (-1)^{n_1 + \dots + n_m} \\ &\times \sum_{k_1=1}^{n_1-1} \binom{2n_1}{2k_1} \dots \sum_{k_m=1}^{n_m-1} \binom{2n_m}{2k_m} \sum_{\substack{\text{directed trees} \\ \text{with } m \text{ labeled edges}}} \prod_{i=1}^{m+1} \bar{\mathcal{V}}_i. \end{aligned} \quad (2.18)$$

Finally, by exactly the same arguments as in our previous paper [19], the last sum can be reduced to a sum over unlabeled trees

$$\begin{aligned} \mathcal{F}_0(\tilde{\lambda}_1, \tilde{\lambda}_2) &= \sum_{m=1}^{\infty} (-2)^m \sum_{n_1, \dots, n_m=2}^{\infty} \frac{\zeta(2n_1 - 1) \dots \zeta(2n_m - 1)}{n_1 \dots n_m} (-1)^{n_1 + \dots + n_m} \\ &\times \sum_{k_1=1}^{n_1-1} \binom{2n_1}{2k_1} \dots \sum_{k_m=1}^{n_m-1} \binom{2n_m}{2k_m} \sum_{\substack{\text{unlabeled trees} \\ \text{with } m \text{ edges}}} \frac{1}{|\text{Aut}(\mathbb{T})|} \prod_{i=1}^{m+1} \bar{\mathcal{V}}_i. \end{aligned} \quad (2.19)$$

Let's mention a further property of $\mathcal{F}_0(\tilde{\lambda}_1, \tilde{\lambda}_2)$. Since $\mathcal{F}_0(\tilde{\lambda}_2, \tilde{\lambda}_1) = \mathcal{F}_0(\tilde{\lambda}_1, \tilde{\lambda}_2)$ and $\mathcal{F}_0(\tilde{\lambda}_1, \tilde{\lambda}_1) = 0$, it follows that $\mathcal{F}_0(\tilde{\lambda}_1, \tilde{\lambda}_2)$ has a double zero,

$$\mathcal{F}_0(\tilde{\lambda}_1, \tilde{\lambda}_2) = (\tilde{\lambda}_1 - \tilde{\lambda}_2)^2 f(\tilde{\lambda}_1, \tilde{\lambda}_2), \quad (2.20)$$

this implies that at the orbifold point $\lambda_1 = \lambda_2$ - see comment below (2.3) - not just the free energy, but also its first derivative with respect to λ coincides with the $\mathcal{N} = 4$ result. To see that this property is implied by our result (2.19), we are going to prove that the contribution of every tree to (2.19) is of the form

$$(\tilde{\lambda}_1 - \tilde{\lambda}_2)^{v_{\text{odd}}} p(\tilde{\lambda}_1, \tilde{\lambda}_2), \quad (2.21)$$

where v_{odd} is the number of vertices of the tree with odd degree, and $p(\tilde{\lambda}_1, \tilde{\lambda}_2)$ is a symmetric polynomial in $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ with positive coefficients. This follows from inspection of the factor attached to each vertex, (2.17). When the degree α of a vertex is odd, $\tilde{\lambda}_1 = \tilde{\lambda}_2$ is a simple root of that factor. After pulling out these factors, what is left is a polynomial with positive coefficients. As a check, notice that v_{odd} is always even: for a tree with $m + 1$ vertices, $\sum_{i=1}^{m+1} \alpha_i = 2m$, and since $\sum_i \alpha_i^{\text{even}}$ is even, $\sum_i \alpha_i^{\text{odd}}$ must be even also, which implies that v_{odd} is even. This concludes the argument for (2.21). Now, since every tree has at least two vertices of degree one, $v_{\text{odd}} \geq 2$, and (2.20) follows.

To illustrate (2.19), let's work out the first terms. The $m = 1$ terms in (2.19) are terms with a single value of ζ [20]. To write them, it is convenient to first recall the definition of the Narayana numbers

$$N(n, k) = \frac{1}{n} \binom{n}{k} \binom{n}{k-1}, \quad (2.22)$$

and the Narayana polynomials

$$\mathfrak{C}_n(t) = \sum_{k=0}^{n-1} N(n, k+1) t^k, \quad (2.23)$$

that satisfy $\mathfrak{C}_n(1) = \mathcal{C}_n$ with \mathcal{C}_n the Catalan numbers. At this order, we have to consider trees with two vertices. There is just one such tree, and both vertices have degree one. Then,

$$\begin{aligned} \mathcal{F}_0(\tilde{\lambda}_1, \tilde{\lambda}_2)|_{\zeta} &= - \sum_{n=2}^{\infty} \frac{\zeta(2n-1)}{n} (-1)^n \sum_{k=1}^{n-1} \binom{2n}{2k} \mathcal{C}_{n-k} \mathcal{C}_k \left(\tilde{\lambda}_1^{n-k} - \tilde{\lambda}_2^{n-k} \right) \left(\tilde{\lambda}_1^k - \tilde{\lambda}_2^k \right) \\ &= - \sum_{n=2}^{\infty} \frac{\zeta(2n-1)}{n} (-1)^n \mathcal{C}_n \tilde{\lambda}_1^n \left[\left(1 + \frac{\tilde{\lambda}_2^n}{\tilde{\lambda}_1^n} \right) \mathcal{C}_{n+1} - 2\mathfrak{C}_{n+1} \left(\frac{\tilde{\lambda}_2}{\tilde{\lambda}_1} \right) \right], \end{aligned} \quad (2.24)$$

where to avoid confusion, the first term in the parenthesis involves the Catalan number \mathcal{C}_{n+1} , and the second one the Narayana polynomial $\mathfrak{C}_{n+1}(\tilde{\lambda}_2/\tilde{\lambda}_1)$. A first question we can ask about this series is what is its domain of convergence in \mathbb{C}^2 . As pointed out in [19, 20], when $\lambda_2 = 0$ it is straightforward to prove that the radius of convergence is $\lambda_1 = \pi^2$, and the same holds, *mutatis mutandi*, when $\lambda_1 = 0$. When both couplings are different from zero, since $\mathcal{F}_0(\lambda_1, \lambda_1) = 0$ the series trivially converges when both couplings are equal. When the two couplings are different, one of them is larger, say λ_1 , applying the quotient criterion it follows that for any $|\lambda_2| < |\lambda_1| \leq \pi^2$, the series is convergent. All in all, this series is convergent in $|\lambda_1| \leq \pi^2$, $|\lambda_2| \leq \pi^2$ plus the $\lambda_1 = \lambda_2$ line.

For $\mathcal{N} = 2$ superconformal field theories with a simple gauge group, terms with a fixed number of values of the ζ function form an infinite series. In [19] we sketched an

argument that all these series have the same radius of convergence. It seems possible that this property extends to quiver theories.

Let's work out a couple more of terms in (2.19). Terms with two values of the ζ function are given by a sum over trees with two edges. There is just one tree with two edges, and its vertices have degrees (1, 2, 1). As a last example, terms with three values of the ζ function are given by a sum over trees with three edges. There are two such unlabeled trees. The degrees are (1, 2, 2, 1) for the first tree, and (3, 1, 1, 1) for the second, all these trees are depicted in figure 1 and 2. Up to this order,

$$\begin{aligned}
 \mathcal{F}_0(\tilde{\lambda}_1, \tilde{\lambda}_2) = & - \sum_{n=2}^{\infty} \frac{\zeta(2n-1)}{n} (-1)^n \sum_{k=1}^{n-1} \binom{2n}{2k} \mathcal{V}(n-k) \mathcal{V}(k) (\tilde{\lambda}_1^{n-k} - \tilde{\lambda}_2^{n-k}) (\tilde{\lambda}_1^k - \tilde{\lambda}_2^k) \\
 & + \frac{1}{2} \sum_{n_i=2}^{\infty} \frac{\zeta(2n_i-1)}{n_1 n_2} (-1)^{n_1+n_2} \sum_{k_i=1}^{n_i-1} \binom{2n_i}{2k_i} 4 \mathcal{V}(k_1) \mathcal{V}(n_1-k_1, n_2-k_2) \mathcal{V}(k_2) \\
 & \times (\tilde{\lambda}_1^{k_1} - \tilde{\lambda}_2^{k_1}) (\tilde{\lambda}_1^{n_1-k_1+n_2-k_2} + \tilde{\lambda}_2^{n_1-k_1+n_2-k_2}) (\tilde{\lambda}_1^{k_2} - \tilde{\lambda}_2^{k_2}) \\
 & - \frac{1}{3!} \sum_{n_i=2}^{\infty} \frac{\zeta(2n_i-1)}{n_1 n_2 n_3} (-1)^{n_1+n_2+n_3} \sum_{k_i=1}^{n_i-1} \binom{2n_i}{2k_i} 8 \left[3 \mathcal{V}(n_1-k_1) \mathcal{V}(k_1, n_2-k_2) \right. \\
 & \times \mathcal{V}(k_2, n_3-k_3) \mathcal{V}(k_3) (\tilde{\lambda}_1^{n_1-k_1} - \tilde{\lambda}_2^{n_1-k_1}) (\tilde{\lambda}_1^{k_1+n_2-k_2} + \tilde{\lambda}_2^{k_1+n_2-k_2}) \\
 & \times (\tilde{\lambda}_1^{k_2+n_3-k_3} + \tilde{\lambda}_2^{k_2+n_3-k_3}) (\tilde{\lambda}_1^{k_3} - \tilde{\lambda}_2^{k_3}) + \mathcal{V}(n_1-k_1, n_2-k_2, n_3-k_3) \mathcal{V}(k_1) \mathcal{V}(k_2) \mathcal{V}(k_3) \\
 & \left. \times (\tilde{\lambda}_1^{n_1-k_1+n_2-k_2+n_3-k_3} - \tilde{\lambda}_2^{n_1-k_1+n_2-k_2+n_3-k_3}) (\tilde{\lambda}_1^{k_1} - \tilde{\lambda}_2^{k_1}) (\tilde{\lambda}_1^{k_2} - \tilde{\lambda}_2^{k_2}) (\tilde{\lambda}_1^{k_3} - \tilde{\lambda}_2^{k_3}) \right] \\
 & + \mathcal{O}(\zeta^4).
 \end{aligned} \tag{2.25}$$

As a first check, when either of the two couplings vanishes, we recover the result of $\mathcal{N} = 2$ SCQD presented in [19]. Also, in this expression we can see rather explicitly that at every order the contribution has at least a double zero $(\tilde{\lambda}_1 - \tilde{\lambda}_2)^2$. In appendix A we have written the outcome of these sums, up to order $\tilde{\lambda}^6$.

2.2 The Lee-Yang property of the planar free energy expansion

We would like to discuss one further property of the perturbative expansion (2.19). Notice that the contribution of a given tree is obtained by summing over all the possible ways to assign one gauge group, 1 or 2, to each vertex in the tree, see figures 1 and 2. This is reminiscent of the Ising model defined on that tree, where on each vertex we can have a spin up or down. It is indeed possible to construct a generalized Ising-type model, with inhomogeneous external magnetic field, whose partition function yields each tree contribution in (2.19). This generalized Ising model is admittedly a bit contrived, but following the classical work by Lee and Yang [21], it motivates the study of the zeros of its partition function.

In more detail, every tree graph contributes to the planar energy in (2.19) a homogeneous polynomial in λ_1 and λ_2 . Being homogeneous, these polynomials can be thought of as polynomials of a single variable λ_2/λ_1 . Inspired by the classical work by Lee and Yang [21] on the ferromagnetic Ising model, we are going to put forward two conjectures regarding the zeros of these polynomials: first, that for a given tree, all the zeros of the

corresponding polynomial are on the unit circle in the complex λ_2/λ_1 plane. Second, that when we sum the contributions from different trees with the same number of nodes, the same property holds.

To provide context, let's start by briefly recalling the definition of the Ising model on a graph and the Lee-Yang theorem. Let G be a finite graph, E its set of edges and V its set of vertices. The Ising model on G is defined by assigning to each vertex $i \in V$, a $\sigma_i = \pm 1$ (spin up/down). The Hamiltonian is

$$\mathcal{H} = -J \sum_{i-j \in E} \sigma_i \sigma_j - H \sum_{i \in V} \sigma_i, \quad (2.26)$$

with J the coupling among spins and H the external magnetic field. The partition function can be written as

$$Z(\beta J, \beta H) = \sum_{\text{all states}} e^{-\beta \mathcal{H}} = e^{\beta J |E| - \beta H |V|} \sum_{\text{all states}} e^{-2\beta J e_{\pm}} e^{2\beta H v_{\uparrow}}, \quad (2.27)$$

where e_{\pm} is the number of edges connecting different spins, and v_{\uparrow} the number of spins up in a given configuration. Define $\tau = e^{-2\beta J}$, $x = e^{2\beta H}$. The last sum defines a polynomial palindromic in x ,

$$P(\tau, x) = \sum_{\text{all states}} \tau^{e_{\pm}} x^{v_{\uparrow}}. \quad (2.28)$$

In [21], Lee and Yang proved that for $\tau \in [-1, 1]$, the polynomials $P(\tau, x)$ have all their x roots on the unit circle. In fact, they proved it for arbitrary ferromagnetic couplings $J_{ij} \geq 0$, and different magnetic fields per site H_i .

To construct an Ising-type model whose partition function yields the polynomials that appear in (2.19), proceed as follows. Take the graph G to be a tree T ,

1. Assign a positive integer n_i to each of the e edges of the tree graph.
2. For every edge, split n_i into two positive integers, $n_i = k_i + (n_i - k_i)$ and assign each of these two integers to one of the vertices at the ends of that edge.
3. Then, if a vertex has degree d_j this procedure assigns to that vertex d_j integers. Let m_j be the sum of these integers at a given vertex; the magnetic field at that vertex is then $m_j H$.

So far, for a fixed partition of all n_i , this is a peculiar way to assign external magnetic fields that are different at each vertex. This defines

$$P(\tau, x, k_i, n_i) = \sum_{\text{all states}} \tau^{e_{\pm}} \prod_{\substack{\text{vertices} \\ \text{with spin up}}} x^{m_j}, \quad (2.29)$$

Lee and Yang already proved (lemma in appendix II of [21]) that all the zeros of these polynomials are on the unit circle. Finally, consider the sum over all the partitions of each of the n_i into two

$$P(\tau, x, n_1, \dots, n_e) = \sum_{k_1=1}^{n_1-1} \cdots \sum_{k_e=1}^{n_e-1} \rho(k_i, n_i) \sum_{\text{all states}} \tau^{e_{\pm}} \prod_{\substack{\text{vertices} \\ \text{with spin up}}} x^{m_j}, \quad (2.30)$$

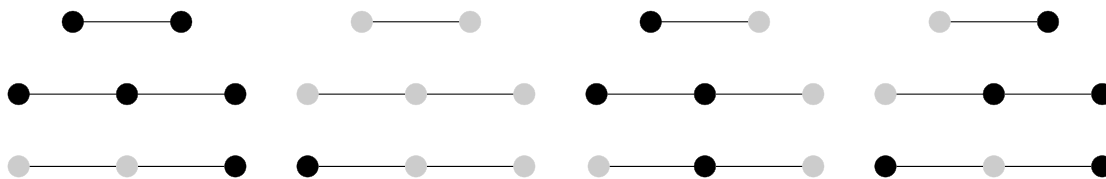


Figure 1. Trees contributing to the first and second order expansion of the free energy.

where $\rho(k_i, n_i)$ is a distribution that weights different configurations. The contribution of every tree to the planar free energy in (2.19) is obtained from the free energy of this Ising-type model, by setting $\tau = -1$, $x = \lambda_2/\lambda_1$, and the distribution

$$\rho(k_i, n_i) = \binom{2n_1}{2k_1} \cdots \binom{2n_m}{2k_m} \prod_{i=1}^m \mathcal{V}_i. \tag{2.31}$$

The main reason we have defined this family of Ising-type models is that there is numerical evidence that suggests that they share the Lee-Yang property with the original Ising model. This leads us to formulate the following two conjectures:

Conjecture 1. For any tree with e edges, any fixed positive integers n_1, \dots, n_e and arbitrary $\rho(k_i, n_i) > 0$ the polynomials $P(\tau, x, n_1, \dots, n_e)$ have all their x roots on the unit circle.

Conjecture 2. If we sum the polynomials of all the trees with the same number of edges, the resulting polynomial still has the Lee-Yang property.

We can prove the first conjecture in the particular case of the simplest tree. In this case, (2.30) is simply

$$P(\tau, x, k, n) = x^n + \tau x^{n-k} + \tau x^k + 1, \tag{2.32}$$

that for $|\tau| \leq 1$ has its roots on the unit circle. Then

$$P(\tau, x, k, n) = \sum_{k=1}^{n-1} \rho(n, k) \left(x^n + \tau x^{n-k} + \tau x^k + 1 \right), \tag{2.33}$$

with arbitrary $\rho(n, k) > 0$. To prove that these polynomials have their roots on the unit circle, we make use of the following theorem [32]: if $P(x) = A_n x^n + A_{n-1} x^{n-1} + \cdots + A_1 x + A_0$ is a palindromic polynomial and $2|A_n| \geq \sum_{j=1}^{n-1} |A_j|$, then all its zeros are in the unit circle. In our case, the inequality in the theorem is satisfied as long as $|\tau| \leq 1$, so the result follows. Back to the free energy of the quiver theory, one can check indeed that the polynomials in the expansion (2.24) have the Lee-Yang property.

We haven't been able to prove these two conjectures for arbitrary tree graphs. After the seminal work [21], the proof of the Lee-Yang unit circle theorem has been extended to many other systems, see e.g. [33, 34]. It would be interesting to see if any of these arguments can be adapted to prove our conjectures.

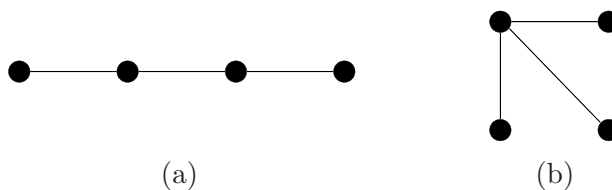


Figure 2. The two trees with three edges: (a) Tree with vertices of degrees (1,2,2,1). (b) Tree with vertices of degrees (3,1,1,1). There are 16 ways to color each of them.

3 Wilson loop in the large N limit

For each of the gauge groups of the quiver theory, we can define a 1/2 BPS Wilson loop, with circular contour in Euclidean signature. The evaluation of its expectation value reduces to a matrix integral thanks to supersymmetric localization. We will now evaluate the planar limit of this expectation value and show that the perturbative series involves a sum over rooted trees. While the Wilson loop can be defined for arbitrary representations of the gauge group, in order to take advantage of the results of [19, 31], we will restrict its study to the fundamental representation

$$\langle W^I \rangle = \left\langle \frac{1}{N} \text{Tr}_F \mathcal{P} \exp \oint_{\mathcal{C}} ds (iA_{\mu}^I(x) \dot{x}^{\mu} + \Phi^I(x) |\dot{x}|) \right\rangle, \quad (3.1)$$

where $I = 1, \dots, n$. The theory can be localized [13] on the sphere with squashing parameter b , where $b = 1$ corresponds to S^4 , in such case the vev of the 1/2 BPS Wilson loop reduces to

$$\langle W_I^{\pm} \rangle = \frac{1}{Z} \int da_I \text{Tr} \left(e^{-2\pi b^{\pm} a_I} \right) e^{-\sum_{I=1}^n \frac{8\pi^2}{g_I^2} \text{Tra}_I^2} \mathcal{Z}_{1\text{-loop}}(a_I, b) |\mathcal{Z}_{\text{inst}}(a_I, b)|^2, \quad (3.2)$$

now \pm represents the two different trajectories in which we can compute the Wilson loop on the squashed sphere [11]; from now on we will avoid the \pm to make the notation less cumbersome, bearing in mind that in order to switch between trajectories we need to make the replacement $b \rightarrow b^{-1}$ in the following results. Once again we will consider the 1-loop contribution as an effective action, given by (2.10), and as discussed on the previous section we will compute the large N limit of this interacting theory while restricting ourselves to the zero-instanton sector. We are interested in observables that are only sensitive to the linear dependence of $\langle W_b \rangle$ in $(b-1)$, and since the dependence of $\mathcal{Z}_{1\text{-loop}}(a_I, b)$ is quadratic in $b-1$, for our purposes we can compute $\langle W_b \rangle$ directly on S^4 [35],

$$\langle W_I^{\pm} \rangle = \frac{1}{Z} \int da_I \text{Tr} \left(e^{-2\pi b^{\pm} a_I} \right) e^{-\sum_{I=1}^n \frac{8\pi^2}{g_I^2} \text{Tra}_I^2} \mathcal{Z}_{1\text{-loop}}(a_I) + \mathcal{O}((b-1)^2). \quad (3.3)$$

Let us expand the Wilson loop insertion

$$\langle W_I \rangle = \sum_{l=0}^{\infty} \frac{(4\pi^2 b^2)^l}{(2l)!} \frac{\langle N^{-1} \text{Tr} a_I^{2l} e^{-S} \rangle}{\langle e^{-S} \rangle}. \quad (3.4)$$

As argued in our recent work [19], the large N expansion of this expectation value scales like N^0 , so given the overall normalization factor $1/N$, the relevant terms to keep from

$\langle \text{Tr } a_I^{2l} S^m \rangle$ are products of $m + 1$ connected correlators. Now there are $2m + 1$ traces to be distributed in $m + 1$ correlators, but since $\langle \text{Tr } a_I^{2l} \rangle$ can't be by itself, we effectively have to distribute $2m$ traces into the $m + 1$ connected correlators, which is the by now familiar sign that the possibilities are given by tree graphs. As in [19], one of the vertices is singled out by the presence of $\langle \text{Tr } a_I^{2l} \rangle$, so these are rooted trees. The correlator that contains $\langle \text{Tr } a_I^{2l} \rangle$ is a correlator of a_I operators, so it involves the λ_I coupling; by convention, the root vertex corresponding to this correlator will be referred as the vertex 1. The remaining m correlators can be either products of a_I traces or a_J traces. As we found in the evaluation of the planar free energy in the previous section, this is accounted for by modifying the numerical factor of the correlator by a weighted sum over the coupling. eq. (2.17). All in all, for the case of \widehat{A}_1

$$\begin{aligned}
 \langle W_1 \rangle - \langle W_1 \rangle_0 &= \sum_{l=1}^{\infty} \frac{(2\pi b)^{2l}}{(2l)!} \sum_{m=1}^{\infty} (-2)^m \sum_{n_1, \dots, n_m=2}^{\infty} \frac{\zeta(2n_1-1) \dots \zeta(2n_m-1)}{n_1 \dots n_m} (-1)^{n_1 + \dots + n_m} \\
 &\quad \times \sum_{k_1=1}^{n_1-1} \binom{2n_1}{2k_1} \dots \sum_{k_m=1}^{n_m-1} \binom{2n_m}{2k_m} \sum_{\substack{\text{unlabeled rooted trees} \\ \text{with } m \text{ edges}}} \frac{1}{|\text{Aut}(\mathbb{T})|} \tilde{\lambda}_1^{d_1} \mathcal{V}_1 \prod_{i=2}^{m+1} \tilde{\mathcal{V}}_i, \quad (3.5)
 \end{aligned}$$

In the language of Ising-type models on trees introduced in the previous section, we can think of the Wilson loop insertion as a spin that is pinned to be up, at the rooted vertex. To illustrate this result, let's expand it up to second order,

$$\begin{aligned}
 \langle W_1 \rangle - \langle W_1 \rangle_0 &= \sum_{l=1}^{\infty} \frac{(4\pi^2 b^2)^l}{(2l)!} \left\{ - \sum_{n=2}^{\infty} \frac{\zeta(2n-1)}{n} (-1)^n \sum_{k=1}^{n-1} \binom{2n}{2k} 2\mathcal{V}(l, n-k) \mathcal{V}(k) \tilde{\lambda}_1^{l+n-k} (\tilde{\lambda}_1^k - \tilde{\lambda}_2^k) \right. \\
 &\quad + \frac{1}{2} \sum_{n_1, n_2=2}^{\infty} \frac{\zeta(2n_1-1) \zeta(2n_2-1)}{n_1 n_2} (-1)^{n_1+n_2} \sum_{k_i=1}^{n_i-1} \binom{2n_1}{2k_1} \binom{2n_2}{2k_2} \\
 &\quad \times \left[8\mathcal{V}(l, n_1-k_1) \mathcal{V}(k_1, n_2-k_2) \mathcal{V}(k_2) \tilde{\lambda}_1^{l+n_1-k_1} (\tilde{\lambda}_1^{k_1+n_2-k_2} + \tilde{\lambda}_2^{k_1+n_2-k_2}) (\tilde{\lambda}_1^{k_2} - \tilde{\lambda}_2^{k_2}) \right. \\
 &\quad \left. \left. + 4\mathcal{V}(l, n_1-k_1, n_2-k_2) \mathcal{V}(k_1) \mathcal{V}(k_2) \tilde{\lambda}_1^{l+n_1-k_1+n_2-k_2} (\tilde{\lambda}_1^{k_1} - \tilde{\lambda}_2^{k_1}) (\tilde{\lambda}_1^{k_2} - \tilde{\lambda}_2^{k_2}) \right] \right\}, \quad (3.6)
 \end{aligned}$$

for which the corresponding rooted trees can be seen in figure 3.

In appendix B, we present the result of these sums up to order $\tilde{\lambda}^7$. We have checked that they reproduce the results of [10, 11]. Contrary to what happened for the free energy, the expectation value of this Wilson loop does not have nice properties under the exchange $\tilde{\lambda}_1 \leftrightarrow \tilde{\lambda}_2$. The reason is obvious, the Wilson loop is defined for one of the two gauge groups in the quiver, thus breaking the \mathbb{Z}_2 symmetry. For this reason, let's consider the linear combinations $\langle W_1 \rangle \pm \langle W_2 \rangle$, which were referred in [7] as twisted and untwisted. These are symmetric and antisymmetric under the $\tilde{\lambda}_1 \leftrightarrow \tilde{\lambda}_2$ exchange, so we can introduce

$$\langle W_1 \rangle + \langle W_2 \rangle - \langle W_1 \rangle_0 - \langle W_2 \rangle_0 = (\tilde{\lambda}_1 - \tilde{\lambda}_2)^2 w_+(\tilde{\lambda}_1, \tilde{\lambda}_2), \quad (3.7)$$

$$\langle W_1 \rangle - \langle W_2 \rangle - \langle W_1 \rangle_0 + \langle W_2 \rangle_0 = (\tilde{\lambda}_1 - \tilde{\lambda}_2) w_-(\tilde{\lambda}_1, \tilde{\lambda}_2), \quad (3.8)$$

with w_{\pm} symmetric under $\tilde{\lambda}_1 \leftrightarrow \tilde{\lambda}_2$. What is more, to the orders we have checked explicitly, again all the polynomials that appear in the expansion of w_{\pm} have all their roots in the unit

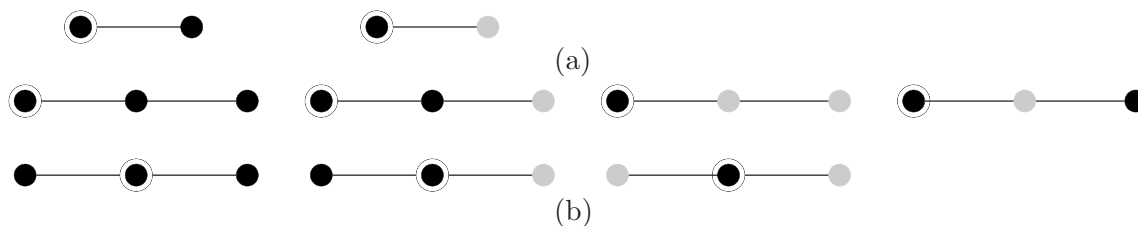


Figure 3. Rooted trees corresponding to the Wilson loop in the large N , we see that inserting the operator selects from figure 1 trees with the same color as the operator that we are inserting, trees containing two different colors arise from interaction terms in (2.10). (a) Terms corresponding to $\mathcal{V}(l, n_1 - k_1)\mathcal{V}(k_1)$. (b) Trees corresponding to $\mathcal{V}(l, n_1 - k_1)\mathcal{V}(k_1, n_2 - k_2)\mathcal{V}(k_2)$ and $\mathcal{V}(l, n_1 - k_1, n_2 - k_2)\mathcal{V}(k_1)\mathcal{V}(k_2)$.

circle of the complex $\tilde{\lambda}_2/\tilde{\lambda}_1$ plane. We again conjecture that this is true for the polynomials generated by every tree.

For the polynomials that appear in $w_+(\tilde{\lambda}_1, \tilde{\lambda}_2)$, this would follow from our first conjecture if it is true. In particular, since in the previous section we proved the first conjecture for the simplest tree, it follows that it holds also for w_+ , for the simplest tree. For w_- the argument does not apply immediately, since $\langle W_1 \rangle - \langle W_2 \rangle - \langle W_1 \rangle_0 + \langle W_2 \rangle_0$ produces antipalindromic polynomials.

To conclude, we can use these results to compute the one-point function of the energy-momentum tensor with these 1/2 BPS Wilson loops. This one-point function is fixed up to a coefficient h_W [36], which can be obtained from the expectation value of the deformed Wilson loop $\langle W_b \rangle$ by the formula [35, 37]

$$h_W = \frac{1}{12\pi^2} \partial_b \ln \langle W_b \rangle|_{b=1}. \quad (3.9)$$

finally, we can also compute the Bremsstrahlung function B [38] using the relation $B = 3h_W$ [35, 39, 40], valid for any $\mathcal{N} = 2$ superconformal field theory [41]. The results we obtain agree with those of [11].

Acknowledgments

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A Planar free energy up to 6th order

Here we present the explicit form of the planar free energy in terms of $\tilde{\lambda}_i = \frac{\lambda_i}{16\pi^2}$

$$\begin{aligned}
 \mathcal{F}_0(\tilde{\lambda}_1, \tilde{\lambda}_2) = & (\tilde{\lambda}_1 - \tilde{\lambda}_2)^2 \left[-3\zeta_3 + 20\zeta_5 \left(\tilde{\lambda}_1 + \tilde{\lambda}_2 \right) - 70\zeta_7 \left(2\tilde{\lambda}_1^2 + 3\tilde{\lambda}_1\tilde{\lambda}_2 + 2\tilde{\lambda}_2^2 \right) \right. \\
 & + 84\zeta_9 \left(\tilde{\lambda}_1 + \tilde{\lambda}_2 \right) \left(13\tilde{\lambda}_1^2 + 10\tilde{\lambda}_1\tilde{\lambda}_2 + 13\tilde{\lambda}_2^2 \right) \\
 & - 154\zeta_{11} \left(61\tilde{\lambda}_1^4 + 116\tilde{\lambda}_1^3\tilde{\lambda}_2 + 141\tilde{\lambda}_1^2\tilde{\lambda}_2^2 + 116\tilde{\lambda}_1\tilde{\lambda}_2^3 + 61\tilde{\lambda}_2^4 \right) \\
 & + 36\zeta_3^2 \left(\tilde{\lambda}_1^2 + \tilde{\lambda}_2^2 \right) - 240\zeta_3\zeta_5 \left(\tilde{\lambda}_1 + \tilde{\lambda}_2 \right) \left(3\tilde{\lambda}_1^2 - 2\tilde{\lambda}_1\tilde{\lambda}_2 + 3\tilde{\lambda}_2^2 \right) \quad (\text{A.1}) \\
 & + 840\zeta_3\zeta_7 \left(8\tilde{\lambda}_1^4 + 5\tilde{\lambda}_1^3\tilde{\lambda}_2 + 2\tilde{\lambda}_1^2\tilde{\lambda}_2^2 + 5\tilde{\lambda}_1\tilde{\lambda}_2^3 + 8\tilde{\lambda}_2^4 \right) \\
 & + 200\zeta_5^2 \left(19\tilde{\lambda}_1^4 + 12\tilde{\lambda}_1^3\tilde{\lambda}_2 + 4\tilde{\lambda}_1^2\tilde{\lambda}_2^2 + 12\tilde{\lambda}_1\tilde{\lambda}_2^3 + 19\tilde{\lambda}_2^4 \right) \\
 & \left. - 144\zeta_3^3 \left(5\tilde{\lambda}_1^4 - 2\tilde{\lambda}_1^3\tilde{\lambda}_2 + 6\tilde{\lambda}_1^2\tilde{\lambda}_2^2 - 2\tilde{\lambda}_1\tilde{\lambda}_2^3 + 5\tilde{\lambda}_2^4 \right) \right] + \mathcal{O}(\tilde{\lambda}^7).
 \end{aligned}$$

Up to the order we have explicitly checked, the polynomials have all unimodular roots.

B Wilson loop up to $\tilde{\lambda}^7$

Here we present the explicit expansion of the circular Wilson loop corresponding to an insertion in the first node of the quiver; it is possible to obtain the insertion in the second node by making the change $\tilde{\lambda}_1 \leftrightarrow \tilde{\lambda}_2$. For simplicity, in the expansion we have set $b = 1$ and $\tilde{\lambda}_i = \frac{\lambda_i}{16\pi^2}$. If one wishes to restore the powers of b that appear in the perturbative expansion of $\langle W_b \rangle$ evaluated on S^4 , one only needs to add in each term as many powers of b as powers of π there are.

$$\begin{aligned}
 \langle W_1 \rangle - \langle W_1 \rangle_0 = & \left(\tilde{\lambda}_1 - \tilde{\lambda}_2 \right) \left[-24\pi^2\zeta_3\tilde{\lambda}_1^2 - 32\pi^4\zeta_3\tilde{\lambda}_1^3 - 16\pi^6\zeta_3\tilde{\lambda}_1^4 - \frac{64}{15}\pi^8\zeta_3\tilde{\lambda}_1^5 - \frac{32}{45}\pi^{10}\zeta_3\tilde{\lambda}_1^6 \right. \\
 & + 80\pi^2\zeta_5\tilde{\lambda}_1^2 \left(3\tilde{\lambda}_1 + \tilde{\lambda}_2 \right) + \frac{80}{3}\pi^4\zeta_5\tilde{\lambda}_1^3 \left(13\tilde{\lambda}_1 + 4\tilde{\lambda}_2 \right) \\
 & + \frac{32}{3}\pi^6\zeta_5\tilde{\lambda}_1^4 \left(17\tilde{\lambda}_1 + 5\tilde{\lambda}_2 \right) + \frac{64}{9}\pi^8\zeta_5\tilde{\lambda}_1^5 \left(7\tilde{\lambda}_1 + 2\tilde{\lambda}_2 \right) \\
 & - 280\pi^2\zeta_7\tilde{\lambda}_1^2 \left(8\tilde{\lambda}_1^2 + 5\tilde{\lambda}_1\tilde{\lambda}_2 + \tilde{\lambda}_2^2 \right) \\
 & - \frac{112}{3}\pi^4\zeta_7\tilde{\lambda}_1^3 \left(91\tilde{\lambda}_1^2 + 55\tilde{\lambda}_1\tilde{\lambda}_2 + 10\tilde{\lambda}_2^2 \right) \\
 & - \frac{112}{3}\pi^6\zeta_7\tilde{\lambda}_1^4 \left(49\tilde{\lambda}_1^2 + 29\tilde{\lambda}_1\tilde{\lambda}_2 + 5\tilde{\lambda}_2^2 \right) \\
 & + 336\pi^2\zeta_9\tilde{\lambda}_1^2 \left(5\tilde{\lambda}_1 + \tilde{\lambda}_2 \right) \left(13\tilde{\lambda}_1^2 + 8\tilde{\lambda}_1\tilde{\lambda}_2 + 3\tilde{\lambda}_2^2 \right) \\
 & + 672\pi^4\zeta_9\tilde{\lambda}_1^3 \left(51\tilde{\lambda}_1^3 + 41\tilde{\lambda}_1^2\tilde{\lambda}_2 + 17\tilde{\lambda}_1\tilde{\lambda}_2^2 + 2\tilde{\lambda}_2^3 \right) \quad (\text{B.1}) \\
 & - 3696\pi^2\zeta_{11}\tilde{\lambda}_1^2 \left(61\tilde{\lambda}_1^4 + 56\tilde{\lambda}_1^3\tilde{\lambda}_2 + 36\tilde{\lambda}_1^2\tilde{\lambda}_2^2 + 11\tilde{\lambda}_1\tilde{\lambda}_2^3 + \tilde{\lambda}_2^4 \right) \\
 & + 288\pi^2\zeta_3^2\tilde{\lambda}_1^2 \left(2\tilde{\lambda}_1^2 - \tilde{\lambda}_1\tilde{\lambda}_2 + \tilde{\lambda}_2^2 \right) \\
 & + 192\pi^4\zeta_3^2\tilde{\lambda}_1^3 \left(5\tilde{\lambda}_1^2 - 3\tilde{\lambda}_1\tilde{\lambda}_2 + 2\tilde{\lambda}_2^2 \right)
 \end{aligned}$$

$$\begin{aligned}
 & + 192\pi^6\zeta_3^2\tilde{\lambda}_1^4 \left(3\tilde{\lambda}_1^2 - 2\tilde{\lambda}_1\tilde{\lambda}_2 + \tilde{\lambda}_2^2 \right) \\
 & - 960\pi^2\zeta_3\zeta_5\tilde{\lambda}_1^2 \left(15\tilde{\lambda}_1^3 - 5\tilde{\lambda}_1^2\tilde{\lambda}_2 + \tilde{\lambda}_1\tilde{\lambda}_2^2 + 5\tilde{\lambda}_2^3 \right) \\
 & - 320\pi^4\zeta_3\zeta_5\tilde{\lambda}_1^3 \left(77\tilde{\lambda}_1^3 - 32\tilde{\lambda}_1^2\tilde{\lambda}_2 + \tilde{\lambda}_1\tilde{\lambda}_2^2 + 20\tilde{\lambda}_2^3 \right) \\
 & + 3360\pi^2\zeta_3\zeta_7\tilde{\lambda}_1^2 \left(48\tilde{\lambda}_1^4 - 7\tilde{\lambda}_1^3\tilde{\lambda}_2 - 7\tilde{\lambda}_1^2\tilde{\lambda}_2^2 + 11\tilde{\lambda}_1\tilde{\lambda}_2^3 + 11\tilde{\lambda}_2^4 \right) \\
 & + 1600\pi^2\zeta_5^2\tilde{\lambda}_1^2 \left(57\tilde{\lambda}_1^4 - 8\tilde{\lambda}_1^3\tilde{\lambda}_2 - 10\tilde{\lambda}_1^2\tilde{\lambda}_2^2 + 14\tilde{\lambda}_1\tilde{\lambda}_2^3 + 13\tilde{\lambda}_2^4 \right) \\
 & - 3456\pi^2\zeta_3^3\tilde{\lambda}_1^2 \left(5\tilde{\lambda}_1^4 - 5\tilde{\lambda}_1^3\tilde{\lambda}_2 + 5\tilde{\lambda}_1^2\tilde{\lambda}_2^2 - 3\tilde{\lambda}_1\tilde{\lambda}_2^3 + 2\tilde{\lambda}_2^4 \right) \Big].
 \end{aligned}$$

Note that we are inserting the operator in only one of the two nodes of the quiver thus breaking the \mathbb{Z}_2 invariance of the theory. This is the reason why the vev (B.1) does not exhibit the same properties as the free energy. It is possible to retain the \mathbb{Z}_2 invariance if we consider the sum and the difference, for the case of the sum we have

$$\begin{aligned}
 w_+(\tilde{\lambda}_1, \tilde{\lambda}_2) = & \left[-24\pi^2\zeta_3 \left(\tilde{\lambda}_1 + \tilde{\lambda}_2 \right) - 32\pi^4\zeta_3 \left(\tilde{\lambda}_1^2 + \tilde{\lambda}_1\tilde{\lambda}_2 + \tilde{\lambda}_2^2 \right) \right. \\
 & - 16\pi^6\zeta_3 \left(\tilde{\lambda}_1 + \tilde{\lambda}_2 \right) \left(\tilde{\lambda}_1^2 + \tilde{\lambda}_2^2 \right) - \frac{64}{15}\pi^8\zeta_3 \left(\tilde{\lambda}_1^4 + \tilde{\lambda}_1^3\tilde{\lambda}_2 + \tilde{\lambda}_1^2\tilde{\lambda}_2^2 + \tilde{\lambda}_1\tilde{\lambda}_2^3 + \tilde{\lambda}_2^4 \right) \\
 & - \frac{32}{45}\pi^{10}\zeta_3 \left(\tilde{\lambda}_1 + \tilde{\lambda}_2 \right) \left(\tilde{\lambda}_1^4 + \tilde{\lambda}_1^2\tilde{\lambda}_2^2 + \tilde{\lambda}_2^4 \right) + 80\pi^2\zeta_5 \left(3\tilde{\lambda}_1^2 + 4\tilde{\lambda}_1\tilde{\lambda}_2 + 3\tilde{\lambda}_2^2 \right) \\
 & + \frac{80}{3}\pi^4\zeta_5 \left(\tilde{\lambda}_1 + \tilde{\lambda}_2 \right) \left(13\tilde{\lambda}_1^2 + 4\tilde{\lambda}_1\tilde{\lambda}_2 + 13\tilde{\lambda}_2^2 \right) \\
 & + \frac{32}{3}\pi^6\zeta_5 \left(17\tilde{\lambda}_1^4 + 22\tilde{\lambda}_1^3\tilde{\lambda}_2 + 22\tilde{\lambda}_1^2\tilde{\lambda}_2^2 + 22\tilde{\lambda}_1\tilde{\lambda}_2^3 + 17\tilde{\lambda}_2^4 \right) \\
 & + \frac{64}{9}\pi^8\zeta_5 \left(\tilde{\lambda}_1 + \tilde{\lambda}_2 \right) \left(7\tilde{\lambda}_1^4 + 2\tilde{\lambda}_1^3\tilde{\lambda}_2 + 7\tilde{\lambda}_1^2\tilde{\lambda}_2^2 + 2\tilde{\lambda}_1\tilde{\lambda}_2^3 + 7\tilde{\lambda}_2^4 \right) \\
 & - 280\pi^2\zeta_7 \left(\tilde{\lambda}_1 + \tilde{\lambda}_2 \right) \left(8\tilde{\lambda}_1^2 + 5\tilde{\lambda}_1\tilde{\lambda}_2 + 8\tilde{\lambda}_2^2 \right) \\
 & - \frac{112}{3}\pi^4\zeta_7 \left(91\tilde{\lambda}_1^4 + 146\tilde{\lambda}_1^3\tilde{\lambda}_2 + 156\tilde{\lambda}_1^2\tilde{\lambda}_2^2 + 146\tilde{\lambda}_1\tilde{\lambda}_2^3 + 91\tilde{\lambda}_2^4 \right) \\
 & - \frac{112}{3}\pi^6\zeta_7 \left(\tilde{\lambda}_1 + \tilde{\lambda}_2 \right) \left(49\tilde{\lambda}_1^4 + 29\tilde{\lambda}_1^3\tilde{\lambda}_2 + 54\tilde{\lambda}_1^2\tilde{\lambda}_2^2 + 29\tilde{\lambda}_1\tilde{\lambda}_2^3 + 49\tilde{\lambda}_2^4 \right) \\
 & + 336\pi^2\zeta_9 \left(65\tilde{\lambda}_1^4 + 118\tilde{\lambda}_1^3\tilde{\lambda}_2 + 138\tilde{\lambda}_1^2\tilde{\lambda}_2^2 + 118\tilde{\lambda}_1\tilde{\lambda}_2^3 + 65\tilde{\lambda}_2^4 \right) \tag{B.2} \\
 & + 672\pi^4\zeta_9 \left(\tilde{\lambda}_1 + \tilde{\lambda}_2 \right) \left(51\tilde{\lambda}_1^4 + 41\tilde{\lambda}_1^3\tilde{\lambda}_2 + 68\tilde{\lambda}_1^2\tilde{\lambda}_2^2 + 41\tilde{\lambda}_1\tilde{\lambda}_2^3 + 51\tilde{\lambda}_2^4 \right) \\
 & - 3696\pi^2\zeta_{11} \left(\tilde{\lambda}_1 + \tilde{\lambda}_2 \right) \left(61\tilde{\lambda}_1^4 + 56\tilde{\lambda}_1^3\tilde{\lambda}_2 + 96\tilde{\lambda}_1^2\tilde{\lambda}_2^2 + 56\tilde{\lambda}_1\tilde{\lambda}_2^3 + 61\tilde{\lambda}_2^4 \right) \\
 & + 288\pi^2\zeta_3^2 \left(\tilde{\lambda}_1 + \tilde{\lambda}_2 \right) \left(2\tilde{\lambda}_1^2 - \tilde{\lambda}_1\tilde{\lambda}_2 + 2\tilde{\lambda}_2^2 \right) \\
 & + 192\pi^4\zeta_3^2 \left(5\tilde{\lambda}_1^4 + 2\tilde{\lambda}_1^3\tilde{\lambda}_2 + 4\tilde{\lambda}_1^2\tilde{\lambda}_2^2 + 2\tilde{\lambda}_1\tilde{\lambda}_2^3 + 5\tilde{\lambda}_2^4 \right) \\
 & + 192\pi^6\zeta_3^2 \left(\tilde{\lambda}_1 + \tilde{\lambda}_2 \right) \left(3\tilde{\lambda}_1^4 - 2\tilde{\lambda}_1^3\tilde{\lambda}_2 + 4\tilde{\lambda}_1^2\tilde{\lambda}_2^2 - 2\tilde{\lambda}_1\tilde{\lambda}_2^3 + 3\tilde{\lambda}_2^4 \right) \\
 & - 960\pi^2\zeta_3\zeta_5 \left(15\tilde{\lambda}_1^4 + 10\tilde{\lambda}_1^3\tilde{\lambda}_2 + 6\tilde{\lambda}_1^2\tilde{\lambda}_2^2 + 10\tilde{\lambda}_1\tilde{\lambda}_2^3 + 15\tilde{\lambda}_2^4 \right) \\
 & - 320\pi^4\zeta_3\zeta_5 \left(\tilde{\lambda}_1 + \tilde{\lambda}_2 \right) \left(77\tilde{\lambda}_1^4 - 32\tilde{\lambda}_1^3\tilde{\lambda}_2 + 78\tilde{\lambda}_1^2\tilde{\lambda}_2^2 - 32\tilde{\lambda}_1\tilde{\lambda}_2^3 + 77\tilde{\lambda}_2^4 \right)
 \end{aligned}$$

$$\begin{aligned}
 & + 3360\pi^2\zeta_3\zeta_7 \left(\tilde{\lambda}_1 + \tilde{\lambda}_2 \right) \left(48\tilde{\lambda}_1^4 - 7\tilde{\lambda}_1^3\tilde{\lambda}_2 + 30\tilde{\lambda}_1^2\tilde{\lambda}_2^2 - 7\tilde{\lambda}_1\tilde{\lambda}_2^3 + 48\tilde{\lambda}_2^4 \right) \\
 & + 1600\pi^2\zeta_5^2 \left(\tilde{\lambda}_1 + \tilde{\lambda}_2 \right) \left(57\tilde{\lambda}_1^4 - 8\tilde{\lambda}_1^3\tilde{\lambda}_2 + 34\tilde{\lambda}_1^2\tilde{\lambda}_2^2 - 8\tilde{\lambda}_1\tilde{\lambda}_2^3 + 57\tilde{\lambda}_2^4 \right) \\
 & - 3456\pi^2\zeta_3^3 \left(\tilde{\lambda}_1 + \tilde{\lambda}_2 \right) \left(5\tilde{\lambda}_1^4 - 5\tilde{\lambda}_1^3\tilde{\lambda}_2 + 8\tilde{\lambda}_1^2\tilde{\lambda}_2^2 - 5\tilde{\lambda}_1\tilde{\lambda}_2^3 + 5\tilde{\lambda}_2^4 \right) \Big].
 \end{aligned}$$

For the case of the difference we have

$$\begin{aligned}
 w_-(\tilde{\lambda}_1, \tilde{\lambda}_2) = & \left[-24\pi^2\zeta_3 \left(\tilde{\lambda}_1^2 + \tilde{\lambda}_2^2 \right) - 32\pi^4\zeta_3 \left(\tilde{\lambda}_1^3 + \tilde{\lambda}_2^3 \right) - 16\pi^6\zeta_3 \left(\tilde{\lambda}_1^4 + \tilde{\lambda}_2^4 \right) \right. \\
 & - \frac{64}{15}\pi^8\zeta_3 \left(\tilde{\lambda}_1^5 + \tilde{\lambda}_2^5 \right) - \frac{32}{45}\pi^{10}\zeta_3 \left(\tilde{\lambda}_1^6 + \tilde{\lambda}_2^6 \right) \\
 & + 80\pi^2\zeta_5 \left(\tilde{\lambda}_1 + \tilde{\lambda}_2 \right) \left(3\tilde{\lambda}_1^2 - 2\tilde{\lambda}_1\tilde{\lambda}_2 + 3\tilde{\lambda}_2^2 \right) \\
 & + \frac{80}{3}\pi^4\zeta_5 \left(13\tilde{\lambda}_1^4 + 4\tilde{\lambda}_1^3\tilde{\lambda}_2 + 4\tilde{\lambda}_1\tilde{\lambda}_2^3 + 13\tilde{\lambda}_2^4 \right) \\
 & + \frac{32}{3}\pi^6\zeta_5 \left(\tilde{\lambda}_1 + \tilde{\lambda}_2 \right) \left(17\tilde{\lambda}_1^4 - 12\tilde{\lambda}_1^3\tilde{\lambda}_2 + 12\tilde{\lambda}_1^2\tilde{\lambda}_2^2 - 12\tilde{\lambda}_1\tilde{\lambda}_2^3 + 17\tilde{\lambda}_2^4 \right) \\
 & + \frac{64}{9}\pi^8\zeta_5 \left(7\tilde{\lambda}_1^6 + 2\tilde{\lambda}_1^5\tilde{\lambda}_2 + 2\tilde{\lambda}_1\tilde{\lambda}_2^5 + 7\tilde{\lambda}_2^6 \right) \\
 & - 280\pi^2\zeta_7 \left(8\tilde{\lambda}_1^4 + 5\tilde{\lambda}_1^3\tilde{\lambda}_2 + 2\tilde{\lambda}_1^2\tilde{\lambda}_2^2 + 5\tilde{\lambda}_1\tilde{\lambda}_2^3 + 8\tilde{\lambda}_2^4 \right) \\
 & - \frac{112}{3}\pi^4 \left(\tilde{\lambda}_1 + \tilde{\lambda}_2 \right) \left(91\tilde{\lambda}_1^4 - 36\tilde{\lambda}_1^3\tilde{\lambda}_2 + 46\tilde{\lambda}_1^2\tilde{\lambda}_2^2 - 36\tilde{\lambda}_1\tilde{\lambda}_2^3 + 91\tilde{\lambda}_2^4 \right) \\
 & - \frac{112}{3}\pi^6\zeta_7 \left(49\tilde{\lambda}_1^6 + 29\tilde{\lambda}_1^5\tilde{\lambda}_2 + 5\tilde{\lambda}_1^4\tilde{\lambda}_2^2 + 5\tilde{\lambda}_1^2\tilde{\lambda}_2^4 + 29\tilde{\lambda}_1\tilde{\lambda}_2^5 + 49\tilde{\lambda}_2^6 \right) \\
 & + 336\pi^2\zeta_9 \left(\tilde{\lambda}_1 + \tilde{\lambda}_2 \right) \left(65\tilde{\lambda}_1^4 - 12\tilde{\lambda}_1^3\tilde{\lambda}_2 + 38\tilde{\lambda}_1^2\tilde{\lambda}_2^2 - 12\tilde{\lambda}_1\tilde{\lambda}_2^3 + 65\tilde{\lambda}_2^4 \right) \quad (B.3) \\
 & + 672\pi^4\zeta_9 \left(51\tilde{\lambda}_1^6 + 41\tilde{\lambda}_1^5\tilde{\lambda}_2 + 17\tilde{\lambda}_1^4\tilde{\lambda}_2^2 + 4\tilde{\lambda}_1^3\tilde{\lambda}_2^3 + 17\tilde{\lambda}_1^2\tilde{\lambda}_2^4 + 41\tilde{\lambda}_1\tilde{\lambda}_2^5 + 51\tilde{\lambda}_2^6 \right) \\
 & - 3696\pi^2\zeta_{11} \left(61\tilde{\lambda}_1^6 + 56\tilde{\lambda}_1^5\tilde{\lambda}_2 + 37\tilde{\lambda}_1^4\tilde{\lambda}_2^2 + 22\tilde{\lambda}_1^3\tilde{\lambda}_2^3 + 37\tilde{\lambda}_1^2\tilde{\lambda}_2^4 + 56\tilde{\lambda}_1\tilde{\lambda}_2^5 + 61\tilde{\lambda}_2^6 \right) \\
 & + 288\pi^2\zeta_3^2 \left(2\tilde{\lambda}_1^4 - \tilde{\lambda}_1^3\tilde{\lambda}_2 + 2\tilde{\lambda}_1^2\tilde{\lambda}_2^2 - \tilde{\lambda}_1\tilde{\lambda}_2^3 + 2\tilde{\lambda}_2^4 \right) \\
 & + 192\pi^4\zeta_3^2 \left(\tilde{\lambda}_1 + \tilde{\lambda}_2 \right) \left(\tilde{\lambda}_1^2 + \tilde{\lambda}_2^2 \right) \left(5\tilde{\lambda}_1^2 - 8\tilde{\lambda}_1\tilde{\lambda}_2 + 5\tilde{\lambda}_2^2 \right) \\
 & + 192\pi^6\zeta_3^2 \left(3\tilde{\lambda}_1^6 - 2\tilde{\lambda}_1^5\tilde{\lambda}_2 + \tilde{\lambda}_1^4\tilde{\lambda}_2^2 + \tilde{\lambda}_1^2\tilde{\lambda}_2^4 - 2\tilde{\lambda}_1\tilde{\lambda}_2^5 + 3\tilde{\lambda}_2^6 \right) \\
 & - 960\pi^2\zeta_3\zeta_5 \left(\tilde{\lambda}_1 + \tilde{\lambda}_2 \right) \left(15\tilde{\lambda}_1^4 - 20\tilde{\lambda}_1^3\tilde{\lambda}_2 + 26\tilde{\lambda}_1^2\tilde{\lambda}_2^2 - 20\tilde{\lambda}_1\tilde{\lambda}_2^3 + 15\tilde{\lambda}_2^4 \right) \\
 & - 320\pi^4\zeta_3\zeta_5 \left(77\tilde{\lambda}_1^6 - 32\tilde{\lambda}_1^5\tilde{\lambda}_2 + \tilde{\lambda}_1^4\tilde{\lambda}_2^2 + 40\tilde{\lambda}_1^3\tilde{\lambda}_2^3 + \tilde{\lambda}_1^2\tilde{\lambda}_2^4 - 32\tilde{\lambda}_1\tilde{\lambda}_2^5 + 77\tilde{\lambda}_2^6 \right) \\
 & + 3360\pi^2\zeta_3\zeta_7 \left(48\tilde{\lambda}_1^6 - 7\tilde{\lambda}_1^5\tilde{\lambda}_2 + 4\tilde{\lambda}_1^4\tilde{\lambda}_2^2 + 22\tilde{\lambda}_1^3\tilde{\lambda}_2^3 + 4\tilde{\lambda}_1^2\tilde{\lambda}_2^4 - 7\tilde{\lambda}_1\tilde{\lambda}_2^5 + 48\tilde{\lambda}_2^6 \right) \\
 & + 1600\pi^2\zeta_5^2 \left(57\tilde{\lambda}_1^6 - 8\tilde{\lambda}_1^5\tilde{\lambda}_2 + 3\tilde{\lambda}_1^4\tilde{\lambda}_2^2 + 28\tilde{\lambda}_1^3\tilde{\lambda}_2^3 + 3\tilde{\lambda}_1^2\tilde{\lambda}_2^4 - 8\tilde{\lambda}_1\tilde{\lambda}_2^5 + 57\tilde{\lambda}_2^6 \right) \\
 & \left. - 3456\pi^2\zeta_3^3 \left(5\tilde{\lambda}_1^6 - 5\tilde{\lambda}_1^5\tilde{\lambda}_2 + 7\tilde{\lambda}_1^4\tilde{\lambda}_2^2 - 6\tilde{\lambda}_1^3\tilde{\lambda}_2^3 + 7\tilde{\lambda}_1^2\tilde{\lambda}_2^4 - 5\tilde{\lambda}_1\tilde{\lambda}_2^5 + 5\tilde{\lambda}_2^6 \right) \right].
 \end{aligned}$$

The series $w_{\pm}(\tilde{\lambda}_1, \tilde{\lambda}_2)$ are symmetric. At the considered orders, the polynomials that appear also have all unimodular roots.

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