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Infrared structure of SU(N) × U(1) gauge theory to three loops

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Abstract: We study the infrared (IR) structure of SU(N) × U(1) (QCD × QED) gauge theory with n_f quarks and n_l leptons within the framework of perturbation theory. In particular, we unravel the IR structure of the form factors and inclusive real emission cross sections that contribute to inclusive production of color neutral states, such as a pair of leptons or single W/Z in Drell-Yan processes and a Higgs boson in bottom quark annihilation, in Large Hadron Collider (LHC) in the threshold limit. Explicit computation of the relevant form factors to third order and the use of Sudakov's $K + G$ equation in SU(N)×U(1) gauge theory demonstrate the universality of the cusp anomalous dimensions ($A_I, I = q, b$). The abelianization rules that relate A_I of SU(N) with those from U(1) and SU(N) × U(1) can be used to predict the soft distribution that results from the soft gluon emission subprocesses in the threshold limit. Using the latter and the third order form factors, we can obtain the collinear anomalous dimensions (B_I) and the renormalisation constant Z_b to third order in perturbation theory. The form factors, the process independent soft distribution functions can be used to predict fixed and resummed inclusive cross sections to third order in couplings and in leading logarithmic approximation respectively.

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1 Introduction

Precision studies at the Large Hadron Collider (LHC) is one of the thrust areas in particle physics. LHC has enormous potential [1] to unravel the details of the standard model (SM) and also to discover new physics. This is possible due to large center of mass energy and high design luminosity options [2]. The discovery of the Higgs boson [3, 4] at the LHC and the on-going studies to understand the nature of the Higgs bosons and its couplings to the other SM particles provide opportunities to probe the SM in great detail [5]. In addition, both ATLAS and CMS collaborations have been engaged in plenty of high precision measurements of variety of observables in the scattering reactions. These include measurement of production cross sections of lepton pairs and vector bosons in Drell-Yan process, pair of top quarks, Higgs bosons, jets etc. These can be used to constrain parameters of beyond the SM (BSM) as well. In general, theoretical predictions based on precise computations within the SM scenarios can be compared against the data from various observables to look for tiny deviations that could hint for BSM. For example, the mass of the W boson at the lowest order in electroweak theory can be predicted in terms of mass of the Z boson, fine structure and Fermi constants. Radiative corrections from the SM alter the predictions and are sensitive to parameters of the SM as well as heavy states from BSM. While leading order process is often electroweak, radiative corrections from quantum chromodynamics (QCD) dominate over those from the electroweak (EW) theory and they improve the predictions significantly. Efforts are also underway to include higher order corrections from EW theory. In addition, the predictions are found to be sensitive to parton distributions of down type quarks which leave large theoretical uncertainties. Hence, the on-going precise measurements of W boson mass by ATLAS and CMS through Drell-Yan process are absolutely necessary to confirm the consistency of the SM and also to set constraints on the parameters of the BSMs. Similarly, understanding the physics of top quarks provide opportunities to probe new physics. Observables related to top quark, being the heaviest particle in the SM, are expected to be sensitive to new physics. Hence, dedicated studies on the production cross sections of top quarks have been topic of interest ever since it was discovered at the Tevtaron. For the top quarks, since the leading order process is through strong interaction, there is a large theoretical uncertainty and hence higher order QCD corrections have been consistently included to stabilize the theoretical predictions.

LHC being the hadron machine, QCD plays an important role in all these studies. Often, the leading order predictions suffer large theoretical uncertaintities and hence, we witnessed plethora of works that include higher order QCD corrections to most of the observables that can be measured at the LHC. Needless to say that the state-of-the art computations not only provided most precise theoretical predictions for the observables at the collider experiments but also generated lot of interest among theorists to study the universal structure of the perturbative series at high energies. Computations of the multi loop and multi leg scattering amplitudes and cross sections in QCD provide laboratory to unravel the underlying infrared dynamics in terms of universal anomalous dimensions.

The precision in the predictions from the dominant QCD corrections has reached the level that requires inclusion of corrections from the electroweak sector as well. The fact that the square of the strong coupling constant (α_s^2) is comparable to fine structure constant α necessitates the inclusion of effects from quantum electrodynamics (QED). In addition, electroweak logarithms in the Sudakov regions need to be included for a consistent prediction at the LHC. In [6–8], predictions for the di-jet productions are improved by including electroweak corrections. One finds that the third order QCD effects for the inclusive production rate for the Higgs bosons at the LHC are comparable to those from the electroweak sector. Also, EW corrections play an important role in the W mass measurements through DY process. While there are already several important works in this direction, there is a surge of efforts now towards estimating these corrections for the scattering processes at the LHC. Note that the next to leading order electro weak corrections to Drell-Yan process was computed in [9–13]. Similarly, for the Higgs boson production, the dominant two loop effects from EW sector [14] plays important role in the theoretical predictions.

Unlike QCD, electroweak sector contains several heavy particles which can make the computations technically challenging. The loop integrals as well as the phase space integrals involve massive particles making them hard to evaluate. However, the subset of radiative corrections from QED resemble those of QCD if lepton masses are set equal to zero, an approximation valid at high energies where the quarks are treated light in most of the perturbative QCD computations.

At hadron colliders, EW corrections affect the evolution of parton distribution functions as well as parton level cross sections. In [15, 16], $\mathcal{O}(\alpha_s\alpha)$ as well as $\mathcal{O}(\alpha^2)$ corrections to the splitting function that govern the evolution of PDFs were obtained using the algorithm called abelianization which is incorporated in the determination of precise photon distributions in the proton within the LUXqed approach [17–19]. For the Drell-Yan process, there have been continuous effort to obtain NNLO EW corrections as NLO EW corrections for both charged [9–11] as well as neutral [12, 13] currents are known for a while. For example, works towards NNLO EW corrections can be found in [20–22]. Mixed QCD and EW/QED effects are known in the pole approximation [23, 24]. In [25, 26], master integrals for double real as well as two virtual corrections relevant for two loop QCD-EW were computed to obtain predictions for W production at NNLO level in QCD-EW couplings. In [27], pure QED as well as QCD$\times$QED corrections at $\mathcal{O}(\alpha^2)$ and $\mathcal{O}(\alpha_s\alpha)$ respectively for Drell-Yan process were obtained using abelianization to study the phenomenological importance of QED effects at LHC energies. For earlier work on this can be found in [28]. NNLO QED as well as QCD$\times$QED corrections to Higgs production in bottom quark annihilation were obtained [29] to estimate the impact of QED radiative corrections. More recently, NNLO EW-QCD effects to single vector boson production were reported in [30, 31].

Thanks to a large number of perturbative results available in SU(N) gauge theory, its IR structure is well understood in terms of universal anomalous dimensions (see [32–38]). Recent results which point to the relevance of electroweak corrections to the LHC observables provide the opportunity to understand the underlying infrared structure of U(1) gauge theory with massless fermions. The infrared structure of amplitudes with mixed gauge groups at two loops was reported in [39]. In [15, 16, 27], abelianizations provide a useful tool to obtain NLO QED effects for the splitting functions of parton distribution functions and NNLO QED corrections to inclusive cross section for the Drell-Yan process. One finds that the abelianization can be used to relate ultraviolate and infrared anomalous dimensions of QCD with those of QED. In [29], explicit computation of form factors of vector and scalar operators in QED and QCD$\times$QED set up to second order in perturbation theory demonstrates the usefulness of abelianization. In addition, the results on inclusive cross sections for DY process, such as di-lepton or W/Z productions and also for Higgs production in bottom quark annihilation support this procedure of abelianization up to NNLO level in QED as well as in QCD$\times$QED. Hence, it is tempting to apply the abelianization to obtain results beyond two loops and also beyond NNLO level for QED and QCD$\times$QED for

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di-lepton or W/Z production in light quark annihilation and for the Higgs boson production in bottom quark annihilation. In this paper, we perform this exercise at three loop level in QED and QCD×QED to find out the scope of abelianization. In addition, the explicit computations at three loop level provide valuable informations on the universality of cusp, collinear and soft anomalous dimensions up to third order in couplings both in QED as well as QCD×QED. We use Sudakov's K plus G (K+G) equation to study the infrared structure of the three loop form factors and the validity of abelianization. We derive the third order corrections in QED and QCD×QED to soft distribution function resulting from those parton level subprocesses where at least one real soft gluon is present. Using these form factors and the soft distribution functions and exploiting the universal property of the inclusive Drell-Yan production, we obtain the infrared safe parton level soft plus virtual contributions to third in QED and QCD×QED. We also derive the resummed threshold enhanced contribution the inclusive DY production up to next to next to next to leading logarithmic $\mathrm{N^3LL}$ approximation. In the following, instead of restricting to $\mathrm{SU}(3) \times \mathrm{U}(1)$, we study $\mathrm{SU}(N) \times \mathrm{U}(1)$ gauge theory where it is transperent to understand abelianisation relations between $\mathrm{SU}(N)$ and $\mathrm{U}(1)$ gauge theories. Setting $N = 3$, one can easily obtain the corresponding results in QCD × QED gauge theory. In addition, the IR structure of the former goes through for QCD × QED straightforwardly. Hence, in the rest of the paper we use QCD × QED interchangebly with $\mathrm{SU}(N) \times \mathrm{U}(1)$ without loss of generality.

2 Theoretical framework

We work with the gauge theory which is invariant under the gauge group $\mathrm{SU}(N) \times \mathrm{U}(1)$. The gauge group $\mathrm{SU}(N)$ corresponds to QCD which describes the strong interaction while $\mathrm{U}(1)$ (QED) describes the electromagnetic interaction. The Lagrangian of $\mathrm{SU}(N) \times \mathrm{U}(1)$ is given as,

$$\mathcal{L} = \bar{\psi}^i \Big(i\gamma_\mu D^\mu_{ij} - m\delta_{ij} \Big) \psi^j - \frac{1}{4} \mathcal{G}^a_{\mu\nu} \mathcal{G}^{a\mu\nu} - \frac{1}{4} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} - \frac{1}{2\xi} \Big(\partial^\mu G^a_\mu \Big)^2 . \tag{2.1}$$

Here ξ is the gauge fixing parameter and ψ^k denotes the fermionic field in the fundamental representation of the $\mathrm{SU}(N)$ group with $k = 1, \cdots, N$. The covariant derivative $D^\mu_{ij} = \partial^\mu \delta_{ij} - i g_s \big(T^c\big)_{ij} G^\mu_c - ieA^\mu \delta_{ij}$. The gluonic and photonic field strength tensors are given respectively as,

$$\begin{aligned} \mathcal{G}^a_{\mu\nu} &= \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + i g_s f^{abc} G^b_\mu G^c_\nu \,, \\ \mathcal{F}_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \,, \end{aligned}$$

where the gluon gauge fields G^a_μ with $a = 1, \cdots, N^2 - 1$ and the photon gauge fields A_μ belong to the adjoint representation. We use the standard perturbation theory to compute various quantities in this theory in powers of coupling constants defined by $a_s = g_s^2/16\pi^2$ and $a_e = e^2/16\pi^2$ where g_s and e are strong and electromagnetic coupling constants respectively. Since we are interested in quantities in the high energy limit, both the quarks and leptons are treated massless throughout. We use dimensional regularisation to perform higher order computations and $\overline{MS}$ to renormalise the fields and the couplings in this theory. In dimensional regularisation the space time dimension is taken to be $d = 4 + \varepsilon$. The field as well as coupling constant renormalisation constants contain poles in ε in the vicinity of $d = 4$ space time dimensions due to ultraviolet (UV) divergences. Higher order radiative corrections are often sensitive to soft divergences due to massless gluons of SU(N) and massless photons of U(1) and also to collinear divergences due to the presence of (almost) massless quarks and leptons. These are called infrared (IR) divergences and they also show up as poles in ε in dimensional regularisation.

We begin with the renormalisation of the coupling constants when both the interactions are simultaneously present. Let us denote the renormalisation constant $Z_{a_c}, c = s, e$ for the QCD and QED coupling constants respectively. Then the unrenormalised coupling constants $\hat{a}_c, c = s, e$ will be related to the renormalised ones through Z_{a_c} as

$$\frac{\hat{a}_c}{(\mu^2)^{\frac{\varepsilon}{2}}} S_\varepsilon = \frac{a_c(\mu_R^2)}{(\mu_R^2)^{\frac{\varepsilon}{2}}} \, Z_{a_c} \left(a_s(\mu_R^2), a_e(\mu_R^2), \varepsilon\right) \,, \tag{2.2}$$

where $a_c = \{a_s, a_e\}$. Here, $S_\varepsilon \equiv \exp[(\gamma_E - \ln 4\pi) \frac{\varepsilon}{2}]$ is the phase-space factor in d-dimensions, $\gamma_E = 0.5772\ldots$ is the Euler-Mascheroni constant and μ is an arbitrary mass scale introduced to make $\hat{a}_s$ and $\hat{a}_e$ dimensionless in d-dimensions and μ_R is the renormalisation scale. The fact that bare coupling constants $\hat{a}_c$ is independent of the renormalisation scale μ_R results in renormalisation group equations for the couplings $a_c(\mu_R^2)$:

$$\mu_R^2 \frac{d}{d\mu_R^2} \ln Z_{a_c} = \frac{\varepsilon}{2} + \beta_{a_c}(a_s(\mu_R^2), a_e(\mu_R^2)) \,. \tag{2.3}$$

In the perturbation theory with both the interactions active, the beta functions β_{a_c} can be expanded in powers of a_s as well as a_e:

$$\beta_{a_s} = - \sum_{i,j=0}^{\infty} \beta_{ij} a_s^{i+2} a_e^j \,, \qquad \beta_{a_e} = - \sum_{i,j=0}^{\infty} \beta'_{ij} a_e^{j+2} a_s^i \,. \tag{2.4}$$

The explicit calculation of *beta*-function begins with renormalising the Lagrangian. This involves renormalising fields, the Lagrangian. This involves renormalising fields, couplings, masses and gauge-fixing parameter. Hence we redefine the fields as,

$$\psi = Z_2^{1/2} \psi_r, \quad G_\mu^a = Z_3^{1/2} G_{\mu r}^a, \quad A_\mu = Z_{3\gamma}^{1/2} A_{\mu r} \,, \tag{2.5}$$

and the parameters as,

$$g_s = Z_g g_r, \quad e = Z_e e_r, \quad m = Z_m m_r, \quad \xi = Z_3 \xi_r \,, \tag{2.6}$$

where the constants $Z_2, Z_3, Z_{3\gamma}$ are called the fermion-field, gluon-field and photon-field renormalisation constants while the constants Z_g, Z_e and Z_m are called the coupling-constant and mass renormalisation constant respectively. All the renormalized fields, masses and parameters are designated with an subscript r. Inserting all of these into the Lagrangian and collecting all the terms involving δZ, where $\delta Z = Z - 1$ for any renormalisation constants, we get the counterterm Lagrangian. Now let us look at the counterterm for the four-gluon vertex to compute the renormalisation constant Z_{a_s}. The leading order starts at a_s^2 and hence the QED contributions to Z_{a_s} will always be proportional to a_s^2. Similarly for the QED β-function, the counterterm for the fermion-fermion-photon vertex is,

$$\sim e_r \bar{\psi}_r \gamma_\mu A_r^\mu \psi_r (Z_2 Z_{3\gamma}^{1/2} Z_e - 1) \equiv e_r \bar{\psi}_r \gamma_\mu A_r^\mu \psi_r (Z_{1e} - 1), \tag{2.7}$$

which implies

$$Z_e = Z_{1e}/(Z_2 \sqrt{Z_{3\gamma}}). \tag{2.8}$$

The Ward identity, which is derived using the conservation of the electromagnetic current, demands that $Z_{1e} = Z_2$ to all orders in perturbation theory. This in turn suggests that Z_{a_e} is fully determined by $Z_{3\gamma}$ which starts at order a_e and hence the QCD corrections to Z_{a_e} will always be proportional to a_e. Substituting eq. (2.4) in eq. (2.3) and solving for the renormalisation constants Z_{a_c} up to third order, we obtain

$$\begin{aligned} Z_{a_s} &= 1 + a_s \left(\frac{2\beta_{00}}{\varepsilon}\right) + a_s a_e \left(\frac{\beta_{01}}{\varepsilon}\right) + a_s a_e^2 \left(\frac{2\beta'_{00}\beta_{01}}{3\varepsilon^2} + \frac{2\beta_{02}}{3\varepsilon}\right) + a_s^2 \left(\frac{4\beta_{00}^2}{\varepsilon^2} + \frac{\beta_{10}}{\varepsilon}\right) \\ &\quad + a_s^2 a_e \left(\frac{4\beta_{00}\beta_{01}}{\varepsilon^2} + \frac{2\beta_{11}}{3\varepsilon}\right) + a_s^3 \left(\frac{8\beta_{00}^3}{\varepsilon^3} + \frac{14\beta_{00}\beta_{10}}{3\varepsilon^2} + \frac{2\beta_{20}}{3\varepsilon}\right) + \cdots, \\ Z_{a_e} &= 1 + a_e \left(\frac{2\beta'_{00}}{\varepsilon}\right) + a_e a_s \left(\frac{\beta'_{10}}{\varepsilon}\right) + a_e a_s^2 \left(\frac{2\beta_{00}\beta'_{10}}{3\varepsilon^2} + \frac{2\beta'_{20}}{3\varepsilon}\right) + a_e^2 \left(\frac{4\beta'^2_{00}}{\varepsilon^2} + \frac{\beta'_{01}}{\varepsilon}\right) \\ &\quad + a_e^2 a_s \left(\frac{4\beta'_{00}\beta'_{10}}{\varepsilon^2} + \frac{2\beta'_{11}}{3\varepsilon}\right) + a_e^3 \left(\frac{8\beta'^3_{00}}{\varepsilon^3} + \frac{14\beta'_{00}\beta'_{01}}{3\varepsilon^2} + \frac{2\beta'_{02}}{3\varepsilon}\right) + \cdots. \end{aligned} \tag{2.9}$$

We have used the symbol $\cdots$ to denote the missing higher order terms of the order $a_s^i a_e^j, i + j > 3$ throughout. While, these constants are sufficient to obtain UV finite observables, the UV divergences resulting from composite operators in the theory beyond leading order require additional overall renormalisation constants. These constants are expanded in power series expansions of both a_s as well as a_e. Similarly, if the fields of QCD and QED couple to external fields, then the corresponding couplings are renormalised by separate renormalisation constants. One such example that we need to study in the present paper is Yukawa coupling that describes the coupling of a Higgs boson with the bottom quarks in this theory. If we denote the bare Yukawa coupling by $\hat{\lambda}_b$, then the corresponding renormalisation constant Z_λ^b relates this to the renormalised one λ_b by

$$\frac{\hat{\lambda}_b}{(\mu^2)^{\frac{\varepsilon}{2}}} S_\varepsilon = \frac{\lambda_b(\mu_R^2)}{(\mu_R^2)^{\frac{\varepsilon}{2}}} \, Z_\lambda^b \left(a_s(\mu_R^2), a_e(\mu_R^2), \varepsilon\right) , \tag{2.10}$$

where $a_c = \{a_s, a_e\}$. The renormalisation constant $Z_\lambda^b(a_s, a_e)$ satisfies the renormalisation group equation:

$$\mu_R^2 \frac{d}{d\mu_R^2} \ln Z_\lambda^b = \frac{\varepsilon}{4} + \gamma_b(a_s(\mu_R^2), a_e(\mu_R^2)) , \tag{2.11}$$

whose solution in terms of the anomalous dimensions $\gamma_b^{(i,j)}$ and β_{ij}, β'_{ij} up to three loops is found to be

$$\begin{aligned}
Z_\lambda^b(a_s,a_e,\varepsilon) &= 1 + a_s\left\{\frac{1}{\varepsilon}\left(2\gamma_b^{(1,0)}\right)\right\} + a_e\left\{\frac{1}{\varepsilon}\left(2\gamma_b^{(0,1)}\right)\right\} + a_s^2\left\{\frac{1}{\varepsilon^2}\left(2\left(\gamma_b^{(1,0)}\right)^2 + 2\beta_{00}\gamma_b^{(1,0)}\right)\right.\\
&\quad\left. + \frac{1}{\varepsilon}\gamma_b^{(2,0)}\right\} + a_e^2\left\{\frac{1}{\varepsilon^2}\left(2\left(\gamma_b^{(0,1)}\right)^2 + 2\beta'_{00}\gamma_b^{(0,1)}\right) + \frac{1}{\varepsilon}\gamma_b^{(0,2)}\right\} + a_s a_e\left\{\frac{1}{\varepsilon^2}\left(4\gamma_b^{(1,0)}\gamma_b^{(0,1)}\right)\right.\\
&\quad\left. + \frac{1}{\varepsilon}\left(\gamma_b^{(1,1)}\right)\right\} + a_s^3\left\{\frac{1}{\varepsilon^3}\left(\frac{4}{3}\left(\gamma_b^{(1,0)}\right)^3 + 4\beta_{00}\left(\gamma_b^{(1,0)}\right)^2 + \frac{8}{3}\beta_{00}^2\left(\gamma_b^{(1,0)}\right)\right)\right.\\
&\quad\left. + \frac{1}{\varepsilon^2}\left(2\gamma_b^{(1,0)}\gamma_b^{(2,0)} + \frac{4}{3}\beta_{10}\gamma_b^{(1,0)} + \frac{4}{3}\beta_{00}\gamma_b^{(2,0)}\right) + \frac{1}{\varepsilon}\left(\frac{2}{3}\gamma_b^{(3,0)}\right)\right\} + a_e^3\left\{\frac{1}{\varepsilon^3}\left(\frac{4}{3}\left(\gamma_b^{(0,1)}\right)^3\right.\right.\\
&\quad\left.\left. + 4\beta'_{00}\left(\gamma_b^{(0,1)}\right)^2 + \frac{8}{3}\beta'^2_{00}\left(\gamma_b^{(0,1)}\right)\right) + \frac{1}{\varepsilon^2}\left(2\gamma_b^{(0,1)}\gamma_b^{(0,2)} + \frac{4}{3}\beta'_{01}\gamma_b^{(0,1)} + \frac{4}{3}\beta'_{00}\gamma_b^{(0,2)}\right)\right.\\
&\quad\left. + \frac{1}{\varepsilon}\left(\frac{2}{3}\gamma_b^{(0,3)}\right)\right\} + a_s a_e^2\left\{\frac{1}{\varepsilon^3}\left(4\gamma_b^{(1,0)}\left(\gamma_b^{(0,1)}\right)^2 + 4\beta'_{00}\gamma_b^{(1,0)}\gamma_b^{(0,1)}\right)\right.\\
&\quad\left. + \frac{1}{\varepsilon^2}\left(2\gamma_b^{(0,1)}\gamma_b^{(1,1)} + 2\gamma_b^{(1,0)}\gamma_b^{(0,2)} + \frac{4}{3}\beta'_{10}\gamma_b^{(0,1)} + \frac{2}{3}\beta'_{00}\gamma_b^{(1,1)}\right) + \frac{1}{\varepsilon}\left(\frac{2}{3}\gamma_b^{(1,2)}\right)\right\}\\
&\quad + a_s^2 a_e\left\{\frac{1}{\varepsilon^3}\left(4\left(\gamma_b^{(1,0)}\right)^2\gamma_b^{(0,1)} + 4\beta_{00}\gamma_b^{(1,0)}\gamma_b^{(0,1)}\right) + \frac{1}{\varepsilon^2}\left(2\gamma_b^{(0,1)}\gamma_b^{(2,0)} + 2\gamma_b^{(1,0)}\gamma_b^{(1,1)}\right.\right.\\
&\quad\left.\left. + \frac{4}{3}\beta_{01}\gamma_b^{(1,0)} + \frac{2}{3}\beta_{00}\gamma_b^{(1,1)}\right) + \frac{1}{\varepsilon}\left(\frac{2}{3}\gamma_b^{(2,1)}\right)\right\} + \cdots. \qquad (2.12)
\end{aligned}$$

Note that while the UV singularities factorize through Z_λ^b, singularities from QCD and QED mix from two loops onwards.

Having expanded the renormalisation constants of a_s, a_e and λ_b in powers of a_s and a_e, our next task is to determine the constants β_{ij}, β'_{ij} that appear in $Z_{a_c}, c = s, e$ as well as $\gamma_b^{(i,j)}$ up to three loops in QED and QCD×QED. The text book approach to this is to compute relevant loop corrections to the truncated n-point off-shell Green's functions of the fermions and gauge fields in the regularised theory. Alternatively [40, 41], one can determine them by studying the on-shell form factors of certain local/composite operators in the theory. For example, computing the form factors of vector as well as scalar operators made up of fermionic fields up to three loops in QCD×QED, and exploiting their universal infrared structure using Sudakov's K+G equation, we demonstrate how we can obtain most of these constants to the desired accuracy. In the process, we confirm some of the results for these constants which are already known in the literature. For QCD, β_{i0} and $\gamma_b^{(i,0)}$ are known to five loops [42–45]. In the following, we elaborate on how we determine them and also discuss the reliability of abelianization at three loop level.

3 Sudakov formalism

3.1 Form factors

In quantum field theory, form factor (FF) of a composite operator is defined by its matrix element between on-shell states. Given a composite operator $\mathcal{O}(x)$, the form factor in the

Fourier space is found to be

$$F_{\mathcal{O}}(q^2)(2\pi)^4\delta^{(4)}(q+p_1-p_2) = \int d^4y e^{iq\cdot y}\langle p_2|\mathcal{O}(y)|p_1\rangle\,. \tag{3.1}$$

We restrict ourselves to two composite operators namely

$$\mathcal{O}^\mu(y) = \overline{\psi}(y)\gamma^\mu\psi(y), \qquad \mathcal{O}(y) = \overline{\psi}(y)\psi(y)\,. \tag{3.2}$$

The corresponding form factors are denoted by F_q and F_b respectively. F_q contributes to di-lepton or W/Z production and F_b contributes to production of Higgs bosons in bottom quark annihilation. In the eq. (3.1), ψ is the fermion field and the states $|p_i\rangle, i = 1,2$ are on-shell fermionic states with momenta p_i. We begin with the bare form factors $\hat{F}_I(Q^2,\mu^2,\varepsilon)$ where $I = q,b$ and the invariant scale is defined by $Q^2 = (p_1-p_2)^2$. They are calculable in perturbation theory in powers of a_s and a_e using dimensional regularisation. Both QCD as well as QED interactions are taken into account simultaneously. Beyond the leading order in perturbation theory, the FFs contain both UV, soft and collinear divergences. UV divergences are removed by coupling constants as well as overall renormalisation constant namely $Z_{a_c}, c = s,e$ and Z^b_λ respectively. The soft divergences arise due to massless gauge bosons and the collinear ones are due to massless fermions. Explicit computation of the form factors shows that the IR singularities, resulting from QCD and QED interactions factorize. In particular, they can be factored out using a universal IR counter term denoted by $Z_{IR}(a_s,a_e,Q^2,\mu_R^2)$ and hence we can write $\hat{F}_I$ in dimensional regularisation as

$$\hat{F}_I(Q^2,\mu^2,\varepsilon) = Z_{IR}(Q^2,\mu^2,\mu_R^2,\varepsilon)\hat{F}_I^{fin}(Q^2,\mu^2,\mu_R^2,\varepsilon)\,, \tag{3.3}$$

where Z_{IR} contains all the IR poles while $\hat{F}_I^{fin}$ is IR finite. In addition, both $\hat{F}_I$ and $\hat{F}_I^{fin}$ can be made UV finite after appropriate UV renormalisation. Differentiating both sides with respect to Q^2, we obtain K+G equation for the form factors $\hat{F}_I$ as

$$Q^2\frac{d}{dQ^2}\ln\hat{F}_I(Q^2,\mu^2,\varepsilon) = \frac{1}{2}\left[K_I\left(\{\hat{a}_c\},\frac{\mu_R^2}{\mu^2},\varepsilon\right) + G_I\left(\{\hat{a}_c\},\frac{Q^2}{\mu_R^2},\frac{\mu_R^2}{\mu^2},\varepsilon\right)\right], \tag{3.4}$$

where

$$\begin{aligned} K_I\left(\{\hat{a}_c\},\frac{\mu_R^2}{\mu^2},\varepsilon\right) &= 2Q^2\frac{d}{dQ^2}\ln Z_{IR}(Q^2,\mu^2,\mu_R^2,\varepsilon)\,, \\ G_I\left(\{\hat{a}_c\},\frac{Q^2}{\mu_R^2},\frac{\mu^2}{\mu_R^2},\varepsilon\right) &= 2Q^2\frac{d}{dQ^2}\ln\hat{F}_I^{fin}(Q^2,\mu^2,\mu_R^2,\varepsilon)\,. \end{aligned} \tag{3.5}$$

Following, [40, 41] we solve Sudakov K+G equation (eq. (3.4)) order by order in perturbation theory.

The radiative corrections resulting from QCD and QED interactions can not be factored out independently. In other words, if we factorize IR singularities from the FFs, neither the IR singular function Z_{IR} nor the finite FF, $\hat{F}_I^{fin}$, can be written as a product of pure QCD and pure QED contributions. More specifically, there will be terms proportional to $a_s^i a_e^j$, where $i,j > 0$, which will not allow factorization of QCD and QED contributions for both Z_{IR} and $\hat{F}_I^{fin}$. One finds that the constant K_I will have IR poles in ε from pure QED and pure QCD in every order in perturbation theory and in addition, from QCD$\times$QED starting from $\mathcal{O}(a_s a_e)$. The constants G_Is are IR finite and they get contributions from both QCD as well as QED and they mix beyond leading order in perturbation theory. Since, the IR singularities of FFs have dipole structure, K_I will be independent of Q^2 while G_Is will be finite in $\varepsilon \to 0$ and also contain only logarithms in Q^2. The fact that $\hat{F}_I$ are renormalisation group (RG) invariant implies the sum $K_I + G_I$ is also RG invariant. This implies

$$\mu_R^2 \frac{d}{d\mu_R^2} K_I\left(\{\hat{a}_c\}, \frac{\mu_R^2}{\mu^2}, \varepsilon\right) = -\mu_R^2 \frac{d}{d\mu_R^2} G_I\left(\{\hat{a}_c\}, \frac{Q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \varepsilon\right) = -A_I(\{a_c(\mu_R^2)\}), \quad (3.6)$$

where A_I are the cusp anomalous dimensions. Since the FFs are dependent on both the couplings a_s as well as a_e, the A_I also depend on them. The solutions to the above RG equations for K_I can be obtained by expanding the cusp anomalous dimensions (A_I) in powers of renormalized coupling constants $a_s(\mu_R^2)$ and $a_e(\mu_R^2)$ as

$$A_I(\{a_c(\mu_R^2)\}) = \sum_{i,j=0} a_s^i(\mu_R^2) a_e^j(\mu_R^2) A_I^{(i,j)}, \qquad A_I^{(0,0)} = 0, \quad (3.7)$$

and K_I as

$$K_I(\{\hat{a}_c\}, \mu_R^2, \varepsilon) = \sum_{i,j=0} \hat{a}_s^i \hat{a}_e^j \left(\frac{\mu_R^2}{\mu^2}\right)^{(i+j)\frac{\varepsilon}{2}} S_\varepsilon^{(i+j)} K_I^{(i,j)}(\varepsilon), \qquad K_I^{(0,0)} = 0, \quad (3.8)$$

where $A_I^{(i,0)}$ and $A_I^{(0,i)}$ result from pure QCD and pure QED interactions respectively and $A_I^{(i,j)}, i,j > 0$ from QCD$\times$QED. The constants $K_I^{(i,j)}$ in eq. (3.8) can be obtained using eq. (3.6) and RG equations for the couplings a_s and a_e, in terms of the cusp anomalous dimensions. They are listed in the appendix A . Since G_Is depend on the finite part of the FFs, they do not contain any IR singularities but depend only on Q^2 and hence we expand them as

$$G_I\left(\{\hat{a}_c\}, \frac{Q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \varepsilon\right) = G_I(\{a_c(Q^2)\}, 1, \varepsilon) + \int_{\frac{Q^2}{\mu_R^2}}^1 \frac{d\lambda^2}{\lambda^2} A_I(\{a_c(\lambda^2 \mu_R^2)\}), \quad (3.9)$$

where the first term results from the boundary condition on each G_I at $\mu_R^2 = Q^2$. Again expanding A_I in powers of a_s and a_e and using RG equations for QCD and QED couplings, we obtain

$$\int_{\frac{Q^2}{\mu_R^2}}^1 \frac{d\lambda^2}{\lambda^2} A_I(\{a_c(\lambda^2 \mu_R^2)\}) = \sum_{i,j} \hat{a}_s^i \hat{a}_e^j \left(\frac{\mu_R^2}{\mu^2}\right)^{(i+j)\frac{\varepsilon}{2}} S_\varepsilon^{(i+j)} \left[\left(\frac{Q^2}{\mu_R^2}\right)^{(i+j)\frac{\varepsilon}{2}} - 1\right] K_I^{(i,j)}(\varepsilon). \quad (3.10)$$

The next step is to integrate eq. (3.4) over Q^2 to obtain the solution to K+G equation for $\hat{F}_I$. For this, we substitute the solutions of K_I and G_I in the right hand side of eq. (3.4) along with the expansion for $G_I(a_s(Q^2), a_e(Q^2), 1, \varepsilon)$ given by

$$G_I(\{a_c(Q^2)\}, 1, \varepsilon) = \sum_{i,j} a_s^i(Q^2) a_e^j(Q^2) G_I^{(i,j)}(\varepsilon). \quad (3.11)$$

We thus obtain,

$$\ln \hat{F}_I = \sum_{i,j} \hat{a}_s^i \hat{a}_e^j \left(\frac{Q^2}{\mu^2}\right)^{(i+j)\frac{\varepsilon}{2}} S_\varepsilon^{(i+j)} \hat{\mathcal{L}}_{F_I}^{(i,j)}(\varepsilon)\,, \tag{3.12}$$

where,

$$\begin{aligned}
\hat{\mathcal{L}}_{F_I}^{(1,0)} &= \frac{1}{\varepsilon^2}\Big(-2A_I^{(1,0)}\Big) + \frac{1}{\varepsilon}\Big(G_I^{(1,0)}(\varepsilon)\Big)\,,\\
\hat{\mathcal{L}}_{F_I}^{(0,1)} &= \frac{1}{\varepsilon^2}\Big(-2A_I^{(0,1)}\Big) + \frac{1}{\varepsilon}\Big(G_I^{(0,1)}(\varepsilon)\Big)\,,\\
\hat{\mathcal{L}}_{F_I}^{(2,0)} &= \frac{1}{\varepsilon^3}\Big(\beta_{00}A_I^{(1,0)}\Big) + \frac{1}{\varepsilon^2}\Big(-\frac{1}{2}A_I^{(2,0)} - \beta_{00}G_I^{(1,0)}(\varepsilon)\Big) + \frac{1}{2\varepsilon}\Big(G_I^{(2,0)}(\varepsilon)\Big)\,,\\
\hat{\mathcal{L}}_{F_I}^{(0,2)} &= \frac{1}{\varepsilon^3}\Big(\beta'_{00}A_I^{(0,1)}\Big) + \frac{1}{\varepsilon^2}\Big(-\frac{1}{2}A_I^{(0,2)} - \beta'_{00}G_I^{(0,1)}(\varepsilon)\Big) + \frac{1}{2\varepsilon}\Big(G_I^{(0,2)}(\varepsilon)\Big)\,,\\
\hat{\mathcal{L}}_{F_I}^{(1,1)} &= \frac{1}{\varepsilon^2}\Big(-\frac{1}{2}A_I^{(1,1)}\Big) + \frac{1}{2\varepsilon}\Big(G_I^{(1,1)}(\varepsilon)\Big)\,,\\
\hat{\mathcal{L}}_{F_I}^{(3,0)} &= \frac{1}{\varepsilon^4}\Big(-\frac{8}{9}\beta_{00}^2A_I^{(1,0)}\Big) + \frac{1}{\varepsilon^3}\Big(\frac{8}{9}\beta_{00}A_I^{(2,0)} + \frac{2}{9}\beta_{10}A_I^{(1,0)} + \frac{4}{3}\beta_{00}^2G_I^{(1,0)}\Big)\\
&\quad + \frac{1}{\varepsilon^2}\Big(-\frac{2}{9}A_I^{(3,0)} - \frac{1}{3}\beta_{10}G_I^{(1,0)} - \frac{4}{3}\beta_{00}G_I^{(2,0)}\Big) + \frac{1}{3\varepsilon}\Big(G_I^{(3,0)}\Big)\,,\\
\hat{\mathcal{L}}_{F_I}^{(1,2)} &= \frac{1}{\varepsilon^3}\Big(\frac{4}{9}\beta'_{00}A_I^{(1,1)} + \frac{2}{9}\beta'_{10}A_I^{(0,1)}\Big) + \frac{1}{\varepsilon^2}\Big(-\frac{2}{9}A_I^{(1,2)} - \frac{1}{3}\beta'_{10}G_I^{(0,1)} - \frac{2}{3}\beta'_{00}G_I^{(1,1)}\Big)\\
&\quad + \frac{1}{3\varepsilon}\Big(G_I^{(1,2)}\Big)\,,\\
\hat{\mathcal{L}}_{F_I}^{(2,1)} &= \frac{1}{\varepsilon^3}\Big(\frac{4}{9}\beta_{00}A_I^{(1,1)} + \frac{2}{9}\beta_{01}A_I^{(1,0)}\Big) + \frac{1}{\varepsilon^2}\Big(-\frac{2}{9}A_I^{(2,1)} - \frac{1}{3}\beta_{01}G_I^{(1,0)} - \frac{2}{3}\beta_{00}G_I^{(1,1)}\Big)\\
&\quad + \frac{1}{3\varepsilon}\Big(G_I^{(2,1)}\Big)\,,\\
\hat{\mathcal{L}}_{F_I}^{(0,3)} &= \frac{1}{\varepsilon^4}\Big(-\frac{8}{9}\beta_{00}'^2A_I^{(0,1)}\Big) + \frac{1}{\varepsilon^3}\Big(\frac{8}{9}\beta'_{00}A_I^{(0,2)} + \frac{2}{9}\beta'_{01}A_I^{(0,1)} + \frac{4}{3}\beta_{00}'^2G_I^{(0,1)}\Big)\\
&\quad + \frac{1}{\varepsilon^2}\Big(-\frac{2}{9}A_I^{(0,3)} - \frac{1}{3}\beta'_{01}G_I^{(0,1)} - \frac{4}{3}\beta'_{00}G_I^{(0,2)}\Big) + \frac{1}{3\varepsilon}\Big(G_I^{(0,3)}\Big)\,.
\end{aligned} \tag{3.13}$$

Our next task is to compute the FFs to third order in the couplings of QCD, QED and QCD×QED. The method of this computation is well documented in the literature [46, 47] and applied to several of the form factors [48–50] in QCD. Following [46, 47], we computed $\hat{F}_I$ for $I = q, b$ up to third order in QCD,QED and QCD×QED. The form factors $\hat{F}_I, I = q, b$ thus obtained in the present paper up to three loop level are listed in the appendix B. We use them to extract the cusp anomalous dimensions ($A_I^{(i,j)}$) by comparing them against eq. (3.13). From the one loop result for $\hat{\mathcal{L}}_{F_I}^{(1,0)}$ we obtain $\{G_I^{(1,0)}, A_I^{(1,0)}(\varepsilon)\}$ and from the result for $\hat{\mathcal{L}}_{F_I}^{(0,1)}$ we get $\{G_I^{(0,1)}, A_I^{(0,1)}(\varepsilon)\}$. Substituting these one loop results for A_I and $G_I(\varepsilon)$ along the two loop results for $\hat{\mathcal{L}}_{F_I}^{(2,0)}$ and $\hat{\mathcal{L}}_{F_I}^{(0,2)}$ in eq. (3.13), we obtain $\{\beta_{00}, G_I^{(2,0)}(\varepsilon), A_I^{(2,0)}\}$ and $\{\beta'_{00}, G_I^{(0,2)}(\varepsilon), A_I^{(0,2)}\}$ respectively. We continue this procedure with three loop results of FF to determine $\{\beta_{10}, G_I^{(3,0)}(\varepsilon), A_I^{(3,0)}\}$ and $\{\beta'_{01}, G_I^{(0,3)}(\varepsilon), A_I^{(0,3)}\}$. From the QCD×QED two and three loops results for the FFs, we obtain $\{G_I^{(1,1)}(\varepsilon), A_I^{(1,1)}\}$

and $\{\beta_{01}, G_I^{(2,1)}(\varepsilon), A_I^{(2,1)}\}$, $\{\beta'_{10}, G_I^{(1,2)}(\varepsilon), A_I^{(1,2)}\}$ respectively. This way we can obtain all the cusp anomalous dimensions, beta function coefficients and $G_I(\varepsilon)$s up to three loops. We find $A_q^{(i,j)} = A_b^{(i,j)}$ up to three loops in QCD,QED and QCD$\times$QED demonstrating the universal nature of these constants. However, we find that the constants $G_I(\varepsilon)$s depend on the type of form factors. The constants $A_I^{(i,0)}$ are known till three loops in [51] and here in appendix C we enlist the new ones $A_I^{(1,2)}, A_I^{(2,1)}$ along with the existing ones in [29]. The βs (see [52] for the leading order ones) are given by

$$\beta_{00} = \frac{11}{3}C_A - \frac{4}{3}n_f T_F\,, \quad \beta'_{00} = -\frac{4}{3}\left(N\sum_q e_q^2 + \sum_l e_l^2\right), \quad \beta_{01} = -2\left(\sum_q e_q^2 + \sum_l e_l^2\right),$$
$$\beta'_{01} = -4\left(N\sum_q e_q^4 + \sum_l e_l^4\right), \quad \beta_{10} = \left(\frac{34}{3}C_A^2 - \frac{20}{3}C_A n_f T_F - 4C_F n_f T_F\right),$$
$$\beta'_{10} = -4C_F\left(N\sum_q e_q^2 + \sum_l e_l^2\right). \tag{3.14}$$

Here, $C_A = N$ is the adjoint Casimir of SU(N) and the fundamental Casimir is $C_F = (N^2-1)/2N$, $T_F = 1/2$ and $n_f(n_l)$ is the number of active quark flavors (leptons). The electric charge of quark q is denoted by e_q while e_l refers to the electric charge of the lepton l.

Our next task is to investigate the structure of the constants $G_I^{(i,j)}(\varepsilon)$ following the observation made in [53] for the $G_I^{(i,0)}, I = q, b, g$ in QCD, namely we expand $G_I^{(i,j)}(\varepsilon)$ around $\varepsilon = 0$ in terms of collinear ($B_I^{(i,j)}$), soft ($f_I^{(i,j)}$) and UV ($\gamma_I^{(i,j)}$) anomalous dimensions as

$$G_I^{(i,j)}(\varepsilon) = 2(B_I^{(i,j)} - \gamma_I^{(i,j)}) + f_I^{(i,j)} + \sum_{k=0}\varepsilon^k g_{I,ij}^k\,, \tag{3.15}$$

with

$$g_{I,10}^0 = 0\,, \quad g_{I,01}^0 = 0\,, \quad g_{I,11}^0 = 0\,, \quad g_{I,20}^0 = -2\beta_{00}g_{I,10}^1\,, \quad g_{I,02}^0 = -2\beta'_{00}g_{I,01}^1\,,$$
$$g_{I,30}^0 = -2\left(\beta_{10}g_{I,10}^1 + \beta_{00}\left(2\beta_{00}g_{I,10}^2 + g_{I,20}^1\right)\right), \quad g_{I,21}^0 = -2\beta_{01}g_{I,10}^1 - \beta_{00}g_{I,11}^1\,,$$
$$g_{I,03}^0 = -2\left(\beta'_{01}g_{I,01}^1 + \beta'_{00}\left(2\beta'_{00}g_{I,01}^2 + g_{I,02}^1\right)\right), \quad g_{I,12}^0 = -2\beta'_{10}g_{I,01}^1 - \beta'_{00}g_{I,11}^1\,. \tag{3.16}$$

It was found in the context of QCD up to three loops and in QED and QCD$\times$QED up to two loops that the constants B_I, f_I depend only on the type of external states, not on the operator. In other words, the constants $B_q(f_q)$ and $B_b(f_b)$ extracted from F_q and F_b respectively were found to be the same and similarly B_gs (f_gs) extracted from $G^a_{\mu\nu}G^{\mu\nu a}$ and $G^a_{\mu\nu}\tilde{G}^{\mu\nu a}$ were also same (see [48, 54]). Hence, we expect that $B_q^{(i,j)} = B_b^{(i,j)}$ and $f_q^{(i,j)} = f_b^{(i,j)}$ in QED as well as in QCD$\times$QED. However, the anomalous dimensions for γ_b and γ_q will be different because they originate from the UV sector. Since $\mathcal{O}_q$ is a conserved operator, γ_q is identically zero to all orders in both a_s and a_e and this is not the case for γ_b which gets contribution from a_s as well as a_e in the perturbative expansion. Using the fact that B_I and f_I are operator independent and that $\gamma_q = 0$ to all orders, we can obtain γ_b up to three loops in QCD, QED and in QCD$\times$QED by computing $G_b(\varepsilon) - G_q(\varepsilon)$. Thus we obtain $\gamma_b^{(i,0)}, \gamma_b^{(0,i)}$ for $i = 1, 2, 3$ from pure QCD , QED and $\gamma_b^{(1,1)}, \gamma_b^{(2,1)}$ and $\gamma_b^{(1,2)}$

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from QCD×QED and they are listed in the appendix D. Substituting these anomalous dimensions in eq. (2.12), we obtain Z^b_λ to third order in the couplings.

3.2 Soft distribution function

In the following, we show how we can determine collinear and soft anomalous dimensions from the soft distribution function. Unlike $A_I^{(i,j)}$, the other anomalous dimensions $B_I^{(i,j)}$, $f_I^{(i,j)}$ and $\gamma_I^{(i,j)}$ ($\gamma_q^{(i,j)}$ is zero) can not be disentangled from $\hat{F}_q$ and $\hat{F}_b$ alone. In order to disentangle $B_I^{(i,j)}$ and $f_I^{(i,j)}$, we study the partonic cross sections resulting from soft gluon and soft photon emissions alone as they are sensitive to only $f_I^{(i,j)}$. The process independent part of soft gluon/photon contributions in the real emission sub-processes can be obtained following the method described in [40, 41], where it was demonstrated up to three loops in QCD, the soft distribution function, denoted by Φ_I, for the inclusive cross section for producing a colorless state can be computed using the FFs and partonic sub-process cross sections involving real emissions of gluons. For the QED and QCD×QED we use the respective FFs and the inclusive cross sections involving photons as well as gluons that contribute in the soft limit. In the case of QCD, the soft distribution functions were found to be dependent on cusp (A_I) and soft (f_I) anomalous dimensions, where $I = q, b, g$. Up to three loops in QCD, one finds $\Phi_b = \Phi_q = C_F/C_A \Phi_g$ [40, 41, 55],where Φ_g was found in [56, 57]. This relation is expected to hold beteween quark and gluon soft distribution functions because they are defined by the expectation value of certain gauge invariant bi-local quark and gluon operators computed between on-shell quark and gluons fields. The Wilson lines made up of guage fields make these bi-local operators gauge invariant. (see [58–64]. Using the partonic sub-processes of either DY process ($\hat{\sigma}_{q\bar{q}}$) or the Higgs boson production in bottom quark annihilation ($\hat{\sigma}_{b\bar{b}}$) normalized by the square of the bare form factor $\hat{F}_q$ or $\hat{F}_b$, we can obtain Φ_I, $i = q, b$. Φ_Is are found to be functions of the scaling variable $z = q^2/s$ and q^2 is invariant mass square of the lepton pair for the DY and $q^2 = m_h^2$ for the Higgs production. Note that Q^2 introduced in the form factors is related to q^2 by $Q^2 = -q^2$. We write,

$$\mathcal{C} \exp\Big(2\Phi_I(z)\Big) = \frac{\hat{\sigma}_{I\bar{I}}(z)}{Z_I^2 |\hat{F}_I|^2}, \qquad I = q, b \tag{3.17}$$

where $Z_q = 1$ for the $\mathcal{O}_q$ as $\gamma_q = 0$ and Z^b_λ can be obtained using eq. (2.12) in terms of $\gamma_b^{(i,j)}$ known up to three loops in QCD, QED and QCD×QED. The symbol $\mathcal{C}$ refers to "ordered exponential" which has the following expansion:

$$\mathcal{C}e^{f(z)} = \delta(1-z) + \frac{1}{1!}f(z) + \frac{1}{2!}(f \otimes f)(z) + \cdots \qquad . \tag{3.18}$$

Here $\otimes$ denotes the Mellin convolution and $f(z)$ is a distribution of the kind $\delta(1-z)$ and $\mathcal{D}_i$. The plus distribution $\mathcal{D}_i$ is defined as,

$$\mathcal{D}_i = \left(\frac{\ln^i(1-z)}{(1-z)}\right)_+ . \tag{3.19}$$

In [29], we computed UV finite $\hat{\sigma}_{I\bar{I}}$ up to NNLO in QCD, QED and QCD×QED , the two loop corrected bare FFs and the overall renormalisation constant Z^b_λ to obtain the soft distribution function Φ_I up to second order in a_s from QCD, in a_e from QED and in $a_s a_e$ from QCD×QED. We found in [29] that the soft distribution functions extracted from two different processes satisfy a remarkable relation, namely $\Phi_q = \Phi_b$ up to second order in both the couplings a_s and a_e demonstrating the universality among QCD, QED and QCD×QED results. It was found in [55] that this relation is valid up to three loops in QCD. Hence, we propose that this relation continues to hold true up to three loops even in QED and in QCD×QED. Our next task is to predict Φ_q (equivalently Φ_b) to third order in QED and QCD×QED.

Following [40, 41, 55] we express the soft distribution function Φ_I in terms of cusp (A_I) and soft (f_I) anomalous dimensions order by order in perturbation theory. It was also shown in [40, 41], that Φ_I satisfy Sudakov K+G equation analogous to FF owing to universal IR structure of these quantities:

$$q^2 \frac{d}{dq^2}\Phi_I = \frac{1}{2}\left[\overline{K}_I\left(\{\hat{a}_c\}, \frac{\mu_R^2}{\mu^2}, \varepsilon, z\right) + \overline{G}_I\left(\{\hat{a}_c\}, \frac{q^2}{\mu_R^2}, \frac{\mu_R^2}{\mu^2}, \varepsilon, z\right)\right], \tag{3.20}$$

where, $\overline{K}_I$ contains all the IR singularities and the IR finite part is denoted by $\overline{G}_I$. One finds that the RG invariance of Φ_I leads to

$$\mu_R^2 \frac{d}{d\mu_R^2}\overline{K}_I = -\mu_R^2 \frac{d}{d\mu_R^2}\overline{G}_I = A_I(\{a_c(\mu_R^2)\})\delta(1-z)\,. \tag{3.21}$$

From the explicit results [55] computed up to second order in QCD, QED and QCD×QED, we had shown that the anomalous dimension A_I are identical to the cusp anomalous dimension that appears in the form factors $\hat{F}_I$ confirming the universality of IR structure of the underlying gauge theory(ies). In other words, A_I are universal and they govern the evolution of both K_I, G_I and $\overline{K}_I, \overline{G}_I$.

Following the method described in [40, 41], we obtain

$$\Phi_I(\{\hat{a}_c\}, q^2, \mu^2, \varepsilon, z) = \sum_{i,j} \hat{a}_s^i \hat{a}_e^j \left(\frac{q^2(1-z)^2}{\mu^2}\right)^{(i+j)\frac{\varepsilon}{2}} S_\varepsilon^{(i+j)} \left(\frac{(i+j)\varepsilon}{1-z}\right) \hat{\phi}_I^{(i,j)}(\varepsilon)\,, \tag{3.22}$$

where,

$$\hat{\phi}_I^{(i,j)}(\varepsilon) = \frac{1}{(i+j)\varepsilon}\left[\overline{K}_I^{(i,j)}(\varepsilon) + \overline{G}_I^{(i,j)}(\varepsilon)\right]. \tag{3.23}$$

To obtain $\overline{G}_I^{(ij)}$, we first expand $\overline{G}_I(\{a_c(q^2)\}, 1, \varepsilon, z)$ in terms of $\overline{G}_I^{(i,j)}(\varepsilon)$ and relate the latter to $\overline{\mathcal{G}}_I^{(i,j)}$ as

$$\begin{aligned} \overline{G}_I(\{a_c(q^2)\}, 1, \varepsilon, z) &= \sum_{i,j} \hat{a}_s^i \hat{a}_e^j \left(\frac{q_z^2}{\mu^2}\right)^{(i+j)\frac{\varepsilon}{2}} S_\varepsilon^{(i+j)} \overline{G}_I^{(i,j)}(\varepsilon) \\ &= \sum_{i,j} a_s^i(q_z^2) a_e^j(q_z^2) \overline{\mathcal{G}}_I^{(i,j)}(\varepsilon)\,, \end{aligned} \tag{3.24}$$

where $q_z^2 = q^2(1-z)^2$ and the IR finite $\overline{\mathcal{G}}_I^{(i,j)}(\varepsilon)$ can be expanded (following [40, 41]) as

$$\overline{\mathcal{G}}_I^{(i,j)}(\varepsilon) = -f_I^{(i,j)} + \sum_{k=0} \varepsilon^k \overline{\mathcal{G}}_{I,ij}^{(k)}\,. \tag{3.25}$$

Up to third order, one finds

$$\begin{aligned}
&\overline{\mathcal{G}}^{(0)}_{I,10} = 0\,, \quad \overline{\mathcal{G}}^{(0)}_{I,01} = 0\,, \quad \overline{\mathcal{G}}^{(0)}_{I,11} = 0\,, \quad \overline{\mathcal{G}}^{(0)}_{I,20} = -2\beta_{00}\overline{\mathcal{G}}^{(1)}_{I,10}\,, \quad \overline{\mathcal{G}}^{(0)}_{I,02} = -2\beta'_{00}\overline{\mathcal{G}}^{(1)}_{I,01}\,,\\
&\overline{\mathcal{G}}^{(0)}_{I,30} = -2\beta_{10}\overline{\mathcal{G}}^{(1)}_{I,10} - 2\beta_{00}\overline{\mathcal{G}}^{(1)}_{I,20} - 4\beta_{00}^2\overline{\mathcal{G}}^{(2)}_{I,10}\,, \quad \overline{\mathcal{G}}^{(0)}_{I,03} = -2\beta'_{01}\overline{\mathcal{G}}^{(1)}_{I,01} - 2\beta'_{00}\overline{\mathcal{G}}^{(1)}_{I,02} - 4\beta'^2_{00}\overline{\mathcal{G}}^{(2)}_{I,01}\,,\\
&\overline{\mathcal{G}}^{(0)}_{I,21} = -2\beta_{01}\overline{\mathcal{G}}^{(1)}_{I,10} - \beta_{00}\overline{\mathcal{G}}^{(1)}_{I,11}\,, \quad \overline{\mathcal{G}}^{(0)}_{I,12} = -2\beta'_{10}\overline{\mathcal{G}}^{(1)}_{I,01} - \beta'_{00}\overline{\mathcal{G}}^{(1)}_{I,11}\,.
\end{aligned} \tag{3.26}$$

As already mentioned, Φ_I is known to third order in QCD and to second order in pure QED and in mixed QCD$\times$QED. To determine third order contribution to Φ_I in QED and in mixed QCD$\times$QED, we require the constants $\{f_I^{(0,3)}, f_I^{(1,2)}, f_I^{(2,1)}\}$ and $\{\overline{\mathcal{G}}^{(0)}_{I,12}, \overline{\mathcal{G}}^{(0)}_{I,21}, \overline{\mathcal{G}}^{(0)}_{I,03}\}$. They can be obtained by computing the N^{3}LO contributions to Drell-Yan production taking into account QED and QCD$\times$QED effects. While this is beyond the scope of our present work, we predict these constants from the corresponding ones in QCD using certain relations that relate QCD cusp anomalous dimension with the corresponding ones in QED and QCD$\times$QED. We find these relations owing to the fact that the cusp anomalous dimensions in QCD, QED and QCD$\times$QED can be extracted unambiguously from the form factors of vector and scalar operators order by order in perturbation theory. Interestingly, in QCD up to three loops, the cusp $A_I^{(i,0)}$, the soft $f_I^{(i,0)}$ and the constants $\overline{\mathcal{G}}^{(k)}_{I,ij}$ contain identical set of color factors, namely at one loop, we have $\{C_F\}$, at two loops $\{C_F C_A, C_F T_f n_f\}$ and at three loops $\{C_F C_A^2, C_F C_A T_f n_f, C_F^2 T_f n_f, C_F T_f^2 n_f^2\}$. In other words, the soft distribution function Φ_I in QCD demonstrate uniform color factor structure at every order in perturbation theory. This is in contrast to the constants A_I, B_I, γ_I and G_I that contribute to the form factors, which contain different sets of color factors. Hence, a uniform and unambiguous relations between QCD, QED and mixed QCD$\times$QED do not exist for the latter ones. Section 5 is devoted to study of these transformation rules in detail. Assuming that uniform color and charge factor structure for Φ_I in pure QED and mixed QCD$\times$QED holds true to third order, we apply the relations that relate their cusp anomalous dimensions, to predict $f_I^{(i,j)}$, $\overline{\mathcal{G}}^{(k)}_{I,ij}$ and hence the entire Φ_I to third order from the corresponding ones in QCD. This can be validated only by explicit computation which is reserved for future publication. Now that we have $f_q^{(i,j)}$ to third order, it is straightforward to obtain $B_q^{(i,j)}$ in eq. (3.15) to the same accuracy in QCD,QED and QCD$\times$QED from the explicit results on $G_q^{(i,j)}$ as $\gamma_q^{(i,j)} = 0$. Similarly, substituting $f_b^{(i,j)}$ and $\gamma_b^{(i,j)}$ in $G_b^{(i,j)}$ (eq. (3.15)) we can determine $B_b^{(i,j)}$ up to third order in QCD, QED and QCD$\times$QED. We find $B_q^{(i,j)}$ obtained from $G_q^{(i,j)}$ is identical to $B_b^{(i,j)}$ from $G_b^{(i,j)}$, namely $B_q^{(i,j)} = B_b^{(i,j)}$ up to third order in QCD,QED and QCD$\times$QED. The constants f_I and B_I for the pure QCD case are known to three loops in [51, 65, 66] and the new ones along with the pre-existing ones are enlisted in appendix E and F respectively .

4 Soft-virtual and resummed cross sections

The results obtained so far have two important phenomenological implications. Firstly third order threshold predictions for DY (di-lepton or W/Z production) as well as for Higgs boson production in bottom quark annihilation in QED and QCD$\times$QED. Secondly, the threshold enhanced resummed predictions in the N Mellin space .

We begin with the threshold predictions for DY and Higgs boson productions. Denoting the mass-factorised finite cross-section by Δ_I^{SV} and following [40, 41], we find

$$\Delta_I^{SV}(\tau, Q^2, \mu_R^2, \mu_F^2) = Z_I^2(\mu_R^2)|\hat{F}_I(Q^2)|^2\delta(1-z) \otimes \mathcal{C}e^{2\Phi_I(q^2)} \otimes \Gamma_{II}(\mu_F^2) \otimes \Gamma_{\overline{II}}(\mu_F^2)\,, \quad (4.1)$$

where $\Gamma_{II} = \Gamma_{\overline{II}}$ are Altarelli-Parisi (AP) kernels [67] that are required to remove the initial state collinear singularities. The scale μ_F is called factorization scale at which collinear singularities are removed from the partonic cross sections. In the above equation, we drop all the regular terms after the convolutions are performed to obtain only threshold contributions, often called soft plus virtual contributions (SV). In above equation for Δ_I^{SV}, the soft and collinear singularities arising from gluons/photons/fermions in the virtual subprocesses are guaranteed to cancel against those from the real sub-processes when all the degenerate states are summed up, thanks to the KLN theorem [68, 69]. The remaining initial state collinear singularities are removed by mass factorization kernels, namely the AP kernels which satisfy renormalisation group equations

$$\mu_F^2 \frac{d}{d\mu_F^2}\Gamma_{II}(z, \mu_F^2) = \frac{1}{2}P_{II}(\mu_F^2)\;\Gamma_{II}(\mu_F^2)\,, \quad (4.2)$$

where $P_{II}(z, \mu_F^2)$ are AP splitting functions known upto three loop level in pure QCD [65, 66]. In [15, 16], these splitting functions up to NNLO level, both in QED and QCD$\times$QED, were obtained using the abelianization procedure. The splitting functions that we have obtained [55] by demanding finite-ness of the mass factorised cross section, agreed with those in [15]. One finds AP splitting functions can be expressed in terms of distribution and regular functions as follows:

$$P_{II}(z, \mu_F^2) = 2\left(\frac{A_I(\{a_c(\mu_F^2)\})}{(1-z)_+} + B_I(\{a_c(\mu_F^2)\}\}\;\delta(1-z)\right) + P_{reg,II}(z, \mu_F^2)\,. \quad (4.3)$$

Since we are interested only in threshold corrections to finite mass-factorised partonic cross-section Δ_I, it is sufficient to drop P_{II}^{reg} in P_{II} when computing Δ_I^{SV} using eq. (4.1). Hence, we need only A_I and B_I from P_{II} to obtain Γ_{II}. Using Z_I, $\hat{F}_I$, Φ_I and Γ_{II} to third order in QCD,QED and QCD$\times$QED, we can readily obtain Δ_I^{SV} to third order. We expand Δ_I^{SV} as

$$\Delta_I^{SV}(z, q^2, \mu_F^2) = \sum_{i,j=0} a_s^i(\mu_R^2)a_e^j(\mu_R^2)\Delta_I^{SV,(i,j)}(z, q^2, \mu_F^2, \mu_R^2)\,, \quad (4.4)$$

and present the results for $\Delta_I^{SV,(i,j)}$ in the appendix H up to third order in QED and QCD$\times$QED for $I = q, b$. Up to two loops, our results for SV agree with those obtained earlier [27, 29] and results at third order are our predictions using the factorisation properties of the scattering cross section and the universal structure of the soft distribution function.

In the following we exploit these properties to systematically resum certain class of logarithms to all orders in perturbation theory. In QCD, it is well known that threshold logarithms of the kind $\mathcal{D}_i(z)$ spoil the reliability of the fixed order perturbation theory when $z = q^2/\hat{s}$ is closer to threshold namely $z \to 1$. These logarithms originate from the soft distribution functions after the cancellation of soft and collinear singularities against those from the FF and the AP (mass factorisation) kernels. These logarithms when convoluted with the appropriate parton distribution functions to compute the production cross sections at the hadronic level, can enhance the cross section provided the latter also dominates. In other words, the interplay between perturbative (threshold logarithms) and non-perturbation (parton distribution functions) terms enhance the cross section at every order in perturbation theory questioning the truncation of the perturbative series. The resolution to this problem was provided in [58, 64] which proposed a systematic way of reorganising the perturbative series through a procedure called threshold resummation. Working in Mellin space parametrised by a complex variable N, one can resum the order one terms of the form $a_s\beta_{00}\log N$ or $a_e\beta'_{00}\log N$ to all orders in perturbation theory. Following [40, 41], it is straightforward to obtain z space result that is required to obtain resummed result in the Mellin space. In order to get the z space result, we write the soft distribution function Φ_I as

$$\begin{aligned}\Phi_I = &\left(\int_{\mu_F^2}^{q_z^2}\frac{d\lambda^2}{\lambda^2(1-z)}A_I\left(a_s(\lambda^2),a_e(\lambda^2)\right)+\frac{1}{(1-z)}\frac{D_I}{2}\left(a_s(q_z^2),a_e(q_z^2)\right)\right)_+\\&+\sum_{i,j=1}^{\infty}\hat{a}_s^i\hat{a}_e^jS_\varepsilon^{i+j}\left(\frac{\mu_F^2}{\mu^2}\right)^{(i+j)\frac{\varepsilon}{2}}\frac{1}{(1-z)_+}\overline{K}_I^{(i,j)}(\varepsilon)\\&+\sum_{i,j=1}^{\infty}a_s^ia_e^jS_\varepsilon^{i+j}\left(\frac{q^2}{\mu^2}\right)^{(i+j)\frac{\varepsilon}{2}}\delta(1-z)\hat{\phi}_I^{(i,j)}(\varepsilon)\,,\end{aligned} \tag{4.5}$$

where D_I is related to $\overline{G}_I$ given in eq. (3.24) through

$$D_I\left(a_s(q_z^2),a_e(q_z^2)\right)=2\overline{G}_I\left(\{a_c(q^2)\},1,\varepsilon,z\right)|_{\varepsilon=0}\,. \tag{4.6}$$

Up to third order, they are listed in the appendix G. The first term in the above expression is finite as $\varepsilon\to 0$ while the second and third terms contain singularities that cancel against those from the AP kernels and the FF. Substituting Eq. (4.5) in Eq. (4.1), and taking a Mellin transform, we obtain

$$\begin{aligned}\Delta_{I,N}^{res}(q^2) &= \int_0^1 d\tau\tau^{N-1}\Delta_I^{SV}(\tau,q^2)\\&= C_{I,0}(q^2,\mu_R^2,\mu_F^2)\exp\left(G_{I,N}(q^2,\mu_R^2,\mu_F^2)\right)\,,\end{aligned} \tag{4.7}$$

where

$$G_{I,N}=\int_0^1 dzz^{N-1}\left(\int_{\mu_F^2}^{q_z^2}\frac{d\lambda^2}{\lambda^2(1-z)}2A_I\left(a_s(\lambda^2),a_e(\lambda^2)\right)+\frac{1}{1-z}D_I\left(a_s(q_z^2),a_e(q_z^2)\right)\right)_+ \tag{4.8}$$

where $C_{I,0}$ gets contributions from terms proportional to $\delta(1-z)$ in FF and the soft distribution function. All the μ_R and μ_F dependent logarithms in C_0 come from renormalisation constant and the AP kernels after the poles in ε cancel against the form factors and soft distribution functions. We have presented $C_{I,0}$ in the appendix I for $I=q,b$.

QCD (a_s^3)	QCD×QED ($a_s^2 a_e$)	QCD×QED ($a_s a_e^2$)	QED (a_e^3)
C_F^3	$3C_F^2 e_I^2$	$3C_F e_I^4$	e_I^6
$C_A C_F^2$	$C_A C_F e_I^2$	0	0
$C_F^2 n_f T_F$	$a \; C_F n_f T_F e_I^2$ $+b \; C_F T_F \Big(\sum\limits_q e_q^2 + \sum\limits_l e_l^2\Big)$	$C_F e_I^2 \Big(N \sum\limits_q e_q^2 + \sum\limits_l e_l^2\Big)$	$a \; e_I^4 \Big(N \sum\limits_q e_q^2 + \sum\limits_l e_l^2\Big)$ $+b \; e_I^2 \Big(N \sum\limits_q e_q^4 + \sum\limits_l e_l^4\Big)$
$C_F C_A^2$	0	0	0
$C_F C_A n_f T_F$	0	0	0
$C_F n_f^2 T_F^2$	0	0	$e_I^2 \Big(N \sum\limits_q e_q^2 + \sum\limits_l e_l^2\Big)^2$

Table 1. The index q is summed over all the quark charges and index l is summed over all the lepton charges. Moreover index $I = q, b$ corresponding to Drell-Yan pair production and Higgs production in bottom quark annihilation.

5 Abelianisation procedure

In a series of works [15, 16, 27], the second order contributions to Altarelli-Parisi splitting functions and inclusive cross section for Drell-Yan production in QED and in mixed QCD×QED were obtained from the existing QCD results using certain transformation rules that relate color and charge factors for the relevant Feynman diagrams. These transformation rules (also called as the Abelianisation rules) were found to hold true [29] for the FFs of vector and scalar operators and the inclusive cross section for the production of a Higgs boson in bottom quark annihilation. In addition, in [29], the infrared structure of QED and mixed QCD×QED at NNLO level were studied thereby obtaining the same set of transformation rules for the anamolous dimensions in QED as well as in QCD×QED from the QCD ones. In the following, we discuss, in detail, the existence of such transformation rules at the three loop level.

The explicit computation of FFs to third order shows that for the color factors C_F^3, $C_F^2 C_A$, $C_F C_A^2$ and $C_F n_f^2 T_F^2$ in pure QCD, there exists definite transformation rules that relate FFs, Δ_I^{SV} and anomalous dimensions of pure QED and mixed QCD×QED to pure QCD at the third order. However for the color factor $C_F^2 n_f T_F$ which arises from topologies with single fermion loop, there is no one-to-one mapping from pure QCD to pure QED at a_e^3 and to mixed QCD×QED at $a_s^2 a_e$ order. But interestingly, at $a_s a_e^2$ order there is again a one-to-one correspondence with QCD color factors. We present a general set of such rules obtained from the explicit calculation of FFs in table 1. The coefficients $\{a, b\}$ against the corresponding color factors depend on the contribution from relevant topologies and are dependent on the FFs. Similar set of transformation rules were obtained for Δ_I^{SV} and anomalous dimensions with different coefficients $\{a', b'\}$. However, strikingly for the cusp anomalous dimension, we find that there exists a one-to-one mapping from pure QCD to those of mixed QCD×QED and to pure QED for all the color factors. This happens because the contributions from $C_F n_f T_F e_I^2$ and $e_I^4(N \sum_q e_q^2 + \sum_l e_l^2)$ are absent in $A_I^{(2,1)}$ and $A_I^{(3,0)}$ respectively. Thus for the cusp anomalous dimension, the color factor $C_F^2 n_f T_F$ maps to $C_F T_F \big(N \sum\limits_q e_q^2 + \sum\limits_l e_l^2\big)$ in $a_s^2 a_e$ and to $e_I^2 \big(N \sum\limits_q e_q^4 + \sum\limits_l e_l^4\big)$ in a_e^3 as can be seen in appendix C.

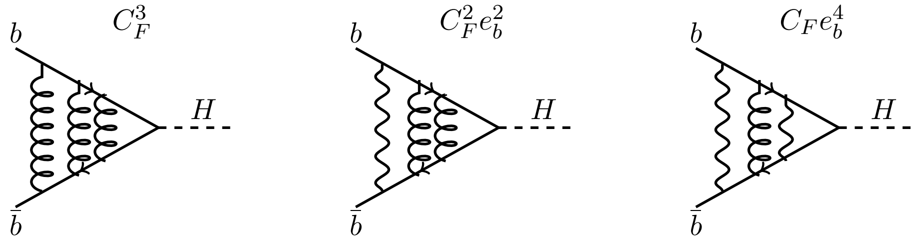


Figure 1. An example of Feynman diagram which contributes to the color factors $C_F^3, C_F^2 e_b^2, C_F e_b^4$.

With these observations at hand it is possible to understand the reason behind the transformation rules. We begin with the transformation rule for C_F^3. This color factor arises from those diagrams where no fermion or gluon loops are present. In the below we show some Feynman diagrams which leads to C_F^3 color factor.

The numerical factor of three at $a_s^2 a_e$ order accounts for the number of ways a gluon field can be replaced by a photon field in a pure QCD Feynman diagram. For instance at the $a_s^2 a_e$ order, the factor of three against $C_F^2 e_I^2$ corresponds to the number of ways a gluon can be replaced by a photon for a particular pure QCD diagram. Having found the reason we anticipate that the transformation rule for C_F^3 can be generalised for higher orders. For this, let N_1 denote the number of gluon fields in any pure QCD diagram, then the number of ways N_2 photon fields and $N_1 - N_2$ gluon fields can be arranged in any QCD×QED diagram is

$$C_f^{N_1} \to \frac{N_1!}{(N_1 - N_2)! N_2!} C_f^{N_1 - N_2} (e_q^2)^{N_2} \tag{5.1}$$

Next we discuss the color factor $C_A^2 C_F$, which arises from diagrams involving three gluon vertices. Such diagrams are absent in the mixed case as well as in pure QED case due to the absence of self-interaction vertices. Similarly the color factor $C_F n_f^2 T_F^2$ corresponds to diagrams shown in table 2 and due to trace-less property of Gell-Mann matrices they will be absent in mixed QCD×QED. Hence, for higher loop configurations, we can anticipate that diagrams where the internal lines are connected only by gluon loops or only by fermion loops will always be absent in the mixed case. But for the color factor $C_F^2 n_f T_F$, the rules are not so definite and cannot be extended unambiguously to higher orders. This is due to the fact that at the third order some of the topologies, which mainly come from the single fermion loop configurations lack the aforementioned one-to-one mapping from QCD to QED and to QCD×QED. In table 2 we show some of those configurations which demonstrate the ambiguous transformations.

From the table we can see that the coefficient corresponding to color factor $C_F^2 n_f T_F$ in QCD splits and gives rise to two different color-charge (charge) factor contributions in QCD×QED (QED). This accounts for the fact that QCD is flavor blind whereas QED is not and hence the above transformations.

Thus at higher orders, more of such loop configurations will open up which will lead to ambiguous mapping from QCD to QED as well as to QCD×QED. Such one-to-many mappings are observed even in case of UV (appendix D) and collinear anomalous dimensions (appendix F) as well as for Δ_I^{SV} (appendix H). But we find that the cusp anomalous dimension is free from such ambiguity and it could be due to the fact that it contains the soft gluon or photon contribution which is universal.

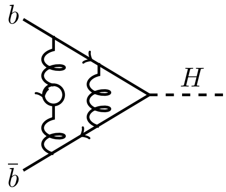
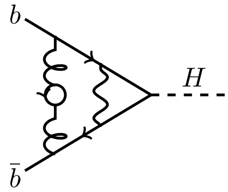
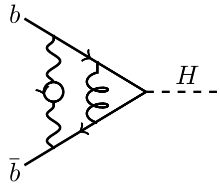
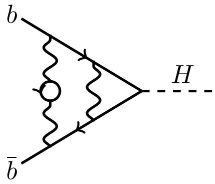
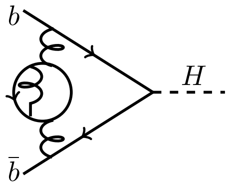
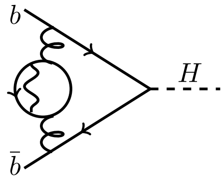
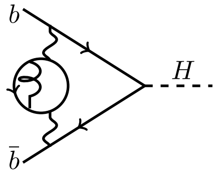
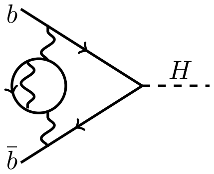
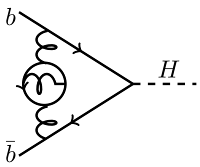
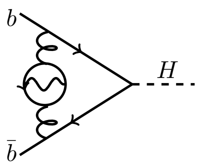
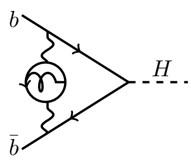
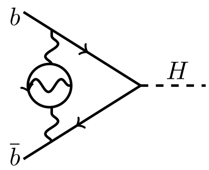
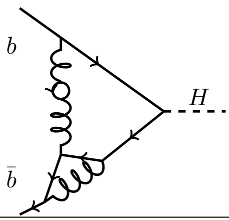
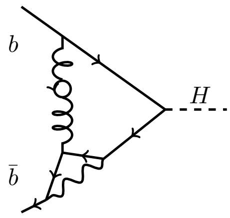
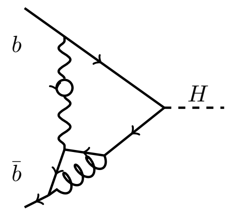
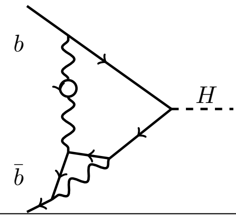
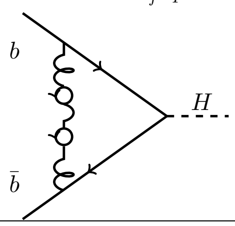
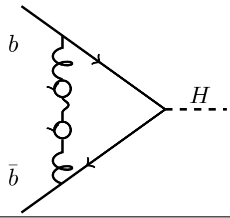
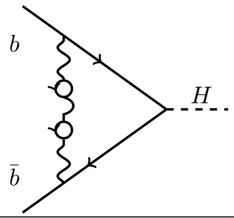
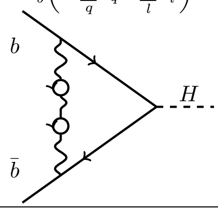
a_s^3	$a_s^2 a_e$	$a_s a_e^2$	a_e^3
$C_F^2 n_f T_F$ 	$C_F n_f T_F e_q^2$ 	$C_F e_b^2\left(N\sum_q e_q^2+\sum_l e_l^2\right)$ 	$e_b^4\left(N\sum_q e_q^2+\sum_l e_l^2\right)$ 
$C_F^2 n_f T_F$ 	$C_F T_F\left(\sum_q e_q^2+\sum_l e_l^2\right)$ 	$C_F e_b^2\left(N\sum_q e_q^2+\sum_l e_l^2\right)$ 	$e_b^2\left(N\sum_q e_q^4+\sum_l e_l^4\right)$ 
$\left(C_F^2-\frac{C_A C_F}{2}\right)n_f T_F$ 	$C_F T_F\left(\sum_q e_q^2+\sum_l e_l^2\right)$ 	$C_F e_b^2\left(N\sum_q e_q^2+\sum_l e_l^2\right)$ 	$e_b^2\left(N\sum_q e_q^4+\sum_l e_l^4\right)$ 
$\left(C_F^2-\frac{C_A C_F}{2}\right)n_f T_F$ 	$C_F n_f T_F e_b^2$ 	$C_F e_b^2\left(N\sum_q e_q^2+\sum_l e_l^2\right)$ 	$e_b^4\left(N\sum_q e_q^2+\sum_l e_l^2\right)$ 
$C_F n_f^2 T_F^2$ 	0 	0 	$e_b^2\left(N\sum_q e_q^2+\sum_l e_l^2\right)^2$ 

Table 2. Flavour and charge distributions for some pure QCD, pure QED and QCD×QED loop configurations. Color factors are obtained for the $b\bar{b}$ channel after conjugating with born amplitude and taking the color average up to an overall $\frac{1}{N}$ factor.

In summary, we infer that the abelianisation procedure [27, 29] which succeeded in giving definite color transformation rules at the two loop level without explicit calculation fails at the three loop level. At two loop level in QCD×QED, the single fermion loop diagrams do not contribute. Hence, taking the abelian limit of the pure QCD result is straightforward. On the other hand, in the case of two loop QED, although the single fermion loop diagrams contribute, still switching off diagrams involving three gluon vertices was sufficient to reproduce pure QED results from pure QCD ones. But at three loops, closed fermion loop configurations map to different charge-color factors in QED and QCD×QED and hence taking abelian limit of the pure QCD FF results does not produce pure QED as well as QCD×QED results. So, the fact that the coefficients $\{a, b\}$ for QCD×QED and pure QED color factors can only be fixed by explicit calculation, limits the use of abelianisation procedure beyond NNLO.

6 Summary and conclusions

In this paper, we have studied the infrared structure of a theory which is invariant under $\mathrm{SU}(N) \times \mathrm{U}(1)$ (QCD × QED), containing n_f number of quarks and n_l number of leptons with their respective anti-particles. We have treated all the quarks and leptons massless throughout. We considered two inclusive reactions at hadron colliders, namely production of a pair of leptons through quark anti-quark annihilation and production of a Higgs boson in bottom quark annihilation as theoretical laboratories. We used the parton model throughout. In the parton model, one factorises the hadron cross section into IR safe partonic cross sections and parton distribution functions. The former ones being computable order by order in perturbation theory, are expanded in double series expansion of the gauge couplings a_s and a_e of QCD and QED respectively to include radiative corrections. The computation of these corrections beyond leading order in perturbation theory provides ample opportunity to understand both UV and IR structures of the underlying gauge theory. In our case, the computation of IR safe partonic cross sections can help to understand the contributions from pure virtual and real Feynman diagrams and virtual-real diagrams both from QCD and QED. While UV divergences go away after including appropriate renormalisation constants, we are left with soft and collinear divergences in virtual and real subprocesses. In order to shed light on the IR structure, we have restricted ourselves to the computation where only soft and virtual contributions to the partonic subprocess are included in a IR safe way. In particular, we have computed those contributions that result from threshold region alone. This is achieved by appropriately combining entire pure virtual contributions with the soft part of the certain partonic subprocesses where at least one real radiation is present and with the AP kernels computed in the threshold limit. Each part of the computation involves careful study of its IR structure. The form factors are shown to satisfy K+G equations up to third order in perturbation theory. The renormalisation group invariance of the form factors can be used to obtain the universal cusp anomalous dimensions A_I of the underlying theory in powers of both a_s and a_e to third order. We find that the abelianization relations hold among the coefficients $A_I^{(i,j)}$ at various orders in couplings irrespective of I. Assuming the universal structure of the single poles of the form factors, we determined for the first time the renormalisation constant for the Yukawa coupling at third order both in pure QED and QCD×QED. Using the abelianization rules obtained from the results of $A_I^{(i,j)}$, we have determined the constants $f_I^{(0,i)}, i = 1, 2, 3$ and $f_I^{(1,1)}, f_I^{(2,1)}, f_I^{(1,2)}$ of QED and QCD×QED. These are our predictions for f_I in QED and QCD×QED up to third order irrespective of I. From the knowledge of $f_I^{(i,j)}$ up to third order, we can determine the corresponding $B_I^{(i,j)}$ for QED and QCD×QED. Interestingly, we find that abelianization rules that we obtained at third order for $A_I^{(i,j)}$ do not work for B_I. Since we have explicitly computed the form factors up to third order, it is easy to find that certain color factors of the form factors at third order in QCD can come from different kinds of topologies while these topologies in pure QED and QCD×QED cases can give different charge and color-charge factors. In other words, there is no one to one mapping between color factors of form factors in QCD and charge or charge-color factors in QED or QCD×QED. However, these topologies do not contribute to A_I allowing us to find consistent abelianization rules for them, but the UV renormalisation constants and collinear anomalous dimensions B_I in QCD get contributions from them. Hence, we fail to find consistent abelianization rules for them. In summary, we have determined β function coefficients, cusp and soft anomalous dimensions from the FFs computed to third order in QCD,QED and QCD×QED. We have predicted the soft distribution function Φ_I by applying the abelianization rules on the corresponding ones in QCD and extracted collinear anomalous dimension B_I from the explicit results of the form factors. Using these ingredients, we have obtained the soft plus virtual cross section to third order and resummed cross section to N^{3}LL accuracy in QED and QCD×QED.

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A $K_I^{(i,j)}$s in the form factor

The constants $K_I^{(i,j)}$ in the form factor are given by,

$$
\begin{aligned}
K_I^{(1,0)} &= \frac{1}{\varepsilon}\Big(- 2A_I^{(1,0)}\Big), \qquad K_I^{(0,1)} = \frac{1}{\varepsilon}\Big(- 2A_I^{(0,1)}\Big), \qquad K_I^{(1,1)} = \frac{1}{\varepsilon}\Big(- A_I^{(1,1)}\Big), \\
K_I^{(2,0)} &= \frac{1}{\varepsilon^2}\Big(2\beta_{00}A_I^{(1,0)}\Big) + \frac{1}{\varepsilon}\Big(- A_I^{(2,0)}\Big), \quad K_I^{(0,2)} = \frac{1}{\varepsilon^2}\Big(2\beta'_{00}A_I^{(0,1)}\Big) + \frac{1}{\varepsilon}\Big(- A_I^{(0,2)}\Big), \\
K_I^{(3,0)} &= \frac{1}{\varepsilon^3}\Big(- \frac{8}{3}\beta_{00}^2A_I^{(1,0)}\Big) + \frac{1}{\varepsilon^2}\Big(\frac{8}{3}\beta_{00}A_I^{(2,0)} + \frac{2}{3}\beta_{10}A_I^{(1,0)}\Big) + \frac{1}{\varepsilon}\Big(- \frac{2}{3}A_I^{(3,0)}\Big), \\
K_I^{(1,2)} &= \frac{1}{\varepsilon^2}\Big(\frac{4}{3}\beta'_{00}A_I^{(1,1)} + \frac{2}{3}\beta'_{10}A_I^{(0,1)}\Big) + \frac{1}{\varepsilon}\Big(- \frac{2}{3}A_I^{(1,2)}\Big), \\
K_I^{(2,1)} &= \frac{1}{\varepsilon^2}\Big(\frac{4}{3}\beta_{00}A_I^{(1,1)} + \frac{2}{3}\beta_{01}A_I^{(1,0)}\Big) + \frac{1}{\varepsilon}\Big(- \frac{2}{3}A_I^{(2,1)}\Big), \\
K_I^{(0,3)} &= \frac{1}{\varepsilon^3}\Big(- \frac{8}{3}\beta_{00}'^2A_I^{(0,1)}\Big) + \frac{1}{\varepsilon^2}\Big(\frac{8}{3}\beta'_{00}A_I^{(0,2)} + \frac{2}{3}\beta'_{01}A_I^{(0,1)}\Big) + \frac{1}{\varepsilon}\Big(- \frac{2}{3}A_I^{(0,3)}\Big).
\end{aligned}
\tag{A.1}
$$

B Form factor

The unrenormalized form factor $(\hat{F}_I)$ can be written as follows in the perturbative expansion of unrenormalized strong coupling constant $(\hat{a}_s)$ and unrenormalized fine structure constant $(\hat{a}_e)$

$$\begin{aligned}
\hat{F}_I =& 1+\hat{a}_s\left(\frac{Q^2}{\mu^2}\right)^{\frac{\varepsilon}{2}}\mathcal{S}_\varepsilon\left[C_F\mathcal{F}_1^I\right]+\hat{a}_e\left(\frac{Q^2}{\mu^2}\right)^{\frac{\varepsilon}{2}}\mathcal{S}_\varepsilon\left[e_I^2\mathcal{F}_1^I\right]+\hat{a}_s^2\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon}\mathcal{S}_\varepsilon^2\left[C_F^2\mathcal{F}_{2,0}^I+C_AC_F\mathcal{F}_{2,1}^I\right.\\
&\left.+C_Fn_fT_F\mathcal{F}_{2,2}^I\right]+\hat{a}_s\hat{a}_e\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon}\mathcal{S}_\varepsilon^2\left[2C_Fe_I^2\mathcal{F}_{2,0}^I\right]+\hat{a}_e^2\left(\frac{Q^2}{\mu^2}\right)^{\varepsilon}\mathcal{S}_\varepsilon^2\left[e_I^4\mathcal{F}_{2,0}^I+e_I^2\left(N\sum_q e_q^2\right)\mathcal{F}_{2,2}^I\right]\\
&+\hat{a}_s^3\left(\frac{Q^2}{\mu^2}\right)^{\frac{3\varepsilon}{2}}\mathcal{S}_\varepsilon^3\left[C_F^3\mathcal{F}_{3,0}^I+C_F^2C_A\mathcal{F}_{3,1}^I+C_FC_A^2\mathcal{F}_{3,2}^I+C_F^2n_fT_F\mathcal{F}_{3,3}^I+C_FC_An_fT_F\mathcal{F}_{3,4}^I\right.\\
&\left.+C_Fn_f^2T_F^2\mathcal{F}_{3,5}^I+C_FN_{F,V}\left(\frac{N^2-4}{N}\right)\mathcal{F}_{3,6}^I\right]+\hat{a}_s^2\hat{a}_e\left(\frac{Q^2}{\mu^2}\right)^{\frac{3\varepsilon}{2}}\mathcal{S}_\varepsilon^3\left[3C_F^2e_I^2\mathcal{F}_{3,0}^I+C_FC_Ae_I^2\mathcal{F}_{3,1}^I\right.\\
&\left.+C_FT_F\left(\sum_q e_q^2+\sum_l e_l^2\right)\mathcal{F}_{3,3a}^I+C_Fn_fT_Fe_I^2\mathcal{F}_{3,3b}^I+12C_Fe_I\left(\sum_q e_q+\sum_l e_l\right)\mathcal{F}_{3,6}^I\right]\\
&+\hat{a}_s\hat{a}_e^2\left(\frac{Q^2}{\mu^2}\right)^{\frac{3\varepsilon}{2}}\mathcal{S}_\varepsilon^3\left[3C_Fe_I^4\mathcal{F}_{3,0}^I+C_Fe_I^2\left(N\sum_q e_q^2+\sum_l e_l^2\right)\mathcal{F}_{3,3}^I\right]+\hat{a}_e^3\left(\frac{Q^2}{\mu^2}\right)^{\frac{3\varepsilon}{2}}\mathcal{S}_\varepsilon^3\left[e_I^6\mathcal{F}_{3,0}^I\right.\\
&+e_I^2\left(N\sum_q e_q^4+\sum_l e_l^4\right)\mathcal{F}_{3,3a}^I+e_I^4\left(N\sum_q e_q^2+\sum_l e_l^2\right)\mathcal{F}_{3,3b}^I+e_I^2\left(N\sum_q e_q^2+\sum_l e_l^2\right)^2\mathcal{F}_{3,5}^I\\
&\left.+8e_I^3\left(N\sum_q e_q^3+\sum_l e_l^3\right)\mathcal{F}_{3,6}^I\right]
\end{aligned}\tag{B.1}$$

$I = q, b$ denotes the Drell-Yan pair production and the Higgs boson production in bottom quark annihilation, respectively. Here, $N_{F,V}$ corresponds to the charge weighted sum of the quark flavors [47]. The Coefficients $\mathcal{F}_{i,j}^I$ for $i, j < 3$ are given in paper [29]. The $\mathcal{F}_{i,j}^q$ for DY at third order are given by,

$$\begin{aligned}
\mathcal{F}_{3,0}^q =& \frac{1}{\varepsilon^6}\left(\frac{-256}{3}\right)+\frac{1}{\varepsilon^5}\left(192\right)+\frac{1}{\varepsilon^4}\left(-400+32\zeta_2\right)+\frac{1}{\varepsilon^3}\left(664+24\zeta_2-\frac{800}{3}\zeta_3\right)+\frac{1}{\varepsilon^2}\left(-1030\right.\\
&\left.-154\zeta_2+\frac{426}{5}\zeta_2^2+552\zeta_3\right)+\frac{1}{\varepsilon}\left(\frac{9073}{6}+467\zeta_2-\frac{1461}{10}\zeta_2^2-\frac{4238}{3}\zeta_3+\frac{428}{3}\zeta_2\zeta_3-\frac{1288}{5}\zeta_5\right)\\
&-\frac{53675}{24}-\frac{9095}{252}\zeta_2^3-\frac{13001}{12}\zeta_2+\frac{12743}{40}\zeta_2^2+2669\zeta_3+61\zeta_2\zeta_3-\frac{1826}{3}\zeta_3^2+\frac{4238}{5}\zeta_5,\\
\mathcal{F}_{3,1}^q =& \frac{1}{\varepsilon^5}\left(-\frac{352}{3}\right)+\frac{1}{\varepsilon^4}\left(\frac{3448}{9}-32\zeta_2\right)+\frac{1}{\varepsilon^3}\left(208\zeta_3-\frac{25660}{27}+\frac{28}{3}\zeta_2\right)+\frac{1}{\varepsilon^2}\left(\frac{158554}{81}+\frac{1487}{9}\zeta_2\right.\\
&\left.-\frac{332}{5}\zeta_2^2-840\zeta_3\right)+\frac{1}{\varepsilon}\left(-\frac{1773839}{486}-\frac{38623}{54}\zeta_2+\frac{9839}{36}\zeta_2^2+\frac{6703}{3}\zeta_3-\frac{430}{3}\zeta_2\zeta_3+284\zeta_5\right)\\
&+\frac{37684115}{5832}+\frac{664325}{324}\zeta_2-\frac{1265467}{2160}\zeta_2^2-\frac{18619}{1260}\zeta_2^3-\frac{96715}{18}\zeta_3+\frac{46}{9}\zeta_2\zeta_3+\frac{1616}{3}\zeta_3^2-\frac{46594}{45}\zeta_5,\\
\mathcal{F}_{3,2}^q =& \frac{1}{\varepsilon^4}\left(-\frac{3872}{81}\right)+\frac{1}{\varepsilon^3}\left(\frac{52168}{243}-\frac{704}{27}\zeta_2\right)-\frac{1}{\varepsilon^2}\left(\frac{161156}{243}+\frac{2212}{81}\zeta_2+\frac{352}{45}\zeta_2^2-\frac{6688}{27}\zeta_3\right)\\
&+\frac{1}{\varepsilon}\left(\frac{3741128}{2187}+\frac{68497}{243}\zeta_2-\frac{1604}{15}\zeta_2^2-\frac{24212}{27}\zeta_3+\frac{176}{9}\zeta_2\zeta_3+\frac{272}{3}\zeta_5\right)-\frac{52268375}{13122}-\frac{6152}{189}\zeta_2^3\\
&+\frac{152059}{540}\zeta_2^2-\frac{1136}{9}\zeta_3^2-\frac{767320}{729}\zeta_2+\frac{1341553}{486}\zeta_3-\frac{710}{9}\zeta_2\zeta_3+\frac{2932}{9}\zeta_5,
\end{aligned}$$

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$$
\begin{aligned}
\mathcal{F}^q_{3,3}=&\frac{1}{\varepsilon^5}\left(\frac{128}{3}\right)-\frac{1}{\varepsilon^4}\left(\frac{1184}{9}\right)+\frac{1}{\varepsilon^3}\left(\frac{8720}{27}+\frac{16}{3}\zeta_2\right)+\frac{1}{\varepsilon^2}\left(-\frac{51992}{81}-\frac{532}{9}\zeta_2+\frac{1168}{9}\zeta_3\right)\\
&+\frac{1}{\varepsilon}\left(\frac{277730}{243}+\frac{5698}{27}\zeta_2-\frac{337}{9}\zeta_2^2-\frac{10228}{27}\zeta_3\right)-\frac{2732173}{1458}-\frac{45235}{81}\zeta_2+\frac{8149}{108}\zeta_2^2+\frac{102010}{81}\zeta_3\\
&-\frac{686}{9}\zeta_2\zeta_3+\frac{556}{45}\zeta_5,\\
\mathcal{F}^q_{3,4}=&\frac{1}{\varepsilon^4}\left(\frac{2816}{81}\right)+\frac{1}{\varepsilon^3}\left(\frac{256}{27}\zeta_2-\frac{36064}{243}\right)+\frac{1}{\varepsilon^2}\left(\frac{109432}{243}+\frac{2528}{81}\zeta_2-\frac{2048}{27}\zeta_3\right)+\frac{1}{\varepsilon}\left(\frac{176}{5}\zeta_2^2\right.\\
&\left.-\frac{44108}{243}\zeta_2-\frac{2495948}{2187}+\frac{25744}{81}\zeta_3\right)+\frac{17120104}{6561}+\frac{442961}{729}\zeta_2-\frac{2186}{27}\zeta_2^2-\frac{90148}{81}\zeta_3\\
&+\frac{736}{9}\zeta_2\zeta_3-\frac{416}{3}\zeta_5,\\
\mathcal{F}^q_{3,5}=&\frac{1}{\varepsilon^4}\left(-\frac{512}{81}\right)+\frac{1}{\varepsilon^3}\left(\frac{6016}{243}\right)+\frac{1}{\varepsilon^2}\left(-\frac{1984}{27}-\frac{64}{9}\zeta_2\right)+\frac{1}{\varepsilon}\left(\frac{399200}{2187}+\frac{752}{27}\zeta_2-\frac{1088}{81}\zeta_3\right)\\
&-\frac{2710864}{6561}-\frac{248}{3}\zeta_2-\frac{332}{135}\zeta_2^2+\frac{12784}{243}\zeta_3,\\
\mathcal{F}^q_{3,6}=&4-\frac{2}{5}\zeta_2^2+10\zeta_2+\frac{14}{3}\zeta_3-\frac{80}{3}\zeta_5,\\
\mathcal{F}^q_{3,3a}=&\frac{1}{\varepsilon^3}\left(-\frac{32}{9}\right)+\frac{1}{\varepsilon^2}\left(\frac{656}{27}-\frac{128}{9}\zeta_3\right)+\frac{1}{\varepsilon}\left(\frac{1472}{27}\zeta_3-\frac{8012}{81}-4\zeta_2+\frac{64}{15}\zeta_2^2\right)+\left(\frac{76781}{243}+\frac{82}{3}\zeta_2\right.\\
&\left.-\frac{736}{45}\zeta_2^2-\frac{14180}{81}\zeta_3-16\zeta_2\zeta_3-\frac{224}{9}\zeta_5\right),\\
\mathcal{F}^q_{3,3b}=&\frac{1}{\varepsilon^5}\left(\frac{128}{3}\right)+\frac{1}{\varepsilon^4}\left(-\frac{1184}{9}\right)+\frac{1}{\varepsilon^3}\left(\frac{8816}{27}+\frac{16}{3}\zeta_2\right)+\frac{1}{\varepsilon^2}\left(144\zeta_3-\frac{53960}{81}-\frac{532}{9}\zeta_2\right)\\
&+\frac{1}{\varepsilon}\left(\frac{301766}{243}+\frac{5806}{27}\zeta_2-\frac{1877}{45}\zeta_2^2-\frac{1300}{3}\zeta_3\right)-\frac{3192859}{1458}-\frac{47449}{81}\zeta_2+\frac{49577}{540}\zeta_2^2+\frac{12910}{9}\zeta_3\\
&-\frac{542}{9}\zeta_2\zeta_3+\frac{1676}{45}\zeta_5. \qquad\qquad (B.2)
\end{aligned}
$$

The FF $\mathcal{F}^b_{i,j}$ for bottom quark annihilation at third order are given by,

$$
\begin{aligned}
\mathcal{F}^b_{3,0}=&\frac{1}{\epsilon^6}\left(-\frac{256}{3}\right)+\frac{1}{\epsilon^4}\left(-64+32\zeta_2\right)+\frac{1}{\epsilon^3}\left(64+96\zeta_2-\frac{800}{3}\zeta_3\right)+\frac{1}{\epsilon^2}\left(-112-104\zeta_2+\frac{426}{5}\zeta_2^2\right.\\
&\left.+240\zeta_3\right)+\frac{1}{\epsilon}\left(\frac{476}{3}+188\zeta_2-\frac{252}{5}\zeta_2^2-\frac{1568}{3}\zeta_3+\frac{428}{3}\zeta_2\zeta_3-\frac{1288}{5}\zeta_5\right)+\left(-\frac{385}{3}-\frac{1085}{3}\zeta_2\right.\\
&\left.+\frac{887}{10}\zeta_2^2-\frac{9095}{252}\zeta_2^3+538\zeta_3+202\zeta_2\zeta_3-\frac{1826}{3}\zeta_3^2+676\zeta_5\right),\\
\mathcal{F}^b_{3,1}=&\frac{1}{\epsilon^5}\left(-\frac{352}{3}\right)+\frac{1}{\epsilon^4}\left(\frac{1072}{9}-32\zeta_2\right)+\frac{1}{\epsilon^3}\left(-\frac{4312}{27}-\frac{44}{3}\zeta_2+208\zeta_3\right)+\frac{1}{\epsilon^2}\left(\frac{18028}{81}+\frac{1262}{9}\zeta_2\right.\\
&\left.-\frac{332}{5}\zeta_2^2-540\zeta_3\right)+\frac{1}{\epsilon}\left(-\frac{76024}{243}-\frac{10199}{27}\zeta_2+\frac{31591}{180}\zeta_2^2+\frac{3442}{3}\zeta_3-\frac{430}{3}\zeta_2\zeta_3+284\zeta_5\right)\\
&+\frac{332065}{729}+\frac{131161}{162}\zeta_2-\frac{305831}{1080}\zeta_2^2-\frac{18619}{1260}\zeta_2^3-\frac{17273}{9}\zeta_3-\frac{1663}{18}\zeta_2\zeta_3+\frac{1616}{3}\zeta_3^2-\frac{27829}{45}\zeta_5,\\
\mathcal{F}^b_{3,2}=&\frac{1}{\epsilon^4}\left(-\frac{3872}{81}\right)+\frac{1}{\epsilon^3}\left(\frac{26032}{243}-\frac{704}{27}\zeta_2\right)+\frac{1}{\epsilon^2}\left(-\frac{38828}{243}-\frac{2212}{81}\zeta_2-\frac{352}{45}\zeta_2^2+\frac{6688}{27}\zeta_3\right)\\
&+\frac{1}{\epsilon}\left(\frac{385325}{2187}+\frac{31966}{243}\zeta_2-\frac{1604}{15}\zeta_2^2-\frac{17084}{27}\zeta_3+\frac{176}{9}\zeta_2\zeta_3+\frac{272}{3}\zeta_5\right)-\frac{1870897}{26244}-\frac{478157}{1458}\zeta_2\\
&+\frac{100597}{540}\zeta_2^2-\frac{6152}{189}\zeta_2^3+\frac{306992}{243}\zeta_3-\frac{980}{9}\zeta_2\zeta_3-\frac{1136}{9}\zeta_3^2+\frac{3472}{9}\zeta_5,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}^b_{3,3} &= \frac{1}{\epsilon^5}\left(\frac{128}{3}\right) - \frac{1}{\epsilon^4}\left(\frac{320}{9}\right) + \frac{1}{\epsilon^3}\left(\frac{1664}{27} + \frac{16}{3}\zeta_2\right) - \frac{1}{\epsilon^2}\left(\frac{6920}{81} + \frac{424}{9}\zeta_2 - \frac{1168}{9}\zeta_3\right) + \frac{1}{\epsilon}\Big(\frac{31022}{243} \\
&\quad + \frac{2944}{27}\zeta_2 - \frac{337}{9}\zeta_2^2 - \frac{7528}{27}\zeta_3\Big) - \frac{307879}{1458} - \frac{16885}{81}\zeta_2 + \frac{15769}{270}\zeta_2^2 + \frac{55624}{81}\zeta_3 - \frac{686}{9}\zeta_2\zeta_3 + \frac{556}{45}\zeta_5\,, \\
\mathcal{F}^b_{3,3a} &= \frac{1}{\epsilon^3}\left(-\frac{32}{9}\right) + \frac{1}{\epsilon^2}\left(\frac{440}{27} - \frac{128}{9}\zeta_3\right) + \frac{1}{\epsilon}\left(-\frac{3638}{81} - 4\zeta_2 + \frac{64}{15}\zeta_2^2 + \frac{608}{27}\zeta_3\right) - \frac{51259}{486} + \frac{55}{3}\zeta_2 \\
&\quad - \frac{304}{45}\zeta_2^2 - \frac{4460}{81}\zeta_3 - 16\zeta_2\zeta_3 - \frac{224}{9}\zeta_5\,, \\
\mathcal{F}^b_{3,3b} &= \frac{1}{\epsilon^5}\left(\frac{128}{3}\right) - \frac{1}{\epsilon^4}\left(\frac{320}{9}\right) + \frac{1}{\epsilon^3}\left(\frac{1760}{27} + \frac{16}{3}\zeta_2\right) + \frac{1}{\epsilon^2}\left(-\frac{8240}{81} - \frac{424}{9}\zeta_2 + 144\zeta_3\right) + \frac{1}{\epsilon}\Big(\frac{41936}{243} \\
&\quad + \frac{3052}{27}\zeta_2 - \frac{1877}{45}\zeta_2^2 - \frac{904}{3}\zeta_3\Big) - \frac{230828}{729} - \frac{18370}{81}\zeta_2 + \frac{17593}{270}\zeta_2^2 + \frac{6676}{9}\zeta_3 - \frac{542}{9}\zeta_2\zeta_3 + \frac{1676}{45}\zeta_5\,, \\
\mathcal{F}^b_{3,4} &= \frac{1}{\epsilon^4}\left(\frac{2816}{81}\right) - \frac{1}{\epsilon^3}\left(\frac{17056}{243} - \frac{256}{27}\zeta_2\right) + \frac{1}{\epsilon^2}\left(\frac{27280}{243} + \frac{2528}{81}\zeta_2 - \frac{2048}{27}\zeta_3\right) + \frac{1}{\epsilon}\Big(-\frac{20132}{243}\zeta_2 \\
&\quad + \frac{176}{5}\zeta_2^2 + \frac{20560}{81}\zeta_3 - \frac{361220}{2187}\Big) + \frac{1451329}{6561} + \frac{127142}{729}\zeta_2 - \frac{7582}{135}\zeta_2^2 - \frac{47524}{81}\zeta_3 + \frac{736}{9}\zeta_2\zeta_3 \\
&\quad - \frac{416}{3}\zeta_5\,, \\
\mathcal{F}^b_{3,5} &= \frac{1}{\epsilon^4}\left(-\frac{512}{81}\right) + \frac{1}{\epsilon^3}\left(\frac{2560}{243}\right) + \frac{1}{\epsilon^2}\left(-\frac{512}{27} - \frac{64}{9}\zeta_2\right) + \frac{1}{\epsilon}\left(\frac{76928}{2187} + \frac{320}{27}\zeta_2 - \frac{1088}{81}\zeta_3\right) \\
&\quad - \frac{438112}{6561} - \frac{64}{3}\zeta_2 - \frac{332}{135}\zeta_2^2 + \frac{5440}{243}\zeta_3\,, \\
\mathcal{F}^b_{3,6} &= 0\,.
\end{aligned}
\tag{B.3}
$$

C Cusp anomalous dimension $A_I^{(i,j)}$s

The cusp anomalous dimensions $A_I^{(i,j)}$, $I = q, b$ up to three loop order are found to be,

$$
\begin{aligned}
A_I^{(1,0)} &= 4C_F\,, \qquad A_I^{(0,1)} = 4e_I^2\,, \qquad A_I^{(1,1)} = 0\,, \\
A_I^{(2,0)} &= \left(8C_AC_F\left(\frac{67}{18} - \zeta_2\right) + 8C_Fn_fT_F\left(-\frac{10}{9}\right)\right), \\
A_I^{(0,2)} &= 8e_I^2\left(N\sum_q e_q^2 + \sum_l e_l^2\right)\left(-\frac{10}{9}\right), \\
A_I^{(3,0)} &= C_A^2C_F\left(\frac{490}{3} - \frac{1072}{9}\zeta_2 + \frac{176}{5}\zeta_2^2 + \frac{88}{3}\zeta_3\right) + C_AC_Fn_fT_F\Big(-\frac{1672}{27} + \frac{320}{9}\zeta_2 \\
&\qquad - \frac{224}{3}\zeta_3\Big) + C_F^2n_fT_F\left(-\frac{220}{3} + 64\zeta_3\right) + C_Fn_f^2T_F^2\left(-\frac{64}{27}\right), \\
A_I^{(1,2)} &= C_Fe_I^2\left(N\sum_q e_q^2 + \sum_l e_l^2\right)\left(-\frac{220}{3} + 64\zeta_3\right), \\
A_I^{(2,1)} &= C_FT_F\left(\sum_q e_q^2 + \sum_l e_l^2\right)\left(-\frac{220}{3} + 64\zeta_3\right), \\
A_I^{(0,3)} &= e_I^2\left(N\sum_q e_q^4 + \sum_l e_l^4\right)\left(-\frac{220}{3} + 64\zeta_3\right) + e_I^2\left(N\sum_q e_q^2 + \sum_l e_l^2\right)^2\left(-\frac{64}{27}\right).
\end{aligned}
\tag{C.1}
$$

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D UV anomalous dimensions $\gamma_b^{(i,j)}$s

The UV anomalous dimensions $\gamma_b^{(i,j)}$ up to three loop order are found to be,

$$
\begin{aligned}
\gamma_b^{(1,0)} &= 3C_F\,, \qquad \gamma_b^{(0,1)} = 3e_b^2\,, \qquad \gamma_b^{(1,1)} = 3C_F e_b^2\,,\\
\gamma_b^{(2,0)} &= \frac{3}{2}C_F^2 + \frac{97}{6}C_A C_F - \frac{10}{3}C_F n_f T_F\,, \quad \gamma_b^{(0,2)} = \frac{3}{2}e_b^4 - \frac{10}{3}e_b^2\Big(N\sum_q e_q^2 + \sum_l e_l^2\Big)\,,\\
\gamma_b^{(3,0)} &= C_A^2 C_F\Big(\frac{11413}{108}\Big) + C_A C_F^2\Big(-\frac{129}{4}\Big) + C_A C_F n_f T_F\Big(-\frac{556}{27} - 48\zeta_3\Big) + C_F^3\Big(\frac{129}{2}\Big)\\
&\quad + C_F n_f^2 T_F^2\Big(-\frac{140}{27}\Big) + C_F^2 n_f T_F\Big(-46 + 48\zeta_3\Big)\,,\\
\gamma_b^{(1,2)} &= 3C_F e_b^4\Big(\frac{129}{2}\Big) + C_F e_b^2\Big(N\sum_q e_q^2 + \sum_l e_l^2\Big)\Big(-46 + 48\zeta_3\Big)\,,\\
\gamma_b^{(2,1)} &= C_A C_F e_b^2\Big(-\frac{129}{4}\Big) + 3C_F^2 e_b^2\Big(\frac{129}{2}\Big) + C_F n_f T_F e_b^2\Big(-1\Big)\\
&\quad + C_F T_F\Big(\sum_q e_q^2 + \sum_l e_l^2\Big)\Big(-45 + 48\zeta_3\Big)\,,\\
\gamma_b^{(0,3)} &= e_b^6\Big(\frac{129}{2}\Big) + e_b^2\Big(N\sum_q e_q^2 + \sum_l e_l^2\Big)^2\Big(-\frac{140}{27}\Big) + e_b^4\Big(N\sum_q e_q^2 + \sum_l e_l^2\Big)\Big(-1\Big)\\
&\quad + e_b^2\Big(N\sum_q e_q^4 + \sum_l e_l^4\Big)\Big(-45 + 48\zeta_3\Big)\,.
\end{aligned}
\tag{D.1}
$$

E Soft anomalous dimensions $f_I^{(i,j)}$s

The soft anomalous dimensions $f_I^{(i,j)}$ up to three loop order are found to be,

$$
\begin{aligned}
f_I^{(1,0)} &= 0\,, \qquad f_I^{(0,1)} = 0\,, \qquad f_I^{(1,1)} = 0\,,\\
f_I^{(2,0)} &= C_A C_F\Big(-\frac{22}{3}\zeta_2 - 28\zeta_3 + \frac{808}{27}\Big) + C_F n_f T_F\Big(\frac{8}{3}\zeta_2 - \frac{224}{27}\Big)\,,\\
f_I^{(0,2)} &= e_I^2\Big(N\sum_q e_q^2 + \sum_l e_l^2\Big)\Big(\frac{8}{3}\zeta_2 - \frac{224}{27}\Big)\,,\\
f_I^{(3,0)} &= C_A^2 C_F\Big(\frac{136781}{729} - \frac{12650}{81}\zeta_2 + \frac{352}{5}\zeta_2^2 - \frac{1316}{3}\zeta_3 + \frac{176}{3}\zeta_2\zeta_3 + 192\zeta_5\Big)\\
&\quad + C_A C_F n_f T_F\Big(-\frac{23684}{729} + \frac{5656}{81}\zeta_2 - \frac{192}{5}\zeta_2^2 + \frac{1456}{27}\zeta_3\Big) + C_F n_f^2 T_F^2\\
&\quad \times\Big(-\frac{8320}{729} - \frac{160}{27}\zeta_2 + \frac{448}{27}\zeta_3\Big) + C_F^2 n_f T_F\Big(-\frac{3422}{27} + 8\zeta_2 + \frac{64}{5}\zeta_2^2 + \frac{608}{9}\zeta_3\Big)\,,\\
f_I^{(1,2)} &= C_F e_I^2\Big(N\sum_q e_q^2 + \sum_l e_l^2\Big)\Big(-\frac{3422}{27} + 8\zeta_2 + \frac{64}{5}\zeta_2^2 + \frac{608}{9}\zeta_3\Big)\,,\\
f_I^{(2,1)} &= C_F T_F\Big(\sum_q e_q^2 + \sum_l e_l^2\Big)\Big(-\frac{3422}{27} + 8\zeta_2 + \frac{64}{5}\zeta_2^2 + \frac{608}{9}\zeta_3\Big)\,,
\end{aligned}
$$

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$$f_I^{(0,3)} = e_I^2\Big(N\sum_q e_q^2 + \sum_l e_l^2\Big)^2\Big(-\frac{8320}{729} - \frac{160}{27}\zeta_2 + \frac{448}{27}\zeta_3\Big) + e_I^2\Big(N\sum_q e_q^4 + \sum_l e_l^4\Big)$$
$$\times\Big(-\frac{3422}{27} + 8\zeta_2 + \frac{64}{5}\zeta_2^2 + \frac{608}{9}\zeta_3\Big). \tag{E.1}$$

F Collinear anomalous dimensions $B_I^{(i,j)}$s

The collinear anomalous dimensions $B_I^{(i,j)}$ up to three loop order are found to be,

$$B_I^{(1,0)} = 3C_F, \qquad B_I^{(0,1)} = 3e_I^2, \qquad B_I^{(1,1)} = C_F e_I^2\Big(3 - 24\zeta_2 + 48\zeta_3\Big),$$
$$B_I^{(2,0)} = \frac{1}{2}\Big\{C_F^2(3 - 24\zeta_2 + 48\zeta_3) + C_A C_F\Big(\frac{17}{3} + \frac{88}{3}\zeta_2 - 24\zeta_3\Big)$$
$$+ C_F n_f T_F\Big(-\frac{4}{3} - \frac{32}{3}\zeta_2\Big)\Big\},$$
$$B_I^{(0,2)} = \frac{1}{2}\Big\{e_I^4(3 - 24\zeta_2 + 48\zeta_3) + e_I^2\Big(N\sum_q e_q^2 + \sum_l e_l^2\Big)\Big(-\frac{4}{3} - \frac{32}{3}\zeta_2\Big)\Big\},$$
$$B_I^{(3,0)} = C_A^2 C_F\Big(-\frac{1657}{36} + \frac{4496}{27}\zeta_2 - 2\zeta_2^2 - \frac{1552}{9}\zeta_3 + 40\zeta_5\Big)$$
$$+ C_A C_F n_f T_F\Big(40 - \frac{2672}{27}\zeta_2 + \frac{8}{5}\zeta_2^2 + \frac{400}{9}\zeta_3\Big) + C_F n_f^2 T_F^2\Big(-\frac{68}{9} + \frac{320}{27}\zeta_2 - \frac{64}{9}\zeta_3\Big)$$
$$+ C_A C_F^2\Big(\frac{151}{4} - \frac{410}{3}\zeta_2 - \frac{988}{15}\zeta_2^2 + \frac{844}{3}\zeta_3 + 16\zeta_2\zeta_3 + 120\zeta_5\Big)$$
$$+ C_F^3\Big(\frac{29}{2} + 18\zeta_2 + \frac{288}{5}\zeta_2^2 + 68\zeta_3 - 32\zeta_2\zeta_3 - 240\zeta_5\Big)$$
$$+ C_F^2 n_f T_F\Big(-46 + \frac{40}{3}\zeta_2 + \frac{464}{15}\zeta_2^2 - \frac{272}{3}\zeta_3\Big),$$
$$B_I^{(1,2)} = 3C_F e_I^4\Big(\frac{29}{2} + 18\zeta_2 + \frac{288}{5}\zeta_2^2 + 68\zeta_3 - 32\zeta_2\zeta_3 - 240\zeta_5\Big) + C_F e_I^2\Big(N\sum_q e_q^2 + \sum_l e_l^2\Big)$$
$$\times\Big(-46 + \frac{40}{3}\zeta_2 + \frac{464}{15}\zeta_2^2 - \frac{272}{3}\zeta_3\Big),$$
$$B_I^{(2,1)} = C_F C_A e_I^2\Big(\frac{151}{4} - \frac{410}{3}\zeta_2 - \frac{988}{15}\zeta_2^2 + \frac{844}{3}\zeta_3 + 16\zeta_2\zeta_3 + 120\zeta_5\Big)$$
$$+ 3C_F^2 e_I^2\Big(\frac{29}{2} + 18\zeta_2 + \frac{288}{5}\zeta_2^2 + 68\zeta_3 - 32\zeta_2\zeta_3 - 240\zeta_5\Big) + C_F T_F\Big(\sum_q e_q^2 + \sum_l e_l^2\Big)$$
$$\times\Big(-37 - 16\zeta_2 + 48\zeta_3\Big) + C_F n_f T_F e_I^2\Big(-9 + \frac{88}{3}\zeta_2 + \frac{464}{15}\zeta_2^2 - \frac{416}{3}\zeta_3\Big),$$

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$$B_I^{(0,3)} = e_I^2\left(N\sum_q e_q^2+\sum_l e_l^2\right)^2\left(-\frac{68}{9}+\frac{320}{27}\zeta_2-\frac{64}{9}\zeta_3\right)+e_I^6\left(\frac{29}{2}+18\zeta_2+\frac{288}{5}\zeta_2^2\right.$$
$$\left.+68\zeta_3-32\zeta_2\zeta_3-240\zeta_5\right)+e_I^2\left(N\sum_q e_q^4+\sum_l e_l^4\right)\left(-37-16\zeta_2+48\zeta_3\right)$$
$$+e_I^4\left(N\sum_q e_q^2+\sum_l e_l^2\right)\left(-9+\frac{88}{3}\zeta_2+\frac{464}{15}\zeta_2^2-\frac{416}{3}\zeta_3\right). \tag{F.1}$$

G $D_I^{(i,j)}$s in soft distribution function

The constants $D_I^{(i,j)}$ in soft distributon functions up to three loop order are found to be,

$$D_I^{(1,0)}=0, \qquad D_I^{(0,1)}=0, \qquad D_I^{(1,1)}=0.$$
$$D_I^{(2,0)}=C_Fn_fT_F\left(\frac{448}{27}-\frac{64}{3}\zeta_2\right)+C_FC_A\left(-\frac{1616}{27}+56\zeta_3+\frac{176}{3}\zeta_2\right)$$
$$D_I^{(0,2)}=e_I^2\left(N\sum_q e_q^2+\sum_l e_l^2\right)\left(\frac{448}{27}-\frac{64}{3}\zeta_2\right),$$
$$D_I^{(3,0)}=C_Fn_f^2T_F^2\left(-\frac{14848}{729}+\frac{1280}{27}\zeta_3+\frac{2560}{27}\zeta_2\right)+C_FC_An_fT_F\left(\frac{250504}{729}-\frac{4960}{9}\zeta_3\right.$$
$$\left.-\frac{58784}{81}\zeta_2+\frac{1472}{15}\zeta_2^2\right)+C_FC_A^2\left(-\frac{594058}{729}-384\zeta_5+\frac{40144}{27}\zeta_3+\frac{98224}{81}\zeta_2-\frac{352}{3}\zeta_2\zeta_3\right.$$
$$\left.-\frac{2992}{15}\zeta_2^2\right)+C_F^2n_fT_F\left(\frac{6844}{27}-\frac{1216}{9}\zeta_3-64\zeta_2-\frac{128}{5}\zeta_2^2\right),$$
$$D_I^{(1,2)}=C_Fe_I^2\left(N\sum_q e_q^2+\sum_l e_l^2\right)\left(\frac{6844}{27}-\frac{1216}{9}\zeta_3-64\zeta_2-\frac{128}{5}\zeta_2^2\right),$$
$$D_I^{(2,1)}=C_FT_F\left(\sum_q e_q^2+\sum_l e_l^2\right)\left(\frac{6844}{27}-\frac{1216}{9}\zeta_3-64\zeta_2-\frac{128}{5}\zeta_2^2\right),$$
$$D_I^{(0,3)}=e_I^2\left(N\sum_q e_q^2+\sum_l e_l^2\right)^2\left(-\frac{14848}{729}+\frac{1280}{27}\zeta_3+\frac{2560}{27}\zeta_2\right),$$
$$+e_I^2\left(N\sum_q e_q^4+\sum_l e_l^4\right)\left(\frac{6844}{27}-\frac{1216}{9}\zeta_3-64\zeta_2-\frac{128}{5}\zeta_2^2\right). \tag{G.1}$$

H $\Delta_I^{(i,j),SV}$ for QCD, QED and QCD-QED up to N^{3}LO

Here we present the soft virtual cross-section $\Delta_I^{(i,j),SV}$ defined in eq. (4.4) at the third order in the strong and electromagnetic coupling constants. The finite cross-section upto two loop is already available in appendix C of [29]. At the third order, $\Delta_I^{(i,j),SV}$ takes the following form:

$$\Delta_{I\bar{I}}^{(3,0),SV}=\Big[C_F^3\Delta_{(3,1)}^{I\bar{I}}+C_F^2C_A\Delta_{(3,2)}^{I\bar{I}}+C_FC_A^2\Delta_{(3,3)}^{I\bar{I}}+C_F^2n_fT_F\Delta_{(3,4)}^{I\bar{I}}+C_FC_An_fT_F\Delta_{(3,5)}^{I\bar{I}}$$
$$+C_Fn_f^2T_F^2\Delta_{(3,6)}^{I\bar{I}}\Big],$$

$$
\begin{aligned}
\Delta^{(0,3),SV}_{I\bar{I}} &= \Bigg[e_I^6\Delta^{I\bar{I}}_{(3,1)}+e_I^2\Bigg(N\sum_q e_q^4+\sum_l e_l^4\Bigg)\Delta^{I\bar{I}}_{(3,4a)}+e_I^4\Bigg(N\sum_q e_q^2+\sum_l e_l^2\Bigg)\Delta^{I\bar{I}}_{(3,4b)} \\
&\quad +e_I^2\Bigg(N\sum_q e_q^2+\sum_l e_l^2\Bigg)^2\Delta^{I\bar{I}}_{(3,6)}\Bigg], \\
\Delta^{(2,1),SV}_{I\bar{I}} &= \Bigg[3C_F^2e_I^2\Delta^{I\bar{I}}_{(3,1)}+C_FC_Ae_I^2\Delta^{I\bar{I}}_{(3,2)}+C_FT_F\Bigg(\sum_q e_q^2+\sum_l e_l^2\Bigg)\Delta^{I\bar{I}}_{(3,4a)}+C_Fn_fT_Fe_I^2\Delta^{I\bar{I}}_{(3,4b)}\Bigg], \\
\Delta^{(1,2),SV}_{I\bar{I}} &= \Bigg[3C_Fe_I^4\Delta^{I\bar{I}}_{(3,1)}+C_Fe_I^2\Bigg(N\sum_q e_q^2+\sum_l e_l^2\Bigg)\Delta^{I\bar{I}}_{(3,4)}\Bigg]. \qquad \text{(H.1)}
\end{aligned}
$$

For $I\bar{I}=q\bar{q},b\bar{b}$ denotes the Drell-Yan pair production and the Higgs boson production in bottom quark annihilation, respectively. The coefficients for the above color factors are given as:

$$
\begin{aligned}
\Delta^{q\bar{q}}_{(3,1)} &= \delta(1-z)\Bigg(94\zeta_2-\frac{2036}{3}+960\zeta_3+\frac{10240}{3}\zeta_3^2-480\zeta_2\zeta_3+\frac{2192}{5}\zeta_2^2-\frac{75968}{105}\zeta_2^3\Bigg) \\
&\quad +\mathcal{D}_0\big(12288\zeta_5-4096\zeta_3-6144\zeta_2\zeta_3\big)+\mathcal{D}_1\Bigg(2044-960\zeta_3+2976\zeta_2-\frac{14208}{5}\zeta_2^2\Bigg) \\
&\quad +\mathcal{D}_2\big(10240\zeta_3\big)+\mathcal{D}_3\big(-2048-3072\zeta_2\big)+\mathcal{D}_5\big(512\big), \\
\Delta^{q\bar{q}}_{(3,2)} &= \delta(1-z)\Bigg(\frac{6140}{3}-\frac{11264}{3}\zeta_5-\frac{37952}{27}\zeta_3+896\zeta_3^2-\frac{18088}{9}\zeta_2+4096\zeta_2\zeta_3-\frac{6928}{45}\zeta_2^2 \\
&\quad +32\zeta_2^3\Bigg)+\mathcal{D}_0\Bigg(\frac{25856}{27}+\frac{26240}{9}\zeta_3-\frac{12416}{27}\zeta_2-1472\zeta_2\zeta_3+\frac{1408}{3}\zeta_2^2\Bigg)+\mathcal{D}_1\Bigg(-\frac{35572}{9} \\
&\quad -5184\zeta_3-\frac{11648}{9}\zeta_2+\frac{3648}{5}\zeta_2^2\Bigg)+\mathcal{D}_2\Bigg(-\frac{4480}{9}+1344\zeta_3+\frac{11264}{3}\zeta_2\Bigg)+\mathcal{D}_3\Bigg(\frac{17152}{9} \\
&\quad -512\zeta_2\Bigg)+\mathcal{D}_4\Bigg(-\frac{7040}{9}\Bigg), \\
\Delta^{q\bar{q}}_{(3,3)} &= \delta(1-z)\Bigg(\frac{1088}{9}+\frac{74422}{27}\zeta_2-1056\zeta_2\zeta_3-\frac{1522}{9}\zeta_2^2+\frac{528}{5}\zeta_2^3\Bigg)+\mathcal{D}_0\Bigg(-\frac{594058}{729}-384\zeta_5 \\
&\quad +\frac{40144}{27}\zeta_3+\frac{98224}{81}\zeta_2-\frac{352}{3}\zeta_2\zeta_3-\frac{2992}{15}\zeta_2^2\Bigg)+\mathcal{D}_1\Bigg(\frac{124024}{81}-704\zeta_3-\frac{12032}{9}\zeta_2 \\
&\quad +\frac{704}{5}\zeta_2^2\Bigg)+\mathcal{D}_2\Bigg(-\frac{28480}{27}+\frac{704}{3}\zeta_2\Bigg)+\mathcal{D}_3\Bigg(\frac{7744}{27}\Bigg), \\
\Delta^{q\bar{q}}_{(3,4)} &= \delta(1-z)\Bigg(-720+\frac{4096}{3}\zeta_5+\frac{256}{27}\zeta_3+\frac{3824}{9}\zeta_2-1088\zeta_2\zeta_3+\frac{448}{9}\zeta_2^2\Bigg)+\mathcal{D}_0\Bigg(-\frac{11456}{9}\zeta_3 \\
&\quad -12+\frac{3904}{27}\zeta_2-\frac{2944}{15}\zeta_2^2\Bigg)+\mathcal{D}_1\Bigg(\frac{8576}{9}+2560\zeta_3+\frac{4096}{9}\zeta_2\Bigg)+\mathcal{D}_2\Bigg(\frac{1088}{9}-\frac{4096}{3}\zeta_2\Bigg) \\
&\quad +\mathcal{D}_3\Bigg(-\frac{5120}{9}\Bigg)+\mathcal{D}_4\Bigg(\frac{2560}{9}\Bigg), \\
\Delta^{q\bar{q}}_{(3,4a)} &= \delta(1-z)\Bigg(-\frac{128}{3}-288\zeta_2+192\zeta_2\zeta_3\Bigg)+\mathcal{D}_0\Bigg(\frac{6844}{27}-\frac{1216}{9}\zeta_3-64\zeta_2-\frac{128}{5}\zeta_2^2\Bigg) \\
&\quad +\mathcal{D}_1\Bigg(-\frac{880}{3}+256\zeta_3\Bigg)+\mathcal{D}_2\big(64\big),
\end{aligned}
$$

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$$\Delta^{q\bar{q}}_{(3,4b)} = \delta(1-z)\Big(-\frac{2032}{3}+\frac{4096}{3}\zeta_5+\frac{256}{27}\zeta_3+\frac{6416}{9}\zeta_2-1280\zeta_2\zeta_3+\frac{448}{9}\zeta_2^2\Big)+\mathcal{D}_0\Big(-\frac{7168}{27} -\frac{10240}{9}\zeta_3+\frac{5632}{27}\zeta_2-\frac{512}{3}\zeta_2^2\Big)+\mathcal{D}_1\Big(\frac{11216}{9}+2304\zeta_3+\frac{4096}{9}\zeta_2\Big)+\mathcal{D}_2\Big(\frac{512}{9}-\frac{4096}{3}\zeta_2\Big) +\mathcal{D}_3\Big(-\frac{5120}{9}\Big)+\mathcal{D}_4\Big(\frac{2560}{9}\Big),$$

$$\Delta^{q\bar{q}}_{(3,5)} = \delta(1-z)\Big(-\frac{640}{9}-\frac{46828}{27}\zeta_2+192\zeta_2\zeta_3-\frac{272}{9}\zeta_2^2\Big)+\mathcal{D}_0\Big(\frac{250504}{729}-\frac{4960}{9}\zeta_3-\frac{58784}{81}\zeta_2 +\frac{1472}{15}\zeta_2^2\Big)+\mathcal{D}_1\Big(-\frac{65632}{81}+768\zeta_2\Big)+\mathcal{D}_2\Big(\frac{18496}{27}-\frac{256}{3}\zeta_2\Big)+\mathcal{D}_3\Big(-\frac{5632}{27}\Big),$$

$$\Delta^{q\bar{q}}_{(3,6)} = \delta(1-z)\Big(\frac{6496}{27}\zeta_2+\frac{224}{9}\zeta_2^2\Big)+\mathcal{D}_0\Big(-\frac{14848}{729}+\frac{1280}{27}\zeta_3+\frac{2560}{27}\zeta_2\Big)+\mathcal{D}_1\Big(\frac{6400}{81}-\frac{1024}{9}\zeta_2\Big) +\mathcal{D}_2\Big(-\frac{2560}{27}\Big)+\mathcal{D}_3\Big(\frac{1024}{27}\Big). \tag{H.2}$$

Similarly $\Delta^{b\bar{b}}_{(i,j)}$ for bottom quark annihilation at third order are given as,

$$\Delta^{b\bar{b}}_{(3,1)} = \delta(1-z)\Big(-\frac{128}{3}+240\zeta_3+\frac{10240}{3}\zeta_3^2-480\zeta_2\zeta_3+\frac{1248}{5}\zeta_2^2-\frac{75968}{105}\zeta_2^3\Big) +\mathcal{D}_0\Big(12288\zeta_5-1024\zeta_3-6144\zeta_2\zeta_3\Big)+\mathcal{D}_1\Big(256-960\zeta_3+1024\zeta_2-\frac{14208}{5}\zeta_2^2\Big) +\mathcal{D}_2\Big(10240\zeta_3\Big)+\mathcal{D}_3\Big(-512-3072\zeta_2\Big)+\mathcal{D}_5\Big(512\Big),$$

$$\Delta^{b\bar{b}}_{(3,2)} = \delta(1-z)\Big(-\frac{664}{9}-\frac{11264}{3}\zeta_5-\frac{24992}{27}\zeta_3+896\zeta_3^2+\frac{400}{9}\zeta_2+3808\zeta_2\zeta_3-\frac{22624}{45}\zeta_2^2 +32\zeta_2^3\Big)+\mathcal{D}_0\Big(\frac{6464}{27}+\frac{32288}{9}\zeta_3+\frac{6592}{27}\zeta_2-1472\zeta_2\zeta_3+\frac{1408}{3}\zeta_2^2\Big)+\mathcal{D}_1\Big(-\frac{544}{3} -5760\zeta_3-\frac{20864}{9}\zeta_2+\frac{3648}{5}\zeta_2^2\Big)+\mathcal{D}_2\Big(-\frac{10816}{9}+1344\zeta_3+\frac{11264}{3}\zeta_2\Big) +\mathcal{D}_3\Big(\frac{17152}{9}-512\zeta_2\Big)+\mathcal{D}_4\Big(-\frac{7040}{9}\Big),$$

$$\Delta^{b\bar{b}}_{(3,3)} = \delta(1-z)\Big(\frac{272}{9}+\frac{23878}{27}\zeta_2-1056\zeta_2\zeta_3-\frac{1522}{9}\zeta_2^2+\frac{528}{5}\zeta_2^3\Big)+\mathcal{D}_0\Big(-\frac{594058}{729}-384\zeta_5 +\frac{40144}{27}\zeta_3+\frac{98224}{81}\zeta_2-\frac{352}{3}\zeta_2\zeta_3-\frac{2992}{15}\zeta_2^2\Big)+\mathcal{D}_1\Big(\frac{124024}{81}-704\zeta_3-\frac{12032}{9}\zeta_2 +\frac{704}{5}\zeta_2^2\Big)+\mathcal{D}_2\Big(-\frac{28480}{27}+\frac{704}{3}\zeta_2\Big)+\mathcal{D}_3\Big(\frac{7744}{27}\Big),$$

$$\Delta^{b\bar{b}}_{(3,4)} = \delta(1-z)\Big(-\frac{160}{9}+\frac{4096}{3}\zeta_5+\frac{5440}{27}\zeta_3-\frac{1532}{9}\zeta_2-1088\zeta_2\zeta_3+\frac{1600}{9}\zeta_2^2\Big) +\mathcal{D}_0\Big(\frac{1684}{9}-\frac{11456}{9}\zeta_3-\frac{3008}{27}\zeta_2-\frac{2944}{15}\zeta_2^2\Big)+\mathcal{D}_1\Big(-\frac{368}{3}+2560\zeta_3+\frac{6400}{9}\zeta_2\Big) +\mathcal{D}_2\Big(\frac{3392}{9}-\frac{4096}{3}\zeta_2\Big)+\mathcal{D}_3\Big(-\frac{5120}{9}\Big)+\mathcal{D}_4\Big(\frac{2560}{9}\Big),$$

$$\Delta^{b\bar{b}}_{(3,4a)} = \delta(1-z)\Big(-\frac{32}{3}-220\zeta_2+192\zeta_2\zeta_3\Big)+\mathcal{D}_0\Big(\frac{6844}{27}-\frac{1216}{9}\zeta_3-64\zeta_2-\frac{128}{5}\zeta_2^2\Big) +\mathcal{D}_1\Big(-\frac{880}{3}+256\zeta_3\Big)+\mathcal{D}_2\Big(64\Big),$$

$$\Delta^{b\bar{b}}_{(3,4b)} = \delta(1-z)\left(-\frac{64}{9}+\frac{4096}{3}\zeta_5+\frac{5440}{27}\zeta_3+\frac{448}{9}\zeta_2-1280\zeta_2\zeta_3+\frac{1600}{9}\zeta_2^2\right)$$
$$+\mathcal{D}_0\left(-\frac{1792}{27}-\frac{10240}{9}\zeta_3-\frac{1280}{27}\zeta_2-\frac{512}{3}\zeta_2^2\right)+\mathcal{D}_1\left(\frac{512}{3}+2304\zeta_3+\frac{6400}{9}\zeta_2\right)$$
$$+\mathcal{D}_2\left(\frac{2816}{9}-\frac{4096}{3}\zeta_2\right)+\mathcal{D}_3\left(-\frac{5120}{9}\right)+\mathcal{D}_4\left(\frac{2560}{9}\right),$$
$$\Delta^{b\bar{b}}_{(3,5)} = \delta(1-z)\left(-\frac{160}{9}-\frac{13816}{27}\zeta_2+192\zeta_2\zeta_3-\frac{272}{9}\zeta_2^2\right)+\mathcal{D}_0\left(\frac{250504}{729}-\frac{4960}{9}\zeta_3-\frac{58784}{81}\zeta_2\right.$$
$$\left.+\frac{1472}{15}\zeta_2^2\right)+\mathcal{D}_1\left(-\frac{65632}{81}+768\zeta_2\right)+\mathcal{D}_2\left(\frac{18496}{27}-\frac{256}{3}\zeta_2\right)+\mathcal{D}_3\left(-\frac{5632}{27}\right),$$
$$\Delta^{b\bar{b}}_{(3,6)} = \delta(1-z)\left(\frac{1600}{27}\zeta_2+\frac{224}{9}\zeta_2^2\right)+\mathcal{D}_0\left(-\frac{14848}{729}+\frac{1280}{27}\zeta_3+\frac{2560}{27}\zeta_2\right)+\mathcal{D}_1\left(\frac{6400}{81}\right.$$
$$\left.-\frac{1024}{9}\zeta_2\right)+\mathcal{D}_2\left(-\frac{2560}{27}\right)+\mathcal{D}_3\left(\frac{1024}{27}\right). \tag{H.3}$$

I $C_{I,0}$ for QCD, QED and QCD×QED up to N^{3}LO

Here we present $C_{I,0}$ in eq. (4.7) with the following expansion in a_s and a_e,

$$C_{I,0} = 1+a_s\left[C_F c^I_{1,1}\right]+a_e\left[e_I^2 c^I_{1,1}\right]+a_s^2\left[C_F^2 c^I_{2,1}+C_A C_F c^I_{2,2}+C_F n_f T_F c^I_{2,3}\right]$$
$$+a_s a_e\left[2C_F e_I^2 c^I_{2,1}\right]+a_e^2\left[e_I^4 c^I_{2,1}+e_I^2\left(N\sum_q e_q^2\right)c^I_{2,3}\right]+a_s^3\left[C_F^3 c^I_{3,1}+C_F^2 C_A c^I_{3,2}\right.$$
$$\left.+C_F C_A^2 c^I_{3,3}+C_F^2 n_f T_F c^I_{3,4}+C_F C_A n_f T_F c^I_{3,5}+C_F n_f^2 T_F^2 c^I_{3,6}\right]$$
$$+a_e^3\left[e_I^6 c^I_{3,1}+e_I^2\left(N\sum_q e_q^4+\sum_l e_l^4\right)c^I_{3,4a}+e_I^4\left(N\sum_q e_q^2+\sum_l e_l^2\right)c^I_{3,4b}+e_I^2\left(N\sum_q e_q^2\right.\right.$$
$$\left.\left.+\sum_l e_l^2\right)^2 c^I_{3,6}\right]+a_s^2 a_e\left[3C_F^2 e_I^2 c^I_{3,1}+C_F C_A e_I^2 c^I_{3,2}+C_F T_F\left(\sum_q e_q^2+\sum_l e_l^2\right)c^I_{3,4a}\right.$$
$$\left.+C_F n_f T_F e_I^2 c^I_{3,4b}\right]+a_s a_e^2\left[3C_F e_I^4 c^I_{3,1}+C_F e_I^2\left(N\sum_q e_q^2+\sum_l e_l^2\right)c^I_{3,4}\right]. \tag{I.1}$$

As before $I{=}q,b$ for the Drell-Yan pair production and the Higgs boson production in bottom quark annihilation, respectively. For brevity we denote $\log\left(\frac{\mu_F^2}{\mu_R^2}\right) = L_{fr}$ and $\log\left(\frac{q^2}{\mu_R^2}\right) = L_{qr}$. The coefficients for the above color factors are,

$$c^q_{1,1} = \left\{-16+8\zeta_2-(6)L_{fr}+(6)L_{qr}\right\},$$
$$c^q_{2,1} = \left\{\frac{511}{4}-70\zeta_2+\frac{72}{5}\zeta_2^2-60\zeta_3+(93-24\zeta_2-48\zeta_3)(L_{fr}-L_{qr})+(-36)L_{fr}L_{qr}\right.$$
$$\left.+(18)(L_{fr}^2+L_{qr}^2)\right\},$$
$$c^q_{2,2} = \left\{-\frac{1535}{12}+\frac{592}{9}\zeta_2-\frac{12}{5}\zeta_2^2+28\zeta_3+\left(-\frac{17}{3}-\frac{88}{3}\zeta_2+24\zeta_3\right)L_{fr}+\left(\frac{193}{3}-24\zeta_3\right)L_{qr}\right.$$
$$\left.+(11)(L_{fr}^2-L_{qr}^2)\right\},$$

$$
\begin{aligned}
c^q_{2,3} &= \Big\{ \frac{127}{3} - \frac{224}{9}\zeta_2 + 16\zeta_3 + \Big(\frac{4}{3} + \frac{32}{3}\zeta_2\Big)L_{fr} + \Big(-\frac{68}{3}\Big)L_{qr} + (4)\big(L_{qr}^2 - L_{fr}^2\big)\Big\}, \\
c^q_{3,1} &= \Big\{ -\frac{2036}{3} + 94\zeta_2 + \frac{1168}{5}\zeta_2^2 - \frac{832}{15}\zeta_2^3 + 960\zeta_3 - 480\zeta_2\zeta_3 + \Big(-\frac{1495}{2} - 24\zeta_2 - \frac{48}{5}\zeta_2^2 \\
&\quad + 992\zeta_3 - 320\zeta_2\zeta_3 + 480\zeta_5\Big)\big(L_{fr} - L_{qr}\big) + \big(-270 + 288\zeta_3\big)\big(L_{fr}^2 + L_{qr}^2\big) \\
&\quad - 36\big(L_{fr}^3 - L_{qr}^3\big) + \big(540 - 576\zeta_3\big)L_{fr}L_{qr} + (-108)\big(L_{fr}L_{qr}^2 - L_{fr}^2L_{qr}\big)\Big\}, \\
c^q_{3,2} &= \Big\{ \frac{6140}{3} - \frac{18088}{9}\zeta_2 + \frac{1648}{45}\zeta_2^2 - \frac{96}{5}\zeta_2^3 - 448\zeta_3 + 1280\zeta_2\zeta_3 + \Big(\frac{2348}{3} + \frac{908}{3}\zeta_2 \\
&\quad - \frac{1328}{15}\zeta_2^2 - \frac{3344}{3}\zeta_3 + 160\zeta_2\zeta_3 - 240\zeta_5\Big)L_{fr} + \Big(-\frac{3439}{2} + \frac{632}{3}\zeta_2 - \frac{256}{15}\zeta_2^2 + \frac{4664}{3}\zeta_3 \\
&\quad - 160\zeta_2\zeta_3 + 240\zeta_5\Big)L_{qr} + \big(-131 + 176\zeta_2 + 32\zeta_3\big)L_{fr}^2 + \big(551 - 320\zeta_3\big)L_{qr}^2 \\
&\quad + \big(-420 - 176\zeta_2 + 288\zeta_3\big)L_{fr}L_{qr} + (66)\big(L_{fr}^2L_{qr} + L_{qr}^2L_{fr} - L_{fr}^3 - L_{qr}^3\big)\Big\}, \\
c^q_{3,3} &= \Big\{ \frac{1088}{9} + \frac{74422}{27}\zeta_2 - \frac{1522}{9}\zeta_2^2 + \frac{528}{5}\zeta_2^3 - 1056\zeta_2\zeta_3 + \Big(\frac{1657}{18} - \frac{8992}{27}\zeta_2 + 4\zeta_2^2 \\
&\quad + \frac{3104}{9}\zeta_3 - 80\zeta_5\Big)L_{fr} + \Big(\frac{3082}{3} - 240\zeta_2 + \frac{68}{5}\zeta_2^2 - \frac{4952}{9}\zeta_3 + 80\zeta_5\Big)L_{qr} \\
&\quad + \Big(\frac{493}{9} + \frac{968}{9}\zeta_2 - 88\zeta_3\Big)L_{fr}^2 + \Big(-\frac{2429}{9} + 88\zeta_3\Big)L_{qr}^2 + \Big(\frac{242}{9}\Big)\big(L_{qr}^3 - L_{fr}^3\big)\Big\}, \\
c^q_{3,4} &= \Big\{ -720 + \frac{3824}{9}\zeta_2 - \frac{64}{9}\zeta_2^2 - 256\zeta_3 - 64\zeta_2\zeta_3 + \Big(-\frac{550}{3} - \frac{112}{3}\zeta_2 + \frac{352}{15}\zeta_2^2 \\
&\quad + \frac{256}{3}\zeta_3\Big)L_{fr} + \Big(460 - \frac{352}{3}\zeta_2 + \frac{224}{15}\zeta_2^2 - \frac{736}{3}\zeta_3\Big)L_{qr} + \big(40 - 64\zeta_2 - 64\zeta_3\big)L_{fr}^2 \\
&\quad - \big(184 - 64\zeta_3\big)L_{qr}^2 + \big(144 + 64\zeta_2\big)L_{fr}L_{qr} + 24\big(L_{qr}^3 + L_{fr}^3 - L_{qr}L_{fr}^2 - L_{qr}^2L_{fr}\big)\Big\}, \\
c^q_{3,5} &= \Big\{ -\frac{640}{9} - \frac{46828}{27}\zeta_2 - \frac{272}{9}\zeta_2^2 + 192\zeta_2\zeta_3 + \Big(-80 + \frac{5344}{27}\zeta_2 - \frac{16}{5}\zeta_2^2 - \frac{800}{9}\zeta_3\Big)L_{fr} \\
&\quad + \Big(-\frac{6104}{9} + \frac{640}{3}\zeta_2 - \frac{16}{5}\zeta_2^2 + \frac{416}{9}\zeta_3\Big)L_{qr} + \Big(-\frac{292}{9} - \frac{704}{9}\zeta_2 + 32\zeta_3\Big)L_{fr}^2 \\
&\quad + \Big(\frac{1700}{9} - 32\zeta_3\Big)L_{qr}^2 + \Big(\frac{176}{9}\Big)\big(L_{fr}^3 - L_{qr}^3\big)\Big\}, \\
c^q_{3,6} &= \Big\{ \frac{6496}{27}\zeta_2 + \frac{224}{9}\zeta_2^2 + \Big(\frac{136}{9} - \frac{640}{27}\zeta_2 + \frac{128}{9}\zeta_3\Big)L_{fr} + \Big(\frac{880}{9} - \frac{128}{3}\zeta_2 + \frac{256}{9}\zeta_3\Big)L_{qr} \\
&\quad + \Big(\frac{16}{9} + \frac{128}{9}\zeta_2\Big)L_{fr}^2 + \Big(-\frac{272}{9}\Big)L_{qr}^2 + \Big(\frac{32}{9}\Big)\big(L_{qr}^3 - L_{fr}^3\big)\Big\}, \\
c^q_{3,4a} &= \Big\{ -\frac{128}{3} - 288\zeta_2 + 192\zeta_2\zeta_3 + \big(74 + 32\zeta_2 - 96\zeta_3\big)L_{fr} + \big(-138 + 96\zeta_3\big)L_{qr} \\
&\quad + (12)\big(L_{qr}^2 - L_{fr}^2\big)\Big\},
\end{aligned}
$$

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$$c^q_{3,4b}=\Big\{-\frac{2032}{3}+\frac{6416}{9}\zeta_2-\frac{64}{9}\zeta_2^2-256\zeta_3-256\zeta_2\zeta_3+\Big(-\frac{772}{3}-\frac{208}{3}\zeta_2+\frac{352}{15}\zeta_2^2$$
$$+\frac{544}{3}\zeta_3\Big)L_{fr}+\Big(598-\frac{352}{3}\zeta_2+\frac{224}{15}\zeta_2^2-\frac{1024}{3}\zeta_3\Big)L_{qr}+\big(144+64\zeta_2\big)L_{qr}L_{fr}$$
$$+\big(52-64\zeta_2-64\zeta_3\big)L_{fr}^2+\big(-196+64\zeta_3\big)L_{qr}^2+24\big(L_{qr}^3+L_{fr}^3$$
$$-L_{fr}^2L_{qr}-L_{qr}^2L_{fr}\big)\Big\}. \tag{I.2}$$

In the following we present $c^b_{i,j}$ for bottom quark annihilation at third order as,

$$c^b_{1,1}=\big\{-4+8\zeta_2\big\}+L_{fr}\big\{-6\big\},$$
$$c^b_{2,1}=\Big\{16-60\zeta_3+\frac{72}{5}\zeta_2^2\Big\}+L_{qr}\big\{48\zeta_3-24\zeta_2\big\}+L_{fr}\big\{21-48\zeta_3-24\zeta_2\big\}+L_{fr}^2\big\{18\big\},$$
$$c^b_{2,2}=\Big\{\frac{166}{9}-8\zeta_3+\frac{232}{9}\zeta_2-\frac{12}{5}\zeta_2^2\Big\}+L_{qr}\big\{-12-24\zeta_3\big\}+L_{fr}\Big\{-\frac{17}{3}+24\zeta_3$$
$$-\frac{88}{3}\zeta_2\Big\}+L_{fr}^2\big\{11\big\},$$
$$c^b_{2,3}=\Big\{\frac{16}{9}+16\zeta_3-\frac{80}{9}\zeta_2\Big\}+L_{fr}\Big\{\frac{4}{3}+\frac{32}{3}\zeta_2\Big\}+L_{fr}^2\big\{-4\big\},$$
$$c^b_{3,1}=\Big\{-\frac{128}{3}+240\zeta_3-480\zeta_2\zeta_3+\frac{992}{5}\zeta_2^2-\frac{832}{15}\zeta_2^3\Big\}+L_{qr}\Big\{-100-480\zeta_5-56\zeta_3$$
$$+132\zeta_2+320\zeta_2\zeta_3-\frac{384}{5}\zeta_2^2\Big\}+L_{fr}\Big\{-113+480\zeta_5+416\zeta_3-156\zeta_2-320\zeta_2\zeta_3$$
$$-\frac{48}{5}\zeta_2^2\Big\}+L_{fr}L_{qr}\big\{-288\zeta_3+144\zeta_2\big\}+L_{fr}^2\big\{-54+288\zeta_3\big\}+L_{fr}^3\big\{-36\big\},$$
$$c^b_{3,2}=\Big\{-\frac{664}{9}+32\zeta_3+\frac{400}{9}\zeta_2+992\zeta_2\zeta_3-\frac{14048}{45}\zeta_2^2-\frac{96}{5}\zeta_2^3\Big\}+L_{qr}\Big\{\frac{388}{3}+240\zeta_5$$
$$+\frac{3296}{3}\zeta_3-604\zeta_2-160\zeta_2\zeta_3-\frac{8}{3}\zeta_2^2\Big\}+L_{qr}^2\big\{-176\zeta_3+88\zeta_2\big\}+L_{fr}\Big\{-\frac{327}{2}$$
$$-240\zeta_5-\frac{1832}{3}\zeta_3+\frac{572}{3}\zeta_2+160\zeta_2\zeta_3-\frac{1328}{15}\zeta_2^2\Big\}+L_{fr}L_{qr}\big\{72+144\zeta_3\big\}$$
$$+L_{fr}^2\big\{1+32\zeta_3+176\zeta_2\big\}+L_{fr}^3\big\{-66\big\},$$
$$c^b_{3,3}=\Big\{\frac{272}{9}+\frac{23878}{27}\zeta_2-1056\zeta_2\zeta_3-\frac{1522}{9}\zeta_2^2+\frac{528}{5}\zeta_2^3\Big\}+L_{qr}\Big\{-\frac{1180}{3}+80\zeta_5$$
$$-\frac{2576}{9}\zeta_3+\frac{160}{3}\zeta_2+\frac{68}{5}\zeta_2^2\Big\}+L_{qr}^2\big\{44+88\zeta_3\big\}+L_{fr}\Big\{\frac{1657}{18}-80\zeta_5+\frac{3104}{9}\zeta_3$$
$$-\frac{8992}{27}\zeta_2+4\zeta_2^2\Big\}+L_{fr}^2\Big\{\frac{493}{9}-88\zeta_3+\frac{968}{9}\zeta_2\Big\}+L_{fr}^3\Big\{-\frac{242}{9}\Big\},$$
$$c^b_{3,4}=\Big\{-\frac{160}{9}-64\zeta_3-\frac{1532}{9}\zeta_2-64\zeta_2\zeta_3+\frac{1088}{9}\zeta_2^2\Big\}+L_{qr}\Big\{\frac{16}{3}-\frac{1312}{3}\zeta_3+144\zeta_2$$
$$+\frac{224}{15}\zeta_2^2\Big\}+L_{qr}^2\big\{64\zeta_3-32\zeta_2\big\}+L_{fr}\Big\{76+\frac{256}{3}\zeta_3-\frac{16}{3}\zeta_2+\frac{352}{15}\zeta_2^2\Big\}+L_{fr}^2\big\{$$
$$-8-64\zeta_3-64\zeta_2\big\}+L_{fr}^3\big\{24\big\},$$

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$$
\begin{aligned}
c^b_{3,4a} &= \left\{-\frac{32}{3} - 220\zeta_2 + 192\zeta_2\zeta_3\right\} + L_{fr}\left\{74 - 96\zeta_3 + 32\zeta_2\right\} + L^2_{fr}\left\{-12\right\}, \\
c^b_{3,4b} &= \left\{-\frac{64}{9} - 64\zeta_3 + \frac{448}{9}\zeta_2 - 256\zeta_2\zeta_3 + \frac{1088}{9}\zeta_2^2\right\} + L_{qr}\left\{\frac{16}{3} - \frac{1312}{3}\zeta_3 + 144\zeta_2\right. \\
&\quad \left. + \frac{224}{15}\zeta_2^2\right\} + L^2_{qr}\left\{64\zeta_3 - 32\zeta_2\right\} + L_{fr}\left\{2 + \frac{544}{3}\zeta_3 - \frac{112}{3}\zeta_2 + \frac{352}{15}\zeta_2^2\right\} \\
&\quad + L^2_{fr}\left\{4 - 64\zeta_3 - 64\zeta_2\right\} + L^3_{fr}\left\{24\right\}, \\
c^b_{3,5} &= \left\{-\frac{160}{9} - \frac{13816}{27}\zeta_2 + 192\zeta_2\zeta_3 - \frac{272}{9}\zeta_2^2\right\} + L_{qr}\left\{\frac{392}{3} + \frac{416}{9}\zeta_3 - \frac{32}{3}\zeta_2 - \frac{16}{5}\zeta_2^2\right\} \\
&\quad + L^2_{qr}\left\{-16 - 32\zeta_3\right\} + L_{fr}\left\{-80 - \frac{800}{9}\zeta_3 + \frac{5344}{27}\zeta_2 - \frac{16}{5}\zeta_2^2\right\} + L^2_{fr}\left\{-\frac{292}{9}\right. \\
&\quad \left. - 64\zeta_2 + 32\zeta_3 - \frac{704}{9}\zeta_2\right\} + L^3_{fr}\left\{\frac{176}{9}\right\}, \\
c^b_{3,6} &= \left\{\frac{1600}{27}\zeta_2 + \frac{224}{9}\zeta_2^2\right\} + L_{qr}\left\{\frac{256}{9}\zeta_3\right\} + L_{fr}\left\{\frac{136}{9} + \frac{128}{9}\zeta_3 - \frac{640}{27}\zeta_2\right\} + L^2_{fr}\left\{\frac{16}{9}\right. \\
&\quad \left. + \frac{128}{9}\zeta_2\right\} + L^3_{fr}\left\{-\frac{32}{9}\right\}. \qquad \text{(I.3)}
\end{aligned}
$$

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