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Weights and recursion relations for ϕ^p tree amplitudes from the positive geometry

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ABSTRACT: Recently, the accordiohedron in kinematic space was proposed as the positive geometry for planar tree-level scattering amplitudes in the ϕ^p theory [1]. The scattering amplitudes are given as a weighted sum over canonical forms of some accordiohedra with appropriate weights. These weights were determined by demanding that the weighted sum corresponds to the scattering amplitudes. It means that we need additional data from the quantum field theory to compute amplitudes from the geometry. It has been an important problem whether scattering amplitudes are completely obtained from only the geometry even in this ϕ^p theory.

In this paper, we show that these weights are completely determined by the factorization property of the accordiohedron. It means that the geometry of the accordiohedron is enough to determine these weights.

In addition to this, we study one-parameter recursion relations for the ϕ^p amplitudes. The one-parameter “BCFW”-like recursion relation for the ϕ^3 amplitudes was obtained from the triangulation of the ABHY-associahedron [2]. After this, a new recursion relation was proposed from the projecting triangulation of the generalized ABHY-associahedron in [3, 4]. We generalize these one-parameter recursion relations to the ϕ^p amplitudes and interpret as triangulations of the accordiohedra.

KEYWORDS: Scattering Amplitudes, Differential and Algebraic Geometry

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1 Introduction

Recent years have revealed an unexpected connection between scattering amplitudes and the positive geometry [2, 3, 5–11]. This connection gives a purely geometric definition of scattering amplitudes. In particular, tree amplitudes for massless bi-adjoint ϕ^3 theory are given as the canonical form of an ABHY-associahedron defined in kinematic space [12]. This geometric picture is generalized for more general interaction cases such as the stokes polytope for ϕ^4 theory [13], the accordiohedron for ϕ^p theory [1], and the case of the polynomial interaction [14]. In the ϕ^p ($p > 3$) case, the geometry is a union of many polytopes of a given dimension. The canonical form of each polytope gives a partial amplitude which is given from the subsets of ϕ^p graphs. We need to sum over all canonical forms for each

polytope with appropriate weight to obtain tree-level planar amplitudes. These weights are determined uniquely from the condition that when summed over all canonical forms, the residue of each pole is unit. Once we choose these weights from this condition, the sum of canonical forms corresponds to the scattering amplitude. This condition is obtained by demanding that the weighted sum corresponds to the scattering amplitudes [1]. The important point is that the derivation of this condition relied on the form of scattering amplitudes. It has been an important problem whether these weights are completely determined from the geometry only even in this ϕ^p theory.

This geometric picture of the amplitudes gives us a new viewpoint of the recursion relations. In [12], a one-parameter recursion relation for tree bi-adjoint ϕ^3 amplitudes is derived from the triangulation of the ABHY-associahedron. This relation of the recursion relation and triangulation is similar to the relation between “BCFW recursion relation” [15, 16] of the tree NMHV amplitude in the $\mathcal{N} = 4$ SYM and the triangulation of the amplituhedron [5, 6]. From this “BCFW-like” recursion for ϕ^3 amplitudes, an all-multiplicity results is obtained in [17]. Recently, a new recursion relation was proposed from the projecting triangulation of the generalized ABHY-associahedron [3]. This “projective recursion” was derived from field-theoretical consideration in [4] and it can be interpreted as a generalization of the BCFW-like recursion. It is known that the special case of this projective recursion: “one-variable recursion” is much more efficient than the BCFW-like recursion [4]. This recursion can be also derived from the general properties of canonical forms of simple polytopes [18].

In this paper, we investigate the weights of the accordiohedra and one-parameter recursion relations of the ϕ^p amplitudes. We show that these weights are completely determined from the factorization property of the accordiohedron. This means that the geometry of the accordiohedron is enough to determine these weights. We can also see that the condition that each pole has unit residue is equivalent to one of the constraints obtained from the factorization.

We generalize the one-parameter recursions for ϕ^p amplitudes. The important point is that the recursion relation can be defined to the subset of ϕ^p graphs, not to all of the graphs. This comes from the fact that in the $\phi^p(p > 3)$ case, the scattering form on the kinematic space can be defined uniquely from the projectivity only for the subset of ϕ^p graphs. This projectivity means that partial amplitudes have no pole at infinity in the kinematic space. We need this property to construct the recursion relation.

This paper is organized as follows: in section 2, we briefly review the positive geometry of the ϕ^p tree amplitude. In section 3, we will see how to determine all the weights from the factorization property of the accordiohedron. In section 4, we will construct the BCFW-like recursion relation for ϕ^p amplitudes. The recursion relation is applied to each subset of graphs. We also see that this recursion can be interpreted as a triangulation of the accordiohedra and the general formula of the triangulation leads all-multiplicity results. Finally, in section 5, we will construct the projective recursion relation for ϕ^p amplitudes. We show some explicit calculations of the one-variable projective recursion which is much more efficient than the BCFW-like recursion.

2 Positive Geometry for ϕ^p tree amplitudes

2.1 Accordiohedron

Here we define a geometric object called the “accordiohedron” [19], which is proposed as the positive geometry for ϕ^p ($p > 3$) interactions [1]. First we introduce the notion of “ Q -compatible diagonal” and “ Q -flip”. Let us consider the decomposition of convex polygon into p -gons and we call this decomposition as p -angulation. We can p -angulate a $(2p - 2)$ -gon into two p -gons and there are $p - 1$ p -angulations. We denote these p -angulations by using the diagonals $\{(1, p), (2, p + 1), \dots, (p - 1, 2p - 2)\}$. We introduce a notion of “ Q -compatible diagonal” to each diagonal (i, j) of a $(2p - 2)$ -gon as:

$$(i, j) \rightarrow (\text{Mod}(i + p - 2, 2p - 2), \text{Mod}(j + p - 2, 2p - 2)). \quad (2.1)$$

This can be generalized to any p -angulation of an n -gon. To do this, we need to consider the unique $(2p - 2)$ -gon which contains the diagonal (i, j) . Let’s consider $p = 4, n = 8$ case and the quadrangulation $\{(1, 4), (1, 6)\}$ as an example. The unique hexagon which includes the diagonal $(1, 6)$ is $\{1, 4, 5, 6, 7, 8\}$ and we label this as $\{1', 2', 3', 4', 5', 6'\}$. Then the Q -compatible diagonal of $(1, 6)$ is given as

$$(\text{Mod}(3, 6), \text{Mod}(8, 6)) \rightarrow (3', 6') = (5, 8). \quad (2.2)$$

The operation of replacing a diagonal with its Q -compatible diagonal is called “ Q -flip”.

Next, we define the accordiohedron $\mathcal{AC}_{p,n}^P$ by using this Q -flip as follows. When we p -angulate an n -gon into $C_p = (n - 2)/(p - 2)$ cells with $D_p = (n - p)/(p - 2)$ diagonals, we call this as complete p -angulation.¹ First we choose a complete p -angulation P of the n -gon. For each D_p diagonals of P , we apply the Q -flip operation until we do not generate any new p -angulation, and we obtain a subset of the p -angulations. We call this subset as Q -compatible set of P “ $Q(P)$ ”. Then the accordiohedron $\mathcal{AC}_{p,n}^P$ for this P is defined as:

Vertices $\leftrightarrow$ p -angulations in the $Q(P)$

Edges $\leftrightarrow$ Q -Flips

k -Facets $\leftrightarrow$ k -partial p -angulations

We can easily verify that $\mathcal{AC}_{p,n}^P$ depends on the p -angulation P and the number of vertices of n -gon.

The accordiohedron contains both associahedron and stokes polytope as special cases. In the case of $p = 3$, (2.1) reduces to $(i, j) \rightarrow (\text{Mod}(i + 1, 4), \text{Mod}(j + 1, 4))$ and this is the mutation rule of the triangulation [12]. Then the $\mathcal{AC}_{3,n}^P$ corresponds to the associahedron. In the case of $p = 4$, (2.1) reduces to $(i, j) \rightarrow (\text{Mod}(i + 2, 6), \text{Mod}(j + 2, 6))$ and this is the Q -compatibility of the quadrangulation [13]. From this, $\mathcal{AC}_{4,n}^P$ corresponds to the stokes polytope.

¹Only $p + m(p - 2)$ -point polygon ($m = 0, 1, \dots$) can be p -angulated completely. If we substitute $p + m(p - 2)$ for n , we can easily verify that these C_p, D_p are integers.

2.2 Planar kinematic variables and the scattering form

Kinematic space of massless momenta $p_i, i = 1, \dots, n$ is spanned by the Mandelstam variables:

$$s_{ij} = (p_i + p_j)^2, \text{ for } 1 \leq i < j \leq n. \quad (2.3)$$

We introduce generalized Mandelstam variables as:

$$s_I = \left(\sum_{i \in I} p_i \right)^2 = \sum_{\substack{i, j \in I \\ i < j}} s_{ij}. \quad (2.4)$$

A convenient basis for this kinematic space is so-called “planar variables”

$$X_{i,j} = (p_i + p_{i+1} + \dots + p_j) = s_{i,i+1,\dots,j-1} \text{ for } 1 \leq i < j \leq n. \quad (2.5)$$

By definition, we can see that $X_{i,i+1} = X_{1,n} = 0$. These variables are identified with the diagonals of an n -gon.

The relation of these variables and Mandelstam variables is

$$s_{ij} = X_{i,j+1} + X_{i+1,j} - X_{i,j} - X_{i+1,j+1}. \quad (2.6)$$

Each planar n -point tree ϕ^p graph has D_p propagators and there is a one-to-one correspondence between this graph and a complete p -angulation of the n -gon with D_p diagonals.

Next, we introduce a planar scattering form for ϕ^p interactions. In the case of ϕ^3 , the scattering form is determined uniquely by demanding the projectivity of the form [12]. However, there is no notion of projectivity for this general interactions, then we need to choose a subset of graphs to determine a scattering form uniquely by its projectivity. This subset of graphs is obtained from the Q -compatible set of the complete p -angulations. Then we can define a planar scattering form for this subset [1].

Let $g(Q(P))$ denote tree graphs for ϕ^p interactions in the Q -compatible set $Q(P)$ with propagators X_{i_a, j_a} for $a = 1, \dots, D_p$. Then a $Q(P)$ dependent planar scattering form Ω_n^P is defined as

$$\Omega_n^P = \sum_{g(Q(P))} \text{sign}(g(Q(P))) \bigwedge_{a=1}^{D_p} d \log X_{i_a, j_a} \quad (2.7)$$

where $\text{sign}(g(Q(P))) = \pm 1$ and we sum over a $d \log$ form for every planar graphs in the subset P . The sign is determined as

$$\text{sign} g(Q(P)) = -\text{sign} g'(Q(P)) \quad (2.8)$$

where g' is a graph which obtained by the Q -flip of g . This rule make the form projective.

The important point is that any Q -compatible set of graphs does not exhaust all the graphs (or p -angulations). Then we need to sum all of these subsets with appropriate weight to obtain the scattering amplitude. To do this, we introduce “a subset of primitive p -angulations” $\{P_1, \dots, P_I\}$:

- no two p -angulation of this subset are related to each other by cyclic permutations σ
- all the other p -angulations are obtained by a cyclic permutations of p -angulation of this subset.

Then the ϕ^p tree amplitude is given as

$$M_{p,n} = \sum_{\sigma} \sum_{\text{primitive } P} \alpha_{p,n}^P m_{p,n}^{P,\sigma} \quad (2.9)$$

where $\alpha_{p,n}^P$ is a weight and $m_{p,n}^{P,\sigma}$ is the ratio part of the scattering form:

$$m_{p,n}^P = \left(\sum_{g(Q(P))} \frac{1}{\prod_{a=1}^{D_p} X_{i_a, j_a}} \right) \quad (2.10)$$

We call this $m_{p,n}^P$ as “ $Q(P)$ -compatible amplitude”.

Since the cyclic permutation of the primitive p -augulation does not change the relative configuration of diagonals, the weights depend only on primitive p -angulations. In the next section, we see how to embed the accordiohedron into the kinematic space and the relation between the canonical form of this kinematic accordiohedron and the scattering form.

2.3 The kinematic accordiohedron

The kinematic accordiohedron $\mathcal{AC}_{p,D_p}^P$ is a D_p -dim polytope defined as the intersection of the positive region:

$$\Delta_n = \{X_{ij} \geq 0 \text{ for all } 1 \leq i < j \leq n\} \quad (2.11)$$

with a hyperplane defined as follows:

$$\begin{aligned} H_n = \{ & C_{ij} = -X_{i,j+1} - X_{i+1,j} + X_{i,j} + X_{i+1,j+1} \text{ for } 1 \leq i < j \leq n-1 \\ & d_{r_i, s_i} = X_{r_i, s_i} \text{ s.t. } (r_i, s_i) \text{ is a complete triangulation} \} \end{aligned} \quad (2.12)$$

where $C_{i,j}$ and d_{r_i, s_i} are positive constants. The dimension of this hyperplane is given as

$$d(H_n) = \frac{(n-2)(np-2n-p)}{2(p-2)}. \quad (2.13)$$

It is easy to check that these dimensions are correct. First, the dimension of Δ_n is $n(n-3)/2$. Next, we consider the dimension of the hyperplane H_n . The number of the first line condition in (2.12) is $(n-3)(n-2)/2$ as same as the associahedron case. When we p -angulate the n -point polygon, this polygon decomposes into $D_p + 1$ p -gons. For each p -gon, we need to consider the complete triangulation. The number of triangulation of the p -gon is $p-2$ and there are $p-3$ diagonals. From this, the number of the second line conditions are

$$(D_p + 1) \times (p-3) = \frac{(n-2)(p-3)}{p-2}. \quad (2.14)$$

Then the total number of the constraints of (2.12) is

$$\frac{(n-2)(p-3)}{p-2} + \frac{(n-3)(n-2)}{2} = \frac{(n-2)(np-2n-p)}{2(p-2)}. \quad (2.15)$$

This is the dimension of the hyperplane H_n . Finally, the dimension of the intersection $\mathcal{AC}_{p,D_p}^P = \Delta_n \cap H_n$ is

$$\frac{n(n-3)}{2} - \frac{(n-2)(np-2n-p)}{2(p-2)} = \frac{n-p}{p-2} = D_p. \quad (2.16)$$

Since the accordiohedron $\mathcal{AC}_{p,D_p}^P$ is a simple polytope, the canonical form is a sum over its vertices [2]. For each vertex Z , we denote its adjacent facets as $X_{i_a,j_a} = 0$ for $a = 1, \dots, D_p$. Then the canonical form of $\mathcal{AC}_{p,D_p}^P$ is written as

$$\Omega(\mathcal{AC}_{p,D_p}^P) = \sum_{\text{vertex } Z} \text{sign}(Z) \bigwedge_{a=1}^{D_p} d \log X_{i_a,j_a} \quad (2.17)$$

where $\text{sign}(Z)$ is evaluated on the ordering of the facets. This canonical form (2.17) is equivalently the pullback of the scattering form (2.7) to the subspace H_n . Then the ratio part of the canonical form (canonical function) corresponds to the $Q(P)$ -compatible amplitude. From (2.9), the tree amplitude of ϕ^p theory is given as the weighted sum of these $Q(P)$ -compatible amplitudes. In the next section, we will see some explicit calculations to obtain the amplitudes from the kinematic accordiohedron.

2.4 Amplitudes from the accordiohedron

First, we see the simplest example: $p = 4, n = 6$ case. In this case, there is only one primitive quadrangulation, and all the others are obtained from the cyclic permutations. We choose $P = 14$ as the reference quadrangulation, here ij denotes the diagonal. The Q -compatible set of this reference quadrangulation is $Q(14) = \{(14, +), (36, -)\}$. The signs are determined that if a quadrangulation P' is related to the reference quadrangulation P by an odd (even) number of Q -flips, we associate $- (+)$ sign. Then the accordiohedron (stokes polytope) is a one-dimensional line as (a) in figure 1. Its canonical form is given by

$$\Omega_{p=4,n=6}^{14} = (d \ln X_{14} - d \ln X_{36}). \quad (2.18)$$

This stokes polytope locates inside the kinematic space with the following constraints

$$\begin{aligned} s_{ij} &= -C_{ij}, \text{ for } 1 \leq i < j \leq 5 \text{ with } |i - j| \geq 2 \\ X_{13} &= d_{13}, X_{15} = d_{15}, \text{ with } d_{13}, d_{15} > 0. \end{aligned} \quad (2.19)$$

From these constraints, we can see that the planar kinematic variables satisfy

$$\begin{aligned} X_{36} &= -X_{14} + \sum_{\substack{1 \leq a < 3 \\ 4 \leq b < 6}} C_{ab} \geq 0 \\ X_{25} &= d_{15} - d_{13} + C_{14} + C_{13} \geq 0. \end{aligned} \quad (2.20)$$

Now we can pull back the canonical form as

$$\Omega_{4,6}^{14} = \left(\frac{1}{X_{14}} + \frac{1}{X_{36}} \right) dX_{14} = m_{4,6}^{14} dX_{14}. \quad (2.21)$$

It is easy to calculate other primitive quadrangulations cases $Q(25), Q(36)$. From (2.9), the six-point amplitude is given as

$$M_{p=4,n=6} = \alpha_{p=4,n=6}^{14} (m_{4,6}^{14} + m_{4,6}^{25} + m_{4,6}^{36}) = 2\alpha_{4,6}^{14} \left(\frac{1}{X_{14}} + \frac{1}{X_{36}} + \frac{1}{X_{25}} \right), \quad (2.22)$$

Figure 1. Stokes polytopes.

where $\alpha_{4,6}^{14}$ is a weight and if we choose $\alpha_{4,6}^{14} = 1/2$, this corresponds to the six-point amplitude.

Let us consider the $n = 8$ case. In this case, there are two primitive quadrangulations $\{P_1, P_2\}$ and we take them to be $\{P_1 = (14, 58), P_2 = (14, 16)\}$. The Q -compatible set $Q(P_1)$ is given by

$$Q(P_1) = \{(14, 58, +), (14, 47, -), (83, 58, -), (83, 47, +)\}. \quad (2.23)$$

Then the stokes polytope $\mathcal{S}^{P_1}$ is a two dimensional square as (b) in figure 1. Similarly, $Q(P_2)$ is given by

$$Q(P_2) = \{(14, 16, +), (14, 58, -), (36, 16, -), (36, 83, +), (58, 83)\}. \quad (2.24)$$

We can see that the stokes polytope $\mathcal{S}^{P_2}$ is a two-dimensional pentagon as (c) in figure 1. The constraints locating these stokes polytopes inside the kinematic space are given by

$$\begin{aligned} \mathcal{S}^{P_1} : \quad & s_{ij} = -C_{ij}, \text{ for } 1 \leq i < j \leq 7 \text{ with } |i - j| \geq 2 \\ & X_{13} = d_{13}, X_{48} = d_{48}, X_{57} = d_{57} \\ \mathcal{S}^{P_2} : \quad & s_{ij} = -C_{ij}, \text{ for } 1 \leq i < j \leq 7 \text{ with } |i - j| \geq 2 \\ & X_{13} = d_{13}, X_{46} = d_{46}, X_{68} = d_{68}. \end{aligned} \quad (2.25)$$

From these constraints, the canonical forms become as

$$\begin{aligned}
 \Omega_{p=4,n=8}^{P_1} &= \left(\frac{1}{X_{14}X_{58}} + \frac{1}{X_{38}X_{47}} + \frac{1}{X_{14}X_{47}} + \frac{1}{X_{38}X_{58}} \right) d \ln X_{14} \wedge d \ln X_{58} \\
 &= m_{4,8}^{P_1} d \ln X_{14} \wedge d \ln X_{58} \\
 \Omega_{4,8}^{P_2} &= \left(\frac{1}{X_{14}X_{16}} + \frac{1}{X_{14}X_{58}} + \frac{1}{X_{36}X_{16}} + \frac{1}{X_{36}X_{83}} + \frac{1}{X_{58}X_{83}} \right) d \ln X_{14} \wedge d \ln X_{16} \\
 &= m_{4,8}^{P_2} d \ln X_{14} \wedge d \ln X_{16}.
 \end{aligned} \tag{2.26}$$

From (2.9), the 8-point amplitude is given as

$$M_{p=4,n=8} = \alpha_{4,8}^{P_1} \left(m_{4,8}^{P_1} + \text{cyclic} \right) + \alpha_{4,8}^{P_2} \left(m_{4,8}^{P_2} + \text{cyclic} \right). \tag{2.27}$$

If we choose these weights as $\alpha_{4,8}^{P_1} = \frac{1}{3}, \alpha_{4,8}^{P_2} = \frac{1}{6}$, this corresponds to the 8-point amplitude.

Similarly we can calculate the amplitudes from the accordiohedron for general p case. For example, we consider $p = 5, n = 11$ case. There are two Q -compatible sets $Q_5(P_1), Q_5(P_2)$:

$$\begin{aligned}
 Q_5(P_1) &= \{(15, 711), (411, 711), (15, 610), (411, 610)\} \\
 Q_5(P_2) &= \{(15, 18), (18, 48), (15, 711), (411, 711), (411, 48)\}.
 \end{aligned} \tag{2.28}$$

The constraints are given as

$$\begin{aligned}
 Q_5(P_1) : \quad & s_{ij} = -C_{ij}, \text{ for } 1 \leq i < j \leq 10 \text{ with } |i - j| \geq 2 \\
 & X_{13} = d_{13}, X_{35} = d_{35}, X_{17} = d_{17}, X_{57} = d_{57}, X_{810} = d_{810}, X_{710} = d_{710}. \\
 Q_5(P_2) : \quad & s_{ij} = -C_{ij}, \text{ for } 1 \leq i < j \leq 10 \text{ with } |i - j| \geq 2 \\
 & X_{13} = d_{13}, X_{35} = d_{35}, X_{16} = d_{16}, X_{68} = d_{68}, X_{810} = d_{810}, X_{110} = d_{110}.
 \end{aligned} \tag{2.29}$$

The shape of the accordiohedra for each Q -compatible set are given in figure 2. The canonical forms for each accordiohedron are

$$\begin{aligned}
 \Omega_{p=5,n=11}^{P_1} &= \left(\frac{1}{X_{15}X_{610}} + \frac{1}{X_{411}X_{610}} + \frac{1}{X_{15}X_{711}} + \frac{1}{X_{411}X_{711}} \right) d \ln X_{15} \wedge d \ln X_{610} \\
 &= m_{5,11}^{P_1} d \ln X_{15} \wedge d \ln X_{610} \\
 \Omega_{5,11}^{P_2} &= \left(\frac{1}{X_{15}X_{18}} + \frac{1}{X_{18}X_{48}} + \frac{1}{X_{15}X_{711}} + \frac{1}{X_{411}X_{711}} + \frac{1}{X_{48}X_{411}} \right) d \ln X_{15} \wedge d \ln X_{18} \\
 &= m_{5,11}^{P_2} d \ln X_{15} \wedge d \ln X_{18}.
 \end{aligned} \tag{2.30}$$

From (2.9), the 11-point amplitude is given as

$$M_{p=5,n=11} = \alpha_{5,11}^{P_1} \left(m_{5,11}^{P_1} + \text{cyclic} \right) + \alpha_{5,11}^{P_2} \left(m_{5,11}^{P_2} + \text{cyclic} \right). \tag{2.31}$$

If we choose these weights as $\alpha_{5,11}^{P_1} = \frac{3}{11}, \alpha_{5,11}^{P_2} = \frac{2}{11}$, this corresponds to the 11-point amplitude.

The important point is that to obtain the amplitudes, we need to choose the appropriate weights. In the next section, we see that these weights are determined by the factorization property of the accordiohedron.

Figure 2. $p = 5, n = 11$ accordiohedra.

3 Weights from the factorization

In this section, we see that the weights can be obtained from the factorization property of the accordiohedron. In the planar $\mathcal{N} = 4$ SYM, it is known that the geometric factorization of the amplituhedron implies physical factorization of scattering amplitudes [5]. This fact also holds for associahedron [12], stokes polytope [13], and accordiohedron [1]. First, we briefly review this factorization property of the accordiohedron. Then we see some explicit calculations of the weights.

3.1 Factorization

For any diagonal (ij) , we consider all primitive p -angulation P which contains (ij) and the kinematic accordiohedron $\mathcal{AC}_{p,n}^P$. The facet $X_{ij} = 0$ of this accordiohedron is given as a product of lower dimensional accordiohedra

$$\mathcal{AC}_{p,n}^P|_{X_{ij}=0} = \mathcal{AC}_{p,m}^{P_1} \times \mathcal{AC}_{p,n+2-m}^{P_2}, \quad (3.1)$$

where P_1 is the p -angulation of the polygon $\{i, i+1, \dots, j\}$ and P_2 is the p -angulation of $\{j, j+1, \dots, n, 1, \dots, i\}$. These p -angulation satisfy $P_1 \cup P_2 \cup (ij) = P$.

Next, we see that this geometric factorization implies the physical factorization of scattering amplitudes. The canonical form of the positive geometry satisfies the following property [2]:

1. For any hyper-surface H containing a boundary $\mathcal{B}$ of $\mathcal{A}$, the residue along H is given by

$$\text{Res}_H \Omega(\mathcal{A}) = \Omega(\mathcal{B}). \quad (3.2)$$

2. For any pair of positive geometries $\mathcal{A}$ and $\mathcal{B}$, we have

$$\Omega(\mathcal{A} \times \mathcal{B}) = \Omega(\mathcal{A}) \wedge \Omega(\mathcal{B}). \quad (3.3)$$

By using these facts, we can obtain the relation

$$\text{Res}_{X_{ij}=0} \Omega(\mathcal{AC}_{p,n}^P) = \Omega(\mathcal{AC}_{p,m}^{P_1}) \wedge \Omega(\mathcal{AC}_{p,n+2-m}^{P_2}). \quad (3.4)$$

This factorization property implies the physical factorization of amplitudes. The diagonal (ij) divides the n -gon into two polygons $\{i, i+1, \dots, j\}$ and $\{j, j+1, \dots, n, 1, \dots, i\}$. By considering the kinematic accordiohedron associated to these polygons, we can obtain two sub-amplitudes $\{M_{|j-i+1|}, M_{n+2-(|j-i+1|)}\} = \{M_L, M_R\}$. Then (3.4) implies that

$$M_n|_{X_{ij}=0} = M_L \frac{1}{X_{ij}} M_R. \quad (3.5)$$

This is the physical factorization of the amplitudes.

3.2 Determination of the weights

In this section, we see that the weights can be determined from the factorization property of the accordiohedron. From (3.1) and (3.4), the weights are constrained as

$$\sum_{P \in (ij)} \alpha_P = \sum_{P_L, P_R} \alpha_{P_L} \alpha_{P_R}. \quad (3.6)$$

The left hand side involves sum over all accordiohedra $\mathcal{AC}_{p,n}^P$ for which $P \in (ij)$ and the right hand side involves sum over P_L and P_R which range over all the p -angulations of the two polygons $\{i, i+1, \dots, j\}$ and $\{j, j+1, \dots, n, 1, \dots, i\}$ respectively.

In addition to this, we can further constraint the weights from the factorization. By choosing the diagonal $(kl) \in \{i, i+1, \dots, j\}$, the polygon $\{i, i+1, \dots, j\}$ is divided into two polygons $\{k, k+1, \dots, l\}$ and $\{l, l+1, \dots, i, j, \dots, k\}$. By repeating the same procedure of the before section, we obtain

$$M_{i,i+1,\dots,j}|_{X_{kl}=0} = M_{L_2} \frac{1}{X_{kl}} M_{R_2}. \quad (3.7)$$

where M_{L_2} and M_{R_2} are sub-amplitudes $\{M_{L_2}, M_{R_2}\} = \{M_{|l-k+1|}, M_{i+j+1-|l-k+1|}\}$. From (3.5) and (3.7), we obtain

$$M_n|_{X_{ij}=0, X_{kl}=0} = M_{L_2} \frac{1}{X_{kl}} M_{R_2} \frac{1}{X_{ij}} M_R. \quad (3.8)$$

Then the weights satisfy

$$\sum_{P \in (ij), (kl)} \alpha_P = \sum_{P_{L_2}, P_{R_2}, P_R} \alpha_{P_{L_2}} \alpha_{P_{R_2}} \alpha_{P_R}. \quad (3.9)$$

The left hand side involves sum over all accordiohedra $\mathcal{AC}_{p,n}^P$ for which $P \in (ij), (kl)$ and the right hand side involves sum over P_{L_2}, P_{R_2} and P_R which range over all the p -angulations of the three polygons $\{k, k+1, \dots, l\}$, $\{l, l+1, \dots, i, j, \dots, k\}$ and $\{j, j+1, \dots, n, 1, \dots, i\}$ respectively. Continuing this procedure, we can obtain

$$M_n|_{X_{i_1 j_1}, X_{i_2 j_2}, \dots, X_{i_k j_k}=0} = M_{L_k} \frac{1}{X_{i_k j_k}} M_{R_k} \frac{1}{X_{i_{k-1} j_{k-1}}} M_{R_{k-1}} \cdots M_{R_2} \frac{1}{X_{i_1 j_1}} M_{R_1}, \quad (3.10)$$

and

$$\sum_{P \in (i_1 j_1), (i_2 j_2), \dots, (i_k j_k)} \alpha_P = \sum_{P_{L_k}, P_{R_k}, \dots, P_{R_1}} \alpha_{P_{L_k}} \alpha_{P_{R_k}} \cdots \alpha_{P_{R_1}}. \quad (3.11)$$

If the diagonals $(i_1 j_1), (i_2 j_2), \dots, (i_k j_k)$ make the complete p -angulation of P , the all weights of the right hand side are 1. In this case, the constraints become simple as

$$\sum_{P \in (i_1 j_1), (i_2 j_2), \dots, (i_k j_k)} \alpha_P = 1. \quad (3.12)$$

Since the complete p -angulation $(i_1 j_1), (i_2 j_2), \dots, (i_k j_k)$ is the vertex of the accordiohedron, the canonical form of the accordiohedron $\mathcal{AC}_{p,n}^P$ for $P \in (i_1 j_1), \dots, (i_k j_k)$ has the term $\prod_{a=1}^k 1/X_{i_a, j_a}$. Then the sum of the left-hand side of (3.12) means that sum over all accordiohedra for which has the term $\prod_{a=1}^k 1/X_{i_a, j_a}$ in its canonical form. From this, we can interpret the equation (3.12) as the constraint that in the weighted sum of all canonical forms, each $\prod_{a=1}^k 1/X_{i_a, j_a}$ appears exactly once. This is corresponding to the constraint of the weights obtained in [1, 13] by demanding that the weighted sum of canonical forms gives the full amplitude. In [1], it was also proven that to satisfy this constraint, it is sufficient to impose these constraints for the primitive p -angulations.

$$\sum_{i=1}^l n_P^i \alpha_P^i = 1 \quad \text{for each primitive } 1 \leq i \leq l \quad (3.13)$$

where i labels each primitive p -angulation, n_P^i is number of times primitive i appears in the vertices of all accordiohedra and α_P^i are the corresponding weights. Here we assumed that there are l primitive p -angulations. The important point is that the derivation of this condition in [1, 13] relied on the form of scattering amplitudes. However, here we derive this formula from the factorization property of the accordiohedron.

By using these constraints, we can determine the weights uniquely. Let us consider $p = 4$ case. In the most simple $n = 4$ case, the weight is trivial $\alpha_{4,4} = 1$. Next we consider $n = 6$ case. There is only way to divide the hexagon $\{1, 2, \dots, 6\}$ into two squares and both these have trivial weights, then (3.11) gives

$$\begin{aligned} 2\alpha_{4,6} &= 1 \\ \alpha_{4,6} &= \frac{1}{2}. \end{aligned} \quad (3.14)$$

In the $n = 8$ case, we choose the diagonals $\{(47), (38)\}$ and $\{(38), (36)\}$. Then the equation (3.11) gives two constraints

$$\begin{aligned} 2\alpha_{4,8}^{P_1} + 2\alpha_{4,8}^{P_2} &= 1 \\ \alpha_{4,8}^{P_1} + 4\alpha_{4,8}^{P_2} &= 1. \end{aligned} \quad (3.15)$$

From this, we obtain the correct weights $\alpha_{4,8}^{P_1} = \frac{1}{3}, \alpha_{4,8}^{P_2} = \frac{1}{6}$. If we choose another diagonals, the result does not change. For example if we choose the diagonals $\{(38)\}$ and

$\{(14), (58)\}$, (3.11) gives

$$\begin{aligned} 2\alpha_{4,8}^{P_1} + 5\alpha_{4,8}^{P_2} &= \frac{3}{2} \\ 2\alpha_{4,8}^{P_1} + 2\alpha_{4,8}^{P_2} &= 1. \end{aligned} \quad (3.16)$$

These give the same weights. Finally we consider $n = 10$ case. We choose the diagonals $\{(14), (510)\}$, $\{(14), (69)\}$, $\{(14), (49)\}$, $\{(14), (58)\}$, $\{(14), (510), (69)\}$, $\{(14), (49), (69)\}$, $\{(14), (510), (58)\}$. Then the equation (3.11) gives

$$\begin{aligned} 2\alpha^{P_1} + 2\alpha^{P_2} + \alpha^{P_3} + \alpha^{P_4} + 2\alpha^{P_5} + 3\alpha^{P_6} + 3\alpha^{P_7} &= \frac{3}{2} \\ 2\alpha^{P_1} + 4\alpha^{P_2} + 3\alpha^{P_3} + 3\alpha^{P_4} + 2\alpha^{P_5} + 2\alpha^{P_6} + 2\alpha^{P_7} &= \frac{3}{2} \\ \alpha^{P_1} + 5\alpha^{P_2} + 3\alpha^{P_3} + 2\alpha^{P_4} + 3\alpha^{P_5} + 3\alpha^{P_6} + 2\alpha^{P_7} &= \frac{3}{2} \\ \alpha^{P_1} + 3\alpha^{P_2} + \alpha^{P_3} + \alpha^{P_4} + 4\alpha^{P_5} + 3\alpha^{P_6} + 3\alpha^{P_7} &= \frac{3}{2} \\ 2\alpha^{P_1} + \alpha^{P_3} + \alpha^{P_4} + 2\alpha^{P_6} + 2\alpha^{P_7} &= 1 \\ \alpha^{P_1} + 4\alpha^{P_2} + 3\alpha^{P_3} + 2\alpha^{P_4} + 2\alpha^{P_5} + 2\alpha^{P_6} &= 1 \\ \alpha^{P_1} + 2\alpha^{P_2} + \alpha^{P_4} + 2\alpha^{P_5} + 2\alpha^{P_6} + 2\alpha^{P_7} &= 1, \end{aligned} \quad (3.17)$$

where $\alpha^{P_i} = \alpha_{4,10}^{P_i}$. From these equations, we can obtain

$$\alpha^{P_1} = \frac{5}{24}, \quad \alpha^{P_2} = \frac{1}{24}, \quad \alpha^{P_3} = \frac{1}{24}, \quad \alpha^{P_4} = \frac{1}{24}, \quad \alpha^{P_5} = \frac{1}{12}, \quad \alpha^{P_6} = \frac{1}{8}, \quad \alpha^{P_7} = \frac{1}{8}. \quad (3.18)$$

These are the correct weights.

We can similarly calculate the weights of the general p case from the factorization. Let us consider $p = 5, n = 11$ case, we choose the diagonals $\{(15)\}$ and $\{(15), (711)\}$, (3.11) gives

$$\begin{aligned} 4\alpha_{5,11}^{P_1} + 5\alpha_{5,11}^{P_2} &= 2 \\ 3\alpha_{5,11}^{P_1} + \alpha_{5,11}^{P_2} &= 1. \end{aligned} \quad (3.19)$$

From this, we obtain the correct weights $\alpha_{5,11}^{P_1} = \frac{3}{11}, \alpha_{5,11}^{P_2} = \frac{2}{11}$.

4 “BCFW”-like recursion relation

4.1 General setups

In this section, we construct a one-parameter recursion relation for ϕ^p amplitudes. This is just the generalization of the BCFW-like recursion relation of ϕ^3 case [12, 17] to the ϕ^p case. The important point is that in the general ϕ^p cases, the recursion relation can be defined only for each Q -compatible set of graphs. We choose the Q -compatible set $Q(P)$ which is obtained from a reference p -angulation P and consider the D_p basis variables X_{A_i} .

Here we construct the recursion relation for this Q -compatible set of graphs explicitly from a one-parameter deformation of the basis variables:

$$X_{A_i} \rightarrow \hat{X}_{A_i} := zX_{A_i}, \quad \text{for } i = 1, 2, \dots, D_p, \quad (4.1)$$

where we do not change the constant C_{ij} . Following the same logic as the case of ϕ^3 , we consider the contour integral:

$$m_{p,n}^P(X, C) = \oint_{|z-1|=\epsilon} \frac{z^{D_p} dz}{z-1} m_{p,n}^P(\hat{X}, C). \quad (4.2)$$

Then the amplitude is given by the residue at $z = 1$. From Cauchy theorem, this integral becomes as

$$m_{p,n}^P(X, C) = - \left(\text{Res}_{z=\infty} + \sum_{\text{finite poles}} \text{Res}_{z=z^*} \right) \frac{z^{D_p} dz}{z-1} m_{p,n}^P(\hat{X}, C). \quad (4.3)$$

Because of a z^{D_p} , the function $\frac{z^{D_p}}{z-1} m_{p,n}^P(zX, C)$ doesn't have a pole at $z = 0$. One can verify that this function has no pole at infinity. We will see this in the later of this section.

Then we consider the residues at each finite pole. We denote the set of all non-basis variables X_{B_i} which depend on the basis variables X_{A_i} as

$$X_{B_i} = C_{B_i} + X_i + \sum_{j=1}^{D_p} \lambda_{i,j} X_{A_j}, \quad (4.4)$$

where $\lambda_{i,j}$ are real numbers and C_{B_i} is a linear combination of constants C and X_i is a linear combination of unshifted planar variables. We denote z_{B_i} as the solutions of $\hat{X}_{B_i}(z) = 0$. The amplitude factorize at each physical pole as

$$\lim_{\hat{X}_{B_i} \rightarrow 0} \hat{X}_{B_i} m_{p,n}^P(zX, C) = m_{a,\dots,b-1,I}^{P_L}(z_{B_i}X, C) \times m_{I,b,\dots,a-1}^{P_R}(z_{B_i}X, C), \quad (4.5)$$

where $P_L \cup P_R \cup B_i = P$. Then the residue becomes as

$$\text{Res}_{\hat{X}_{B_i}} \frac{z^{D_p}}{z-1} m_{p,n}^P(zX, C) = \frac{z_{B_i}^{D_p}}{\sum_{j=1}^{D_p} \lambda_{i,j} X_{A_j}(z_{B_i} - 1)} m_{a,\dots,b-1,I}^{P_L}(z_{B_i}X, C) \times m_{I,b,\dots,a-1}^{P_R}(z_{B_i}X, C). \quad (4.6)$$

By using $\hat{X}_{B_i}(z) = 0$, the denominator becomes as

$$\sum_{j=1}^{D_p} \lambda_{i,j} X_{A_j}(z_{B_i} - 1) = -X_{B_i}. \quad (4.7)$$

Then we can obtain the recursion relation formula

$$m_{p,n}^P(X, C) = \sum_{B_i} \frac{z_{B_i}^{D_p}}{X_{B_i}} m_{a,\dots,b-1,I}^{P_L}(z_{B_i}X, C) \times m_{I,b,\dots,a-1}^{P_R}(z_{B_i}X, C) \quad (4.8)$$

where B_i runs over all the shifted planar variables. This is the BCFW-like recursion relation for the Q -compatible set of the amplitudes. For each Q -compatible set, we can apply this recursion relation. To obtain the amplitude, we need to sum over all primitive p -angulation and all cyclic permutation with the weight as (2.9).

Next, we prove that there is no pole at infinity. We consider the behavior of function $\frac{z^{D_p}}{z-1} m_{p,n}^P(zX, C)$ at $z \rightarrow \infty$. The canonical function $m_{p,n}^P(X, C)$ of the accordiohedron have the form

$$m_{p,n}^P(X, C) = \sum_{\text{vertices}} \frac{1}{\prod^{D_p} X}. \quad (4.9)$$

Since the dimension of the accordiohedron is D_p , there are no planar variables that are linear independent with basis variables. Then the canonical function has only terms of $\mathcal{O}\left(\frac{1}{z^{D_p}}\right)$. Then at $z \rightarrow \infty$,

$$\lim_{z \rightarrow \infty} m_{p,n}^P(X, C) \sim \frac{1}{z^{D_p}} m_{p,n}^P(X, 0). \quad (4.10)$$

When we fix i and set $D_p C_{ij} = 0$, the canonical function of the accordiohedron vanishes as,

$$m_{p,n}^P(zX, 0) \rightarrow 0. \quad (4.11)$$

We can easily verify this from a property of the canonical form of the accordiohedron. After we fix i and set $D_p C_{ij} = 0$, all facets of the accordiohedron pass through the origin. It means that the accordiohedron shrinks to the origin and the canonical form vanish. Without using the geometric property of the accordiohedron, we can also prove this from the factorization. First, we can see that all $2p - 2$ -point amplitudes vanish at $C = 0$,

$$m_{p,2p-2}^{P=(1p)}(zX, 0) = \frac{1}{X_{1p}} + \frac{1}{X_{p-1,2p-2}} = \frac{1}{X_{1p}} + \frac{1}{0 - X_{1p}} = 0. \quad (4.12)$$

Here we use the equation

$$X_{1p} + X_{p-1,2p-2} = \sum_{\substack{1 \leq a < p-1 \\ p \leq b < 2p-2}} C_{ab} \rightarrow 0. \quad (4.13)$$

We can see this for the other case obtained from the cyclic permutation. When lower-point amplitudes vanish with $C = 0$, any residue of the amplitude in the factorization vanishes. From these facts, we can verify that the canonical function of the accordiohedron vanishes at $C = 0$. Then the function $\frac{z^{D_p}}{z-1} m_{p,n}^P(zX, C)$ has no pole at $z \rightarrow \infty$.

In [4], this condition is proved for general ABHY polytopes. In this proof, they used the fact that the ABHY-polytopes are “ C -independent” polytopes. Since the accordiohedra are not C -independent, we cannot apply that proof to this accordiohedra.

The absence of the pole at infinity is related to the interesting properties of amplitudes. For example, in Yang-Mills theory, this fact is explained from the “dual conformal symmetry”. In this ϕ^p case, the Q -compatible sets of amplitudes have this property. It is thus interesting to see an analog of this hidden symmetry for each Q -compatible set of amplitudes, not all amplitudes.

Here we compare this BCFW-like recursion with the BCFW recursion relation [15, 16]. Under the BCFW shift of the external momenta, there are two contributions: residue parts of the finite pole and a boundary term that comes from the pole at $z \rightarrow \infty$. Since the boundary term does not vanish in general ϕ^p case, we need to consider not only the finite pole parts but also the boundary term. For example, the BCFW recursion relation with the boundary term for the ϕ^4 case has constructed in [20]. The systematic way to determine this boundary term has constructed in [21]. Unlike this BCFW recursion, the BCFW-like recursion which has constructed from the positive geometry has no contribution from the boundary term. From this, the BCFW-like recursion is simpler than the usual BCFW recursion relation for the general ϕ^p case. We see some examples of this recursion relation for $p = 4, 5, 6$ cases.

4.2 Explicit calculations

$p = 4, n = 6$. We choose $P = (36)$ as the primitive quadrangulation and denote X_{36} as the basis variable. Other variables are given as

$$\begin{aligned} X_{14} &= -X_{36} + C_{14} + C_{24} + C_{15} + C_{25}, \\ X_{25} &= d_{15} + C_{14} - d_{13} + C_{13}. \end{aligned} \quad (4.14)$$

The term $m_{345I}^{36L} \times m_{I612}^{36R}$ is the residues at the pole $\hat{X}_{14} := C_{14} + C_{24} + C_{15} + C_{25} - z_{14}X_{36} = 0$. Here z_{ab} is the solution of the equation $\hat{X}_{ab}(z) = 0$. Then

$$z_{14} = \frac{C_{14} + C_{24} + C_{15} + C_{25}}{X_{36}}. \quad (4.15)$$

Since four-point amplitudes are trivial $m_4^{36L,R} = \pm 1$, we can obtain the $Q(36)$ -compatible amplitude:

$$m_{p=4,6}^{36} = \frac{z_{14}}{X_{14}} = \frac{(C_{14} + C_{24} + C_{15} + C_{25})}{X_{36}X_{14}} = \frac{1}{X_{14}} + \frac{1}{X_{36}}. \quad (4.16)$$

If we choose X_{14} as the basis, we can obtain the same result. Similarly, we can compute other primitive quadrangulations obtained from the cyclic permutations. The results are given as

$$\begin{aligned} m_{4,6}^{25} &= \frac{1}{X_{25}} + \frac{1}{X_{14}} \\ m_{4,6}^{14} &= \frac{1}{X_{36}} + \frac{1}{X_{25}}. \end{aligned} \quad (4.17)$$

Sum of these terms with appropriate weight corresponds to the six-point amplitude,

$$M_{4,6} = \alpha_{4,6}^{14}(m_{4,6}^{36} + m_{4,6}^{25} + m_{4,6}^{14}). \quad (4.18)$$

$p = 4, n = 8$. Next, we obtain the eight-point amplitude from six-point amplitudes. We have seen in the section 2.4 that there are two primitive P_1, P_2 . First, we choose $P_1 = (14, 58)$ as the primitive quadrangulation, then the Q -compatible set is given as (2.23).

If we choose X_{14}, X_{58} as the basis, we need to sum over residues at $\hat{X}_{38} = 0, \hat{X}_{47} = 0$. They correspond to the factorizations:

$$\hat{X}_{38} = 0 : m_{34567I}^{P_{1L}} \times m_{I812}^{P_{1R}}, \quad \hat{X}_{47} = 0 : m_{456I}^{P_{1L}} \times m_{I78123}^{P_{1R}}. \quad (4.19)$$

From the results of the four-point and six-point amplitude, we have:

$$\begin{aligned} m_{I812}^{P_{1R}} &= 1, \quad m_{34567I}^{P_{1L}} = \frac{1}{\hat{X}_{58}(z_{38})} + \frac{1}{\hat{X}_{47}(z_{38})} \\ m_{456I}^{P_{1L}} &= 1, \quad m_{I78123}^{P_{1R}} = \frac{1}{\hat{X}_{14}(z_{47})} + \frac{1}{\hat{X}_{38}(z_{47})}. \end{aligned} \quad (4.20)$$

This gives the result for $Q(P_1)$ -compatible amplitude:

$$\begin{aligned} m_{4,8}^{P_1} &= \frac{z_{38}^2}{X_{38}} \left(\frac{1}{\hat{X}_{58}(z_{38})} + \frac{1}{\hat{X}_{47}(z_{38})} \right) + \frac{z_{47}^2}{X_{47}} \left(\frac{1}{\hat{X}_{14}(z_{47})} + \frac{1}{\hat{X}_{38}(z_{47})} \right) \\ &= \frac{A_{38}^2}{X_{38}X_{14}} \left(\frac{1}{X_{58}A_{38}} + \frac{1}{(A_{47}X_{14} - A_{38}X_{58})} \right) \\ &\quad + \frac{A_{47}^2}{X_{47}X_{58}} \left(\frac{1}{X_{14}A_{47}} + \frac{1}{(A_{38}X_{58} - A_{47}X_{14})} \right) \end{aligned} \quad (4.21)$$

where A_{38}, A_{47} are given as

$$A_{38} = \sum_{i=4}^7 (C_{1i} + C_{2i}), \quad A_{47} = d_{57} + d_{48} + C_{47}. \quad (4.22)$$

From the straightforward calculation, we can see that

$$m_{4,8}^{P_1} = \frac{1}{X_{14}X_{47}} + \frac{1}{X_{38}X_{47}} + \frac{1}{X_{14}X_{58}} + \frac{1}{X_{38}X_{58}}. \quad (4.23)$$

Next, we consider the $P_2 = (14, 16)$ and the $Q(P_2)$ -compatible set is given as (2.24). We choose X_{14}, X_{16} as the basis. We need to sum over residues at $\hat{X}_{58} = 0, \hat{X}_{38} = 0, \hat{X}_{36} = 0$. Each residue is given as

$$\begin{aligned} \hat{X}_{58} = 0 : \quad & m_{567I}^{P_{2L}} \times m_{I81234}^{P_{2R}} \\ &= \frac{A_{58}A_{38}}{X_{14}(A_{38}X_{16} - A_{58}X_{14})(A_{58} - X_{16})}, \\ \hat{X}_{38} = 0 : \quad & m_{34567I}^{P_{2L}} \times m_{I812}^{P_{2R}} \\ &= \frac{A_{36}A_{38}^2}{(X_{14} - A_{38})(A_{38}X_{16} - A_{58}X_{14})(A_{36}X_{14} - A_{38}X_{14} + A_{38}X_{16})}, \\ \hat{X}_{36} = 0 : \quad & m_{345I}^{P_{2L}} \times m_{I67812}^{P_{2R}} \\ &= \frac{(A_{36} - A_{58})(A_{36} + A_{38} - A_{58})}{X_{16}(A_{36} + A_{38} - A_{58} - X_{14} + X_{16})(A_{36}X_{14} - A_{38}X_{14} + A_{38}X_{16})} \end{aligned} \quad (4.24)$$

where A_{58}, A_{36} are given as

$$A_{58} = \sum_{\substack{1 \leq a < 5 \\ 6 \leq b < 8}} C_{ab}, \quad A_{36} = \sum_{\substack{3 \leq a < 5 \\ 6 \leq b < 8}} C_{ab}. \quad (4.25)$$

We can see that

$$m_{4,8}^{P_2} = \frac{1}{X_{14}X_{16}} + \frac{1}{X_{14}X_{58}} + \frac{1}{X_{36}X_{16}} + \frac{1}{X_{36}X_{83}} + \frac{1}{X_{58}X_{83}}. \quad (4.26)$$

From these results, we can obtain the eight-point amplitude

$$M_{4,8} = \alpha_{4,8}^{P_1} m_{4,8}^{P_1} + \alpha_{4,8}^{P_2} m_{4,8}^{P_2} + \text{cyclic} \quad (4.27)$$

where $\alpha_{4,8}^{P_1} = \frac{1}{3}$ and $\alpha_{4,8}^{P_2} = \frac{1}{6}$. These weights are obtained from the factorization.

$p = 4, n = 10$. In this case, there are 7 primitive $\{P_1, P_2, \dots, P_7\}$. For each P_i , we need to apply the recursion relation to obtain the 10-point amplitude. First we consider $P_1 = (14, 510, 69)$, the Q -compatible set is given as

$$Q(P_1) = \{(14, 510, 69, +), (310, 510, 69, -), (14, 49, 69, -), (14, 510, 58, -), \\ (14, 49, 58, +), (310, 510, 58, +), (310, 49, 69, +), (310, 49, 58, -)\}. \quad (4.28)$$

Once we choose X_{14}, X_{510}, X_{69} as the basis, we need to sum over residues at $\hat{X}_{310} = 0$, $\hat{X}_{49} = 0$, $\hat{X}_{58} = 0$. Here we see the one example: the residue of $\hat{X}_{310} = 0$. The shift parameter is given as

$$\hat{X}_{310} = z_{310}X_{14} + \sum_{\substack{1 \leq a < 3 \\ 4 \leq b < 10}} C_{ab} = 0 \rightarrow z_{310} = \sum_{\substack{1 \leq a < 3 \\ 4 \leq b < 10}} C_{ab}/X_{14} \quad (4.29)$$

This residue corresponds to the factorization:

$$m_{3456789I}^{P_{1L}} \times m_{I1012}^{P_{1R}}. \quad (4.30)$$

By using the eight point result,

$$\begin{aligned} & \frac{z_{310}^3}{X_{310}} m_{3456789I}^{P_1} \times m_{I1012}^{P_1} \\ &= \frac{A_{310}^3}{X_{310}X_{14}^3} \left(\frac{1}{\hat{X}_{510}\hat{X}_{58}} + \frac{1}{\hat{X}_{49}\hat{X}_{58}} + \frac{1}{\hat{X}_{510}\hat{X}_{69}} + \frac{1}{\hat{X}_{49}\hat{X}_{69}} \right) \\ &= \frac{A_{310}A_{49}A_{58}X_{14}}{X_{310}X_{510}(A_{49}X_{14} - A_{310}X_{510})X_{69}(A_{58}X_{14} - A_{310}X_{69})} \end{aligned} \quad (4.31)$$

where

$$A_{310} = \sum_{\substack{1 \leq a < 3 \\ 4 \leq b < 10}} C_{ab}, \quad A_{58} = d_{68} + d_{59} + C_{59}, \quad A_{49} = d_{59} + d_{410} + C_{410}. \quad (4.32)$$

Now it is easy to compute other residues.

$p = 5, n = 8$. In this case, there is one primitive p -angulation and we choose it as $P = (15)$. Once we choose X_{15} as the basis, then X_{48} is given as

$$X_{48} = \sum_{i=1}^3 (C_{i5} + C_{i6} + C_{i7}). \quad (4.33)$$

The only term $m_{1234I}^{P_L} \times m_{I5678}^{P_R}$ is the residues at the pole $\hat{X}_{48} = 0$. As same as the case of ϕ^4 , we can obtain the $Q(15)$ -compatible amplitude:

$$m_{p=5, n=8}^{15} = \frac{z_{48}}{X_{48}} = \frac{1}{X_{15}} + \frac{1}{X_{48}}, \quad (4.34)$$

where $z_{48} = \sum_{i=1}^3 (C_{i5} + C_{i6} + C_{i7})/X_{15}$. The eight-point amplitude is given as the sum of all cyclic permutations of (4.34).

$p = 5, n = 11$. There are two primitive p -angulations $P_1 = (15, 711), P_2 = (15, 18)$ and two Q -compatible sets given as (2.28). First we consider P_1 and we choose X_{15}, X_{711} as the basis. We need to sum over residues at $\hat{X}_{411} = 0, \hat{X}_{610} = 0$. The residues at the pole $\hat{X}_{411} = 0$ is $m_{45678910I}^{P_{1L}} \times m_{I11123}^{P_{1R}}$ and at the pole $\hat{X}_{610} = 0$ is $m_{6789I}^{P_{1L}} \times m_{I101112345}^{P_{1R}}$. These residues are given as

$$\begin{aligned} \hat{X}_{411} = 0 : & \quad \frac{z_{411}^2}{X_{411}} \left(\frac{1}{\hat{X}_{117}(z_{411})} + \frac{1}{\hat{X}_{610}(z_{411})} \right) \\ & = \frac{A_{411}^2}{X_{411}X_{15}} \left(\frac{1}{A_{411}X_{117}} + \frac{1}{(A_{610}X_{15} - A_{411}X_{711})} \right) \\ \hat{X}_{610} = 0 : & \quad \frac{z_{610}^2}{X_{610}} \left(\frac{1}{\hat{X}_{15}(z_{610})} + \frac{1}{\hat{X}_{411}(z_{610})} \right) \\ & = \frac{A_{610}^2}{X_{711}X_{610}} \left(\frac{1}{A_{610}X_{15}} + \frac{1}{(A_{411}X_{711} - A_{610}X_{15})} \right), \end{aligned} \quad (4.35)$$

where A_{411}, A_{610} are given as

$$A_{411} = \sum_{\substack{1 \leq a < 4 \\ 5 \leq b < 11}} C_{ab}, \quad A_{610} = \sum_{\substack{1 \leq a < 6 \\ 7 \leq b < 11}} C_{ab} + C_{610} - d_{17} + d_{710}. \quad (4.36)$$

The $Q(P_1)$ -compatible amplitude $m_{p=5, n=11}^{P_1}$ is given as a sum of these two residues. From the straightforward calculation, we can see that

$$m_{5,11}^{P_1} = \frac{1}{X_{15}X_{610}} + \frac{1}{X_{411}X_{610}} + \frac{1}{X_{15}X_{711}} + \frac{1}{X_{411}X_{711}}. \quad (4.37)$$

Similarly we can compute the P_2 case. We choose X_{15}, X_{18} as the basis. There are three residues at poles $\hat{X}_{48} = 0, \hat{X}_{411} = 0, \hat{X}_{711} = 0$. These residues are given as

$$\begin{aligned}
 \hat{X}_{48} = 0 : & \frac{z_{48}^2}{X_{48}} \left(\frac{1}{\hat{X}_{18}(z_{48})} + \frac{1}{\hat{X}_{411}(z_{48})} \right) \\
 &= \frac{A_{48}(A_{48} - A_{411})}{X_{18}X_{48}(A_{48}X_{15} + A_{411}X_{18} - A_{411}X_{18})} \\
 \hat{X}_{411} = 0 : & \frac{z_{411}^2}{X_{411}} \left(\frac{1}{\hat{X}_{48}(z_{411})} + \frac{1}{\hat{X}_{711}(z_{411})} \right) \\
 &= \frac{A_{411}^2(A_{711} + A_{48} - A_{411})}{(A_{711}X_{15} - A_{411}X_{18})(A_{48}X_{15} + A_{411}X_{18} - A_{411}X_{15})X_{411}} \\
 \hat{X}_{711} = 0 : & \frac{z_{711}^2}{X_{711}} \left(\frac{1}{\hat{X}_{15}(z_{711})} + \frac{1}{\hat{X}_{411}(z_{711})} \right) \\
 &= \frac{A_{411}A_{711}}{X_{15}X_{711}(A_{411}X_{18} - A_{711}X_{15})}, \tag{4.38}
 \end{aligned}$$

where A_{48}, A_{711} are given as

$$A_{48} = \sum_{\substack{1 \leq a < 4 \\ 5 \leq b < 8}} C_{ab}, \quad A_{711} = \sum_{\substack{1 \leq a < 7 \\ 8 \leq b < 11}} C_{ab}. \tag{4.39}$$

The $Q(P_2)$ -compatible amplitude $m_{p=5, n=11}^{P_2}$ is

$$m_{5,11}^{P_2} = \frac{1}{X_{15}X_{18}} + \frac{1}{X_{18}X_{48}} + \frac{1}{X_{15}X_{711}} + \frac{1}{X_{411}X_{711}} + \frac{1}{X_{48}X_{411}}. \tag{4.40}$$

Then the 10-point amplitude is given as a sum of these two compatible amplitudes with weights.

$p = 6, n = 14$. There are three primitive p -angulations $P_1 = (16, 914), P_2 = (16, 110), P_3 = (16, 813)$ and three Q -compatible sets are given as

$$\begin{aligned}
 Q_6(P_1) &= \{(16, 914), (514, 914), (16, 813), (514, 813)\} \\
 Q_6(P_2) &= \{(16, 813), (514, 813), (16, 712), (514, 712)\} \\
 Q_6(P_3) &= \{(16, 110), (16, 914), (110, 510), (510, 514), (514, 914)\}. \tag{4.41}
 \end{aligned}$$

Even in this $p = 6$ case, we can compute each residue as same as lower p case. Here we consider only P_1 case. There are two residues at $\hat{X}_{514} = 0, \hat{X}_{813} = 0$. These residues are given as

$$\begin{aligned}
 \hat{X}_{514} = 0 : & \quad m_{5678910111213I}^{P_{1L}} \times m_{I141234}^{P_{1R}} = \left(\frac{1}{\hat{X}_{914}(z_{514})} + \frac{1}{\hat{X}_{813}(z_{514})} \right), \\
 \hat{X}_{813} = 0 : & \quad m_{89101112I}^{P_{1L}} \times m_{I13141234567}^{P_{1R}} = \left(\frac{1}{\hat{X}_{16}(z_{813})} + \frac{1}{\hat{X}_{514}(z_{813})} \right). \tag{4.42}
 \end{aligned}$$

Then the $Q_6(P_1)$ -compatible amplitude is

$$\begin{aligned}
 m_{5,11}^{P_1} &= \frac{z_{514}^2}{X_{514}} \left(\frac{1}{\hat{X}_{914}(z_{514})} + \frac{1}{\hat{X}_{813}(z_{514})} \right) + \frac{z_{813}^2}{X_{813}} \left(\frac{1}{\hat{X}_{16}(z_{813})} + \frac{1}{\hat{X}_{514}(z_{813})} \right) \\
 &= \frac{A_{514}^2}{X_{514}X_{16}} \left(\frac{1}{A_{514}X_{914}} + \frac{1}{A_{813}X_{16} - A_{514}X_{914}} \right) \\
 &\quad + \frac{A_{813}^2}{X_{813}X_{914}} \left(\frac{1}{A_{914}X_{16}} + \frac{1}{A_{514}X_{914} - A_{813}X_{16}} \right) \\
 &= \frac{1}{X_{16}X_{914}} + \frac{1}{X_{514}X_{914}} + \frac{1}{X_{16}X_{813}} + \frac{1}{X_{514}X_{813}}
 \end{aligned} \tag{4.43}$$

where

$$A_{514} = \sum_{\substack{1 \leq a < 5 \\ 6 \leq b < 14}} C_{ab}, \quad A_{813} = d_{814} + d_{913} + C_{813}. \tag{4.44}$$

There is no difficulty to compute other compatible amplitudes.

4.3 Recursion from the triangulation

In this section, we interpret the results of the recursion relation as triangulations of accordiohedra. We see some examples for $p = 4, 5, 6$ cases and obtain the general n -point formula.

$p = 4, n = 8$. We have seen in section 2.4 that there are two stokes polytopes: one is a two dimensional square $\mathcal{S}^{P_1}$ and the other is a two-dimensional pentagon $\mathcal{S}^{P_2}$. First, we consider the triangulation of the $\mathcal{S}^{P_1}$. The constraints defining it inside the kinematic space are given in (2.25). To consider the triangulation, we use the coordinate for vertices of the stokes polytope. Once we choose a basis, the coordinates of any vertex can be obtained by solving D_p linear equations of the form $X_{a,b} = 0$. We choose the basis as $\{X_{14}, X_{58}\}$ and let us denote the vertices

$$\{X_{58}, X_{14}\}, \{X_{38}, X_{58}\}, \{X_{38}, X_{47}\}, \{X_{47}, X_{14}\} \rightarrow \mathcal{Z}_*^{P_1}, \mathcal{Z}_1^{P_1}, \mathcal{Z}_2^{P_1}, \mathcal{Z}_3^{P_1}. \tag{4.45}$$

For example, a vertex $\mathcal{Z}_2^{P_1}$ can be obtained by solving

$$-X_{14} + \sum_{\substack{1 \leq a < 3 \\ 1 \leq b < 8}} C_{ab} = 0, \quad -X_{58} + d_{57} + d_{48} + C_{47} = 0. \tag{4.46}$$

These equations give $\mathcal{Z}_2^{P_1} = (\sum_{\substack{1 \leq a < 3 \\ 1 \leq b < 8}} C_{ab}, d_{57} + d_{48} + C_{47})$. The direct computation gives

$$\begin{aligned}
 \mathcal{Z}_*^{P_1} &= (0, 0), & \mathcal{Z}_1^{P_1} &= \left(\sum_{\substack{1 \leq a < 3 \\ 1 \leq b < 8}} C_{ab}, 0 \right), \\
 \mathcal{Z}_2^{P_1} &= \left(\sum_{\substack{1 \leq a < 3 \\ 1 \leq b < 8}} C_{ab}, d_{57} + d_{48} + C_{47} \right), & \mathcal{Z}_3^{P_1} &= (0, d_{57} + d_{48} + C_{47}).
 \end{aligned} \tag{4.47}$$

Figure 3. Triangulation of $p = 4, n = 8$ stokes polytopes.

Then we introduce affine coordinate, $Z := (1, \mathcal{Z})$ and $Y = (1, \mathbf{X})$, where $\mathbf{X}$ denotes the basis. We triangulate $\mathcal{S}^{P_1}$ into two triangles as (a) in figure 3. The canonical function of a simplex of dimension $(n - 4)/2$ is given as

$$[Z_0, Z_1, \dots, Z_{\frac{n-4}{2}}]_P := \frac{\langle Z_0 Z_1 \dots Z_{\frac{n-4}{2}} \rangle_P^{\frac{n-4}{2}}}{\prod_{i=0}^{\frac{n-4}{2}} \langle Y Z_0 Z_1 \dots \hat{Z}_i \dots Z_{\frac{n-4}{2}} \rangle_P}, \quad (4.48)$$

where $\hat{Z}_i$ means the omission of Z_i . We use the notation

$$\langle Z_0 Z_1 \dots Z_i \rangle_P := \epsilon_{IJ\dots K} (Z_0^P)^I (Z_1^P)^J \dots (Z_i^P)^K. \quad (4.49)$$

From this formula, the canonical function of the $\mathcal{S}^{P_1}$ is given as

$$\begin{aligned} & [Z_*, Z_1, Z_2]_{P_1} + [Z_*, Z_2, Z_3]_{P_1} \\ &= \frac{\langle Z_* Z_1 Z_2 \rangle_{P_1}^2}{\langle Y Z_* Z_1 \rangle_{P_1} \langle Y Z_* Z_2 \rangle_{P_1} \langle Y Z_1 Z_2 \rangle_{P_1}} + \frac{\langle Z_* Z_2 Z_3 \rangle_{P_1}^2}{\langle Y Z_* Z_2 \rangle_{P_1} \langle Y Z_* Z_3 \rangle_{P_1} \langle Y Z_2 Z_3 \rangle_{P_1}} \\ &= \frac{A_{38} A_{47}}{X_{58} (A_{38} X_{58} - A_{47} X_{14}) (A_{38} - X_{14})} - \frac{A_{38} A_{47}}{X_{14} (A_{38} X_{58} - A_{47} X_{14}) (A_{47} - X_{58})}, \end{aligned} \quad (4.50)$$

where we use (4.47) and A_{38}, A_{47} are given as (4.22). We can check this corresponds to the canonical function of the $\mathcal{S}^{P_1}$. We can also check that the terms $[Z_*, Z_1, Z_2]_{P_1}, [Z_*, Z_2, Z_3]_{P_1}$ are corresponding to the residues of the recursion relation $m_{34567I}^{P_{1L}} \times m_{I812}^{P_{1R}}, m_{456I}^{P_{1L}} \times m_{I78123}^{P_{1R}}$ respectively. Next, we consider the $\mathcal{S}^{P_2}$. The constraints defining it inside the kinematic space are given in (2.25). We choose $\{X_{14}, X_{16}\}$ as the basis and denote the vertices of this stokes polytope as

$$\begin{aligned} & \{X_{14}, X_{16}\}, \{X_{14}, X_{58}\}, \{X_{58}, X_{83}\}, \{X_{83}, X_{36}\}, \{X_{36}, X_{16}\} \\ & \rightarrow \mathcal{Z}_*^{P_2}, \mathcal{Z}_1^{P_2}, \mathcal{Z}_2^{P_2}, \mathcal{Z}_3^{P_2}, \mathcal{Z}_4^{P_2}. \end{aligned} \quad (4.51)$$

Similarly we can triangulate this polytope as (b) in figure 3. Then the canonical function is given as

$$\begin{aligned}
 & [Z_*, Z_1, Z_2]_{P_2} + [Z_*, Z_2, Z_3]_{P_2} + [Z_*, Z_3, Z_4]_{P_2} \\
 &= \frac{A_{58}A_{38}}{X_{14}(A_{38}X_{16} - A_{58}X_{14})(A_{58} - X_{16})} \\
 &+ \frac{A_{36}A_{38}^2}{(X_{14} - A_{38})(A_{38}X_{16} - A_{58}X_{14})(A_{36}X_{14} - A_{38}X_{14} + A_{38}X_{16})} \\
 &+ \frac{(A_{36} - A_{58})(A_{36} + A_{38} - A_{58})}{X_{16}(A_{36} + A_{38} - A_{58} - X_{14} + X_{16})(A_{36}X_{14} - A_{38}X_{14} + A_{38}X_{16})}
 \end{aligned} \tag{4.52}$$

where A_{58}, A_{36} are given as (4.25). We can also check that these three terms are corresponding to the residues of the recursion relation $m_{567I}^{P_{2L}} \times m_{I81234}^{P_{2R}}, m_{34567I}^{P_{2L}} \times m_{I812}^{P_{2R}}, m_{345I}^{P_{2L}} \times m_{I67812}^{P_{2R}}$ respectively. The sum of these functions with appropriate weight corresponds to the eight-point amplitude:

$$M_{p=4, n=8} = \alpha_{P_1} \sum_{\sigma} \sum_{i=1}^2 [Z_*, Z_i, Z_{i+1}]_{P_1, \sigma} + \alpha_{P_2} \sum_{\sigma'} \sum_{i=1}^3 [Z_*, Z_i, Z_{i+1}]_{P_2, \sigma'}, \tag{4.53}$$

where σ, σ' range over all the cyclic permutations and $\alpha_{P_1} = \frac{1}{3}$ and $\alpha_{P_2} = \frac{1}{6}$.

This representation is familiar from the BCFW representation of tree NMHV amplitudes for $\mathcal{N} = 4$ SYM or ϕ^3 tree amplitudes. Similarly, we can obtain this representation for more higher point case. In the $n = 10$ case, the stokes polytope becomes three-dimensional polytope and we can triangulate explicitly. We put the results in the appendix.

$p = 5, n = 11$ case: there are two primitive p -angulations $P_1 = (15, 711), P_2 = (15, 18)$. Two Q -compatible sets and the constraints are given as (2.28), (2.29). The $Q_5(P_1)$ accordiohedron is a two-dimensional square. We can triangulate the accordiohedron as (a) in figure 4 and the canonical function is given as

$$\begin{aligned}
 & [Z_*, Z_1, Z_2]_{P_1} + [Z_*, Z_2, Z_3]_{P_1} \\
 &= \frac{A_{411}A_{610}}{X_{15}(X_{15} + X_{711} - A_{610})(A_{610}X_{15} - A_{411}X_{15} - A_{411}X_{711})} \\
 &+ \frac{A_{411}(A_{411} - A_{610})}{(A_{411} - X_{15})X_{711}(-A_{610}X_{15} + A_{411}X_{15} + A_{411}X_{711})}
 \end{aligned} \tag{4.54}$$

where A_{411}, A_{610} are given as (4.36). We can see that these two terms are corresponding to the residues at $\hat{X}_{411}, \hat{X}_{610} = 0$ respectively. Similarly, we can compute the canonical function from the triangulation of the $Q_5(P_2)$ accordiohedron which is given as (b) in figure 4.

$$\begin{aligned}
 & [Z_*, Z_1, Z_2]_{P_2} + [Z_*, Z_2, Z_3]_{P_2} + [Z_*, Z_3, Z_4]_{P_2} \\
 &= \frac{A_{411}A_{711}}{X_{15}(X_{18} - A_{711})(A_{711}X_{15} - A_{411}X_{18})} \\
 &+ \frac{A_{411}^2(A_{711} + A_{48} - A_{411})}{(A_{411} - X_{15})(X_{15}A_{411} - X_{18}A_{411} - A_{48}X_{15})(-A_{711}X_{15} + A_{411}X_{18})} \\
 &+ \frac{A_{48}(A_{48} - A_{411})}{X_{18}(X_{15}A_{411} - X_{18}A_{411} - A_{48}X_{15})(A_{48} - X_{15} + X_{18})},
 \end{aligned} \tag{4.55}$$

Figure 4. Triangulation of $p = 5, n = 11$ accordiohedra.

where A_{48}, A_{711} are given as (4.39). These three terms are corresponding to the residues at $\hat{X}_{48}, \hat{X}_{411}, \hat{X}_{711} = 0$ respectively. Then the 11-point amplitude is given as

$$M_{p=5, n=11} = \alpha_{5,11}^{P_1} \sum_{\sigma} \sum_{i=1}^2 [Z_*, Z_i, Z_{i+1}]_{P_1, \sigma} + \alpha_{5,11}^{P_2} \sum_{\sigma'} \sum_{i=1}^3 [Z_*, Z_i, Z_{i+1}]_{P_2, \sigma'}, \quad (4.56)$$

where $\alpha_{5,11}^{P_1} = \frac{3}{11}$ and $\alpha_{5,11}^{P_2} = \frac{2}{11}$.

$p = 6, n = 14$ case: there are three Q -compatible sets $Q_6(P_1), Q_6(P_2), Q_6(P_3)$ which given in (4.41). We can compute the canonical function from the triangulation and the 14-point amplitude is given as

$$\begin{aligned} M_{p=6, n=14} = & \alpha_{6,14}^{P_1} \sum_{\sigma} \sum_{i=1}^2 [Z_*, Z_i, Z_{i+1}]_{P_1, \sigma} + \alpha_{6,14}^{P_2} \sum_{\sigma} \sum_{i=1}^2 [Z_*, Z_i, Z_{i+1}]_{P_2, \sigma'} \\ & + \alpha_{6,14}^{P_3} \sum_{\sigma''} \sum_{i=1}^3 [Z_*, Z_i, Z_{i+1}]_{P_3, \sigma''}, \end{aligned} \quad (4.57)$$

where $\alpha_{6,14}^{P_1} = \frac{1}{3}$, $\alpha_{6,14}^{P_2} = \frac{1}{6}$ and $\alpha_{6,14}^{P_3} = \frac{1}{3}$. The explicit representation for each team is written in the appendix.

4.4 General formula for ϕ^p tree amplitudes

The triangulation of the general associahedron is given in [17]. Since the accordiohedron is the simple polytope which is same as the associahedron, we can triangulate it by applying the same formula of [17]. The triangulation of the D_p dimensional accordiohedron $\mathcal{AC}_{p, D_p}^P$ is given as

$$\sum_{i_1, j_1} \sum_{i_2, j_2} \cdots \sum_{i_{D_p}, j_{D_p}} [Z^{(0)}, Z_{i_1, j_1}^{(1)}, \dots, Z_{i_{D_p-1}, j_{D_p-1}}^{(D_p-1)}, Z_{i_{D_p}, j_{D_p}}^{(D_p)}]_P \quad (4.58)$$

Here we briefly see how to obtain this formula following with [17]. First we choose an origin $Z^{(0)}$ and connect to all co-dimension one facets that are not adjacent to $Z^{(0)}$. We denote these facets as X_{i_1,j_1} . Next, for each facet X_{i_1,j_1} , we triangulate it by connecting an origin $Z_{i_1,j_1}^{(1)}$ to all of its facets X_{i_2,j_2} . Continuing this for D_p step and sum over all pairs $(i_1, j_1), (i_2, j_2), \dots$, we can triangulate this polytope.

By using this formula, the general n -point amplitude for ϕ^p is given as

$$M_{p,n} = \sum_{a=1}^{p_n} \sum_{\sigma} \sum_{i_1,j_1} \cdots \sum_{i_{D_p},j_{D_p}} \alpha_{p,n}^{P_a} [Z^{(0)}, Z_{i_1,j_1}^{(1)}, \dots, Z_{i_{D_p-1},j_{D_p-1}}^{(D_p-1)}, Z_{i_{D_p},j_{D_p}}^{(D_p)}]_{P_a,\sigma}. \quad (4.59)$$

where p_n is a number of the primitive p -angulation and $\alpha_{p,n}^{P_a}$ is the weight.

The number p_n for ϕ^p case is obtained in [1]. The explicit formula for this number is given as

$$p_n = \begin{cases} \frac{1}{n} F_{D_p,n} + \frac{1}{2} F_{(D_p+1)/2,p} + \frac{1}{n} \sum_{d=\text{Gcd}(D_p,p)} \phi(p)^{\frac{1}{d}} \binom{(p-1)D_p + \frac{p}{d}}{D_p} & (\text{if } n \text{ is odd}) \\ \frac{1}{n} F_{D_p,n} + \frac{1}{n} \sum_{d=\text{Gcd}(D_p,p)} \phi(p)^{\frac{1}{d}} \binom{(p-1)D_p + \frac{p}{d}}{D_p} & (\text{if } n \text{ is even}) \end{cases}. \quad (4.60)$$

where $F_{D_p,n}$ is the Fuss Catalan number

$$F_{D_p,n} = \frac{1}{n(n-p)+p} \binom{(p-1)D_p}{D_p}. \quad (4.61)$$

And $\phi(d)$ is given by:

$$\phi(d) = d \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_r}\right) \quad (4.62)$$

where each p_i comes from the prime factorization of d :

$$d = p_a^{a_1} p_2^{a_2} \cdots p_r^{a_r}, \quad p_i \text{ are the prime numbers.} \quad (4.63)$$

As we have seen in section 3, these weights α_{P_a} are determined from the factorization of the accordiohedron.

5 Recursion from the projective triangulation

In section 4, we introduced the “BCFW”-like recursion relation for the ϕ^p amplitudes. This recursion is interpreted as a triangulation of accordiohedra. Recently, in [3] a new triangulation called “projective triangulation” for generalized ABHY-associahedra was introduced. From this new triangulation, we can obtain a new recursion relation called “projective recursion relation”. The field-theoretical derivation of the projective recursion was obtained in [4]. In this section, we apply this new recursion for ϕ^p amplitudes and interpret this as a projective triangulation of the accordiohedra.

5.1 Projective recursion relation

Here we briefly review the projective recursion relation by following [4]. First we consider a rescaling of k variables

$$X_{A_i} \rightarrow zX_{A_i}, \quad i = 1, 2, \dots, k, \quad (5.1)$$

here we choose X_{A_i} as part of the basis of planar variables and $1 \leq k \leq D_p$. Similar to the BCFW-like recursion, we consider the integral

$$\oint \frac{z^k}{z-1} m_{p,n}^P(zX, C), \quad (5.2)$$

where $m_{p,n}^P(zX, C)$ is the shifted n -point amplitude. By repeating same procedure of section 4, we obtain the recursion relation formula

$$m_{p,n}^P(X, C) = \sum_{B_i} \frac{z_{B_i}^k}{X_{B_i}} m_{a, \dots, b-1, I}^{P_L}(z_{B_i} X, C) m_{I, b, \dots, a-1}^{P_R}(z_{B_i} X, C), \quad (5.3)$$

where the sum of B_i runs over all the shifted planar variables. Here we denote the planar variables which depend on basis variables X_{A_i} as

$$X_{B_i} = C_{B_i} + X_i + \sum_{j=1}^k \lambda_{i,j} X_{A_j}, \quad (5.4)$$

where $\lambda_{i,j}$ are real numbers, C_{B_i} and X_i are linear combinations of constants C s and undeformed variable X_{A_j} , $j = k+1, \dots, D_p$. When $k = D_p$, this corresponds to (4.8). Then we can interpret this as a generalization of the BCFW-like recursion relation.

In this derivation, we use the fact that the function $\frac{z^k}{z-1} m_{p,n}^P(zX, C)$ has no pole at $z = 0$ and infinity. We prove these following with [4].

Proof of no pole at $z = 0$. First, we consider the pole at $z = 0$. The canonical form of simple polytopes is given in [2]:

$$\Omega = \sum_{v \in \text{vertices}} \prod_{\text{facet } f \in v} \frac{1}{X_f}, \quad (5.5)$$

where the facets f are determined from $X_f = 0$. Since a d -dimensional simple polytope has d facets adjacent to any vertex v , the order of the product of $\frac{1}{X_f}$ becomes as

$$\prod_{\text{facet } f \in v} \frac{1}{X_f} = O\left(\frac{1}{X^d}\right). \quad (5.6)$$

If we rescale k basis variables as

$$X_{A_i} \rightarrow zX_{A_i}, \quad i = 1, \dots, k, \quad (5.7)$$

at most k of the d X_{ij} 's that appear in the product (5.6) have a z dependence. Then the product (5.6) can at most have an order k pole at $z = 0$, and this is cancelled by the z^k factor of the numerator of the function $\frac{z^k}{z-1} m_{p,n}^P(zX, C)$. This is what we want to prove.

Proof of no pole at infinity. Here we rescale k basis variables (5.7) and consider the behavior of function $\frac{z^k}{z-1}m_{p,n}^P(zX, C)$ at $z \rightarrow \infty$. To see this, we need to consider the $O(\frac{1}{z^k})$ contribution of the canonical function (5.5). This can be obtained that k of the basis variables X_{A_i} in the denominator of (5.7) shifted and $(D_p - k)$ of them unshifted. Here we denote the unshifted variables as $\bar{X}_j$ and group the terms of (5.5) having the same $(D_p - k)$ $\bar{X}_j$'s;

$$\left(\sum_{\substack{\text{vertices shared by} \\ D_p - k \ \bar{X}_j}} \frac{1}{\prod^k X} \right) \frac{1}{\prod_{j=1}^{D_p - k} \bar{X}_j}. \quad (5.8)$$

Only the quantity in the bracket is affected by the rescaling. When all these $\bar{X}_j \rightarrow 0$, this quantity is the canonical function of the polytope which is the intersection of all these facets $\bar{X}_j$ and we denote as $m_{p,k}^P(X, C)$. When $z \rightarrow \infty$,

$$\lim_{z \rightarrow \infty} m_{p,k}^P(X, C) \sim \frac{1}{z^k} m_{p,k}^P(X, 0) \rightarrow 0. \quad (5.9)$$

Here we used the soft condition of the $m_{p,k}^P(X, C)$. We can easily verify this similarly as (4.11). Then the function $\frac{z^k}{z-1}m_{p,n}^P(zX, C)$ has no pole at infinity.

In [4], it was proven that this recursion relation (5.3) for the ABHY polytope is equivalent to the “projective triangulation”. We will see that in the accordiohedron case, this recursion relation can be interpreted as the projective triangulation of the accordiohedron.

5.2 One-variable projective recursion

In this section, we consider a interesting case: one-variable rescaling

$$X_A \rightarrow zX_A. \quad (5.10)$$

Under the rescaling (5.10), the shift parameter is determined by solving the equation

$$\hat{X}_{B_i} = C_{B_i} + X_i + \sum_{j=1}^{k-1} \lambda_{i,j} X_{A_j} + z_{B_i} \lambda_{i,k} X_{A_k} = 0. \quad (5.11)$$

The result is

$$z_{B_i} = -\frac{C_{B_i} + X_i + \sum_{j=1}^{k-1} \lambda_{i,j} X_{A_j}}{\lambda_i Z_A} = 1 - \frac{X_{B_i}}{\lambda_{i,k} X_{A_k}}. \quad (5.12)$$

Then the shifted basis variables are written as

$$X_A \rightarrow z_{B_i} X_A = X_A - \frac{1}{\lambda_{i,k}} X_{B_i}. \quad (5.13)$$

When we determine z_{B_i} from the equation $\hat{X}_{B_i} = 0$, the another variable $X_C = C_C + \lambda X_A + \sum_{j=1}^{D_p-1} \lambda_{i,j} X_{A_j}$ which depends on the shifted basis variable becomes as

$$\hat{X}_{C_i} = C_{C_i} + X_i + \sum_{j=1}^{k-1} \lambda_{i,j} X_{A_j} + z_{B_i} \lambda_{i,k} X_{A_k} = X_{C_i} - X_{B_i}. \quad (5.14)$$

From these results and

$$\frac{z_{B_i}}{X_{B_i}} = \left(\frac{1}{X_{B_i}} - \frac{1}{\lambda_i X_A} \right), \quad (5.15)$$

we can obtain the one-variable rescaling recursion

$$m_{p,n}^P(X, C) = \sum_{B_i} \left(\frac{1}{X_{B_i}} - \frac{1}{\lambda_i X_A} \right) \hat{m}_{a,\dots,b-1,I}^{P_L} \times \hat{m}_{I,b,\dots,a-1}^{P_R}, \quad (5.16)$$

where $\hat{m}$ means that before sum over B_i , we need to make the replacement (5.13) and (5.14). This recursion relation was also derived from the general properties of canonical forms of simple polytopes [18]. Next, we see some examples of this one-variable projective recursion.

$p = 4, n = 8$ case: let's consider a one-variable rescaling

$$X_{14} \rightarrow z X_{14}. \quad (5.17)$$

In this case, there are two Q -compatible sets (2.23) and (2.24). In the $Q(P_1)$ set, only X_{38} will be shifted. From (5.16),

$$\begin{aligned} m_{p=4,n=8}^{P_1}(X, C) &= \left(\frac{1}{X_{38}} + \frac{1}{X_{14}} \right) \hat{m}_{34567I}^{P_{1L}} \times \hat{m}_{I812}^{P_{1R}} \\ &= \left(\frac{1}{X_{38}} + \frac{1}{X_{14}} \right) \left(\frac{1}{X_{58}} + \frac{1}{X_{47}} \right). \end{aligned} \quad (5.18)$$

This is corresponding to the $p = 4, n = 8, Q(P_1)$ -compatible amplitude. This is the canonical function of the rectangle form by X_{38}, X_{58} as its edges. We can interpret this result as the prism formed by projecting the facet X_{38} onto line X_{14} .

In the $Q(P_2)$ set, X_{38} and X_{36} will be shifted. Then from the recursion formula,

$$\begin{aligned} m_{4,8}^{P_2}(X, C) &= \left(\frac{1}{X_{38}} + \frac{1}{X_{14}} \right) \hat{m}_{34567I}^{P_{2L}} \times \hat{m}_{I812}^{P_{2R}} + \left(\frac{1}{X_{36}} + \frac{1}{X_{14}} \right) \hat{m}_{345I}^{P_{2L}} \times \hat{m}_{I67812}^{P_{2R}} \\ &= \left(\frac{1}{X_{38}} + \frac{1}{X_{14}} \right) \left(\frac{1}{X_{36} - X_{38}} + \frac{1}{X_{58}} \right) \\ &\quad + \left(\frac{1}{X_{36}} + \frac{1}{X_{14}} \right) \left(\frac{1}{X_{38} - X_{36}} + \frac{1}{X_{16}} \right). \end{aligned} \quad (5.19)$$

It is easy to check that this corresponds to the $p = 4, n = 8, Q(P_2)$ -compatible amplitude. These two terms are the canonical function of the two 2-dimensional prisms as (a) in figure 5.

$p = 4, n = 10$ case: in this case, there are 7 primitive $\{P_1, P_2, \dots, P_7\}$. Here we consider $P_1 = (14, 510, 69)$, the Q -compatible set is given as (4.28). Let's consider a one-variable rescaling

$$X_{14} \rightarrow z X_{14}. \quad (5.20)$$

Only X_{310} will be shifted and the recursion formula becomes as

$$\begin{aligned} m_{p=4,n=10}^{P_1}(X, C) &= \left(\frac{1}{X_{14}} + \frac{1}{X_{310}} \right) \hat{m}_{3456789I}^{P_{1L}} \times \hat{m}_{I1012}^{P_{1R}} \\ &= \left(\frac{1}{X_{14}} + \frac{1}{X_{310}} \right) \left(\frac{1}{X_{510}X_{58}} + \frac{1}{X_{49}X_{58}} + \frac{1}{X_{510}X_{69}} + \frac{1}{X_{49}X_{69}} \right). \end{aligned} \quad (5.21)$$

Figure 5. Projecting triangulation for $p = 4, n = 8$ and $p = 5, n = 11$.

This corresponds to the $p = 4, n = 10, Q(P_1)$ -compatible amplitude. This is the product of the canonical functions of the line $\{X_{14}, X_{310}\}$ and the rectangle form by $\{X_{58}, X_{510}, X_{69}, X_{49}\}$. We can interpret this result as the prism formed by projecting the facet X_{310} onto line X_{14} .

$p = 5, n = 11$ case: let's consider a one-variable rescaling

$$X_{15} \rightarrow zX_{15}. \quad (5.22)$$

In this case, there are two Q -compatible sets (2.28). In the $Q(P_1)$ set, only X_{411} will be shifted and the recursion formula becomes as

$$\begin{aligned} m_{p=5, n=11}^{P_1}(X, C) &= \left(\frac{1}{X_{411}} + \frac{1}{X_{15}} \right) \hat{m}_{45678910I}^{P_{1L}} \times \hat{m}_{I11123}^{P_{1R}} \\ &= \left(\frac{1}{X_{411}} + \frac{1}{X_{15}} \right) \left(\frac{1}{X_{117}} + \frac{1}{X_{610}} \right). \end{aligned} \quad (5.23)$$

This corresponds to the $p = 5, n = 11, Q(P_1)$ -compatible amplitude. This is the canonical function of the prism formed by projecting the facet X_{411} onto line X_{15} . Similarly, in the $Q(P_2)$ set X_{48} and X_{411} will be shifted. Then from the recursion formula,

$$\begin{aligned} m_{5,11}^{P_2}(X, C) &= \left(\frac{1}{X_{48}} + \frac{1}{X_{15}} \right) \hat{m}_{4567I}^{P_{2L}} \times \hat{m}_{I891011123}^{P_{2R}} \\ &\quad + \left(\frac{1}{X_{411}} + \frac{1}{X_{15}} \right) \hat{m}_{45678910I}^{P_{2L}} \times \hat{m}_{I11123}^{P_{2R}} \\ &= \left(\frac{1}{X_{48}} + \frac{1}{X_{15}} \right) \left(\frac{1}{X_{411} - X_{48}} + \frac{1}{X_{18}} \right) \\ &\quad + \left(\frac{1}{X_{411}} + \frac{1}{X_{15}} \right) \left(\frac{1}{X_{48} - X_{411}} + \frac{1}{X_{711}} \right). \end{aligned} \quad (5.24)$$

This corresponds to the $p = 5, n = 11, Q(P_2)$ -compatible amplitude. We can also see that these two terms are the canonical functions of the two 2-dimensional prisms as (b) in figure 5.

6 Discussions

In this paper, we have investigated the weights of the accordiohedron and the one-parameter recursion relations of the ϕ^p tree amplitude. The main difference with the ϕ^3 case is that there is no single polytope which represents complete scattering amplitude. For each Q -compatible set of graphs of ϕ^p , we can define the accordiohedron as the positive geometry and the full scattering amplitudes are given as a weighted sum over all of these accordiohedra.

We have determined these weights from the factorization property of the accordiohedra. This means that even in this ϕ^p case, the geometry of the accordiohedron is enough to calculate scattering amplitudes.

We also have constructed one-parameter recursion relations of the ϕ^p tree amplitudes. In the case of ϕ^p , we need to consider these recursion relations for each Q -compatible set of graphs. After some examples of the BCFW-like recursion, we have obtained the all-multiplicity result for the ϕ^p tree amplitudes. In addition to this, we constructed the projective recursion relation of the ϕ^p tree amplitudes. We calculated some examples of the one-variable projective recursion. This can be interpreted as the projective triangulation of the accordiohedron.

There are many open questions for future studies. One of the issues is to consider the polynomial interaction case. In the case of $\lambda_3\phi^3 + \lambda_4\phi^4$, it is known that the positive geometry is the accordiohedron [14]. Even in this case, there are some accordiohedra for each amplitude and we need to determine the weights to obtain the amplitude. Then is it possible to determine the weights from the factorization of the accordiohedra for this polynomial interaction case? Another question is whether is it possible to apply the one-parameter recursions to this polynomial case. We plan to address these questions in the future.

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A Some details for $p = 4, n = 10$ case

Triangulation of the stokes polytope. There are 7 primitive quadrangulation

$$\begin{aligned} P_1 &= (14, 510, 69), & P_2 &= (14, 16, 18), & P_3 &= (14, 16, 69), & P_4 &= (14, 49, 69), \\ P_5 &= (14, 47, 710), & P_6 &= (14, 510, 710), & P_7 &= (14, 16, 710). \end{aligned} \quad (\text{A.1})$$

First, we consider P_1 , the Q -compatible set is given as (4.28). The stokes polytope for this case is 8-vertices three dimensional polytope. The constraints defining it in the kinematic space are given as

$$\begin{aligned} s_{ij} &= -C_{ij}, \text{ for } i \leq i < j \leq 9 \text{ with } |i - j| \geq 2 \\ X_{13} &= d_{13}, X_{410} = d_{410}, X_{59} = d_{59}, X_{68} = d_{68}. \end{aligned} \quad (\text{A.2})$$

We label all of the vertices by the three adjacent facets as

$$\begin{aligned} Z^{(0)} &= \{X_{14}, X_{510}, X_{69}\}, & Z_1 &= \{X_{58}, X_{49}, X_{310}\}, \\ Z_2 &= \{X_{58}, X_{510}, X_{310}\}, & Z_3 &= \{X_{69}, X_{510}, X_{310}\}, \\ Z_4 &= \{X_{49}, X_{14}, X_{58}\}, & Z_{310}^{(1)} &= \{X_{69}, X_{310}, X_{49}\}, \\ Z_{49}^{(1)} &= \{X_{49}, X_{69}, X_{14}\}, & Z_{58}^{(1)} &= \{X_{58}, X_{14}, X_{510}\}. \end{aligned} \quad (\text{A.3})$$

The direct computation gives

$$\begin{aligned} Z^{(0)} &= (1, 0, 0, 0), & Z_1 &= (1, A_{310}, A_{49}, A_{58}), \\ Z_2 &= (1, A_{310}, 0, A_{58}), & Z_3 &= (1, A_{310}, 0, 0), \\ Z_4 &= (1, 0, A_{49}, A_{58}), & Z_{310}^{(1)} &= (1, A_{310}, A_{49}, 0), \\ Z_{49}^{(1)} &= (1, 0, A_{49}, 0), & Z_{58}^{(1)} &= (1, 0, 0, A_{58}). \end{aligned} \quad (\text{A.4})$$

where

$$A_{310} = \sum_{\substack{1 \leq a < 3 \\ 4 \leq b < 10}} C_{ab}, \quad A_{58} = d_{68} + d_{59} + C_{59}, \quad A_{49} = d_{59} + d_{410} + C_{410}, \quad (\text{A.5})$$

and we omitted the label of $Q(P_1)$ ($Z^{P_1} \rightarrow Z$). From the general formula, the triangulation of this polytope is given as

$$\begin{aligned} &[Z^{(0)}, Z_{310}^{(1)}, Z_1, Z_2]_{P_1} + [Z^{(0)}, Z_{310}^{(1)}, Z_2, Z_3]_{P_1} + [Z^{(0)}, Z_{49}^{(1)}, Z_4, Z_1]_{P_1} \\ &+ [Z^{(0)}, Z_{49}^{(1)}, Z_1, Z_{310}^{(1)}]_{P_1} + [Z^{(0)}, Z_{58}^{(1)}, Z_2, Z_1]_{P_1} + [Z^{(0)}, Z_{58}^{(1)}, Z_1, Z_4]_{P_1}, \end{aligned} \quad (\text{A.6})$$

We can compute each term, for example:

$$\begin{aligned} &[Z^{(0)}, Z_{310}^{(1)}, Z_1, Z_2]_{P_1} \\ &= \frac{A_{310}^3 A_{49} A_{58}}{(A_{310} - X_{14})(A_{310} X_{510} - A_{49} X_{14})(A_{310} X_{69} - A_{58} X_{14})(A_{310} A_{49} X_{69} + A_{310} A_{58} X_{510} - A_{49} A_{58} X_{14})}, \end{aligned} \quad (\text{A.7})$$

$$\begin{aligned} &[Z^{(0)}, Z_{310}^{(1)}, Z_2, Z_3]_{P_1} \\ &= \frac{A_{310} A_{49} A_{58}}{(-A_{310} + X_{14}) X_{510} X_{69} (A_{310} A_{49} X_{69} + A_{310} A_{58} X_{510} - A_{49} A_{58} X_{14})}, \end{aligned} \quad (\text{A.8})$$

$$\begin{aligned} &[Z^{(0)}, Z_{49}^{(1)}, Z_4, Z_1]_{P_1} \\ &= \frac{A_{310} A_{49} A_{58}}{X_{14} (A_{49} - X_{510}) (A_{58} X_{14} - A_{310} X_{69}) (A_{49} X_{69} - A_{58} X_{510})}, \end{aligned} \quad (\text{A.9})$$

$$\begin{aligned}
 & [Z^{(0)}, Z_{49}^{(1)}, Z_1, Z_{310}^{(1)}]_{P_1} \\
 &= \frac{A_{310}A_{49}A_{58}}{X_{14}X_{69}(-A_{49} + X_{510})(A_{310}A_{49}X_{69} + A_{310}A_{58}X_{510} - A_{49}A_{58}X_{14})},
 \end{aligned} \tag{A.10}$$

$$\begin{aligned}
 & [Z^{(0)}, Z_{58}^{(1)}, Z_2, Z_1]_{P_1} \\
 &= \frac{A_{310}A_{49}A_{58}}{X_{510}(-A_{49}X_{14} + A_{310}X_{510})(A_{58} - X_{69})(A_{58}X_{14} - A_{310}X_{69})},
 \end{aligned} \tag{A.11}$$

$$\begin{aligned}
 & [Z^{(0)}, Z_{58}^{(1)}, Z_1, Z_2]_{P_1} \\
 &= \frac{A_{310}A_{49}A_{58}}{X_{14}(-A_{49}X_{14} + A_{310}X_{510})(A_{58} - X_{69})(-A_{58}X_{510} + A_{49}X_{69})}
 \end{aligned} \tag{A.12}$$

Then the sum of these terms is

$$\begin{aligned}
 & \frac{1}{X_{14}X_{49}X_{58}} + \frac{1}{X_{310}X_{49}X_{58}} + \frac{1}{X_{14}X_{510}X_{58}} + \frac{1}{X_{310}X_{510}X_{58}} \\
 & + \frac{1}{X_{14}X_{49}X_{69}} + \frac{1}{X_{310}X_{49}X_{69}} + \frac{1}{X_{14}X_{510}X_{69}} + \frac{1}{X_{310}X_{510}X_{69}}.
 \end{aligned} \tag{A.13}$$

This corresponds to the canonical function of the stokes polytope.

B Explicit results of the triangulation for $p = 6$, $n = 14$ case

$$\begin{aligned}
 & [Z_*, Z_1, Z_2]_{P_1} + [Z_*, Z_2, Z_3]_{P_1} \\
 &= \frac{A_{514}A_{813}}{(A_{514} - X_{16})X_{914}(-X_{914}A_{514} + X_{16}A_{813})} \\
 & + \frac{A_{514}A_{813}}{(X_{914} - A_{813})X_{16}(X_{16}A_{813} - X_{914}A_{514})}, \\
 & [Z_*, Z_1, Z_2]_{P_2} + [Z_*, Z_2, Z_3]_{P_2} \\
 &= \frac{A_{514}A_{712}}{(X_{914} - A_{712})X_{914}(X_{914}A_{514} - X_{16}A_{712})} \\
 & + \frac{A_{514}A_{712}}{(X_{16} - A_{514})X_{16}(X_{16}A_{712} - X_{914}A_{514})}, \\
 & [Z_*, Z_1, Z_2]_{P_3} + [Z_*, Z_2, Z_3]_{P_3} + [Z_*, Z_3, Z_4]_{P_3} \\
 &= \frac{A_{510}(A_{510} - A_{514})}{X_{110}(A_{510} + X_{110} - X_{16})(A_{514}X_{110} - A_{514}X_{16} + A_{510}X_{16})} \\
 & + \frac{A_{514}^2(A_{510} - A_{514} + A_{914})}{(A_{514} - X_{16})(A_{514}X_{110} - A_{914}X_{16})(A_{514}X_{16} - A_{514}X_{110} - A_{510}X_{16})} \\
 & + \frac{A_{514}A_{914}}{(A_{914} - X_{110})X_{16}(X_{110}A_{514} - X_{16}A_{914})}
 \end{aligned} \tag{B.1}$$

where

$$\begin{aligned}
 A_{514} &= \sum_{\substack{1 \leq a < 5 \\ 6 \leq b < 14}} C_{ab}, & A_{510} &= \sum_{\substack{1 \leq a < 5 \\ 6 \leq b < 10}} C_{ab} \\
 A_{813} &= d_{814} + d_{913} + C_{813}, & A_{712} &= d_{812} + d_{713} + C_{712}.
 \end{aligned} \tag{B.2}$$

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