Hyperbolic geometry and closed bosonic string field theory. Part II. The rules for evaluating the quantum BV master action

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ABSTRACT: The quantum Batalian-Vilkovisky master action for closed string field theory consists of kinetic term and infinite number of interaction terms. The interaction strengths (coupling constants) are given by integrating the off-shell string measure over the distinct string diagrams describing the elementary interactions of the closed strings. In the first paper of this series, it was shown that the string diagrams describing the elementary interactions can be characterized using the Riemann surfaces endowed with the hyperbolic metric with constant curvature -1 . In this paper, we construct the off-shell bosonic-string measure as a function of the Fenchel-Nielsen coordinates of the Teichmüller space of hyperbolic Riemann surfaces. We also describe an explicit procedure for integrating the off-shell string measure over the region inside the moduli space corresponding to the elementary interactions of the closed strings.

KEYWORDS: String Field Theory, Bosonic Strings

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We dedicate this paper to the memory of Maryam Mirzakhani who tragically passed away recently, and whose seminal ideas about the space of hyperbolic Riemann surfaces form some of the basic tools that we use in this work.

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1 Introduction

The basic object in string theory is a one-dimensional extended object, the string. The harmonics of the vibrating string correspond to the elementary particles with different masses and quantum numbers. String can support infinitely many harmonics, and hence string theory contains infinite number of elementary particles. Therefore, we can consider string theory as a framework for describing the interactions of infinitely many elementary particles. String field theory describe the dynamics of this system of infinite number of elementary particles using the language of quantum field theory [1–3]. Compared to the

conventional formulation of string perturbation theory [19], string field theory has the advantage that the latter provide us with the standard tools in quantum field theory for computing the S-matrix elements that are free from infrared divergences [20–25, 31, 32, 34]. Furthermore, the S-matrix computed using string field is unitary [26–30]. Since string field theory is based on a Lagrangian, it also has the potential to open the door towards the non-perturbative regime of string theory [15], even though no one has succeeded in studying the non-perturbative behaviour of closed strings using closed string field theory yet [16, 33]. Moreover, string field theory can be used for the first principle construction for the effective actions describing the low energy dynamics of strings [31–33].

The gauge transformations of closed string field theory form a complicated infinite dimensional gauge group. Consequently, the quantization of closed string field theory requires the sophisticated machinery of Batalian-Vilkovisky formalism (BV formalism) [7–13]. The quantum BV master action for closed string field theory can be obtained by solving the quantum BV master equation. The perturbative solution of quantum BV master action for the closed bosonic string field theory has been already constructed [3]. The striking feature of closed string field theory is that, albeit, the quantum BV master action contains a kinetic term and infinite number of interaction terms, the theory has only one independent parameter, the closed string coupling. The interaction strengths (coupling constants) of the elementary interactions in closed string field theory are expressed as integrals over the distinct two dimensional world-sheets describing the elementary interactions of the closed strings.

The collection of world-sheets describing the elementary interactions of the closed strings are called as the string vertex. A consistent set of string vertices provide a cell decomposition of the moduli space of Riemann surfaces [3]. The main challenge in constructing string field theory is to find a consistent set of string vertices that give rise to a suitable cell decomposition of the moduli spaces of Riemann surfaces. In principle, all the string vertices that provide such a cell decomposition of the moduli space can be constructed using the Riemann surfaces endowed with the metric solving the generalized minimal area problem [3]. Unfortunately, our current understanding of minimal area metrics is insufficient to obtain a calculable formulation of closed string field theory.¹ However, there exist an alternate construction of the string vertices using Riemann surfaces endowed with metrics having constant curvature -1 [5, 6]. They can be characterized using the Fenchel-Nielsen coordinates for the Teichmüller space and the local coordinates around the punctures on the world-sheets in terms of the hyperbolic metric. They can be used to construct a closed string field theory with approximate gauge invariance.

The interaction strengths in closed string field theory are obtained by integrating the off-shell string measure over the region in the moduli space that corresponds to the distinct two dimensional world-sheets describing the elementary interactions of the closed strings.

¹Recently, the cubic vertex of heterotic string field theory has constructed by using $SL(2, \mathbb{C})$ local coordinate maps which in turn has been used to construct the one loop tadpole string vertex in heterotic string field theory [18]. The cubic string vertex defined this way differ from the cubic string vertex defined by the minimal area metric.

The explicit evaluation of the interaction strength requires:

1. A convenient choice of parametrization of the Teichmüller space and the conditions on them that specify the region of the moduli space inside the Teichmüller space.
2. An explicit description for the region inside moduli space that corresponds to the string vertex and a consistent choice of local coordinates around the punctures on the Riemann surfaces belong to the string vertex.
3. An explicit procedure for constructing the off-shell string measure in terms of the chosen coordinates of the moduli space.
4. Finally, an explicit procedure for integrating the off-shell string measure over the region inside the moduli space that corresponds to the string vertex.

In this paper, we provide detailed descriptions for each of them.

Summary of the results. The main results of this paper are as follows:

- We explicitly construct the off-shell string measure in terms of the Fenchel-Nielsen coordinates for the Teichmüller space using a specific choice of local coordinates that is encoded in the definition of the string vertices.
- The interaction strengths in closed string field theory are obtained by integrating the off-shell string measure, which is an MCG-invariant object, over the region in the moduli space that corresponds to the Riemann surfaces describing the elementary interactions of the closed strings. The moduli space is the quotient of the Teichmüller space with the action of the mapping class group (MCG). However, in the generic case, an explicit fundamental region for the action of the MCG inside the Teichmüller space is not known. Therefore, integrating an MCG-invariant function over a region in the moduli space of the Riemann surfaces is not a straightforward operation to perform. In this paper, we discuss a way to bypass this difficulty and obtain an effective expression for the integral using the prescription for performing the integration over the moduli space of hyperbolic Riemann surfaces, parametrized using the Fenchel-Nielsen coordinates, introduced by M.Mirzakhani [39].
- We show that this integration method has an important property when we restrict the integration to a thin region around the boundary of the moduli space. Using this property, we find an effective expression for the integral of the off-shell string measure over the region inside the moduli space that corresponds to the string vertex.

In short, we describe a systematic method for evaluating the quantum BV master action for closed bosonic string field theory.

Organization of the paper. This paper is organized as follows. In section 2, we briefly review the general construction of the quantum BV master action for closed bosonic string field theory and explain what do we mean by the explicit evaluation of the quantum

action. In section 3 we discuss the construction string vertices using hyperbolic Riemann surfaces described in [6]. In section 4, we describe the explicit construction of the off-shell string measure in terms of the Fenchel-Nielsen coordinates of the Teichmüller space. In section 5 we discuss the concept of effective string vertices and the practical procedure of evaluating the corrected interaction vertices. In section 6 we provide a brief summary of the paper and mention some of the future directions. In appendix A, we review the theory of hyperbolic Riemann surfaces. In appendix B and appendix C, we discuss two classes of non-trivial identities satisfied by the lengths of the simple closed geodesics on a hyperbolic Riemann surface.

2 The quantum BV master action

The quantum BV master action for closed string field theory is a functional of the fields and the antifields in the theory. The *fields* and *antifields* are specified by splitting the string field $|\Psi\rangle$, which is an arbitrary element in the Hilbert space of the worldsheet CFT [2], as

$$|\Psi\rangle = |\Psi_-\rangle + |\Psi_+\rangle. \quad (2.1)$$

Both $|\Psi_-\rangle$ and $|\Psi_+\rangle$ are annihilated by b_0^- and L_0^- . The string field $|\Psi_-\rangle$ contains all the fields and the string field $|\Psi_+\rangle$ contains all the antifields. They can be decomposed as follows

$$\begin{aligned} |\Psi_-\rangle &= \sum'_{G(\Phi_s) \leq 2} |\Phi_s\rangle \psi^s, \\ |\Psi_+\rangle &= \sum'_{G(\Phi_s) \leq 2} |\tilde{\Phi}_s\rangle \psi_s^*, \end{aligned} \quad (2.2)$$

where $|\tilde{\Phi}_s\rangle = b_0^- |\Phi_s^c\rangle$, such that $\langle \Phi_r^c | \Phi_s \rangle = \delta_{rs}$. The state $\langle \Phi_r^c |$ is the conjugate state of $|\Phi_r\rangle$. The sum in (2.2) extends over the basis states $|\Phi_s\rangle$ with ghost number less than or equal to two. The prime over the summation sign reminds us that the sum is only over those states that are annihilated by L_0^- . The target space field ψ_s^* is the antifield that corresponds to the target space field ψ^s . The target space ghost number of the fields $g^t(\psi^s)$ takes all possible non-negative values and that of antifields $g^t(\psi_s^*)$ takes all possible negative values. They are related via the following relation

$$g^t(\psi_s^*) + g^t(\psi^s) = -1. \quad (2.3)$$

Therefore, the statistics of the antifield is opposite to that of the field. Moreover, it is possible to argue that corresponding to each target space field ψ^s there is a unique antifield ψ_s^* [2].

The quantum BV master action must be a solution of the following quantum BV master equation for the closed bosonic string theory

$$\frac{\partial_r S}{\partial \psi^s} \frac{\partial_l S}{\partial \psi_s^*} + \hbar \frac{\partial_r}{\partial \psi^s} \frac{\partial_l S}{\partial \psi_s^*} = 0, \quad (2.4)$$

where the target space field ψ_s^* is the antifield corresponding to the field ψ^s and ∂_r, ∂_l denote the right and left derivatives respectively. The perturbative solution of this equation in the closed string coupling g_s is given by [3]:

$$S(\Psi) = g_s^{-2} \left[\frac{1}{2} \langle \Psi | c_0^- Q_B | \Psi \rangle + \sum_{g \geq 0} (\hbar g_s^2)^g \sum_{n \geq 1} \frac{g_s^n}{n!} \{ \Psi^n \}_g \right], \quad (2.5)$$

where Ψ denotes the string field (2.1) having arbitrary ghost number that is built using target space fields and antifields carrying arbitrary ghost numbers. $\{ \Psi^n \}_g$ denotes the g -loop elementary interaction vertex $\{ \Psi_1, \dots, \Psi_n \}_g$ for n closed string fields with $\Psi_i = \Psi$ for $i = 1, \dots, n$. The g -loop elementary interaction vertex $\{ \Psi_1, \dots, \Psi_n \}_g$ for n closed string fields can be defined as the integral of the off-shell string measure $\Omega_{6g-6+2n}^{(g,n)}(|\Psi_1\rangle, \dots, |\Psi_n\rangle)$ over the string vertex $\mathcal{V}_{g,n}$:

$$\{ \Psi_1, \dots, \Psi_n \}_g \equiv \int_{\mathcal{V}_{g,n}} \Omega_{6g-6+2n}^{(g,n)}(|\Psi_1\rangle, \dots, |\Psi_n\rangle), \quad (2.6)$$

where $\Psi_1, \dots, \Psi_n$ denotes the off-shell closed string states $|\Psi_1\rangle, \dots, |\Psi_n\rangle$. The definition of the string vertices and the construction of off-shell measure is discussed below.

2.1 The string vertex $\mathcal{V}_{g,n}$

The string vertex $\mathcal{V}_{g,n}$ for the closed strings can be understood as a collection of genus g Riemann surfaces with n punctures that belong to a specific region inside the compactified moduli space $\overline{\mathcal{M}}_{g,n}$. We can define the string vertices by stating the properties that they must satisfy [3]:

- The string vertices must not contain Riemann surfaces that are arbitrarily close to the degeneration.
- The Riemann surfaces that belong to the string vertices must be equipped with a specific choice of local coordinates around each of its punctures. The coordinates are only defined up to a constant phase and they are defined continuously over the set $\mathcal{V}_{g,n}$.
- The local coordinates around the punctures on the Riemann surfaces that belong to the string vertices must be independent of the labeling of the punctures. Moreover, if a Riemann surface $\mathcal{R}$ with labeled punctures is in $\mathcal{V}_{g,n}$ then copies of $\mathcal{R}$ with all other inequivalent labelings of the punctures also must be included in $\mathcal{V}_{g,n}$.
- If a Riemann surface belongs to the string vertex, then its complex conjugate also must be included in the string vertex. A complex conjugate Riemann surface of a Riemann surface $\mathcal{R}$ with coordinate z can be obtained by using the anti-conformal map $z \rightarrow -\bar{z}$.

The string vertices with the above mentioned properties must also satisfy the following geometric identity. This identity can be understood as the geometric realization of the quantum BV master equation (2.4):

$$\partial\mathcal{V}_{g,n} = -\frac{1}{2} \sum_{\substack{g_1, g_2 \\ g_1+g_2=g}} \sum_{\substack{n_1, n_2 \\ n_1+n_2=n}} \mathbf{S}[\{\mathcal{V}_{g_1, n_1}, \mathcal{V}_{g_2, n_2}\}] - \Delta\mathcal{V}_{g-1, n+2}, \quad (2.7)$$

where $\partial\mathcal{V}_{g,n}$ denotes the boundary of the string vertex $\mathcal{V}_{g,n}$ and $\mathbf{S}$ represents the operation of summing over all inequivalent permutations of the external punctures. $\{\mathcal{V}_{g_1, n_1}, \mathcal{V}_{g_2, n_2}\}$ denotes the set of Riemann surfaces obtained by taking a Riemann surface from the string vertex $\mathcal{V}_{g_1, n_1}$ and a Riemann surface from the string vertex $\mathcal{V}_{g_2, n_2}$ and gluing them by identifying the regions around one of the puncture from each via the special plumbing fixture relation:

$$zw = e^{i\theta}, \quad 0 \leq \theta \leq 2\pi, \quad (2.8)$$

where z and w denote the local coordinates around the punctures that are being glued. The special plumbing fixture corresponds to the locus $|t| = 1$ of the plumbing fixture relation

$$zw = t, \quad t \in \mathbb{C}, \quad 0 \leq |t| \leq 1. \quad (2.9)$$

The resulting surface has genus $g = g_1 + g_2$ and $n = n_1 + n_2 - 2$. Δ denotes the operation of taking a pair of punctures on a Riemann surface that belongs to the string vertex $\mathcal{V}_{g-1, n+2}$ and gluing them via the special plumbing fixture relation (2.8). Therefore, the first term of (2.7) represents the gluing of two distinct surfaces via the special plumbing fixture and the second terms represents the special plumbing fixture applied to a single surface.

The geometric condition (2.7) demands that the set of Riemann surfaces that belong to the boundary of a string vertex having dimension, say d , must agree with the set of union of surfaces having dimension d obtained by applying the special plumbing fixture construction (2.8) to the surfaces belong to the lower dimensional string vertices only once, both in their moduli parameters and in their local coordinates around the punctures.

2.2 The off-shell string measure $\Omega_{6g-6+2n}^{(g,n)}$

The off-shell string measure $\Omega_{6g-6+2n}^{(g,n)}(|\Psi_1\rangle, \dots, |\Psi_n\rangle)$ is constructed using n number of vertex operators with arbitrary conformal dimensions. Consequently, the off-shell string measure depends on the choice of local coordinates around the punctures on the Riemann surface. Therefore, the integration measure of an off-shell amplitude is not a genuine differential form on the moduli space $\mathcal{M}_{g,n}$, because the moduli spaces do not know about the various choices of local coordinates around the punctures. Instead, we need to consider it as a differential form defined on a section of a larger space $\widehat{\mathcal{P}}_{g,n}$. This space is defined as a fiber bundle over $\mathcal{M}_{g,n}$. The fiber direction of the fiber bundle $\pi : \widehat{\mathcal{P}}_{g,n} \rightarrow \mathcal{M}_{g,n}$ contains the information about different choices of local coordinates around each of the n punctures that differ only by a phase factor. The section of our interest corresponds to the choice of a specific set of local coordinates around the punctures for each point $\mathcal{R}_{g,n} \in \mathcal{M}_{g,n}$. Therefore, in order to construct a differential form on such a section, we only need to

consider the tangent vectors of $\widehat{\mathcal{P}}_{g,n}$ that are the tangent vectors of the moduli space of Riemann surfaces equipped with the choice local coordinates that defines the section. They are given by the Beltrami differentials spanning the tangent space of the moduli space of Riemann surfaces [49].

Let us denote the coordinates of $\mathcal{M}_{g,n}$ by $(t_1, \dots, t_{6g-6+2n})$. Consider B_p , an operator-valued p -form defined on the section of the space $\widehat{\mathcal{P}}_{g,n}$. The contraction of B_p with $\{V_1, \dots, V_p\}$, p tangent vectors of the section, is given by

$$B_p[V_1, \dots, V_p] = b(V_1) \cdots b(V_p), \quad (2.10)$$

where

$$b(V_k) = \int d^2z \left(b_{zz} \mu_{k\bar{z}}^z + b_{\bar{z}\bar{z}} \mu_{kz}^{\bar{z}} \right). \quad (2.11)$$

Here μ_k denotes the Beltrami differential associated with the moduli t_k of the Riemann surfaces belong to the section of the fiber space $\widehat{\mathcal{P}}_{g,n}$ in which we are interested. The p -form on the section can be obtained by taking the expectation value of the operator valued p -form B_p between the surface state $\langle \mathcal{R} |$ and the state $|\Phi\rangle$:

$$\Omega_p^{(g,n)}(|\Phi\rangle) = (2\pi i)^{-(3g-3+n)} \langle \mathcal{R} | B_p | \Phi \rangle. \quad (2.12)$$

The state $|\Phi\rangle$ is the tensor product of external off-shell states $|\Psi_i\rangle$, $i = 1, \dots, n$ inserted at the punctures and the state $\langle \mathcal{R} |$ is the surface state associated with the surface $\mathcal{R}_{g,n}$. It describes the state that is created on the boundaries of the discs D_i , $i = 1, \dots, n$ by performing a functional integral over the fields of CFT on $\mathcal{R} - \sum_i D_i$. The inner product between $\langle \mathcal{R} |$ and a state $|\Psi_1\rangle \otimes \dots \otimes |\Psi_n\rangle \in \mathcal{H}^{\otimes n}$

$$\langle \mathcal{R} | (|\Psi_1\rangle \otimes \dots \otimes |\Psi_n\rangle), \quad (2.13)$$

can be understood as the n -point correlation function on $\mathcal{R}$ with the vertex operator for $|\Psi_i\rangle$ inserted at the i^{th} puncture using the local coordinate around that puncture.

The path integral representation of $\Omega_{6g-6+2n}^{(g,n)}(|\Psi_1\rangle, \dots, |\Psi_n\rangle)$ is given by

$$\begin{aligned} & \Omega_{6g-6+2n}^{(g,n)}(|\Psi_1\rangle, \dots, |\Psi_n\rangle) \\ &= \frac{dt_1 \cdots dt_{6g-6+2n}}{(2\pi i)^{(3g-3+n)}} \int \mathcal{D}x^\mu \int \mathcal{D}c \mathcal{D}\bar{c} \mathcal{D}b \mathcal{D}\bar{b} e^{-I_m(x) - I_{gh}(b,c)} \prod_{j=1}^{6g-6+2n} b(V_j) \prod_{i=1}^n [\bar{c}\bar{c} V_i(k_i)]_{w_i}, \end{aligned} \quad (2.14)$$

where $[\bar{c}\bar{c} V_i(k_i)]_{w_i}$ denotes the vertex operator corresponds to the state $|\Psi_i\rangle$ inserted using the local coordinate w_i . $I_m(x)$ is the action for matter fields and $I_{gh}(b, c)$ is the actions for ghost fields. z is the global coordinate on $\mathcal{R}$.

2.3 The explicit evaluation of the quantum master action

In this subsection, we explain, what we mean by the explicit evaluation of the quantum BV master action for the closed string field theory. Let us denote the vertex operator

corresponds to the basis state $|\Phi_s\rangle$ by $\mathcal{A}(\Phi_s)$. Then the string field entering in the quantum BV master action can be expressed as

$$|\Psi\rangle = \sum'_{G(\Phi_s) \leq 2} \sum_p \psi^s(p) \mathcal{A}(\Phi_s) |\mathbf{1}, p\rangle + \sum'_{G(\Phi_s) \leq 2} \sum_p \psi_s^*(p) \mathcal{A}(\tilde{\Phi}_s) |\mathbf{1}, p\rangle, \quad (2.15)$$

where $|\mathbf{1}, p\rangle$ denotes the $\text{SL}(2, \mathbb{C})$ invariant family of vacua for the worldsheet CFT for the closed bosonic string theory, parameterized by p . The expression for the quantum BV master action in terms of the target space fields and the target space antifields can be obtained by substituting this expansion of the string field Ψ in the quantum BV master action (2.5):

$$\begin{aligned} S(\Psi) = & \frac{1}{2g_s^2} \sum'_{G(\Phi_{s_j}) \leq 2} \sum_{\substack{\phi^{s_i} \in \mathcal{S}_i \\ i=1,2}} \sum_{p_1, p_2} \phi^{s_1}(p_1) \mathbf{P}_{s_1 s_2}(p_1, p_2) \phi^{s_2}(p_2) \\ & + \sum_{\substack{g \geq 0 \\ n \geq 1}} \frac{\hbar^g g_s^{2g+n-2}}{n!} \sum'_{G(\Phi_{s_j}) \leq 2} \sum_{\substack{\phi^{s_i} \in \mathcal{S}_i \\ i=1, \dots, n}} \sum_{p_1, \dots, p_n} \mathbf{V}_{s_1 \dots s_n}^{g,n}(p_1, \dots, p_n) \phi^{s_1}(p_1) \dots \phi^{s_n}(p_n), \end{aligned} \quad (2.16)$$

where $\mathcal{S}_i = \{\psi^{s_i}, \psi_{s_i}^*\}$ is the set of all fields and antifields of the closed bosonic string field theory spectrum. $\mathbf{P}_{s_1 s_2}(p_1, p_2)$, the inverse of the propagator, is given by

$$\mathbf{P}_{s_1 s_2}(p_1, p_2) = \langle \mathcal{Y}_{s_1}, p_1 | c_0^- Q_B | \mathcal{Y}_{s_2}, p_2 \rangle, \quad (2.17)$$

and $\mathbf{V}_{s_1 \dots s_n}^{g,n}(p_1, \dots, p_n)$, the g loop interaction vertex of n target spacetime fields/antifields $\{\phi^{s_1}(p_1), \dots, \phi^{s_n}(p_n)\}$, is given by

$$\mathbf{V}_{s_1 \dots s_n}^{g,n}(p_1, \dots, p_n) = \int_{\mathcal{V}_{g,n}} \Omega_{6g-6+2n}^{(g,n)}(|\mathcal{Y}_{s_1}, p_1\rangle, \dots, |\mathcal{Y}_{s_n}, p_n\rangle). \quad (2.18)$$

Here, $|\mathcal{Y}_{s_i}, p_i\rangle$ is the state associated with the string field/antifield $\phi^{s_i}(p_i)$. The state $|\mathcal{Y}_{s_i}, p_i\rangle$ is annihilated by both b_0^- and L_0^- . By the explicit evaluation of the quantum master action, we mean the explicit evaluation of $\mathbf{V}_{s_1 \dots s_n}^{g,n}(p_1, \dots, p_n)$. The explicit evaluation requires:

1. A convenient choice of parametrization of the Teichmüller space and the conditions on them that specify the region of the moduli space inside the Teichmüller space.
2. An explicit procedure for constructing the off-shell string measure in terms of the chosen coordinates of the moduli space.
3. An explicit description for the region inside moduli space that corresponds to the string vertex and a consistent choice of local coordinates around the punctures on the Riemann surfaces belong to the string vertex.
4. Finally, an explicit procedure for integrating the off-shell string measure over the region inside moduli space that corresponds to the string vertex.

In the remaining sections of this paper, we provide a detailed description for each of these steps.

3 The string vertices using hyperbolic metric

The main challenge in constructing string field theory is to find a suitable cell decomposition of the moduli spaces of closed Riemann surfaces. In principle, the string vertices satisfying the conditions listed in the section 2 that provide such a cell decomposition of the moduli space can be constructed using the Riemann surfaces endowed with the metric solving the generalized minimal area problem [3]. Unfortunately, the current understanding of minimal area metrics is not sufficient enough to obtain a calculable formulation of closed string field theory. In our previous paper [6], we described an alternate construction of the string vertices using Riemann surfaces endowed with metric having constant curvature -1 . We briefly review this construction below. For a brief review of the theory of hyperbolic Riemann surfaces, read appendix A.

A hyperbolic Riemann surface can be represented as a quotient of the upper half-plane $\mathbb{H}$ by a Fuchsian group. A puncture on a hyperbolic Riemann surface corresponds to the fixed point of a parabolic element (element having trace ± 2) of the Fuchsian group acting on the upper half-plane $\mathbb{H}$. For a puncture p on a hyperbolic Riemann surface, there is a natural local conformal coordinate w with $w(p) = 0$, induced from the hyperbolic metric. The local expression for the hyperbolic metric around the puncture is given by

$$ds^2 = \left(\frac{|dw|}{|w| \ln |w|} \right)^2. \quad (3.1)$$

We can naively define the string vertices by means of the Riemann surfaces endowed with the hyperbolic metric as below.

The naive string vertex $\mathcal{V}_{g,n}^0$. Consider a genus- g hyperbolic Riemann surface $\mathcal{R}$ having n punctures with no simple closed geodesics that has geodesic length $l \leq c_*$, where c_* is some positive real number such that $c_* \ll 1$. The local coordinates around the punctures on $\mathcal{R}$ are chosen to be $e^{\frac{\pi^2}{c_*}} w$, where w is the natural local coordinate induced from the hyperbolic metric on $\mathcal{R}$. The set of all such inequivalent hyperbolic Riemann surfaces forms the string vertex $\mathcal{V}_{g,n}^0$.

It was shown in [6] that the string vertices $\mathcal{V}_{g,n}^0$ fails to provide a single cover of the moduli space for any non-vanishing value of c_* . The argument goes as follows. We can claim that the string vertex $\mathcal{V}_{g,n}^0$ together with the Feynman diagrams provide a cell decomposition of the moduli space only if the Fenchel-Nielsen length parameters and the local coordinates around the punctures on the surfaces at the boundary of the string vertex region matches exactly with the Fenchel-Nielsen length parameters and the local coordinates around the punctures on the surface obtained by the special plumbing fixture construction

$$\tilde{z} \cdot \tilde{w} = e^{i\theta}, \quad 0 \leq \theta \leq 2\pi, \quad (3.2)$$

where $\tilde{z}$ and $\tilde{w}$ denote the local coordinates around the punctures that are being glued. However, the metric on the surface obtained by the plumbing fixture of a set of hyperbolic Riemann surface fails to be exactly hyperbolic all over the surface [40–42].

Consider the Riemann surface $\mathcal{R}_t$, for $t = (t_1, \dots, t_m)$ obtained via plumbing fixture around m nodes of a hyperbolic surface $\mathcal{R}_{t=0} \equiv \mathcal{R}_0$ with m nodes. We denote the set of Riemann surfaces obtained by removing the nodes from $\mathcal{R}_0$ by $\hat{\mathcal{R}}$, i.e., $\hat{\mathcal{R}}_0 = \mathcal{R}_0 - \{\text{nodes}\}$. The Riemann surfaces $\hat{\mathcal{R}}_0$ have a pair of punctures (a_j, b_j) in place of the j^{th} node of $\mathcal{R}_0$, $j = 1, \dots, m$. Assume that $w_j^{(1)}$ and $w_j^{(2)}$ are the local coordinates around the punctures a_j and b_j with the property that $w_j^{(1)}(a_j) = 0$ and $w_j^{(2)}(b_j) = 0$. Let us choose the local coordinates $w_j^{(1)}$ and $w_j^{(2)}$ such that, in terms of these local coordinates, the hyperbolic metric around the punctures of $\hat{\mathcal{R}}_0$ has the local expression

$$ds^2 = \left(\frac{|d\zeta|}{|\zeta| \ln |\zeta|} \right)^2, \quad \zeta = w_j^{(1)} \text{ or } w_j^{(2)}. \quad (3.3)$$

Let us call the metric on the glued surface $\mathcal{R}_t$ as the *grafted metric* ds_{graft}^2 . The grafted metric has curvature -1 except at the collar boundaries, where the interpolation leads to a deviation of magnitude $(\ln|t|)^{-2}$ [40]. This deviation makes the resulting surface almost hyperbolic except at the boundaries of the plumbing collar.

However, we can compute the hyperbolic metric on $\mathcal{R}_t$ by solving the *curvature correction equation* [40, 41]. To describe the curvature correction equation, consider a compact Riemann surface having metric ds^2 with Gauss curvature² $\mathbf{C}$. Then, another metric $e^{2f} ds^2$ on this surface has constant curvature -1 provided

$$Df - e^{2f} = \mathbf{C}, \quad (3.4)$$

where D is the Laplace-Beltrami operator on the surface. Therefore, in order to get the hyperbolic metric on $\mathcal{R}_t$, we need to solve this curvature correction equation perturbatively around the grafted metric by adding a Weyl factor. Then we can invert this expression for hyperbolic metric on $\mathcal{R}_t$ in terms of the grafted metric to obtain the grafted metric in terms of the hyperbolic metric.

To the second order the hyperbolic metric on $\mathcal{R}_t$, Riemann surface at the boundary of the string vertex $\mathcal{V}_{g,n}^0$ obtained by the special plumbing fixture (3.2) of the hyperbolic Riemann surfaces, is related to the grafted metric as follows

$$ds_{\text{hyp}}^2 = ds_{\text{graft}}^2 \left(1 + \sum_{i=1}^m \frac{c_*^2}{3} \left(E_{i,1}^\dagger + E_{i,2}^\dagger \right) + \mathcal{O}(c_*^3) \right). \quad (3.5)$$

The functions $E_{i,1}^\dagger$ and $E_{i,2}^\dagger$ are the melding of the Eisenstein series $E(\cdot; 2)$ associated to the pair of cusps plumbed to form the i^{th} collar. For the definition of these functions, see [6]. The details of these functions are not very important for our discussions.

Using this relation, we modify the definition of the string vertices by changing the choice of local coordinates on the surfaces which belong to the boundary region of the string vertices as follows [6]. The boundary of the string vertex with m plumbing collar is defined as the locus in the moduli space of the hyperbolic Riemann surfaces with m non-homotopic and disjoint non trivial simple closed curves having length equal to that of

²In two dimension, the Gaussian curvature is half of the Ricci curvature of the surface.

the length of the simple geodesic on any plumbing collar of a Riemann surface obtained by gluing m pair of punctures on a set of hyperbolic Riemann surfaces via the special plumbing fixture relation (3.2). To the second order in c_* , there is no correction to the hyperbolic length of the geodesics on the plumbing collars. Therefore, to second order in c_* , we don't have to correct the definition of the region corresponding to the string vertex in the moduli space for the hyperbolic Riemann surfaces parametrized using the Fenchel-Nielsen coordinates. However, the choice of local coordinates around the punctures must be modified to make it gluing compatible to second order in c_* . In order to modify the assignment of local coordinates in the string vertex $\mathcal{V}_{g,n}^0$, we divide it into subregions. Let us denote the subregion in the region corresponds to the string vertex $\mathcal{V}_{g,n}^0$ consists of surfaces with m simple closed geodesics (none of them are related to each other by the action of any elements in MCG) of length between c_* and $(1 + \delta)c_*$ by $\mathbf{W}_{g,n}^{(m)}$, where δ is an infinitesimal real number. Then we modify the local coordinates as follows:

- For surfaces belong to the subregion $\mathbf{W}_{g,n}^{(0)}$, we choose the local coordinate around the j^{th} puncture to be $e^{\frac{\pi^2}{c_*}} w_j$. In terms of w_j , the hyperbolic metric in the neighbourhood of the puncture takes the following form

$$\left(\frac{|dw_j|}{|w_j| \ln |w_j|} \right)^2, \quad j = 1, \dots, n. \quad (3.6)$$

- For surfaces belong to the region $\mathbf{W}_{g,n}^{(m)}$ with $m \neq 0$, we choose the local coordinates around the j^{th} puncture to be $e^{\frac{\pi^2}{c_*}} \tilde{w}_{j,m}$, where $\tilde{w}_{j,m}$, up to a phase ambiguity, is given by

$$\tilde{w}_{j,m} = e^{\frac{c_*^2}{6} \sum_{i=1}^m f(l_i) Y_{ij}} w_j. \quad (3.7)$$

We found $\tilde{w}_{j,m}$ by solving the following equation

$$\left(\frac{|d\tilde{w}_{j,m}|}{|\tilde{w}_{j,m}| \ln |\tilde{w}_{j,m}|} \right)^2 = \left(\frac{|dw_j|}{|w_j| \ln |w_j|} \right)^2 \left\{ 1 - \frac{c_*^2}{3 \ln |w_j|} \sum_{i=1}^m f(l_i) Y_{ij} \right\}, \quad (3.8)$$

where l_i denotes the length of the i^{th} degenerating simple closed geodesic and the function $f(l_i)$ is an arbitrary smooth real function of the geodesic length l_i defined in the interval $(c_*, c_* + \delta c_*)$, such that $f(c_*) = 1$ and $f(c_* + \delta c_*) = 0$. The coefficient Y_{ij} is given by

$$Y_{ij} = \sum_{q=1}^2 \sum_{c_i^q, d_i^q} \pi^2 \frac{\epsilon(j, q)}{|c_i^q|^4}$$

$$c_i^q > 0 \quad d_i^q \bmod c_i^q \quad \begin{pmatrix} * & * \\ c_i^q & d_i^q \end{pmatrix} \in (\sigma_i^q)^{-1} \Gamma_i^q \sigma_j \quad (3.9)$$

Here, Γ_i^q denotes the Fuchsian group for the component Riemann surface with the cusp denoted by the index q that is being glued via plumbing fixture to obtain the i^{th} collar. The transformation σ_j^{-1} maps the cusp corresponding to the j^{th} cusp to ∞ and $(\sigma_j^q)^{-1}$

maps the cusp denoted by the index q that is being glued via plumbing fixture to obtain the i^{th} collar to ∞ . The factor $\epsilon(j, q)$ is one if both the j^{th} cusp and the cusp denoted by the index q that is being glued via plumbing fixture to obtain the i^{th} collar belong to the same component surface otherwise $\epsilon(j, q)$ is zero.

The string vertices corrected in this way are denoted as $\mathcal{V}_{g,n}^2$. They provide an improved approximate cell decomposition of the moduli space that has no mismatch up to the order c_*^2 .

4 The off-shell string measure and Fenchel-Nielsen parameters

In this section, we describe the explicit construction of the off-shell string measure in terms of the Fenchel-Nielsen coordinates of the Teichmüller space. As explained in subsection 2.2, the off-shell string measure can be defined using a specific choice of local coordinates, that is encoded in the definition of the string vertices, and the Beltrami differentials associated with the moduli parameters.

A flow in $\mathcal{T}_{g,n}$, the Teichmüller space of $\mathcal{R}$, hyperbolic Riemann surfaces with g handles and n borders, can be generated by a twist field defined with respect a simple closed curve on the Riemann surface [44–47]. The twist field t_α , where α is a simple closed geodesic, generates a flow in $\mathcal{T}_{g,n}$ that can be understood as the *Fenchel-Nielsen deformation* of $\mathcal{R}$ with respect to α . The Fenchel-Nielsen deformation is the operation of cutting the hyperbolic surface along α and attaching the boundaries after rotating one boundary relative to the other by some amount δ . The magnitude δ parametrizes the flow on $\mathcal{T}_{g,n}$.

Assume that $\mathcal{R}$ is uniformized as $\mathbb{H}/\Gamma$. Suppose that the element of Γ that corresponds to a simple closed geodesic α is the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then, the Beltrami differential corresponds to the twist vector field t_α is given by [45]

$$\mathbf{t}_\alpha = \frac{i}{\pi} (\text{Im} z)^2 \bar{\Theta}_\alpha. \quad (4.1)$$

Θ_α is the following relative Poincaré series

$$\Theta_\alpha = \sum_{B \in \langle A \rangle \setminus \Gamma} \omega_{B^{-1}AB}, \quad (4.2)$$

where $\langle A \rangle$ denote the infinite cyclic group generated by the element A , and ω_A is given by

$$\omega_A = \frac{(a+d)^2 - 4}{(cz^2 + (d-a)z - b)^2}. \quad (4.3)$$

Consider the Fenchel-Nielsen coordinates of the Teichmüller space (τ_i, ℓ_i) , $i = 1, \dots, 3g - 3 + n$ defined with respect to the pants decomposition $\mathcal{P} = \{C_1, \dots, C_{3g-3+n}\}$, where C_i denotes a simple geodesic on $\mathcal{R}$. By definition, for $i \neq j$, the curves C_i and C_j are disjoint and non-homotopic to each other. The tangent space at a point in the Teichmüller space is

spanned by the Fenchel-Nielsen coordinate vector fields $\left\{ \frac{\partial}{\partial \tau_i}, \frac{\partial}{\partial \ell_i} \right\}$, $i = 1, \dots, 3g - 3 + n$. The Fenchel-Nielsen coordinate vector field $\frac{\partial}{\partial \tau_i}$ can be identified with the twist vector field t_{C_i} defined with respect to the curve C_i . Hence, the *Beltrami differential corresponds to the Fenchel-Nielsen coordinate vector field* $\frac{\partial}{\partial \tau_i}$ is given by $\mathbf{t}_{C_i}$. The Beltrami differential for the Fenchel-Nielsen coordinate vector field $\frac{\partial}{\partial \ell_i}$ can also be constructed by noting the that with respect to the WP symplectic form $\frac{\partial}{\partial \ell_i}$ is dual to the twist vector field $\frac{\partial}{\partial \tau_i}$ [46]. We denote the Beltrami differential for the Fenchel-Nielsen coordinate vector field $\frac{\partial}{\partial \ell_i}$ as $\mathbf{l}_{C_i}$.

Putting these together, the off-shell bosonic-string measure can be written as

$$\begin{aligned} & \Omega_{6g-6+2n}^{(g,n)}(|\Psi_1\rangle \otimes \dots \otimes |\Psi_n\rangle) \\ &= \frac{\prod_{i=1}^{3g-3+n} d\ell_i d\tau_i}{(2\pi i)^{(3g-3+n)}} \int \mathcal{D}x^\mu \int \mathcal{D}c \mathcal{D}\bar{c} \mathcal{D}b \mathcal{D}\bar{b} e^{-I_m(x) - I_{gh}(b,c)} \prod_{j=1}^{3g-3+n} b(\mathbf{t}_{C_j}) b(\mathbf{l}_{C_j}) \prod_{i=1}^n [c\bar{c} V_i(k_i)]_{w_i}, \end{aligned} \quad (4.4)$$

where $[c\bar{c}V_i(k_i)]_{w_i}$ denote the vertex operator for the state $|\Psi_i\rangle$ inserted at i^{th} puncture using the local coordinate w_i and

$$\begin{aligned} b(\mathbf{t}_{C_i}) &= \int_{\mathcal{F}} d^2z (b_{zz} \mathbf{t}_{C_i} + b_{\bar{z}\bar{z}} \bar{\mathbf{t}}_{C_i}), \\ b(\mathbf{l}_{C_i}) &= \int_{\mathcal{F}} d^2z (b_{zz} \mathbf{l}_{C_i} + b_{\bar{z}\bar{z}} \bar{\mathbf{l}}_{C_i}). \end{aligned} \quad (4.5)$$

Here $\mathcal{F}$ denotes the fundamental domain in the upper half-plane for the action of the Fuchsian group Γ that corresponds to $\mathcal{R}$. Here, we assumed that $\mathcal{R}$ belongs to the string vertex $\mathcal{V}_{g,n}$. Remember that, the Riemann surfaces belong to the string vertices carry a specific choice of local coordinates around its punctures which is consistent with the geometrical identity (2.7).

Assume that the vertex operator $V_i(k_i)$ has conformal dimension h_i with no ghost fields in it, for $i = 1, \dots, n$. Then we have

$$\begin{aligned} & \Omega_{6g-6+2n}^{(g,n)}(|\Psi_1\rangle \otimes \dots \otimes |\Psi_n\rangle) \\ &= \frac{\prod_{i=1}^{3g-3+n} d\ell_i d\tau_i}{(2\pi i)^{(3g-3+n)}} z \left| \frac{\partial z}{\partial w_i} \right|^{2h_i-2} \sqrt{\det' P_1^\dagger P_1} \left(\frac{2\pi^2}{\int d^2z \sqrt{g}} \det' \Delta \right)^{-13} \int \mathcal{D}x^\mu e^{-I_m(x)} \prod_{i=1}^n V_i(k_i), \end{aligned} \quad (4.6)$$

where, Δ is the Laplacian acting on scalars defined on $\mathcal{R}$ a genus g hyperbolic Riemann surface with n punctures. The prime indicates that we do not include contributions from zero modes while computing the determinant of Δ . The operator $P_1 = \nabla_z^1 \oplus \nabla_{-1}^z$ and $P_1^\dagger = -(\nabla_z^2 \oplus \nabla_{-2}^z)$. Operators ∇_z^n and ∇_n^z are defined by their action on $T(dz)^n$, which is given by

$$\begin{aligned} \nabla_z^n (T(dz)^n) &= (g_{z\bar{z}})^n \frac{\partial}{\partial z} ((g^{z\bar{z}})^n T) (dz)^{n+1}, \\ \nabla_n^z (T(dz)^n) &= g^{z\bar{z}} \frac{\partial}{\partial \bar{z}} T(dz)^{n-1}. \end{aligned} \quad (4.7)$$

Interestingly, the determinant $\det' P_1^\dagger P_1$ and $\det' \Delta$ can be evaluated on any hyperbolic Riemann surface in terms of Selberg zeta functions [60–64]. For instance, $\det' \Delta$ on a genus

g hyperbolic Riemann surface with n punctures can be expressed as follows [65]

$$\det' \Delta = 2^{\frac{n}{2} + \frac{1}{2} \text{tr} \Phi(\frac{1}{2})} (2\pi)^{g-1 + \frac{n}{2}} e^{(2g-2+n)(2\zeta'(-1) - \frac{1}{4})} \frac{d}{ds} Z(s) \Big|_{s=1} \quad (4.8)$$

where $\zeta(s)$ is the Riemann zeta function and $\Phi(s) = (\phi_{ij}(s))_{1 \leq i, j \leq n}$. The elements ϕ_{ij} can be found by expanding the Eisenstein series defined with respect to the i^{th} puncture around the j^{th} puncture. The expansion can be obtained by taking the limit ($y = \text{Im}(z)$) $\rightarrow \infty$

$$E_i(\sigma_j z, s) = \delta_{ij} y^s + \phi_{ij}(s) y^{1-s} + \dots, \quad (4.9)$$

where $\sigma_i^{-1} \kappa_i \sigma_i = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. κ_i is the parabolic generator associated with the i^{th} puncture. Finally $Z(s)$ is the Selberg zeta function given by

$$Z(s) = \prod_{\gamma \in S} \prod_{k=1}^{\infty} \left[1 - e^{-(s+k)\ell_\gamma} \right], \quad (4.10)$$

where γ is a simple closed geodesic on $\mathcal{R}$ and S is the set of all simple closed geodesics on $\mathcal{R}$. A simple closed geodesic on $\mathcal{R}$ corresponds to a primitive element in the Fuchsian group Γ . A hyperbolic element of Γ is said to be a primitive element if it can not be written as a power of any hyperbolic element in Γ . However, a primitive element can be an inverse of another primitive element in Γ . If $g \in \Gamma$ represents the simple closed geodesic γ , then the length of γ is given by

$$\ell_\gamma = \cosh^{-1} \left(\frac{1}{2} |\text{tr } g| \right). \quad (4.11)$$

Therefore the Selberg zeta function can be expressed as a product over all the primitive elements in Γ . The $\det P_1^\dagger P_1$ on $\mathcal{R}$ also can be similarly expressed in terms of the Selberg zeta functions.

The matter sector path integral can be expressed in terms of the Green's function G for the Laplacian acting on the scalars on $\mathcal{R}$. To demonstrate this, let us consider the case where all the external states are tachyons, ie. $V_i(k_i) = e^{ik_i \cdot X_i}$. Then we have

$$\begin{aligned} \Omega_{6g-6+2n}^{(g,n)}(|T_1\rangle \otimes \dots \otimes |T_n\rangle) &= \frac{\prod_{i=1}^{3g-3+n} d\ell_i d\tau_i}{(2\pi i)^{(3g-3+n)}} \left| \frac{\partial z}{\partial w_i} \right|^{2h_i-2} \sqrt{\det' P_1^\dagger P_1} \left(\frac{2\pi^2}{\int d^2 z \sqrt{g}} \det' \Delta \right)^{-13} \\ &\times e^{\frac{1}{2} \sum_{i,j} k_i \cdot k_j G(x_i, x_j)} (2\pi)^{26} \delta(k_1 + \dots + k_n), \end{aligned} \quad (4.12)$$

where x_i denotes the fixed point corresponds to the i^{th} puncture. The Green's function on $\mathcal{R}$ can be constructed by first constructing the Green's function on $\mathbb{H}$ and then by considering the sum over all the elements of Γ , which is same as the idea of method of images [63].

Assume that the hyperbolic Riemann surface $\mathcal{R}$ corresponds to a point in the Teichmüller space with coordinate $(\ell_1, \tau_1, \dots, \ell_{3g-3+n}, \tau_{3g-3+n})$. Then by following the general algorithm described in [66], it is possible to express the matrix elements of the generators of Γ as functions of $(\ell_1, \tau_1, \dots, \ell_{3g-3+n}, \tau_{3g-3+n})$. Using these generators it is in

principle possible to construct all the primitive elements of Γ . Therefore we can express the determinants of the Laplacians and the Green's functions on $\mathcal{R}$ as functions of the Fenchel-Nielsen coordinates. Finally we get an expression of the off-shell string measure in terms of the Fenchel-Nielsen coordinates.

5 The effective string vertices

The interaction vertices in closed string field theory is obtained by integrating the off-shell bosonic string measure constructed in the previous section over the region in the compactified moduli space $\overline{\mathcal{M}}_{g,n}$ that corresponds to the string vertex $\mathcal{V}_{g,n}$, which is denoted as $\mathcal{W}_{g,n}$. The modification of the local coordinates requires dividing $\mathcal{W}_{g,n}$ into different sub-regions. The moduli space $\mathcal{M}_{g,n}$ can be understood as the quotient of the Teichmüller space $\mathcal{T}_{g,n}$ with the action of the MCG (mapping class group). Unfortunately, in generic cases, an explicit fundamental region for the action of MCG is not known in terms of the Fenchel-Nielsen coordinates. This is due to the fact that the form of the action of MCG on the Fenchel-Nielsen coordinates is not yet known [36, 37]. Therefore, modifying the naive string vertex, to make it consistent to $\mathcal{O}(c_*^2)$, appears to be impractical. In this section, we discuss a way to overcome this difficulty by following the prescription for performing integrations in the moduli space introduced by M.Mirzakhani [39].

5.1 The effective calculations

Consider the space $\mathcal{M}$ with a covering space $\mathcal{N}$. The covering map is given by

$$\pi : \mathcal{N} \rightarrow \mathcal{M}.$$

If $dv_{\mathcal{M}}$ is a volume form for $\mathcal{M}$, then

$$dv_{\mathcal{N}} \equiv \pi^{-1} * (dv_{\mathcal{M}}),$$

defines the volume form for the covering space $\mathcal{N}$. Assume that h is a smooth function defined in the space $\mathcal{N}$. Then the push forward of the function h at a point x in the space $\mathcal{M}$, which is denoted by $\pi_* h(x)$, can be obtained by the summation over the values of the function h at all points in the fiber of the point x in $\mathcal{N}$:

$$(\pi_* h)(x) \equiv \sum_{y \in \pi^{-1}\{x\}} h(y). \quad (5.1)$$

This relation defines a smooth function on the space $\mathcal{M}$. As a result, the integral of this pull-back function over the space $\mathcal{M}$ can be lifted to the covering space $\mathcal{N}$ as follows:

$$\int_{\mathcal{M}} dv_{\mathcal{M}} (\pi_* h) = \int_{\mathcal{N}} dv_{\mathcal{N}} h. \quad (5.2)$$

Integration over $\mathbb{S}^1$ as an integration over $\mathbb{R}$. In order to elucidate the basic logic behind the integration method, let us discuss a simple and explicit example. Consider the real line $\mathbb{R} = (-\infty, \infty)$ as the covering space of circle $\mathbb{S}^1 = [0, 1)$. We denote the covering map by

$$\pi : \mathbb{R} \rightarrow \mathbb{S}^1.$$

Assume that $f(x)$ is a function living in $\mathbb{S}^1$, i.e. $f(x+k) = f(x)$, $k \in \mathbb{Z}$. Then we can convert the integration over $\mathbb{S}^1$ into an integration over $\mathbb{R}$ with the help of the identity

$$1 = \sum_{k=-\infty}^{\infty} \frac{\sin^2(\pi[x-k])}{\pi^2(x-k)^2}, \quad (5.3)$$

as follows:

$$\begin{aligned} \int_0^1 dx f(x) &= \int_0^1 dx \left(\sum_{k=-\infty}^{\infty} \frac{\sin^2(\pi[x-k])}{\pi^2(x-k)^2} \right) f(x) \\ &= \int_0^1 dx \sum_{k=-\infty}^{\infty} \left(\frac{\sin^2(\pi[x-k])}{\pi^2(x-k)^2} f(x-k) \right) \\ &= \sum_{k=-\infty}^{\infty} \int_0^1 dx \frac{\sin^2(\pi[x-k])}{\pi^2(x-k)^2} f(x-k) \\ &= \int_{-\infty}^{\infty} dx \frac{\sin^2(\pi x)}{\pi^2 x^2} f(x). \end{aligned} \quad (5.4)$$

In the last step, we absorbed the summation over k and converted the integration over $\mathbb{S}^1$ to the integration over $\mathbb{R}$. For instance, choosing $f(x)$ to be the one, gives the following well-known result

$$1 = \int_{-\infty}^{\infty} dx \frac{\sin^2(\pi x)}{\pi^2 x^2}.$$

5.2 Effective regions in the Teichmüller spaces

The discussion in the previous subsection suggest that, if we have a region in the Teichmüller space that can be identified as a covering space of a region in the moduli space, then the integration of a differential form defined in the moduli space can be performed by expressing the differential form as a push-forward of a differential form in the Teichmüller space using the covering map. In the remaining part of this section, we shall explain that it is indeed possible to find such a covering map and express the off-shell string measure as a push-forward of a differential form defined in the Teichmüller space.

Naive interaction vertex $\mathbf{S}_{1,2}$. Let us start by constructing the naive one-loop interaction vertex $\mathbf{S}_{1,2}$ with two external states external states represented by the unintegrated vertex operators V_1 and V_2 . It is given by

$$\mathbf{S}_{1,2} = (2\pi i)^{-2} \int_{\mathcal{W}_{1,2}} d\ell_{\gamma_1} d\tau_{\gamma_1} d\ell_{\gamma_2} d\tau_{\gamma_2} \langle \mathcal{R}_{1,2} | b(\mathbf{t}_{\gamma_1}) b(\mathbf{l}_{\gamma_1}) b(\mathbf{t}_{\gamma_2}) b(\mathbf{l}_{\gamma_2}) | V_1 \rangle_{w_1} \otimes | V_2 \rangle_{w_2}, \quad (5.5)$$

Figure 1. Curves $\gamma_1, \gamma_2, \gamma_3$ are different non-self intersecting closed geodesics on twice-punctured torus. By shrinking these curves we can reach the boundaries of the string vertex $\mathcal{V}_{1,2}$.

where $|\mathcal{R}_{1,2}\rangle$ is the surface state associated with the twice-punctured torus, and $|V_i\rangle_{w_i}$ denotes the state inserted at the i^{th} puncture of the torus using the coordinate $e^{\frac{\pi^2}{c_*} w_i}$ induced from the hyperbolic metric on $\mathcal{R}_{1,2}$. The parameters $(\tau_{\gamma_j}, \ell_{\gamma_j})$, $j = 1, 2$ denote the Fenchel-Nielsen coordinates for the Teichmüller space $\mathcal{T}_{1,2}$ of twice-punctured tori defined with respect to the curves γ_1 and γ_2 , see figure 1. And

$$\begin{aligned} b(\mathbf{t}_{\gamma_i}) &= \int_{\mathcal{F}} d^2 z \left(b_{zz} \mathbf{t}_{\gamma_i} + b_{\bar{z}\bar{z}} \bar{\mathbf{t}}_{\gamma_i} \right), \\ b(\mathbf{l}_{\gamma_i}) &= \int_{\mathcal{F}} d^2 z \left(b_{zz} \mathbf{l}_{\gamma_i} + b_{\bar{z}\bar{z}} \bar{\mathbf{l}}_{\gamma_i} \right), \end{aligned} \quad (5.6)$$

where $\mathcal{F}$ denotes the fundamental domain of the action of $\Gamma_{1,2}$, the Fuchsian group associated with $\mathcal{R}_{1,2}$, in $\mathbb{H}$. $\mathbf{t}_{\gamma_i}$ and $\mathbf{l}_{\gamma_i}$ are the Beltrami differentials associated with the Fenchel-Nielsen coordinates $(\tau_{\gamma_i}, \ell_{\gamma_i})$. Finally, $\mathcal{W}_{1,2}$ is the region covered by the naive string vertex $\mathcal{V}_{1,2}^0$ in the moduli space. Although a copy of $\mathcal{W}_{1,2}$ is a subspace in $\mathcal{T}_{1,2}$, it has no simple description in terms of the Fenchel-Nielsen coordinates.

In order to evaluate $\mathbf{S}_{1,2}$ we must specify $\mathcal{W}_{1,2}$ in terms of the Fenchel-Nielsen coordinates. This seems impossible, since there is no simple description of $\mathcal{W}_{1,2}$ or even $\mathcal{M}_{1,2}$ in terms of $(\tau_{\gamma_1}, \ell_{\gamma_1}, \tau_{\gamma_2}, \ell_{\gamma_2})$. However, there is an interesting resolution to this issue. The lengths of the non-self intersecting closed geodesics on $\mathcal{R}_{1,2}$ satisfy the following

curious identity [38]:

$$\sum_{g_1 \in \text{MCG}(\mathcal{R}_{1,2}, \gamma_1 + \gamma_3)} \frac{2}{1 + e^{\frac{\ell_{g_1 \cdot \gamma_1} + \ell_{g_1 \cdot \gamma_3}}{2}}} + \sum_{g_2 \in \text{MCG}(\mathcal{R}_{1,2}, \gamma_2)} \frac{2}{1 + e^{\frac{\ell_{g_2 \cdot \gamma_2}}{2}}} = 1, \quad (5.7)$$

where γ_1, γ_2 and γ_3 are the non-self intersecting closed geodesics on $\mathcal{R}_{1,2}$ as shown in figure 1, and ℓ_{γ_i} denotes the hyperbolic length of γ_i . $\text{MCG}(\mathcal{R}_{1,2}, \gamma_1 + \gamma_3)$ denotes the subgroup of mapping class group (MCG) of $\mathcal{R}_{1,2}$ that acts non-trivially only on the curve $\gamma_1 + \gamma_3$. Similarly, $\text{MCG}(\mathcal{R}_{1,2}, \gamma_2)$ denotes the subgroup of MCG of $\mathcal{R}_{1,2}$ that acts non-trivially only on the curve γ_2 .

The MCG group $\text{MCG}(\mathcal{R}_{1,2})$ can be factorized in different ways as follows:

$$\begin{aligned} \text{MCG}(\mathcal{R}_{1,2}) &= \text{MCG}(\mathcal{R}_{1,2}, \gamma_1 + \gamma_3) \times \text{Dehn}(\gamma_1) \times \text{Dehn}(\gamma_3), \\ \text{MCG}(\mathcal{R}_{1,2}) &= \text{MCG}(\mathcal{R}_{1,2}, \gamma_2) \times \text{Dehn}^*(\gamma_2) \times \text{MCG}(\mathcal{R}_{1,1}(\ell_{\gamma_2})), \end{aligned} \quad (5.8)$$

where $\text{MCG}(\mathcal{R}_{1,1}(\ell_{\gamma_2}))$ denotes the MCG of the torus $\mathcal{R}_{1,1}(\ell_{\gamma_2})$ with a border having length ℓ_{γ_2} . $\text{Dehn}(\gamma_i)$ denotes the group generated by the Dehn twist $\tau_{\gamma_i} \rightarrow \tau_{\gamma_i} + \ell_{\gamma_i}$ and $\text{Dehn}^*(\gamma_i)$ denotes the group generated by the half Dehn twist $\tau_{\gamma_i} \rightarrow \tau_{\gamma_i} + \frac{1}{2}\ell_{\gamma_i}$. Interestingly, the lengths of the non-self intersecting closed geodesics on $\mathcal{R}_{1,1}(\ell_{\gamma_2})$ also satisfy an identity of the kind (5.7) [39]:

$$\sum_{g \in \text{MCG}(\mathcal{R}_{1,1}(\ell_{\gamma_2}))} \left[1 - \frac{1}{\ell_{\gamma_2}} \ln \left(\frac{\cosh(\frac{\ell_{g \cdot \gamma_1}}{2}) + \cosh(\frac{\ell_{\gamma_2} + \ell_{g \cdot \gamma_1}}{2})}{\cosh(\frac{\ell_{g \cdot \gamma_1}}{2}) + \cosh(\frac{\ell_{\gamma_2} - \ell_{g \cdot \gamma_1}}{2})} \right) \right] = 1. \quad (5.9)$$

We also have an identity that involves the sum over all images of the elements in the group $\text{Dehn}(\gamma_i)$, and is given by

$$\sum_{g \in \text{Dehn}(\gamma_i)} \text{sinc}^2 \left(\frac{\tau_{g \cdot \gamma_i}}{\ell_{g \cdot \gamma_i}} \right) = \sum_{g \in \text{Dehn}^*(\gamma_i)} \text{sinc}^2 \left(\frac{2\tau_{g \cdot \gamma_i}}{\ell_{g \cdot \gamma_i}} \right) = 1, \quad (5.10)$$

where $\text{sinc}(x) = \frac{\sin \pi x}{\pi x}$. The identity (5.10) can be verified using the following well known identity

$$\sum_{k=-\infty}^{\infty} \text{sinc}^2(x - k) = 1 \quad x \in \mathbb{R}. \quad (5.11)$$

Combining the identities (5.7), (5.9), (5.10) give the following identity

$$\sum_{g \in \text{MCG}(\mathcal{R}_{1,2})} G_1(\ell_{g \cdot \gamma_1}, \tau_{g \cdot \gamma_1}, \ell_{g \cdot \gamma_3}, \tau_{g \cdot \gamma_3}) + \sum_{g \in \text{MCG}(\mathcal{R}_{1,2})} G_2(\ell_{g \cdot \gamma_1}, \tau_{g \cdot \gamma_1}, \ell_{g \cdot \gamma_2}, \tau_{g \cdot \gamma_2}) = 1, \quad (5.12)$$

where G_1 and G_2 are given by

$$\begin{aligned} G_1(\ell_{\gamma_1}, \tau_{\gamma_1}, \ell_{\gamma_3}, \tau_{\gamma_3}) &= \frac{2 \text{sinc}^2 \left(\frac{\tau_{\gamma_1}}{\ell_{\gamma_1}} \right) \text{sinc}^2 \left(\frac{\tau_{\gamma_3}}{\ell_{\gamma_3}} \right)}{1 + e^{\frac{\ell_{\gamma_1} + \ell_{\gamma_3}}{2}}}, \\ G_2(\ell_{\gamma_1}, \tau_{\gamma_1}, \ell_{\gamma_2}, \tau_{\gamma_2}) &= \frac{2 \text{sinc}^2 \left(\frac{2\tau_{\gamma_2}}{\ell_{\gamma_2}} \right) \left[1 - \frac{1}{\ell_{\gamma_2}} \ln \left(\frac{\cosh(\frac{\ell_{\gamma_1}}{2}) + \cosh(\frac{\ell_{\gamma_2} + \ell_{\gamma_1}}{2})}{\cosh(\frac{\ell_{\gamma_1}}{2}) + \cosh(\frac{\ell_{\gamma_2} - \ell_{\gamma_1}}{2})} \right) \right]}{1 + e^{\frac{\ell_{\gamma_2}}{2}}}. \end{aligned} \quad (5.13)$$

Notice that the functions G_1 and G_2 have the following decaying property

$$\lim_{\ell_{\gamma_3} \rightarrow \frac{1}{c_*}} G_1(\ell_{\gamma_1}, \tau_{\gamma_1}, \ell_{\gamma_3}, \tau_{\gamma_3}) = \mathcal{O}(e^{-1/c_*}), \quad \lim_{\ell_{\gamma_2} \rightarrow \frac{1}{c_*}} G_2(\ell_{\gamma_1}, \tau_{\gamma_1}, \ell_{\gamma_2}, \tau_{\gamma_2}) = \mathcal{O}(e^{-1/c_*}). \quad (5.14)$$

Using the identity (5.12) we can express the amplitude $\mathbf{S}_{1,2}$ as an integral over the Teichmüller space $\mathcal{T}_{1,2}$ of twice-punctured tori as follows:

$$\begin{aligned} \mathbf{S}_{1,2} = & (2\pi i)^{-2} \int_{\mathcal{TW}_{1,2}^{\mathbf{P}_1}} d\ell_{\gamma_1} d\tau_{\gamma_1} d\ell_{\gamma_2} d\tau_{\gamma_2} G_1(\ell_{\gamma_1}, \tau_{\gamma_1}, \ell_{\gamma_2}, \tau_{\gamma_2}) \langle \mathcal{R}_{1,2} | b(\mathbf{t}_{\gamma_1}) b(\mathbf{l}_{\gamma_1}) b(\mathbf{t}_{\gamma_2}) b(\mathbf{l}_{\gamma_2}) | V_1 \rangle_{w_1} \otimes | V_2 \rangle_{w_2} \\ & + (2\pi i)^{-2} \int_{\mathcal{TW}_{1,2}^{\mathbf{P}_2}} d\ell_{\gamma_1} d\tau_{\gamma_1} d\ell_{\gamma_3} d\tau_{\gamma_3} G_2(\ell_{\gamma_1}, \tau_{\gamma_1}, \ell_{\gamma_3}, \tau_{\gamma_3}) \langle \mathcal{R}_{1,2} | b(\mathbf{t}_{\gamma_1}) b(\mathbf{l}_{\gamma_1}) b(\mathbf{t}_{\gamma_3}) b(\mathbf{l}_{\gamma_3}) | V_1 \rangle_{w_1} \otimes | V_2 \rangle_{w_2}, \end{aligned} \quad (5.15)$$

where $\mathcal{TW}_{1,2}^{\mathbf{P}_1}$ is the image of $\mathcal{W}_{1,2}$ in the Teichmüller space defined with respect to the pair of pants decomposition $\mathbf{P}_1$ given by the curves γ_1 and γ_2 . $\mathcal{TW}_{1,2}^{\mathbf{P}_2}$ is the union of all the images of $\mathcal{W}_{1,2}$ in the Teichmüller space defined with respect to the pair of pants decomposition $\mathbf{P}_2$ given by the curves γ_1 and γ_3 . Although $\mathcal{TW}_{1,2}^{\mathbf{P}_1}$ and $\mathcal{TW}_{1,2}^{\mathbf{P}_2}$ do not have a nice description, the decay behaviour of the functions G_1 and G_2 (5.14) allows us to replace them with the effective regions $EW_{1,2}^{\mathbf{P}_1}$ and $EW_{1,2}^{\mathbf{P}_2}$ without changing the value of $\mathbf{S}_{1,2}$. The string vertex region $\mathcal{W}_{1,2}$ has the property that it does not contain any hyperbolic Riemann surface having simple closed geodesics with length less than c_* . Consequently, $\mathbf{S}_{1,2}$ computed by integrating the off-shell string measure over $\mathcal{W}_{1,2}$ does not receive any contribution from hyperbolic Riemann surfaces having a simple closed geodesics with length less than c_* . Therefore, $\mathbf{S}_{1,2}$ computed by integrating the differential form in $\mathcal{T}_{1,2}$ over $EW_{1,2}^{\mathbf{P}_1}$ must also not receive any finite contribution from such surfaces. This is true if we identify $EW_{1,2}^{\mathbf{P}_1}$ with the following region in $\mathcal{T}_{1,2}$

$$EW_{1,2}^{\mathbf{P}_1} : \quad \ell_{\gamma_1} \in [c_*, \infty) \quad \ell_{\gamma_2} \in [c_*, \infty), \quad \tau_{\gamma_1} \in (-\infty, \infty), \quad \tau_{\gamma_2} \in (-\infty, \infty), \quad (5.16)$$

and $EW_{1,2}^{\mathbf{P}_2}$ with the following region

$$EW_{1,2}^{\mathbf{P}_2} : \quad \ell_{\gamma_1} \in [c_*, \infty) \quad \ell_{\gamma_3} \in [c_*, \infty), \quad \tau_{\gamma_1} \in (-\infty, \infty), \quad \tau_{\gamma_3} \in (-\infty, \infty). \quad (5.17)$$

Notice that the region $EW_{1,2}^{\mathbf{P}_1}$ includes hyperbolic Riemann surfaces with simple closed geodesic γ_3 having length less than c_* . Interestingly, when $\ell_{\gamma_3} \rightarrow c_*$ the length of γ_2 decay very fast and the function G_2 exponentially decays. As a result, the integration over region $EW_{1,2}^{\mathbf{P}_1}$ does not include any finite contribution from hyperbolic Riemann surfaces with simple closed geodesic γ_3 having length less than c_* . Similar statement is true for the integration over $EW_{1,2}^{\mathbf{P}_2}$. Then we can write $\mathbf{S}_{1,2}$ as

$$\begin{aligned} \mathbf{S}_{1,2} = & (2\pi i)^{-2} \int_{EW_{1,2}^{\mathbf{P}_1}} d\ell_{\gamma_1} d\tau_{\gamma_1} d\ell_{\gamma_2} d\tau_{\gamma_2} G_1(\ell_{\gamma_1}, \tau_{\gamma_1}, \ell_{\gamma_2}, \tau_{\gamma_2}) \langle \mathcal{R}_{1,2} | b(\mathbf{t}_{\gamma_1}) b(\mathbf{l}_{\gamma_1}) b(\mathbf{t}_{\gamma_2}) b(\mathbf{l}_{\gamma_2}) | V_1 \rangle_{w_1} \otimes | V_2 \rangle_{w_2} \\ & + (2\pi i)^{-2} \int_{EW_{1,2}^{\mathbf{P}_2}} d\ell_{\gamma_1} d\tau_{\gamma_1} d\ell_{\gamma_3} d\tau_{\gamma_3} G_2(\ell_{\gamma_1}, \tau_{\gamma_1}, \ell_{\gamma_3}, \tau_{\gamma_3}) \langle \mathcal{R}_{1,2} | b(\mathbf{t}_{\gamma_1}) b(\mathbf{l}_{\gamma_1}) b(\mathbf{t}_{\gamma_3}) b(\mathbf{l}_{\gamma_3}) | V_1 \rangle_{w_1} \otimes | V_2 \rangle_{w_2}, \end{aligned} \quad (5.18)$$

Corrected interaction vertex $\tilde{\mathbf{S}}_{1,2}$. The naive interaction vertex $\mathbf{S}_{1,2}$ must be modified to make them suitable for constructing a string field theory with approximate gauge invariance. The modification can be implemented once we specify the subregions $\mathbf{W}_{1,2}^{(0)}$, $\mathbf{W}_{1,2}^{(1)}$ and $\mathbf{W}_{1,2}^{(2)}$ inside $\mathcal{W}_{1,2}$.

The subregion $\mathbf{W}_{1,2}^{(0)}$ has the property that it does not include any hyperbolic Riemann surface with one or more simple closed geodesic having length less than $c_*(1 + \delta)$. Let us denote the union of all the images of $\mathbf{W}_{1,2}^{(0)}$ in $\mathcal{T}_{1,2}$ defined with respect to the pants decomposition $\mathbf{P}_1$ as $\mathcal{TW}_{1,2}^{\mathbf{P}_1, (0)}$. For $\mathcal{T}_{1,2}$ defined with respect to the pants decomposition $\mathbf{P}_2$, the union of all images of $\mathbf{W}_{1,2}^{(0)}$ is denoted as $\mathcal{TW}_{1,2}^{\mathbf{P}_2, (0)}$. Then by repeating the arguments in the previous paragraph we can identify the effective region $E\mathbf{W}_{1,2}^{\mathbf{P}_1, (0)}$ in $\mathcal{T}_{1,2}$ that corresponds to $\mathcal{TW}_{1,2}^{\mathbf{P}_1, (0)}$ with the following region

$$E\mathbf{W}_{1,2}^{\mathbf{P}_1, (0)} : \quad \ell_{\gamma_1} \in [c_*(1 + \delta), \infty), \quad \ell_{\gamma_2} \in [c_*(1 + \delta), \infty), \quad \tau_{\gamma_1} \in (-\infty, \infty), \quad \tau_{\gamma_2} \in (-\infty, \infty). \quad (5.19)$$

Similarly, we can identify the effective region $E\mathbf{W}_{1,2}^{\mathbf{P}_2, (0)}$ that corresponds to $\mathcal{TW}_{1,2}^{\mathbf{P}_2, (0)}$ with the following region

$$E\mathbf{W}_{1,2}^{\mathbf{P}_2, (0)} : \quad \ell_{\gamma_1} \in [c_*(1 + \delta), \infty), \quad \ell_{\gamma_3} \in [c_*(1 + \delta), \infty), \quad \tau_{\gamma_1} \in (-\infty, \infty), \quad \tau_{\gamma_3} \in (-\infty, \infty). \quad (5.20)$$

Now let us analyze the subregion $\mathbf{W}_{1,2}^{(1)}$. It has the property that any hyperbolic Riemann surface in this region has only one simple closed geodesic having length between c_* and $c_*(1 + \delta)$. Let us denote the union of all the images of $\mathbf{W}_{1,2}^{(1)}$ in $\mathcal{T}_{1,2}$ defined with respect to the pants decomposition $\mathbf{P}_1$ as $\mathcal{TW}_{1,2}^{\mathbf{P}_1, (1)}$ and that defined with respect to the pants decomposition $\mathbf{P}_2$ as $\mathcal{TW}_{1,2}^{\mathbf{P}_2, (1)}$. We can identify the effective regions correspond to $\mathcal{TW}_{1,2}^{\mathbf{P}_1, (1)}$ and $\mathcal{TW}_{1,2}^{\mathbf{P}_2, (1)}$ as follows:

$$\begin{aligned} E\mathbf{W}_{1,2}^{\mathbf{P}_1, (1)} &= E\mathbf{W}_{1,2}^{\mathbf{P}_1, \gamma_1} \cup E\mathbf{W}_{1,2}^{\mathbf{P}_1, \gamma_2} \\ E\mathbf{W}_{1,2}^{\mathbf{P}_2, (1)} &= E\mathbf{W}_{1,2}^{\mathbf{P}_2, \gamma_1} \cup E\mathbf{W}_{1,2}^{\mathbf{P}_2, \gamma_3} \end{aligned} \quad (5.21)$$

where

$$\begin{aligned} E\mathbf{W}_{1,2}^{\mathbf{P}_1, \gamma_1} &= \ell_{\gamma_1} \in [c_*, c_*(1 + \delta)), \quad \ell_{\gamma_2} \in [c_*(1 + \delta), \infty), \quad \tau_{\gamma_1} \in (-\infty, \infty), \quad \tau_{\gamma_2} \in (-\infty, \infty), \\ E\mathbf{W}_{1,2}^{\mathbf{P}_1, \gamma_2} &= \ell_{\gamma_1} \in [c_*(1 + \delta), \infty), \quad \ell_{\gamma_2} \in [c_*, c_*(1 + \delta)), \quad \tau_{\gamma_1} \in (-\infty, \infty), \quad \tau_{\gamma_2} \in (-\infty, \infty), \\ E\mathbf{W}_{1,2}^{\mathbf{P}_2, \gamma_1} &= \ell_{\gamma_1} \in [c_*, c_*(1 + \delta)), \quad \ell_{\gamma_3} \in [c_*(1 + \delta), \infty), \quad \tau_{\gamma_1} \in (-\infty, \infty), \quad \tau_{\gamma_3} \in (-\infty, \infty), \\ E\mathbf{W}_{1,2}^{\mathbf{P}_2, \gamma_3} &= \ell_{\gamma_1} \in [c_*(1 + \delta), \infty), \quad \ell_{\gamma_3} \in [c_*, c_*(1 + \delta)), \quad \tau_{\gamma_1} \in (-\infty, \infty), \quad \tau_{\gamma_3} \in (-\infty, \infty). \end{aligned} \quad (5.22)$$

Finally, the effective regions $E\mathbf{W}_{1,2}^{\mathbf{P}_1, \gamma_1, \gamma_2}$ and $E\mathbf{W}_{1,2}^{\mathbf{P}_2, \gamma_1, \gamma_3}$ for the subregion $\mathbf{W}_{1,2}^{(2)}$ in $\mathcal{T}_{1,2}$ defined with respect to the pants decomposition $\mathbf{P}_1$ and $\mathbf{P}_2$ respectively are given by

$$\begin{aligned} E\mathbf{W}_{1,2}^{\mathbf{P}_1, \gamma_1, \gamma_2} &= \ell_{\gamma_1} \in [c_*, c_*(1 + \delta)), \quad \ell_{\gamma_2} \in [c_*c_*(1 + \delta)), \quad \tau_{\gamma_1} \in (-\infty, \infty), \quad \tau_{\gamma_2} \in (-\infty, \infty), \\ E\mathbf{W}_{1,2}^{\mathbf{P}_2, \gamma_1, \gamma_3} &= \ell_{\gamma_1} \in [c_*, c_*(1 + \delta)), \quad \ell_{\gamma_3} \in [c_*c_*(1 + \delta)), \quad \tau_{\gamma_1} \in (-\infty, \infty), \quad \tau_{\gamma_3} \in (-\infty, \infty). \end{aligned} \quad (5.23)$$

Given, that we have identified the effective regions for the subregions in $\mathcal{W}_{1,2}$, let us construct the corrected interaction vertex $\tilde{\mathbf{S}}_{1,2}$. It is given by

$$\begin{aligned}
 \tilde{\mathbf{S}}_{1,2} = & \int_{E\mathbf{W}_{1,2}^{\mathbf{P}^{1,(0)}}} \frac{d\ell_{\gamma_1} d\tau_{\gamma_1} d\ell_{\gamma_2} d\tau_{\gamma_2}}{(2\pi i)^2} G_1(\ell_{\gamma_1}, \tau_{\gamma_1}, \ell_{\gamma_2}, \tau_{\gamma_2}) \langle \mathcal{R}_{1,2} | b(\mathbf{t}_{\gamma_1}) b(\mathbf{l}_{\gamma_1}) b(\mathbf{t}_{\gamma_2}) b(\mathbf{l}_{\gamma_2}) | V_1 \rangle_{\tilde{w}_1} \otimes | V_2 \rangle_{\tilde{w}_2} \\
 & + \int_{E\mathbf{W}_{1,2}^{\mathbf{P}^{2,(0)}}} \frac{d\ell_{\gamma_1} d\tau_{\gamma_1} d\ell_{\gamma_3} d\tau_{\gamma_3}}{(2\pi i)^2} G_2(\ell_{\gamma_1}, \tau_{\gamma_1}, \ell_{\gamma_3}, \tau_{\gamma_3}) \langle \mathcal{R}_{1,2} | b(\mathbf{t}_{\gamma_1}) b(\mathbf{l}_{\gamma_1}) b(\mathbf{t}_{\gamma_3}) b(\mathbf{l}_{\gamma_3}) | V_1 \rangle_{\tilde{w}_1} \otimes | V_2 \rangle_{\tilde{w}_2} \\
 & + \int_{E\mathbf{W}_{1,2}^{\mathbf{P}^{1,\gamma_1}}} \frac{d\ell_{\gamma_1} d\tau_{\gamma_1} d\ell_{\gamma_2} d\tau_{\gamma_2}}{(2\pi i)^2} G_1(\ell_{\gamma_1}, \tau_{\gamma_1}, \ell_{\gamma_2}, \tau_{\gamma_2}) \langle \mathcal{R}_{1,2} | b(\mathbf{t}_{\gamma_1}) b(\mathbf{l}_{\gamma_1}) b(\mathbf{t}_{\gamma_2}) b(\mathbf{l}_{\gamma_2}) | V_1 \rangle_{\tilde{w}_1^{\gamma_1}} \otimes | V_2 \rangle_{\tilde{w}_2^{\gamma_1}} \\
 & + \int_{E\mathbf{W}_{1,2}^{\mathbf{P}^{1,\gamma_2}}} \frac{d\ell_{\gamma_1} d\tau_{\gamma_1} d\ell_{\gamma_2} d\tau_{\gamma_2}}{(2\pi i)^2} G_1(\ell_{\gamma_1}, \tau_{\gamma_1}, \ell_{\gamma_2}, \tau_{\gamma_2}) \langle \mathcal{R}_{1,2} | b(\mathbf{t}_{\gamma_1}) b(\mathbf{l}_{\gamma_1}) b(\mathbf{t}_{\gamma_2}) b(\mathbf{l}_{\gamma_2}) | V_1 \rangle_{\tilde{w}_1^{\gamma_2}} \otimes | V_2 \rangle_{\tilde{w}_2^{\gamma_2}} \\
 & + \int_{E\mathbf{W}_{1,2}^{\mathbf{P}^{2,\gamma_1}}} \frac{d\ell_{\gamma_1} d\tau_{\gamma_1} d\ell_{\gamma_3} d\tau_{\gamma_3}}{(2\pi i)^2} G_2(\ell_{\gamma_1}, \tau_{\gamma_1}, \ell_{\gamma_3}, \tau_{\gamma_3}) \langle \mathcal{R}_{1,2} | b(\mathbf{t}_{\gamma_1}) b(\mathbf{l}_{\gamma_1}) b(\mathbf{t}_{\gamma_3}) b(\mathbf{l}_{\gamma_3}) | V_1 \rangle_{\tilde{w}_1^{\gamma_1}} \otimes | V_2 \rangle_{\tilde{w}_2^{\gamma_1}} \\
 & + \int_{E\mathbf{W}_{1,2}^{\mathbf{P}^{2,\gamma_3}}} \frac{d\ell_{\gamma_1} d\tau_{\gamma_1} d\ell_{\gamma_3} d\tau_{\gamma_3}}{(2\pi i)^2} G_2(\ell_{\gamma_1}, \tau_{\gamma_1}, \ell_{\gamma_3}, \tau_{\gamma_3}) \langle \mathcal{R}_{1,2} | b(\mathbf{t}_{\gamma_1}) b(\mathbf{l}_{\gamma_1}) b(\mathbf{t}_{\gamma_3}) b(\mathbf{l}_{\gamma_3}) | V_1 \rangle_{\tilde{w}_1^{\gamma_3}} \otimes | V_2 \rangle_{\tilde{w}_2^{\gamma_3}} \\
 & + \int_{E\mathbf{W}_{1,2}^{\mathbf{P}^{1,\gamma_1,\gamma_2}}} \frac{d\ell_{\gamma_1} d\tau_{\gamma_1} d\ell_{\gamma_2} d\tau_{\gamma_2}}{(2\pi i)^2} G_1(\ell_{\gamma_1}, \tau_{\gamma_1}, \ell_{\gamma_2}, \tau_{\gamma_2}) \langle \mathcal{R}_{1,2} | b(\mathbf{t}_{\gamma_1}) b(\mathbf{l}_{\gamma_1}) b(\mathbf{t}_{\gamma_2}) b(\mathbf{l}_{\gamma_2}) | V_1 \rangle_{\tilde{w}_1^{\gamma_1\gamma_2}} \otimes | V_2 \rangle_{\tilde{w}_2^{\gamma_1\gamma_2}} \\
 & + \int_{E\mathbf{W}_{1,2}^{\mathbf{P}^{2,\gamma_1,\gamma_3}}} \frac{d\ell_{\gamma_1} d\tau_{\gamma_1} d\ell_{\gamma_3} d\tau_{\gamma_3}}{(2\pi i)^2} G_2(\ell_{\gamma_1}, \tau_{\gamma_1}, \ell_{\gamma_3}, \tau_{\gamma_3}) \langle \mathcal{R}_{1,2} | b(\mathbf{t}_{\gamma_1}) b(\mathbf{l}_{\gamma_1}) b(\mathbf{t}_{\gamma_3}) b(\mathbf{l}_{\gamma_3}) | V_1 \rangle_{\tilde{w}_1^{\gamma_1\gamma_3}} \otimes | V_2 \rangle_{\tilde{w}_2^{\gamma_1\gamma_3}}.
 \end{aligned} \tag{5.24}$$

The local coordinates are as follows

$$\begin{aligned}
 \tilde{w}_j &= e^{\frac{\pi^2}{c_*}} w_j, \\
 \tilde{w}_j^{\gamma_1} &= e^{\frac{c_*}{6} f(\ell_{\gamma_1}) Y_{1j}^{(1)}} e^{\frac{\pi^2}{c_*}} w_j, \\
 \tilde{w}_j^{\gamma_2} &= e^{\frac{c_*}{6} f(\ell_{\gamma_2}) Y_{2j}^{(1)}} e^{\frac{\pi^2}{c_*}} w_j, \\
 \tilde{w}_j^{\gamma_3} &= e^{\frac{c_*}{6} f(\ell_{\gamma_3}) Y_{3j}^{(1)}} e^{\frac{\pi^2}{c_*}} w_j, \\
 \tilde{w}_j^{\gamma_1\gamma_2} &= e^{\frac{c_*}{6} [f(\ell_{\gamma_1}) Y_{1j}^{(2)} + f(\ell_{\gamma_2}) Y_{2j}^{(2)}]} e^{\frac{\pi^2}{c_*}} w_j \\
 \tilde{w}_j^{\gamma_1\gamma_3} &= e^{\frac{c_*}{6} [f(\ell_{\gamma_1}) Y_{1j}^{(2)} + f(\ell_{\gamma_3}) Y_{3j}^{(2)}]} e^{\frac{\pi^2}{c_*}} w_j
 \end{aligned}$$

where f is an arbitrary smooth real function of the geodesic length defined in the interval $(c_*, c_* + \delta c_*)$, such that $f(c_*) = 1$ and $f(c_* + \delta c_*) = 0$. The coefficient $Y_{1j}^{(1)}$ is given by

$$\begin{aligned}
 Y_{1j}^{(1)} &= \sum_{q=1}^2 \sum_{c_i^q, d_i^q} \frac{\pi^2}{|c_i^q|^4} \\
 c_i^q &> 0 \quad d_i^q \bmod c_i^q \quad \begin{pmatrix} * & * \\ c_i^q & d_i^q \end{pmatrix} \in (\sigma_{\gamma_1}^q)^{-1} \Gamma_{0,4} \sigma_j,
 \end{aligned} \tag{5.25}$$

where the transformation σ_j^{-1} maps the cusp corresponding to the j^{th} puncture to ∞ and $(\sigma_{\gamma_1}^q)^{-1}$ maps the cusp corresponds to the one of the two punctures, marked as q , obtained

by degenerating the curve γ_1 , to ∞ . $\Gamma_{0,4}$ is the Fuchisan group of a four punctured hyperbolic Riemann surface with Fenchel-Nielsen parameters $(\ell_{\gamma_1}, \tau_{\gamma_1}, \ell_{\gamma_2}, \tau_{\gamma_2})$. $Y_{1j}^{(1)}$ is given by

$$Y_{2j}^{(1)} = \sum_{q=1}^2 \sum_{c_i^q, d_i^q} \frac{\pi^2}{|c_i^q|^4}$$

$$c_i^q > 0 \quad d_i^q \bmod c_i^q \quad \begin{pmatrix} * & * \\ c_i^q & d_i^q \end{pmatrix} \in (\sigma_{\gamma_2}^q)^{-1} \Gamma_{0,4} \sigma_j, \quad (5.26)$$

where $(\sigma_{\gamma_2}^q)^{-1}$ maps the cusp corresponds to the one of the two punctures, marked as q , obtained by degenerating the curve γ_2 , to ∞ . $Y_{3j}^{(1)}$ is given by

$$Y_{3j}^{(1)} = \sum_{q=1}^2 \sum_{c_i^q, d_i^q} \frac{\pi^2}{|c_i^q|^4}$$

$$c_i^q > 0 \quad d_i^q \bmod c_i^q \quad \begin{pmatrix} * & * \\ c_i^q & d_i^q \end{pmatrix} \in (\sigma_{\gamma_3}^q)^{-1} \Gamma_{0,4} \sigma_j, \quad (5.27)$$

where $(\sigma_{\gamma_3}^q)^{-1}$ maps the cusp corresponds to the one of the two punctures, marked as q , obtained by degenerating the curve γ_3 , to ∞ . $Y_{1j}^{(2)}$ is given by

$$Y_{1j}^{(2)} = \sum_{q=1}^2 \sum_{c_i^q, d_i^q} \pi^2 \frac{\epsilon(j, q)}{|c_i^q|^4}$$

$$c_i^q > 0 \quad d_i^q \bmod c_i^q \quad \begin{pmatrix} * & * \\ c_i^q & d_i^q \end{pmatrix} \in (\sigma_{\gamma_1}^q)^{-1} \Gamma_{0,3} \sigma_j, \quad (5.28)$$

where $\Gamma_{0,3}$ is the Fuchsian group of a thrice punctured hyperbolic sphere. The factor $\epsilon(j, q)$ is one if both the j^{th} puncture and the puncture denoted by the index q obtained by degenerating the curve γ_1 belong to the same thrice punctured sphere, other wise $\epsilon(j, q)$ is zero. $Y_{2j}^{(2)}$ is given by

$$Y_{2j}^{(2)} = \sum_{q=1}^2 \sum_{c_i^q, d_i^q} \pi^2 \frac{\epsilon(j, q)}{|c_i^q|^4}$$

$$c_i^q > 0 \quad d_i^q \bmod c_i^q \quad \begin{pmatrix} * & * \\ c_i^q & d_i^q \end{pmatrix} \in (\sigma_{\gamma_2}^q)^{-1} \Gamma_{0,3} \sigma_j, \quad (5.29)$$

and $Y_{3j}^{(2)}$ is given by

$$Y_{3j}^{(2)} = \sum_{q=1}^2 \sum_{c_i^q, d_i^q} \pi^2 \frac{\epsilon(j, q)}{|c_i^q|^4}$$

$$c_i^q > 0 \quad d_i^q \bmod c_i^q \quad \begin{pmatrix} * & * \\ c_i^q & d_i^q \end{pmatrix} \in (\sigma_{\gamma_3}^q)^{-1} \Gamma_{0,3} \sigma_j. \quad (5.30)$$

The last ingredient that one need for computing of the corrected interaction vertex $\tilde{S}_{1,2}$ is the explicit form of the generators of the Fuchsian groups $\Gamma_{0,3}$, associated with the thrice punctured hyperbolic sphere and $\Gamma_{0,4}$, associated with the four punctured hyperbolic Riemann surface with specific Fenchel-Nielsen parameters. Interestingly, following the algorithm given in [66], it is possible to construct the Fuchsian group of any hyperbolic Riemann surface having specific Fenchel-Nielsen parameters. For example, the group $\Gamma_{0,3}$ is generated by the transformations

$$z \rightarrow \frac{z}{2z+1} \quad z \rightarrow z+2.$$

The Fuchsian group $\Gamma_{0,4}(\ell, \tau)$ that produces a four punctured sphere with Fenchel-Nielsen parameter (ℓ, τ) can be generated using the following three elements

$$\begin{aligned} a_1 &= \begin{pmatrix} 1+\beta & -\beta \\ \beta & 1-\beta \end{pmatrix} \\ a_2 &= \begin{pmatrix} (1-\beta) & -\beta e^{2\tau} \\ \beta e^{-2\tau} & (1+\beta) \end{pmatrix} \\ a_3 &= - \begin{pmatrix} (1+\beta)e^\ell & \beta e^{-\ell+2\tau} \\ -\beta e^{\ell-2\tau} & (1-\beta)e^{-\ell} \end{pmatrix}, \end{aligned} \quad (5.31)$$

where $\beta = -\frac{\cosh \ell + 1}{\sinh \ell}$.

Arbitrary interaction vertex. It is straightforward to generalize this discussion to the case of a general interaction vertex in closed string field theory. This is because the lengths of simple closed geodesics on a general hyperbolic Riemann surface with borders also satisfies identities of the kind (5.7). Assume that $\mathcal{R}(L_1, \dots, L_n)$ is a Riemann surface with g handles and n boundaries $\beta_1, \dots, \beta_n$ having hyperbolic lengths $L_1, \dots, L_n$. In the limit $L_i \rightarrow 0$, $i = 1, \dots, n$, the bordered surface $\mathcal{R}(L_1, \dots, L_n)$ becomes $\mathcal{R}$, a genus g Riemann surface with n punctures. The lengths of the non-self intersecting closed geodesics on $\mathcal{R}(L_1, \dots, L_n)$ satisfy the following identity [39]:

$$\sum_{i=1}^n \sum_k \sum_{g \in \text{MCG}(\mathcal{R}(L_1, \dots, L_n), \mathcal{C}_i^k)} \mathcal{Q}_i(L_1, L_i, \ell_{g \cdot \mathcal{C}_i^k}) = 1, \quad (5.32)$$

where

$$\begin{aligned} \mathcal{Q}_i(L_1, L_i, \ell_{\mathcal{C}_i^k}) &= \delta_{1i} \mathcal{D}(L_1, \ell_{\alpha_1^k}, \ell_{\alpha_2^k}) + (1 - \delta_{1i}) \mathcal{E}(L_1, L_i, \ell_{\mathcal{C}_i^k}) \\ \mathcal{D}(x_1, x_2, x_3) &= 1 - \frac{1}{x_1} \ln \left(\frac{\cosh(\frac{x_2}{2}) + \cosh(\frac{x_1+x_3}{2})}{\cosh(\frac{x_2}{2}) + \cosh(\frac{x_1-x_3}{2})} \right), \\ \mathcal{E}(x_1, x_2, x_3) &= \frac{2}{x_1} \ln \left(\frac{e^{\frac{x_1}{2}} + e^{\frac{x_2+x_3}{2}}}{e^{-\frac{x_1}{2}} + e^{\frac{x_2+x_3}{2}}} \right). \end{aligned}$$

$\mathcal{C}_1^k$ is the multi-curve $\alpha_1^k + \alpha_2^k$, where the simple closed geodesics α_1^k and α_2^k together with β_1 bounds a pair of pants, see figure 2. $\mathcal{C}_i^k$, $i \in \{2, \dots, n\}$ is a simple closed

Figure 2. Cutting a surface along a curve $\mathcal{C}_1^k = \alpha_1^k + \alpha_2^k$ produces a surface with borders.

geodesic γ_i^k which together with β_1 and β_i bounds a pair of pants. The index k distinguishes curves that are not related to each other via the action of elements in $\text{MCG}(\mathcal{R}(L_1, \dots, L_n))$, the mapping class group of $\mathcal{R}(L_1, \dots, L_n)$. The summation over k add contributions from all such distinct classes of curves. By $\ell_{\mathcal{C}_1^k}$ we mean the pair $(\ell_{\alpha_1^k}, \ell_{\alpha_2^k})$. $\text{MCG}(\mathcal{R}(L_1, \dots, L_n), \mathcal{C}_i^k)$ is the subgroup of $\text{MCG}(\mathcal{R}(L_1, \dots, L_n))$ that acts non-trivially only on the curve $\mathcal{C}_i^k$. Remember that a Dehn twist performed with respect to $\mathcal{C}_i^k$ is not an element of $\text{MCG}(\mathcal{R}(L_1, \dots, L_n), \mathcal{C}_i)$. We also have an identity for the group Dehn twists, and is given by

$$\sum_{g \in \text{Dehn}(\mathcal{C}_i^k)} \mathcal{Y}_i(\ell_{g \cdot \mathcal{C}_i^k}, \tau_{g \cdot \mathcal{C}_i^k}) = 1, \quad (5.33)$$

where $\text{Dehn}(\mathcal{C}_1^k)$ denotes the product group $\text{Dehn}(\alpha_1^k) \times \text{Dehn}(\alpha_2^k)$, and

$$\mathcal{Y}_i(\ell_{\mathcal{C}_i^k}, \tau_{\mathcal{C}_i^k}) = \delta_{i1} \text{sinc}^2\left(\frac{\tau_{\alpha_1^k}}{\ell_{\alpha_1^k}}\right) \text{sinc}^2\left(\frac{\tau_{\alpha_2^k}}{\ell_{\alpha_2^k}}\right) + (1 - \delta_{i1}) \text{sinc}^2\left(\frac{\tau_{\gamma_i^k}}{\ell_{\gamma_i^k}}\right). \quad (5.34)$$

Combining the identity (5.32) with the identity (5.33) gives us the following identity

$$\sum_{i=1}^n \sum_k \sum_{g \in \text{MCG}(\mathcal{R}(L_1, \dots, L_n), \mathcal{C}_i^k) \times \text{Dehn}(\mathcal{C}_i^k)} \mathcal{Z}_i(L_1, L_i, \ell_{g \cdot \mathcal{C}_i^k}, \tau_{g \cdot \mathcal{C}_i^k}) = 1, \quad (5.35)$$

where

$$\mathcal{Z}_i(L_1, L_i, \ell_{\mathcal{C}_i^k}, \tau_{\mathcal{C}_i^k}) = \mathcal{Q}_i(L_1, L_i, \ell_{\mathcal{C}_i^k}) \mathcal{Y}_i(\ell_{\mathcal{C}_i^k}, \tau_{\mathcal{C}_i^k}). \quad (5.36)$$

Now consider cutting $\mathcal{R}(L_1, \dots, L_n)$ along $\mathcal{C}_i^k$. Let us denote the surface obtained as a result of this cutting by $\mathcal{R}(L_1, \dots, L_n; \ell_{\mathcal{C}_i^k})$. Notice that the group $\text{MCG}(\mathcal{R}(L_1, \dots, L_n; \ell_{\mathcal{C}_i^k}) \times \text{Dehn}(\mathcal{C}_i^k)$ has no non-trivial action on $\mathcal{R}(L_1, \dots, L_n; \ell_{\mathcal{C}_i^k})$. Therefore, we can repeat the whole exercise by considering the identity (5.32) on $\mathcal{R}(L_1, \dots, L_n; \ell_{\mathcal{C}_i^k})$. At the end we obtain an identity of the following kind

$$\sum_{g \in \text{MCG}(\mathcal{R}(L_1, \dots, L_n))} \sum_s \mathcal{G}_s(\ell_{g \cdot \gamma_s}, \tau_{g \cdot \gamma_s}) = 1, \quad (5.37)$$

where $\mathcal{G}_s$ are functions of the Fenchel-Nielsen coordinates of $\mathcal{R}(L_1, \dots, L_n)$ defined with respect to a multi-curves $\gamma_s = \sum_{i=1}^{3g-3+n} \gamma_s^i$. The collection of curves $\{\gamma_s^1, \dots, \gamma_s^{3g-3+n}\}$

form a system of non-self intersecting geodesics that define a pair of pants decomposition $\mathbf{P}^s$ of $\mathcal{R}(L_1, \dots, L_n)$. The sum over s represents the sum over pair of pants decompositions that are not related to each other via any MCG transformation.

The function $\mathcal{G}_s$ has an important property. To demonstrate this property a non-self intersecting closed geodesic γ on $\mathcal{R}(L_1, \dots, L_n)$ that can not be mapped to any element in the set $\{\gamma_s^1, \dots, \gamma_s^{3g-3+n}\}$ by the action of any element in $\text{MCG}(\mathcal{R}(L_1, \dots, L_n))$. The hyperbolic metric on $\mathcal{R}(L_1, \dots, L_n)$ has the property that if $\ell_\gamma \rightarrow c_*$, then the length of at least one of the curve in the set $\{\gamma_s^1, \dots, \gamma_s^{3g-3+n}\}$ will have length of the order $e^{\frac{1}{c_*}}$. Moreover, the function $\mathcal{G}_s$ is a function of all the curves in the set $\{\gamma_s^1, \dots, \gamma_s^{3g-3+n}\}$ constructed by multiplying the functions $\mathcal{D}(x, y, z)$ and $\mathcal{E}(x, y, z)$. Note that the function $\mathcal{D}(x, y, z)$ appearing in the Mirzakhani-McShane identity (B.3) given by

$$\mathcal{D}(x, y, z) = 2 \ln \left(\frac{e^{\frac{x}{2}} + e^{\frac{y+z}{2}}}{e^{\frac{-x}{2}} + e^{\frac{y+z}{2}}} \right), \quad (5.38)$$

vanishes in the limits $y \rightarrow \infty$ keeping x, z fixed and $z \rightarrow \infty$ keeping x, y fixed:

$$\lim_{y, z \rightarrow \infty} \mathcal{D}(x, y, z) = \lim_{y, z \rightarrow \infty} \mathcal{O} \left(e^{-\frac{y+z}{2}} \right). \quad (5.39)$$

$\mathcal{E}(x, y, z)$ given by

$$\mathcal{E}(x, y, z) = x - \ln \left(\frac{\cosh \left(\frac{y}{2} \right) + \cosh \left(\frac{x+z}{2} \right)}{\cosh \left(\frac{y}{2} \right) + \cosh \left(\frac{x-z}{2} \right)} \right), \quad (5.40)$$

becomes 0 in the limit $z \rightarrow \infty$ keeping x, y fixed:

$$\lim_{z \rightarrow \infty} \mathcal{E}(x, y, z) = \lim_{z \rightarrow \infty} \mathcal{O} \left(e^{-\frac{z}{2}} \right). \quad (5.41)$$

This can be easily verified by using the following relation

$$\mathcal{E}(x, y, z) = \frac{\mathcal{D}(x, y, z) + \mathcal{D}(x, -y, z)}{2}. \quad (5.42)$$

Combining these observations suggests that the function $\mathcal{G}_s$ has the following property

$$\lim_{\ell_\gamma \rightarrow c_*} \mathcal{G}_s = \mathcal{O}(e^{-1/c_*}). \quad (5.43)$$

Naive interaction vertex $\mathbf{S}_{g,n}$. The naive g -loop n -point interaction vertex $\mathbf{S}_{g,n}$ for n external off-shell states $|V_1\rangle, \dots, |V_n\rangle$ represented by the vertex operators $V_1, \dots, V_n$ constructed using the naive string vertex $\mathcal{V}_{g,n}^0$ is given by

$$\mathbf{S}_{g,n} = \int_{\mathcal{W}_{g,n}} \frac{d\ell_{\gamma_1^s} d\tau_{\gamma_1^s} \dots d\ell_{\gamma_Q^s} d\tau_{\gamma_Q^s}}{(2\pi i)^Q} \langle \mathcal{R} | b(\mathbf{t}_{\gamma_1^s}) b(\mathbf{l}_{\gamma_1^s}) \dots b(\mathbf{t}_{\gamma_Q^s}) b(\mathbf{l}_{\gamma_Q^s}) | V_1 \rangle_{w_1} \otimes \dots \otimes | V_n \rangle_{w_n}, \quad (5.44)$$

where $Q = 3g - 3 + n$ and the states $|V_1\rangle, \dots, |V_n\rangle$ are inserted on the Riemann surface $\mathcal{R}$ using the set of local coordinates $(e^{\frac{\pi^2}{c_*}} w_1, \dots, e^{\frac{\pi^2}{c_*}} w_n)$ induced from the hyperbolic metric.

$|\mathcal{R}\rangle$ is the surface state associated with the Riemann surface $\mathcal{R}$. $(\tau_{\gamma_j^s}, \ell_{\gamma_j^s})$, $1 \leq j \leq Q$ denote the Fenchel-Nielsen coordinates for the Teichmüller space $\mathcal{T}_{g,n}$ defined with respect to the pants decomposition $\mathbf{P}_s$ of $\mathcal{R}$. Using the identity (5.37), we can decompose (5.24) into sum over all possible distinct pants decompositions of $\mathcal{R}$ with each term expressed as an integral over $\mathcal{T}_{g,n}$:

$$\mathbf{S}_{g,n} = \sum_s \int_{\mathcal{TW}_{g,n}} \frac{d\ell_{\gamma_1^s} d\tau_{\gamma_1^s} \cdots d\ell_{\gamma_Q^s} d\tau_{\gamma_Q^s}}{(2\pi i)^Q} \mathcal{G}_s \langle \mathcal{R} | b(\mathbf{t}_{\gamma_1^s}) b(\mathbf{l}_{\gamma_1^s}) \cdots b(\mathbf{t}_{\gamma_Q^s}) b(\mathbf{l}_{\gamma_Q^s}) | V_1 \rangle_{w_1} \otimes \cdots \otimes | V_n \rangle_{w_n}, \quad (5.45)$$

where $\mathcal{TW}_{g,n}$ is the union of all images $\mathcal{W}_{g,n}$ in $\mathcal{T}_{g,n}$. Since the set local coordinates induced from hyperbolic metric do not satisfy the geometrical identity induced from the quantum BV master equation, the closed string field theory action constructed using the naive interaction vertex $\mathbf{S}_{g,n}$ is not gauge invariant.

Corrected interaction vertex $\tilde{\mathbf{S}}_{g,n}$. In order to correct the interaction vertex $\tilde{\mathbf{S}}_{g,n}$ the set of local coordinates on the world-sheets in $\mathcal{W}_{g,n}$ induced from the hyperbolic metric used to construct $\mathbf{S}_{g,n}$ must be modified. The set of local coordinates has to be modified if $\mathcal{R}$ belongs to the regions $\mathbf{W}_{g,n}^{(m)}$ for $m \neq 0$. Although the regions inside $\mathcal{M}_{g,n}$ where we need to modify the local coordinates have a simple description in terms of the length of the simple closed geodesics, it is impossible to specify them as explicit regions inside $\mathcal{T}_{g,n}$ using the Fenchel-Nielsen coordinates. This is due to the fact that there are infinitely many simple closed geodesics on a Riemann surface.

Interestingly, the effective expression (5.44) has a noteworthy feature due to the decay property of the weight factors $\mathcal{G}_s$ (5.43). Assume that the length of a simple closed geodesic α which does not belong to the set of curves $\{\gamma_s^1, \dots, \gamma_s^{3g-3+n}\}$ associated with the pants decomposition $\mathbf{P}^s$ becomes c_* . Then the weight factor $\mathcal{G}_s$ decays to $\mathcal{O}(e^{-1/c_*})$. As a result, by correcting interaction vertex by modifying the local coordinates within each term in the effective expression (5.44) independently it is possible to make them approximately satisfy the quantum BV master equation.

Consider the s^{th} term in the effective expression. The effective region $E\mathcal{W}_{g,n}^{\mathbf{P}^s}$ for $\mathcal{TW}_{g,n}$ is given by

$$E\mathcal{W}_{g,n}^{\mathbf{P}^s} : \ell_{\gamma_s^1} \in [c_*, \infty) \quad \cdots \quad \ell_{\gamma_s^Q} \in [c_*, \infty) \quad \tau_1 \in (-\infty, \infty) \cdots \tau_Q \in (-\infty, \infty). \quad (5.46)$$

In order to modify the local coordinates we must divide $E\mathcal{W}_{g,n}^{\mathbf{P}^s}$ into subregions $E\mathbf{W}_{g,n}^{\mathbf{P}^s, (m)}$, $m = 0, \dots, Q$. Divide the subregion $E\mathbf{W}_{g,n}^{\mathbf{P}^s, (m)}$ further into $\frac{Q!}{m!(Q-m)!}$ number of regions $E\mathbf{W}_{g,n}^{\mathbf{P}^s, \gamma_{j_1}^{i_1} \cdots \gamma_{j_m}^{i_m}}$, where $i_1, \dots, i_m \in \{1, \dots, Q\}$. The number $\frac{Q!}{m!(Q-m)!}$ counts the inequivalent ways of choosing m curves from the set $\{\gamma_s^1, \dots, \gamma_s^Q\}$. For surfaces that belong to the region $E\mathbf{W}_{g,n}^{\mathbf{P}^s, \gamma_s^{i_1} \cdots \gamma_s^{i_m}}$ with $m \neq 0$, we choose the local coordinates around the j^{th} puncture, up to a phase ambiguity, is given by

$$\tilde{w}_j^{i_1 \cdots i_m} = e^{\frac{c_*^2}{6} \sum_{k=1}^m f(\ell_{\gamma_{i_k}}) Y_{i_k j}^{\gamma_{i_1}^{i_1} \cdots \gamma_{i_m}^{i_m}}} e^{\frac{\pi^2}{c_*}} w_j, \quad (5.47)$$

where

$$Y_{i_k j}^{\gamma_{i_1} \dots \gamma_{i_m}} = \sum_{q=1}^2 \sum_{c_j^q, d_j^q} \pi^2 \frac{\epsilon(j, q)}{|c_j^q|^4}$$

$$c_j^q > 0 \quad d_j^q \bmod c_j^q \quad \begin{pmatrix} * & * \\ c_j^q & d_j^q \end{pmatrix} \in (\sigma_i^q)^{-1} \Gamma_{\gamma_{i_1} \dots \gamma_{i_m}}^{j q} \sigma_j \quad (5.48)$$

Here, $\Gamma_{\gamma_{i_1} \dots \gamma_{i_m}}^{j q}$ denotes the Fuchsian group for the component Riemann surface obtained from $\mathcal{R}$ by degenerating the curves $\gamma_{i_1}, \dots, \gamma_{i_m}$ carrying the j^{th} puncture and the puncture denoted by the index q which is obtained by degenerating the curve γ_{i_k} . The transformation σ_j^{-1} maps the cusp corresponding to the j^{th} cusp to ∞ and $(\sigma_j^q)^{-1}$ maps the puncture denoted by the index q obtained by degenerating the curve γ_{i_k} to ∞ . The factor $\epsilon(j, q)$ is one if both the j^{th} puncture and the puncture denoted by the index q belong to the same component surface, other wise $\epsilon(j, q)$ is zero.

Then the corrected interaction vertex $\tilde{\mathbf{S}}_{g,n}$ is given by

$$\tilde{\mathbf{S}}_{g,n} = \sum_s \sum_{m=0}^Q \sum_{\{i_1, \dots, i_m\}} \int_{E \mathbf{W}_{g,n}^{\mathbf{P}^s, \gamma_1^{i_1} \dots \gamma_s^{i_m}}} \frac{\prod_{j=1}^Q d\ell_{\gamma_j^s} d\tau_{\gamma_j^s}}{(2\pi i)^Q} \mathcal{G}_s$$

$$\times \langle \mathcal{R} | b(\mathbf{t}_{\gamma_1^s}) b(\mathbf{l}_{\gamma_1^s}) \dots b(\mathbf{t}_{\gamma_Q^s}) b(\mathbf{l}_{\gamma_Q^s}) | V_1 \rangle_{\tilde{w}_1^{\gamma_1^{i_1} \dots \gamma_s^{i_m}}} \otimes \dots \otimes | V_n \rangle_{\tilde{w}_n^{\gamma_1^{i_1} \dots \gamma_s^{i_m}}}, \quad (5.49)$$

where the sum over the sets $\{i_1, \dots, i_m\}$ denotes the sum of $\frac{Q!}{m!(Q-m)!}$ inequivalent ways of choosing m curves from the set $\{\gamma_1^1, \dots, \gamma_s^Q\}$. The expression for corrected interaction vertex $\tilde{\mathbf{S}}_{g,n}$ is true for any values of g and n such that $3g - 3 + n \geq 0$, and the closed string field theory master action constructed using these corrected interaction vertices will have approximate gauge invariance.

6 Discussion

In this paper we completed the construction of quantum closed string field theory with approximate gauge invariance by exploring the hyperbolic geometry of Riemann surfaces initiated in [6]. In [6] it was shown that although the string vertices constructed using Riemann surfaces with local coordinates induced from hyperbolic Riemann surfaces -1 constant curvature fails to provide gauge invariant quantum closed string field theory, the corrected string vertices obtained by modifying these local coordinates on Riemann surfaces belongs to the boundary region of string vertices give rise to quantum closed string field theory with approximate gauge invariance. Unfortunately, due to the complicated action of mapping class group on the Fenchel-Nielsen coordinates the implementing the suggested modification seemed to be impractical. However, in this paper we argued that by using the non-trivial identities satisfied by the lengths of simple closed geodesics on hyperbolic Riemann surfaces it is indeed possible to implement the modifications in very convenient fashion. The identities that we explored in this paper are due to McShane and Mirzakhani [38, 39]. Although they are very convenient to use, they have a very important

drawback. They are applicable only for the case of hyperbolic Riemann surfaces with at least one border or puncture. For instance, for computing the contributions from vacuum graphs to the string field theory action, we can not use them. Interestingly, there exists another class of such identities due to Luo and Tan [52–59] that are applicable for kinds of hyperbolic Riemann surfaces with no elliptic fixed points. For a quick introduction read appendix C. But they have one disadvantage, the functions involved in these identities are significantly more complicated than the functions appearing in the identities due to McShane and Mirzakhani.

There are many interesting direction that deserve further study. It would be very useful to check whether it is possible to construct the string vertices in closed superstring field theory that avoids the occurrence of any unphysical singularities due to the picture changing operators by exploring hyperbolic geometry. It might be worth exploring hyperbolic geometry of super Riemann surfaces to construct closed superstring field theory using the supergeometric formulation of superstring theory. This is particularly interesting due to the fact that there exist generalization of McShane-Mirzakhani identities for the case of super Riemann surfaces [67]. Another interesting direction is to use the formalism discussed in this paper to systematically compute the field theory limit of string amplitudes. We hope to report on this in the near future.

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A Brief review of hyperbolic geometry

A Riemann surface is a one-dimensional complex manifold. The map carrying the structure of the complex plane to the Riemann surface is called a chart. The transition maps between two overlapping charts are required to be holomorphic. The charts together with the transition functions between them define a complex structure on the Riemann surface [35]. The topological classification of the compact Riemann surfaces is done with the pair (g, n) , where g denotes the genus and n denotes the number of boundaries of the Riemann surface. However, within each topological class determined by (g, n) , there are different surfaces endowed with different complex structures. Two such complex structure are equivalent if there is a conformal (i.e complex analytic) map between them. The set of all such equivalent complex structures define a *conformal class* in the set of all complex structures on the surface.

The uniformization theorem [50] asserts that for a Riemann surface $\mathcal{R}$:

$$\left\{ \begin{array}{c} \text{Hyperbolic Structure} \\ \text{on } \mathcal{R} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Complex Structure} \\ \text{on } \mathcal{R} \end{array} \right\} \quad (\text{A.1})$$

Therefore, the space of all conformal classes is the same as the classification space of all inequivalent hyperbolic structures. A hyperbolic structure on a Riemann surface $\mathcal{S}$ is a diffeomorphism $\phi : \mathcal{S} \rightarrow \mathcal{R}$, where $\mathcal{R}$ is a surface with finite-area hyperbolic metric and geodesic boundary components. Here, hyperbolic metric on a Riemann surface refers to the metric having constant curvature -1 all over the surface. Two hyperbolic structures on a Riemann surface $\mathcal{S}$, given by $\phi_1 : \mathcal{S} \rightarrow \mathcal{R}_1$ and $\phi_2 : \mathcal{S} \rightarrow \mathcal{R}_2$, are equivalent if there is an isometry $I : \mathcal{R}_1 \rightarrow \mathcal{R}_2$ such that the maps $I \circ \phi_1 : \mathcal{S} \rightarrow \mathcal{R}_1$ and $\phi_2 : \mathcal{S} \rightarrow \mathcal{R}_2$ are homotopic i.e. the map $I \circ \phi_1$ can be continuously deformed into ϕ_2 by a homotopy map.

The classification space of the homotopy classes of the hyperbolic structures on a Riemann surface of a given topological type i.e. the space of the hyperbolic structures up to the homotopies, is called *the Teichmüller space of Riemann surfaces* of a given topological type (g, n) and it is denoted by $\mathcal{T}_{g,n}(\mathcal{R})$ [49]:

$$\mathcal{T}_{g,n}(\mathcal{R}) = \{\text{hyperbolic structures on } \mathcal{R}\} / \text{homotopy}. \quad (\text{A.2})$$

A.1 Hyperbolic Riemann surfaces and the Fuchsian uniformization

A hyperbolic Riemann surface is a Riemann surface with a metric having constant curvature -1 defined on it. According to the uniformization theorem, any Riemann surface with negative Euler characteristic can be made hyperbolic. The hyperbolic Riemann surface $\mathcal{R}$ can be represented as a quotient of the upper half-plane $\mathbb{H}$ by a Fuchsian group Γ , which is a subgroup of the isometries of $\mathbb{H}$ endowed with hyperbolic metric on it. The hyperbolic metric on the upper half-plane

$$\mathbb{H} = \{z : \text{Im } z > 0\}, \quad (\text{A.3})$$

is given by

$$ds_{\text{hyp}}^2 = \frac{dzd\bar{z}}{(\text{Im}z)^2}. \quad (\text{A.4})$$

Each hyperbolic Riemann surface $\mathcal{R}$ inherits, by projection from $\mathbb{H}$, its own hyperbolic geometry.

The isometries of the hyperbolic metric on $\mathbb{H}$ form the $\text{PSL}(2, \mathbb{R})$ group:

$$z \rightarrow \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{R}, \quad ad - bc = 1. \quad (\text{A.5})$$

The hyperbolic distance, $\rho(z, w)$, between two points z and w is defined to be the length of the geodesic in $\mathbb{H}$ that join z to w . A geodesic on the hyperbolic upper half-plane is a segment of the half-circle with origin on the real axis or the line segment parallel to the imaginary axis which passes through z and w that is orthogonal to the boundary of $\mathbb{H}$. This distance is given by

$$\rho(z, w) = \ln \left(\frac{1 + \tau(z, w)}{1 - \tau(z, w)} \right) = 2 \tanh^{-1} (\tau(z, w)), \quad (\text{A.6})$$

where $\tau(z, w) = \left| \frac{z-w}{\bar{z}-w} \right|$. An open set $\mathcal{F}$ of the upper half plane $\mathbb{H}$ satisfies the following three conditions is called the fundamental domain for Γ in $\mathbb{H}$:

1. $\gamma(\mathcal{F}) \cap \mathcal{F} = \emptyset$ for every $\gamma \in \Gamma$ with $\gamma \neq id$.

Figure 3. The fundamental domain of Fuchsian uniformization corresponding to the genus 2 surface. The boundary curves of the fundamental domain with red colour represent the a -cycles and those with blue colours represent the b -cycles of the Riemann surface.

2. If $\bar{\mathcal{F}}$ is the closure of $\mathcal{F}$ in $\mathbb{H}$, then $\mathbb{H} = \bigcup_{\gamma \in \Gamma} \gamma(\bar{\mathcal{F}})$.
3. The relative boundary $\partial\mathcal{F}$ of $\mathcal{F}$ in $\mathbb{H}$ has measure zero with respect to the two dimensional Lebesgue measure.

The Riemann surface $\mathcal{R} = \mathbb{H}/\Gamma$ is represented as $\bar{\mathcal{F}}$ with points in $\partial\mathcal{F}$ identified under the action of elements in the group Γ . For instance, a fundamental domain of Fuchsian uniformization corresponding to the genus 2 surface is as shown in figure 3. Let $\pi : \mathbb{H} \rightarrow \mathcal{R}$ be the map that projects $\mathbb{H}$ onto $\mathcal{R} = \mathbb{H}/\Gamma$. Since ds_{hyp}^2 is invariant under the action of elements in Γ , we obtain a Riemannian metric $ds_{\mathcal{R}}^2$ on $\mathcal{R}$. The metric $ds_{\mathcal{R}}^2$ is the hyperbolic metric on $\mathcal{R}$. Moreover, every $\gamma \in \Gamma$ corresponds to an element $[C_\gamma]$ of the fundamental group $\pi_1(\mathcal{R})$ of $\mathcal{R}$. In particular, γ determines the free homotopy class of C_γ , where C_γ is a representative of the class $[C_\gamma]$. For hyperbolic element $\gamma \in \Gamma$, i.e. $(\text{tr}(\gamma))^2 > 4$, the closed curve $L_\gamma = A_\gamma/\langle\gamma\rangle$, the image on $\mathcal{R}$ of the axis A_γ by π , is the unique geodesic with respect to the hyperbolic metric on $\mathcal{R}$ belonging to the same homotopy class of C_γ . The axis of a hyperbolic element γ is the geodesic on $\mathbb{H}$ that connects the fixed points of γ . Let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{R}, \quad ad - bc = 1, \quad (\text{A.7})$$

be a hyperbolic element of Γ and L_γ be the closed geodesic corresponding to γ . Then the hyperbolic length $l(L_\gamma)$ of L_γ satisfies following relation

$$\text{tr}^2(\gamma) = (a + d)^2 = 4\cosh^2\left(\frac{l(L_\gamma)}{2}\right). \quad (\text{A.8})$$

A.2 The Fenchel-Nielsen coordinates for the Teichmüller space

Let $\mathcal{R}$ be a hyperbolic Riemann surface. Consider cutting $\mathcal{R}$ along mutually disjoint simple closed geodesics with respect to the hyperbolic metric $ds_{\mathcal{R}}^2$ on $\mathcal{R}$. If there are no more closed geodesics of $\mathcal{R}$ contained in the remaining open set that are non-homotopic to the boundaries of the open set, then every piece should be a pair of pants of $\mathcal{R}$. The

complex structure of each pair of pants on $\mathcal{R}$ is uniquely determined by a triple of the hyperbolic lengths of boundary geodesics of it. To see this, consider a pair of pants P with boundary components L_1, L_2 and L_3 . Assume that Γ_P is the Fuchsian group associated with P . Then Γ_P is a free group generated by two hyperbolic transformations γ_1 and γ_2 . The action of γ_1 and γ_2 on $\mathbb{H}$ is given by

$$\begin{aligned}\gamma_1(z) &= \lambda^2 z, & 0 < \lambda < 1, \\ \gamma_2(z) &= \frac{az + b}{cz + d}, & ad - bc = 1 \quad 0 < c.\end{aligned}\tag{A.9}$$

We assume that 1 is the attractive fixed point of γ_2 , or equivalently, $a + b = c + d$, $0 < -\frac{b}{c} < 1$. We also assume that

$$(\gamma_3)^{-1}(z) = \gamma_2 \circ \gamma_1(z) = \frac{\tilde{a}z + \tilde{b}}{\tilde{c}z + \tilde{d}}, \quad \tilde{a}\tilde{d} - \tilde{b}\tilde{c} = 1.\tag{A.10}$$

Then using (A.8) we obtain the following relations

$$\begin{aligned}\left(\lambda + \frac{1}{\lambda}\right)^2 &= 4\cosh^2\left(\frac{a_1}{2}\right), \\ (a + d)^2 &= 4\cosh^2\left(\frac{a_2}{2}\right), \\ (\tilde{a} + \tilde{d})^2 &= 4\cosh^2\left(\frac{a_3}{2}\right).\end{aligned}\tag{A.11}$$

This means that the action of the generators of the group Γ_P , γ_1 and γ_2 are uniquely determined by the hyperbolic lengths of the boundary components.

The basic idea behind the Fenchel-Nielsen construction is to decompose $\mathcal{R}$, a genus g hyperbolic Riemann surface with n boundaries $B_1, \dots, B_n$ with fixed lengths, into pairs of pants using $3g - 3 + n$ simple closed curves (See figure 4). Then, there are $3g - 3 + n$ length parameters that determine the hyperbolic structure on each pair of pants, and there are $3g - 3 + n$ twist parameters that determine how the pairs of pants are glued together. Taken together, these $6g - 6 + 2n$ coordinates are the Fenchel-Nielsen coordinates [51]. Below we describe this more precisely.

In order to define the Fenchel-Nielsen coordinates, we must first choose a *coordinate system of curves* on $\mathcal{R}$ [51]. For this, we need to choose:

- a pants decomposition $\{\gamma_1, \dots, \gamma_{3g-3+n}\}$ of oriented simple closed geodesics (boundary curves are not included) and
- a set $\{\beta_1, \dots, \beta_m\}$ of *seams*; that is, a collection of disjoint simple closed geodesics in $\mathcal{R}$ so that the intersection of the union $\cup \beta_i$ with any pair of pants P determined by the $\{\gamma_j\}$ is a union of three disjoint arcs connecting the boundary components of P pairwise. See figure 4.

Fix once and for all a coordinate system of curves on $\mathcal{R}$. The $3g - 3 + n$ number of length parameters of a point $\tilde{\mathcal{R}} \in \mathcal{T}_{g,n}(\mathcal{R})$ are defined to be the ordered $(3g - 3 + n)$ -tuple of positive real numbers

$$\left(\ell_1(\tilde{\mathcal{R}}), \dots, \ell_{3g-3+n}(\tilde{\mathcal{R}})\right),\tag{A.12}$$

Figure 4. A pairs of pants decomposition of a genus 2 Riemann surface with three boundaries. The gray dashed curves provide a choice of seams.

Figure 5. The effect of the twisting on geodesic arcs. If the twist parameter was zero, the geodesic arc at the end would be the union of the two geodesic arcs from the original pairs of pants.

where $\ell_i(\tilde{\mathcal{R}})$ is the hyperbolic length of γ_i in $\tilde{\mathcal{R}}$. The length parameters determine the hyperbolic structure of the $2g - 2 + n$ number of pairs of pants obtained by cutting $\mathcal{R}$ along the curves $\{\gamma_1, \dots, \gamma_{3g-3+n}\}$. The information about how these pants are glued together is recorded in the twist parameters $\tau_i(\tilde{\mathcal{R}})$.

In order to understand the information recorded in the twist parameter, let us consider two hyperbolic pairs of pants with three geodesic boundaries. If these pairs of pants have boundary components of the same length, then we can glue them together to obtain a compact hyperbolic surface $\mathcal{R}_{0,4}$ of genus zero with four boundary components. The hyperbolic structure of $\mathcal{R}_{0,4}$ depends on how much we rotate the pairs of pants before gluing. For instance, the shortest arc connecting two boundary components of $\mathcal{R}_{0,4}$ changes as we change the gluing instruction, (see figure 5). Thus we have a circle's worth of choices for the hyperbolic structure of $\mathcal{R}_{0,4}$. Therefore, the twist parameters we define on the Teichmüller space will be real numbers, but modulo 2π , they are simply recording the angles at which we glue pairs of pants. Below we explain the precise definition of the twist parameters.

Assume that β is an arc in a hyperbolic pair of pants P connecting the boundary components γ_1 and γ_2 of P . Let δ be the unique shortest arc connecting γ_1 and γ_2 . Let $\mathcal{N}_1$

Figure 6. Modifying an arc on a pair of pants so that it agrees with a perpendicular arc except near its endpoints.

Figure 7. The hyperbolic surfaces $\mathcal{R}_1$ and $\mathcal{R}_2$ are not the same point of $\mathcal{T}_{g,n}(\mathcal{R})$ due to the difference in the pants decomposition associated with them. However, both represents the same hyperbolic Riemann surface.

and $\mathcal{N}_2$ be the neighbourhoods of γ_1 and γ_2 . Modify β by isotopy, leaving the endpoints fixed, so that it agrees with δ outside of $\mathcal{N}_1 \cup \mathcal{N}_2$; see figure 6. The *twisting number* of β at γ_1 is the signed horizontal displacement of the endpoints $\beta \cap \partial\mathcal{N}_1$. The orientation of γ_1 determines the sign. Similarly, the twisting number of β at γ_2 is the signed horizontal displacement of the endpoints $\beta \cap \partial\mathcal{N}_2$.

Then we define the i^{th} twist parameter $\tau_i(\tilde{\mathcal{R}})$ of a given Riemann surface $\tilde{\mathcal{R}} \in \mathcal{T}_{g,n}(\mathcal{R})$ as follows. Assume that β_j is one of the two seams that cross γ_i . On each side of γ_i there is a pair of pants, and β_j gives an arc in each of these. Let t_L and t_R be the twisting numbers of each of these arcs on the left and right sides of γ_i , respectively. The i^{th} twist parameter of $\tilde{\mathcal{R}}$ is defined to be

$$\tau_i(\tilde{\mathcal{R}}) = 2\pi \frac{t_L - t_R}{\ell_i(\tilde{\mathcal{R}})}. \quad (\text{A.13})$$

Note that, there were two choices of seams β_j . It is possible to show that the twist parameters computed from the two seams are the same [51].

A.3 Fundamental domain for the MCG inside the Teichmüller space

Let $\mathcal{S}$ be a bordered Riemann surface. Denote the group of orientation-preserving diffeomorphisms of $\mathcal{S}$ that restrict to the identity on $\partial\mathcal{S}$ by $\text{Diff}^+(\mathcal{S}, \partial\mathcal{S})$. Then, the *mapping class group* (MCG) of $\mathcal{S}$, denoted by $\text{Mod}(\mathcal{S})$, is the group

$$\text{MCG}(\mathcal{S}) = \text{Diff}^+(\mathcal{S}, \partial\mathcal{S}) / \text{Diff}_0(\mathcal{S}, \partial\mathcal{S}), \quad (\text{A.14})$$

Figure 8. The S and the A moves.

where $\text{Diff}_0(\mathcal{S}, \partial\mathcal{S})$ denotes the components of $\text{Diff}^+(\mathcal{S}, \partial\mathcal{S})$ that can be continuously connected to the identity. The *moduli space* of hyperbolic surfaces homomorphic to $\mathcal{S}$ is defined to be the quotient space

$$\mathcal{M}(\mathcal{S}) = \mathcal{T}(\mathcal{S})/\text{MCG}(\mathcal{S}), \quad (\text{A.15})$$

where $\mathcal{T}(\mathcal{S})$ is the Teichmüller space of hyperbolic Riemann surfaces homomorphic to $\mathcal{S}$ with a specific pants decomposition associated with it. The elements in the group $\text{Mod}(\mathcal{S})$ act on $\mathcal{T}(\mathcal{S})$ simply by changing the pants decomposition, see figure 7.

A presentation of the mapping class groups in terms of Dehn twists is given in [36]. A pants decomposition of $\mathcal{R}$, a hyperbolic Riemann surface, decomposes $\mathcal{R}$ into a set of pairs of pants. If $\mathcal{R}$ is not itself a pair of pants, then there are infinitely many different isotopy classes of pants decompositions of $\mathcal{R}$. Interestingly, any two isotopy classes of pants decompositions can be joined by a finite sequence of elementary moves in which only one closed curve changes at a time [37]; (see figure 8). The different types of elementary moves are as follows:

- Let $\mathcal{P}$ be a pants decomposition, and assume that one of the simple closed curve γ of $\mathcal{P}$ is such that deleting γ from $\mathcal{R}$ produces a one holed torus as a complementary component. This is equivalent to saying that there is a simple closed curve β in $\mathcal{R}$ which intersects γ in one point transversely and is disjoint from all other circles in $\mathcal{P}$. In this case, replacing γ by β in $\mathcal{P}$ produces a new pants decomposition $\mathcal{P}'$. This replacement is known as a simple move or S-move.
- If $\mathcal{P}$ contains a simple closed curve γ such that deleting γ from $\mathcal{P}$ produces a four holed sphere as a complementary component, then we obtain a new pants decomposition $\mathcal{P}'$ by replacing γ with a simple closed curve β which intersects γ transversely in two points and is disjoint from the other curves of $\mathcal{P}$. The transformation $\mathcal{P} \rightarrow \mathcal{P}'$ in this case is called an associative move or A-move.

The inverse of an S-move is again an S-move, and the inverse of an A-move is again an A-move. Unfortunately, the presentation so obtained is rather complicated, and stands in need of considerable simplification before much light can be shed on the structure of the MCG. As a result, *in the generic situation, it seems hopeless to obtain the explicit description of a fundamental domain for the action of the MCG on the Teichmüller space* [54–59].

Figure 9. The complete geodesics that are disjoint from B_2, B_3 and orthogonal to B_1 .

B The Mirzakhani-McShane identity

Before stating the Mirzakhani-McShane identity, let us discuss some aspects of infinite simple geodesic rays on a hyperbolic pair of pants. Consider $P(x_1, x_2, x_3)$, the unique hyperbolic pair of pants with the geodesic boundary curves (B_1, B_2, B_3) such that

$$l_{B_i}(P) = x_i \quad i = 1, 2, 3.$$

$P(x_1, x_2, x_3)$ is constructed by pasting two copies of the (unique) right hexagons along the three edges. The edges of the right hexagons are the geodesics that meet perpendicularly with the non-adjacent sides of length $\frac{x_1}{2}, \frac{x_2}{2}$ and $\frac{x_3}{2}$. Consequently, $P(x_1, x_2, x_3)$ admits a reflection involution symmetry $\mathcal{J}$ which interchanges the two hexagons. Such a hyperbolic pair of pants has five complete geodesics disjoint from B_1, B_2 and B_3 . Two of these geodesics meet the border B_1 , say at positions z_1 and z_2 , and spiral around the border B_2 , see figure 9. Similarly, the other two geodesics meet the border B_1 , say at y_1 and y_2 , and spiral around the border B_3 . The simple geodesic that emanates perpendicularly from the border B_1 to itself meet the border B_1 perpendicularly at two points, say at w_1 and w_2 , is the fifth complete geodesic. The involution symmetry $\mathcal{J}$ relates the two points w_1 and w_2 , i.e. $\mathcal{J}(w_1) = w_2$. Likewise, $\mathcal{J}(z_1) = z_2$ and $\mathcal{J}(y_1) = y_2$. The geodesic length of the interval between z_1 and z_2 along the boundary β_1 containing w_1 and w_2 is given by [39]

$$\mathcal{D}(x_1, x_2, x_3) = x_1 - \ln \left(\frac{\cosh(\frac{x_2}{2}) + \cosh(\frac{x_1+x_3}{2})}{\cosh(\frac{x_2}{2}) + \cosh(\frac{x_1-x_3}{2})} \right). \quad (\text{B.1})$$

Twice of the geodesic distance between the two geodesics that are perpendicular to the boundary B_1 and spiralling around the boundary B_2 and the boundary B_3 is given by [39]

$$\mathcal{E}(x_1, x_2, x_3) = 2 \ln \left(\frac{e^{\frac{x_1}{2}} + e^{\frac{x_2+x_3}{2}}}{e^{-\frac{x_1}{2}} + e^{\frac{x_2+x_3}{2}}} \right). \quad (\text{B.2})$$

We say that three isotopy classes of the connected simple closed curves $(\alpha_1, \alpha_2, \alpha_3)$ on $\mathcal{R}$, a genus g hyperbolic Riemann surface with n borders $\mathbf{L} = (L_1, \dots, L_n)$ having lengths

Figure 10. Scissoring the surface along the curve $\alpha_1 + \alpha_2$ produces a pair of pants, a genus 1 surface with 3 borders and a genus 1 surface with 2 borders.

Figure 11. Scissoring the surface along the curve $\alpha_1 + \alpha_2$ produces a pair of pants and genus 1 surface with 5 borders.

$(l_1, \dots, l_n)$, bound a pair of pants if there exists an embedded pair of pants $P \subset \mathcal{R}$ such that $\partial P = \{\alpha_1, \alpha_2, \alpha_3\}$. The boundary curves can have vanishing lengths, which turns a boundary to a puncture. Following definitions are useful:

- For $1 \leq i \leq n$, we define $\mathcal{F}_i$ be the set of unordered pairs of isotopy classes of the simple closed curves $\{\alpha_1, \alpha_2\}$ bounding a pairs of pants with L_i such that $\alpha_1, \alpha_2 \notin \partial(\mathcal{R})$ (see figures 10 and 11).
- For $1 \leq i \neq j \leq n$, we define $\mathcal{F}_{i,j}$ be the set of isotopy classes of the simple closed curves γ bounding a pairs of pants containing the borders L_i and L_j (see figure 12).

Let us state the Mirzakhani-McShane identity for hyperbolic bordered surfaces. For any genus g hyperbolic Riemann surface $\mathcal{R} \in \mathcal{T}_{g,n}(l_1, \dots, l_n)$ with n borders $L_1, \dots, L_n$ having lengths $l_1, \dots, l_n$, satisfying $3g - 3 + n > 0$, we have

$$\sum_{\{\alpha_1, \alpha_2\} \in \mathcal{F}_1} \mathcal{D}(l_1, l_{\alpha_1}(\mathcal{R}), l_{\alpha_2}(\mathcal{R})) + \sum_{i=2}^n \sum_{\gamma \in \mathcal{F}_{1,i}} \mathcal{E}(l_1, l_i, l_\gamma(\mathcal{R})) = l_1. \quad (\text{B.3})$$

We use this identity write down a decomposition of unity:

$$1 = \sum_k \sum_{\{\alpha_1^k, \alpha_2^k\} \in \mathcal{F}_1^k} \frac{\mathcal{D}(l_1, l_{\alpha_1^k}(\mathcal{R}), l_{\alpha_2^k}(\mathcal{R}))}{l_1} + \sum_{i=2}^n \sum_{\gamma \in \mathcal{F}_{1,i}} \frac{\mathcal{E}(l_1, l_i, l_\gamma(\mathcal{R}))}{l_1}. \quad (\text{B.4})$$

Figure 12. Scissoring the surface along the curve γ_2 produces a pair of pants and a genus 2 surface with 3 borders.

Figure 13. The pair of pants P_1 with boundaries $\partial P_1 = \{C_1, B_1, B_2\}$ is a properly embedded pair of pants on the torus with two borders B_1 and B_2 . The pair of pants P_2 with boundaries $\partial P_2 = \{C_1, C_2, C_2\}$ is a quasi properly embedded pair of pants on the torus.

Here, we decomposed the summation over each simple closed geodesics that are disjoint from the boundaries into a sum over a set of simple closed geodesic that are related each other by the action of MCG and a sum over a discrete variable which differentiate the class of simple closed geodesics that are not related by the action of MCG.

C The Luo-Tan dilogarithm identity

Like the Mirzakhani-McShane identity (B.4) the Luo-Tan identity for simple closed geodesics on hyperbolic Riemann surfaces with or without borders [52–59] also provides a decomposition of unity. Although the functions appear in the Mirzakhani-McShane identities are significantly more simpler than those appear in the Luo-Tan identities, the latter have the following advantage that unlike the Mirzakhani-McShane identities the Luo-Tan identities are also valid for Riemann surfaces with no borders.

C.1 Properly and quasi-properly embedded geometric surfaces

Consider $\mathcal{R}_{g,n}^r$, a genus g hyperbolic Riemann surface with n geodesic boundary components and r cusps (punctures). We say that a compact embedded subsurface $\mathcal{R}_{g_1,n_1}^{r_1} \subset \mathcal{R}_{g,n}^r$ with

$g_1 \leq g, n_1 \leq n, r_1 \leq r$ is *geometric*, if the boundaries of $\mathcal{R}_{g_1, n_1}^{r_1}$ are geodesics. We say $\mathcal{R}_{g_1, n_1}^{r_1}$ *proper*, if the inclusion map

$$\iota : \mathcal{R}_{g_1, n_1}^{r_1} \rightarrow \mathcal{R}_{g, n}^r, \quad (\text{C.1})$$

is injective. Therefore, the subsurface is $\mathcal{R}_{g_1, n_1}^{r_1}$ is said to be a *properly embedded geometric surface*, if its boundaries are geodesics on $\mathcal{R}_{g, n}^r$ and the inclusion map is one-to one.

For example, each pair of pants in the pants decomposition of a genus zero hyperbolic Riemann surface using $n - 3$ non-homotopic disjoint simple closed geodesics is a properly embedded geometric surface. However, it is impossible to obtain the pants decomposition of genus 1 hyperbolic Riemann surface with one border using properly embedded geometric pair of pants. This is due to the following fact. We obtain a torus with one border from a hyperbolic pair of pants P by identifying the two boundaries of it. The identification of the boundary makes the inclusion map

$$\iota : P \rightarrow \mathcal{R}_{1, 1}^0, \quad (\text{C.2})$$

non-injective. Therefore, the only properly embedded hyperbolic surface inside a hyperbolic torus with one border is the surface itself.

We define a *quasi-embedded geometric* pair of pants P in $\mathcal{R}_{g, n}^r$ to be an immersion P into $\mathcal{R}_{g, n}^r$ which is injective on the interior $\text{int}(P)$ of P such that the boundaries are mapped to geodesics, but two of the boundaries are mapped to the same geodesic. Hence, a quasi-embedded geometric pair of pants is contained in a unique embedded geometric 1-holed torus. Conversely, every embedded geometric 1-holed torus together with a non-trivial simple closed geodesic on the torus that is not parallel to its boundary geodesic determines a quasi-embedded geometric pair of pants (see figure 13).

C.2 The Roger's dilogarithm functions

The dilogarithm function Li_2 is defined for $z \in \mathbb{C}$ with $|z| < 1$ by the following Taylor series

$$\text{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}. \quad (\text{C.3})$$

It is straightforward to verify that, for $x \in \mathbb{R}$ with $|x| < 1$, the dilogarithm function has the following expression

$$\text{Li}_2(x) = - \int_0^x \frac{\ln(1-z)}{z} dz. \quad (\text{C.4})$$

The Roger's dilogarithm function $\mathcal{L}$ is defined by

$$\mathcal{L}(x) \equiv \text{Li}_2(x) + \frac{1}{2} \ln(|x|) \ln(1-x). \quad (\text{C.5})$$

The Roger's $\mathcal{L}$ -function has the following special values

$$\begin{aligned}\mathcal{L}(0) &= 0, \\ \mathcal{L}\left(\frac{1}{2}\right) &= \frac{\pi^2}{12}, \\ \mathcal{L}(1) &= \frac{\pi^2}{6}, \\ \mathcal{L}(-1) &= -\frac{\pi^2}{12}.\end{aligned}\tag{C.6}$$

The first derivative of the Roger's $\mathcal{L}$ -function has the following simple form

$$\mathcal{L}'(z) = -\frac{1}{2} \left(\frac{\ln(1-z)}{z} + \frac{\ln(z)}{1-z} \right).\tag{C.7}$$

The $\mathcal{L}$ -function satisfies the following Euler relations

$$\mathcal{L}(x) + \mathcal{L}(1-x) = \frac{\pi^2}{6},\tag{C.8}$$

for $x \in (0, 1)$, and

$$\mathcal{L}(-x) + \mathcal{L}\left(-\frac{1}{x}\right) = 2\mathcal{L}(-1) = -\frac{\pi^2}{6},\tag{C.9}$$

for $x > 0$. It satisfies the Landen's identity given by

$$\mathcal{L}\left(\frac{x}{x-1}\right) = -\mathcal{L}(x),\tag{C.10}$$

for $x \in (0, 1)$. If we define $y \equiv \frac{x}{x-1}$, then from the Landen's identity, we get:

$$\lim_{y \rightarrow \infty} \mathcal{L}(y) = -\mathcal{L}(1) = -\frac{\pi^2}{6}.\tag{C.11}$$

It also satisfies the following pentagon relation, for $x, y \in [0, 1]$ and $xy \neq 1$

$$\mathcal{L}(x) + \mathcal{L}(y) + \mathcal{L}(1-xy) + \mathcal{L}\left(\frac{1-x}{1-xy}\right) + \mathcal{L}\left(\frac{1-y}{1-xy}\right) = \frac{\pi^2}{2}.\tag{C.12}$$

Finally, we define the Lasso function $La(x, y)$ in terms of the Roger's $\mathcal{L}$ -function as follows

$$La(x, y) \equiv \mathcal{L}(y) + \mathcal{L}\left(\frac{1-y}{1-xy}\right) - \mathcal{L}\left(\frac{1-x}{1-xy}\right).\tag{C.13}$$

C.3 The length invariants of a pair of pants and a 1-holed tori

Consider a hyperbolic pair of pants P with geodesic boundaries B_1, B_2, B_3 having lengths l_1, l_2, l_3 respectively. Let M_i be the shortest geodesic arc between B_j and B_k having length m_i , for $\{i, j, k\} = \{1, 2, 3\}$. Denote by D_i , the shortest non-trivial geodesic arc from B_i to itself having length p_i . The hyperbolic metric on the pair of pants makes the geodesic arcs M_i and D_i orthogonal to ∂P , the boundaries of the pair of pants P (see figure 14).

Figure 14. The geodesic arcs connecting the boundaries of the pair of pants.

By cutting along the geodesic arcs M_i , $i = 1, 2, 3$, we can decompose the pair of pants P into two right-angled hyperbolic hexagons with cyclicly ordered side-lengths $\left\{\frac{l_1}{2}, m_3, \frac{l_2}{2}, m_1, \frac{l_3}{2}, m_2\right\}$. Then, using the following sine and cosine rules for the right angled hexagons and pentagons

$$\begin{aligned} \frac{\sinh m_i}{\sinh\left(\frac{l_i}{2}\right)} &= \frac{\sinh m_j}{\sinh\left(\frac{l_j}{2}\right)} = \frac{\sinh m_k}{\sinh\left(\frac{l_k}{2}\right)}, \\ \cosh m_i \sinh\left(\frac{l_j}{2}\right) \sinh\left(\frac{l_k}{2}\right) &= \cosh\left(\frac{l_i}{2}\right) + \cosh\left(\frac{l_j}{2}\right) \cosh\left(\frac{l_k}{2}\right), \\ \cosh\left(\frac{p_k}{2}\right) &= \sinh\left(\frac{l_i}{2}\right) \sinh m_j. \end{aligned} \quad (\text{C.14})$$

We can express the lengths of the geodesics arcs $D_1, D_2, D_3, M_1, M_2, M_3$ in terms of the lengths of the boundary geodesics B_1, B_2, B_3 :

$$\begin{aligned} \cosh m_i &= \cosh\left(\frac{l_i}{2}\right) \operatorname{cosech}\left(\frac{l_j}{2}\right) \operatorname{cosech}\left(\frac{l_k}{2}\right) + \coth\left(\frac{l_j}{2}\right) \coth\left(\frac{l_k}{2}\right), \\ \cosh\left(\frac{p_k}{2}\right) &= \sinh\left(\frac{l_i}{2}\right) \sinh m_j. \end{aligned} \quad (\text{C.15})$$

Let us discuss the length invariants of T , a hyperbolic 1-holed torus with boundary component C . Consider an arbitrary closed simple geodesic A that is not parallel to the boundary of T . We obtain a hyperbolic pair of pants P_A with boundary geodesics C, A^+ and A^- by cutting T along the geodesic A , see figure 15. The pair of pants P_A is a quasi-properly embedded geometric surface. We denote the length of the geodesics C and A by c and a respectively. The length of the shortest geodesic M_A between C and A^+ in P_A (or A^-) by m_A . Finally, p_A denotes the length of the shortest non-trivial geodesic arc from C to C in P_A and q_A denotes the length of the shortest non-trivial geodesic arc from A^+ to A^- in P_A . Again, using the sine and cosine rules for the hexagons and the

Figure 15. The image of geodesic arcs on 1 holed torus connecting the boundaries of the pair of pants in the pair of pants obtained by cutting along the geodesic A .

pentagons correspond to the hyperbolic pair of pants P_A associated with T , we can express the lengths m_A, p_A and q_A of geodesic arcs in terms of the lengths c and a of boundary geodesics as follows

$$\begin{aligned} \cosh m_A &= \cosh\left(\frac{a}{2}\right) \operatorname{cosech}\left(\frac{a}{2}\right) \operatorname{cosech}\left(\frac{c}{2}\right) + \coth\left(\frac{a}{2}\right) \coth\left(\frac{c}{2}\right), \\ \cosh q_A &= \cosh\left(\frac{c}{2}\right) \operatorname{cosech}\left(\frac{a}{2}\right) \operatorname{cosech}\left(\frac{c}{2}\right) + \coth\left(\frac{a}{2}\right) \coth\left(\frac{a}{2}\right), \\ \cosh\left(\frac{p_A}{2}\right) &= \sinh\left(\frac{a}{2}\right) \sinh m_A. \end{aligned} \quad (\text{C.16})$$

C.4 The Luo-Tan identity

Consider $\mathcal{R}_{g,n}^0$, a hyperbolic Riemann surface of genus g with n geodesic boundary components and no cusps. On $\mathcal{R}_{g,n}^0$ there are two types of geometric pairs of pants: quasi-properly embedded geometric pairs of pants and three kinds of properly embedded geometric pairs of pants. Below, we list some useful functions associated with each type of geometric pairs of pants.

Properly embedded geometric pairs of pants.

1. Assume that P_3 is a properly embedded geometric pair of pants with no boundaries in common with that of the surface $\mathcal{R}_{g,n}^0$. Let the length invariants of P_3 be l_i, m_i and p_i for $i = 1, 2, 3$. We define the function $\mathcal{K}_3(P_3)$ in terms of the length invariants of P_3 as follows

$$\mathcal{K}_3(P_3) \equiv 4\pi^2 - 8 \left\{ \sum_{i=1}^3 \left(\mathcal{L}\left(\operatorname{sech}^2\left(\frac{m_i}{2}\right)\right) + \mathcal{L}\left(\operatorname{sech}^2\left(\frac{p_i}{2}\right)\right) \right) + \sum_{i \neq j} L_a\left(e^{-l_i}, \tanh^2\left(\frac{m_i}{2}\right)\right) \right\}. \quad (\text{C.17})$$

2. Assume that P_2 is a properly embedded geometric pair of pants with only one boundary in common with the boundaries of the surface $\mathcal{R}_{g,n}^0$. Let L_1 be the boundary that is also the boundary of $\mathcal{R}_{g,n}^0$ and the length invariants of P_2 be l_i, m_i and p_i

for $i = 1, 2, 3$. We define the function $\mathcal{K}_2(P_2)$ in terms of the length invariants of P_2 as follows

$$\mathcal{K}_2(P_2) \equiv \mathcal{K}_3(P_2) + 8 \left\{ \mathcal{L} \left(\text{sech}^2 \left(\frac{p_1}{2} \right) \right) + La \left(e^{-l_2}, \tanh^2 \left(\frac{m_3}{2} \right) \right) + La \left(e^{-l_3}, \tanh^2 \left(\frac{m_2}{2} \right) \right) \right\}. \quad (\text{C.18})$$

3. Assume that P_1 is a properly embedded geometric pair of pants with only two boundaries in common with the boundaries of the surface $\mathcal{R}_{g,n}^0$. Let L_1 and L_2 be the boundaries that are also the boundaries of $\mathcal{R}_{g,n}^0$ and the length invariants of P_1 be l_i, m_i and p_i for $i = 1, 2, 3$. We define the function $\mathcal{K}_1(P_1)$ in terms of the length invariants of P_1 as follows

$$\begin{aligned} \mathcal{K}_1(P_1) \equiv \mathcal{K}_3(P_1) + 8 \left\{ \mathcal{L} \left(\text{sech}^2 \left(\frac{p_1}{2} \right) \right) + \mathcal{L} \left(\text{sech}^2 \left(\frac{p_2}{2} \right) \right) + \mathcal{L} \left(\text{sech}^2 \left(\frac{m_3}{2} \right) \right) \right. \\ \left. + La \left(e^{-l_2}, \tanh^2 \left(\frac{m_3}{2} \right) \right) + La \left(e^{-l_3}, \tanh^2 \left(\frac{m_2}{2} \right) \right) \right. \\ \left. + La \left(e^{-l_3}, \tanh^2 \left(\frac{m_1}{2} \right) \right) + La \left(e^{-l_1}, \tanh^2 \left(\frac{m_3}{2} \right) \right) \right\}. \quad (\text{C.19}) \end{aligned}$$

Quasi-properly embedded geometric pairs of pants. Assume that P_T is a quasi-properly embedded geometric pair of pants in a 1-holed torus T which is a properly geometric embedded surface on $\mathcal{R}_{g,n}^0$. Let C is the boundary of T and A be the simple closed geodesic along which we cut T to obtain P_T . Assume that the length invariants of T be c, a, m_A, q_A and p_A as explained in the previous subsection. We define the function $\mathcal{K}_T(P_T)$ in terms of the length invariants of P_T as follows

$$\begin{aligned} \mathcal{K}_T(P_T) \equiv 8 \left\{ \mathcal{L} \left(\tanh^2 \left(\frac{q_A}{2} \right) \right) + 2\mathcal{L} \left(\tanh^2 \left(\frac{m_A}{2} \right) \right) - \mathcal{L} \left(\text{sech}^2 \left(\frac{p_A}{2} \right) \right) \right. \\ \left. - 2La \left(e^{-a}, \tanh^2 \left(\frac{m_A}{2} \right) \right) - 2La \left(e^{-\frac{c}{2}}, \tanh^2 \left(\frac{m_A}{2} \right) \right) \right\}. \quad (\text{C.20}) \end{aligned}$$

Now, we are in a position to state the Luo-Tan dilogarithm identity.

The Luo-Tan identity. Let $\mathcal{R}_{g,n}^0$ be a hyperbolic Riemann surface with n boundaries. Then the functions $\mathcal{K}_1(P_1), \mathcal{K}_2(P_2), \mathcal{K}_3(P_3)$ and $\mathcal{K}_T(P_T)$ satisfies the identity given by

$$\sum_{P_1} \mathcal{K}_1(P_1) + \sum_{P_2} \mathcal{K}_2(P_2) + \sum_{P_3} \mathcal{K}_3(P_3) + \sum_{P_T} \mathcal{K}_T(P_T) = 4\pi^2(2g - 2 + n), \quad (\text{C.21})$$

where the first sum is over all properly embedded geometric pairs of pants sum is over all properly embedded geometric pair of pants $P_1 \subset \mathcal{R}_{g,n}^0$ with exactly two boundary component in $\partial\mathcal{R}_{g,n}^0$, the second sum is over all properly embedded geometric pairs of pants $P_2 \subset \mathcal{R}_{g,n}^0$ with exactly one boundary component in $\partial\mathcal{R}_{g,n}^0$, the third sum is over all properly embedded geometric pairs of pants $P_3 \subset \mathcal{R}_{g,n}^0$ such that $\partial P_3 \cap \partial\mathcal{R}_{g,n}^0 = \emptyset$, and the fourth sum is over all quasi-properly embedded geometric pairs of pants $P_T \subset \mathcal{R}_{g,n}^0$. Moreover, if the lengths of r boundary components of $\mathcal{R}_{g,n}^0$ tends to zero, then, we obtain the identity for the hyperbolic surfaces $\mathcal{R}_{g,n-r}^r$ of genus g with $n - r$ geodesic boundary and r cusps.

Let us elaborate on the Luo-Tan identity. A properly embedded geometric pair of pants inside $\mathcal{R}_{g,n}^0$ can be understood as a set of three simple closed geodesics on $\mathcal{R}_{g,n}^0$

that bound P . Therefore, we can replace the sum over all properly embedded geometric pair of pants P_1 with the sum over all tuple of simple closed geodesics $\{L_i, L_j, A_{i,j}\}$ on $\mathcal{R}_{g,n}^0$ that bound the pair of pants P_1 . Here, L_i and L_j are two boundaries of $\mathcal{R}_{g,n}^0$ for $i, j = 1, \dots, n$ and $A_{i,j}$ is a simple closed geodesic which is disjoint from the boundaries of $\mathcal{R}_{g,n}^0$. The sum over all properly embedded geometric pair of pants P_2 can be replaced with the sum over all tuple of simple closed geodesics $\{L_i, B_i, C_i\}$ on $\mathcal{R}_{g,n}^0$ that bound the pair of pants P_2 , for $i = 1, \dots, n$. Here B_i and C_i are two simple closed geodesic disjoint from the boundaries. The sum over all properly embedded geometric pair of pants P_3 can be replaced with the sum over all tuple of simple closed geodesics $\{X, Y, Z\}$ on $\mathcal{R}_{g,n}^0$ that bound the pair of pants P_3 . Here, X, Y and Z are three simple closed geodesic disjoint from the boundaries. Similarly, the sum over all quasi-properly embedded pairs of pants P_T can be replaced by the sum over pairs of simple closed geodesics $\{U, V\}$ on $\mathcal{R}_{g,n}^0$ that bound P_T , i.e. $\partial P_T = \{U, V, V\}$. Then we can express the Luo-Tan identity as a decomposition of unity, as follows:

$$\begin{aligned}
 1 = & \sum_{i < j; i,j=1}^n \sum_{A_{i,j} \in \mathcal{G}_{i,j}} \frac{\mathcal{K}_1(L_i, L_j, A_{i,j})}{4\pi^2(2g-2+n)} + \sum_{i=1}^n \sum_k \sum_{\{B_{i,k}, C_{i,k}\} \in \mathcal{G}_i^k} \frac{\mathcal{K}_2(L_i, B_{i,k}, C_{i,k})}{4\pi^2(2g-2+n)} \\
 & + \sum_{q,m} \sum_{\{X_q, Y_m, Z_{q,m}\} \in \mathcal{G}^{q,m}} \frac{\mathcal{K}_3(X_q, Y_m, Z_{q,m})}{4\pi^2(2g-2+n)} + \sum_r \sum_{\{U_r, V_r\} \in \mathcal{G}_T^r} \frac{\mathcal{K}_T(U_r, V_r)}{4\pi^2(2g-2+n)}. \quad (\text{C.22})
 \end{aligned}$$

Here, we decomposed the summation over simple closed geodesics γ that are disjoint from the boundaries into a summation over a set of simple closed geodesics that are related each other by the action of elements in $\text{Mod}_{g,n}/\text{Stab}(\gamma)$ and a summation over a discrete variables which differentiate the class of simple closed geodesics γ that are not related by the action of elements in $\text{Mod}_{g,n}/\text{Stab}(\gamma)$. For the first term in the right hand side of (C.22) $\gamma = A_{i,j}$, for the second term $\gamma = B_{i,k} + C_{i,k}$, for the third term $\gamma = X_q + Y_m + Z_{q,m}$ and for the last term $\gamma = U_r + V_r$. An arbitrary element in the set $\mathcal{G}_{i,j}$ together with the boundaries L_i and L_j of $\mathcal{R}_{g,n}^0$ form tuple of simple closed geodesics that can be identified as the boundary of a properly embedded geometric pairs of pants inside $\mathcal{R}_{g,n}^0$. An arbitrary element in the set $\mathcal{G}_i^k$, which is a pair of closed simple geodesics disjoint from the boundary, together with the boundary L_i of $\mathcal{R}_{g,n}^0$ form tuple of simple closed geodesics that can be identified as the boundary of a properly embedded geometric pairs of pants inside $\mathcal{R}_{g,n}^0$. An arbitrary element in the set $\mathcal{G}^{q,m}$, which is a tuple of simple closed geodesics disjoint from the boundary, can be identified as the boundary of a properly embedded geometric pair of pants with respect to $\mathcal{R}_{g,n}^0$. Finally, an arbitrary element in the set $\mathcal{G}_T^r$, which is a pair of simple closed geodesics disjoint from the boundary, can be identified as the boundary of a quasi-properly embedded geometric pair of pants inside $\mathcal{R}_{g,n}^0$.

The functions $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_T$ appearing in the Luo-Tan identity (C.21) have the following important property:

$$\lim_{l_i \rightarrow \infty} \mathcal{K}_I(P_I) = \lim_{l_i \rightarrow \infty} \mathcal{O}\left(e^{-l_i}\right) = 0, \quad I \in \{1, 2, 3, T\}, \quad i \in \{1, 2, 3\}, \quad (\text{C.23})$$

where l_i is the length of the i^{th} boundary of the pair of pants P_I . Let us verify this. The

function $\mathcal{K}_3(P_3)$ can be written as

$$\mathcal{K}_3(P_3) = 4 \sum_{i \neq j} \left\{ 2\mathcal{L} \left(\frac{1-x_i}{1-x_i y_j} \right) - 2\mathcal{L} \left(\frac{1-y_j}{1-x_i y_j} \right) - \mathcal{L}(y_j) - \mathcal{L} \left(\frac{(1-y_j)^2 x_i}{(1-x_i)^2 y_j} \right) \right\}, \quad (\text{C.24})$$

where

$$x_i \equiv e^{-l_i}, \quad y_i \equiv \tanh^2 \left(\frac{m_i}{2} \right), \quad (\text{C.25})$$

with m_i given by

$$\cosh m_i = \frac{\cosh \left(\frac{l_i}{2} \right) + \cosh \left(\frac{l_j}{2} \right) \cosh \left(\frac{l_k}{2} \right)}{\sinh \left(\frac{l_j}{2} \right) \sinh \left(\frac{l_k}{2} \right)}. \quad (\text{C.26})$$

For $i \neq j, k$ and $\{i, j, k\} = \{1, 2, 3\}$, it is straightforward to derive that

$$\begin{aligned} \lim_{l_i \rightarrow \infty} y_i &= 1, \\ \lim_{l_i \rightarrow \infty} y_j &= x_k, \\ \lim_{l_i \rightarrow \infty} y_k &= x_j. \end{aligned} \quad (\text{C.27})$$

Then using (C.27), (C.6) and (C.8), we can show that $\lim_{l_i \rightarrow \infty} \mathcal{K}_3(P_3) = 0$ for $i \in \{1, 2, 3\}$. Again, by repeating the same analysis, we can see that $\lim_{l_i \rightarrow \infty} \mathcal{K}_I(P_I) = 0$ for $i \in \{1, 2, 3\}$ and $I \in \{1, 2, T\}$.

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References

- [1] E. Witten, *Noncommutative Geometry and String Field Theory*, *Nucl. Phys. B* **268** (1986) 253 [[INSPIRE](#)].
- [2] C.B. Thorn, *String Field Theory*, *Phys. Rept.* **175** (1989) 1 [[INSPIRE](#)].
- [3] B. Zwiebach, *Closed string field theory: Quantum action and the B-V master equation*, *Nucl. Phys. B* **390** (1993) 33 [[hep-th/9206084](#)] [[INSPIRE](#)].
- [4] K. Strebel, *Quadratic Differentials*, Springer, Berlin Heidelberg, Germany (1984).
- [5] S.F. Moosavian and R. Pius, *Hyperbolic Geometry of Superstring Perturbation Theory*, [arXiv:1703.10563](#) [[INSPIRE](#)].
- [6] S.F. Moosavian and R. Pius, *Hyperbolic Geometry and Closed Bosonic String Field Theory I: The String Vertices Via Hyperbolic Riemann Surfaces*, [arXiv:1706.07366](#) [[INSPIRE](#)].
- [7] I.A. Batalin and G.A. Vilkovisky, *Gauge Algebra and Quantization*, *Phys. Lett.* **102B** (1981) 27 [[INSPIRE](#)].
- [8] I.A. Batalin and G.A. Vilkovisky, *Quantization of Gauge Theories with Linearly Dependent Generators*, *Phys. Rev. D* **28** (1983) 2567 [Erratum *ibid.* **D 30** (1984) 508] [[INSPIRE](#)].

- [9] G. Barnich, F. Brandt and M. Henneaux, *Local BRST cohomology in the antifield formalism. 1. General theorems*, *Commun. Math. Phys.* **174** (1995) 57 [[hep-th/9405109](#)] [[INSPIRE](#)].
- [10] G. Barnich, F. Brandt and M. Henneaux, *Local BRST cohomology in the antifield formalism. II. Application to Yang-Mills theory*, *Commun. Math. Phys.* **174** (1995) 93 [[hep-th/9405194](#)] [[INSPIRE](#)].
- [11] M. Henneaux, *Lectures on the Antifield-BRST Formalism for Gauge Theories*, *Nucl. Phys. Proc. Suppl.* **A 18** (1990) 47.
- [12] M. Henneaux and C. Teitelboim, *Quantization of gauge systems*, Princeton University Press, U.S.A. (1992).
- [13] J. Gomis, J. Paris and S. Samuel, *Antibracket, antifields and gauge theory quantization*, *Phys. Rept.* **259** (1995) 1 [[hep-th/9412228](#)] [[INSPIRE](#)].
- [14] J.M.L. Fisch and M. Henneaux, *Homological Perturbation Theory and the Algebraic Structure of the Antifield-Antibracket Formalism for Gauge Theories*, *Commun. Math. Phys.* **128** (1990) 627 [[INSPIRE](#)].
- [15] M. Schnabl, *Analytic solution for tachyon condensation in open string field theory*, *Adv. Theor. Math. Phys.* **10** (2006) 433 [[hep-th/0511286](#)] [[INSPIRE](#)].
- [16] H. Yang and B. Zwiebach, *A closed string tachyon vacuum?*, *JHEP* **09** (2005) 054 [[hep-th/0506077](#)] [[INSPIRE](#)].
- [17] N. Moeller, *Closed bosonic string field theory at quartic order*, *JHEP* **11** (2004) 018 [[hep-th/0408067](#)] [[INSPIRE](#)].
- [18] T. Erler, S. Konopka and I. Sachs, *One Loop Tadpole in Heterotic String Field Theory*, *JHEP* **11** (2017) 056 [[arXiv:1704.01210](#)] [[INSPIRE](#)].
- [19] D. Friedan, E.J. Martinec and S.H. Shenker, *Conformal Invariance, Supersymmetry and String Theory*, *Nucl. Phys.* **B 271** (1986) 93 [[INSPIRE](#)].
- [20] R. Pius, A. Rudra and A. Sen, *Mass Renormalization in String Theory: Special States*, *JHEP* **07** (2014) 058 [[arXiv:1311.1257](#)] [[INSPIRE](#)].
- [21] R. Pius, A. Rudra and A. Sen, *Mass Renormalization in String Theory: General States*, *JHEP* **07** (2014) 062 [[arXiv:1401.7014](#)] [[INSPIRE](#)].
- [22] A. Sen, *Off-shell Amplitudes in Superstring Theory*, *Fortsch. Phys.* **63** (2015) 149 [[arXiv:1408.0571](#)] [[INSPIRE](#)].
- [23] R. Pius, A. Rudra and A. Sen, *String Perturbation Theory Around Dynamically Shifted Vacuum*, *JHEP* **10** (2014) 70 [[arXiv:1404.6254](#)] [[INSPIRE](#)].
- [24] A. Sen, *Supersymmetry Restoration in Superstring Perturbation Theory*, *JHEP* **12** (2015) 075 [[arXiv:1508.02481](#)] [[INSPIRE](#)].
- [25] A. Sen, *BV Master Action for Heterotic and Type II String Field Theories*, *JHEP* **02** (2016) 087 [[arXiv:1508.05387](#)] [[INSPIRE](#)].
- [26] R. Pius and A. Sen, *Cutkosky rules for superstring field theory*, *JHEP* **10** (2016) 024 [*Erratum ibid.* **09** (2018) 122] [[arXiv:1604.01783](#)] [[INSPIRE](#)].
- [27] R. Pius and A. Sen, *Unitarity of the Box Diagram*, *JHEP* **11** (2018) 094 [[arXiv:1805.00984](#)] [[INSPIRE](#)].

- [28] A. Sen, *Equivalence of Two Contour Prescriptions in Superstring Perturbation Theory*, *JHEP* **04** (2017) 025 [[arXiv:1610.00443](#)] [[INSPIRE](#)].
- [29] A. Sen, *Reality of Superstring Field Theory Action*, *JHEP* **11** (2016) 014 [[arXiv:1606.03455](#)] [[INSPIRE](#)].
- [30] A. Sen, *Unitarity of Superstring Field Theory*, *JHEP* **12** (2016) 115 [[arXiv:1607.08244](#)] [[INSPIRE](#)].
- [31] A. Sen, *Gauge Invariant 1PI Effective Action for Superstring Field Theory*, *JHEP* **06** (2015) 022 [[arXiv:1411.7478](#)] [[INSPIRE](#)].
- [32] A. Sen, *Gauge Invariant 1PI Effective Superstring Field Theory: Inclusion of the Ramond Sector*, *JHEP* **08** (2015) 025 [[arXiv:1501.00988](#)] [[INSPIRE](#)].
- [33] A. Sen, *Wilsonian Effective Action of Superstring Theory*, *JHEP* **01** (2017) 108 [[arXiv:1609.00459](#)] [[INSPIRE](#)].
- [34] C. de Lacroix, H. Erbin, S.P. Kashyap, A. Sen and M. Verma, *Closed Superstring Field Theory and its Applications*, *Int. J. Mod. Phys. A* **32** (2017) 1730021 [[arXiv:1703.06410](#)] [[INSPIRE](#)].
- [35] H.M. Farkas and M.I. Kra, *Riemann surfaces*, Springer, New York, U.S.A. (1992).
- [36] A. Hatcher and W. Thurston, *A presentation for the mapping class group of a closed orientable surface*, *Topology* **19** (1980) 221.
- [37] A. Hatcher, *Pants decompositions of surfaces*, [math/9906084](#).
- [38] G. McShane, *Simple geodesics and a series constant over Teichmuller space*, *Invent. Math.* **132** (1998) 607.
- [39] M. Mirzakhani, *Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces*, *Invent. Math.* **167** (2006) 179.
- [40] S.A. Wolpert, *The hyperbolic metric and the geometry of the universal curve*, *J. Diff. Geom.* **31** (1990) 417.
- [41] K. Obitsu and S.A. Wolpert, *Grafting hyperbolic metrics and Eisenstein series*, *Math. Ann.* **341** (2008) 685.
- [42] R. Melrose and X. Zhu, *Boundary Behaviour of Weil-Petersson and Fiber Metrics for Riemann Moduli Spaces*, [arXiv:1606.01158](#).
- [43] S.A. Wolpert, *Families of Riemann surfaces and Weil-Petersson geometry*, No. 113, American Mathematical Society, (2010).
- [44] S.P. Kerckhoff, *The Nielsen realization problem*, *Annals Math.* **117** (1983) 235.
- [45] S.A. Wolpert, *The Fenchel-Nielsen deformation*, *Annals Math.* **115** (1982) 501.
- [46] S.A. Wolpert, *On the symplectic geometry of deformations of a hyperbolic surface*, *Annals Math.* **117** (1983) 207.
- [47] S.A. Wolpert, *On the Weil-Petersson geometry of the moduli space of curves*, *Am. J. Math.* **107** (1985) 969.
- [48] W. Fenchel and J. Nielsen, *Discontinuous groups of non-Euclidean motions*, Walter De Gruyter Inc (2002).

- [49] Y. Iwayoshi and M. Taniguchi, *An Introduction to Teichmüller spaces*, Springer Science & Business Media (2012).
- [50] W. Abikoff, *The Uniformization theorem*, *Am. Math. Mon.* **88** (1981) 574.
- [51] F. Benson and D. Margalit, *A Primer on Mapping Class Groups*, Princeton University Press (2011).
- [52] F. Luo and S.P. Tan, *A dilogarithm identity on moduli spaces of curves*, *J. Diff. Geom.* **97** (2014) 255.
- [53] H. Hengnan and S.P. Tan, *New identities for small hyperbolic surfaces*, *Bull. London Math. Soc.* **46** (2014) 1021.
- [54] R.C. Penner, *Decorated Teichmüller theory*, European Mathematical Society (2012).
- [55] R.C. Penner, *Weil-Petersson volumes*, *J. Diff. Geom.* **35** (1992) 559.
- [56] S.A. Wolpert, *Lectures and notes: Mirzakhani's volume recursion and approach for the Witten-Kontsevich theorem on moduli tautological intersection numbers*, [arXiv:1108.0174](#) [[INSPIRE](#)].
- [57] S.A. Wolpert, *On the Kähler form of the moduli space of once punctured tori*, *Comment. Math. Helv.* **58** (1983) 246.
- [58] R. Fricke and F. Klein, *Vorlesungen über die Theorie der automorphen Functionen*, Volume 1, BG Teubner (1897).
- [59] L. Keen, *On fundamental domains and the Teichmüller modular group*, *Contributions to Analysis*, Academic Press, (1974) pp. 185–194.
- [60] P. Sarnak, *Determinants of Laplacians*, *Commun. Math. Phys.* **110** (1987) 113.
- [61] E. D'Hoker and D.H. Phong, *On Determinants of Laplacians on Riemann Surfaces*, *Commun. Math. Phys.* **104** (1986) 537.
- [62] J. Bolte and F. Steiner, *Determinants of Laplace-like operators on Riemann surfaces*, *Commun. Math. Phys.* **130** (1990) 581.
- [63] D.A. Hejhal, *The Selberg trace formula for $\mathrm{PSL}(2, R)$* , Volume 2, Springer, (2006).
- [64] E. D'Hoker and D.H. Phong, *Multiloop Amplitudes for the Bosonic Polyakov String*, *Nucl. Phys. B* **269** (1986) 205 [[INSPIRE](#)].
- [65] L.P. Teo, *Ruelle zeta function for cofinite hyperbolic Riemann surfaces with ramification points*, [arXiv:1901.07898](#). [[arXiv:1901.07898](#)].
- [66] B. Maskit, *Matrices for Fenchel-Nielsen coordinates*, RECON no. 20010088230, *Annales Academiae Scientiarum Fennicae: Mathematica* **26** (2001) 267.
- [67] D. Stanford and E. Witten, *JT Gravity and the Ensembles of Random Matrix Theory*, [arXiv:1907.03363](#) [[INSPIRE](#)].