RECEIVED: May 24, 2019

REVISED: July 20, 2019

ACCEPTED: August 19, 2019

PUBLISHED: August 29, 2019

Abelian dualities of $\mathcal{N} = (0, 4)$ boundary conditions

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ABSTRACT: We propose dual pairs of $\mathcal{N} = (0, 4)$ half-BPS boundary conditions for 3d $\mathcal{N} = 4$ Abelian gauge theories related to mirror symmetry and S-duality by showing the matching of boundary 't Hooft anomalies and supersymmetric indices. We find simple $\mathcal{N} = (0, 4)$ mirror symmetry between 2d $\mathcal{N} = (0, 4)$ Abelian gauge theories and free Fermi multiplets that generalizes $\mathcal{N} = (0, 2)$ Abelian duality. We also propose a prescription for computing half-index of enriched Neumann boundary condition including 2d boundary bosonic matters by gauging the 2d boundary flavor symmetry of Dirichlet boundary condition. By coupling $\mathcal{N} = (0, 4)$ half-BPS boundary configurations of 3d $\mathcal{N} = 4$ gauge theories to quarter-BPS corner configurations of 4d $\mathcal{N} = 4$ Super Yang-Mills theories, we further obtain a new type of 4d-3d-2d duality that may involve 3d non-Abelian gauge symmetry.

KEYWORDS: Brane Dynamics in Gauge Theories, Duality in Gauge Field Theories, Supersymmetric Gauge Theory, Supersymmetry and Duality

ARXIV EPRINT: [1905.07425](https://arxiv.org/abs/1905.07425)

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1 Introduction and conclusions

Quantum field theories with boundaries can serve as various frameworks in studying a wide range of physical and mathematical problems. Supersymmetric boundary conditions in two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric field theories can determine D-branes in the target space that play a significant role in the study of mirror symmetry [1–3]. The half-BPS boundary conditions and interfaces in four-dimensional $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory are studied in [4, 5]. These boundary conditions are realized as the brane configuration in Type IIB string theory and admit singular half-BPS boundary condition, aka Nahm pole boundary condition. The action of S-duality beautifully produces dualities of these boundary conditions [5] as well as 3d $\mathcal{N} = 4$ mirror symmetry [6]. These half-BPS boundary conditions can also play a role in the Geometric Langlands Program as they can be mapped to mathematical objects in the categories relevant for the Geometric Langlands duality [7, 8]. Moreover, such half-BPS interfaces in four-dimensional $\mathcal{N} = 2$ gauge theories have played a central role in finding new duality, that is 3d-3d relation [9–11]. While AGT relation [12] is related to the compactification of M5-branes on Riemann surface, the 3d-3d relation is concerned with the compactification of M5-branes on 3-manifold.

The analysis of half-BPS boundary conditions in three-dimensional $\mathcal{N} = 2$ supersymmetric field theories was initiated in [13, 14]. The half-BPS boundary conditions in three-dimensional supersymmetric field theories are more fruitful as they are classified by the chirality of preserved supercharges. While the $\mathcal{N} = (1, 1)$ boundary conditions flow to the special Lagrangian branes, the $\mathcal{N} = (0, 2)$ boundary conditions flow to the holomorphic branes in the IR sigma-models [14]. The $\mathcal{N} = (0, 2)$ boundary conditions in 3d $\mathcal{N} = 2$ gauge theories are also important in the study of 4d-2d relation [13, 15–20], which is involved with the compactification of M5-branes on 4-manifold. The half-BPS boundary conditions in three-dimensional $\mathcal{N} = 4$ supersymmetric gauge theories are studied in [21, 22]. It is argued in [21] that the $\mathcal{N} = (2, 2)$ boundary conditions which flow to the holomorphic Lagrangian branes in the IR sigma-models can produce a certain set of modules in the categories so that mirror symmetry of these boundary conditions are realized as Symplectic Duality [23, 24] that is an equivalence of categories of modules. The $\mathcal{N} = (2, 2)$ and $\mathcal{N} = (0, 4)$ half-BPS boundary conditions can be realized in brane configurations [22]. The BPS boundary conditions in three-dimensional $\mathcal{N} \geq 4$ Chern-Simons theories, including BLG model, ABJM model are analyzed in [25–28]. These boundary conditions can describe the M5-brane, M9-brane and M-theory wave interacting with M2-branes.

If one has a pair of UV gauge theories which flow to the same superconformal field theory, one may find a pair of UV boundary conditions¹ for these theories which flow to the same superconformal boundary condition after the RG flow. Recently the dualities of boundary conditions and interfaces have been confirmed by computing supersymmetric indices. In [30] various dualities of $\mathcal{N} = (0, 2)$ supersymmetric boundary conditions in 3d $\mathcal{N} = 2$ gauge theories have been demonstrated. More recently, [31] have presented dualities of half-BPS interfaces and quarter-BPS corner configurations in four-dimensional $\mathcal{N} = 4$ SYM theory. Subsequently, [32] have shown more general dualities of half-BPS boundary conditions and interfaces in 4d $\mathcal{N} = 4$ SYM theory conjectured by Gaiotto and Witten [5]. When the local operators are supported at two-dimensional boundary and corner configuration of superconformal field theories, one can embed the Vertex Operator Algebras (VOAs) into the algebras of local operators so that the correlation functions of certain protected operators are encoded in the VOAs [33, 34]. Consequently the indices reduce to characters of the VOAs in the appropriate limit.

1.1 Main results

In this paper we extend the duality network of 4d $\mathcal{N} = 4$, 3d $\mathcal{N} = 4$ and 2d $\mathcal{N} = (0, 4)$ gauge theories [31, 32] by examining half-BPS $\mathcal{N} = (0, 4)$ boundary conditions in 3d $\mathcal{N} = 4$ gauge theories that can be constructed from the brane setup in Type IIB string theory [22, 31, 35]. Unlike the cases studied in [31, 32], the $\mathcal{N} = (0, 4)$ configurations discussed in this paper involve 3d $\mathcal{N} = 4$ gauge theories with Neumann boundary condition $\mathcal{N}'$ or/and 2d $\mathcal{N} = (0, 4)$ gauge theories.² Accordingly, as we argue in the main text, they need more careful treatment in the computation of supersymmetric indices. We propose the appropriate prescription for evaluating the half-indices of enriched Neumann b.c. which includes 2d bosonic matter fields by gauging the 2d boundary flavor symmetry of Dirichlet boundary condition. We find $\mathcal{N} = (0, 4)$ mirror symmetry of boundary conditions for 3d $\mathcal{N} = 4$ Abelian gauge theories. Besides, we propose simple $\mathcal{N} = (0, 4)$ mirror symmetry between 2d $\mathcal{N} = (0, 4)$ Abelian gauge theories and free Fermi multiplets which is reminiscent of $\mathcal{N} = (0, 2)$ Abelian duality [15, 16]. They are descended from the action of S-duality in string theory. We also find that these dualities are further generalized by coupling to quarter-BPS corners of 4d $\mathcal{N} = 4$ SYM theories studied in [31]. They can involve 3d $\mathcal{N} = 4$ non-Abelian gauge theories with Neumann boundary condition $\mathcal{N}'$. The dual configurations are characterized by the Dirichlet/Nahm pole boundary conditions for 3d $\mathcal{N} = 4$ twisted gauge theories with vector multiplets and twisted hypermultiplets as well as corners or boundaries of 4d $\mathcal{N} = 4$ gauge theories. We confirm these dualities by expanding the supersymmetric indices that include half-indices of 3d $\mathcal{N} = 4$ gauge theories and then checking several terms of series expansion. As a byproduct, our supersymmetric

¹The half-BPS boundary conditions or interfaces in dimensions higher than four behave differently since they flow to the UV in the RG flow as discussed in [29] for the half-BPS interface in five-dimensional $\mathcal{N} = 1$ gauge theories.

²In [31] the gauge symmetry stems from 4d gauge theories while in [32] it also results from 3d gauge theories.

indices should reduce to the vacuum characters of the VOAs arising from the $\mathcal{N} = (0, 4)$ supersymmetric configurations when setting the fugacities to special values.

The paper contains many confirmed examples to gain a bottom-up understanding, however, the general formulae for Abelian dualities of $\mathcal{N} = (0, 4)$ boundary conditions of 3d theories are summarized in subsection 4.4 and those for dualities of 4d-3d-2d configurations are summarized in subsubsection 5.1.8, 5.2.11, 5.3.5, 5.4.2, 5.5.8, 5.6.5, and 5.7.6.

1.2 Structure

The paper is organized as follows. In section 2 we introduce main tools to test dualities of $\mathcal{N} = (0, 4)$ boundary conditions. We define half-indices of 3d $\mathcal{N} = 4$ gauge theories and full-indices of 2d $\mathcal{N} = (0, 4)$ gauge theories. In section 3 we study simple examples of brane box models in [35]. We propose the minimal $\mathcal{N} = (0, 4)$ mirror symmetry by computing the indices. In section 4 we analyze $\mathcal{N} = (0, 4)$ boundary conditions for 3d $\mathcal{N} = 4$ Abelian gauge theories. We confirm the dualities of $\mathcal{N} = (0, 4)$ boundary conditions by checking the matching of boundary anomalies and half-indices. We also discuss the half-indices for enriched Neumann boundary conditions which include 2d bosonic matter fields. In section 5 we propose the dualities of $\mathcal{N} = (0, 4)$ supersymmetric configurations which are composed of quarter-BPS corners of 4d $\mathcal{N} = 4$ SYM theory and half-BPS boundaries of 3d $\mathcal{N} = 4$ gauge theories. The configurations include non-Abelian 3d gauge theories with Neumann b.c. which map to the dual configurations with no gauge symmetry. We present the matching of quarter-indices as powerful evidences of proposed dualities of these corner configurations.

1.3 Future works

There still remain a variety of interesting open questions which we leave for future works.

- **Non-Abelian $\mathcal{N} = (0, 4)$ configurations:** most of our $\mathcal{N} = (0, 4)$ configurations are limited to the cases with Abelian gauge symmetry. They should be promoted to more general $\mathcal{N} = (0, 4)$ mirror symmetry of 2d non-Abelian gauge theories and dualities of $\mathcal{N} = (0, 4)$ boundary conditions including enriched Neumann b.c. and Nahm pole b.c. for 3d $\mathcal{N} = 4$ non-Abelian gauge theories. In particular, it is interesting to explore $\mathcal{N} = (0, 4)$ mirror symmetry for general brane box models in [35] with several boxes of multiple D3-branes. We hope to report a detailed analysis in the future.
- **Boundary/corner VOAs:** the Vertex Operator Algebras (VOAs) in 3d $\mathcal{N} = 4$ gauge theories are constructed as the algebras of local operators obeying the holomorphic boundary conditions for their topologically twisted theories, which can be obtained by deforming $\mathcal{N} = (0, 4)$ boundary conditions [36]. These VOAs are motivated as powerful computational tools to study the bulk topological field theory (TFT) and potential applications to the gauge theory interpretation of the Geometric Langlands Program [4, 5, 37, 38]. Also the VOAs arising from corners of 4d $\mathcal{N} = 4$ SYM theory are constructed in [39]. It would be intriguing to figure out the proposed dualities in terms of VOAs appearing from the combined configurations of boundaries of 3d theories and corners of 4d theories studied in this paper.

Also the relation between 3d supersymmetric field theories and VOAs has been recently explored in the context of 3d Modularity [40] and study of 4-manifolds [41]. It would be also interesting to study these issues related to $\mathcal{N} = (0, 4)$ boundary conditions.³

- **Line and surface defects:** our dualities should be generalized by introducing non-local operators, such as line or surface operators in $\mathcal{N} = (0, 4)$ supersymmetric boundary/corner configurations. It would be interesting to examine the difference equations satisfied by the supersymmetric indices [9, 30, 42, 43] which correlate with the half-BPS line or surface operators. From the perspective of the VOAs, including half-BPS line or surface operators can produce non-trivial modules in VOAs [44]. The analysis of these modules may be fruitfully addressed by computing the supersymmetric indices as they can be viewed as promotions of characters of these modules in that they reduce to these characters in the special fugacity limits.
- **Holographic duals:** the gravitational solutions for D3-brane ending on fivebranes are argued in [45–48]. It would be interesting to explore the Type IIB supergravity solutions corresponding to our brane configurations described by quarter BPS corners of 4d $\mathcal{N} = 4$ SYM theory, half-BPS boundaries of 3d $\mathcal{N} = 4$ gauge theories, and 2d $\mathcal{N} = (0, 4)$ gauge theories in the low-energy limit. The indices on the gravity side that count the Kaluza-Klein excitation of the massless fields in the Type IIB supergravity may be compared with the supersymmetric indices in the large N limit.
- **$\mathcal{N} = (2, 2)$ boundaries and corners:** boundaries and junctions with $\mathcal{N} = (2, 2)$ supersymmetry also admit brane constructions [22] which lead to conjectural dualities under the action of S-duality. While for $\mathcal{N} = (0, 4)$ supersymmetric case one finds boundary/corner Fermi multiplets which may carry topological symmetry [31, 35], for $\mathcal{N} = (2, 2)$ supersymmetric case no Fermi multiplet can exist at the boundary/corner. We hope to report progress in that direction in the near future.
- **M2-M5 brane systems:** the 3d $\mathcal{N} = 4$ $U(N)$ gauge theory with an adjoint hypermultiplet and a fundamental hypermultiplet is known to flow to the same IR fixed point as the $U(N)_1 \times U(N)_{-1}$ ABJM theory describing multiple N M2-branes [49]. It would be interesting to examine the $\mathcal{N} = (0, 4)$ boundary conditions in this theory as they may describe the M5-branes on which M2-branes end on. In the special fugacity limit, the half-indices would reduce to the vacuum characters of the associated VOAs.
- **Boundary conditions for $\mathcal{N} \geq 4$ Chern-Simons matter theories:** it would be intriguing to understand $\mathcal{N} = (0, 4)$ boundary conditions for 3d $\mathcal{N} \geq 4$ Chern-Simons matter theories [50–52]. The analysis of $\mathcal{N} = (0, 2)$ boundary conditions for 3d $\mathcal{N} = 2$ Yang Mills-Chern-Simons theories in [30], which argues the level-rank duality of boundary conditions, may be generalized by including two-dimensional chiral superalgebras [28, 53], as the construction of these theories is intimately related to the supergroup structure.

³The author thanks Sergei Gukov for discussing relevant issues.

- **Identities of q -series:** although we have confirmed the matching of supersymmetric indices by checking several terms in the expansions, some are not still analytically proven. The resulting cute identities of q -series may have potential applications to number theory, representation theory, and combinatorics. In particular, we conjecture “the minimal mirror identity” between the half-indices (4.14) and (4.15) that essentially captures the $\mathcal{N} = (0, 4)$ mirror symmetry of the boundary conditions for 3d $\mathcal{N} = 4$ Abelian gauge theories. It would be interesting to explore the analytic proof of this identity.

2 Indices

We define the quarter-index as the trace over the cohomology of the preserved supercharges [31]

$$\mathbb{IV}(t, x; q) := \text{Tr}_{\text{Op}}(-1)^F q^{J + \frac{H+C}{4}} t^{H-C} x^f \quad (2.1)$$

where F is the Fermion number, J is the generator of the $U(1)_J$ rotational symmetry in the two-dimensional space-time, H and C are the Cartan generators of the $SU(2)_H$ and $SU(2)_C$ R-symmetry groups respectively. f is the Cartan generator of the global symmetry.

When the bulk 4d theory is empty, the quarter-index (2.1) can be identified with the half-index of 3d theory, and when the bulk 4d and boundary 3d theories are absent, the quarter-index (2.1) can be viewed as the full-index of 2d theory.

We use the q -shifted factorial defined by

$$\begin{aligned} (a; q)_0 &:= 1, & (a; q)_n &:= \prod_{k=0}^{n-1} (1 - aq^k), & (q)_n &:= \prod_{k=1}^n (1 - q^k), & n &\geq 1, \\ (a; q)_\infty &:= \prod_{k=0}^{\infty} (1 - aq^k), & (q)_\infty &:= \prod_{k=1}^{\infty} (1 - q^k), \\ (a^\pm; q)_\infty &:= (a; q)_\infty (a^{-1}; q)_\infty \end{aligned} \quad (2.2)$$

where a and q are complex variables with $|q| < 1$.

We construct the $\mathcal{N} = (0, 4)$ supersymmetric configurations from brane configurations in Type IIB string theory whose world-volumes span the following directions:

- D3-branes extended along $x^0 x^1 x^2 x^6$,
- NS5-branes extended along $x^0 x^1 x^2 x^3 x^4 x^5$,
- D5-branes extended along $x^0 x^1 x^2 x^7 x^8 x^9$,
- NS5'-branes extended along $x^0 x^1 x^6 x^7 x^8 x^9$,
- D5'-branes extended along $x^0 x^1 x^3 x^4 x^5 x^6$

The finite segments of D3-branes in the x^6 direction between NS5-branes introduce 3d $\mathcal{N} = 4$ vector multiplet and D5-branes intersecting with the D3-branes introduce 3d $\mathcal{N} = 4$ hypermultiplets [54]. The $\mathcal{N} = (0, 4)$ boundaries in 3d $\mathcal{N} = 4$ gauge theories are realized

by further adding NS5'- and D5'-branes on which the D3-branes terminate [22, 35]. The NS5'-brane leads to the Neumann b.c. $\mathcal{N}'$ for 3d vector multiplet and Neumann b.c. $\mathcal{N}'$ for 3d hypermultiplet, whereas the D5'-brane can introduce the Dirichlet b.c. $\mathcal{D}'$ and Nahm' b.c. for 3d vector multiplet and Dirichlet $\mathcal{D}'$ and Nahm pole boundary conditions for 3d hypermultiplet.

In the following, we identify the dual $\mathcal{N} = (0, 4)$ configurations by applying S-duality in the brane setup. We then test the inspired dualities by evaluating boundary 't Hooft anomalies and supersymmetric indices.

2.1 Half-indices of 3d $\mathcal{N} = 4$ gauge theories

While the half-indices for $\mathcal{N} = (0, 2)$ Neumann b.c. $\mathcal{N}$ of three-dimensional gauge theories were firstly considered in [13], the half-indices for $\mathcal{N} = (0, 2)$ Dirichlet b.c. $\mathcal{D}$ of three-dimensional gauge theories were proposed in [30]. In this subsection we present the half-indices for $\mathcal{N} = (0, 4)$ Neumann and Dirichlet boundary conditions which can be viewed as specializations of quarter-index (2.1).

The 3d $\mathcal{N} = 4$ hypermultiplet includes a pair of complex scalars $\mathbb{H}, \tilde{\mathbb{H}}$ forming a doublet of $SU(2)_H$ and a pair of complex fermions $\psi_+^{\mathbb{H}}, \tilde{\psi}_+^{\mathbb{H}}$ forming a doublet of $SU(2)_C$. The charges of 3d $\mathcal{N} = 4$ hypermultiplet is

	$\mathbb{H}$	$\tilde{\mathbb{H}}$	$\psi_+^{\mathbb{H}}$	$\tilde{\psi}_+^{\mathbb{H}}$	$\bar{\psi}_-^{\mathbb{H}}$	$\bar{\tilde{\psi}}_-^{\mathbb{H}}$
$U(1)_C$	0	0	-	-	+	+
$U(1)_H$	+	+	0	0	0	0

(2.3)

The 3d $\mathcal{N} = 4$ Abelian vector multiplet includes a 3d gauge field A_μ^{3d} , three scalars, which we denote by real and complex scalars σ, φ forming the $SU(2)_C$ triplet, and two complex fermions $(\lambda_\alpha^{3d}, \eta_\alpha^{3d})$. The charges of 3d $\mathcal{N} = 4$ vector multiplet is given by

	A_μ^{3d}	σ	φ	$\lambda_\pm^{3d}$	$\bar{\lambda}_\pm^{3d}$	$\eta_\pm^{3d}$	$\bar{\eta}_\pm^{3d}$
$U(1)_C$	0	0	2	+	-	+	-
$U(1)_H$	0	0	0	+	-	-	+

(2.4)

We introduce the half-indices of $\mathcal{N} = (0, 4)$ half-BPS boundary conditions for 3d $\mathcal{N} = 4$ gauge theories by physical consideration.

2.1.1 3d matter multiplets

The simplest example of $\mathcal{N} = (0, 4)$ half-BPS boundary conditions for 3d $\mathcal{N} = 4$ hypermultiplet are the Neumann b.c. $\mathcal{N}'$ and Dirichlet b.c. $\mathcal{D}'$

$$\begin{aligned}
 \mathcal{N}' : \partial_2 \mathbb{H}|_\partial = 0, \partial_2 \tilde{\mathbb{H}}|_\partial = 0, \\
 \mathcal{D}' : \partial_\mu \mathbb{H}|_\partial = 0, \partial_\mu \tilde{\mathbb{H}}|_\partial = 0,
 \end{aligned}
 \quad \mu = 0, 1
 \tag{2.5}$$

For Neumann b.c. $\mathcal{N}'$ the complex scalar fields $\mathbb{H}, \tilde{\mathbb{H}}$ and their superpartners, i.e. the right-moving fermions can fluctuate while for Dirichlet b.c. $\mathcal{D}'$ the complex scalar fields $\mathbb{H}, \tilde{\mathbb{H}}$ are fixed and the left-moving fermions can survive at the boundary [22].

We introduce the fugacity x for the flavor symmetry. The half-index of Neumann b.c. N' for 3d $\mathcal{N} = 4$ hypermultiplet is

$$\mathbb{III}_{N'}^{3d \text{ HM}}(t, x; q) = \frac{1}{(q^{\frac{1}{4}}tx; q)_{\infty}(q^{\frac{1}{4}}tx^{-1}; q)_{\infty}}. \quad (2.6)$$

It has an expansion

$$\mathbb{III}_{N'}^{3d \text{ HM}}(t, x; q) = \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{1}{(q)_k (q)_{n-k}} x^{n-2k} q^{\frac{n}{4}} t^n. \quad (2.7)$$

One can deform the Neumann b.c. N' for 3d hypermultiplet so that it is compatible with the H-twist (mirror Rozansky-Witten twist). The resulting deformed boundary condition supports the VOA Sb of symplectic bosons $X(z)$ and $Y(z)$ of conformal dimension $\frac{1}{2}$ which obey the OPE [36, 39]

$$X(z)Y(w) \sim \frac{1}{z-w}. \quad (2.8)$$

Correspondingly, the vacuum character for the symplectic boson VOA Sb can be obtained from the Neumann half-index (2.6) by taking the H-twist limit $t \rightarrow q^{\frac{1}{4}}$

$$\begin{aligned} \mathbb{III}_{N'}^{3d \text{ HM}}(t = q^{\frac{1}{4}}, x; q) &= \frac{1}{(q^{\frac{1}{2}}x; q)_{\infty}(q^{\frac{1}{2}}x^{-1}; q)_{\infty}} \\ &= \chi_{\text{Sb}}(x; q). \end{aligned} \quad (2.9)$$

The half-index of Dirichlet b.c. D' for 3d $\mathcal{N} = 4$ hypermultiplet takes the form

$$\mathbb{III}_{D'}^{3d \text{ HM}}(t, x; q) = (q^{\frac{3}{4}}t^{-1}x; q)_{\infty}(q^{\frac{3}{4}}t^{-1}x^{-1}; q)_{\infty}. \quad (2.10)$$

It is expanded as

$$\mathbb{III}_{D'}^{3d \text{ HM}} = \frac{1}{(q)_{\infty}^2} \sum_{n=0}^{\infty} \sum_{k=0}^n (q^{1+k}; q)_{\infty} (q^{1+n-k}; q)_{\infty} q^{\frac{n^2-n}{2} + k(k-n) + \frac{3n}{4}} (-1)^n x^{-n+2k} t^{-n}. \quad (2.11)$$

The Dirichlet b.c. D' for 3d hypermultiplet can be deformed in such a way that it is compatible with the C-twist. The deformed boundary condition supports the VOA Fc of fermionic currents $x(z)$ and $y(z)$ of conformal dimension 1 with the OPE

$$x(z)y(w) \sim \frac{1}{(z-w)^2}. \quad (2.12)$$

In fact, in the C-twist limit $t \rightarrow q^{-\frac{1}{4}}$, the Dirichlet half-index (2.10) reduces to the vacuum character for the fermionic current VOA Fc

$$\begin{aligned} \mathbb{III}_{D'}^{3d \text{ HM}}(t = q^{-\frac{1}{4}}, x; q) &= (qx; q)_{\infty}(qx^{-1}; q)_{\infty} \\ &= \chi_{\text{Fc}}(x; q). \end{aligned} \quad (2.13)$$

The half-indices for 3d twisted hypermultiplet can be obtained by setting $t \rightarrow t^{-1}$. The half-index of Neumann b.c. N for 3d $\mathcal{N} = 4$ twisted hypermultiplet is

$$\mathbb{I}_N^{3d \text{ tHM}}(t, x; q) = \frac{1}{(q^{\frac{1}{4}} t^{-1} x; q)_\infty (q^{\frac{1}{4}} t^{-1} x^{-1}; q)_\infty} \quad (2.14)$$

and the half-index of Dirichlet b.c. D for 3d $\mathcal{N} = 4$ twisted hypermultiplet is

$$\mathbb{I}_D^{3d \text{ tHM}}(t, x; q) = (q^{\frac{3}{4}} t x; q)_\infty (q^{\frac{3}{4}} t x^{-1}; q)_\infty. \quad (2.15)$$

2.1.2 3d gauge multiplets

The $\mathcal{N} = (0, 4)$ half-BPS boundary conditions for 3d $\mathcal{N} = 4$ U(1) vector multiplet admit the Neumann b.c. $\mathcal{N}'$ and Dirichlet b.c. $\mathcal{D}'$

$$\begin{aligned} \mathcal{N}' : F_{2\mu}|_\partial = 0, D_\mu \sigma|_\partial = 0, D_\mu \varphi|_\partial = 0, \\ \mathcal{D}' : F_{\mu\nu}|_\partial = 0, D_2 \sigma|_\partial = 0, D_2 \varphi|_\partial = 0, \end{aligned} \quad \mu, \nu = 0, 1 \quad (2.16)$$

For Neumann b.c. $\mathcal{N}'$, the 3d gauge fields and the left-moving gauginos can fluctuate at the boundary, while for Dirichlet b.c. $\mathcal{D}'$, the gauge symmetry is broken and one keeps scalar fields σ, φ and their superpartners, the right-moving fermions at the boundary [22].

The half index of Neumann b.c. $\mathcal{N}'$ for 3d $\mathcal{N} = 4$ U(1) vector multiplet takes the form

$$\mathbb{I}_{\mathcal{N}'}^{3d \text{ U}(1)}(t; q) = (q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty \oint \frac{ds}{2\pi i s}. \quad (2.17)$$

When there is no charged 2d bosonic matter field at the two-dimensional boundary, the integration contour for variable s can be taken as a unit circle. Except for the enriched Neumann boundary condition discussed in section 4.5, we choose the contour of gauge fugacity plane as a unit circle or torus in this paper. This allows us to expand the integrand as a series in q and then to project onto the gauge invariants. However, as we discuss in section 4.5, when we try to compute half-indices of 3d gauge theories with Neumann b.c. coupled to 2d bosonic boundary matter, field configuration of 2d bosonic matter fields may support some kind of boundary monopole operator, which requires to deform the contour.

Dirichlet b.c. $\mathcal{D}'$ for 3d $\mathcal{N} = 4$ U(1) vector multiplet leads to the following perturbative contributions to the half-index

$$\mathbb{I}_{\mathcal{D}'}^{3d \text{ pert U}(1)}(t; q) = \frac{1}{(q)_\infty (q^{\frac{1}{2}} t^{-2}; q)_\infty}. \quad (2.18)$$

In addition, the Dirichlet half-index receives a non-perturbative contributions from monopole operators as Dirichlet b.c. $\mathcal{D}'$ allows for the boundary monopole operator [21]. The non-perturbative completion would be achieved by the formula proposed in [30]⁴

$$\mathbb{I}_{\mathcal{D}'}^{3d \text{ U}(1)}(t; q) = \frac{1}{(q)_\infty (q^{\frac{1}{2}} t^{-2}; q)_\infty} \sum_{m \in \mathbb{Z}} q^{\frac{1}{2} k_{\text{eff}} m^2} u^{k_{\text{eff}} m} x^{k_{\text{eff}} m} \times [\text{matter indices}] (q^m u). \quad (2.21)$$

⁴More precisely, the formula (2.21) is written as

$$\mathbb{I}_{\mathcal{D}'}^{3d \text{ U}(1)}(t; q) = \frac{1}{(q)_\infty (q^{\frac{1}{2}} t^{-2}; q)_\infty} \sum_{m \in \mathbb{Z}} q^{\frac{1}{2} (m \mathbf{0}) k_{\text{eff}} \binom{m}{\mathbf{0}}} e^{(m \mathbf{0}) k_{\text{eff}} \left(\frac{\log u}{\log x} \right)} \times [\text{matter indices}] (q^m u) \quad (2.19)$$

where $\mathbf{0}$ is a zero vector with dimension being the number of the global fugacities x and k_{eff} takes the form

Here u is the fugacity for the boundary $U(1)_\partial$ global symmetry resulting from the broken $U(1)$ gauge symmetry and x is for some other global symmetry F with a boundary anomaly. The $\sum_{m \in \mathbb{Z}}$ stands for the summation over magnetic flux sectors. This prescription is distinguished from the half-index of Dirichlet b.c. for 4d gauge theories [31, 32].

The effective Chern-Simons coupling k_{eff} encodes the total boundary $U(1)_\partial$ and F anomalies in such a way that the boundary anomaly can be viewed as the exterior derivative of a Chern-Simons form at level k_{eff} [30]. The terms $q^{\frac{1}{2}k_{\text{eff}}m^2} u^{k_{\text{eff}}m} x^{k_{\text{eff}}m}$ are the contributions from bare monopole operators with electric charge $k_{\text{eff}}m$, global charge $k_{\text{eff}}m$ and spin $\frac{k_{\text{eff}}}{2} \text{Tr}(m^2)$. Since electrically charged states get effective spin in the presence of magnetic flux, the matter indices receive the shift of fugacity $u \rightarrow q^m s$.

For 3d $\mathcal{N} = 4$ non-Abelian vector multiplet the $\mathcal{N} = (0, 4)$ Dirichlet boundary condition $\mathcal{D}'$ would become more general boundary condition which allows the bulk fields to have a singular profile, that is Nahm pole boundary condition [22]:

$$\begin{aligned} \mathcal{N}' : \quad & F_{2\mu}|_\partial = 0, \quad D_\mu \sigma|_\partial = 0, \quad D_\mu \varphi|_\partial = 0, \quad \mu, \nu = 0, 1 \\ \text{Nahm}' : \quad & F_{\mu\nu}|_\partial = 0, \quad D_2 \vec{\phi} + \vec{\phi} \times \vec{\phi}|_\partial = 0, \end{aligned} \quad (2.22)$$

where $\vec{\phi}$ is the $SU(2)_C$ triplet consisting of a real scalar σ and a complex scalar φ .

The half-index of Neumann b.c. $\mathcal{N}'$ for 3d $\mathcal{N} = 4$ $U(2)$ vector multiplet is

$$\begin{aligned} \mathbb{III}_{\mathcal{N}'}^{3d \ U(2)}(t; q) &= \frac{1}{2} (q)_\infty^2 (q^{\frac{1}{2}} t^2; q)_\infty^2 \oint \frac{ds_1}{2\pi i s_1} \frac{ds_2}{2\pi i s_2} \\ &\times \left(\frac{s_1}{s_2}; q \right)_\infty \left(\frac{s_2}{s_1}; q \right)_\infty \left(q^{\frac{1}{2}} t^2 \frac{s_1}{s_2}; q \right)_\infty \left(q^{\frac{1}{2}} t^2 \frac{s_2}{s_1}; q \right)_\infty. \end{aligned} \quad (2.23)$$

Again the integration contour for variable s_i can be taken as a unit torus $\mathbb{T}^2$ when there is no 2d bosonic matter field at the two-dimensional boundary.

Choosing the integration contour as a unit torus $\mathbb{T}^2$, one can evaluate the half-index (2.23) as

$$\begin{aligned} \mathbb{III}_{\mathcal{N}'}^{3d \ U(2)}(t; q) &= \frac{1}{2} \frac{(q^{\frac{1}{2}} t^2; q)_\infty^2}{(q)_\infty^2} \sum_{n=0}^{\infty} \sum_{k=0}^n \sum_{l=0}^n (q^{1+k}; q)_\infty (q^{1+n-k}; q)_\infty (q^{1+l}; q)_\infty (q^{1+n-l}; q)_\infty \\ &\times q^{n^2 - n + k(k-n) + l(l-n) + \frac{1}{2}(k+l)} \cdot t^{2(k+l)}. \end{aligned} \quad (2.24)$$

The first term in the sum takes the form of square of half-index $\mathbb{III}_{\mathcal{N}'}^{4d \ U(1)}$ for Neumann b.c. $\mathcal{N}'$ for 3d $U(1)$ gauge theory. We may associate this expansion to a Higgsing process that leads to a decomposition $U(2) \rightarrow U(1) \times U(1)$.

of matrix that encodes the anomaly coefficients:

$$k_{\text{eff}} = \begin{pmatrix} \text{broken gauge anomaly coeff.} & \text{mixed anomaly coeff.} \\ \text{mixed anomaly coeff.} & * \end{pmatrix}. \quad (2.20)$$

Since the anomaly coefficient for the broken $U(1)_\partial$ gauge symmetry is only contributed from the 3d bosonic matter and it is always positive for the Dirichlet b.c. [31, 35], the term $q^{\frac{1}{2}k_{\text{eff}}m^2}$ involves only positive powers of q so that the formula (2.21) gives well-defined q -series.

The half-index of Neumann b.c. $\mathcal{N}'$ for 3d $\mathcal{N} = 4$ $U(N)$ vector multiplet is

$$\mathbb{III}_{\mathcal{N}'}^{3d \ U(N)}(t; q) = \frac{1}{N!} (q)_\infty^N (q^{\frac{1}{2}} t^2; q)_\infty^N \oint \prod_{i=1}^N \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty \left(q^{\frac{1}{2}} t^2 \frac{s_i}{s_j}; q \right)_\infty. \quad (2.25)$$

Again when there is no charged 2d bosonic matter field at the two-dimensional boundary, the integration contour for gauge fugacities s_i can be taken as a unit torus $\mathbb{T}^N$.

The Dirichlet b.c. $\mathcal{D}'$ for 3d $\mathcal{N} = 4$ non-Abelian vector multiplet should be realized as a trivial Nahm' boundary condition. The half-index of Dirichlet b.c. $\mathcal{D}'$ for 3d $\mathcal{N} = 4$ vector multiplet of non-Abelian gauge group G should receive the non-perturbative monopole contributions with the form

$$\begin{aligned} \mathbb{III}_{\mathcal{D}'}^{3d \ G}(t, u; q) &= \left[\frac{1}{(q)_\infty (q^{\frac{1}{2}} t^{-2}; q)_\infty} \right]^{\text{rank}(G)} \sum_{m \in \text{cochar}(G)} \\ &\times \left[\prod_{\alpha \in \text{roots}(G)} \frac{1}{(q^{1+m \cdot \alpha} u^\alpha; q)_\infty (q^{\frac{1}{2}+m \cdot \alpha} t^{-2} u^\alpha; q)_\infty} \right] \\ &\times q^{\frac{1}{2} k_{\text{eff}} \text{Tr}(m^2)} \cdot u^{k_{\text{eff}} m} \cdot x^{k_{\text{eff}} m} \times [\text{matter indices}] (q^m u) \end{aligned} \quad (2.26)$$

where u is the fugacity for the boundary G_∂ global symmetry and x is that for some other global symmetry F .

Similarly to the Abelian case, the effective Chern-Simons coupling k_{eff} encodes the boundary 't Hooft anomaly. $\sum_{m \in \text{cochar}(G)}$ is now the summation over cocharacter of G . The terms in the second line include the contributions from bare monopole and the fugacity u in the matter indices are shifted due to the effective spin carried by charged states.

2.2 Elliptic genera of 2d $\mathcal{N} = (0, 4)$ gauge theories

When the both 4d theory and 3d theory are trivial, the quarter-index (2.1) is simply the elliptic genus of 2d $\mathcal{N} = (0, 4)$ gauge theory [55–58].

2.2.1 2d matter multiplets

The full-index of 2d $\mathcal{N} = (0, 2)$ chiral multiplet is

$$C(x; q) = \frac{1}{(x; q)_\infty (qx^{-1}; q)_\infty} \quad (2.27)$$

and the full-index of 2d $\mathcal{N} = (0, 2)$ Fermi multiplet is

$$F(x; q) = (x; q)_\infty (qx^{-1}; q)_\infty. \quad (2.28)$$

The $\mathcal{N} = (0, 4)$ hypermultiplet consists of a pair of $\mathcal{N} = (0, 2)$ chiral multiplets. Under the R-symmetry $SU(2)_C \times SU(2)_H$ the chiral field and the anti-chiral field transform as $(\mathbf{1}, \mathbf{2})$.

The full-index of 2d $\mathcal{N} = (0, 4)$ hypermultiplet is given by

$$\mathbb{I}^{2d \ \text{HM}}(t, x; q) = \frac{1}{(q^{\frac{1}{4}} t x^\pm; q)_\infty (q^{\frac{3}{4}} t^{-1} x^\pm; q)_\infty} = C(q^{\frac{1}{4}} t x; q) \times C(q^{\frac{1}{4}} t x^{-1}; q). \quad (2.29)$$

This can be expanded as

$$\begin{aligned} \mathbb{I}^{2d \text{ HM}}(t, x; q) & \\ &= \frac{1}{(q)_\infty^4} \sum_{n=0}^{\infty} \sum_{k=0}^n \sum_{l=0}^n \frac{(q^{1+k}; q)_\infty (q^{1+l}; q)_\infty (q^{1+n-k}; q)_\infty (q^{1+n-l}; q)_\infty}{(q^{1+n}; q)_\infty} q^{\frac{3n}{4} - \frac{k}{2}} \cdot t^{2k-n} \cdot x^{2l-n}. \end{aligned} \quad (2.30)$$

In the H-twist limit, the hypermultiplet index (2.29) becomes the square of the vacuum character (2.9) for the symplectic boson VOA Sb.

Similarly, the $\mathcal{N} = (0, 4)$ twisted hypermultiplet consists of a pair of $\mathcal{N} = (0, 2)$ chiral multiplets. Under the R-symmetry $SU(2)_C \times SU(2)_H$ the chiral field and the anti-chiral field transform as $(\mathbf{2}, \mathbf{1})$.

By setting $t \rightarrow t^{-1}$, we can obtain the index for 2d $\mathcal{N} = (0, 4)$ twisted hypermultiplet

$$\mathbb{I}^{2d \text{ tHM}}(t, x; q) = \frac{1}{(q^{\frac{1}{4}} t^{-1} x^\pm; q)_\infty (q^{\frac{3}{4}} t x^\pm; q)_\infty} = C(q^{\frac{1}{4}} t^{-1} x; q) \times C(q^{\frac{1}{4}} t^{-1} x^{-1}; q). \quad (2.31)$$

In the C-twist limit, the twisted hypermultiplet index (2.31) becomes the square of the vacuum character (2.9) for the symplectic boson VOA Sb.

2.2.2 2d vector multiplets

The index of 2d $\mathcal{N} = (0, 4)$ U(1) vector multiplet takes the form

$$\mathbb{I}^{2d \text{ U}(1)}(t; q) = (q)_\infty^2 (q^{\frac{1}{2}} t^2; q)_\infty (q^{\frac{1}{2}} t^{-2}; q)_\infty \oint_{\text{JK}} \frac{ds}{2\pi i s}. \quad (2.32)$$

Here the subscript “JK” in the contour integral indicates that the prescription for computing the 2d elliptic genus is not the naive zero charge projection given by a contour integral along the unit circle $|s| = 1$ in the gauge fugacity plane.

Instead, the integrand for a non-anomalous 2d symmetry is a well-defined function on the fugacity torus, i.e. invariant under $s \rightarrow qs$. The Jeffrey-Kirwan (JK) residue prescription involves a contour integral around poles associated to $\mathcal{N} = (0, 2)$ chiral fields of, say, negative charge [59–61]. In the case of hypers or twisted hypers, this should pick up poles of the form $x = q^{\frac{1}{4}} t^{\pm 1}$.

Notice that the naive prescription cannot be right, as it is not modular invariant. On the other hand, the naive prescription clearly accounts for gauge-invariant operators built from elementary fields. The full prescription must include corrections due to non-local operators from twisted sectors under gauge symmetries.

3 $\mathcal{N} = (0, 4)$ box models

In this section, we analyze simple examples of $\mathcal{N} = (0, 4)$ brane box models in [35]. In particular, we propose simple $\mathcal{N} = (0, 4)$ mirror symmetry that may be related to the 3d $\mathcal{N} = 4$ mirror symmetry between 3d SQED₁ and free twisted hypermultiplet [62].

Figure 1. The brane box configurations of 2d $\mathcal{N} = (0, 4)$ U(1) gauge theory with a charged hypermultiplet and two charged Fermi multiplets and of its mirror $\mathcal{N} = (0, 4)$ Fermi multiplet. We take the horizontal and vertical directions along the x^6 and x^2 respectively.

3.1 Abelian $\mathcal{N} = (0, 4)$ theories

3.1.1 2d U(1) + hyper + 2 Fermi

We start from a 2d $\mathcal{N} = (0, 4)$ U(1) gauge theory with a charged hypermultiplet and two charged Fermi multiplets. Let $\mathbf{s}$ be the field strength of U(1) gauge symmetry. The gauge anomaly polynomial is⁵

$$\mathcal{I}^{2d \text{ U(1) + 1HM+2FM}} = \underbrace{-2\mathbf{s}^2}_{2d \text{ hyper}} + \underbrace{2 \cdot \mathbf{s}^2}_{2d \text{ Fermi}} = 0 \quad (3.1)$$

so that the theory is free from gauge anomaly.

The theory is constructed by the brane box configuration in figure 1. Here and hereafter we take the horizontal and vertical directions along the x^6 and x^2 respectively. The 2d $\mathcal{N} = (0, 4)$ U(1) vector multiplet appears from the D3-brane box configuration surrounded by NS5- and NS5'-branes, that is a 3d $\mathcal{N} = 4$ U(1) vector multiplet on a segment with Neumann b.c. $\mathcal{N}'$ at each end. Correspondingly, we have a schematic relation

$$\mathbb{I}^{2d \text{ U(1)}} = \frac{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U(1)}} \times \mathbb{III}_{\mathcal{N}'}^{3d \text{ U(1)}}}{\mathbb{I}^{3d \text{ pert U(1)}}} \quad (3.2)$$

where $\mathbb{I}^{3d \text{ pert U(1)}} = (q^{\frac{1}{2}}t^2; q)_\infty (q^{\frac{1}{2}}t^{-2}; q)_\infty^{-1}$ is the perturbative full-index of 3d $\mathcal{N} = 4$ U(1) vector multiplet [32].

The D5-brane gives rise to a 2d $\mathcal{N} = (0, 4)$ fundamental hypermultiplet and two Fermi multiplets. The Fermi multiplets appear at the two intersections of D3- and D5-branes [35], which are depicted as red dots in figure 1.

⁵See [31, 35] for conventions of (boundary) anomalies.

The index of the theory 2d U(1) + hyper + 2 Fermi is evaluated as

$$\begin{aligned} \mathbb{I}^{2d \text{ U}(1)+\text{HM}+2\text{FM}}(t, z; q) \\ = \underbrace{(q)_\infty^2 (q^{\frac{1}{2}} t^2; q)_\infty (q^{\frac{1}{2}} t^{-2}; q)_\infty}_{\mathbb{I}^{2d \text{ U}(1)}} \oint_{\text{JK}} \frac{ds}{2\pi i s} \cdot \underbrace{\frac{(q^{\frac{1}{2}} s^\pm z^\pm; q)_\infty (q^{\frac{1}{2}} s^\pm z^\mp; q)_\infty}{(q^{\frac{1}{4}} t s^\pm; q)_\infty (q^{\frac{3}{4}} t^{-1} s^\pm; q)_\infty}}_{\mathbb{I}^{2d \text{ HM}(s) \cdot F(q^{\frac{1}{2}} s z) \cdot F(q^{\frac{1}{2}} s z^{-1})}}. \end{aligned} \quad (3.3)$$

The choice of Fermi multiplet fugacities makes the integrand modular. Picking the pole of hypermultiplet at $s = q^{\frac{1}{4}} t$ gives cancellations and we find that

$$\mathbb{I}^{2d \text{ U}(1)+\text{HM}+2\text{FM}}(t, z; q) = \underbrace{(q^{\frac{3}{4}} t z; q)_\infty (q^{\frac{1}{4}} t^{-1} z^{-1}; q)_\infty}_{F(q^{\frac{3}{4}} t z)} \cdot \underbrace{(q^{\frac{3}{4}} t z^{-1}; q)_\infty (q^{\frac{1}{4}} t^{-1} z; q)_\infty}_{F(q^{\frac{3}{4}} t z^{-1})}. \quad (3.4)$$

This is identified with the index of the sort of $\mathcal{N} = (0, 4)$ Fermi which arises from a 3d twisted hypermultiplet on a segment with Dirichlet boundary conditions D .

On the other hand, the original 2d $\mathcal{N} = (0, 4)$ gauge theory can be interpreted as 3d $\mathcal{N} = 4$ SQED₁ on a segment with Neumann b.c. $\mathcal{N}'$. Therefore this is compatible with 3d $\mathcal{N} = 4$ mirror symmetry. Our index computation indicates the minimal $\mathcal{N} = (0, 4)$ mirror symmetry:

$$2d \mathcal{N} = (0, 4) \text{ U}(1) \text{ gauge theory with 1 hyper} + 2 \text{ Fermi} \leftrightarrow 2d \text{ free } \mathcal{N} = (0, 4) \text{ Fermi}. \quad (3.5)$$

We note that the duality (3.5) can be viewed as a sort of $\mathcal{N} = (0, 4)$ generalization of the duality between 2d $\mathcal{N} = (0, 2)$ SQED and free fermions proposed in [15, 16].

3.1.2 2d U(1) + 2 hypers + 4 Fermi

Next consider a 2d $\mathcal{N} = (0, 4)$ U(1) gauge theory with two charged hypermultiplets and four charged Fermi multiplets. The gauge anomaly polynomial is

$$\mathcal{I}^{2d \text{ U}(1) + 2\text{HM} + 4\text{FM}} = \underbrace{-2 \cdot 2\mathbf{s}^2}_{2d \text{ hypers}} + \underbrace{4 \cdot \mathbf{s}^2}_{2d \text{ Fermi}} = 0 \quad (3.6)$$

and the theory is free from gauge anomaly.

The brane construction is depicted in figure 2.

The index takes the form

$$\begin{aligned} \mathbb{I}^{2d \text{ U}(1) + 2\text{HM} + 4\text{FM}}(t, x, z_\alpha; q) \\ = \underbrace{(q)_\infty^2 (q^{\frac{1}{2}} t^2; q)_\infty (q^{\frac{1}{2}} t^{-2}; q)_\infty}_{\mathbb{I}^{2d \text{ U}(1)}} \oint_{\text{JK}} \frac{ds}{2\pi i s} \\ \times \prod_{\alpha=1}^4 \underbrace{\frac{(q^{\frac{1}{2}} s^\pm z_\alpha^\pm; q)_\infty}{(q^{\frac{1}{4}} t s^\pm x^\pm; q)_\infty (q^{\frac{3}{4}} t^{-1} s^\pm x^\pm; q)_\infty (q^{\frac{1}{4}} t s^\pm x^\mp; q)_\infty (q^{\frac{3}{4}} t^{-1} s^\pm x^\mp; q)_\infty}}_{\mathbb{I}^{2d \text{ HM}(sx) \cdot \mathbb{I}^{2d \text{ HM}(sx^{-1}) \cdot F(q^{\frac{1}{2}} s z_\alpha)}} \end{aligned} \quad (3.7)$$

Figure 2. The brane box configuration of 2d $\mathcal{N} = (0,4)$ U(1) gauge theory with two charged hypermultiplets and four charged Fermi multiplets and the S-dual configuration.

with $\prod_{\alpha=1}^4 z_{\alpha} = 1$. Here we have a total of 8 factors both in the denominator and in the numerator.

Picking the pole for 2d hypermultiplet at $s = q^{\frac{1}{4}}tx$, we get

$$\mathbb{I}_1^{2d \text{ U}(1) + 2\text{HM} + 4\text{FM}}(t, x, z_{\alpha}; q) = \prod_{\alpha=1}^4 \frac{(q^{\frac{3}{4}}txz_{\alpha}; q)_{\infty}(q^{\frac{1}{4}}t^{-1}x^{-1}z_{\alpha}^{-1}; q)_{\infty}}{(q^{\frac{1}{2}}t^2x^2; q)_{\infty}(q^{\frac{1}{2}}t^{-2}x^{-2}; q)_{\infty}(qx^2; q)_{\infty}(x^{-2}; q)_{\infty}}. \quad (3.8)$$

$\mathbb{I}^{2d \text{ HM}}(q^{\frac{1}{4}}tx^2) \cdot F(q^{\frac{3}{4}}txz_{\alpha})$

The another pole for 2d hypermultiplet at $s = q^{\frac{1}{4}}tx^{-1}$ gives

$$\mathbb{I}_2^{2d \text{ U}(1) + 2\text{HM} + 4\text{FM}}(t, x, z_{\alpha}; q) = \prod_{\alpha=1}^4 \frac{(q^{\frac{3}{4}}tx^{-1}z_{\alpha}; q)_{\infty}(q^{\frac{1}{4}}t^{-1}xz_{\alpha}^{-1}; q)_{\infty}}{(q^{\frac{1}{2}}t^2x^{-2}; q)_{\infty}(q^{\frac{1}{2}}t^{-2}x^2; q)_{\infty}(x^2; q)_{\infty}(qx^{-2}; q)_{\infty}}. \quad (3.9)$$

$\mathbb{I}^{2d \text{ HM}}(q^{\frac{1}{4}}tx^{-2}) \cdot F(q^{\frac{3}{4}}tx^{-1}z_{\alpha})$

The total index is the sum of the two contributions:

$$\begin{aligned} & \mathbb{I}^{2d \text{ U}(1) + 2\text{HM} + 4\text{FM}}(t, x, z_{\alpha}; q) \\ &= \sum_{i=1}^2 \mathbb{I}_i^{2d \text{ U}(1) + 2\text{HM} + 4\text{FM}}(t, x, z_{\alpha}; q) \\ &= 1 + \left[t^2(1 + x^2 + x^{-2}) + t^{-2} \left(1 + \sum_{\alpha < \beta} z_{\alpha}^{-1} z_{\beta}^{-1} \right) \right] q^{\frac{1}{2}} + \dots \end{aligned} \quad (3.10)$$

The pole at $x^2 = 1$ cancels out and we get a nice fractional power series in q .

Unfortunately, we cannot immediately make contact with a mirror symmetry, in the sense that the original 2d gauge theory can be interpreted as 3d SQED₂ on a segment with Neumann b.c. ($\mathcal{N}', \mathcal{N}'$) and the mirror is 3d $\widetilde{\text{SQED}}_2$ on a segment with Dirichlet b.c. ($\mathcal{D}, \mathcal{D}$) and the dual photon gives a circle-valued scalar which needs to be treated with extra care.

Still, the Neumann b.c. of 3d SQED₂ preserves the SU(2) topological symmetry enhancement of the Coulomb branch of 3d SQED₂. The same enhancement should happen

Figure 3. The brane box configuration of 2d $\mathcal{N} = (0, 4)$ U(1) gauge theory with a charged hypermultiplet, a charged twisted hypermultiplet, four charged Fermi multiplets and a neutral Fermi multiplet and the S-dual configuration.

then in the 2d setup. If we look at the block-diagonal $SU(2) \times SU(2) \times U(1)$ subgroup of the $SU(4)$ rotating the Fermi multiplet, the $U(1)$ should enhance to a third $SU(2)$. This would imply that $SU(4) \simeq SO(6)$ is enhanced to $SO(7)$. This is indeed a symmetry of the index.

3.1.3 2d U(1) + hyper + twisted hyper + 4 Fermi + neutral Fermi

Let us consider a 2d $\mathcal{N} = (0, 4)$ U(1) gauge theory with a charged hypermultiplet, a charged twisted hypermultiplet, four charged Fermi multiplets and a neutral Fermi multiplet. The gauge anomaly polynomial is

$$\mathcal{I}^{2d \text{ U}(1)+\text{HM}+\text{tHM}+4\text{FM}} = \underbrace{-2s^2}_{2d \text{ hyper}} \underbrace{-2s^2}_{2d \text{ thyper}} + \underbrace{4 \cdot s^2}_{2d \text{ Fermi}} = 0 \quad (3.11)$$

and the theory is free from gauge anomaly. The coexistence of hyper and twisted hyper multiplets requires the neutral Fermi multiplet which achieve their cubic coupling [63].

The brane construction is illustrated in figure 3. The $\mathcal{N} = (0, 4)$ U(1) vector multiplet is realized as a box of D3-brane, the hyper and twisted hyper multiplets are produced by the D5- and D5'-branes respectively. There are two intersections of D5- and NS'-branes and two intersections of D5'- and NS-branes. They lead to four charged Fermi multiplets. In addition, a neutral Fermi multiplet appears at the D5-D5' junction.

The index is given by

$$\begin{aligned} & \mathbb{I}^{2d \text{ U}(1)+\text{HM}+\text{tHM}+4\text{FM}+\text{nFM}}(t, x, z_\alpha; q) \\ &= \underbrace{(q)_\infty^2 (q^{\frac{1}{2}} t^2; q)_\infty (q^{\frac{1}{2}} t^{-2}; q)_\infty \oint_{JK} \frac{ds}{2\pi i s}}_{\mathbb{I}^{2d \text{ U}(1)}} \\ & \times \prod_{\alpha=1}^4 \underbrace{\frac{(q^{\frac{1}{2}} s^\pm z_\alpha^\pm; q)_\infty \cdot (q^{\frac{1}{2}} x^{\pm 2}; q)_\infty}{(q^{\frac{1}{4}} t s^\pm x^\mp; q)_\infty (q^{\frac{3}{4}} t^{-1} s^\pm x^\mp; q)_\infty \cdot (q^{\frac{1}{4}} t^{-1} s^\pm x^\pm; q)_\infty (q^{\frac{3}{4}} t s^\pm x^\pm; q)_\infty}}_{\mathbb{I}^{2d \text{ HM}}(sx^{-1}) \cdot \mathbb{I}^{2d \text{ tHM}}(sx) \cdot F(q^{\frac{1}{2}} s z_\alpha) \cdot F(q^{\frac{1}{2}} x^2)}. \end{aligned} \quad (3.12)$$

Here we have a total of 8 factors in the denominator and 8 factors in the numerator for the integrand. We need $\prod_{\alpha=1}^4 z_{\alpha} = 1$. We gave the neutral Fermi multiplet a fugacity needed for a cubic coupling to the hyper and twisted hyper.

Picking the pole for 2d hypermultiplet at $s = q^{\frac{1}{4}}tx$ gives

$$\mathbb{I}_1^{2d \text{ U}(1)+\text{HM}+\text{tHM}+4\text{FM}+\text{nFM}}(t, x, z_{\alpha}; q) = \prod_{\alpha=1}^4 \frac{(q^{\frac{3}{4}}tz_{\alpha}^{\pm}x^{\pm}; q)_{\infty}}{\underbrace{(t^{-2}x^{-2}; q)_{\infty}(qx^2; q)_{\infty}}_{C(t^{-2}x^2) \cdot F(q^{\frac{3}{4}}tz_{\alpha}x)}} \quad (3.13)$$

where the Fermi multiplet cancels part of the twisted hypermultiplet contribution.

On the other hand, picking the pole for 2d twisted hypermultiplet at $s = q^{\frac{1}{4}}t^{-1}x^{-1}$, we obtain

$$\mathbb{I}_2^{2d \text{ U}(1)+\text{HM}+\text{tHM}+4\text{FM}+\text{nFM}}(t, x, z_{\alpha}; q) = \prod_{\alpha=1}^4 \frac{(q^{\frac{3}{4}}t^{-1}z_{\alpha}^{\pm}x^{\mp}; q)_{\infty}}{\underbrace{(t^2x^2; q)_{\infty}(qt^{-2}x^{-2}; q)_{\infty}}_{C(t^2x^2) \cdot F(q^{\frac{3}{4}}t^{-1}z_{\alpha}x^{-1})}} \quad (3.14)$$

with the Fermi multiplet canceling part of the hypermultiplet contribution.

The total index is the sum of the two contributions

$$\begin{aligned} & \mathbb{I}^{2d \text{ U}(1)+\text{HM}+\text{tHM}+4\text{FM}+\text{nFM}}(t, x, z_{\alpha}; q) \\ &= \sum_{i=1}^2 \mathbb{I}_i^{2d \text{ U}(1)+\text{HM}+\text{tHM}+4\text{FM}+\text{nFM}}(t, x, z_{\alpha}; q) \\ &= 1 - \frac{(z_1 + z_2 + z_3 + z_1z_2^2z_3^2 + z_1^2z_2z_3^2 + z_1^2z_2^2z_3)}{z_1z_2z_3} q^{\frac{1}{2}} + \dots \end{aligned} \quad (3.15)$$

We see that the resulting index (3.15) has no dependence on the fugacities x and t .

From the brane picture in figure 3, the dual theory has no gauge symmetry so that the index is expected to take the form of product of full-indices. In fact, we find that this index matches a very simple answer:

$$\mathbb{I}^{2d \text{ U}(1)+\text{HM}+\text{tHM}+4\text{FM}+\text{nFM}}(t, x, z_{\alpha}; q) = \underbrace{(q^{\frac{1}{2}}z_1^{\pm}z_2^{\pm}; q)_{\infty} \cdot (q^{\frac{1}{2}}z_2^{\pm}z_3^{\pm}; q)_{\infty} \cdot (q^{\frac{1}{2}}z_3^{\pm}z_1^{\pm}; q)_{\infty}}_{F(q^{\frac{1}{2}}z_1z_2) \cdot F(q^{\frac{1}{2}}z_2z_3) \cdot F(q^{\frac{1}{2}}z_3z_1)} \quad (3.16)$$

This is identified with the index of six real Fermi multiplets transforming in the fundamental representation under the $\text{SO}(6)$ flavor symmetry. Our index computation indicates the $\mathcal{N} = (0, 4)$ mirror symmetry:

$$2d \mathcal{N} = (0, 4) \text{ U}(1) \text{ with } 1 \text{ hyper} + 1 \text{ thyper} + 4 \text{ Fermi} \leftrightarrow 2d \text{ free SO}(6) \text{ Fermi}. \quad (3.17)$$

In order to give a 3d or 4d interpretation of this, we would need to consider enriched Neumann boundary conditions for 3d gauge theories.

Figure 4. The brane box configuration of 2d $\mathcal{N} = (0,4)$ U(1) gauge theory with two charged hypermultiplets, a charged twisted hypermultiplet, six charged Fermi multiplets and two neutral Fermi multiplets and the S-dual configuration.

3.1.4 2d U(1) + 2 hypers + twisted hyper + 6 Fermi + 2 neutral Fermi

Let us consider a 2d $\mathcal{N} = (0,4)$ U(1) gauge theory with two charged hypermultiplets, twisted hypermultiplet, six charged Fermi multiplets and two neutral Fermi multiplets. The gauge anomaly polynomial is

$$\mathcal{I}^{2d \text{ U}(1) + 2\text{HM} + \text{tHM} + 6\text{FM} + 2\text{nFM}} = \underbrace{-2 \cdot 2s^2}_{2d \text{ hyper}} \underbrace{-2s^2}_{2d \text{ thyper}} + \underbrace{6 \cdot s^2}_{2d \text{ Fermi}} = 0 \quad (3.18)$$

and the theory is gauge anomaly free.

The brane construction is shown in figure 4. The two D5-branes and a D5'-brane lead to the two 3d charged hypermultiplets and a twisted hyper multiplet. Corresponding to the four NS5'-D5 junctions and two NS5-D5' junctions, the effective theory has six charged Fermi multiplets. In addition, there are two D5-D5' junction, which yields two neutral Fermi multiplets.

We can compute the index as

$$\begin{aligned} & \mathbb{I}^{2d \text{ U}(1) + 2\text{HM} + \text{tHM} + 6\text{FM} + 2\text{nFM}}(t, x, z_\alpha; q) \\ &= \underbrace{(q)_\infty^2 (q^{\frac{1}{2}} t^2; q)_\infty (q^{\frac{1}{2}} t^{-2}; q)_\infty \oint_{\text{JK}} \frac{ds}{2\pi i s}}_{\mathbb{I}^{2d \text{ U}(1)}} \\ & \times \prod_{\alpha=1}^6 \underbrace{(q^{\frac{1}{2}} s^\pm z_\alpha^\pm; q)_\infty \cdot (q^{\frac{1}{2}} x^{\pm 3} u^\mp; q)_\infty \cdot (q^{\frac{1}{2}} x^{\pm 3} u^\pm; q)_\infty}_{F(q^{\frac{1}{2}} s z_\alpha) \cdot F(q^{\frac{1}{2}} x^3 u^{-1}) \cdot F(q^{\frac{1}{2}} x^3 u)} \\ & \times \frac{1}{\underbrace{(q^{\frac{1}{4}} t s^\pm x^\mp u^\pm; q)_\infty (q^{\frac{3}{4}} t^{-1} s^\pm x^\mp u^\pm; q)_\infty \cdot (q^{\frac{1}{4}} t s^\pm x^\mp u^\mp; q)_\infty (q^{\frac{3}{4}} t^{-1} s^\pm x^\mp u^\mp; q)_\infty}_{\mathbb{I}^{2d \text{ HM}}(sx^{-1}u) \cdot \mathbb{I}^{2d \text{ HM}}(sx^{-1}u^{-1})}} \\ & \times \frac{1}{\underbrace{(q^{\frac{1}{4}} t^{-1} s^\pm x^{\pm 2}; q)_\infty (q^{\frac{3}{4}} t s^\pm x^{\pm 2}; q)_\infty}_{\mathbb{I}^{2d \text{ tHM}}(sx^2)}} \end{aligned} \quad (3.19)$$

where the integrand includes a total of 12 factors in the denominator and 12 factors in the numerator. Correspondingly, we have three contributions to the index.

The first pole for 2d hypermultiplet at $s = q^{\frac{1}{4}}txu^{-1}$ gives

$$\begin{aligned} \mathbb{I}_1^{2d \text{ U}(1) + 2\text{HM} + \text{tHM} + 6\text{FM} + 2\text{nFM}}(t, x, z_\alpha; q) \\ = \prod_{\alpha=1}^4 \frac{(q^{\frac{3}{4}}tz_\alpha xu^{-1}; q)_\infty (q^{\frac{1}{4}}t^{-1}z_\alpha^{-1}x^{-1}u; q)_\infty \cdot (q^{\frac{1}{2}}x^{\pm 3}u^\pm; q)_\infty}{\underbrace{(q^{\frac{1}{2}}t^2u^{-2}; q)_\infty (q^{\frac{1}{2}}t^{-2}u^2; q)_\infty \cdot (u^2; q)_\infty (qu^{-2}; q)_\infty \cdot (t^{-2}x^{-3}u; q)_\infty (qt^2x^3u^{-1}; q)_\infty}_{F(q^{\frac{3}{4}}tz_\alpha xu^{-1}) \cdot F(q^{\frac{1}{2}}x^3u) \cdot C(q^{\frac{1}{2}}t^2u^{-2}) \cdot C(u^2) \cdot C(t^{-2}x^{-3}u)}}. \end{aligned} \quad (3.20)$$

Picking the another pole for 2d hypermultiplet at $s = q^{\frac{1}{4}}txu$, we obtain

$$\begin{aligned} \mathbb{I}_2^{2d \text{ U}(1) + 2\text{HM} + \text{tHM} + 6\text{FM} + 2\text{nFM}}(t, x, z_\alpha; q) \\ = \prod_{\alpha=1}^4 \frac{(q^{\frac{3}{4}}tz_\alpha xu; q)_\infty (q^{\frac{1}{4}}t^{-1}z_\alpha^{-1}x^{-1}u^{-1}; q)_\infty \cdot (q^{\frac{1}{2}}x^{\pm 3}u^\mp; q)_\infty}{\underbrace{(q^{\frac{1}{2}}t^2u^2; q)_\infty (q^{\frac{1}{2}}t^{-2}u^{-2}; q)_\infty \cdot (u^{-2}; q)_\infty (qu^2; q)_\infty \cdot (t^{-2}x^{-3}u^{-1}; q)_\infty (qt^2x^3u; q)_\infty}_{F(q^{\frac{3}{4}}tz_\alpha xu) \cdot F(q^{\frac{1}{2}}x^3u^{-1}) \cdot C(q^{\frac{1}{2}}t^2u^2) \cdot C(u^{-2}) \cdot C(t^{-2}x^{-3}u^{-1})}}. \end{aligned} \quad (3.21)$$

Picking the pole for twisted hypermultiplet at $s = q^{\frac{1}{4}}t^{-1}x^{-2}$, we get

$$\begin{aligned} \mathbb{I}_3^{2d \text{ U}(1) + 2\text{HM} + \text{tHM} + 6\text{FM} + 2\text{nFM}}(t, x, z_\alpha; q) \\ = \prod_{\alpha=1}^4 \frac{(q^{\frac{3}{4}}t^{-1}z_\alpha x^{-2}; q)_\infty (q^{\frac{1}{4}}tz_\alpha^{-1}x^2; q)_\infty}{\underbrace{(t^2x^3u; q)_\infty (qt^{-2}x^{-3}u^{-1}; q)_\infty \cdot (t^2x^3u^{-1}; q)_\infty (qt^{-2}x^{-3}u; q)_\infty}_{F(q^{\frac{3}{4}}t^{-1}z_\alpha x^{-2}) \cdot C(t^2x^3u) \cdot C(t^2x^3u^{-1})}}. \end{aligned} \quad (3.22)$$

The sum simplifies, which is nicely compatible with the expectation. The total index has no $q^{\frac{1}{4}}$ term but has $q^{\frac{1}{2}}$ and $q^{\frac{3}{4}}$ terms which do not cancel. However we have not understood so far the expression in terms of a simple product of free fields. It would be interesting to figure out the appropriate expression and identify the matter fields in the S-dual configuration of figure 4.

4 $\mathcal{N} = (0, 4)$ boundaries of 3d theories

The $\mathcal{N} = (0, 4)$ boundary conditions for 3d $\mathcal{N} = 4$ Abelian gauge theories can be realized in the brane setup [22, 35]. We check the matching of boundary anomalies and half-indices for dual pairs of boundary conditions.

4.1 3d SQED₁

Consider 3d $\mathcal{N} = 4$ U(1) gauge theory with a single charged hypermultiplet, SQED₁. It has a U(1)_t topological symmetry. The mirror of SQED₁ is the theory of a free twisted hypermultiplet [62]. Under mirror symmetry the topological symmetry for SQED₁ maps to a U(1) flavor symmetry for a twisted hypermultiplet.

Figure 5. The brane constructions of the Neumann b.c. $(\mathcal{N}', N') + \Gamma$ for SQED_1 and its dual Dirichlet b.c. D for twisted hypermultiplet.

4.1.1 $(\mathcal{N}', N' + \text{Fermi}) \leftrightarrow D$

We begin with the boundary condition which includes Neumann b.c. $\mathcal{N}'$ for $U(1)$ vector multiplet and Neumann b.c. N' for a charged hypermultiplet which contains a complex scalar $\mathbb{H}$. The corresponding brane construction is depicted in figure 5. The Neumann b.c. for gauge multiplet requires that the gauge symmetry is preserved at the boundary. Therefore the boundary gauge anomaly must be cancelled. From the brane box analysis, there would be a boundary charged Fermi multiplet [35]. In fact, a gauge anomaly free boundary condition can be obtained by introducing boundary charged Fermi multiplet Γ .

From the S-dual configuration in figure 5, we expect that the b.c. $(\mathcal{N}', N' + \Gamma)$ for SQED_1 is dual to Dirichlet b.c. D for twisted hypermultiplet which contains a complex scalar field $\mathbb{T}$.

The charges of matter fields are given by

matter \ symmetry	$G = U(1)$	$U(1)_t$
$\mathbb{H}$	+	0
Γ	+	+
$\mathbb{T}$	0	+

(4.1)

Let $\mathbf{s}$ and $\mathbf{z}$ be the field strengths of $G = U(1)$ gauge symmetry and $U(1)_t$ topological symmetry for SQED_1 . The boundary anomaly polynomial for Neumann b.c. $(\mathcal{N}', N')$ together with boundary Fermi multiplet Γ is

$$\mathcal{I}_{\mathcal{N}', N' + \Gamma}^{\text{3d SQED}_1} = \underbrace{-\mathbf{s}^2}_{N'} + \underbrace{(\mathbf{s} + \mathbf{z})^2}_{\Gamma} - \underbrace{2\mathbf{s} \cdot \mathbf{z}}_{\text{FI}} = \mathbf{z}^2 \quad (4.2)$$

where the first term is contributed from the charged hyper with Neumann b.c. N' , the second is contributed from the boundary Fermi Γ and the last is the contribution from the FI term [9].

The boundary anomaly polynomial for Dirichlet b.c. D of free twisted hypermultiplet is

$$\mathcal{I}_D^{\text{tHM}} = \mathbf{z}^2, \quad (4.3)$$

which agrees with the boundary anomaly (4.2) for SQED_1 with b.c. $(\mathcal{N}', N', \Gamma)$.

Figure 6. Higgsing precess of Neumann b.c. $(\mathcal{N}', N' + \Gamma)$ for SQED_1 that separates the D5-brane.

In addition, one can check the duality by computing half-indices. Let s and z be the fugacity for $\text{U}(1)$ gauge symmetry and $\text{U}(1)$ topological symmetry respectively. The half-index for the boundary condition $(\mathcal{N}', N' + \Gamma)$ for SQED_1 takes the form

$$\mathbb{I}_{\mathcal{N}', N' + \Gamma}^{\text{3d SQED}_1}(t, z; q) = \underbrace{(q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty}_{\mathbb{I}_{\mathcal{N}'}^{\text{3d U}(1)}} \oint \frac{ds}{2\pi i s} \underbrace{\frac{(q^{\frac{1}{2}} s z; q)_\infty (q^{\frac{1}{2}} s^{-1} z^{-1}; q)_\infty}{(q^{\frac{1}{4}} t s; q)_\infty (q^{\frac{1}{4}} t s^{-1}; q)_\infty}}_{\mathbb{I}_{N'}^{\text{3d HM}}(s) \cdot F(q^{\frac{1}{2}} s z)}. \quad (4.4)$$

We can evaluate the half-index (4.4) by picking up the residues at poles $s = q^{\frac{1}{4} + m} t$, $m = 0, 1, \dots$ for the fundamental hypermultiplet

$$\mathbb{I}_{\mathcal{N}', N' + \Gamma}^{\text{3d SQED}_1}(t, z; q) = \frac{(q^{\frac{3}{4}} t z^{-1}; q)_\infty (q^{\frac{1}{4}} t^{-1} z; q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty}{(q)_\infty} \sum_{m=0}^{\infty} \frac{(q^{1+m}; q)_\infty}{(q^{\frac{1}{2} + m} t^2; q)_\infty} q^{\frac{m}{4}} t^{-m} z^m. \quad (4.5)$$

Making use of the q -binomial theorem, we find that the half-index (4.5) is equal to

$$\mathbb{I}_D^{\text{3d tHM}}(t, z; q) = (q^{\frac{3}{4}} t z; q)_\infty (q^{\frac{3}{4}} t z^{-1}; q)_\infty. \quad (4.6)$$

As we expect, this is the half-index of Dirichlet b.c. D for a free twisted hypermultiplet.

We can alternatively expand the half-index (4.4) as

$$\mathbb{I}_{\mathcal{N}', N' + \Gamma}^{\text{3d SQED}_1}(t, z; q) = \frac{(q^{\frac{1}{2}} t^2; q)_\infty}{(q)_\infty} (q^{\frac{1}{4}} t^{-1} z; q)_\infty (q^{\frac{1}{4}} t^{-1} z^{-1}; q)_\infty \sum_{m=0}^{\infty} \frac{(q^{1+m}; q)_\infty^2}{(q^{\frac{1}{4} + m} t^{-1} z^\pm; q)_\infty}. \quad (4.7)$$

The first term in the sum (4.7) is identified with the half-index $\mathbb{I}_{\mathcal{N}'}^{\text{3d U}(1)}$ of Neumann b.c. $\mathcal{N}'$ for 3d $\text{U}(1)$ vector multiplet. The associated Higgsing precess would separate the D5-brane from the D3-NS5-NS5' system (see figure 6).

We can add a Wilson line operator of charge n ending on a boundary. The half-index reads

$$\mathbb{I}_{\mathcal{N}', N' + \Gamma + \mathcal{W}_n}^{\text{3d SQED}_1}(t, z; q) = (q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty \oint \frac{ds}{2\pi i s} \frac{(q^{\frac{1}{2}} s z; q)_\infty (q^{\frac{1}{2}} s^{-1} z^{-1}; q)_\infty}{(q^{\frac{1}{4}} t s; q)_\infty (q^{\frac{1}{4}} t s^{-1}; q)_\infty} s^n. \quad (4.8)$$

Figure 7. The brane constructions of the Dirichlet b.c. $(\mathcal{D}', D')$ for SQED_1 and its dual Neumann b.c. N for twisted hypermultiplet with a boundary Fermi multiplet.

By taking the sum over residues at hypermultiplet poles $s = q^{\frac{1}{4}+m}t$, we get

$$\mathbb{I}_{\mathcal{N}', N' + \Gamma + \mathcal{W}_n}^{3d \text{ SQED}_1} = \frac{(q^{\frac{1}{4}}t^{-1}z^{-1}; q)_n}{(q^{\frac{3}{4}}tz^{-1}; q)_n} q^{\frac{n}{4}}t^n \times \mathbb{I}_D^{3d \text{ tHM}}(t, z; q). \quad (4.9)$$

4.1.2 $(\mathcal{D}', D') \leftrightarrow (N + \text{Fermi})$

Next consider the boundary condition consisting of Dirichlet b.c. $\mathcal{D}'$ for $U(1)$ vector multiplet and Dirichlet b.c. D for a charged hypermultiplet. According to the Dirichlet b.c. $\mathcal{D}$, the gauge symmetry is broken to a boundary global $U(1)_\partial$ symmetry. The brane setup is illustrated in figure 7.

From the brane picture, the dual boundary condition would include Neumann b.c. N for twisted hypermultiplet corresponding to the NS5-brane on which the D3-brane terminate.

The boundary 't Hooft anomaly of the boundary condition $(\mathcal{D}', D')$ for SQED_1 is

$$\mathcal{I}_{\mathcal{D}', D'}^{3d \text{ SQED}_1} = \underbrace{\mathbf{u}^2}_{D'} - \underbrace{2\mathbf{u} \cdot \mathbf{z}}_{\text{FI}} \quad (4.10)$$

where $\mathbf{u}$ is the field strength for $U(1)_\partial$.

On the other hand, the boundary 't Hooft anomaly of the Neumann b.c. N for twisted hypermultiplet is

$$\mathcal{I}_N^{3d \text{ tHM}} = -\mathbf{z}^2. \quad (4.11)$$

The difference

$$\mathcal{I}_{\mathcal{D}', D'}^{3d \text{ SQED}_1} - \mathcal{I}_N^{3d \text{ tHM}} = (\mathbf{u} - \mathbf{z})^2 \quad (4.12)$$

of the boundary anomalies (4.10) and (4.11) indicates the presence of a boundary Fermi multiplet $\tilde{\Gamma}$ which carries the charges $(+, -)$ under the $U(1)_\partial \times U(1)_t$.

The charges of matter fields are summarized as The charges of matter fields are given by

matter \ symmetry	$U(1)_\partial$	$U(1)_t$
$\mathbb{H}$	+	0
$\mathbb{T}$	0	+
$\tilde{\Gamma}$	+	-

(4.13)

Figure 8. The brane constructions of the Neumann b.c. $(\mathcal{N}', N' + \{\Gamma^{(i)}\})$ for $T[\text{SU}(2)]$ and its dual Dirichlet b.c. $(\mathcal{D}, D)$ for $T[\text{SU}(2)]$.

Now compute the half-indices. Let u be the fugacity for the boundary global $\text{U}(1)_\partial$ symmetry. From the boundary 't Hooft anomaly (4.10), the half-index of boundary condition $(\mathcal{D}', D')$ for SQED_1 reads

$$\begin{aligned} \mathbb{I}_{\mathcal{D}', D'}^{\text{3d SQED}_1}(t, u; q) &= \frac{1}{(q)_\infty (q^{\frac{1}{2}} t^{-2}; q)_\infty} \sum_{m \in \mathbb{Z}} \underbrace{(q^{\frac{3}{4}+m} t^{-1} u; q)_\infty (q^{\frac{3}{4}-m} t^{-1} u^{-1}; q)_\infty}_{\mathbb{I}_{D'}^{\text{3d HM}}(q^m u; q)} \\ &\times q^{\frac{m^2}{2}} (-1)^m u^m z^{-m}. \end{aligned} \quad (4.14)$$

On the other hand, the half-index of boundary condition N for twisted hypermultiplet involving the boundary Fermi multiplet $\tilde{\Gamma}$ which takes the form

$$\mathbb{I}_{N+\tilde{\Gamma}}^{\text{3d tHM}}(t, u, z; q) = \frac{(q^{\frac{1}{2}} z u^{-1}; q)_\infty (q^{\frac{1}{2}} z^{-1} u; q)_\infty}{(q^{\frac{1}{4}} t^{-1} z; q)_\infty (q^{\frac{1}{4}} t^{-1} z^{-1}; q)_\infty} \quad (4.15)$$

where the numerator is the Fermi index $F(q^{\frac{1}{2}} u z^{-1})$ and the denominator is the Neumann half-index $\mathbb{I}_N^{\text{3d tHM}}(z)$ for 3d twisted hypermultiplet. We have checked that the half-indices (4.14) and (4.15) agree with each other.

4.2 $T[\text{SU}(2)]$

Consider the 3d $\mathcal{N} = 4$ $\text{U}(1)$ gauge theory with two charged hypermultiplets, which we call $T[\text{SU}(2)]$. This is the self-mirror theory with $\text{SU}(2)$ flavor symmetry and $\text{SU}(2)$ topological symmetry.

4.2.1 $(\mathcal{N}', N' + \text{Fermi}) \leftrightarrow (\mathcal{D}, D)$

One can find $\mathcal{N} = (0, 4)$ boundary condition for $T[\text{SU}(2)]$ from the brane setup depicted in figure 8. The presence of $\text{NS5}'$ -brane requires the Neumann b.c. $\mathcal{N}'$ for $\text{U}(1)$ vector multiplet and Neumann b.c. N' for two charged hypermultiplets involving two complex scalar fields $\mathbb{H}^{(1)}, \mathbb{H}^{(2)}$. From the brane box analysis [35], there would be two boundary charged Fermi multiplets $\Gamma^{(1)}, \Gamma^{(2)}$ which cancel the gauge anomaly. We denote this boundary condition by $(\mathcal{N}', N' + \{\Gamma^{(i)}\}_{i=1,2})$.

Under the action of S-duality, we find the dual boundary condition for $T[\widetilde{\text{SU}(2)}]$, which is the twisted version of $T[\text{SU}(2)]$ involving two complex scalar fields $\mathbb{T}^{(1)}, \mathbb{T}^{(2)}$, required from the D5-brane (see figure 8). We expect that the b.c. $(\mathcal{N}', N' + \{\Gamma^{(i)}\}_{i=1,2})$ for $T[\text{SU}(2)]$ is dual to Dirichlet b.c. $(\mathcal{D}, D)$ for $T[\widetilde{\text{SU}(2)}]$.

The charges of matter fields read

matter \ symmetry	U(1)	F	$\widetilde{\text{U}(1)}_\partial$	$\widetilde{F}$
$\mathbb{H}^{(1)}$	+	+	0	0
$\mathbb{H}^{(2)}$	+	−	0	0
$\Gamma^{(1)}$	+	+	+	+
$\Gamma^{(2)}$	+	−	−	+
$\mathbb{T}^{(1)}$	0	0	+	+
$\mathbb{T}^{(2)}$	0	0	+	−

(4.16)

where F and $\widetilde{F}$ are the charges for $\text{SU}(2)$ flavor and topological symmetries of $T[\text{SU}(2)]$.

Let $\mathbf{x}$ and $\mathbf{z}$ be the field strengths of the flavor and topological symmetries. The boundary anomaly polynomial of Neumann b.c. $(\mathcal{N}', N' + \Gamma^{(i)})$ for $T[\text{SU}(2)]$ is calculated as

$$\begin{aligned} \mathcal{I}_{\mathcal{N}', N' + \Gamma^{(i)}}^{T[\text{SU}(2)]} &= \underbrace{-(\mathbf{s} + \mathbf{x})^2 - (\mathbf{s} - \mathbf{x})^2}_{N'} + \underbrace{(\mathbf{s} + \mathbf{x} + \mathbf{u} + \mathbf{z})^2 + (\mathbf{s} - \mathbf{x} - \mathbf{u} + \mathbf{z})^2}_{\Gamma^{(i)}} \underbrace{- 2\mathbf{s} \cdot 2\mathbf{z}}_{\text{FI}} \\ &= 2\mathbf{u}^2 + 2\mathbf{z}^2 + 4\mathbf{u} \cdot \mathbf{x}. \end{aligned} \quad (4.17)$$

On the other hand, the boundary anomaly polynomial of Dirichlet b.c. $(\mathcal{D}, D)$ for $T[\widetilde{\text{SU}(2)}]$ is

$$\mathcal{I}_{\mathcal{D}, D}^{T[\widetilde{\text{SU}(2)}]} = \underbrace{(\mathbf{u} + \mathbf{z})^2 + (\mathbf{u} - \mathbf{z})^2}_D + \underbrace{2\mathbf{u} \cdot 2\mathbf{x}}_{\text{FI}} = 2\mathbf{u}^2 + 2\mathbf{z}^2 + 4\mathbf{u} \cdot \mathbf{x}. \quad (4.18)$$

As required for anomaly matching in the RG flows, (4.17) and (4.18) agree with each other.

Let us compute the half-indices for the dual boundary conditions. We can evaluate the half-index of Neumann b.c. $(\mathcal{N}', N' + \Gamma^{(i)})$ for $T[\text{SU}(2)]$ as

$$\begin{aligned} \mathbb{III}_{\mathcal{N}', N' + \Gamma^{(i)}}^{T[\text{SU}(2)]}(t, x, u, z; q) &= \underbrace{(q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(1)}} \oint \frac{ds}{2\pi i s} \underbrace{\frac{(q^{\frac{1}{2}} s^\mp x^\pm u^\pm z^\pm; q)_\infty}{(q^{\frac{1}{4}} t s^\mp x^\pm; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}(s^{-1}x) \cdot F(q^{\frac{1}{2}} s^{-1} x u z)}} \cdot \underbrace{\frac{(q^{\frac{1}{2}} s^\pm x^\pm u^\pm z^\mp; q)_\infty}{(q^{\frac{1}{4}} t s^\pm x^\pm; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}(s x) \cdot F(q^{\frac{1}{2}} s x u z^{-1})}}. \end{aligned} \quad (4.19)$$

From the boundary 't Hooft anomaly (4.18), we obtain the half-index of Dirichlet b.c. $(\mathcal{D}, D)$ for $T[\widetilde{\text{SU}(2)}]$

$$\begin{aligned} \mathbb{III}_{\mathcal{D}, D}^{T[\widetilde{\text{SU}(2)}]}(t, x, u, z; q) &= \frac{1}{(q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty} \sum_{m \in \mathbb{Z}} q^{m^2} u^{2m} x^{2m} \underbrace{(q^{\frac{3}{4} \pm m} t u^\pm z^\pm; q)_\infty}_{\mathbb{III}_D^{3d \text{ tHM}(q^m u z)}} \cdot \underbrace{(q^{\frac{3}{4} \pm m} t u^\pm z^\mp; q)_\infty}_{\mathbb{III}_D^{3d \text{ tHM}(q^m u z^{-1})}}. \end{aligned} \quad (4.20)$$

In fact, the half-indices (4.19) and (4.20) coincide.

4.3 3d SQED₃

Next we turn to the 3d $\mathcal{N} = 4$ $\widetilde{\text{U}(1)}$ gauge theory with three charged hypermultiplets, SQED₃. The mirror theory is $\text{U}(1) \times \widetilde{\text{U}(1)}$ quiver gauge theory with three twisted hypermultiplets, which we denote by $[1] - (1) - (1) - [1]$.⁶

4.3.1 $(\mathcal{N}', N' + \text{Fermi}) \leftrightarrow (\mathcal{D}, D)$

The $\mathcal{N} = (0, 4)$ boundary condition can be realized in the brane setup. When the single D3-brane stretched between two NS5-branes which terminates on the NS5'-brane and the three D5-branes intersect with NS'-brane, one finds Neumann b.c. $\mathcal{N}'$ for $\text{U}(1)$ vector multiplet and Neumann b.c. N' for hypermultiplets involving complex scalar fields $\{\mathbb{H}^{(i)}\}_{i=1,2,3}$. Besides, the configuration should have three charged Fermi multiplets that cancel the boundary gauge anomaly.

By applying S-duality, we find the mirror boundary condition as Dirichlet b.c. $\mathcal{D}$ for $\text{U}(1) \times \widetilde{\text{U}(1)}$ twisted vector multiplet and Dirichlet b.c. D for twisted hypermultiplets containing complex scalar fields $\{\mathbb{T}^{(i)}\}_{i=1,2,3}$.

The charges of matter fields are given by

matter \ symmetry	$\text{U}(1)$	F	$\widetilde{\text{U}(1) \times \text{U}(1)}_{\partial}$	$\tilde{F}$
$\mathbb{H}^{(1)}$	+	$(+, 0, 0)$	$(0, 0)$	$(0, 0)$
$\mathbb{H}^{(2)}$	+	$(0, +, 0)$	$(0, 0)$	$(0, 0)$
$\mathbb{H}^{(3)}$	+	$(0, 0, +)$	$(0, 0)$	$(0, 0)$
$\Gamma^{(1)}$	+	$(+, 0, 0)$	$(-, 0)$	$(+, 0)$
$\Gamma^{(2)}$	+	$(0, +, 0)$	$(+, -)$	$(0, 0)$
$\Gamma^{(3)}$	+	$(0, 0, +)$	$(0, +)$	$(0, -)$
$\mathbb{T}^{(1)}$	0	$(0, 0, 0)$	$(-, 0)$	$(+, 0)$
$\mathbb{T}^{(2)}$	0	$(0, 0, 0)$	$(+, -)$	$(0, 0)$
$\mathbb{T}^{(3)}$	0	$(0, 0, 0)$	$(0, +)$	$(0, -)$

(4.21)

where F is the charges for $\text{SU}(3)$ flavor symmetry of SQED₃ and $\tilde{F}$ is the charges for $\text{U}(1) \times \widetilde{\text{U}(1)}$ topological symmetry of SQED₃.

Let $\mathbf{s}$, $\mathbf{x}_i$, $\mathbf{u}_i$, $\mathbf{z}_i$ be the Abelian field strengths for $\text{U}(1)$, F , $\widetilde{\text{U}(1) \times \text{U}(1)}_{\partial}$ and $\tilde{F}$ respectively. The boundary anomaly polynomial of Neumann b.c. $(\mathcal{N}', N' + \Gamma^{(i)})$ for SQED₃

⁶Here and in the following the round brackets (N) represent $\text{U}(N)$ gauge symmetry and the square brackets $[M]$ denote $\text{U}(M)$ global symmetry. The dash lines are bi-fundamental (twisted)hypermultiplets. The tildes indicate that the corresponding gauge theories are constructed by twisted supermultiplets.

is

$$\begin{aligned}
 \mathcal{I}_{\mathcal{N}', N' + \Gamma^{(i)}}^{\text{3d SQED}_3} &= \underbrace{-(\mathbf{s} + \mathbf{x}_1)^2 - (\mathbf{s} + \mathbf{x}_2)^2 - (\mathbf{s} + \mathbf{x}_3)^2}_{N'} \\
 &\quad + \underbrace{(\mathbf{s} + \mathbf{x}_1 - \mathbf{u}_1 + \mathbf{z}_1)^2 + (\mathbf{s} + \mathbf{x}_2 + \mathbf{u}_1 - \mathbf{u}_2)^2 + (\mathbf{s} + \mathbf{x}_3 + \mathbf{u}_2 - \mathbf{z}_2)^2}_{\Gamma} \\
 &\quad - \underbrace{2\mathbf{s} \cdot (\mathbf{z}_1 - \mathbf{z}_2) - 2\mathbf{x}_1 \cdot \mathbf{z}_1 + 2\mathbf{x}_3 \cdot \mathbf{z}_2}_{\text{FI}} \\
 &= 2\mathbf{u}_1^2 + 2\mathbf{u}_2^2 - 2\mathbf{u}_1 \cdot \mathbf{u}_2 - 2\mathbf{u}_1 \cdot (\mathbf{x}_1 - \mathbf{x}_2 + \mathbf{z}_1) - 2\mathbf{u}_2 \cdot (\mathbf{x}_2 - \mathbf{x}_3 + \mathbf{z}_2) + \mathbf{z}_1^2 + \mathbf{z}_2^2.
 \end{aligned} \tag{4.22}$$

For the Dirichlet b.c. $(\mathcal{D}, D)$ of the mirror quiver gauge theory $[1] - \widetilde{(1)} - (1) - [1]$ we have the boundary 't Hooft anomaly polynomial

$$\begin{aligned}
 \mathcal{I}_{\mathcal{D}, D}^{\text{3d } [1] - \widetilde{(1)} - (1) - [1]} &= \underbrace{(-\mathbf{u}_1 + \mathbf{z}_1)^2 + (\mathbf{u}_1 - \mathbf{u}_2)^2 + (\mathbf{u}_2 - \mathbf{z}_2)^2}_D \\
 &\quad - \underbrace{2\mathbf{u}_1 \cdot (\mathbf{x}_1 - \mathbf{x}_2) - 2\mathbf{u}_2 \cdot (\mathbf{x}_2 - \mathbf{x}_3)}_{\text{FI}} \\
 &= 2\mathbf{u}_1^2 + 2\mathbf{u}_2^2 - 2\mathbf{u}_1 \cdot \mathbf{u}_2 - 2\mathbf{u}_1 \cdot (\mathbf{x}_1 - \mathbf{x}_2 + \mathbf{z}_1) - 2\mathbf{u}_2 \cdot (\mathbf{x}_2 - \mathbf{x}_3 + \mathbf{z}_2) + \mathbf{z}_1^2 + \mathbf{z}_2^2.
 \end{aligned} \tag{4.23}$$

As expected, the anomaly polynomials (4.22) and (4.23) agree with each other.

We can compute the half-index of Neumann b.c. $(\mathcal{N}', N' + \Gamma^{(i)})$ for SQED₃ as

$$\begin{aligned}
 \mathbb{I}_{\mathcal{N}', N' + \Gamma^{(i)}}^{\text{3d SQED}_3}(t, x_\alpha, u_i, z_\alpha; q) &= \underbrace{(q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty \oint \frac{ds}{2\pi i s}}_{\mathbb{I}_{\mathcal{N}'}^{\text{3d U}(1)}} \\
 &\times \underbrace{\frac{(q^{\frac{1}{2}} s^\pm x_1^\pm z_1^\pm u_1^\mp; q)_\infty}{(q^{\frac{1}{4}} t s^\pm x_1^\pm; q)_\infty}}_{\mathbb{I}_{\mathcal{N}'}^{\text{3d HM}}(sx_1) \cdot F(q^{\frac{1}{2}} s x_1 z_1 u_1^{-1})} \cdot \underbrace{\frac{(q^{\frac{1}{2}} s^\pm x_2^\pm u_1^\pm u_2^\mp; q)_\infty}{(q^{\frac{1}{4}} t s^\pm x_2^\pm; q)_\infty}}_{\mathbb{I}_{\mathcal{N}'}^{\text{3d HM}}(sx_2) \cdot F(q^{\frac{1}{2}} s x_2 u_1 u_2^{-1})} \cdot \underbrace{\frac{(q^{\frac{1}{2}} s^\pm x_3^\pm u_2^\pm z_2^\mp; q)_\infty}{(q^{\frac{1}{4}} t s^\pm x_3^\pm; q)_\infty}}_{\mathbb{I}_{\mathcal{N}'}^{\text{3d HM}}(sx_3) \cdot F(q^{\frac{1}{2}} s x_3 u_2 z_2^{-1})}
 \end{aligned} \tag{4.24}$$

where $x_1 x_2 x_3 = 1$.

From the boundary 't Hooft anomaly (4.23), we obtain the half-index of Dirichlet b.c. $(\mathcal{D}, D)$ for the mirror quiver gauge theory $[1] - \widetilde{(1)} - (1) - [1]$

$$\begin{aligned}
 \mathbb{I}_{\mathcal{D}, D}^{\text{3d } [1] - \widetilde{(1)} - (1) - [1]}(t, x_\alpha, u_i, z_\alpha; q) &= \frac{1}{(q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty} \sum_{m_1 \in \mathbb{Z}} \cdot \frac{1}{(q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty} \sum_{m_2 \in \mathbb{Z}} \\
 &\times q^{\frac{m_1^2}{2} + \frac{m_2^2}{2} + \frac{(m_1 - m_2)^2}{2}} \cdot u_1^{2m_1 - m_2} \cdot u_2^{2m_2 - m_1} z_1^{-m_1} z_2^{-m_2} x_1^{-m_1} x_2^{m_1 - m_2} x_3^{m_2} \\
 &\times \underbrace{(q^{\frac{3}{4} \pm m_1} t u_1^\pm z_1^\mp; q)_\infty}_{\mathbb{I}_D^{\text{3d tHM}}(q^{m_1} u_1 z_1^{-1})} \cdot \underbrace{(q^{\frac{3}{4} \pm m_1 \mp m_2} t u_1^\pm u_2^\mp; q)_\infty}_{\mathbb{I}_D^{\text{3d tHM}}(q^{m_1 - m_2} u_1 u_2^{-1})} \cdot \underbrace{(q^{\frac{3}{4} \pm m_2} t u_2^\pm z_2^\mp; q)_\infty}_{\mathbb{I}_D^{\text{3d tHM}}(q^{m_2} u_2 z_2^{-1})}.
 \end{aligned} \tag{4.25}$$

We have confirmed that the half-indices (4.24) and (4.25) coincide to each other.

4.3.2 $(\mathcal{D}', D') \leftrightarrow (\mathcal{N}, N + \text{Fermi})$

When the single D3-brane suspended between two NS5-branes ends on the D5'-brane, one finds Dirichlet b.c. $\mathcal{D}'$ for U(1) vector multiplet and Dirichlet b.c. D' for hypermultiplets.

The S-dual configuration is described by Neumann b.c. $\mathcal{N}$ for twisted vector multiplet and Neumann b.c. N for twisted hypermultiplet in the mirror quiver gauge theory $[1] - \widetilde{(1)} - (1) - [1]$. Conversely, the Fermi multiplets $\widetilde{\Gamma}^{(i)}$ appear at the boundary of the mirror quiver gauge theory to cancel the gauge anomaly.

The charges of matter content are

matter \ symmetry	U(1) _{∂}	F	$\widetilde{\text{U}(1) \times \text{U}(1)}$	$\widetilde{F}$
$\mathbb{H}^{(1)}$	+	(+, 0, 0)	(0, 0)	(0, 0)
$\mathbb{H}^{(2)}$	+	(0, +, 0)	(0, 0)	(0, 0)
$\mathbb{H}^{(3)}$	+	(0, 0, +)	(0, 0)	(0, 0)
$\mathbb{T}^{(1)}$	0	(0, 0, 0)	(-, 0)	(+, 0)
$\mathbb{T}^{(2)}$	0	(0, 0, 0)	(+, -)	(0, 0)
$\mathbb{T}^{(3)}$	0	(0, 0, 0)	(0, +)	(0, -)
$\widetilde{\Gamma}^{(1)}$	-	(-, 0, 0)	(-, 0)	(+, 0)
$\widetilde{\Gamma}^{(2)}$	-	(0, -, 0)	(+, -)	(0, 0)
$\widetilde{\Gamma}^{(3)}$	-	(0, 0, -)	(0, +)	(0, -)

(4.26)

Let $\mathbf{u}$, $\mathbf{x}_i$, $\mathbf{s}_i$ and $\mathbf{z}_i$ be the Abelian field strengths for U(1) _{∂} , F , $\widetilde{\text{U}(1) \times \text{U}(1)}$ and $\widetilde{F}$ symmetries respectively.

The boundary anomaly polynomial of the Dirichlet b.c. $(\mathcal{D}', D')$ for SQED₃ is

$$\begin{aligned}
 \mathcal{I}_{\mathcal{D}', D'}^{\text{3d SQED}_3} &= \underbrace{(\mathbf{u} + \mathbf{x}_1)^2 + (\mathbf{u} + \mathbf{x}_2)^2 + (\mathbf{u} + \mathbf{x}_3)^2}_{D'} \\
 &\quad - \underbrace{2\mathbf{u} \cdot (\mathbf{z}_1 - \mathbf{z}_2) - 2\mathbf{x}_1 \cdot \mathbf{z}_1 + 2\mathbf{x}_3 \cdot \mathbf{z}_2}_{\text{FI}} \\
 &= 3\mathbf{u}^2 + 2\mathbf{u} \cdot (\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 - \mathbf{z}_1 + \mathbf{z}_2) \\
 &\quad + \mathbf{x}_1^2 + \mathbf{x}_2^2 + \mathbf{x}_3^2 - 2\mathbf{x}_1 \cdot \mathbf{z}_1 + 2\mathbf{x}_3 \cdot \mathbf{z}_2.
 \end{aligned}
 \tag{4.27}$$

For the Neumann b.c. $(\mathcal{N}, N + \widetilde{\Gamma}^{(i)})$ of the mirror quiver gauge theory

$[1] - \widetilde{(1)} - (1) - [1]$ we have the boundary anomaly polynomial

$$\begin{aligned}
 \mathcal{I}_{\mathcal{N}, N + \widetilde{\Gamma}^{(i)}}^{3d [1] - \widetilde{(1)} - (1) - [1]} &= \underbrace{-(-\mathbf{s}_1 + \mathbf{z}_1)^2 - (\mathbf{s}_1 - \mathbf{s}_2)^2 - (\mathbf{s}_2 - \mathbf{z}_2)^2}_N \\
 &\quad + \underbrace{(-\mathbf{s}_1 + \mathbf{z}_1 - \mathbf{u} - \mathbf{x}_1)^2 + (\mathbf{s}_1 - \mathbf{s}_2 - \mathbf{u} - \mathbf{x}_2)^2 + (\mathbf{s}_2 - \mathbf{z}_2 - \mathbf{u} - \mathbf{x}_3)^2}_{\widetilde{\Gamma}} \\
 &\quad - \underbrace{2\mathbf{s}_1(\mathbf{x}_1 - \mathbf{x}_2) - 2\mathbf{s}_2(\mathbf{x}_2 - \mathbf{x}_3)}_{\text{FI}} \\
 &= 3\mathbf{u}^2 + 2\mathbf{u} \cdot (\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 - \mathbf{z}_1 + \mathbf{z}_2) \\
 &\quad + \mathbf{x}_1^2 + \mathbf{x}_2^2 + \mathbf{x}_3^2 - 2\mathbf{x}_1 \cdot \mathbf{z}_1 + 2\mathbf{x}_3 \cdot \mathbf{z}_2.
 \end{aligned} \tag{4.28}$$

As expected, the anomaly polynomials (4.27) and (4.28) agree with each other.

From the anomaly polynomial (4.27), we get the half-index of Dirichlet b.c. $(\mathcal{D}', D')$ for SQED₃

$$\begin{aligned}
 \mathbb{III}_{\mathcal{D}', D'}^{3d \text{ SQED}_3}(t, x_\alpha, u, z_\alpha; q) &= \frac{1}{(q)_\infty (q^{\frac{1}{2}} t^{-2}; q)_\infty} \sum_{m \in \mathbb{Z}} (-1)^m q^{\frac{3m^2}{2}} u^{3m} z_1^{-m} z_2^m x_1^m x_2^m x_3^m \\
 &\quad \times \prod_{\alpha=1}^3 \underbrace{(q^{\frac{3}{4} \pm m} t^{-1} u^\pm x_\alpha^\pm; q)_\infty}_{\mathbb{III}_{D'}^{3d \text{ HM}}(q^m u x_\alpha)}
 \end{aligned} \tag{4.29}$$

The half-index of the Neumann b.c. $(\mathcal{N}, N + \widetilde{\Gamma}^{(i)})$ of the mirror quiver gauge theory $[1] - \widetilde{(1)} - (1) - [1]$ is given by

$$\begin{aligned}
 &\mathbb{III}_{\mathcal{N}, N + \widetilde{\Gamma}^{(i)}}^{3d [1] - \widetilde{(1)} - (1) - [1]}(t, x_\alpha, u, z_\alpha; q) \\
 &= \underbrace{(q)_\infty (q^{\frac{1}{2}} t^{-2}; q)_\infty \oint \frac{ds_1}{2\pi i s_1}}_{\mathbb{III}_{\mathcal{N}}^{3d \widetilde{U}(1)}} \cdot \underbrace{(q)_\infty (q^{\frac{1}{2}} t^{-2}; q)_\infty \oint \frac{ds_2}{2\pi i s_2}}_{\mathbb{III}_{\mathcal{N}}^{3d \widetilde{U}(1)}} \\
 &\quad \times \underbrace{\frac{(q^{\frac{1}{2}} z_1^\pm s_1^\mp u^\mp x_1^\mp; q)_\infty}{(q^{\frac{1}{4}} t^{-1} z_1^\pm s_1^\mp; q)_\infty}}_{\mathbb{III}_N^{3d \text{ tHM}}\left(\frac{z_1}{s_1}\right) \cdot F\left(q^{\frac{1}{2}} \frac{z_1}{s_1 u x_1}\right)} \cdot \underbrace{\frac{(q^{\frac{1}{2}} s_1^\pm s_2^\mp u^\mp x_2^\mp; q)_\infty}{(q^{\frac{1}{4}} t^{-1} s_1^\pm s_2^\mp; q)_\infty}}_{\mathbb{III}_N^{3d \text{ tHM}}\left(\frac{s_1}{s_2}\right) \cdot F\left(q^{\frac{1}{2}} \frac{s_1}{s_2 u x_2}\right)} \cdot \underbrace{\frac{(q^{\frac{1}{2}} s_2^\pm z_2^\mp u^\mp x_3^\mp; q)_\infty}{(q^{\frac{1}{4}} t^{-1} s_2^\pm z_2^\mp; q)_\infty}}_{\mathbb{III}_N^{3d \text{ tHM}}\left(\frac{s_2}{z_2}\right) \cdot F\left(q^{\frac{1}{2}} \frac{s_2}{z_2 u x_3}\right)}
 \end{aligned} \tag{4.30}$$

We have checked that the half-indices (4.29) and (4.30) coincide to each other.

4.4 3d SQED_{N_f}

Let us propose the generalization of duality of $\mathcal{N} = (0, 4)$ boundary conditions for 3d $\mathcal{N} = 4$ SQED_{N_f}. One can realize the $\mathcal{N} = (0, 4)$ boundary conditions by introducing a fivebrane on which the single D3-brane ends (see figure 9).

Figure 9. (a) The brane constructions of the Neumann b.c. $(\mathcal{N}', N' + \{\Gamma^{(i)}\})$ for SQED_{N_f} and its dual Dirichlet b.c. $(\mathcal{D}, D)$ for $[1] - \widetilde{(1)^{N_f-1}} - [1]$. (b) The brane constructions of the Dirichlet b.c. $(\mathcal{D}', D')$ for SQED_{N_f} and its dual Neumann b.c. $(\mathcal{N}, N + \{\tilde{\Gamma}^{(i)}\})$ for $[1] - \widetilde{(1)^{N_f-1}} - [1]$.

4.4.1 $(\mathcal{N}', N' + \text{Fermi}) \leftrightarrow (\mathcal{D}, D)$

The 3d $\mathcal{N} = 4$ SQED_{N_f} admits the $\mathcal{N} = (0, 4)$ boundary condition that requires Neumann b.c. $\mathcal{N}'$ for U(1) vector multiplet and Neumann b.c. N' for the charged hypermultiplets. The Neumann b.c. $(\mathcal{N}', N')$ for SQED_{N_f} is obtained when the single D3-brane whose world-volume theory is SQED_{N_f} ends on the NS5'-brane. The cancellation of gauge anomaly is achieved by the boundary Fermi multiplets $\{\Gamma^{(i)}\}_{i=1, \dots, N_f}$.

Under the action of S-duality, the NS5'-brane maps to the D5-brane so that we find the Dirichlet b.c. $(\mathcal{D}, D)$ for the mirror quiver gauge theory $[1] - \widetilde{(1)^{N_f-1}} - [1]$.

Let $\{\mathbb{H}^{(i)}\}_{i=1, \dots, N_f}$ and $\{\mathbb{T}^{(i)}\}_{i=1, \dots, N_f}$ be the complex scalar fields in the hypermultiplets of SQED_{N_f} and those in the twisted hypermultiplets of $[1] - \widetilde{(1)^{N_f-1}} - [1]$ respec-

tively. The charges of the matter fields are given by

matter \ symmetry	U(1)	F	$\widetilde{\text{U}(1)^{N_f-1}_\partial}$	$\widetilde{F}$
$\mathbb{H}^{(1)}$	+	$(+, 0, 0, \dots, 0)$	$(0, 0, 0, \dots, 0)$	$(0, 0)$
$\mathbb{H}^{(2)}$	+	$(0, +, 0, \dots, 0)$	$(0, 0, 0, \dots, 0)$	$(0, 0)$
$\mathbb{H}^{(3)}$	+	$(0, 0, +, \dots, 0)$	$(0, 0, 0, \dots, 0)$	$(0, 0)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbb{H}^{(N_f)}$	+	$(0, 0, 0, \dots, +)$	$(0, 0, 0, \dots, 0)$	$(0, 0)$
$\Gamma^{(1)}$	+	$(+, 0, 0, \dots, 0)$	$(-, 0, 0, \dots, 0)$	$(+, 0)$
$\Gamma^{(2)}$	+	$(0, +, 0, \dots, 0)$	$(+, -, 0, \dots, 0)$	$(0, 0)$
$\Gamma^{(3)}$	+	$(0, 0, +, \dots, 0)$	$(0, +, -, \dots, 0)$	$(0, 0)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\Gamma^{(N_f)}$	+	$(0, 0, 0, \dots, +)$	$(0, 0, 0, \dots, +)$	$(0, -)$
$\mathbb{T}^{(1)}$	0	$(0, 0, 0, \dots, 0)$	$(-, 0, 0, \dots, 0)$	$(+, 0)$
$\mathbb{T}^{(2)}$	0	$(0, 0, 0, \dots, 0)$	$(+, -, 0, \dots, 0)$	$(0, 0)$
$\mathbb{T}^{(3)}$	0	$(0, 0, 0, \dots, 0)$	$(0, +, -, \dots, 0)$	$(0, 0)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbb{T}^{(N_f)}$	0	$(0, 0, 0, \dots, 0)$	$(0, 0, 0, \dots, +)$	$(0, -)$

(4.31)

Let us denote the Abelian field strengths for U(1), F , $\widetilde{\text{U}(1)^{N_f-1}_\partial}$ and $\widetilde{F}$ by $\mathbf{s}$, $\mathbf{x}_i$, $\mathbf{u}_i$ and $\mathbf{z}_i$ respectively.

The boundary anomaly polynomial of Neumann b.c. $(\mathcal{N}', N' + \Gamma^{(i)})$ for SQED_{N_f} is

$$\begin{aligned}
 \mathcal{I}_{\mathcal{N}', N' + \Gamma^{(i)}}^{\text{3d SQED}_{N_f}} = & - \underbrace{\sum_{i=1}^{N_f} (\mathbf{s} + \mathbf{x}_i)^2}_{N'} \\
 & + \underbrace{(\mathbf{s} + \mathbf{x}_1 - \mathbf{u}_1 + \mathbf{z}_1)^2 + \sum_{i=1}^{N_f-2} (\mathbf{s} + \mathbf{x}_{i+1} + \mathbf{u}_i - \mathbf{u}_{i+1})^2 + (\mathbf{s} + \mathbf{x}_{N_f} + \mathbf{u}_{N_f-1} - \mathbf{z}_2)^2}_{\Gamma} \\
 & - \underbrace{2\mathbf{s} \cdot (\mathbf{z}_1 - \mathbf{z}_2) - 2\mathbf{x}_1 \cdot \mathbf{z}_1 + 2\mathbf{x}_{N_f} \cdot \mathbf{z}_2}_{\text{FI}}.
 \end{aligned} \tag{4.32}$$

The boundary anomaly polynomial of Dirichlet b.c. $(\mathcal{D}, D)$ for $[1] - \widetilde{(1)^{N_f-1}} - [1]$ is computed as

$$\begin{aligned} \mathcal{I}_{\mathcal{D}, D}^{3d} [1] - \widetilde{(1)^{N_f-1}} - [1] &= \underbrace{(-\mathbf{u}_1 + \mathbf{z}_1)^2 + \sum_{i=1}^{N_f-2} (\mathbf{u}_i - \mathbf{u}_{i+1})^2 + (\mathbf{u}_{N_f-1} - \mathbf{z}_2)^2}_D \\ &\quad - 2 \underbrace{\sum_{i=1}^{N_f-1} \mathbf{u}_i \cdot (\mathbf{x}_i - \mathbf{x}_{i+1})}_{\text{FI}}. \end{aligned} \quad (4.33)$$

This matches with the boundary 't Hooft anomaly (4.32) of Neumann b.c. $(\mathcal{N}', N' + \Gamma^{(i)})$ for SQED $_{N_f}$.

The half-index of Neumann b.c. $(\mathcal{N}', N' + \Gamma^{(i)})$ for SQED $_{N_f}$ is given by

$$\begin{aligned} \mathbb{I}_{\mathcal{N}', N' + \Gamma^{(i)}}^{3d \text{ SQED}_{N_f}}(t, x_\alpha, u_i, z_\alpha; q) &= \underbrace{(q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty}_{\mathbb{I}_{\mathcal{N}'}^{3d \text{ U}(1)}} \oint \frac{ds}{2\pi i s} \times \underbrace{\frac{(q^{\frac{1}{2}} s^\pm x_1^\pm z_1^\pm u_1^\mp; q)_\infty}{(q^{\frac{1}{4}} t s^\pm x_1^\pm; q)_\infty}}_{\mathbb{I}_{\mathcal{N}'}^{3d \text{ HM}}(sx_1) \cdot F(q^{\frac{1}{2}} s x_1 z_1 u_1^{-1})} \\ &\quad \cdot \prod_{i=1}^{N_f-2} \underbrace{\frac{(q^{\frac{1}{2}} s^\pm x_{i+1}^\pm u_i^\pm u_{i+1}^\mp; q)_\infty}{(q^{\frac{1}{4}} t s^\pm x_{i+1}^\pm; q)_\infty}}_{\mathbb{I}_{\mathcal{N}'}^{3d \text{ HM}}(sx_{i+1}) \cdot F(q^{\frac{1}{2}} s x_{i+1} u_i u_{i+1}^{-1})} \\ &\quad \cdot \underbrace{\frac{(q^{\frac{1}{2}} s^\pm x_{N_f}^\pm u_{N_f-1}^\pm z_2^\mp; q)_\infty}{(q^{\frac{1}{4}} t s^\pm x_{N_f}^\pm; q)_\infty}}_{\mathbb{I}_{\mathcal{N}'}^{3d \text{ HM}}(sx_{N_f}) \cdot F(q^{\frac{1}{2}} s x_{N_f} u_{N_f-1} z_2^{-1})}. \end{aligned} \quad (4.34)$$

The half-index of Dirichlet b.c. $(\mathcal{D}, D)$ for $[1] - \widetilde{(1)^{N_f-1}} - [1]$ is computed as

$$\begin{aligned} \mathbb{I}_{\mathcal{D}, D}^{3d} [1] - \widetilde{(1)^{N_f-1}} - [1](t, x_\alpha, u_i, z_\alpha; q) &= \frac{1}{(q)_\infty^{N_f-1} (q^{\frac{1}{2}} t^2; q)_\infty^{N_f-1}} \sum_{m_1, \dots, m_{N_f-1} \in \mathbb{Z}} \\ &\quad \times q^{\frac{m_1^2}{2} + \sum_{i=1}^{N_f-2} \frac{(m_i - m_{i+1})^2}{2} + \frac{m_{N_f-1}^2}{2}} \cdot u_1^{2m_1 - m_2} \cdot \prod_{i=1}^{N_f-3} u_{i+1}^{2m_{i+1} - m_i - m_{i+2}} \cdot u_{N_f-1}^{2m_{N_f-1} - m_{N_f-2}} \\ &\quad \times z_1^{-m_1} \cdot z_2^{-m_{N_f-1}} \cdot x_1^{-m_1} \cdot \prod_{i=1}^{N_f-2} x_{i+1}^{m_i - m_{i+1}} \cdot x_{N_f}^{m_{N_f-1}} \\ &\quad \times \underbrace{(q^{\frac{3}{4} \pm m_1} t u_1^\pm z_1^\mp; q)_\infty}_{\mathbb{I}_D^{3d \text{ tHM}}(q^{m_1} u_1 z_1^{-1})} \cdot \prod_{i=1}^{N_f-2} \underbrace{(q^{\frac{3}{4} \pm m_i \mp m_{i+1}} t u_i^\pm u_{i+1}^\mp; q)_\infty}_{\mathbb{I}_D^{3d \text{ tHM}}(q^{m_i - m_{i+1}} u_i u_{i+1}^{-1})} \cdot \underbrace{(q^{\frac{3}{4} \pm m_{N_f-1}} t u_{N_f-1}^\pm z_2^\mp; q)_\infty}_{\mathbb{I}_D^{3d \text{ tHM}}(q^{m_{N_f-1}} u_{N_f-1} z_2^{-1})}. \end{aligned} \quad (4.35)$$

Making use of the minimal mirror identity between (4.14) and (4.15), one finds that the half-indices (4.34) and (4.35) give the same answer.

4.4.2 $(\mathcal{D}', D') \leftrightarrow (\mathcal{N}, N + \text{Fermi})$

Next consider the Dirichlet boundary condition $(\mathcal{D}', D')$ for SQED_{N_f} , which is realized by the D5'-brane on which the single D3-brane terminates.

In the dual configuration, the mirror quiver gauge theory $[1] - \widetilde{(1)^{N_f-1}} - [1]$ has Neumann b.c. $\mathcal{N}$ for twisted vector multiplet and Neumann b.c. N for twisted hypermultiplets. Again the gauge anomaly contributed from the twisted hypermultiplets is compensated by the boundary Fermi multiplets $\{\tilde{\Gamma}^{(i)}\}_{i=1, \dots, N_f}$.

The charges of the matter fields are

matter \ symmetry	$U(1)_\partial$	F	$\widetilde{U(1)^{N_f-1}}$	$\tilde{F}$
$\mathbb{H}^{(1)}$	+	$(+, 0, 0, \dots, 0)$	$(0, 0, 0, \dots, 0)$	$(0, 0)$
$\mathbb{H}^{(2)}$	+	$(0, +, 0, \dots, 0)$	$(0, 0, 0, \dots, 0)$	$(0, 0)$
$\mathbb{H}^{(3)}$	+	$(0, 0, +, \dots, 0)$	$(0, 0, 0, \dots, 0)$	$(0, 0)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbb{H}^{(N_f)}$	+	$(0, 0, 0, \dots, +)$	$(0, 0, 0, \dots, 0)$	$(0, 0)$
$\mathbb{T}^{(1)}$	0	$(0, 0, 0, \dots, 0)$	$(-, 0, 0, \dots, 0)$	$(+, 0)$
$\mathbb{T}^{(2)}$	0	$(0, 0, 0, \dots, 0)$	$(+, -, 0, \dots, 0)$	$(0, 0)$
$\mathbb{T}^{(3)}$	0	$(0, 0, 0, \dots, 0)$	$(0, +, -, \dots, 0)$	$(0, 0)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbb{T}^{(N_f)}$	0	$(0, 0, 0, \dots, 0)$	$(0, 0, 0, \dots, +)$	$(0, -)$
$\tilde{\Gamma}^{(1)}$	-	$(-, 0, 0, \dots, 0)$	$(-, 0, 0, \dots, 0)$	$(+, 0)$
$\tilde{\Gamma}^{(2)}$	-	$(0, -, 0, \dots, 0)$	$(+, -, 0, \dots, 0)$	$(0, 0)$
$\tilde{\Gamma}^{(3)}$	-	$(0, 0, -, \dots, 0)$	$(0, +, -, \dots, 0)$	$(0, 0)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\tilde{\Gamma}^{(N_f)}$	-	$(0, 0, 0, \dots, -)$	$(0, 0, 0, \dots, +)$	$(0, -)$

(4.36)

Let $\mathbf{u}$, $\mathbf{x}_i$, $\mathbf{s}_i$ and $\mathbf{z}_i$ be the Abelian field strengths for $U(1)_\partial$, F , $\widetilde{U(1)^{N_f-1}}$ and $\tilde{F}$ respectively.

The boundary 't Hooft anomaly polynomial of the Dirichlet b.c. $(\mathcal{D}', D')$ for SQED_{N_f} is

$$\mathcal{I}_{\mathcal{D}', D'}^{\text{3d SQED}_{N_f}} = \underbrace{\sum_{i=1}^{N_f} (\mathbf{u} - \mathbf{x}_i)^2}_{D'} \underbrace{- 2\mathbf{u} \cdot (\mathbf{z}_1 - \mathbf{z}_2) - 2\mathbf{x}_1 \cdot \mathbf{z}_1 + 2\mathbf{x}_{N_f} \cdot \mathbf{z}_2}_{\text{FI}}. \quad (4.37)$$

This matches with the boundary anomaly polynomial of the Neumann b.c. $(\mathcal{N}, N + \widetilde{\Gamma}^{(i)})$ for the mirror quiver gauge theory $[1] - (1)^{\widetilde{N_f-1}} - [1]$

$$\begin{aligned}
 \mathcal{I}_{\mathcal{N}, N + \widetilde{\Gamma}^{(i)}}^{\text{3d}} [1] - (1)^{\widetilde{N_f-1}} - [1] &= \underbrace{-(-\mathbf{s}_1 + \mathbf{z}_1)^2 - \sum_{i=1}^{N_f-2} (\mathbf{s}_i - \mathbf{s}_{i+1})^2 - (\mathbf{s}_{N_f-1} - \mathbf{z}_2)^2}_N \\
 &+ \underbrace{(-\mathbf{s}_1 + \mathbf{z}_1 - \mathbf{u} - \mathbf{x}_1)^2 + \sum_{i=1}^{N_f-2} (\mathbf{s}_i - \mathbf{s}_{i+1} - \mathbf{u} - \mathbf{x}_{i+1})^2 + (\mathbf{s}_{N_f-1} - \mathbf{z}_2 - \mathbf{u} - \mathbf{x}_{N_f})^2}_{\widetilde{\Gamma}} \\
 &- 2 \underbrace{\sum_{i=1}^{N_f-1} \mathbf{s}_i \cdot (\mathbf{x}_i - \mathbf{x}_{i+1})}_{\text{FI}}. \tag{4.38}
 \end{aligned}$$

From the boundary 't Hooft anomaly (4.37) we obtain the half-index of the Dirichlet b.c. $(\mathcal{D}', D')$ for SQED $_{N_f}$

$$\begin{aligned}
 \mathbb{III}_{\mathcal{D}', D'}^{\text{3d SQED}_{N_f}}(t, x_\alpha, u, z_\alpha; q) &\tag{4.39} \\
 &= \frac{1}{(q)_\infty (q^{\frac{1}{2}} t^{-2}; q)_\infty} \sum_{m \in \mathbb{Z}} (-1)^{N_f m} q^{\frac{N_f m^2}{2}} \cdot u^{N_f m} \cdot z_1^{-m} \cdot z_2^m \cdot \prod_{\alpha=1}^{N_f} x_\alpha^m \cdot \underbrace{(q^{\frac{3}{4} \pm m} t^{-1} u^\pm x_\alpha^\pm; q)_\infty}_{\mathbb{III}_{\mathcal{D}'}^{\text{3d tHM}}(q^m u x_\alpha)}.
 \end{aligned}$$

On the other hand, the half-index of the Neumann b.c. $(\mathcal{N}, N + \widetilde{\Gamma}^{(i)})$ for the mirror quiver gauge theory $[1] - (1)^{\widetilde{N_f-1}} - [1]$ is calculated as

$$\begin{aligned}
 \mathbb{III}_{\mathcal{N}, N + \widetilde{\Gamma}^{(i)}}^{\text{3d}} [1] - (1)^{\widetilde{N_f-1}} - (1) &\tag{4.40} \\
 &= \underbrace{(q)_\infty^{N_f-1} (q^{\frac{1}{2}} t^{-2}; q)_\infty^{N_f-1} \oint \prod_{i=1}^{N_f-1} \frac{ds_i}{2\pi i s_i}}_{\mathbb{III}_{\mathcal{N}}^{\text{3d U}(1)^{\widetilde{N_f-1}}}} \\
 &\times \underbrace{\frac{(q^{\frac{1}{2}} z_1^\pm s_1^\mp u^\mp x_1^\mp; q)_\infty}{(q^{\frac{1}{4}} t^{-1} z_1^\pm s_1^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}}^{\text{3d tHM}}\left(\frac{z_1}{s_1}\right) \cdot F\left(q^{\frac{1}{2}} \frac{z_1}{s_1 u x_1}\right)} \cdot \prod_{i=1}^{N_f-2} \underbrace{\frac{(q^{\frac{1}{2}} s_i^\pm s_{i+1}^\mp u^\mp x_{i+1}^\mp; q)_\infty}{(q^{\frac{1}{4}} t^{-1} s_i^\pm s_{i+1}^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}}^{\text{3d tHM}}\left(\frac{s_i}{s_{i+1}}\right) \cdot F\left(q^{\frac{1}{2}} \frac{s_i}{s_{i+1} u x_{i+1}}\right)} \cdot \underbrace{\frac{(q^{\frac{1}{2}} s_{N_f-1}^\pm z_2^\mp u^\mp x_{N_f}^\mp; q)_\infty}{(q^{\frac{1}{4}} t^{-1} s_{N_f-1}^\pm z_2^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}}^{\text{3d tHM}}\left(\frac{s_{N_f-1}}{z_2}\right) \cdot F\left(q^{\frac{1}{2}} \frac{s_{N_f-1}}{z_2 u x_{N_f}}\right)}.
 \end{aligned}$$

The half-indices (4.39) and (4.40) would agree with each other. In fact, given that the minimal mirror identity holds, one can show that they lead to the same answer.

We therefore propose the following dualities of $\mathcal{N} = (0, 4)$ boundary conditions for 3d $\mathcal{N} = 4$ Abelian gauge theories:

$$\begin{aligned}
 (\mathcal{N}', N' + \{\Gamma^{(i)}\}) \text{ for SQED}_{N_f} &\leftrightarrow (\mathcal{D}, D) \text{ for } [1] - (1)^{\widetilde{N_f-1}} - [1] \\
 (\mathcal{D}', D') \text{ for SQED}_{N_f} &\leftrightarrow (\mathcal{N}, N + \{\widetilde{\Gamma}^{(i)}\}) \text{ for } [1] - (1)^{\widetilde{N_f-1}} - [1] \tag{4.41}
 \end{aligned}$$

In general, a pair of $\mathcal{N} = (0, 4)$ Neumann b.c. for 3d $\mathcal{N} = 4$ U(1) vector multiplet and hypermultiplets should be accompanied with boundary Fermi multiplets that cancel the gauge anomaly. We have confirmed that it is dual to a pair of $\mathcal{N} = (0, 4)$ Dirichlet b.c. for 3d $\mathcal{N} = 4$ U(1) twisted vector multiplet and twisted hypermultiplets in the mirror theory.

4.5 Enriched Neumann from Dirichlet

To compute the half-index of 3d gauge theories with enriched Neumann b.c. coupled to 2d boundary matter other than Fermi multiplets, we expect that monopole operators exist at the boundary in such a way that the boundary condition $*F = J_{2d}$ admits a boundary monopole supported by an appropriate field configuration of the 2d bosonic fields.

We propose the prescription to derive the half-index of 3d gauge theories with enriched Neumann b.c. by adding 2d matter to Dirichlet half-index and gauging 2d boundary system in the Jeffrey-Kirwan residue method.

4.5.1 3d SQED₁

Let us reconsider the half-index for Neumann b.c. $(\mathcal{N}', N')$ of SQED₁

$$\mathbb{III}_{\mathcal{N}', N'+\Gamma}^{3d \text{ SQED}_1}(t, z; q) = \underbrace{(q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(1)}} \oint \frac{ds}{2\pi i s} \underbrace{\frac{(q^{\frac{1}{2}} s z; q)_\infty (q^{\frac{1}{2}} s^{-1} z^{-1}; q)_\infty}{(q^{\frac{1}{4}} t s; q)_\infty (q^{\frac{1}{4}} t s^{-1}; q)_\infty}}_{\mathbb{III}_{N'}^{3d \text{ HM}(s) \cdot F(q^{\frac{1}{2}} s z)}}. \quad (4.42)$$

This is the same as the Dirichlet half-index $\mathbb{III}_D^{3d \text{ tHM}}(z)$ for twisted hyper, however, we would like to manipulate in a different way. We can tentatively rewrite the contour integral as a sum over residues of poles

$$\mathbb{III}_{\mathcal{N}', N'+\Gamma}^{3d \text{ SQED}_1}(t, z; q) = (q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty \sum_{n=0}^{\infty} \oint_{s=q^{\frac{1}{4}+n} t} \frac{ds}{2\pi i s} \frac{(q^{\frac{1}{2}} s z; q)_\infty (q^{\frac{1}{2}} s^{-1} z^{-1}; q)_\infty}{(q^{\frac{1}{4}} t s; q)_\infty (q^{\frac{1}{4}} t s^{-1}; q)_\infty}. \quad (4.43)$$

We can now shift the integration variables to rewrite

$$\begin{aligned} \mathbb{III}_{\mathcal{N}', N'+\Gamma}^{3d \text{ SQED}_1}(t, z; q) &= \underbrace{(q)_\infty^2 (q^{\frac{1}{2}} t^2; q)_\infty (q^{\frac{1}{2}} t^{-2}; q)_\infty}_{\mathbb{II}^{2d \text{ U}(1)}} \oint_{s=q^{\frac{1}{4}} t} \frac{ds}{2\pi i s} \\ &\quad \times \left[\underbrace{\frac{1}{(q)_\infty (q^{\frac{1}{2}} t^{-2}; q)_\infty}}_{\mathbb{III}_{D'}^{3d \text{ U}(1)}} \sum_{n \in \mathbb{Z}} \frac{(q^{\frac{1}{2}+n} s z; q)_\infty (q^{\frac{1}{2}-n} s^{-1} z^{-1}; q)_\infty}{(q^{\frac{1}{4}+n} t s; q)_\infty (q^{\frac{1}{4}-n} t s^{-1}; q)_\infty} \right]. \end{aligned} \quad (4.44)$$

Here we have extended the summation to negative n because the integrand has no residue there. This can be interpreted as 2d gauging of the boundary global symmetry of Dirichlet boundary conditions for the gauge field.

Making use of the relation

$$F(q^{\frac{1}{2}+n} s) = (-1)^n q^{-\frac{n^2}{2}} s^{-n} F(q^{\frac{1}{2}} s), \quad (4.45)$$

Figure 10. The brane constructions of the enriched Neumann b.c. involving 2d twisted hyper, 3 charged Fermi and a neutral Fermi for SQED_1 and its dual configuration. The dual configuration would be described by Neumann b.c. for 3d twisted hyper as well as three Fermi multiplets.

the Fermi index can be also brought out of the sum and we get

$$\begin{aligned}
 & \mathbb{I}_{\mathcal{N}', N'+\Gamma}^{\text{3d SQED}_1}(t, z; q) \\
 &= \underbrace{(q)_\infty^2 (q^{\frac{1}{2}} t^2; q)_\infty (q^{\frac{1}{2}} t^{-2}; q)_\infty \oint_{s=q^{\frac{1}{4}} t} \frac{ds}{2\pi i s}}_{\mathbb{I}^{\text{2d U}(1)}} \cdot \underbrace{(q^{\frac{1}{2}} s x; q)_\infty (q^{\frac{1}{2}} s^{-1} x^{-1}; q)_\infty}_{F(q^{\frac{1}{2}} s x)} \\
 &\times \left[\underbrace{\frac{1}{(q)_\infty (q^{\frac{1}{2}} t^{-2}; q)_\infty}}_{\mathbb{I}_{\mathcal{D}'}^{\text{3d U}(1)}} \sum_{n \in \mathbb{Z}} (-1)^n q^{-\frac{n^2}{2}} s^{-n} z^{-n} \underbrace{\frac{1}{(q^{\frac{1}{4}+n} t s; q)_\infty (q^{\frac{1}{4}-n} t s^{-1}; q)_\infty}}_{\mathbb{I}_{\mathcal{N}'}^{\text{3d HM}(q^n s)}} \right]. \quad (4.46)
 \end{aligned}$$

We can now propose the correct form for the half-index of enriched Neumann b.c. in such a way that one firstly adds the 2d matter to the Dirichlet half-index and then gauge the 2d boundary symmetry in a standard JK-like way.

4.5.2 3d SQED_1 +2d twisted hyper+3 Fermi+neutral Fermi

For example, consider adding a 2d twisted hypermultiplet at the boundary, together with a total of three Fermi multiplets of gauge charge 1 and one neutral to couple to the twisted hyper and the boundary value of the 3d hyper.

The brane construction is illustrated in figure 10. The defect of $\text{D5}'$ -brane would couple a 2d twisted hypermultiplet to 3d $\text{U}(1)$ vector multiplet. At the NS5 - $\text{D5}'$ -junctions and $\text{NS5}'$ - D5 junction, three charged Fermi multiplets would appear while at the D5 - $\text{D5}'$ junction a neutral Fermi multiplet would show up.

The half-index reads

$$\begin{aligned}
 & \mathbb{III}_{\mathcal{N}', N' + \Gamma + T}^{\text{3d SQED}_1}(t, z, u, x_\alpha; q) \\
 &= \underbrace{(q)_\infty^2 (q^{\frac{1}{2}} t^2; q)_\infty (q^{\frac{1}{2}} t^{-2}; q)_\infty}_{\mathbb{I}^{2d} \text{ U}(1)} \oint_{\text{JK}} \frac{ds}{2\pi i s} \cdot \\
 & \times \underbrace{(q^{\frac{1}{2}} u^{\pm 3} z^\pm; q)_\infty}_{F(q^{\frac{1}{2}} u^3 z)} \underbrace{\frac{1}{(q^{\frac{1}{4}} t^{-1} s^\pm u^{\pm 3} z^\pm; q)_\infty (q^{\frac{3}{4}} t s^\pm u^{\pm 3} z^\pm; q)_\infty}}_{\mathbb{I}^{2d} \text{ tHM}(su^3 z)} \cdot \prod_{\alpha=1}^3 \underbrace{(q^{\frac{1}{2}} s^\pm z^\pm u^{\pm 2} x_\alpha^\pm; q)_\infty}_{F(q^{\frac{1}{2}} s z u^2 x_\alpha)} \\
 & \times \left[\frac{1}{(q)_\infty (q^{\frac{1}{2}} t^{-2}; q)_\infty} \sum_{n \in \mathbb{Z}} (-1)^n q^{-\frac{n^2}{2}} s^{-n} z^{-n} \underbrace{\frac{1}{(q^{\frac{1}{4} \pm n} t s^\pm; q)_\infty}}_{\mathbb{III}_{N'}^{\text{3d HM}}(q^n s)} \right] \quad (4.47)
 \end{aligned}$$

with $\prod_{\alpha=1}^3 x_\alpha = 1$.

The JK prescription should pick two poles. One is the pole of 3d hyper at $s = q^{\frac{1}{4}} t$ and the other is the pole for 2d twisted hyper at $s = q^{\frac{1}{4}} t^{-1} z^{-1} u^{-3}$.

The contributions from the first pole of 3d hyper can be shifted back to

$$\begin{aligned}
 & \mathbb{III}_{1, \mathcal{N}', N' + \Gamma + T}^{\text{3d SQED}_1}(t, z, u, x_\alpha; q) \\
 &= (q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty \cdot \underbrace{(q^{\frac{1}{2}} u^{\pm 3} z^\pm; q)_\infty}_{F(q^{\frac{1}{2}} u^3 z)} \sum_{n=0}^{\infty} \oint_{s=q^{\frac{1}{4}} + n t} \frac{ds}{2\pi i s} \\
 & \times \prod_{\alpha=1}^3 \underbrace{\frac{(q^{\frac{1}{2}} s^\pm z^\pm u^{\pm 2} x_\alpha^\pm; q)_\infty}{(q^{\frac{1}{4}} t s^\pm; q)_\infty \cdot (q^{\frac{1}{4}} t^{-1} s^\pm u^{\pm 3} z^\pm; q)_\infty (q^{\frac{3}{4}} t s^\pm u^{\pm 3} z^\pm; q)_\infty}}_{\mathbb{III}_{N'}^{\text{3d HM}}(s) \cdot \mathbb{I}^{2d} \text{ tHM}(su^3 z) \cdot F(q^{\frac{1}{2}} s z u^2 x_\alpha)} \\
 &= \frac{(q^{\frac{1}{2}} t^2; q)_\infty}{(q)_\infty} \prod_{\alpha=1}^3 \underbrace{\frac{(q^{\frac{3}{4}} t z u^2 x_\alpha; q)_\infty (q^{\frac{1}{4}} t^{-1} z^{-1} u^{-2} x_\alpha^{-1}; q)_\infty}{(q t^2 u^3 z; q)_\infty (t^{-2} u^{-3} z^{-1}; q)_\infty}}_{F(q^{\frac{3}{4}} t z u^2 x_\alpha) \cdot C(q t^2 u^3 z)} \\
 & \times \sum_{n=0}^{\infty} \frac{(q^{1+n}; q)_\infty}{(q^{\frac{1}{2} + n} t^2; q)_\infty} q^{\frac{n}{4}} \cdot t^{-n} \cdot z^{-n} \cdot \prod_{\alpha=1}^3 x_\alpha^{-n}. \quad (4.48)
 \end{aligned}$$

The contributions from the second pole of 2d twisted hyper become

$$\begin{aligned}
 & \mathbb{I}_{2, \mathcal{N}', \mathcal{N}' + \Gamma + T}^{\text{3d SQED}_1}(t, z, u, x_\alpha; q) \\
 &= (q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty \cdot \underbrace{(q^{\frac{1}{2}} u^{\pm 3} z^\pm; q)_\infty}_{F(q^{\frac{1}{2}} u^3 z)} \sum_{n=-\infty}^{\infty} \oint_{s=q^{\frac{1}{4}+n} t^{-1} z^{-1} u^{-3}} \frac{ds}{2\pi i s} \\
 & \times \prod_{\alpha=1}^3 \frac{(q^{\frac{1}{2}} s^\pm z^\pm u^{\pm 2} x_\alpha^\pm; q)_\infty}{\underbrace{(q^{\frac{1}{4}} t s^\pm; q)_\infty \cdot (q^{\frac{1}{4}} t^{-1} s^\pm u^{\pm 3} z^\pm; q)_\infty (q^{\frac{3}{4}} t s^\pm u^{\pm 3} z^\pm; q)_\infty}_{\mathbb{I}_{\mathcal{N}'}^{\text{3d HM}}(s) \cdot \mathbb{I}^{\text{2d tHM}}(su^3 z) \cdot F(q^{\frac{1}{2}} s z u^2 x_\alpha)}} \\
 &= \frac{1}{(q)_\infty (q^{\frac{1}{2}} t^{-2}; q)_\infty} \prod_{\alpha=1}^3 \frac{(q^{\frac{3}{4}} t^{-1} u^{-1} x_\alpha; q)_\infty (q^{\frac{1}{4}} t u x_\alpha^{-1}; q)_\infty \cdot (q^{\frac{1}{2}} u^3 z; q)_\infty (q^{\frac{1}{2}} u^{-3} z^{-1}; q)_\infty}{\underbrace{(t^2 u^3 z; q)_\infty (q t^{-2} u^{-3} z^{-1}; q)_\infty}_{F(q^{\frac{3}{4}} t^{-1} u^{-1} x_\alpha) \cdot F(q^{\frac{1}{2}} u^3 z) \cdot C(t^2 u^3 z)}} \\
 & \times \sum_{n=-\infty}^{\infty} \frac{(q^{1+n} t^{-2} u^{-3} z^{-1}; q)_\infty}{(q^{\frac{1}{2}+n} u^{-3} z^{-1}; q)_\infty} q^{\frac{n}{4}} \cdot t^{-n} \cdot z^{-n} \cdot \prod_{\alpha=1}^3 x_\alpha^{-n}. \tag{4.49}
 \end{aligned}$$

The above argument means that the correct prescription for enriched Neumann half-index would be to pick the standard semi-infinite sequence of poles for 3d matter fields obeying Neumann b.c. However, one needs a full infinite sequence of poles of only one charge/type for 2d bosonic matter fields.

The final answer is surprisingly simple:

$$\begin{aligned}
 \mathbb{I}_{\mathcal{N}', \mathcal{N}' + \Gamma + T}^{\text{3d SQED}_1}(t, z, u, x_\alpha; q) &= \sum_{i=1}^2 \mathbb{I}_{i, \mathcal{N}', \mathcal{N}' + \Gamma + T}^{\text{3d SQED}_1}(t, z, u, x_\alpha; q) \\
 &= \frac{1}{\underbrace{(q^{\frac{1}{4}} t^{-1} z^\pm; q)_\infty}_{\mathbb{I}_{\mathcal{N}'}^{\text{3d tHM}}(z)}} \cdot \prod_{\alpha=1}^3 \underbrace{(q^{\frac{1}{2}} u^\pm z^\pm x_\alpha^\pm; q)_\infty}_{F(q^{\frac{1}{2}} u z x_\alpha^{-1})}. \tag{4.50}
 \end{aligned}$$

It includes the half-index of Neumann b.c. $\mathcal{N}$ for 3d twisted hypermultiplet of fugacity z and three full-indices of Fermi multiplets of fugacities $u z x_\alpha^{-1}$ with $\alpha = 1, 2, 3$. This beautifully agrees with the brane box picture!

The result supports the duality:

$$\begin{aligned}
 & (\mathcal{N}', \mathcal{N}' + 2\text{d twisted hyper} + 3\text{ Fermi} + \text{neutral Fermi}) \text{ for } 3\text{d SQED}_1 \\
 & \leftrightarrow \mathcal{N} \text{ for } 3\text{d twisted hyper} + 3\text{ Fermi}. \tag{4.51}
 \end{aligned}$$

5 $\mathcal{N} = (0, 4)$ boundaries of 3d theories and corners of 4d theories

We examine the configurations which involve the quarter-BPS corners of 4d $\mathcal{N} = 4$ SYM theory and the half-BPS boundary of 3d $\mathcal{N} = 4$ gauge theories. We show the dualities between the configurations of gauge theories, which involve local operators in 4d, 3d and 2d theories, by computing the quarter-indices.⁷

⁷See [31] for the half-indices and quarter-indices of 4d $\mathcal{N} = 4$ gauge theory.

Figure 11. The brane constructions of the configuration 4d $U(N)|3d U(M)$ with Neumann b.c. $\mathcal{N}'$ and its mirror corner configuration that has no gauge symmetry.

5.1 4d $U(N)|3d U(M)$ with $\mathcal{N}'$

Consider the configuration where the corner of 4d $\mathcal{N} = 4 U(N)$ gauge theory obeying a pair of Neumann b.c. $(\mathcal{N}, \mathcal{N}')$ coupled to the boundary of 3d $\mathcal{N} = 4 U(M)$ gauge theory satisfying Neumann b.c. $\mathcal{N}'$ through the 3d hypermultiplet with Neumann b.c. N' transforming as $(\mathbf{N}, \overline{\mathbf{M}}) \oplus (\overline{\mathbf{N}}, \mathbf{M})$ under the $U(N) \times U(M)$ gauge symmetry. The corresponding brane configuration is shown in figure 11. From the brane box analysis [35], we expect that the Abelian part of gauge anomaly is cancelled by the 2d cross-determinant Fermi multiplet living at the junction. We denote this configuration by 4d $U(N)|3d U(M)$ with Neumann b.c. $\mathcal{N}'$. The dual configuration can be obtained by applying S-duality to the brane configuration in figure 11. As it does not have gauge symmetry, we find that the corresponding quarter-index is expressed as a product of indices of free fields. These dual configurations are the generalizations of NS5-NS5' and D5-D5' junctions studied in [31].

5.1.1 4d $U(1)|3d U(1)$ with $\mathcal{N}'$

Let us begin with the configuration 4d $U(1)|3d U(1)$ with $\mathcal{N}'$. The junction keeps 4d $U(1)$ gauge symmetry and 3d $U(1)$ gauge symmetry and involves the bi-fundamental 3d hypermultiplet and the 2d cross-determinant Fermi multiplet.

The quarter-index reads

$$\mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d U(1)|(1)|} = \underbrace{(q)_\infty \oint \frac{ds_1}{2\pi i s_1}}_{\mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d U(1)}} \cdot \underbrace{(q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty \oint \frac{ds_2}{2\pi i s_2}}_{\mathbb{III}_{\mathcal{N}'}^{3d U(1)}} \cdot \underbrace{\frac{(q^{\frac{1}{2}} s_1^\pm s_2^\mp z^\pm; q)_\infty}{(q^{\frac{1}{4}} t s_1^\pm s_2^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d HM} \left(\frac{s_1}{s_2} \right) \cdot F \left(q^{\frac{1}{2}} \frac{s_1}{s_2} z \right)}. \quad (5.1)$$

In the S-dual brane configuration there exists a single D3-brane for each side of one of the D5'-branes so that the 3d twisted hypermultiplet arising from D3-D5' string. According to the presence of D5-brane on which D3-brane ends, the 3d twisted hypermultiplet should obey the Dirichlet b.c. D .

In fact, the quarter-index (5.1) coincides with

$$\mathbb{IV}_{\mathcal{D}\mathcal{D}'}^{1|[1]|} = \underbrace{(q)_\infty}_{\mathbb{IV}_{\mathcal{D}\mathcal{D}'}^{4d U(1)}} \cdot \underbrace{(q^{\frac{3}{4}} t z; q)_\infty (q^{\frac{3}{4}} t z^{-1}; q)_\infty}_{\mathbb{III}_D^{3d tHM}(z)}. \quad (5.2)$$

5.1.2 4d U(2)|3d U(1) with $\mathcal{N}'$

Next consider the junction 4d U(2)|3d U(1) with $\mathcal{N}'$. The configuration has 4d U(2) gauge symmetry and 3d U(1) gauge symmetry.

One can evaluate the quarter-index as

$$\begin{aligned} \mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(2)|(1)|} &= \underbrace{\frac{1}{2}(q)_\infty \oint \frac{ds_1}{2\pi i s_1} \frac{ds_2}{2\pi i s_2} \left(\frac{s_1}{s_2}; q\right)_\infty \left(\frac{s_2}{s_1}; q\right)_\infty}_{\mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(2)}} \cdot \underbrace{(q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty \oint \frac{ds_3}{2\pi i s_3}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(1)}} \\ &\times \prod_{i=1}^2 \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_i^\pm s_3^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}}\left(\frac{s_i}{s_3}\right)} \cdot \underbrace{(q^{\frac{1}{2}} s_1^\pm s_2^\pm s_3^\mp z^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_1 s_2}{s_3} z\right)}. \end{aligned} \quad (5.3)$$

Unlike the previous case, the number of D3-branes jumps across the D5'-brane in the dual configuration and therefore there is no twisted hypermultiplet.

We find that the quarter-index (5.3) agrees with

$$\mathbb{IV}_{\mathcal{DD}'}^{2|[1]|} = \underbrace{(q)_\infty}_{\mathbb{IV}_{\mathcal{DD}'}^{4d \text{ U}(1)}}. \quad (5.4)$$

This is identified with the quarter-index of 4d U(1) gauge theory obeying a pair of Dirichlet b.c. ($\mathcal{D}, \mathcal{D}'$).

5.1.3 4d U(3)|3d U(1) with $\mathcal{N}'$

The quarter-index is

$$\begin{aligned} \mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(3)|(1)|} &= \underbrace{\frac{1}{3!}(q)_\infty^3 \oint \prod_{i=1}^3 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q\right)_\infty}_{\mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(3)}} \cdot \underbrace{(q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty \oint \frac{ds_4}{2\pi i s_4}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(1)}} \\ &\times \prod_{i=1}^3 \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_i^\pm s_4^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}}\left(\frac{s_i}{s_4}\right)} \cdot \underbrace{(q^{\frac{1}{2}} s_1^\pm s_2^\pm s_3^\pm s_4^\mp z^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_1 s_2 s_3}{s_4} z\right)}. \end{aligned} \quad (5.5)$$

The quarter-index (5.5) agrees with

$$\mathbb{IV}_{\mathcal{DD}'}^{2|[1]|} = \underbrace{(q)_\infty}_{\mathbb{IV}_{\mathcal{DD}'}^{4d \text{ U}(1)}}. \quad (5.6)$$

5.1.4 4d U(N)|3d U(1) with $\mathcal{N}'$

Now we are led to propose the generalization of quarter-index for the junction 4d U(N)|3d U(1) with $\mathcal{N}'$ for $N > 1$. The junction preserves 4d U(N) gauge symmetry and 3d U(1)

gauge symmetry. It also contains 3d bi-fundamental hyper satisfying Neumann b.c. N' and 2d cross-determinant Fermi multiplet.

The quarter-index is given by

$$\begin{aligned}
 \mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(N)|(1)|} &= \underbrace{\frac{1}{N!}(q)_\infty \oint \prod_{i=1}^N \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(N)}} \cdot \underbrace{(q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty \oint \frac{ds_{N+1}}{2\pi i s_{N+1}}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(1)}} \\
 &\times \underbrace{\prod_{i=1}^N \frac{1}{(q^{\frac{1}{4}} t s_i^\pm s_{N+1}^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}} \left(\frac{s_i}{s_{N+1}} \right)} \cdot \underbrace{(q^{\frac{1}{2}} \prod_{i=1}^N s_i^\pm s_{N+1}^\mp z^\pm; q)_\infty}_{F \left(q^{\frac{1}{2}} \frac{\prod_{i=1}^N s_i}{s_{N+1}} z \right)}. \quad (5.7)
 \end{aligned}$$

The quarter-index (5.7) would be equal to the quarter-index of 4d U(1) gauge theory with a pair of Dirichlet b.c. $(\mathcal{D}, \mathcal{D}')$

$$\mathbb{IV}_{\mathcal{DD}'}^{N|[1]|} = \underbrace{(q)_\infty}_{\mathbb{IV}_{\mathcal{DD}'}^{4d \text{ U}(1)}}. \quad (5.8)$$

5.1.5 4d U(2)|3d U(2) with $\mathcal{N}'$

Consider the case with both 4d and 3d gauge symmetries are non-Abelian. For the configuration 4d U(2)|3d U(2) with $\mathcal{N}'$, we have 4d U(2) gauge symmetry and 3d U(2) gauge symmetry. At the junction there are 3d bi-fundamental hyper obeying Neumann b.c. N' and 2d cross-determinant Fermi multiplet.

The quarter-index is evaluated as

$$\begin{aligned}
 \mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(2)|(2)|} &= \underbrace{\frac{1}{2}(q)_\infty^2 \oint \prod_{i=1}^2 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(2)}} \\
 &\times \underbrace{\frac{1}{2}(q)_\infty^2 (q^{\frac{1}{2}} t^2; q)_\infty^2 \oint \prod_{i=3}^4 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty \left(q^{\frac{1}{2}} t^2 \frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(2)}} \\
 &\times \underbrace{\prod_{i=1}^2 \prod_{j=3}^4 \frac{1}{(q^{\frac{1}{4}} t s_i^\pm s_j^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}} \left(\frac{s_i}{s_j} \right)} \cdot \underbrace{(q^{\frac{1}{2}} s_1^\pm s_2^\pm s_3^\mp s_4^\mp z^\pm; q)_\infty}_{F \left(q^{\frac{1}{2}} \frac{s_1 s_2}{s_3 s_4} z \right)}. \quad (5.9)
 \end{aligned}$$

In the S-dual configuration, there are two D3-branes in each side of one of the D5'-branes, which would lead to the 3d $\mathcal{N} = 4$ fundamental twisted hypermultiplets. They should obey the boundary condition corresponding to the D5-brane which is characterized by the Nahm pole with and embedding $\rho: \mathfrak{su}(2) \rightarrow \mathfrak{u}(2)$.

We have confirmed that the quarter-index (5.9) agrees with

$$\mathbb{IV}_{\mathcal{D}\mathcal{D}'}^{2|[2]|} = \underbrace{(q)_\infty}_{\mathbb{IV}_{\mathcal{D}\mathcal{D}'}^{4d \text{ U}(1)}} \cdot (qt^2 z; q)_\infty (qt^2 z^{-1}; q)_\infty. \quad (5.10)$$

As discussed in [31], the contributions $(qt^2 z^\pm; q)_\infty$ from 3d $\mathcal{N} = 4$ fundamental twisted hyper can be obtained by the Higgsing procedure that produces a Nahm pole. Identifying the fugacities z_1, z_2 for the U(2) global symmetry with $q^{-\frac{1}{4}} z, q^{\frac{1}{4}} z$ gives $(q^{\frac{1}{2}} z^\pm; q)_\infty (qt^2 z^\pm; q)_\infty$. Then stripping off the Fermi index, we get the terms in (5.10).

5.1.6 4d U(3)|3d U(2) with $\mathcal{N}'$

For the junction 4d U(3)|3d U(2) with $\mathcal{N}'$, we have 4d U(3) gauge symmetry and 3d U(2) gauge symmetry. There are 3d $\mathcal{N} = 4$ bi-fundamental hyper with Neumann b.c. N' and 2d cross-determinant Fermi multiplet.

We have the quarter-index

$$\begin{aligned} \mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d \text{ U}(3)|(2)|} &= \underbrace{\frac{1}{3!} (q)_\infty^3 \oint \prod_{i=1}^3 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d \text{ U}(3)}} \\ &\times \underbrace{\frac{1}{2} (q)_\infty^2 (q^{\frac{1}{2}} t^2; q)_\infty^2 \oint \prod_{i=4}^5 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty \left(q^{\frac{1}{2}} t^2 \frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(2)}} \\ &\times \prod_{i=1}^3 \prod_{j=4}^5 \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_i^\pm s_j^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}} \left(\frac{s_i}{s_j} \right)} \cdot \underbrace{(q^{\frac{1}{2}} s_1^\pm s_2^\pm s_3^\pm s_4^\mp s_5^\mp z^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_1 s_2 s_3}{s_4 s_5} z\right)}. \end{aligned} \quad (5.11)$$

In the dual configuration, according to the unequal numbers of D3-branes in each side of D5'-branes, there is no 3d fundamental twisted hypermultiplet.

In fact, the quarter-index (5.11) coincides with the quarter-index of 4d $\mathcal{N} = 4$ U(1) gauge theory with a pair of Dirichlet b.c. $(\mathcal{D}, \mathcal{D}')$

$$\mathbb{IV}_{\mathcal{D}\mathcal{D}'}^{3|(2)|} = \underbrace{(q)_\infty}_{\mathbb{IV}_{\mathcal{D}\mathcal{D}'}^{4d \text{ U}(1)}}. \quad (5.12)$$

5.1.7 4d U(3)|3d U(3) with $\mathcal{N}'$

Similarly, we can compute the quarter-index for the configuration 4d U(3)|3d U(3) with $\mathcal{N}'$ as

$$\begin{aligned}
 \mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(3)|(3)|} &= \underbrace{\frac{1}{3!}(q)_\infty^3 \oint \prod_{i=1}^3 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{IV}_{\mathcal{NN}'}^{3d \text{ U}(3)}} \\
 &\times \underbrace{\frac{1}{3!}(q)_\infty^3 (q^{\frac{1}{2}} t^2; q)_\infty^3 \oint \prod_{i=4}^6 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty \left(q^{\frac{1}{2}} t^2 \frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(3)}} \\
 &\times \prod_{i=1}^3 \prod_{j=4}^6 \frac{1}{\underbrace{(q^{\frac{1}{4}} t s_i^\pm s_j^\mp; q)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}} \left(\frac{s_i}{s_j} \right)}} \cdot \underbrace{(q^{\frac{1}{2}} s_1^\pm s_2^\pm s_3^\pm s_4^\mp s_5^\mp s_6^\mp z^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_1 s_2 s_3}{s_4 s_5 s_6} z\right)}. \quad (5.13)
 \end{aligned}$$

Although the dual junction has no gauge symmetry, it includes 3d $\mathcal{N} = 4$ fundamental twisted hyper since the equal numbers of D3-branes exist in both sides of D5'-brane.

We find that the quarter-index (5.13) coincides with

$$\mathbb{IV}_{\mathcal{DD}'}^{3|(3)|} = \underbrace{(q)_\infty}_{\mathbb{IV}_{\mathcal{DD}'}^{4d \text{ U}(1)}} \cdot (q^{\frac{5}{4}} t^3 z; q)_\infty (q^{\frac{5}{4}} t^3 z^{-1}; q)_\infty. \quad (5.14)$$

Here the contributions from 3d twisted hyper are not Dirichlet b.c. D but rather the Nahm pole boundary condition with an embedding $\rho: \mathfrak{su}(2) \rightarrow \mathfrak{u}(3)$. To achieve the RG flow from Dirichlet to a Nahm pole, the corresponding Higgsing procedure fixes the fugacities z_1, z_2, z_3 for U(3) global symmetry to $q^{\frac{1}{2}} t^2 z, q^{\frac{1}{2}} t^2 z^{-1}, z$, which yields $(q^{\frac{5}{4}} t^3 z^\pm; q)_\infty (q^{\frac{1}{4}} t^{-1} z^\pm; q)_\infty (q^{\frac{3}{4}} t z^\pm; q)_\infty$. Stripping off the Fermi indices $F(q^{\frac{1}{4}} t^{-1} z) F(q^{\frac{1}{4}} t^{-1} z^{-1})$, we get the correct contributions in (5.14).

5.1.8 4d U(N)|3d U(M) with $\mathcal{N}'$

Now we would like to propose the generalization. The configuration keeps 4d U(N) gauge symmetry and 3d U(M) gauge symmetry. At the junction, there is a 3d bi-fundamental hypermultiplet transforming as $(\mathbf{N}, \overline{\mathbf{M}}) \oplus (\overline{\mathbf{N}}, \mathbf{M})$ with Neumann b.c. $\mathcal{N}'$. In addition, there is a cross-determinant Fermi multiplet which cancels the Abelian part of gauge anomaly.

We have the quarter-index

$$\begin{aligned}
 \mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(N)| (M)} &= \underbrace{\frac{1}{N!} (q)_\infty^N \oint \prod_{i=1}^N \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(N)}} \\
 &\times \underbrace{\frac{1}{M!} (q)_\infty^M (q^{\frac{1}{2}} t^2; q)_\infty^M \oint \prod_{i=N+1}^{N+M} \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty \left(q^{\frac{1}{2}} t^2 \frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(M)}} \\
 &\times \prod_{i=1}^N \prod_{j=N+1}^{N+M} \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_i^\pm s_j^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}} \left(\frac{s_i}{s_j} \right)} \cdot \underbrace{(q^{\frac{1}{2}} \prod_{i=1}^N s_i^\pm \prod_{j=N+1}^{N+M} s_j^\mp z^\pm; q)_\infty}_{F \left(q^{\frac{1}{2}} \frac{\prod_{i=1}^N s_i}{\prod_{j=N+1}^{N+M} s_j} z \right)}. \quad (5.15)
 \end{aligned}$$

On the other hand, the dual junction has no gauge symmetry. For $N = M$, $3d \mathcal{N} = 4$ twisted hypermultiplet transforming as fundamental representation under the $U(N)$ gauge symmetry should arise from D3-D5' string. It would obey the Nahm pole boundary condition with and embedding $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{u}(N)$. For $N \neq M$ we have no twisted hypermultiplet.

The quarter-index (5.15) would be equal to

$$\mathbb{IV}_{\mathcal{DD}'}^{N|[M]} = \underbrace{(q)_\infty}_{\mathbb{IV}_{\mathcal{DD}'}^{4d \text{ U}(1)}} \cdot (q^{\frac{3}{4} + \frac{N-1}{4}} t^{1+(N-1)} z^\pm; q)_\infty^{\delta^N M}. \quad (5.16)$$

The contributions include the quarter-index of $4d \mathcal{N} = 4 \text{ U}(1)$ gauge theory with a pair of Dirichlet b.c. $(\mathcal{D}, \mathcal{D}')$ and the half-index of Nahm pole b.c. for $3d$ twisted hyper with and embedding $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{u}(N)$ [31].

5.2 4d $U(N)|3d (M_1) - \dots - (M_{k-1})$ with $\mathcal{N}'$

Let us take the quarter-BPS corner of $4d \mathcal{N} = 4 \text{ U}(N)$ gauge theory with a pair of Neumann b.c. $(\mathcal{N}, \mathcal{N}')$ and the half-BPS boundary of $3d \mathcal{N} = 4$ linear quiver gauge theory $(M_1) - \dots - (M_{k-1})$ obeying Neumann b.c. $(\mathcal{N}', N')$. The corresponding brane setup is illustrated in figure 12. At the NS5-NS5' junction for which the NS5-brane has both-sided D3-branes, there appears a cross-determinant Fermi multiplet.

5.2.1 4d $U(1)|3d U(1) \times U(1)$ with $\mathcal{N}'$

For the junction $4d U(1)|3d U(1) \times U(1)$ with $\mathcal{N}'$, we have $4d U(1)$ gauge theory and $3d U(1) \times U(1)$ gauge symmetry.

Figure 12. The brane constructions of the 4d $U(N)|3d (M_1) - \dots - (M_{k-1})$ with Neumann b.c. $\mathcal{N}'$ and its mirror corner configuration that has no gauge symmetry.

The quarter-index is

$$\begin{aligned}
 \mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(1)|(1)-(1)} &= (q)_\infty \oint \frac{ds_1}{2\pi i s_1} \cdot (q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty \oint \frac{ds_2}{2\pi i s_2} \cdot (q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty \oint \frac{ds_3}{2\pi i s_3} \quad (5.17) \\
 &\quad \underbrace{\mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(1)}}_{\mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(1)}} \underbrace{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(1)}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(1)}} \underbrace{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(1)}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(1)}} \\
 &\quad \times \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_1^\pm s_2^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}}\left(\frac{s_1}{s_2}\right)} \cdot \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_2^\pm s_3^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}}\left(\frac{s_2}{s_3}\right)} \cdot \underbrace{(q^{\frac{1}{2}} s_1^\pm s_2^\mp z_1^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_1}{s_2}\right)} \cdot \underbrace{(q^{\frac{1}{2}} s_2^\pm s_3^\mp z_2^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_2}{s_3}\right)}.
 \end{aligned}$$

In the last line there are contributions from two bi-fundamental hypermultiplets as well as two cross-determinant Fermi multiplets.

The S-dual configuration includes two $D5'$ -branes which have a single $D3$ -brane in their both sides. They should give rise to two fundamental twisted hypermultiplets. In addition, they should obey the Dirichlet b.c. D as the $D3$ -brane terminates on a single $D5$ -brane.

We have checked that the quarter-index (5.17) agrees with

$$\mathbb{IV}_{\mathcal{DD}'}^{1|[1]-[1]} = \underbrace{(q)_\infty}_{\mathbb{IV}_{\mathcal{DD}'}^{4d \text{ U}(1)}} \cdot \underbrace{(q^{\frac{3}{4}} t z_1^\pm; q)_\infty}_{\mathbb{III}_D^{3d \text{ tHM}}(z_1)} \cdot \underbrace{(q^{\frac{3}{4}} t z_2^\pm; q)_\infty}_{\mathbb{III}_D^{3d \text{ tHM}}(z_2)}. \quad (5.18)$$

5.2.2 4d $U(2)|3d U(1) \times U(1)$ with $\mathcal{N}'$

In the configuration 4d $U(2)|3d U(1) \times U(1)$ with $\mathcal{N}'$, 4d $U(2)$ gauge theory satisfies a pair of Neumann b.c. $(\mathcal{N}, \mathcal{N}')$ and 3d $U(1) \times U(1)$ gauge theory obeys Neumann b.c. $\mathcal{N}'$. It has two

3d bi-fundamental hyper multiplets with Neumann b.c. N' and two 2d cross-determinant Fermi multiplets.

The quarter-index is evaluated as

$$\begin{aligned}
 & \mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(2)|(1)-(1)} \\
 &= \underbrace{\frac{1}{2}(q)_\infty^2 \oint \prod_{i=1}^2 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(2)}} \cdot \underbrace{(q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty \oint \frac{ds_3}{2\pi i s_3}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(1)}} \cdot \underbrace{(q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty \oint \frac{ds_4}{2\pi i s_4}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(1)}} \\
 & \times \prod_{i=1}^2 \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_i^\pm s_3^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}}\left(\frac{s_i}{s_3}\right)} \cdot \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_3^\pm s_4^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}}\left(\frac{s_3}{s_4}\right)} \cdot \underbrace{(q^{\frac{1}{2}} s_1^\pm s_2^\pm s_3^\mp z_1^\pm; q)_\infty}_{F\left(\frac{s_1 s_2}{s_3} z_1\right)} \cdot \underbrace{(q^{\frac{1}{2}} s_3^\pm s_4^\mp z_2^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_3}{s_4} z_2\right)}. \quad (5.19)
 \end{aligned}$$

In the S-dual junction, the gauge symmetry is completely broken. As one of the D5' branes has a single D3-brane in its both sides, there would be a 3d charged twisted hypermultiplet. Due to the presence of D5-brane, it should obey the Dirichlet b.c. D .

The quarter-index (5.19) agrees with

$$\mathbb{IV}_{\mathcal{DD}'}^{2|[1]-[1]} = \underbrace{(q)_\infty}_{\mathbb{IV}_{\mathcal{DD}'}^{4d \text{ U}(1)}} \cdot \underbrace{(q^{\frac{3}{4}} t z_2^\pm; q)_\infty}_{\mathbb{III}_D^{3d \text{ tHM}}(z_2)}. \quad (5.20)$$

5.2.3 4d U(2)|3d U(2) × U(1) with $\mathcal{N}'$

Consider the junction 4d U(2)|3d U(2) × U(1) with $\mathcal{N}'$. The configuration has 4d U(2) gauge symmetry and 3d U(2) × U(1) gauge symmetry. There are two 3d bi-fundamental hypermultiplets with Neumann b.c. N' and two 2d cross-determinant Fermi multiplets.

The quarter-index is

$$\begin{aligned}
 & \mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(2)|(2)-(1)} \\
 &= \underbrace{\frac{1}{2}(q)_\infty^2 \oint \prod_{i=1}^2 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(2)}} \\
 & \cdot \underbrace{\frac{1}{2}(q)_\infty^2 (q^{\frac{1}{2}} t^2; q)_\infty^2 \oint \prod_{i=3}^4 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty \left(q^{\frac{1}{2}} t^2 \frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(2)}} \times \underbrace{(q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty \oint \frac{ds_5}{2\pi i s_5}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(1)}} \\
 & \times \prod_{i=1}^2 \prod_{j=3}^4 \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_i^\pm s_j^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}}\left(\frac{s_i}{s_j}\right)} \cdot \prod_{i=3}^4 \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_i^\pm s_5^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}}\left(\frac{s_i}{s_5}\right)} \cdot \underbrace{(q^{\frac{1}{2}} s_1^\pm s_2^\pm s_3^\mp s_4^\mp z_1^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_1 s_2}{s_3 s_4} z_1\right)} \cdot \underbrace{(q^{\frac{1}{2}} s_3^\pm s_4^\pm s_5^\mp z_2^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_3 s_4}{s_5}\right)}. \quad (5.21)
 \end{aligned}$$

The quarter-index (5.21) agrees with

$$\mathbb{IV}_{\mathcal{D}\mathcal{D}'}^{2|[2]-[1]} = \underbrace{(q)_\infty}_{\mathbb{IV}_{\mathcal{D}\mathcal{D}'}^{4d \text{ U}(1)}} \cdot (qt^2 z_1^\pm; q)_\infty. \quad (5.22)$$

This involves the contribution from 3d fundamental twisted hypermultiplet satisfying the Nahm pole boundary condition with an embedding $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{u}(2)$.

5.2.4 4d U(3)|3d U(2) × U(1) with $\mathcal{N}'$

Next consider the configuration 4d U(3)|3d U(2) × U(1) with $\mathcal{N}'$. It keeps 4d U(3) gauge symmetry and 3d U(2) × U(1) gauge symmetry. There are two 3d bi-fundamental hypermultiplets obeying Neumann b.c. $\mathcal{N}'$ and two 2d cross-determinant Fermi multiplets.

The quarter-index reads

$$\begin{aligned} & \mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d \text{ U}(3)|(2)-(1)} \\ &= \underbrace{\frac{1}{3!}(q)_\infty \oint \prod_{i=1}^3 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d \text{ U}(3)}} \cdot \underbrace{\frac{1}{2}(q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty^2 \oint \prod_{i=4}^5 \frac{ds_i}{2\pi i s_i} \left(\frac{s_i}{s_j}; q \right)_\infty \left(q^{\frac{1}{2}} t^2 \frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(2)}} \\ & \times \underbrace{(q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty \oint \frac{ds_6}{2\pi i s_6}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(1)}} \\ & \times \prod_{i=1}^3 \prod_{j=4}^5 \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_i^\pm s_j^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}} \left(\frac{s_i}{s_j} \right)} \cdot \prod_{i=4}^5 \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_i^\pm s_6^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}} \left(\frac{s_i}{s_6} \right)} \cdot \underbrace{(q^{\frac{1}{2}} s_1^\pm s_2^\pm s_3^\pm s_4^\mp s_5^\mp z_1^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_1 s_2 s_3}{s_4 s_5} z_1\right)} \cdot \underbrace{(q^{\frac{1}{2}} s_4^\pm s_5^\pm s_6^\mp z_2^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_4 s_5}{s_6} z_2\right)}. \end{aligned} \quad (5.23)$$

The S-dual corner configuration is viewed as a corner of 4d U(3) gauge theory. It is obtained from the Dirichlet b.c. $\mathcal{D}'$ for 4d U(3) gauge theory further satisfying boundary condition $\mathcal{D}$ corresponding to the D5-brane.

The quarter-index (5.23) agrees with the quarter-index of 4d U(1) gauge theory with a pair of Dirichlet b.c. $(\mathcal{D}, \mathcal{D}')$

$$\mathbb{IV}_{\mathcal{D}\mathcal{D}'}^{3|[2]-[1]} = \underbrace{(q)_\infty}_{\mathbb{IV}_{\mathcal{D}\mathcal{D}'}^{4d \text{ U}(1)}}. \quad (5.24)$$

This reflects the fact that the unequal numbers of D3-branes across D5-branes do not admit the 3d twisted hypermultiplet.

5.2.5 4d U(2)|3d U(2) × U(2) with $\mathcal{N}'$

Let us consider the junction 4d U(2)|3d U(2) × U(2) with $\mathcal{N}'$. This includes a corner of 4d U(2) gauge theory with a pair of Neumann b.c. $(\mathcal{N}, \mathcal{N}')$ and 3d U(2) × U(2) gauge theory with Neumann b.c. $\mathcal{N}'$. It also has two 3d bi-fundamental hypermultiplets transforming

as $(\mathbf{2}, \bar{\mathbf{2}}, \mathbf{1}) \oplus (\bar{\mathbf{2}}, \mathbf{2}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{2}, \bar{\mathbf{2}}) \oplus (\mathbf{1}, \bar{\mathbf{2}}, \mathbf{2})$ under the $U(2) \times U(2) \times U(2)$ gauge symmetry and 2d two cross-determinant Fermi multiplets transforming as $(\det, \det^{-1}, \mathbf{1})$ and $(\mathbf{1}, \det, \det^{-1})$.

The quarter-index takes the form

$$\begin{aligned}
 & \mathbb{IV}_{\mathcal{NN}'}^{4d \ U(2)|(2)-(2)} \quad (5.25) \\
 &= \underbrace{\frac{1}{2}(q)_\infty^2 \oint \prod_{i=1}^2 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{IV}_{\mathcal{NN}'}^{4d \ U(2)}} \cdot \underbrace{\frac{1}{2}(q)_\infty^2 (q^{\frac{1}{2}} t^2; q)_\infty^2 \oint \prod_{i=3}^4 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty \left(q^{\frac{1}{2}} t^2 \frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d \ U(2)}} \\
 &\quad \times \underbrace{\frac{1}{2}(q)_\infty^2 (q^{\frac{1}{2}} t^2; q)_\infty^2 \oint \prod_{i=5}^6 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty \left(q^{\frac{1}{2}} t^2 \frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d \ U(2)}} \\
 &\quad \times \underbrace{\prod_{i=1}^2 \prod_{j=3}^4 \frac{1}{(q^{\frac{1}{4}} t s_i^\pm s_j^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \ HM} \left(\frac{s_i}{s_j} \right)} \cdot \underbrace{\prod_{i=3}^4 \prod_{j=5}^6 \frac{1}{(q^{\frac{1}{4}} t s_i^\pm s_j^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \ HM} \left(\frac{s_i}{s_j} \right)} \\
 &\quad \cdot \underbrace{(q^{\frac{1}{2}} s_1^\pm s_2^\pm s_3^\mp s_4^\mp z_1^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_1 s_2}{s_3 s_4} z_1\right)} \cdot \underbrace{(q^{\frac{1}{2}} s_3^\pm s_4^\pm s_5^\mp s_6^\mp z_2^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_3 s_4}{s_5 s_6} z_2\right)}.
 \end{aligned}$$

The S-dual configuration has no gauge symmetry. As the numbers of D3-branes are the same across the two D5'-branes, the junction includes the contributions from two 3d fundamental twisted hypers. According to the D5-brane, they should obey the Nahm pole boundary condition characterized by an embedding $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{u}(2)$.

We find that the quarter-index (5.25) coincides with

$$\mathbb{IV}_{\mathcal{DD}'}^{2|[2]-[2]} = \underbrace{(q)_\infty}_{\mathbb{IV}_{\mathcal{DD}'}^{4d \ U(1)}} \cdot (qt^2 z_1^\pm; q)_\infty \cdot (qt^2 z_2^\pm; q)_\infty. \quad (5.26)$$

As expected, it involves the two contributions which capture the two 3d fundamental twisted hypers obeying Nahm pole boundary condition with an embedding $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{u}(2)$.

5.2.6 4d $U(3)|3d \ U(3) \times U(1)$ with $\mathcal{N}'$

To gain more insight, let us examine the junction 4d $U(3)|3d \ U(3) \times U(1)$ with $\mathcal{N}'$. The surviving gauge symmetry is 4d $U(3)$ and 3d $U(3) \times U(1)$. We have two 3d bi-fundamental hypers with Neumann b.c. N' and two 2d cross-determinant Fermi multiplets.

The quarter-index is given by

$$\begin{aligned}
 & \mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(3)|(3)-(1)} \\
 &= \underbrace{\frac{1}{3!}(q)_\infty^3 \oint \prod_{i=1}^3 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(3)}} \cdot \underbrace{\frac{1}{3!}(q)_\infty^3 (q^{\frac{1}{2}} t^2; q)_\infty^3 \oint \prod_{i=4}^6 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty \left(q^{\frac{1}{2}} t^2 \frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(3)}} \\
 & \times \underbrace{(q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty \oint \frac{ds_7}{2\pi i s_7}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(1)}} \times \underbrace{\prod_{i=1}^3 \prod_{j=4}^6 \frac{1}{(q^{\frac{1}{4}} t s_i^\pm s_j^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}\left(\frac{s_i}{s_j}\right)}} \cdot \underbrace{\prod_{i=4}^6 \frac{1}{(q^{\frac{1}{4}} t s_i^\pm s_7^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}\left(\frac{s_i}{s_7}\right)}} \\
 & \cdot \underbrace{(q^{\frac{1}{2}} s_1^\pm s_2^\pm s_3^\pm s_4^\mp s_5^\mp s_6^\mp z_1^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_1 s_2 s_3}{s_4 s_5 s_6} z_1\right)} \cdot \underbrace{(q^{\frac{1}{2}} s_4^\pm s_5^\pm s_6^\pm s_7^\mp z_2^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_4 s_5 s_6}{s_7} z_2\right)}.
 \end{aligned} \tag{5.27}$$

In the S-dual junction three D3-branes live in each side of one of the D5'-branes so that one obtains a 3d fundamental twisted hypermultiplet obeying the Nahm pole boundary condition specified by an embedding $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{u}(3)$ corresponding to the D5-brane.

We have checked that the quarter-index (5.27) agrees with

$$\mathbb{IV}_{\mathcal{DD}'}^{3|[3]-[1]} = \underbrace{(q)_\infty}_{\mathbb{IV}_{\mathcal{DD}'}^{4d \text{ U}(1)}} \cdot (q^{\frac{5}{4}} t^3 z_1^\pm; q)_\infty. \tag{5.28}$$

The factors $(q^{\frac{5}{4}} t^3 z_1^\pm; q)_\infty$ are now familiar contributions of 3d fundamental twisted hypermultiplet obeying Nahm pole boundary condition associated to an embedding $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{u}(3)$.

5.2.7 4d U(3)|3d U(3) × U(2) with $\mathcal{N}'$

As a further check, consider the junction 4d U(3)|3d U(3) × U(2) with $\mathcal{N}'$.

Similarly, we can compute the quarter-index as

$$\begin{aligned}
 & \mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(3)|(3)-(2)} \\
 &= \underbrace{\frac{1}{3!}(q)_\infty^3 \oint \prod_{i=1}^3 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(3)}} \cdot \underbrace{\frac{1}{3!}(q)_\infty^3 (q^{\frac{1}{2}} t^2; q)_\infty^3 \oint \prod_{i=4}^6 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty \left(q^{\frac{1}{2}} t^2 \frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(3)}} \\
 & \times \underbrace{\frac{1}{2}(q)_\infty^2 (q^{\frac{1}{2}} t^2; q)_\infty^2 \oint \prod_{i=7}^8 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty \left(q^{\frac{1}{2}} t^2 \frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(2)}}
 \end{aligned}$$

$$\begin{aligned}
 & \times \prod_{i=1}^3 \prod_{j=4}^6 \frac{1}{(q^{\frac{1}{4}} t s_i^{\pm} s_j^{\mp}; q)_{\infty}} \cdot \prod_{i=4}^6 \prod_{j=7}^8 \frac{1}{(q^{\frac{1}{4}} t s_i^{\pm} s_j^{\mp}; q)_{\infty}} \\
 & \quad \quad \quad \mathbb{I}\mathbb{I}\mathbb{I}_{N'}^{3d \text{ HM}} \left(\frac{s_i}{s_j} \right) \quad \quad \quad \mathbb{I}\mathbb{I}\mathbb{I}_{N'}^{3d \text{ HM}} \left(\frac{s_i}{s_j} \right) \\
 & \times \underbrace{(q^{\frac{1}{2}} s_1^{\pm} s_2^{\pm} s_3^{\pm} s_4^{\mp} s_5^{\mp} s_6^{\mp} z_1^{\pm}; q)_{\infty}}_{F\left(\frac{s_1 s_2 s_3}{s_4 s_5 s_6} z_1\right)} \cdot \underbrace{(q^{\frac{1}{2}} s_4^{\pm} s_5^{\pm} s_6^{\pm} s_7^{\mp} s_8^{\mp} z_2^{\pm}; q)_{\infty}}_{F\left(q^{\frac{1}{2}} \frac{s_4 s_5 s_6}{s_7 s_8} z_2\right)}. \tag{5.29}
 \end{aligned}$$

For the dual junction, gauge symmetry is completely broken. Again three D3-branes in each side of one of the D5'-branes yield a 3d fundamental twisted hypermultiplet obeying the Nahm pole boundary condition with an embedding $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{u}(3)$ due to the D5-brane.

In fact, the quarter-index (5.29) agrees with

$$\mathbb{I}\mathbb{V}_{\mathcal{D}\mathcal{D}'}^{3|[3]-[2]} = \underbrace{(q)_{\infty}}_{\mathbb{I}\mathbb{V}_{\mathcal{D}\mathcal{D}'}^{4d \text{ U}(1)}} \cdot (q^{\frac{5}{4}} t^3 z_1^{\pm}; q)_{\infty}. \tag{5.30}$$

5.2.8 4d $\text{U}(N)|3d \text{U}(N)^{k-1}$ with $\mathcal{N}'$

Let us consider the junction 4d $\text{U}(N)|3d \text{U}(N)^{k-1}$ with $\mathcal{N}'$, consisting of a corner of 4d $\text{U}(N)$ gauge theory with a pair of Neumann b.c. $(\mathcal{N}, \mathcal{N}')$ and a boundary of 3d quiver gauge theory of gauge group $\text{U}(N)^{k-1}$ with $(k-2)$ bi-fundamental hypermultiplets. The 3d vector and hypers obey Neumann b.c. $\mathcal{N}'$ and N' respectively. We have an additional 3d bi-fundamental hypermultiplet that couples to 4d $\text{U}(N)$ and 3d $\text{U}(N)$ vector multiplets and that has Neumann b.c. N' . Besides we have $(k-1)$ 2d cross-determinant Fermi multiplets which cancel the Abelian part of boundary gauge anomaly. In the brane construction, this junction is realized by N D3-branes intersecting with k NS5-branes and a single NS5'-brane (see figure 12).

For the junction 4d $\text{U}(N)|3d \text{U}(N)^{k-1}$ with Neumann b.c. $\mathcal{N}'$, we can evaluate the quarter-index as

$$\begin{aligned}
 \mathbb{I}\mathbb{V}_{\mathcal{N}\mathcal{N}'}^{4d \text{ U}(N)|(N)^{k-1}} &= \underbrace{\frac{1}{N!} (q)_{\infty} \oint \prod_{i=1}^N \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_{\infty}}_{\mathbb{I}\mathbb{V}_{\mathcal{N}\mathcal{N}'}^{4d \text{ U}(N)}} \\
 & \times \underbrace{(q)_{\infty}^{(k-1)N} (q^{\frac{1}{2}} t^2; q)_{\infty}^{(k-1)N} \prod_{l=1}^{k-1} \oint \prod_{i=1}^N \frac{ds_i^{(l)}}{2\pi i s_i^{(l)}} \prod_{i \neq j} \left(\frac{s_i^{(l)}}{s_j^{(l)}}; q \right)_{\infty} \left(q^{\frac{1}{2}} t^2 \frac{s_i^{(l)}}{s_j^{(l)}}; q \right)_{\infty}}_{\mathbb{I}\mathbb{I}\mathbb{I}_{N'}^{3d \text{ U}(N)^{k-1}}} \\
 & \times \prod_{i=1}^N \prod_{j=1}^N \frac{1}{(q^{\frac{1}{4}} t s_i^{\pm} s_j^{(1)\mp}; q)_{\infty}} \cdot \prod_{l=1}^{k-2} \prod_{i=1}^N \prod_{j=1}^N \frac{1}{(q^{\frac{1}{4}} t s_i^{(l)\pm} s_j^{(l+1)\mp}; q)_{\infty}} \\
 & \quad \quad \quad \mathbb{I}\mathbb{I}\mathbb{I}_{N'}^{3d \text{ HM}} \left(\frac{s_i}{s_j^{(1)}} \right) \quad \quad \quad \mathbb{I}\mathbb{I}\mathbb{I}_{N'}^{3d \text{ HM}} \left(\frac{s_i^{(l)}}{s_j^{(l+1)}} \right)
 \end{aligned} \tag{5.31}$$

$$\times \underbrace{\left(q^{\frac{1}{2}} \prod_{i=1}^N s_i^{\pm} \prod_{j=1}^N s_j^{(1)\mp} z_1^{\pm}; q \right)_{\infty}}_{F\left(q^{\frac{1}{2}} \frac{\prod_{i=1}^N s_i}{\prod_{j=1}^N s_j^{(1)}} z_1 \right)} \cdot \underbrace{\prod_{l=1}^{k-2} \left(q^{\frac{1}{2}} \prod_{i=1}^N s_i^{(l)\pm} \prod_{j=1}^N s_j^{(l+1)\mp} z_{l+1}^{\pm}; q \right)_{\infty}}_{F\left(q^{\frac{1}{2}} \frac{\prod_{i=1}^N s_i^{(l)}}{\prod_{j=1}^N s_j^{(l+1)}} z_{l+1}; q \right)_{\infty}}.$$

The gauge symmetry is completely broken in the S-dual junction. As the numbers of D3-branes remain the same across the k D5'-branes, the junction would include the contributions from k 3d fundamental twisted hypermultiplets. Since the number of D3-branes jumps from N to 0 across the D5-brane, they should obey the Nahm pole boundary condition characterized by an embedding $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{u}(N)$.

We expect that the quarter-index (5.31) would be equal to

$$\mathbb{IV}_{\mathcal{DD}'}^{N|[N]^{k-1}|} = \underbrace{(q)_{\infty}}_{\mathbb{IV}_{\mathcal{DD}'}^{4d \text{ U}(1)}} \cdot \prod_{l=1}^k \left(q^{\frac{3}{4} + \frac{N-1}{4}} t^{1+(N-1)} z_l^{\pm}; q \right)_{\infty}, \quad (5.32)$$

which contains the quarter-index of 4d U(1) gauge theory with a pair of Dirichlet b.c. $(\mathcal{D}, \mathcal{D}')$ and the contributions from k 3d fundamental twisted hypermultiplets with the Nahm pole boundary condition with an embedding $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{u}(N)$.

5.2.9 4d U(2)|3d U(2) × U(1) × U(1) with $\mathcal{N}'$

Let us examine the case where several same numbers of D3-branes are arranged in both sides of D5'-branes. For the junction 4d U(2)|3d U(2) × U(1) × U(1) with $\mathcal{N}'$, we have 4d U(2) gauge symmetry and 3d U(2) × U(1) × U(1) gauge symmetry. There are three 3d bi-fundamental hypermultiplets and three 2d cross-determinant Fermi multiplets.

The quarter-index takes the form

$$\begin{aligned} & \mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(2)|(2)-(1)-(1)|} \\ &= \underbrace{\frac{1}{2} (q)_{\infty}^2 \oint \prod_{i=1}^2 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_{\infty}}_{\mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(2)}} \cdot \underbrace{\frac{1}{2} (q)_{\infty}^2 (q^{\frac{1}{2}} t^2; q)_{\infty}^2 \oint \prod_{i=3}^4 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_{\infty} \left(q^{\frac{1}{2}} t^2 \frac{s_i}{s_j}; q \right)_{\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(2)}} \\ & \times \underbrace{(q)_{\infty} (q^{\frac{1}{2}} t^2; q)_{\infty} \oint \frac{ds_5}{2\pi i s_5}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(1)}} \cdot \underbrace{(q)_{\infty} (q^{\frac{1}{2}} t^2; q)_{\infty} \oint \frac{ds_6}{2\pi i s_6}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(1)}} \\ & \times \prod_{i=1}^2 \prod_{j=3}^4 \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_i^{\pm} s_j^{\mp}; q)_{\infty}}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}} \left(\frac{s_i}{s_j} \right)} \cdot \prod_{i=3}^4 \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_i^{\pm} s_5^{\mp}; q)_{\infty}}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}} \left(\frac{s_i}{s_5} \right)} \cdot \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_5^{\pm} s_6^{\mp}; q)_{\infty}}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}} \left(\frac{s_5}{s_6} \right)} \\ & \times \underbrace{(q^{\frac{1}{2}} s_1^{\pm} s_2^{\pm} s_3^{\mp} s_4^{\mp} z_1^{\pm}; q)_{\infty}}_{F\left(q^{\frac{1}{2}} \frac{s_1 s_2}{s_3 s_4} z_1 \right)} \cdot \underbrace{(q^{\frac{1}{2}} s_3^{\pm} s_4^{\pm} s_5^{\mp} z_2^{\pm}; q)_{\infty}}_{F\left(q^{\frac{1}{2}} \frac{s_3 s_4}{s_5} z_2 \right)} \cdot \underbrace{(q^{\frac{1}{2}} s_5^{\pm} s_6^{\mp} z_3^{\pm}; q)_{\infty}}_{F\left(q^{\frac{1}{2}} \frac{s_5}{s_6} z_3 \right)} \end{aligned} \quad (5.33)$$

In the dual junction, there are four D5'-branes. For each side of the first D5'-brane, there are two D3-branes. Across the second one, the numbers of D3-branes jumps from two to one. In both sides of the third one, there is a single D3-brane. Across the last one, the number of D3-branes jumps from one to zero. Therefore we expect that there are two kinds of 3d fundamental twisted hypermultiplets associated to the first and third D5'-branes.

We find that the quarter-index (5.33) agrees with

$$\mathbb{IV}_{\mathcal{DD}'}^{2|[2]-[1]-[1]]} = \underbrace{(q)_\infty}_{\mathbb{IV}_{\mathcal{DD}'}^{4d \text{ U}(1)}} \cdot (qt^2 z_1^\pm; q)_\infty \cdot \underbrace{(q^{\frac{3}{4}} t z_3^\pm; q)_\infty}_{\mathbb{III}_D^{3d \text{ tHM}(z_3)}}. \quad (5.34)$$

This includes two factors; $(qt^2 z_1^\pm; q)_\infty$ and $(q^{\frac{3}{4}} t z_3^\pm; q)_\infty$ as well as the quarter-index of 4d U(1) gauge theory with a pair of Dirichlet b.c. $(\mathcal{D}, \mathcal{D}')$. The first factor is the contribution from the 3d fundamental twisted hyper with Nahm pole b.c. specified by an embedding $\rho: \mathfrak{su}(2) \rightarrow \mathfrak{u}(2)$ corresponding to the first D5'-brane, and the second describes the contribution from the 3d fundamental twisted hyper with Dirichlet b.c. D corresponding to the third D5'-brane.

5.2.10 4d U(3)|3d U(3) × U(1) × U(1) with $\mathcal{N}'$

For the configuration 4d U(3)|3d U(3) × U(1) × U(1) with $\mathcal{N}'$, there are 4d U(3) gauge symmetry and 3d U(3) × U(1) × U(1) gauge symmetry. Again there are three 3d bi-fundamental hypermultiplets with Neumann b.c. N' and three 2d cross-determinant Fermi multiplets which cancel the Abelian part of gauge anomaly.

The quarter-index is evaluated as

$$\begin{aligned} & \mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(3)|(3)-(1)-(1)} \\ &= \underbrace{\frac{1}{3!} (q)_\infty^3 \oint \prod_{i=1}^3 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(3)}} \cdot \underbrace{\frac{1}{3!} (q)_\infty^3 (q^{\frac{1}{2}} t^2; q)_\infty^3 \oint \prod_{i=4}^6 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty \left(q^{\frac{1}{2}} t^2 \frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(3)}} \\ & \times \underbrace{(q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty \oint \frac{ds_7}{2\pi i s_7}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(1)}} \cdot \underbrace{(q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty \oint \frac{ds_8}{2\pi i s_8}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(1)}} \\ & \times \prod_{i=1}^3 \prod_{j=4}^6 \frac{1}{(q^{\frac{1}{4}} t s_i^\pm s_j^\mp; q)_\infty} \cdot \prod_{i=4}^6 \frac{1}{(q^{\frac{1}{4}} t s_i^\pm s_7^\mp; q)_\infty} \cdot \frac{1}{(q^{\frac{1}{4}} t s_7^\pm s_8^\mp; q)_\infty} \\ & \quad \underbrace{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}} \left(\frac{s_i}{s_j} \right)}_{\substack{i=1, j=4 \\ \text{---} \\ i=2, j=5 \\ \text{---} \\ i=3, j=6}} \cdot \underbrace{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}} \left(\frac{s_i}{s_7} \right)}_{i=4,5,6} \cdot \underbrace{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}} \left(\frac{s_7}{s_8} \right)}_{\text{---}} \\ & \times \underbrace{(q^{\frac{1}{2}} s_1^\pm s_2^\pm s_3^\pm s_4^\mp s_5^\mp s_6^\mp z_1^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_1 s_2 s_3}{s_4 s_5 s_6} z_1\right)} \cdot \underbrace{(q^{\frac{1}{2}} s_4^\pm s_5^\pm s_6^\pm s_7^\mp z_2^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_4 s_5 s_6}{s_7} z_2\right)} \cdot \underbrace{(q^{\frac{1}{2}} s_7^\pm s_8^\mp z_3^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_7}{s_8} z_3\right)}. \end{aligned} \quad (5.35)$$

The S-dual junction would involve the two types of 3d fundamental twisted hypermultiplets. One should obey the Nahm pole b.c. with a homomorphism $\rho: \mathfrak{su}(2) \rightarrow \mathfrak{u}(3)$ and the other should satisfy the Dirichlet b.c. D .

In fact, we have confirmed that the quarter-index (5.35) coincides with

$$\mathbb{IV}_{\mathcal{D}\mathcal{D}'}^{3|[3]-[1]-[1]]} = \underbrace{(q)_\infty}_{\mathbb{IV}_{\mathcal{D}\mathcal{D}'}^{4d \text{ U}(1)}} \cdot (q^{\frac{5}{4}} t^3 z_1^\pm; q)_\infty \cdot \underbrace{(q^{\frac{3}{4}} t z_3^\pm; q)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ tHM}}(z_3)}. \quad (5.36)$$

5.2.11 4d U(N)|3d (M₁) − ⋯ − (M_{k−1}) with N'

Now we would like to propose the generalization. We consider the junction 4d U(N)|3d (M₁) − ⋯ − (M_{k−1}) with N', which is constructed by the quarter-BPS corner of 4d N = 4 U(N) SYM theory with a pair of Neumann b.c. (N, N') and the half-BPS boundary of 3d linear quiver gauge theory (M₁) − ⋯ − (M_{k−1}) whose vector and (k − 2) hypers obey Neumann b.c. N' and N' respectively. Furthermore the junction contains a 3d bi-fundamental hypermultiplet that couples to 4d U(N) and 3d U(M₁) gauge fields and (k − 1) 2d cross-determinant Fermi multiplets.

The quarter-index takes the form

$$\begin{aligned} & \mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d \text{ U}(N)|(M_1) - (M_2) - \dots - (M_{k-1})|} \\ &= \underbrace{\frac{1}{N!} (q)_\infty^N \oint \prod_{i=1}^N \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d \text{ U}(N)}} \\ & \times \underbrace{\prod_{l=1}^{k-1} \frac{1}{M_l!} (q)_\infty^{M_l} (q^{\frac{1}{2}} t^2; q)_\infty^{M_l} \oint \prod_{i=1}^{M_l} \frac{ds_i^{(l)}}{2\pi i s_i^{(l)}} \prod_{i \neq j} \left(\frac{s_i^{(l)}}{s_j^{(l)}}; q \right)_\infty \left(q^{\frac{1}{2}} t^2 \frac{s_i^{(l)}}{s_j^{(l)}}; q \right)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(M_l)}} \\ & \times \prod_{i=1}^N \prod_{j=1}^{M_1} \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_i^\pm s_j^{(1)\mp}; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}}\left(\frac{s_i}{s_j^{(1)}}\right)} \cdot \prod_{l=1}^{k-2} \prod_{i=1}^{M_l} \prod_{j=1}^{M_{l+1}} \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_i^{(l)\pm} s_j^{(l+1)\mp}; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}}\left(\frac{s_i^{(l)}}{s_j^{(l+1)}}\right)} \\ & \times \underbrace{(q^{\frac{1}{2}} \prod_{i=1}^N s_i^\pm \prod_{j=1}^{M_1} s_j^{(1)\mp} z_1^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{\prod_{i=1}^N s_i}{\prod_{j=1}^{M_1} s_j^{(1)}} z_1\right)} \cdot \underbrace{\prod_{l=1}^{k-2} (q^{\frac{1}{2}} \prod_{i=1}^{M_l} s_i^{(l)\pm} \prod_{j=1}^{M_{l+1}} s_j^{(l+1)\mp} z_{l+1}^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{\prod_{i=1}^{M_l} s_i^{(l)}}{\prod_{j=1}^{M_{l+1}} s_j^{(l+1)}} z_{l+1}\right)}. \end{aligned} \quad (5.37)$$

Under the action of S-duality, we obtain the corner of 4d N = 4 gauge theory with a pair of Dirichlet b.c. (D, D') as the gauge symmetry is completely broken according to the D5- and D5'-branes. The junction may contain the 3d fundamental twisted hypermultiplets which are associated to the D5'-branes with equal numbers of D3-branes in their both sides. Due to the D5-brane on which D3-branes end, these twisted hypers should receive the Dirichlet/Nahm pole boundary conditions.

Figure 13. The brane constructions of the configuration 4d $U(N)|3d (M) - [1]$ with Neumann b.c. $(\mathcal{N}', N')$ and its mirror corner configuration that has no gauge symmetry.

We expect that the quarter-index (5.37) agrees with

$$\begin{aligned} & \mathbb{IV}_{\mathcal{D}\mathcal{D}'}^{N|[M_1]-[M_2]-\dots-[M_{k-1}]} \\ &= \underbrace{(q)_\infty}_{\mathbb{IV}_{\mathcal{D}\mathcal{D}'}^{4d U(1)}} \cdot (q^{\frac{3}{4}+\frac{N-1}{4}} t^{1+(N-1)} z_1^\pm; q)_\infty^{\delta^N M_1} \cdot \prod_{l=1}^{k-2} (q^{\frac{3}{4}+\frac{M_l-1}{4}} t^{1+(M_l-1)} z_{l+1}; q)_\infty^{\delta^{M_l} M_{l+1}}. \end{aligned} \quad (5.38)$$

This contains the quarter-index of 4d $U(1)$ gauge theory with a pair of Dirichlet b.c. $(\mathcal{D}, \mathcal{D}')$ and the contributions from 3d fundamental twisted hypers with Dirichlet/Nahm pole b.c. specified by an embedding $\rho: \mathfrak{su}(2) \rightarrow \mathfrak{u}(N)$.

5.3 4d $U(N)|3d (M) - [1]$ with $(\mathcal{N}', N')$

Now consider the junctions which are constructed from a corner of 4d $\mathcal{N} = 4 U(N)$ gauge theory with a pair of Neumann b.c. $(\mathcal{N}, \mathcal{N}')$ and a boundary of 3d $\mathcal{N} = 4 U(M)$ gauge theory with a fundamental hypermultiplet obeying Neumann b.c. $(\mathcal{N}', N')$. The corresponding brane configuration is illustrated in figure 13. The junction contains 3d bi-fundamental hypermultiplet which couples to 4d $U(N)$ and 3d $U(M)$ gauge fields, 2d cross-determinant Fermi multiplet at the NS5-NS5' junction, and 2d fundamental Fermi multiplet corresponding to the D5-NS5' intersection. We denote this junction by 4d $U(N)|3d (M) - [1]$ with Neumann b.c. $(\mathcal{N}', N')$.

By applying S-duality, we can find the dual junction in figure 13. It has a corner of 4d gauge theory with a pair of boundary conditions $(\mathcal{N}', \mathcal{D})$. It may include two types of 3d twisted hypermultiplets associated to the D3-D5' string and D3-D3 string across the NS5'-brane. They should obey the Dirichlet/Nahm pole boundary conditions due to the D5-brane.

5.3.1 4d $U(1)|3d (1) - [1]$ with $(\mathcal{N}', N')$

Our first example is the junction 4d $U(1)|3d (1) - [1]$ with $(\mathcal{N}', N')$. It has a corner of 4d $U(1)$ gauge theory with a pair of Neumann b.c. $(\mathcal{N}, \mathcal{N}')$ and 3d $U(1)$ gauge theory with a charged hypermultiplet obeying Neumann b.c. $(\mathcal{N}', N')$. The 4d $U(1)$ gauge theory and 3d

U(1) gauge theory are coupled through 3d bi-fundamental hypermultiplet satisfying Neumann b.c. N' . There are two 2d Fermi multiplets. One is the determinant Fermi multiplet with charge $+$ and $-$ under the 4d U(1) and 3d U(1) gauge symmetries. The another is the fundamental Fermi multiplet which carries charge under the 3d gauge symmetry.

The quarter-index is

$$\begin{aligned} \mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(1)|(1) - [1]} &= \underbrace{(q)_\infty \oint \frac{ds_1}{2\pi i s_1}}_{\mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(1)}} \cdot \underbrace{(q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty \oint \frac{ds_2}{2\pi i s_2}}_{\mathbb{III}_{N'}^{3d \text{ U}(1)}} \\ &\times \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_1^\pm s_2^\mp; q)_\infty}}_{\mathbb{III}_{N'}^{3d \text{ HM}}\left(\frac{s_1}{s_2}\right)} \cdot \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_2^\pm; q)_\infty}}_{\mathbb{III}_{N'}^{3d \text{ HM}}(s_2)} \cdot \underbrace{(q^{\frac{1}{2}} s_1^\pm s_2^\mp z^\pm; q)_\infty}_{F\left(\frac{s_1}{s_2} z\right)} \cdot \underbrace{(q^{\frac{1}{2}} s_2^\pm x^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} s_2 x\right)} \end{aligned} \quad (5.39)$$

In the dual junction we have a corner of 4d U(1) gauge theory with a pair of boundary condition $(\mathcal{N}', \mathcal{D})$. There would be two kinds of 3d twisted hypermultiplets with Dirichlet b.c. D . As there is a single D3-brane in both sides of one of the D5'-branes, 3d fundamental twisted hyper appears from D3-D5' string. The another twisted hyper arises from D3-D3 string across the NS5'-brane.

As expected, we find that the quarter-index (5.39) agrees with

$$\mathbb{IV}_{\mathcal{N}'\mathcal{D}'\mathcal{D}}^{1|[1]} = \underbrace{\frac{1}{(q^{\frac{1}{2}} t^2; q)_\infty}}_{\mathbb{IV}_{\mathcal{N}'\mathcal{D}}^{4d \text{ U}(1)}} \cdot \underbrace{(q^{\frac{3}{4}} t z^\pm; q)_\infty}_{\mathbb{III}_D^{3d \text{ tHM}}(z)} \cdot \underbrace{(q^{\frac{3}{4}} t x^\pm; q)_\infty}_{\mathbb{III}_D^{3d \text{ tHM}}(x)}. \quad (5.40)$$

5.3.2 4d U(2)|3d (1) – [1] with $(\mathcal{N}', N')$

Next consider the junction 4d U(2)|3d (1) – [1] with $(\mathcal{N}', N')$. It keeps 4d U(2) gauge symmetry and 3d U(1) gauge symmetry. There are two 3d hypermultiplets; one transforms as the bi-fundamental representation under the $U(2) \times U(1)$ gauge symmetry, while the other transforms as the fundamental representation under the 3d U(1) gauge symmetry. The configuration also includes two 2d Fermi multiplets; one is the cross-determinant Fermi at the NS5-NS5' junction and the other is the fundamental Fermi at the D5-NS5' junction.

The quarter-index is written as

$$\begin{aligned} \mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(2)|(1) - [1]} &= \underbrace{\frac{1}{2} (q)_\infty^2 \oint \frac{ds_1}{2\pi i s_1} \frac{ds_2}{2\pi i s_2} \left(\frac{s_1}{s_2}; q\right)_\infty \left(\frac{s_2}{s_1}; q\right)_\infty}_{\mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(2)}} \cdot \underbrace{(q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty \oint \frac{ds_3}{2\pi i s_3}}_{\mathbb{III}_{N'}^{3d \text{ U}(1)}} \\ &\times \prod_{i=1}^2 \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_i^\pm s_3^\mp; q)_\infty}}_{\mathbb{III}_{N'}^{3d \text{ HM}}\left(\frac{s_i}{s_3}\right)} \cdot \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_3^\pm; q)_\infty}}_{\mathbb{III}_{N'}^{3d \text{ HM}}(s_3)} \cdot \underbrace{(q^{\frac{1}{2}} s_1^\pm s_2^\mp s_3^\mp z^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_1 s_2}{s_3} z\right)} \cdot \underbrace{(q^{\frac{1}{2}} s_3^\pm x^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} s_3 x\right)}. \end{aligned} \quad (5.41)$$

The dual junction has no gauge symmetry. Unequal numbers of D3-branes across the D5'-brane would not admit the 3d fundamental twisted hypermultiplet. However, there

still exists a 3d twisted hypermultiplet arising from D3-D3 string across the NS5'-brane. It should receive Dirichlet b.c. D due to the D5-brane.

Actually we have checked that the quarter-index (5.41) coincides with

$$\mathbb{IV}_{\mathcal{N}'\mathcal{D}'\mathcal{D}}^{2|[1]|} = \underbrace{\frac{1}{(q^{\frac{1}{2}}t^2; q)_{\infty}}}_{\mathbb{IV}_{\mathcal{N}'\mathcal{D}}^{4d \text{ U}(1)}} \cdot \underbrace{(q^{\frac{3}{4}}tx^{\pm}; q)_{\infty}}_{\mathbb{III}_D^{3d \text{ tHM}(x)}}. \quad (5.42)$$

5.3.3 4d U(2)|3d (2) – [1] with $(\mathcal{N}', N')$

Let us investigate the case where the 3d gauge symmetry is non-Abelian. The simplest example is the configuration 4d U(2)|3d (2) – [1] with $(\mathcal{N}', N')$, in which we have a corner of 4d U(2) gauge theory with a pair of Neumann b.c. $(\mathcal{N}, \mathcal{N}')$ and a boundary of 3d U(2) gauge theory with a fundamental hyper subject to Neumann b.c. $(\mathcal{N}', N')$. At the NS5-NS5' junction there appears a 2d cross-determinant Fermi multiplet and at the NS5'-D5 junction one finds a 2d fundamental Fermi multiplet.

The quarter-index is

$$\begin{aligned} \mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d \text{ U}(2)|(2) - [1]} &= \underbrace{\frac{1}{2}(q)_{\infty}^2 \oint \frac{ds_1}{2\pi i s_1} \frac{ds_2}{2\pi i s_2} \left(\frac{s_1}{s_2}; q\right)_{\infty} \left(\frac{s_2}{s_1}; q\right)_{\infty}}_{\mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d \text{ U}(2)}} \\ &\times \underbrace{\frac{1}{2}(q)_{\infty}^2 (q^{\frac{1}{2}}t^2; q)_{\infty}^2 \oint \frac{ds_3}{2\pi i s_3} \frac{ds_4}{2\pi i s_4} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q\right)_{\infty} \left(q^{\frac{1}{2}}t^2 \frac{s_i}{s_j}; q\right)_{\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(2)}} \\ &\times \prod_{i=1}^2 \prod_{j=3}^4 \underbrace{\frac{1}{(q^{\frac{1}{4}}ts_i^{\pm}s_j^{\mp}; q)_{\infty}}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}\left(\frac{s_i}{s_j}\right)}} \cdot \prod_{i=3}^4 \underbrace{\frac{1}{(q^{\frac{1}{4}}ts_i^{\pm}; q)_{\infty}}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}(s_i)}} \cdot \underbrace{(q^{\frac{1}{2}}s_1^{\pm}s_2^{\pm}s_3^{\mp}s_4^{\mp}z^{\pm}; q)_{\infty}}_{F\left(q^{\frac{1}{2}}\frac{s_1s_2}{s_3s_4}z\right)} \cdot \prod_{i=3}^4 \underbrace{(q^{\frac{1}{2}}s_i^{\pm}x^{\pm}; q)_{\infty}}_{F\left(q^{\frac{1}{2}}s_ix\right)}. \end{aligned} \quad (5.43)$$

In the S-dual junction the gauge symmetry is completely broken. The junction would involve two kinds of 3d twisted hypermultiplets. One is the 3d fundamental twisted hyper with Nahm pole b.c. with an embedding $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{u}(2)$. The other is the 3d bi-fundamental hyper associated to the string across the NS5'-brane.

The quarter-index (5.43) turns out to coincide with

$$\mathbb{IV}_{\mathcal{N}'\mathcal{D}'\mathcal{D}}^{2|[2]|} = \underbrace{\frac{1}{(q^{\frac{1}{2}}t^2; q)_{\infty}}}_{\mathbb{IV}_{\mathcal{N}'\mathcal{D}}^{4d \text{ U}(1)}} \cdot (qt^2z^{\pm}; q)_{\infty} \cdot \underbrace{(q^{\frac{3}{4}}tx^{\pm}; q)_{\infty}}_{\mathbb{III}_D^{3d \text{ tHM}(x)}}. \quad (5.44)$$

The first term is the quarter-index of 4d U(1) gauge theory with a pair of boundary conditions $(\mathcal{N}', \mathcal{D})$. The second term is identified with the contributions from 3d fundamental twisted hyper satisfying the Nahm pole b.c. of rank 2. The last term is the contribution

from the 3d bi-fundamental hyper. Unlike the 3d fundamental twisted hyper, it identifies with the Dirichlet half-index of 3d twisted hyper. The fact that we get the quarter index $\mathbb{IV}_{\mathcal{N}'\mathcal{D}}^{4d\ U(1)}$ and the Dirichlet half-index $\mathbb{III}_D^{3d\ tHM}$ of 3d twisted hyper would be caused by the D5'-brane on which two D3-branes terminate.

5.3.4 4d U(3)|3d (3) – [1] with $(\mathcal{N}', N')$

Consider the junction 4d U(3)|3d (3) – [1] with $(\mathcal{N}', N')$ in which 4d U(3) gauge symmetry and 3d U(3) gauge symmetry are preserved.

The quarter-index is

$$\begin{aligned}
 & \mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d\ U(3)|(3) - [1]} \\
 &= \underbrace{\frac{1}{3!}(q)_\infty^3 \oint \prod_{i=1}^3 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d\ U(3)}} \cdot \underbrace{\frac{1}{3!}(q)_\infty^3 (q^{\frac{1}{2}}t^2; q)_\infty^3 \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty \left(q^{\frac{1}{2}}t^2 \frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d\ U(3)}} \\
 & \times \prod_{i=1}^3 \prod_{j=4}^6 \underbrace{\frac{1}{(q^{\frac{1}{4}}ts_i^\pm s_j^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d\ HM}\left(\frac{s_i}{s_j}\right)} \cdot \prod_{i=4}^6 \underbrace{\frac{1}{(q^{\frac{1}{4}}ts_i^\pm; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d\ HM}(s_i)} \\
 & \times \underbrace{(q^{\frac{1}{2}}s_1^\pm s_2^\pm s_3^\pm s_4^\mp s_5^\mp s_6^\mp z^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}}\frac{s_1s_2s_3}{s_4s_5s_6}z\right)} \cdot \prod_{i=4}^6 \underbrace{(q^{\frac{1}{2}}s_i^\pm x^\pm; q)_\infty}_{F(q^{\frac{1}{2}}s_i x)}. \tag{5.45}
 \end{aligned}$$

We have confirmed that the quarter-index (5.45) agrees with

$$\mathbb{IV}_{\mathcal{N}'\mathcal{D}'\mathcal{D}}^{3|[3]} = \underbrace{\frac{1}{(q^{\frac{1}{2}}t^2; q)_\infty}}_{\mathbb{IV}_{\mathcal{N}'\mathcal{D}}^{4d\ U(1)}} \cdot (q^{\frac{5}{4}}t^3z^\pm; q)_\infty \cdot \underbrace{(q^{\frac{3}{4}}tx^\pm; q)_\infty}_{\mathbb{III}_D^{3d\ tHM}(x)}. \tag{5.46}$$

Again it includes the quarter-index of 4d U(1) gauge theory with a pair of boundary conditions $(\mathcal{N}', \mathcal{D})$. The second term is the contributions from 3d fundamental twisted hyper with the Nahm pole b.c. of rank 3. The last term is the Dirichlet half-index of 3d twisted hyper.

5.3.5 4d U(N)|3d (M) – [1] with $(\mathcal{N}', N')$

Let us discuss the general junction 4d U(N)|3d (M) – [1] with $(\mathcal{N}', N')$, which is constructed in the brane setup in figure 13.

It has a corner of 4d $\mathcal{N} = 4$ U(N) SYM theory subject to a pair of Neumann b.c. $(\mathcal{N}, \mathcal{N}')$ and a boundary of 3d $\mathcal{N} = 4$ U(M) gauge theory with a fundamental hypermultiplet obeying Neumann b.c. $(\mathcal{N}', N')$. The 4d U(N) and 3d U(M) vector multiplets are coupled through the 3d bi-fundamental hypermultiplet. In addition, there are 2d cross-determinant Fermi multiplet appearing at the NS5-NS5' junction and fundamental Fermi

multiplet at the NS5'-D5 junction. These Fermi multiplets are required to cancel the boundary gauge anomaly.

The quarter-index is given by

$$\begin{aligned}
 \mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(N)|(M) - [1]} &= \underbrace{\frac{1}{N!} (q)_\infty^N \oint \prod_{i=1}^N \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(N)}} \\
 &\times \underbrace{\frac{1}{M!} (q)_\infty^M (q^{\frac{1}{2}} t^2; q)_\infty^M \oint \prod_{i=N+1}^{N+M} \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty \left(q^{\frac{1}{2}} t^2 \frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(M)}} \\
 &\times \prod_{i=1}^N \prod_{j=N+1}^{N+M} \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_i^\pm s_j^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}} \left(\frac{s_i}{s_j} \right)} \cdot \prod_{i=N+1}^{N+M} \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_i^\pm; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}} (s_i)} \\
 &\times \underbrace{\left(q^{\frac{1}{2}} \prod_{i=1}^N s_i^\pm \prod_{j=N+1}^{N+M} s_j^\mp z^\pm; q \right)_\infty}_{F \left(q^{\frac{1}{2}} \frac{\prod_{i=1}^N s_i}{\prod_{j=N+1}^{N+M} s_j} z \right)} \cdot \underbrace{\prod_{i=N+1}^{N+M} (q^{\frac{1}{2}} s_i^\pm x^\pm; q)_\infty}_{F(q^{\frac{1}{2}} s_i x)}. \tag{5.47}
 \end{aligned}$$

The dual junction has no gauge symmetry. It contains a corner of 4d gauge theory with a pair of boundary conditions $(\mathcal{N}', \mathcal{D})$. There would be the 3d twisted hypermultiplet arising from D3-D3 string across the NS5'-brane obeying the boundary condition due to the D5-brane. For $N = M$, the junction would involve 3d fundamental twisted hypermultiplet arising from D3-D5' string. Due to the D5-brane, it should satisfy the Dirichlet/Nahm pole boundary condition specified by a homomorphism $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{u}(N)$.

The quarter-index (5.47) would lead to the same answer as

$$\mathbb{IV}_{\mathcal{N}'\mathcal{D}'\mathcal{D}}^{N|[M]} = \underbrace{\frac{1}{(q^{\frac{1}{2}} t^2; q)_\infty}}_{\mathbb{IV}_{\mathcal{N}'\mathcal{D}}^{4d \text{ U}(1)}} \cdot (q^{\frac{3}{4} + \frac{N-1}{4}} t^{1+(N-1)} z^\pm; q)_\infty^{\delta_N M} \cdot \underbrace{(q^{\frac{3}{4}} t x^\pm; q)_\infty}_{\mathbb{III}_D^{3d \text{ tHM}}(x)}. \tag{5.48}$$

The first term is the quarter-index of 4d U(1) gauge theory with a pair of boundary conditions $(\mathcal{N}', \mathcal{D})$. The second is a sort of half-index of Dirichlet/Nahm pole b.c. for 3d fundamental twisted hypermultiplet which only appears for $N = M$. The last is the Dirichlet half-index for 3d twisted hypermultiplet which is associated to the 3d twisted hypermultiplet arising from D3-D3 string across the NS5'-brane.

5.4 4d U(N)|3d (M) − [2] with $(\mathcal{N}', N')$

Let us consider the junction 4d U(N)|3d (M) − [2] with $(\mathcal{N}', N')$. It can be obtained by further adding a fundamental hypermultiplet to the junction 4d U(N)|3d (M) − [1] with $(\mathcal{N}', N')$ in section 5.3 and the corresponding brane construction is shown in figure 14.

Figure 14. The brane constructions of the configuration 4d $U(N)|3d (M) - [2]$ with Neumann b.c. $(\mathcal{N}', N')$ and its mirror corner configuration which involves Dirichlet b.c. $\mathcal{D}$ or Nahm pole b.c. for a three-dimensional $U(M)$ gauge theory.

Unlike the previous cases, the S-dual junction has 3d $U(M)$ gauge theory subject to Dirichlet/Nahm pole b.c. Hence the evaluation of the quarter-indices for the dual junctions may include the non-perturbative monopole sum.

5.4.1 4d $U(1)|3d (1) - [2]$ with $(\mathcal{N}', N')$

We begin with the simplest example, that is the junction 4d $U(1)|3d (1) - [2]$ with $(\mathcal{N}', N')$. It has 4d $U(1)$ gauge symmetry and 3d $U(1)$ gauge symmetry. There are two 3d fundamental hypermultiplets with charges under the 3d $U(1)$ gauge symmetry and a 3d bi-fundamental hypermultiplet coupled to 4d $U(1)$ and 3d $U(1)$ gauge fields. In addition, there are a 2d cross-determinant Fermi multiplet at the NS5-NS5' junction and two fundamental Fermi multiplet at the NS5'-D5 junction.

The quarter-index is calculated as

$$\begin{aligned}
 \mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d U(1)|(1) - [2]} &= \underbrace{(q)_\infty \oint \frac{ds_1}{2\pi i s_1}}_{\mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d U(1)}} \cdot \underbrace{(q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty \oint \frac{ds_2}{2\pi i s_2}}_{\mathbb{III}_{\mathcal{N}'}^{3d U(1)}} \\
 &\times \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_1^\pm s_2^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d HM}(\frac{s_1}{s_2})} \cdot \prod_{\alpha=1}^2 \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_2^\pm x_\alpha^\pm; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d HM}(s_2 x_\alpha)} \\
 &\times \underbrace{(q^{\frac{1}{2}} s_1^\pm s_2^\mp z^\pm; q)_\infty}_{F(q^{\frac{1}{2}} \frac{s_1}{s_2} z)} \cdot \underbrace{(q^{\frac{1}{2}} s_2^\pm x_1^\pm u^\pm; q)_\infty}_{F(q^{\frac{1}{2}} s_2 x_1 u)} \cdot \underbrace{(q^{\frac{1}{2}} s_2^\pm x_2^\pm u^\mp; q)_\infty}_{F(q^{\frac{1}{2}} s_2 x_2 u^{-1})} \quad (5.49)
 \end{aligned}$$

where $x_1 x_2 = 1$.

The S-dual configuration contains a corner of 4d $U(1)$ gauge theory with a pair of boundary conditions $(\mathcal{N}', \mathcal{D})$. This 4d gauge theory has a defect of D5'-brane that couples

3d fundamental twisted hyper to 4d gauge theory. It should obey the Dirichlet b.c. $\widetilde{D}$ required from the D5-brane. The junction should have a 3d $\widetilde{U(1)}$ twisted vector multiplet arising from a single D3-brane suspended between NS5'-branes and two 3d bi-fundamental twisted hypers arising strings across the two NS5'-branes. The D5-brane again requires the twisted vector and twisted hypers to obey Dirichlet b.c. $\mathcal{D}$ and D respectively.

We then obtain the quarter-index for the dual junction

$$\begin{aligned} \mathbb{IV}_{\mathcal{N}'\mathcal{D}}^{4d \ U(1)|(1)-[1]} &= \frac{1}{\underbrace{(q^{\frac{1}{2}}t^2; q)_{\infty}}_{\mathbb{IV}_{\mathcal{N}'\mathcal{D}}^{4d \ U(1)}}} \cdot \frac{1}{(q)_{\infty}(q^{\frac{1}{2}}t^2; q)_{\infty}} \sum_{m \in \mathbb{Z}} \\ &\times \underbrace{(q^{\frac{3}{4}}tz^{\pm}; q)_{\infty}}_{\mathbb{III}_D^{3d \ tHM}(z)} \cdot \underbrace{(q^{\frac{3}{4} \pm m}tu^{\pm}; q)_{\infty}}_{\mathbb{III}_D^{3d \ tHM}(q^m u)} \cdot \underbrace{(q^{\frac{3}{4} \pm m}tu^{\pm}; q)_{\infty}}_{\mathbb{III}_D^{3d \ tHM}(q^m u)} \\ &\times q^{m^2} u^{2m} x_1^m x_2^{-m}. \end{aligned} \quad (5.50)$$

In fact, we have confirmed that this matches with the quarter-index (5.49).

5.4.2 4d $U(2)|3d \ (1) - [2]$ with $(\mathcal{N}', N')$

Let us turn to the junction 4d $U(2)|3d \ (1) - [2]$ with $(\mathcal{N}', N')$. We have 4d $U(2)$ gauge symmetry and 3d $U(1)$ gauge symmetry.

Similarly, we can compute the quarter-index as

$$\begin{aligned} \mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d \ U(2)|(1) - [2]} &= \underbrace{\frac{1}{2}(q)_{\infty}^2 \oint \frac{ds_1}{2\pi i s_1} \frac{ds_2}{2\pi i s_2} \left(\frac{s_1}{s_2}; q\right)_{\infty} \left(\frac{s_2}{s_1}; q\right)_{\infty}}_{\mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d \ U(2)}} \cdot \underbrace{(q)_{\infty}(q^{\frac{1}{2}}t^2; q)_{\infty} \oint \frac{ds_3}{2\pi i s_3}}_{\mathbb{III}_{\mathcal{N}'}^{3d \ U(1)}} \\ &\times \prod_{i=1}^2 \underbrace{\frac{1}{(q^{\frac{1}{4}}ts_i^{\pm}s_3^{\mp}; q)_{\infty}}}_{\mathbb{III}_{\mathcal{N}'}^{3d \ HM}\left(\frac{s_i}{s_3}\right)} \prod_{\alpha=1}^2 \underbrace{\frac{1}{(q^{\frac{1}{4}}ts_3^{\pm}x_{\alpha}^{\pm}; q)_{\infty}}}_{\mathbb{III}_{\mathcal{N}'}^{3d \ HM}(s_3x_{\alpha})} \\ &\times \underbrace{(q^{\frac{1}{2}}s_1^{\pm}s_2^{\pm}s_3^{\mp}z_1^{\pm}; q)_{\infty}}_{F\left(q^{\frac{1}{2}}\frac{s_1s_2}{s_3}z_1\right)} \cdot \underbrace{(q^{\frac{1}{2}}s_3^{\pm}x_1^{\pm}u^{\pm}; q)_{\infty}}_{F(q^{\frac{1}{2}}s_3x_1u)} \cdot \underbrace{(q^{\frac{1}{2}}s_3^{\pm}x_2^{\pm}u^{\mp}; q)_{\infty}}_{F(q^{\frac{1}{2}}s_3x_2u^{-1})} \end{aligned} \quad (5.51)$$

with $x_1x_2 = 1$.

In the S-dual configuration 4d $U(2) \times U(1)$ gauge symmetry is firstly broken down to $U(1)$ due to the D5'-brane and we have a corner of 4d $U(1)$ gauge theory with a pair of boundary conditions $(\mathcal{N}', \mathcal{D})$. The 4d gauge theory includes a defect of D5'-brane, however, the unequal numbers of D3-branes across the D5'-brane should not lead to the 3d fundamental twisted hypermultiplet. As a single D3-brane is stretched between the NS5'-branes, there are 3d $\widetilde{U(1)}$ twisted vector multiplet and two twisted hypers arising strings across the NS5'-branes. They should obey the Dirichlet b.c. $\mathcal{D}$ and D .

Figure 15. The brane constructions of the configuration 4d $U(L)|3d U(M)|4d U(N)$ with Neumann b.c. $\mathcal{N}'$ and its mirror corner configuration that has no gauge symmetry.

We have checked that the quarter-index (5.51) agrees with

$$\begin{aligned} \mathbb{IV}_{\mathcal{N}'\mathcal{D}}^{4d U(2) \rightarrow U(1)|(1)-[1]} &= \frac{1}{(q^{\frac{1}{2}}t^2; q)_{\infty}} \cdot \frac{1}{(q)_{\infty}(q^{\frac{1}{2}}t^2; q)_{\infty}} \sum_{m \in \mathbb{Z}} \\ &\quad \underbrace{\mathbb{IV}_{\mathcal{N}'\mathcal{D}}^{4d U(1)}}_{\times (q^{\frac{3}{4} \pm m}tu^{\pm}; q)_{\infty}} \cdot \underbrace{(q^{\frac{3}{4} \pm m}tu^{\pm}; q)_{\infty}}_{\mathbb{III}_D^{3d tHM}(q^mu)} \cdot q^{m^2} \cdot u^{2m} \cdot x_1^m x_2^{-m}. \end{aligned} \quad (5.52)$$

More challenging classes are the junction 4d $U(N)|3d (M) - [2]$ with $(\mathcal{N}', N')$ where M is larger than one. To study them, one would need to figure out the Nahm pole half-indices for 3d $\mathcal{N} = 4$ gauge theories. We leave this for future work.

5.5 4d $U(L)|3d U(M)|4d U(N)$ with $\mathcal{N}'$

Now we would like to examine the $\mathcal{N} = (0, 4)$ junction which involves a pair of corners of 4d $\mathcal{N} = 4$ gauge theories and a boundary of 3d $\mathcal{N} = 4$ gauge theory.

We begin with the junction 4d $U(L)|3d U(M)|4d U(N)$ with $\mathcal{N}'$, which has a pair of 4d $U(L)$ and $U(N)$ gauge theories obeying a pair of Neumann b.c. $(\mathcal{N}, \mathcal{N}')$ and 3d $U(M)$ gauge theory obeying Neumann b.c. $\mathcal{N}'$. There are two 3d hypermultiplets. Under $U(L) \times U(M) \times U(N)$ one transforms as $(\mathbf{L}, \overline{\mathbf{M}}, \mathbf{1}) \oplus (\overline{\mathbf{L}}, \mathbf{M}, \mathbf{1})$ and the other transforming as $(\mathbf{1}, \mathbf{M}, \overline{\mathbf{N}}) \oplus (\mathbf{1}, \overline{\mathbf{M}}, \mathbf{N})$. They obey the Neumann b.c. $\mathcal{N}'$. In addition, we have two 2d cross-determinant Fermi multiplets. One transforms as $(\det, \det^{-1}, \mathbf{1})$ and the other transforms as $(\mathbf{1}, \det, \det^{-1})$. It is constructed from the brane configuration in figure 15.

By applying S-duality, we can address the dual junction. Due to the $D5'$ -branes, 4d $U(L) \times U(M) \times U(N)$ gauge symmetry should be broken down to $U(\min\{L, M, N\})$. Furthermore, the $D5$ -brane breaks down the gauge symmetry $U(\min\{L, M, N\})$ by requiring the Dirichlet/Nahm pole boundary condition specified by an embedding $\rho: \mathfrak{su}(2) \rightarrow \mathfrak{u}(\min\{L, M, N\})$. We check the dualities of these junctions by evaluating the quarter-indices.

5.5.1 4d U(1)|3d U(1)|4d U(1) with $\mathcal{N}'$

Let us start with the junction 4d U(1)|3d U(1)|4d U(1) with $\mathcal{N}'$. This has two corners of 4d U(1) gauge theories with a pair of Neumann b.c. $(\mathcal{N}, \mathcal{N}')$ and 3d U(1) gauge theory between them obeying Neumann b.c. $\mathcal{N}'$. We have two 3d bi-fundamental hypermultiplets with Neumann b.c. $\mathcal{N}'$ and two 2d determinant Fermi multiplets cancelling the gauge anomaly.

The quarter-index reads

$$\begin{aligned} \mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d\text{ U}(1)|(1)|4d\text{ U}(1)} &= \underbrace{(q)_\infty \oint \frac{ds_1}{2\pi i s_1}}_{\mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d\text{ U}(1)}} \cdot \underbrace{(q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty \oint \frac{ds_2}{2\pi i s_2}}_{\mathbb{III}_{\mathcal{N}'}^{3d\text{ U}(1)}} \cdot \underbrace{(q)_\infty \oint \frac{ds_3}{2\pi i s_3}}_{\mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d\text{ U}(1)}} \\ &\times \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_1^\pm s_2^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d\text{ HM}}\left(\frac{s_1}{s_2}\right)} \cdot \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_3^\pm s_4^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d\text{ HM}}\left(\frac{s_3}{s_4}\right)} \cdot \underbrace{(q^{\frac{1}{2}} s_1^\pm s_2^\mp z_1^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_1}{s_2} z_1\right)} \cdot \underbrace{(q^{\frac{1}{2}} s_2^\pm s_3^\mp z_2^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_2}{s_3} z_2\right)}. \end{aligned} \quad (5.53)$$

The dual configuration has 4d U(1) gauge theory with Dirichlet b.c. $\mathcal{D}$. It also contains two defects of D5'-branes. As the numbers of D3-branes are the same across them, they couple two 3d fundamental twisted hypermultiplets to the 4d U(1) gauge theory satisfying Dirichlet b.c. $\mathcal{D}$.

Then the quarter-index would take the form

$$\mathbb{IV}_{\mathcal{D}\mathcal{D}'}^{1|[1]|1} = \underbrace{\frac{(q)_\infty}{(q^{\frac{1}{2}} t^2; q)_\infty}}_{\mathbb{III}_{\mathcal{D}}^{4d\text{ U}(1)}} \cdot \underbrace{(q^{\frac{3}{4}} t z_1^\pm; q)_\infty}_{\mathbb{III}_{\mathcal{D}}^{3d\text{ tHM}}(z_1)} \cdot \underbrace{(q^{\frac{3}{4}} t z_2^\pm; q)_\infty}_{\mathbb{III}_{\mathcal{D}}^{3d\text{ tHM}}(z_2)}. \quad (5.54)$$

In fact, the quarter-indices (5.53) and (5.54) agree with each other.

5.5.2 4d U(2)|3d U(1)|4d U(1) with $\mathcal{N}'$

For the configuration 4d U(2)|3d U(1)|4d U(1) with $\mathcal{N}'$, we have two corners of 4d U(2) and U(1) gauge theories with Neumann boundary conditions $(\mathcal{N}, \mathcal{N}')$ and 3d U(1) gauge theory with Neumann b.c. $\mathcal{N}'$. There are two 3d bi-fundamental hypermultiplets subject to Neumann b.c. $\mathcal{N}'$ together with two 2d cross-determinant Fermi multiplets.

The quarter-index reads

$$\begin{aligned} \mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d\text{ U}(2)|(1)|4d\text{ U}(1)} &= \underbrace{\frac{1}{2}(q)_\infty^2 \oint \frac{ds_1}{2\pi i s_1} \frac{ds_2}{2\pi i s_2} \left(\frac{s_1}{s_2}; q\right)_\infty \left(\frac{s_2}{s_1}; q\right)_\infty}_{\mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d\text{ U}(2)}} \cdot \underbrace{(q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty \oint \frac{ds_3}{2\pi i s_3}}_{\mathbb{III}_{\mathcal{N}'}^{3d\text{ U}(1)}} \cdot \underbrace{(q)_\infty \oint \frac{ds_4}{2\pi i s_4}}_{\mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d\text{ U}(1)}} \\ &\times \prod_{i=1}^2 \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_i^\pm s_3^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d\text{ HM}}\left(\frac{s_i}{s_3}\right)} \cdot \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_3^\pm s_4^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d\text{ HM}}\left(\frac{s_3}{s_4}\right)} \cdot \underbrace{(q^{\frac{1}{2}} s_1^\pm s_2^\pm s_3^\mp z_1^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_1 s_2}{s_3} z_1\right)} \cdot \underbrace{(q^{\frac{1}{2}} s_3^\pm s_4^\mp z_2^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_3}{s_4} z_2\right)}. \end{aligned} \quad (5.55)$$

In the dual configuration, the $4d \text{ U}(2) \times \text{U}(1) \times \text{U}(1)$ gauge symmetry is broken down to $\text{U}(1)$ by the $D5'$ -branes, which is further broken down by the $D5$ -brane. As the number of $D3$ -branes jumps across one of the $D5'$ -brane, there would be just a single twisted hypermultiplet obeying Dirichlet b.c. D .

The quarter-index for the dual junction then takes the form

$$\mathbb{IV}_{\mathcal{D}\mathcal{D}'}^{2|[1]|1} = \underbrace{\frac{(q)_\infty}{(q^{\frac{1}{2}}t^2; q)_\infty}}_{\mathbb{III}_D^{4d \text{ U}(1)}} \cdot \underbrace{(q^{\frac{3}{4}}tz_2^\pm; q)_\infty}_{\mathbb{III}_D^{3d \text{ tHM}(z_2)}}. \quad (5.56)$$

As expected, the quarter-indices (5.55) and (5.56) coincide to each other.

5.5.3 4d U(2)|3d U(1)|4d U(2) with $\mathcal{N}'$

Next consider the junction $4d \text{ U}(2)|3d \text{ U}(1)|4d \text{ U}(2)$ with $\mathcal{N}'$. In this case the $3d$ gauge symmetry is smaller than each of $4d$ gauge symmetries.

We can similarly calculate the quarter-index as

$$\begin{aligned} & \mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d \text{ U}(2)|(1)|4d \text{ U}(2)} \\ &= \underbrace{\frac{1}{2}(q)_\infty^2 \oint \frac{ds_1}{2\pi i s_1} \frac{ds_2}{2\pi i s_2} \left(\frac{s_1}{s_2}; q\right)_\infty \left(\frac{s_2}{s_1}; q\right)_\infty}_{\mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d \text{ U}(2)}} \cdot \underbrace{(q)_\infty (q^{\frac{1}{2}}t^2; q)_\infty \oint \frac{ds_3}{2\pi i s_3}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(1)}} \\ & \times \underbrace{\frac{1}{2}(q)_\infty^2 \oint \frac{ds_4}{2\pi i s_4} \frac{ds_5}{2\pi i s_5} \left(\frac{s_4}{s_5}; q\right)_\infty \left(\frac{s_5}{s_4}; q\right)_\infty}_{\mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d \text{ U}(2)}} \\ & \times \prod_{i=1}^2 \underbrace{\frac{1}{(q^{\frac{1}{4}}ts_i^\pm s_3^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}\left(\frac{s_i}{s_3}\right)}} \prod_{i=4}^5 \underbrace{\frac{1}{(q^{\frac{1}{4}}ts_3^\pm s_i^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}\left(\frac{s_3}{s_i}\right)}} \cdot \underbrace{(q^{\frac{1}{2}}s_1^\pm s_2^\pm s_3^\mp z_1^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}}\frac{s_1 s_2}{s_3} z_1\right)} \cdot \underbrace{(q^{\frac{1}{2}}s_3^\pm s_4^\mp s_5^\mp z_2^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}}\frac{s_3}{s_4 s_5} z_2\right)}. \end{aligned} \quad (5.57)$$

In the S-dual configuration the $D5'$ -branes break the $\text{U}(2) \times \text{U}(1) \times \text{U}(2)$ gauge symmetry down to $\text{U}(1)$ and they do not lead to the $3d$ fundamental twisted hypers. In addition, the $\text{U}(1)$ gauge symmetry is broken by the $D5$ -brane as Dirichlet b.c. D .

The quarter-index for the dual configuration is then simply the Dirichlet half-index of $4d \text{ U}(1)$ gauge theory:

$$\mathbb{IV}_{\mathcal{D}\mathcal{D}'}^{2|[1]|2} = \underbrace{\frac{(q)_\infty}{(q^{\frac{1}{2}}t^2; q)_\infty}}_{\mathbb{III}_D^{4d \text{ U}(1)}}. \quad (5.58)$$

In fact, this coincides with the quarter-index (5.57).

5.5.4 4d U(3)|3d U(1)|4d U(1) with $\mathcal{N}'$

To get more insight, let us consider the junction $4d \text{ U}(3)|3d \text{ U}(1)|4d \text{ U}(1)$ with $\mathcal{N}'$. A pair of corners of $4d \text{ U}(3)$ and $\text{U}(1)$ gauge theories with boundary conditions $(\mathcal{N}, \mathcal{N}')$ are

coupled to 3d U(1) gauge theory through 3d bi-fundamental hypermultiplets which obey Neumann b.c. $\mathcal{N}'$. The junction also has two 2d cross-determinant Fermi multiplets which cancel the boundary gauge anomaly.

The quarter-index is

$$\begin{aligned}
 & \mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(3)|(1)|4d \text{ U}(1)} \\
 &= \underbrace{\frac{1}{3!} (q)_\infty \oint \prod_{i=1}^3 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(3)}} \cdot \underbrace{(q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty \oint \frac{ds_4}{2\pi i s_4}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(1)}} \cdot \underbrace{(q)_\infty \oint \frac{ds_5}{2\pi i s_5}}_{\mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(1)}} \\
 &\times \prod_{i=1}^3 \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_i^\pm s_4^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}} \left(\frac{s_i}{s_4} \right)} \cdot \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_4^\pm s_5^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}} \left(\frac{s_4}{s_5} \right)} \cdot \underbrace{(q^{\frac{1}{2}} s_1^\pm s_2^\pm s_3^\pm s_4^\mp z_1^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_1 s_2 s_3}{s_4} z_1\right)} \cdot \underbrace{(q^{\frac{1}{2}} s_4^\pm s_5^\mp z_2^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_4}{s_5} z_2\right)}.
 \end{aligned} \tag{5.59}$$

Under the action of S-duality, we find the configuration in which the D5'-branes break 4d U(3) $\times$ U(1) $\times$ U(1) gauge symmetry down to U(1) and the D5-brane further breaks the U(1) gauge symmetry. One of the D5'-branes has a single D3-brane in its both sides, which gives rise to 3d fundamental twisted hypermultiplet obeying Dirichlet b.c. $\mathcal{D}$.

We have checked that the quarter-index (5.59) agrees with

$$\mathbb{IV}_{\mathcal{DD}'}^{3|[1]|1} = \underbrace{\frac{(q)_\infty}{(q^{\frac{1}{2}} t^2; q)_\infty}}_{\mathbb{III}_{\mathcal{D}}^{4d \text{ U}(1)}} \cdot \underbrace{(q^{\frac{3}{4}} t z_2^\pm; q)_\infty}_{\mathbb{III}_{\mathcal{D}}^{3d \text{ tHM}}(z_2)}, \tag{5.60}$$

which is the product of a half-index of Dirichlet b.c. $\mathcal{D}$ for 4d U(1) gauge theory and a Dirichlet half-index of 3d twisted hyper.

5.5.5 4d U(1)|3d U(2)|4d U(2) with $\mathcal{N}'$

Now let us consider the junction whose 3d gauge symmetry is non-Abelian. For the junction 4d U(1)|3d U(2)|4d U(2) with $\mathcal{N}'$, we have corners of 4d U(1) and U(2) gauge theories and a boundary of 3d U(2) gauge theory between them. Again the 4d and 3d gauge theories are connected by 3d bi-fundamental hypers together with 2d cross-determinant Fermi multiplets.

The quarter-index takes the form

$$\begin{aligned}
 & \mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(1)|(2)|4d \text{ U}(2)} \\
 &= \underbrace{(q)_\infty \oint \frac{ds_1}{2\pi i s_1}}_{\mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(1)}} \cdot \underbrace{\frac{1}{2}(q)_\infty^2 (q^{\frac{1}{2}} t^2; q)_\infty^2 \oint \prod_{i=1}^2 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty \left(q^{\frac{1}{2}} t^2 \frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(2)}} \\
 &\quad \times \underbrace{\frac{1}{2}(q)_\infty^2 \oint \prod_{i=4}^5 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(2)}} \\
 &\quad \times \prod_{i=2}^3 \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_1^\pm s_i^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}} \left(\frac{s_1}{s_i} \right)} \cdot \prod_{i=2}^3 \prod_{j=4}^5 \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_i^\pm s_j^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}} \left(\frac{s_i}{s_j} \right)} \cdot \underbrace{(q^{\frac{1}{2}} s_1^\pm s_2^\mp s_3^\mp z_1^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_1}{s_2 s_3} z_1\right)} \cdot \underbrace{(q^{\frac{1}{2}} s_2^\pm s_3^\pm s_4^\mp s_5^\mp z_2^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_2 s_3}{s_4 s_5} z_2\right)}.
 \end{aligned} \tag{5.61}$$

In the dual configuration, the 4d $\text{U}(1) \times \text{U}(2) \times \text{U}(2)$ gauge symmetry is broken down to $\text{U}(1)$ according to the D5'-branes. The 4d $\text{U}(1)$ gauge theory further should satisfy Dirichlet b.c. $\mathcal{D}$ due to the D5-brane. In addition, we would have 3d fundamental twisted hypermultiplet as one of the D5'-branes has two D3-branes in its both sides, which should obey Nahm pole b.c. with an embedding $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{u}(2)$.

The quarter-index (5.61) agrees with

$$\mathbb{IV}_{\mathcal{DD}'}^{1|[2]|^2} = \frac{(q)_\infty}{\underbrace{(q^{\frac{1}{2}} t^2; q)_\infty}_{\mathbb{III}_D^{4d \text{ U}(1)}}} \cdot (q t^2 z_2^\pm; q)_\infty. \tag{5.62}$$

This involves the half-index of 4d $\text{U}(1)$ gauge theory subject to Dirichlet b.c. and the half-index of 3d twisted hyper obeying Nahm pole b.c. with an embedding $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{u}(2)$.

5.5.6 4d $\text{U}(2)|3d \text{ U}(2)|4d \text{ U}(2)$ with $\mathcal{N}'$

Consider the configuration 4d $\text{U}(2)|3d \text{ U}(2)|4d \text{ U}(2)$ with $\mathcal{N}'$. We have a pair of corners of 4d $\text{U}(2)$ gauge theories with Neumann b.c. $(\mathcal{N}, \mathcal{N}')$ coupled to a boundary of 3d $\text{U}(2)$ gauge theory through 3d bi-fundamental hypers and 2d cross-determinant Fermi multiplets.

The quarter-index takes the form

$$\begin{aligned}
 \mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(2)|(2)|4d \text{ U}(2)} &= \underbrace{\frac{1}{2}(q)_\infty^2 \oint \frac{ds_1}{2\pi i s_1} \frac{ds_2}{2\pi i s_2} \left(\frac{s_1}{s_2}; q\right)_\infty \left(\frac{s_2}{s_1}; q\right)_\infty}_{\mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(2)}} \\
 &\times \underbrace{\frac{1}{2}(q)_\infty^2 (q^{\frac{1}{2}} t^2; q)_\infty^2 \oint \prod_{i=3}^4 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q\right)_\infty \left(q^{\frac{1}{2}} t^2 \frac{s_i}{s_j}; q\right)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(2)}} \\
 &\times \underbrace{\frac{1}{2}(q)_\infty^2 \oint \frac{ds_5}{2\pi i s_5} \frac{ds_6}{2\pi i s_6} \left(\frac{s_5}{s_6}; q\right)_\infty \left(\frac{s_6}{s_5}; q\right)_\infty}_{\mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(2)}} \\
 &\times \prod_{i=1}^2 \prod_{j=3}^4 \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_i^\pm s_j^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}}\left(\frac{s_i}{s_j}\right)} \cdot \prod_{i=3}^4 \prod_{j=5}^6 \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_i^\pm s_j^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}}\left(\frac{s_i}{s_j}\right)} \\
 &\cdot \underbrace{(q^{\frac{1}{2}} s_1^\pm s_2^\pm s_3^\mp s_4^\mp z_1^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_1 s_2}{s_3 s_4} z_1\right)} \cdot \underbrace{(q^{\frac{1}{2}} s_3^\pm s_4^\pm s_5^\mp s_6^\mp z_2^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_3 s_4}{s_5 s_6}\right)}.
 \end{aligned} \tag{5.63}$$

The $\text{U}(2) \times \text{U}(2) \times \text{U}(2)$ gauge symmetry in the dual configuration is broken down to $\text{U}(2)$ due to the $\text{D5}'$ -branes. The $\text{U}(2)$ gauge symmetry is further broken because of Nahm pole boundary condition of rank 2. As the numbers of D3-branes do not change across the $\text{D5}'$ -branes, the dual junction admits two 3d fundamental twisted hypermultiplets obeying Nahm pole b.c. specified by a homomorphism $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{u}(2)$.

The quarter-index (5.63) agrees with

$$\mathbb{IV}_{\mathcal{D}\mathcal{D}'}^{2|[2]|2} = \underbrace{\frac{(q)_\infty (q^{\frac{3}{2}} t^2; q)_\infty}{(q^{\frac{1}{2}} t^2; q)_\infty (qt^4; q)_\infty}}_{\mathbb{III}_{\text{Nahm}}^{4d \text{ U}(2)}} \cdot (qt^2 z_1^\pm; q)_\infty \cdot (qt^2 z_2^\pm; q)_\infty. \tag{5.64}$$

The first terms are the half-index of Nahm pole boundary condition for 4d $\text{U}(2)$ gauge theory and the other two terms are the two contributions from 3d twisted hypers obeying the Nahm pole b.c.

5.5.7 4d $\text{U}(3)|3d \text{U}(3)|4d \text{U}(3)$ with $\mathcal{N}'$

As a further example of non-Abelian cases, let us examine the junction 4d $\text{U}(3)|3d \text{U}(3)|4d \text{U}(3)$ with $\mathcal{N}'$. The junction preserves 4d $\text{U}(3)$ and $\text{U}(3)$ gauge symmetries and 3d $\text{U}(3)$ gauge symmetry. The 4d $\text{U}(3)$ gauge theories are coupled to 3d $\text{U}(3)$ gauge theory through 3d bi-fundamental hypers and 2d cross-determinant Fermi multiplets.

The quarter-index takes the form

$$\begin{aligned}
 \mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(3)|(3)|4d \text{ U}(3)} &= \underbrace{\frac{1}{3!}(q)_\infty^3 \oint \prod_{i=1}^3 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(3)}} \\
 &\times \underbrace{\frac{1}{3!}(q)_\infty^3 (q^{\frac{1}{2}} t^2; q)_\infty^3 \oint \prod_{i=4}^6 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty \left(q^{\frac{1}{2}} t^2 \frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(3)}} \\
 &\times \underbrace{\frac{1}{3!}(q)_\infty^3 \oint \prod_{i=7}^9 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(3)}} \\
 &\times \prod_{i=1}^3 \prod_{j=4}^6 \frac{1}{(q^{\frac{1}{4}} t s_i^\pm s_j^\mp; q)_\infty} \cdot \prod_{i=4}^6 \prod_{j=7}^9 \frac{1}{(q^{\frac{1}{4}} t s_i^\pm s_j^\mp; q)_\infty} \\
 &\quad \underbrace{\qquad\qquad\qquad}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}} \left(\frac{s_i}{s_j} \right)} \quad \underbrace{\qquad\qquad\qquad}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}} \left(\frac{s_i}{s_j} \right)} \\
 &\times \underbrace{(q^{\frac{1}{2}} s_1^\pm s_2^\pm s_3^\pm s_4^\mp s_5^\mp s_6^\mp z_1^\pm; q)_\infty}_{F(q^{\frac{1}{2}} \frac{s_1 s_2 s_3}{s_4 s_5 s_6} z_1)} \cdot \underbrace{(q^{\frac{1}{2}} s_4^\pm s_5^\pm s_6^\pm s_7^\mp s_8^\mp s_9^\mp z_2^\pm; q)_\infty}_{F(q^{\frac{1}{2}} \frac{s_4 s_5 s_6}{s_7 s_8 s_9} z_2)}. \quad (5.65)
 \end{aligned}$$

According to the D5'-branes, the dual configuration keeps 4d U(3) gauge theory in a half-space. It should satisfy Nahm pole boundary condition of rank 3 required from D5-brane. In addition, there would be two 3d twisted hypermultiplets with Nahm pole b.c. with an embedding $\rho: \mathfrak{su}(2) \rightarrow \mathfrak{u}(3)$.

The quarter-index (5.65) agrees with

$$\mathbb{IV}_{\mathcal{DD}'}^{3|[3]|3} = \underbrace{\frac{(q)_\infty (q^{\frac{3}{2}} t^2; q)_\infty (q^2 t^4; q)_\infty}{(q^{\frac{1}{2}} t^2; q)_\infty (q t^4; q)_\infty (q^{\frac{3}{2}} t^6; q)_\infty}}_{\mathbb{III}_{\text{Nahm}}^{4d \text{ U}(3)}} \cdot (q^{\frac{5}{4}} t^3 z_1^\pm; q)_\infty \cdot (q^{\frac{5}{4}} t^3 z_2^\pm; q)_\infty. \quad (5.66)$$

Again the first terms are identified with the half-index of Nahm pole boundary condition for 4d U(3) gauge theory and the other two are the two contributions from 3d twisted hypers subject to the Nahm pole b.c.

5.5.8 4d U(L)|3d U(M)|4d U(N) with $\mathcal{N}'$

Let us discuss the generalization of the dualities for the junction 4d U(L)|3d U(M)|4d U(N) with $\mathcal{N}'$. The junction is composed of two corners of 4d $\mathcal{N} = 4 \text{ U}(L)$ and $\text{U}(N)$ gauge theories with a pair of Neumann b.c. ($\mathcal{N}, \mathcal{N}'$) and a boundary of 3d $\mathcal{N} = 4 \text{ U}(M)$ gauge theory with Neumann b.c. $\mathcal{N}'$. There are two 3d bi-fundamental hypermultiplets with Neumann b.c. N' and two 2d cross-determinant Fermi multiplets.

The quarter-index takes the form

$$\begin{aligned}
 & \mathbb{I}\mathbb{V}_{\mathcal{N}\mathcal{N}'}^{4d \text{ U}(L)|(M)|4d \text{ U}(N)} \\
 &= \underbrace{\frac{1}{L!}(q)_\infty \oint \prod_{i=1}^L \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{I}\mathbb{V}_{\mathcal{N}\mathcal{N}'}^{4d \text{ U}(L)}} \\
 & \times \underbrace{\frac{1}{M!}(q)_\infty (q^{\frac{1}{2}}t^2; q)_\infty \oint \prod_{i=L+1}^{L+M} \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty \left(q^{\frac{1}{2}}t^2 \frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{I}\mathbb{I}_{\mathcal{N}'}^{3d \text{ U}(M)}} \\
 & \times \underbrace{\frac{1}{N!}(q)_\infty \oint \prod_{i=L+M+1}^{L+M+N} \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{I}\mathbb{V}_{\mathcal{N}\mathcal{N}'}^{4d \text{ U}(N)}} \\
 & \times \prod_{i=1}^L \prod_{j=L+1}^{L+M} \underbrace{\frac{1}{(q^{\frac{1}{4}}ts_i^\pm s_j^\mp; q)_\infty}}_{\mathbb{I}\mathbb{I}_{\mathcal{N}'}^{3d \text{ HM}}\left(\frac{s_i}{s_j}\right)} \cdot \prod_{i=L+1}^{L+M} \prod_{j=L+M+1}^{L+M+N} \underbrace{\frac{1}{(q^{\frac{1}{4}}ts_i^\pm s_j^\mp; q)_\infty}}_{\mathbb{I}\mathbb{I}_{\mathcal{N}'}^{3d \text{ HM}}\left(\frac{s_i}{s_j}\right)} \\
 & \times (q^{\frac{1}{2}} \prod_{i=1}^L s_i^\pm \prod_{j=L+1}^{L+M} s_j^\mp z_1^\pm; q)_\infty \cdot (q^{\frac{1}{2}} \prod_{i=L+1}^{L+M} s_i^\pm \prod_{j=L+M+1}^{L+M+N} s_j^\mp z_2^\pm; q)_\infty. \quad (5.67)
 \end{aligned}$$

In the dual junction $4d \text{ U}(L) \times \text{U}(M) \times \text{U}(N)$ gauge symmetry is broken down to $\text{U}(\min\{L, M, N\})$ due to the D5'-branes so that there is a $4d \text{ U}(\min\{L, M, N\})$ gauge theory in a half-space. According to the D5-brane, it should obey Dirichlet/Nahm pole b.c. specified by an embedding $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{u}(\min\{L, M, N\})$. Besides, the junction may include 3d twisted hypermultiplets. For $L = M$, there would be 3d twisted hypermultiplet obeying the Dirichlet/Nahm pole b.c. with an embedding $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{u}(L)$. Analogously, for $N = M$, we would have 3d twisted hypermultiplet satisfying the Dirichlet/Nahm pole b.c. with an embedding $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{u}(N)$.

Then the quarter-indices for the dual junction is written as

$$\begin{aligned}
 \mathbb{I}\mathbb{V}_{\mathcal{D}\mathcal{D}'}^{L|[M]|N} &= \underbrace{\prod_{k=1}^{\min\{L, M, N\}} \frac{(q^{\frac{k+1}{2}}t^{2(k-1)}; q)_\infty}{(q^{\frac{k}{2}}t^{2k}; q)_\infty}}_{\mathbb{I}\mathbb{I}_{\text{Nahm}}^{4d \text{ U}(\min\{L, M, N\})}} \\
 & \times (q^{\frac{3}{4} + \frac{L-1}{4}}t^{1+(L-1)}z_1^\pm; q)_\infty^{\delta^L M} \cdot (q^{\frac{3}{4} + \frac{M-1}{4}}t^{1+(M-1)}z_2^\pm; q)_\infty^{\delta^M N}. \quad (5.68)
 \end{aligned}$$

We expect that the quarter-indices (5.67) and (5.68) agree with each other.

5.6 4d U(L)|3d (M₁) – (M₂) – ⋯ – (M_{k-1})|4d U(N) with N'

Let us generalize the junction in section 5.5 by including 3d linear quiver gauge theory $(M_1) - \cdots - (M_{k-1})$ with a gauge group $\text{U}(M_1) \times \cdots \times \text{U}(M_{k-1})$ and $(k-2)$ bi-fundamental

Figure 16. The brane constructions of the configuration $4d\ U(L)|3d\ (M_1)-(M_2)-\cdots-(M_{k-1})|4d\ U(N)$ with Neumann b.c. $\mathcal{N}'$ and its mirror configuration.

hypermultiplets. The brane construction is shown in figure 16. The junction has $4d\ U(L)$ and $U(N)$ gauge symmetry and $3d\ U(M_1) \times U(M_{k-1})$ gauge symmetry. In addition to the $(k-2)$ bi-fundamental hypers in the 3d quiver gauge theory, there are two bi-fundamental hypers; one couples to $4d\ U(L)$ and $3d\ U(M_1)$, the other couples to $4d\ U(N)$ and $3d\ U(M_{k-1})$. Furthermore, there are k cross-determinant Fermi multiplets corresponding to the k NS5-NS5'-junction.

Contrast to the original junction, the dual junction is rather simple in that it can be described in terms of free fields as it has no gauge symmetry. In particular, it has the $4d$ gauge theory with Dirichlet/Nahm pole boundary conditions. We confirm the dualities of these junctions by showing the matching of indices.

5.6.1 $4d\ U(1)|3d\ (1)-(1)|4d\ U(1)$ with $\mathcal{N}'$

For the junction $4d\ U(1)|3d\ (1)-(1)|4d\ U(1)$ with $\mathcal{N}'$, we have a pair of corners of $4d\ U(1)$ gauge theories obeying the boundary conditions $(\mathcal{N}, \mathcal{N}')$ and a boundary of $3d\ U(1) \times U(1)$ gauge theory with Neumann b.c. $\mathcal{N}'$. In total, we have three $3d$ bi-fundamental hypermultiplets and three $2d$ cross-determinant Fermi multiplets.

The quarter-index takes the form

$$\begin{aligned}
 & \mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(1)|(1)-(1)|4d \text{ U}(1)} \\
 &= \underbrace{(q)_\infty \oint \frac{ds_1}{2\pi i s_1}}_{\mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(1)}} \cdot \underbrace{(q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty \oint \frac{ds_2}{2\pi i s_2}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(1)}} \cdot \underbrace{(q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty \oint \frac{ds_3}{2\pi i s_3}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(1)}} \cdot \underbrace{(q)_\infty \oint \frac{ds_4}{2\pi i s_4}}_{\mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(1)}} \\
 &\times \frac{1}{\underbrace{(q^{\frac{1}{4}} t s_1^\pm s_2^\mp; q)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}}\left(\frac{s_1}{s_2}\right)}} \cdot \frac{1}{\underbrace{(q^{\frac{1}{4}} t s_2^\pm s_3^\mp; q)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}}\left(\frac{s_2}{s_3}\right)}} \cdot \frac{1}{\underbrace{(q^{\frac{1}{4}} t s_3^\pm s_4^\mp; q)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}}\left(\frac{s_3}{s_4}\right)}} \\
 &\times \underbrace{(q^{\frac{1}{2}} s_1^\pm s_2^\mp z_1^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_1}{s_2} z_1\right)} \cdot \underbrace{(q^{\frac{1}{2}} s_2^\pm s_3^\mp z_2^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_2}{s_3} z_2\right)} \cdot \underbrace{(q^{\frac{1}{2}} s_3^\pm s_4^\mp z_3^\pm; q)_\infty}_{F\left(\frac{s_3}{s_4} z_3\right)}. \tag{5.69}
 \end{aligned}$$

The dual junction has three defects of D5'-branes, which break down $U(1) \times U(1) \times U(1) \times U(1)$ gauge symmetry to $U(1)$. The 4d $U(1)$ gauge theory in a half-space should be subject to Dirichlet b.c. $\mathcal{D}$. The equal numbers of D3-branes across three D5'-branes yield three 3d twisted hypermultiplets, which should satisfy Dirichlet b.c. D .

We then obtain the quarter-index for the dual junction

$$\mathbb{IV}_{\mathcal{DD}'}^{1|[1]-[1]|1} = \frac{(q)_\infty}{\underbrace{(q^{\frac{1}{2}} t^2; q)_\infty}_{\mathbb{III}_{\mathcal{D}}^{4d \text{ U}(1)}}} \cdot \prod_{\alpha=1}^3 \underbrace{(q^{\frac{3}{4}} t z_\alpha^\pm; q)_\infty}_{\mathbb{III}_{\mathcal{D}}^{3d \text{ tHM}}(z_\alpha)}. \tag{5.70}$$

We have checked that the quarter-indices (5.69) and (5.70) coincide to each other.

5.6.2 4d $U(2)|3d (1)-(1)|4d U(1)$ with $\mathcal{N}'$

Next consider the junction 4d $U(2)|3d (1)-(1)|4d U(1)$ with $\mathcal{N}'$. This is obtained from the previous example by replacing one of the 4d $U(1)$ gauge theories with 4d $U(2)$ gauge theory.

In a similar manner, the quarter-index takes the form

$$\begin{aligned}
 & \mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(2)|(1)-(1)|4d \text{ U}(1)} = \underbrace{\frac{1}{2} (q)_\infty^2 \oint \prod_{i=1}^2 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(2)}} \cdot \underbrace{(q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty \oint \frac{ds_3}{2\pi i s_3}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(1)}} \\
 &\times \underbrace{(q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty \oint \frac{ds_4}{2\pi i s_4}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(1)}} \cdot \underbrace{(q)_\infty \oint \frac{ds_5}{2\pi i s_5}}_{\mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(1)}} \\
 &\times \prod_{i=1}^2 \frac{1}{\underbrace{(q^{\frac{1}{4}} t s_i^\pm s_3^\mp; q)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}}\left(\frac{s_i}{s_3}\right)}} \cdot \frac{1}{\underbrace{(q^{\frac{1}{4}} t s_3^\pm s_4^\mp; q)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}}\left(\frac{s_3}{s_4}\right)}} \cdot \frac{1}{\underbrace{(q^{\frac{1}{4}} t s_4^\pm s_5^\mp; q)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}}\left(\frac{s_4}{s_5}\right)}} \\
 &\times \underbrace{(q^{\frac{1}{2}} s_1^\pm s_2^\mp s_3^\mp z_1^\pm; q)_\infty}_{F\left(\frac{s_1 s_2}{s_3} z_1\right)} \cdot \underbrace{(q^{\frac{1}{2}} s_3^\pm s_4^\mp z_2^\pm; q)_\infty}_{F\left(\frac{s_3}{s_4} z_2\right)} \cdot \underbrace{(q^{\frac{1}{2}} s_4^\pm s_5^\mp z_3^\pm; q)_\infty}_{F\left(\frac{s_4}{s_5} z_3\right)}. \tag{5.71}
 \end{aligned}$$

In the dual configuration $U(1) \times U(1) \times U(1) \times U(1)$ gauge symmetry is broken down to $U(1)$ by $D5'$ -branes, which gives rise to 4d $U(1)$ gauge theory with Dirichlet b.c. $\mathcal{D}$. The equal numbers of D3-branes across two $D5'$ -branes yield two 3d twisted hypermultiplets, which should satisfy Dirichlet b.c. D .

In fact, we find that the quarter-index (5.71) agrees with

$$\mathbb{IV}_{\mathcal{DD}'}^{2|[1]-[1]|1} = \underbrace{\frac{(q)_\infty}{(q^{\frac{1}{2}}t^2; q)_\infty}}_{\mathbb{III}_{\mathcal{D}}^{4d U(1)}} \cdot \underbrace{(q^{\frac{3}{4}}tz_2^\pm; q)_\infty}_{\mathbb{III}_{\mathcal{D}}^{3d tHM}(z_2)} \cdot \underbrace{(q^{\frac{3}{4}}tz_3^\pm; q)_\infty}_{\mathbb{III}_{\mathcal{D}}^{3d tHM}(z_3)}. \quad (5.72)$$

5.6.3 4d $U(2)|3d (2) - (1)|4d U(1)$ with $\mathcal{N}'$

Consider the junction 4d $U(2)|3d (2) - (1)|4d U(1)$ with $\mathcal{N}'$. In this case the junction has a pair of $U(2)$ gauge symmetries and a pair of $U(1)$ gauge symmetries.

The quarter-index takes the form

$$\begin{aligned} \mathbb{IV}_{\mathcal{NN}'}^{4d U(2)|(2) - (1)|4d U(1)} &= \underbrace{\frac{1}{2}(q)_\infty^2 \oint \prod_{i=1}^2 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{IV}_{\mathcal{NN}'}^{4d U(2)}} \\ &\cdot \underbrace{\frac{1}{2}(q)_\infty^2 (q^{\frac{1}{2}}t^2; q)_\infty^2 \oint \prod_{i=3}^4 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty \left(q^{\frac{1}{2}}t^2 \frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d U(2)}} \\ &\times \underbrace{(q)_\infty (q^{\frac{1}{2}}t^2; q)_\infty \oint \frac{ds_5}{2\pi i s_5}}_{\mathbb{III}_{\mathcal{N}'}^{3d U(1)}} \cdot \underbrace{(q)_\infty \oint \frac{ds_6}{2\pi i s_6}}_{\mathbb{IV}_{\mathcal{NN}'}^{4d U(1)}} \\ &\times \prod_{i=1}^2 \prod_{j=3}^4 \underbrace{\frac{1}{(q^{\frac{1}{4}}ts_i^\pm s_j^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d HM}\left(\frac{s_i}{s_j}\right)} \cdot \prod_{i=3}^4 \underbrace{\frac{1}{(q^{\frac{1}{4}}ts_i^\pm s_5^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d HM}\left(\frac{s_i}{s_5}\right)} \cdot \underbrace{\frac{1}{(q^{\frac{1}{4}}ts_5^\pm s_6^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d HM}\left(\frac{s_5}{s_6}\right)} \\ &\times \underbrace{(q^{\frac{1}{2}}s_1^\pm s_2^\pm s_3^\mp s_4^\mp z_1^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_1 s_2}{s_3 s_4}\right)} \cdot \underbrace{(q^{\frac{1}{2}}s_3^\pm s_4^\pm s_5^\mp z_2^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_3 s_4}{s_5} z_2\right)} \cdot \underbrace{(q^{\frac{1}{2}}s_5^\pm s_6^\mp z_3^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_5}{s_6} z_3\right)}. \end{aligned} \quad (5.73)$$

From S-duality, one find the configuration in which the three $D5'$ -branes break down $U(2) \times U(2) \times U(1) \times U(1)$ to $U(1)$ and the $D5$ -brane imposes Dirichlet b.c. $\mathcal{D}$ for 4d $U(1)$ gauge theory. As there are two $D5'$ -branes with equal numbers of D3-branes in their each side, there would be two 3d twisted hypermultiplets. Corresponding to the numbers of D3-branes, one has Nahm pole of rank 2 and the other receives Dirichlet b.c. D .

We get the quarter-index

$$\mathbb{IV}_{\mathcal{DD}'}^{2|[2]-[1]|1} = \underbrace{\frac{(q)_\infty}{(q^{\frac{1}{2}}t^2; q)_\infty}}_{\mathbb{III}_{\mathcal{D}}^{4d U(1)}} \cdot (qt^2 z_1^\pm; q)_\infty \cdot \underbrace{(q^{\frac{3}{4}}tz_3^\pm; q)_\infty}_{\mathbb{III}_{\mathcal{D}}^{3d HM}(z_3)}. \quad (5.74)$$

In fact, the quarter-indexes (5.73) and (5.74) agree with each other.

5.6.4 4d U(2)|3d (2) – (2)|4d U(2) with $\mathcal{N}'$

Let us consider the junction in which both 3d and 4d gauge symmetries are non-Abelian. For the junction 4d U(2)|3d (2) – (2)|4d U(2) with $\mathcal{N}'$ we have four factors of U(2) gauge symmetry, three 3d bi-fundamental hypermultiplets and three 2d cross-determinant Fermi multiplets.

The quarter-index is

$$\begin{aligned}
 \mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(2)|(2) - (2)|4d \text{ U}(2)} & \quad (5.75) \\
 &= \underbrace{\frac{1}{2}(q)_\infty^2 \oint \prod_{i=1}^2 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(2)}} \\
 &\quad \cdot \underbrace{\frac{1}{2}(q)_\infty^2 (q^{\frac{1}{2}} t^2; q)_\infty^2 \oint \prod_{i=3}^4 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty \left(q^{\frac{1}{2}} t^2 \frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(2)}} \\
 &\quad \times \underbrace{\frac{1}{2}(q)_\infty^2 (q^{\frac{1}{2}} t^2; q)_\infty^2 \oint \prod_{i=5}^6 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty \left(q^{\frac{1}{2}} t^2 \frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(2)}} \\
 &\quad \cdot \underbrace{\frac{1}{2}(q)_\infty^2 \oint \prod_{i=7}^8 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{IV}_{\mathcal{NN}'}^{4d \text{ U}(2)}} \\
 &\quad \times \prod_{i=1}^2 \prod_{j=3}^4 \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_i^\pm s_j^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}} \left(\frac{s_i}{s_j} \right)} \cdot \prod_{i=3}^4 \prod_{j=5}^6 \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_i^\pm s_j^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}} \left(\frac{s_i}{s_j} \right)} \cdot \prod_{i=5}^6 \prod_{j=7}^8 \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_i^\pm s_j^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}} \left(\frac{s_i}{s_j} \right)} \\
 &\quad \times \underbrace{(q^{\frac{1}{2}} s_1^\pm s_2^\pm s_3^\mp s_4^\mp z_1^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_1 s_2}{s_3 s_4} z_1\right)} \cdot \underbrace{(q^{\frac{1}{2}} s_3^\pm s_4^\pm s_5^\mp s_6^\mp z_2^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_3 s_4}{s_5 s_6} z_2\right)} \cdot \underbrace{(q^{\frac{1}{2}} s_5^\pm s_6^\pm s_7^\mp s_8^\mp z_3^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_5 s_6}{s_7 s_8} z_3\right)}.
 \end{aligned}$$

In the dual configuration there remains 4d U(2) gauge theory in a half-space with Nahm pole boundary condition with a homomorphism $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{u}(2)$. As the numbers of D3-branes are constant across the three D5'-branes, we find three 3d twisted hypermultiplets. Due to the D5-brane, they should satisfy Nahm pole boundary condition characterized by the same homomorphism.

Then we have the quarter-index for the dual junction

$$\mathbb{IV}_{\mathcal{DD}'}^{2|[2]-[2]|2} = \underbrace{\frac{(q)_\infty (q^{\frac{3}{2}} t^2; q)_\infty}{(q^{\frac{1}{2}} t^2; q)_\infty (qt^4; q)_\infty}}_{\mathbb{III}_{\text{Nahm}}^{4d \text{ U}(2)}} \cdot \prod_{\alpha=1}^3 (qt^2 z_\alpha^\pm; q)_\infty. \quad (5.76)$$

We have confirmed that the quarter-indices (5.75) and (5.76) coincides with each other.

5.6.5 4d $U(L)|3d (M_1) - (M_2) - \dots - (M_{k-1})|4d U(N)$ with $\mathcal{N}'$

Now we would like to propose the generalization. For the junction $4d U(L)|3d (M_1) - (M_2) - \dots - (M_{k-1})|4d U(N)$ with $\mathcal{N}'$, we have two corners of $4d \mathcal{N} = 4 U(L)$ and $U(N)$ gauge theories with a pair of Neumann b.c. $(\mathcal{N}, \mathcal{N}')$. Between these $4d$ gauge theories we have $3d \mathcal{N} = 4$ linear quiver gauge theory $(M_1) - (M_2) - \dots - (M_{k-1})$ which has a gauge group $U(M_1) \times \dots \times U(M_{k-1})$ and $(k-2)$ bi-fundamental hypermultiplets. The $4d$ and $3d$ gauge theories are coupled by additional two $3d$ bi-fundamental hypermultiplets. In total, there are k $3d$ bi-fundamental hypermultiplets. Besides, there are k cross-determinant Fermi multiplets.

The quarter-index can be computed as

$$\begin{aligned}
 & \mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d U(L)|(M_1) - (M_2) - \dots - (M_{k-1})|4d U(N)} \\
 &= \underbrace{\frac{1}{L!} (q)_\infty^L \oint \prod_{i=1}^L \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d U(L)}} \\
 & \times \underbrace{\prod_{l=1}^{k-1} \frac{1}{M_l!} (q)_\infty^{M_l} (q^{\frac{1}{2}} t^2; q)_\infty^{M_l} \oint \prod_{i=1}^{M_l} \frac{ds_i^{(l)}}{2\pi i s_i^{(l)}} \prod_{i \neq j} \left(\frac{s_i^{(l)}}{s_j^{(l)}}; q \right)_\infty \left(q^{\frac{1}{2}} t^2 \frac{s_i^{(l)}}{s_j^{(l)}}; q \right)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d U(M_l)}} \\
 & \times \underbrace{\frac{1}{N!} (q)_\infty^N \oint \prod_{i=L+1}^{L+N} \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d U(N)}} \\
 & \times \prod_{i=1}^L \prod_{j=1}^{M_1} \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_i^\pm s_j^{(1)\mp}; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d HM} \left(\frac{s_i}{s_j^{(1)}} \right)} \cdot \prod_{l=1}^{k-2} \prod_{i=1}^{M_l} \prod_{j=1}^{M_{l+1}} \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_i^{(l)\pm} s_j^{(l+1)\mp}; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d HM} \left(\frac{s_i^{(l)}}{s_j^{(l+1)}} \right)} \\
 & \cdot \prod_{i=1}^{M_{k-1}} \prod_{j=L+1}^{L+N} \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_i^{(k-1)\pm} s_j^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d HM} \left(\frac{s_i^{(k-1)}}{s_j} \right)} \\
 & \times \underbrace{\left(q^{\frac{1}{2}} \prod_{i=1}^L s_i^\pm \prod_{j=1}^{M_1} s_j^{(1)\mp} z_1^\pm; q \right)_\infty}_{F \left(q^{\frac{1}{2}} \frac{\prod_{i=1}^L s_i}{\prod_{j=1}^{M_1} s_j^{(1)}} z_1 \right)} \cdot \underbrace{\left(q^{\frac{1}{2}} \prod_{i=1}^{M_l} s_i^{(l)\pm} \prod_{j=1}^{M_{l+1}} s_j^{(l+1)\mp} z_{l+1}^\pm; q \right)_\infty}_{F \left(q^{\frac{1}{2}} \frac{\prod_{i=1}^{M_l} s_i^{(l)}}{\prod_{j=1}^{M_{l+1}} s_j^{(l+1)}} z_{l+1} \right)}
 \end{aligned} \tag{5.77}$$

$$\cdot \underbrace{\left(q^{\frac{1}{2}} \prod_{i=1}^{M_{k-1}} s_i^{(k-1)\pm} \prod_{j=L+1}^{L+N} s_j^{\mp} z_k^{\pm}; q \right)}_{F\left(q^{\frac{1}{2}} \frac{\prod_{i=1}^{M_{k-1}} s_i^{(k-1)}}{\prod_{j=L+1}^{L+N} s_j} z_k \right)} \cdot$$

Under the action of S-duality, we find the dual junction in which the k defects of D5'-branes break the gauge symmetry down to $U(\min\{L, M_i, N\})$ and the D5-brane requires the $U(\min\{L, M_i, N\})$ gauge theory to satisfy Dirichlet/Nahm pole boundary condition characterized by an embedding $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{u}(\min\{L, M_i, N\})$. For D5'-branes which have equal numbers of D3-branes in their both sides, we find 3d twisted hypermultiplets. Due to the D5-brane, they should obey Dirichlet/Nahm pole boundary condition with an embedding specified by the corresponding numbers of D3-branes.

The quarter-index would be given by

$$\begin{aligned} & \mathbb{W}_{\mathcal{D}\mathcal{D}'}^{L|[M_1]-[M_2]-\dots-[M_{k-1}]|N} \\ &= \underbrace{\prod_{k=1}^{\min\{L, M_i, N\}} \frac{(q^{\frac{k+1}{2}} t^{2(k-1)}; q)_{\infty}}{(q^{\frac{k}{2}} t^{2k}; q)_{\infty}}}_{\mathbb{I}_{\text{Nahm}}^{U(\min\{L, M_i, N\})}} \\ & \quad \times (q^{\frac{3}{4} + \frac{L-1}{4}} t^{1+(L-1)} z_1^{\pm}; q)_{\infty}^{\delta^L M_1} \cdot \prod_{l=1}^{k-2} (q^{\frac{3}{4} + \frac{M_l-1}{4}} t^{1+(M_l-1)} z_{l+1}^{\pm}; q)_{\infty}^{\delta^{M_l} M_{l+1}} \\ & \quad \cdot (q^{\frac{3}{4} + \frac{N-1}{4}} t^{1+(N-1)} z_k^{\pm}; q)_{\infty}^{\delta^N M_{k-1}}. \end{aligned} \tag{5.78}$$

We expect that the quarter-indices (5.77) and (5.78) give the same answer.

5.7 4d $U(L)|3d(M) - [1]|4d U(N)$ with $\mathcal{N}'$

Now turn to the junction 4d $U(L)|3d(M) - [1]|4d U(N)$ with $\mathcal{N}'$, which has a 3d fundamental hypermultiplet. This is a generalization of the junction in section 5.3 by adding another corner of 4d theory. The 4d $U(L)$ and $U(N)$ gauge theories are coupled to 3d $U(M)$ gauge theory through two 3d bi-fundamental hypermultiplets. There are two cross-determinant Fermi multiplets and a fundamental Fermi multiplet. The brane configuration is illustrated in figure 17.

By applying S-duality we can obtain the dual junction. Unlike the junction in section 5.3, the D3-branes do not terminate on D5'-brane. This fact admits non-trivial Nahm pole boundary conditions for 3d bi-fundamental twisted hypermultiplets as well as a pair of boundary conditions ($\mathcal{N}', \mathcal{D}/\text{Nahm}$) for 4d gauge theory.

5.7.1 4d $U(1)|3d(1) - [1]|4d U(1)$ with $\mathcal{N}'$

Let us start with the junction 4d $U(1)|3d(1) - [1]|4d U(1)$ with $\mathcal{N}'$. This junction is obtained from the junction in section 5.5.1 by adding a fundamental hypermultiplet with Neumann b.c. N' and a 2d fundamental Fermi multiplet. There is a pair of corners of

Figure 17. The brane constructions of the configuration $4d \text{ U}(L)|3d (M) - [1]|4d \text{ U}(N)$ with Neumann b.c. $\mathcal{N}'$ and its mirror corner configuration that has no gauge symmetry.

$4d \text{ U}(1)$ gauge theories with Neumann b.c. $(\mathcal{N}, \mathcal{N}')$ and a boundary of $3d \text{ U}(1)$ gauge theory with Neumann b.c. $\mathcal{N}'$. Again there are two $3d$ bi-fundamental hypers and two cross-determinant Fermi multiplets.

The quarter-index takes the form

$$\begin{aligned}
 \mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d \text{ U}(1)|(1) - [1]|4d \text{ U}(1)} &= (q)_\infty \underbrace{\oint \frac{ds_1}{2\pi i s_1}}_{\mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d \text{ U}(1)}} \cdot (q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty \underbrace{\oint \frac{ds_2}{2\pi i s_2}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(1)}} \cdot (q)_\infty \underbrace{\oint \frac{ds_3}{2\pi i s_3}}_{\mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d \text{ U}(1)}} \\
 &\times \frac{1}{\underbrace{(q^{\frac{1}{4}} t s_1^\pm s_2^\mp; q)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}}\left(\frac{s_1}{s_2}\right)}} \cdot \frac{1}{\underbrace{(q^{\frac{1}{4}} t s_2^\pm; q)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}}(s_2)}} \cdot \frac{1}{\underbrace{(q^{\frac{1}{4}} t s_2^\pm s_3^\mp; q)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}}\left(\frac{s_2}{s_3}\right)}} \\
 &\times \underbrace{(q^{\frac{1}{2}} s_1^\pm s_2^\mp z_1^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_1}{s_2} z_1\right)} \cdot \underbrace{(q^{\frac{1}{2}} s_2^\pm x^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} s_2 x\right)} \cdot \underbrace{(q^{\frac{1}{2}} s_2^\pm s_3^\mp z_2^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_2}{s_3} z_2\right)}. \quad (5.79)
 \end{aligned}$$

The dual junction has a pair of corners of $4d \text{ U}(1)$ gauge theories with sets of boundary conditions $(\mathcal{N}', \mathcal{D})$. From the D3-D5' strings, we have two $3d$ fundamental hypermultiplets with Dirichlet b.c. $\mathcal{D}$. In addition, from the D3-D3 strings across the NS5'-brane, we have a $3d$ bi-fundamental twisted hypermultiplet, which again obeys Dirichlet b.c. $\mathcal{D}$.

Then one obtains the quarter-index for the dual junction

$$\mathbb{IV}_{\mathcal{N}'\mathcal{D}}^{1|[1]|1} = \frac{1}{\underbrace{(q^{\frac{1}{2}} t^2; q)_\infty}_{\mathbb{IV}_{\mathcal{N}'\mathcal{D}}^{4d \text{ U}(1)}}} \cdot \frac{1}{\underbrace{(q^{\frac{1}{2}} t^2; q)_\infty}_{\mathbb{IV}_{\mathcal{N}'\mathcal{D}}^{4d \text{ U}(1)}}} \cdot \underbrace{(q^{\frac{3}{4}} t z_1^\pm; q)_\infty}_{\mathbb{III}_{\mathcal{D}}^{3d \text{ tHM}}(z_1)} \cdot \underbrace{(q^{\frac{3}{4}} t x^\pm; q)_\infty}_{\mathbb{III}_{\mathcal{D}}^{3d \text{ tHM}}(x)} \cdot \underbrace{(q^{\frac{3}{4}} t z_2^\pm; q)_\infty}_{\mathbb{III}_{\mathcal{D}}^{3d \text{ tHM}}(z_2)}. \quad (5.80)$$

In fact, we find that the quarter-indices (5.79) and (5.80) coincide with each other.

5.7.2 $4d \text{ U}(2)|3d (1) - [1]|4d \text{ U}(1)$ with $\mathcal{N}'$

Consider the junction $4d \text{ U}(2)|3d (1) - [1]|4d \text{ U}(1)$ with $\mathcal{N}'$, which is obtained from the junction in section 5.5.2 by adding a $3d$ charged hypermultiplet with Neumann b.c. $\mathcal{N}'$ together with $2d$ charged Fermi multiplet in the $3d$ gauge theory.

The quarter-index takes the form

$$\begin{aligned}
 & \mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d \text{ U}(2)|(1) - [1]|4d \text{ U}(1)} \\
 &= \underbrace{\frac{1}{2}(q)_\infty^2 \oint \prod_{i=1}^2 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d \text{ U}(2)}} \cdot \underbrace{(q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty \oint \frac{ds_3}{2\pi i s_3}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(1)}} \cdot \underbrace{(q)_\infty \oint \frac{ds_4}{2\pi i s_4}}_{\mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d \text{ U}(1)}} \\
 &\times \prod_{i=1}^2 \frac{1}{\underbrace{(q^{\frac{1}{4}} t s_i^\pm s_3^\mp; q)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}}\left(\frac{s_i}{s_3}\right)}} \cdot \frac{1}{\underbrace{(q^{\frac{1}{4}} t s_3^\pm; q)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}}(s_3)}} \cdot \frac{1}{\underbrace{(q^{\frac{1}{4}} t s_3^\pm s_4^\mp; q)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}}\left(\frac{s_3}{s_4}\right)}} \\
 &\times \underbrace{(q^{\frac{1}{2}} s_1^\pm s_2^\pm s_3^\mp z_1^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_1 s_2}{s_3} z_1\right)} \cdot \underbrace{(q^{\frac{1}{2}} s_3^\pm x^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} s_3 x\right)} \cdot \underbrace{(q^{\frac{1}{2}} s_3^\pm s_4^\mp z_2^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_3}{s_4} z_2\right)}. \tag{5.81}
 \end{aligned}$$

In the dual configuration there is a pair of corners of 4d gauge theories. Due to the D5'-brane, the U(2) gauge symmetry should be broken down to U(1) so that one finds a pair of corners of 4d U(1) gauge theories with two kinds of boundary conditions ($\mathcal{N}'$, $\mathcal{D}$). For the D5'-brane with equal numbers of D3-branes in its both sides, we have 3d fundamental hypermultiplet with Dirichlet b.c. D . In addition, from the D3-D3 strings across the NS5'-brane, we have a 3d bi-fundamental twisted hypermultiplet, which again obeys Dirichlet b.c. D .

As expected, we have checked that the quarter-index (5.81) matches with

$$\mathbb{IV}_{\mathcal{N}'\mathcal{D}}^{2|[1]|1} = \underbrace{\frac{1}{(q^{\frac{1}{2}} t^2; q)_\infty}}_{\mathbb{IV}_{\mathcal{N}'\mathcal{D}}^{4d \text{ U}(1)}} \cdot \underbrace{\frac{1}{(q^{\frac{1}{2}} t^2; q)_\infty}}_{\mathbb{IV}_{\mathcal{N}'\mathcal{D}}^{4d \text{ U}(1)}} \cdot \underbrace{(q^{\frac{3}{4}} t z_2^\pm; q)_\infty}_{\mathbb{III}_D^{3d \text{ tHM}}(z_2)} \cdot \underbrace{(q^{\frac{3}{4}} t x^\pm; q)_\infty}_{\mathbb{III}_D^{3d \text{ tHM}}(x)}. \tag{5.82}$$

5.7.3 4d U(2)|3d (1) – [1]|4d U(2) with $\mathcal{N}'$

As a further example, let us consider the junction 4d U(2)|3d (1) – [1]|4d U(2) with $\mathcal{N}'$. This is obtained by adding a 3d fundamental hypermultiplet with Neumann b.c. N' and a 2d fundamental Fermi multiplet.

Analogously, we can compute the quarter-index as

$$\begin{aligned}
 \mathbb{I}\mathbb{V}_{\mathcal{N}\mathcal{N}'}^{4d \text{ U}(2)|(1) - [1]|4d \text{ U}(2)} &= \underbrace{\frac{1}{2}(q)_\infty^2 \oint \prod_{i=1}^2 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{I}\mathbb{V}_{\mathcal{N}\mathcal{N}'}^{4d \text{ U}(2)}} \\
 &\cdot \underbrace{(q)_\infty (q^{\frac{1}{2}} t^2; q)_\infty \oint \frac{ds_3}{2\pi i s_3}}_{\mathbb{I}\mathbb{I}_{\mathcal{N}'}^{3d \text{ U}(1)}} \cdot \underbrace{\frac{1}{2}(q)_\infty^2 \oint \prod_{i=4}^5 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{I}\mathbb{V}_{\mathcal{N}\mathcal{N}'}^{4d \text{ U}(2)}} \\
 &\times \prod_{i=1}^2 \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_i^\pm s_3^\mp; q)_\infty}}_{\mathbb{I}\mathbb{I}_{\mathcal{N}'}^{3d \text{ HM}} \left(\frac{s_i}{s_j} \right)} \cdot \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_3^\pm; q)_\infty}}_{\mathbb{I}\mathbb{I}_{\mathcal{N}'}^{3d \text{ HM}} (s_i)} \cdot \prod_{i=4}^5 \underbrace{\frac{1}{(q^{\frac{1}{4}} t s_3^\pm s_i^\mp; q)_\infty}}_{\mathbb{I}\mathbb{I}_{\mathcal{N}'}^{3d \text{ HM}} \left(\frac{s_i}{s_j} \right)} \\
 &\times \underbrace{(q^{\frac{1}{2}} s_1^\pm s_2^\pm s_3^\pm z_1^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_1 s_2}{s_3} z_1\right)} \cdot \underbrace{(q^{\frac{1}{2}} s_3^\pm x^\pm; q)_\infty}_{F(q^{\frac{1}{2}} s_3 x)} \cdot \underbrace{(q^{\frac{1}{2}} s_3^\pm s_4^\mp s_5^\mp z_2^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_3}{s_4 s_5} z_2\right)}.
 \end{aligned} \tag{5.83}$$

There are two corners of 4d gauge theories in the dual configuration. According to the D5'-branes, each U(2) gauge symmetry in the two corners is broken down to U(1). As a consequence, there remains a pair of corners of 4d U(1) gauge theories. Since the numbers of D3-branes jump across the D5-branes, the junction would not include 3d fundamental twisted hypermultiplet. Instead, there would be 3d bi-fundamental twisted hypermultiplet arising from D3-D3 string across the NS5'-brane.

The quarter-index for the dual junction gets simplified as

$$\mathbb{I}\mathbb{V}_{\mathcal{N}'\mathcal{D}}^{2|[1]|^2} = \underbrace{\frac{1}{(q^{\frac{1}{2}} t^2; q)_\infty}}_{\mathbb{I}\mathbb{V}_{\mathcal{N}'\mathcal{D}}^{4d \text{ U}(1)}} \cdot \underbrace{\frac{1}{(q^{\frac{1}{2}} t^2; q)_\infty}}_{\mathbb{I}\mathbb{V}_{\mathcal{N}'\mathcal{D}}^{4d \text{ U}(1)}} \cdot (q^{\frac{3}{4}} t x^\pm; q)_\infty. \tag{5.84}$$

In fact, we have confirmed that the quarter-indices (5.83) and (5.84) coincide to each other.

5.7.4 4d U(2)|3d (2) – [1]|4d U(2) with $\mathcal{N}'$

Let us turn to the non-Abelian case. For the junction 4d U(2)|3d (2) – [1]|4d U(2) with $\mathcal{N}'$, we have three factors of U(2) gauge symmetry. The 4d and 3d gauge theories are coupled through two 3d bi-fundamental hypers and two 2d cross-determinant Fermi multiplets. The junction further contains a 3d hypermultiplet with Neumann b.c. N' and a 2d fundamental Fermi multiplet transforming as the fundamental representation under the 3d U(2) gauge symmetry.

Then the quarter-index is evaluated as

$$\begin{aligned}
 \mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d \text{ U}(2)|(2)-[1]|4d \text{ U}(2)} &= \underbrace{\frac{1}{2}(q)_\infty^2 \oint \prod_{i=1}^2 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d \text{ U}(2)}} \\
 &\cdot \underbrace{\frac{1}{2}(q)_\infty^2 (q^{\frac{1}{2}}t^2; q)_\infty^2 \oint \prod_{i=3}^4 \frac{ds_i}{2\pi i s_i} \left(\frac{s_i}{s_j}; q \right)_\infty \left(q^{\frac{1}{2}}t^2 \frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ U}(2)}} \\
 &\times \underbrace{\frac{1}{2}(q)_\infty^2 \oint \prod_{i=5}^6 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d \text{ U}(2)}} \\
 &\times \prod_{i=1}^2 \prod_{j=3}^4 \underbrace{\frac{1}{(q^{\frac{1}{4}}ts_i^\pm s_j^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}}\left(\frac{s_i}{s_j}\right)} \cdot \prod_{i=3}^4 \underbrace{\frac{1}{(q^{\frac{3}{4}}ts_i^\pm; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}}(s_i)} \cdot \prod_{i=3}^4 \prod_{j=5}^6 \underbrace{\frac{1}{(q^{\frac{1}{4}}ts_i^\pm s_j^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d \text{ HM}}\left(\frac{s_i}{s_j}\right)} \\
 &\times \underbrace{(q^{\frac{1}{2}}s_1^\pm s_2^\pm s_3^\mp s_4^\mp z_1^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}}\frac{s_1s_2}{s_3s_4}z_1\right)} \cdot \prod_{i=3}^4 \underbrace{(q^{\frac{1}{2}}s_i^\pm x^\pm; q)_\infty}_{F(q^{\frac{1}{2}}s_i x)} \cdot \underbrace{(q^{\frac{1}{2}}s_3^\pm s_4^\pm s_5^\mp s_6^\mp z_2^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}}\frac{s_3s_4}{s_5s_6}z_2\right)}.
 \end{aligned} \tag{5.85}$$

Taking S-duality, we get the junction which has two corners of 4d U(2) gauge theories with a pair of boundary conditions ($\mathcal{N}'$, Nahm). As the numbers of D3-branes do not change across the two D5'-branes, we would get two 3d fundamental twisted hypers obeying Nahm pole boundary condition with an embedding $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{u}(2)$. In addition, there would be a 3d bi-fundamental twisted hyper satisfying Nahm pole boundary condition with an embedding $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{u}(2)$.

We find that the quarter-index (5.85) coincides with

$$\begin{aligned}
 \mathbb{IV}_{\mathcal{N}'\mathcal{D}}^{2|[2]|^2} &= \underbrace{\frac{1}{(q^{\frac{1}{2}}t^2; q)_\infty (qt^4; q)_\infty}}_{\mathbb{IV}_{\mathcal{N}'\mathcal{D}}^{4d \text{ U}(2)}} \cdot \underbrace{\frac{1}{(q^{\frac{1}{2}}t^2; q)_\infty (qt^4; q)_\infty}}_{\mathbb{IV}_{\mathcal{N}'\mathcal{D}}^{4d \text{ U}(2)}} \\
 &\times (qt^2z_1^\pm; q)_\infty \cdot (qt^2z_2^\pm; q)_\infty \cdot (q^{\frac{3}{4}}tx^\pm; q)_\infty \cdot (q^{\frac{5}{4}}t^3x^\pm; q)_\infty.
 \end{aligned} \tag{5.86}$$

It contains two quarter-indices of 4d U(2) gauge theory with a pair of boundary conditions ($\mathcal{N}'$, $\mathcal{D}$). The factors $(qt^2z_1^\pm; q)_\infty$ and $(qt^2z_2^\pm; q)_\infty$ are now familiar contributions from the two 3d fundamental twisted hypers obeying Nahm pole boundary condition. We interpret the factors $(q^{\frac{3}{4}}tx^\pm; q)_\infty$ $(q^{\frac{5}{4}}t^3x^\pm; q)_\infty$ as the contributions from the 3d bi-fundamental twisted hyper obeying Nahm pole boundary condition.

5.7.5 4d U(3)|3d (3) – [1]|4d U(3) with $\mathcal{N}'$

In order to gain more insight, let us examine the configuration 4d U(3)|3d (3) – [1]|4d U(3) with $\mathcal{N}'$. The junction has a pair of corners of 4d U(3) gauge theories with Neumann b.c.

$(\mathcal{N}, \mathcal{N}')$ coupled to a boundary of 3d $U(3)$ gauge theory with Neumann b.c. $\mathcal{N}'$. There are two 3d bi-fundamental hypermultiplets which couple to both 4d and 3d gauge theories and a 3d fundamental hypermultiplet that couples to 3d gauge theory. In addition, there are two 2d cross-determinant Fermi multiplets and a 2d fundamental Fermi multiplet.

The quarter-index is given by

$$\begin{aligned}
 \mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d\ U(3)|(3)-[1]|4d\ U(3)} & \quad (5.87) \\
 &= \underbrace{\frac{1}{3!}(q)_\infty^3 \oint \prod_{i=1}^3 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d\ U(3)}} \\
 &\quad \times \underbrace{\frac{1}{3!}(q)_\infty^3 (q^{\frac{1}{2}}t^2; q)_\infty^3 \oint \prod_{i=4}^6 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty \left(q^{\frac{1}{2}}t^2 \frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{III}_{\mathcal{N}'}^{3d\ U(3)}} \\
 &\quad \times \underbrace{\frac{1}{3!}(q)_\infty^3 \oint \prod_{i=7}^9 \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_\infty}_{\mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d\ U(3)}} \\
 &\quad \times \prod_{i=1}^3 \prod_{j=4}^6 \underbrace{\frac{1}{(q^{\frac{1}{4}}ts_i^\pm s_j^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d\ HM}\left(\frac{s_i}{s_j}\right)} \cdot \prod_{i=4}^6 \underbrace{\frac{1}{(q^{\frac{1}{4}}ts_i^\pm; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d\ HM}(s_i)} \cdot \prod_{i=4}^6 \prod_{j=7}^9 \underbrace{\frac{1}{(q^{\frac{1}{4}}ts_i^\pm s_j^\mp; q)_\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d\ HM}\left(\frac{s_i}{s_j}\right)} \\
 &\quad \times \underbrace{(q^{\frac{1}{2}}s_1^\pm s_2^\pm s_3^\pm s_4^\mp s_5^\mp s_6^\mp z_1^\pm; q)_\infty}_{F\left(q^{\frac{1}{2}} \frac{s_1 s_2 s_3}{s_4 s_5 s_6} z_1\right)} \cdot \prod_{i=4}^6 \underbrace{(q^{\frac{1}{2}}s_i^\pm x^\pm; q)_\infty}_{F(q^{\frac{1}{2}}s_i x)} \cdot \underbrace{(q^{\frac{1}{2}}s_4^\pm s_5^\pm s_6^\pm s_7^\mp s_8^\mp s_9^\mp z_2^\pm; q)_\infty}_{F\left(\frac{s_4 s_5 s_6}{s_7 s_8 s_9} z_2\right)}.
 \end{aligned}$$

Similarly, the dual junction has two corners of 4d $U(3)$ gauge theories with a pair of boundary conditions $(\mathcal{N}', \text{Nahm})$. The equal numbers of D3-branes across the two D5'-branes give rise to two 3d fundamental twisted hypers obeying Nahm pole boundary condition with a homomorphism $\rho: \mathfrak{su}(2) \rightarrow \mathfrak{u}(3)$. In addition, there would exist a 3d bi-fundamental twisted hyper obeying Nahm pole boundary condition with an embedding $\rho: \mathfrak{su}(2) \rightarrow \mathfrak{u}(3)$.

We find that the quarter-index (5.87) agrees with

$$\begin{aligned}
 \mathbb{IV}_{\mathcal{N}'\mathcal{D}}^{3|[3]|3} &= \frac{1}{\underbrace{(q^{\frac{1}{2}}t^2; q)_\infty (qt^4; q)_\infty (q^{\frac{3}{2}}t^6; q)_\infty}_{\mathbb{IV}_{\mathcal{N}'\mathcal{D}}^{4d\ U(3)}}} \cdot \frac{1}{\underbrace{(q^{\frac{1}{2}}t^2; q)_\infty (qt^4; q)_\infty (q^{\frac{3}{2}}t^6; q)_\infty}_{\mathbb{IV}_{\mathcal{N}'\mathcal{D}}^{4d\ U(3)}}} \quad (5.88) \\
 &\quad \times (q^{\frac{5}{4}}t^3 z_1^\pm; q)_\infty \cdot (q^{\frac{5}{4}}t^3 z_2^\pm; q)_\infty \cdot (q^{\frac{3}{4}}tx^\pm; q)_\infty \cdot (q^{\frac{5}{4}}t^3 x^\pm; q)_\infty \cdot (q^{\frac{7}{4}}t^5 x^\pm; q)_\infty.
 \end{aligned}$$

As expected from brane picture, this has two quarter-indices of 4d $U(3)$ gauge theory with a pair of boundary conditions $(\mathcal{N}', \mathcal{D})$. The factors $(q^{\frac{5}{4}}t^3 z_1^\pm; q)_\infty$ and $(q^{\frac{3}{4}}t^3 z_2^\pm; q)_\infty$

are contributions from the two 3d fundamental twisted hypers obeying Nahm pole boundary condition. We would like to identify the remaining factors $(q^{\frac{3}{4}}tx^{\pm};q)_{\infty}(q^{\frac{5}{4}}t^3x^{\pm};q)_{\infty}(q^{\frac{7}{4}}t^5x^{\pm};q)_{\infty}$ with the contributions from the 3d bi-fundamental twisted hyper obeying Nahm pole boundary condition specified by an embedding $\rho: \mathfrak{su}(2) \rightarrow \mathfrak{u}(3)$.

5.7.6 4d $U(L)|3d(M) - [1]|4d U(N)$ with $\mathcal{N}'$

Now we would like to give the dualities for the junctions $4d U(L)|3d(M) - [1]|4d U(N)$ with $\mathcal{N}'$ where we assume that $L \geq M$ and $N \geq M$. This junction has a pair of 4d $\mathcal{N} = 4$ $U(L)$ and $U(N)$ gauge theories which obey a pair of Neumann b.c. $(\mathcal{N}, \mathcal{N}')$. Between these corners we have a boundary of 3d $\mathcal{N} = 4$ $U(M)$ gauge theory with Neumann b.c. $\mathcal{N}'$.

There are three 3d hypermultiplets. Under the $U(L) \times U(M) \times U(N)$ gauge symmetry, one transforms as $(\mathbf{L}, \overline{\mathbf{M}}, \mathbf{1}) \oplus (\overline{\mathbf{L}}, \mathbf{M}, \mathbf{1})$ and the other transforms as $(\mathbf{1}, \mathbf{M}, \overline{\mathbf{N}}) \oplus (\mathbf{1}, \overline{\mathbf{M}}, \mathbf{N})$ and the last transforms as $(\mathbf{1}, \mathbf{M}, \mathbf{1}) \oplus (\mathbf{1}, \overline{\mathbf{M}}, \mathbf{1})$. All of them obey the Neumann b.c. $\mathcal{N}'$ due to the NS5'-brane.

There are also three 3d Fermi multiplets. Under the $U(L) \times U(M) \times U(N)$ gauge symmetry, one transforms as $(\det, \det^{-1}, \mathbf{1})$ and the other transforms as $(\mathbf{1}, \det, \det^{-1})$ and the last transforms as $(\mathbf{1}, \mathbf{M}, \mathbf{1})$.

The quarter-index is given by

$$\begin{aligned}
 & \mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d U(L)|(M) - [1]|4d U(N)} \\
 &= \underbrace{\frac{1}{L!}(q)_{\infty}^L \oint \prod_{i=1}^L \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_{\infty}}_{\mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d U(L)}} \\
 & \cdot \underbrace{\frac{1}{M!}(q)_{\infty}^M (q^{\frac{1}{2}}t^2; q)_{\infty}^M \oint \prod_{i=L+1}^{L+M} \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_{\infty} \left(q^{\frac{1}{2}}t^2 \frac{s_i}{s_j}; q \right)_{\infty}}_{\mathbb{III}_{\mathcal{N}'}^{3d U(M)}} \\
 & \times \underbrace{\frac{1}{N!}(q)_{\infty}^N \oint \prod_{i=L+M+1}^{L+M+N} \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{s_i}{s_j}; q \right)_{\infty}}_{\mathbb{IV}_{\mathcal{N}\mathcal{N}'}^{4d U(N)}} \\
 & \times \prod_{i=1}^L \prod_{j=L+1}^{L+M} \underbrace{\frac{1}{(q^{\frac{1}{4}}ts_i^{\pm}s_j^{\mp}; q)_{\infty}}}_{\mathbb{III}_{\mathcal{N}'}^{3d HM}\left(\frac{s_i}{s_j}\right)} \cdot \prod_{i=L+1}^{L+M} \underbrace{\frac{1}{(q^{\frac{1}{4}}ts_i^{\pm}; q)_{\infty}}}_{\mathbb{III}_{\mathcal{N}'}^{3d HM}(s_i)} \cdot \prod_{i=L+1}^{L+M} \prod_{j=L+M+1}^{L+M+N} \underbrace{\frac{1}{(q^{\frac{1}{4}}ts_i^{\pm}s_j^{\mp}; q)_{\infty}}}_{\mathbb{III}_{\mathcal{N}'}^{3d HM}\left(\frac{s_i}{s_j}\right)} \\
 & \times \underbrace{\left(q^{\frac{1}{2}} \prod_{i=1}^L s_i^{\pm} \prod_{j=L+1}^{L+M} s_j^{\mp} z_1^{\pm}; q \right)_{\infty}}_{F\left(q^{\frac{1}{2}} \frac{\prod_{i=1}^L s_i}{\prod_{j=L+1}^{L+M} s_j} z_1\right)} \cdot \underbrace{\prod_{i=L+1}^{L+M} \left(q^{\frac{1}{2}} s_i^{\pm} x^{\pm}; q \right)_{\infty}}_{F(q^{\frac{1}{2}} s_i x)} \cdot \underbrace{\left(q^{\frac{1}{2}} \prod_{i=L+1}^{L+M} s_i^{\pm} \prod_{j=L+M+1}^{L+M+N} s_j^{\mp} z_2^{\pm}; q \right)_{\infty}}_{F\left(q^{\frac{1}{2}} \frac{\prod_{i=L+1}^{L+M} s_i}{\prod_{j=L+M+1}^{L+M+N} s_j} z_2\right)} .
 \end{aligned}
 \tag{5.89}$$

Under the action of S-duality, we obtain the dual configuration. It consists of a pair of corners of 4d gauge theories. According to the D5'-branes, the 4d $U(L)$ gauge symmetry in one corner is broken to $U(M)$ and the 4d $U(N)$ gauge symmetry in the other corner is broken to $U(M)$. As a result, we have a pair of corners of 4d $U(M)$ gauge theories with two boundary conditions $(\mathcal{N}', \text{Nahm})$.

When $L = M$, we have a 3d fundamental twisted hyper with Dirichlet/Nahm b.c. specified by an embedding $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{u}(M)$. Likewise, for $N = M$, we have a 3d fundamental twisted hyper with Dirichlet/Nahm b.c. specified by an embedding $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{u}(M)$.

The junction also would have a 3d twisted hypermultiplet arising from D3-D3 string across the NS5'-brane. Due to the D5-brane, it should be subject to Dirichlet/Nahm b.c. specified by an embedding $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{u}(M)$.

The quarter-index for the dual junction would take the form

$$\begin{aligned} \mathbb{IV}_{\mathcal{N}'\mathcal{D}}^{L|[M]|N} &= \underbrace{\prod_{k=1}^M \frac{1}{(q^{\frac{k}{2}} t^{2k}; q)_{\infty}}}_{\mathbb{IV}_{\mathcal{N}'\mathcal{D}}^{4d \ U(M)}} \cdot \underbrace{\prod_{k=1}^M \frac{1}{(q^{\frac{k}{2}} t^{2k}; q)_{\infty}}}_{\mathbb{IV}_{\mathcal{N}'\mathcal{D}}^{4d \ U(M)}} \\ &\times \left(q^{\frac{3}{4} + \frac{M-1}{4}} t^{1+(M-1)} z_1^{\pm}; q \right)_{\infty}^{\delta^L M} \cdot \left(q^{\frac{3}{4} + \frac{M-1}{4}} t^{1+(M-1)} z_2^{\pm}; q \right)_{\infty}^{\delta^N M} \\ &\cdot \prod_{k=1}^M \left(q^{\frac{1}{4} + \frac{k}{2}} t^{-1+2k} x^{\pm}; q \right)_{\infty}. \end{aligned} \quad (5.90)$$

The expression includes two quarter-indices of 4d $U(M)$ gauge theories with a pair of boundary conditions $(\mathcal{N}', \mathcal{D}/\text{Nahm})$. The factors $(q^{\frac{3}{4} + \frac{M-1}{4}} t^{1+(M-1)} z_1^{\pm}; q)_{\infty}^{\delta^L M}$ and $(q^{\frac{3}{4} + \frac{M-1}{4}} t^{1+(M-1)} z_2^{\pm}; q)_{\infty}^{\delta^N M}$ describe the contributions from the possible 3d fundamental twisted hypers with Dirichlet/Nahm b.c. specified by an embedding $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{u}(M)$. The factors $\prod_{k=1}^M (q^{\frac{1}{4} + \frac{k}{2}} t^{-1+2k} x^{\pm}; q)_{\infty}$ are the expected contributions from a 3d twisted hypermultiplet subject to Dirichlet/Nahm b.c. specified by an embedding $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{u}(M)$.

We expect that the quarter-indices (5.89) and (5.90) would give the same result.

Acknowledgments

The author would like to thank Tudor Dimofte, Sergei Gukov, Miroslav Rapcak, Junya Yagi and Masahito Yamazaki for useful discussions and comments. Especially he is grateful to Davide Gaiotto for sharing many useful ideas, discussing and suggesting the problems. This work is supported in part by Perimeter Institute for Theoretical Physics and JSPS Overseas Research fellowships. Research at Perimeter Institute is supported by the Government of Canada through the Department of Innovation, Science and Economic Development and by the Province of Ontario through the Ministry of Research, Innovation and Science.

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References

- [1] H. Ooguri, Y. Oz and Z. Yin, *D-branes on Calabi-Yau spaces and their mirrors*, *Nucl. Phys. B* **477** (1996) 407 [[hep-th/9606112](#)] [[INSPIRE](#)].
- [2] K. Hori, A. Iqbal and C. Vafa, *D-branes and mirror symmetry*, [hep-th/0005247](#) [[INSPIRE](#)].
- [3] M. Herbst, K. Hori and D. Page, *Phases of $N = 2$ theories in $1 + 1$ dimensions with boundary*, [arXiv:0803.2045](#) [[INSPIRE](#)].
- [4] D. Gaiotto and E. Witten, *Supersymmetric boundary conditions in $N = 4$ super Yang-Mills theory*, *J. Statist. Phys.* **135** (2009) 789 [[arXiv:0804.2902](#)] [[INSPIRE](#)].
- [5] D. Gaiotto and E. Witten, *S-duality of boundary conditions in $N = 4$ super Yang-Mills theory*, *Adv. Theor. Math. Phys.* **13** (2009) 721 [[arXiv:0807.3720](#)] [[INSPIRE](#)].
- [6] K.A. Intriligator and N. Seiberg, *Mirror symmetry in three-dimensional gauge theories*, *Phys. Lett. B* **387** (1996) 513 [[hep-th/9607207](#)] [[INSPIRE](#)].
- [7] M. Henningson, *Boundary conditions for geometric-Langlands twisted $N = 4$ supersymmetric Yang-Mills theory*, *Phys. Rev. D* **86** (2012) 085003 [[arXiv:1106.3845](#)] [[INSPIRE](#)].
- [8] D. Gaiotto, *S-duality of boundary conditions and the geometric Langlands program*, *Proc. Symp. Pure Math.* **98** (2018) 139 [[arXiv:1609.09030](#)] [[INSPIRE](#)].
- [9] T. Dimofte, D. Gaiotto and S. Gukov, *Gauge theories labelled by three-manifolds*, *Commun. Math. Phys.* **325** (2014) 367 [[arXiv:1108.4389](#)] [[INSPIRE](#)].
- [10] Y. Terashima and M. Yamazaki, *$SL(2, R)$ Chern-Simons, Liouville and gauge theory on duality walls*, *JHEP* **08** (2011) 135 [[arXiv:1103.5748](#)] [[INSPIRE](#)].
- [11] S. Cecotti, C. Cordova and C. Vafa, *Braids, walls and mirrors*, [arXiv:1110.2115](#) [[INSPIRE](#)].
- [12] L.F. Alday, D. Gaiotto and Y. Tachikawa, *Liouville correlation functions from four-dimensional gauge theories*, *Lett. Math. Phys.* **91** (2010) 167 [[arXiv:0906.3219](#)] [[INSPIRE](#)].
- [13] A. Gadde, S. Gukov and P. Putrov, *Walls, lines and spectral dualities in 3d gauge theories*, *JHEP* **05** (2014) 047 [[arXiv:1302.0015](#)] [[INSPIRE](#)].
- [14] T. Okazaki and S. Yamaguchi, *Supersymmetric boundary conditions in three-dimensional $N = 2$ theories*, *Phys. Rev. D* **87** (2013) 125005 [[arXiv:1302.6593](#)] [[INSPIRE](#)].
- [15] A. Gadde, S. Gukov and P. Putrov, *Fivebranes and 4-manifolds*, [arXiv:1306.4320](#) [[INSPIRE](#)].
- [16] A. Gadde, S. Gukov and P. Putrov, *$(0, 2)$ trialities*, *JHEP* **03** (2014) 076 [[arXiv:1310.0818](#)] [[INSPIRE](#)].
- [17] S. Chun, S. Gukov and D. Roggenkamp, *Junctions of surface operators and categorification of quantum groups*, [arXiv:1507.06318](#) [[INSPIRE](#)].
- [18] S. Gukov, P. Putrov and C. Vafa, *Fivebranes and 3-manifold homology*, *JHEP* **07** (2017) 071 [[arXiv:1602.05302](#)] [[INSPIRE](#)].
- [19] M. Dedushenko, S. Gukov and P. Putrov, *Vertex algebras and 4-manifold invariants*, in *Proceedings, Nigel Hitchin's 70th birthday conference: geometry and physics. A festschrift in honour of Nigel Hitchin*, volume 1, Oxford University Press, Oxford, U.K. (2018), pg. 249 [[arXiv:1705.01645](#)] [[INSPIRE](#)].

- [20] T. Dimofte and N.M. Paquette, $(0, 2)$ dualities and the 4-simplex, [arXiv:1905.05173](#) [[INSPIRE](#)].
- [21] M. Bullimore, T. Dimofte, D. Gaiotto and J. Hilburn, *Boundaries, mirror symmetry and symplectic duality in 3d $N = 4$ gauge theory*, *JHEP* **10** (2016) 108 [[arXiv:1603.08382](#)] [[INSPIRE](#)].
- [22] H.-J. Chung and T. Okazaki, $(2, 2)$ and $(0, 4)$ supersymmetric boundary conditions in 3d $N = 4$ theories and type IIB branes, *Phys. Rev. D* **96** (2017) 086005 [[arXiv:1608.05363](#)] [[INSPIRE](#)].
- [23] T. Braden, N. Proudfoot and B. Webster, *Quantizations of conical symplectic resolutions I: local and global structure*, *Astérisque* **384** (2016) 1 [[arXiv:1208.3863](#)].
- [24] T. Braden, A. Licata, N. Proudfoot and B. Webster, *Quantizations of conical symplectic resolutions II: category $\mathcal{O}$ and symplectic duality*, [arXiv:1407.0964](#) [[INSPIRE](#)].
- [25] D.S. Berman and D.C. Thompson, *Membranes with a boundary*, *Nucl. Phys. B* **820** (2009) 503 [[arXiv:0904.0241](#)] [[INSPIRE](#)].
- [26] D.S. Berman, M.J. Perry, E. Sezgin and D.C. Thompson, *Boundary conditions for interacting membranes*, *JHEP* **04** (2010) 025 [[arXiv:0912.3504](#)] [[INSPIRE](#)].
- [27] K. Hosomichi and S. Lee, *Self-dual strings and 2D SYM*, *JHEP* **01** (2015) 076 [[arXiv:1406.1802](#)] [[INSPIRE](#)].
- [28] T. Okazaki and D.J. Smith, *Topological M-strings and supergroup Wess-Zumino-Witten models*, *Phys. Rev. D* **94** (2016) 065016 [[arXiv:1512.06646](#)] [[INSPIRE](#)].
- [29] D. Gaiotto and H.-C. Kim, *Duality walls and defects in 5d $N = 1$ theories*, *JHEP* **01** (2017) 019 [[arXiv:1506.03871](#)] [[INSPIRE](#)].
- [30] T. Dimofte, D. Gaiotto and N.M. Paquette, *Dual boundary conditions in 3d SCFT's*, *JHEP* **05** (2018) 060 [[arXiv:1712.07654](#)] [[INSPIRE](#)].
- [31] D. Gaiotto and T. Okazaki, *Dualities of corner configurations and supersymmetric indices*, [arXiv:1902.05175](#) [[INSPIRE](#)].
- [32] T. Okazaki, *Mirror symmetry of 3d $N = 4$ gauge theories and supersymmetric indices*, [arXiv:1905.04608](#) [[INSPIRE](#)].
- [33] C. Beem, M. Lemos, P. Liendo, W. Peelaers, L. Rastelli and B.C. van Rees, *Infinite chiral symmetry in four dimensions*, *Commun. Math. Phys.* **336** (2015) 1359 [[arXiv:1312.5344](#)] [[INSPIRE](#)].
- [34] C. Beem, W. Peelaers, L. Rastelli and B.C. van Rees, *Chiral algebras of class S*, *JHEP* **05** (2015) 020 [[arXiv:1408.6522](#)] [[INSPIRE](#)].
- [35] A. Hanany and T. Okazaki, $(0, 4)$ brane box models, *JHEP* **03** (2019) 027 [[arXiv:1811.09117](#)] [[INSPIRE](#)].
- [36] K. Costello and D. Gaiotto, *Vertex operator algebras and 3d $N = 4$ gauge theories*, *JHEP* **05** (2019) 018 [[arXiv:1804.06460](#)] [[INSPIRE](#)].
- [37] A. Kapustin and E. Witten, *Electric-magnetic duality and the geometric Langlands program*, *Commun. Num. Theor. Phys.* **1** (2007) 1 [[hep-th/0604151](#)] [[INSPIRE](#)].
- [38] D. Gaiotto, *Twisted compactifications of 3d $N = 4$ theories and conformal blocks*, *JHEP* **02** (2019) 061 [[arXiv:1611.01528](#)] [[INSPIRE](#)].

- [39] D. Gaiotto and M. Rapčák, *Vertex algebras at the corner*, *JHEP* **01** (2019) 160 [[arXiv:1703.00982](#)] [[INSPIRE](#)].
- [40] M.C.N. Cheng, S. Chun, F. Ferrari, S. Gukov and S.M. Harrison, *3d modularity*, [arXiv:1809.10148](#) [[INSPIRE](#)].
- [41] B. Feigin and S. Gukov, *VOA[M₄]*, [arXiv:1806.02470](#) [[INSPIRE](#)].
- [42] C. Beem, T. Dimofte and S. Pasquetti, *Holomorphic blocks in three dimensions*, *JHEP* **12** (2014) 177 [[arXiv:1211.1986](#)] [[INSPIRE](#)].
- [43] T. Dimofte, D. Gaiotto and S. Gukov, *3-manifolds and 3d indices*, *Adv. Theor. Math. Phys.* **17** (2013) 975 [[arXiv:1112.5179](#)] [[INSPIRE](#)].
- [44] K. Costello, T. Creutzig and D. Gaiotto, *Higgs and Coulomb branches from vertex operator algebras*, *JHEP* **03** (2019) 066 [[arXiv:1811.03958](#)] [[INSPIRE](#)].
- [45] E. D'Hoker, J. Estes and M. Gutperle, *Exact half-BPS type IIB interface solutions I. Local solution and supersymmetric Janus*, *JHEP* **06** (2007) 021 [[arXiv:0705.0022](#)] [[INSPIRE](#)].
- [46] E. D'Hoker, J. Estes and M. Gutperle, *Exact half-BPS type IIB interface solutions II. Flux solutions and multi-Janus*, *JHEP* **06** (2007) 022 [[arXiv:0705.0024](#)] [[INSPIRE](#)].
- [47] O. Aharony, L. Berdichevsky, M. Berkooz and I. Shamir, *Near-horizon solutions for D3-branes ending on 5-branes*, *Phys. Rev. D* **84** (2011) 126003 [[arXiv:1106.1870](#)] [[INSPIRE](#)].
- [48] B. Assel, C. Bachas, J. Estes and J. Gomis, *Holographic duals of $D = 3$ $N = 4$ superconformal field theories*, *JHEP* **08** (2011) 087 [[arXiv:1106.4253](#)] [[INSPIRE](#)].
- [49] D. Bashkirov and A. Kapustin, *Supersymmetry enhancement by monopole operators*, *JHEP* **05** (2011) 015 [[arXiv:1007.4861](#)] [[INSPIRE](#)].
- [50] D. Gaiotto and E. Witten, *Janus configurations, Chern-Simons couplings, and the θ -angle in $N = 4$ super Yang-Mills theory*, *JHEP* **06** (2010) 097 [[arXiv:0804.2907](#)] [[INSPIRE](#)].
- [51] K. Hosomichi, K.-M. Lee, S. Lee, S. Lee and J. Park, *$N = 4$ superconformal Chern-Simons theories with hyper and twisted hyper multiplets*, *JHEP* **07** (2008) 091 [[arXiv:0805.3662](#)] [[INSPIRE](#)].
- [52] K. Hosomichi, K.-M. Lee, S. Lee, S. Lee and J. Park, *$N = 5, 6$ superconformal Chern-Simons theories and M2-branes on orbifolds*, *JHEP* **09** (2008) 002 [[arXiv:0806.4977](#)] [[INSPIRE](#)].
- [53] T. Okazaki and D.J. Smith, *Mock modular index of M2-M5 brane systems*, *Phys. Rev. D* **96** (2017) 026017 [[arXiv:1612.07565](#)] [[INSPIRE](#)].
- [54] A. Hanany and E. Witten, *Type IIB superstrings, BPS monopoles and three-dimensional gauge dynamics*, *Nucl. Phys. B* **492** (1997) 152 [[hep-th/9611230](#)] [[INSPIRE](#)].
- [55] A. Gadde, B. Haghighat, J. Kim, S. Kim, G. Lockhart and C. Vafa, *6d string chains*, *JHEP* **02** (2018) 143 [[arXiv:1504.04614](#)] [[INSPIRE](#)].
- [56] J. Kim, S. Kim, K. Lee, J. Park and C. Vafa, *Elliptic genus of E-strings*, *JHEP* **09** (2017) 098 [[arXiv:1411.2324](#)] [[INSPIRE](#)].
- [57] P. Putrov, J. Song and W. Yan, *(0, 4) dualities*, *JHEP* **03** (2016) 185 [[arXiv:1505.07110](#)] [[INSPIRE](#)].
- [58] J. Kim, K. Lee and J. Park, *On elliptic genera of 6d string theories*, *JHEP* **10** (2018) 100 [[arXiv:1801.01631](#)] [[INSPIRE](#)].

- [59] L.C. Jeffrey and F.C. Kirwan, *Localization for non-Abelian group actions*, *Topology* **34** (1995) 291.
- [60] F. Benini, R. Eager, K. Hori and Y. Tachikawa, *Elliptic genera of two-dimensional $N = 2$ gauge theories with rank-one gauge groups*, *Lett. Math. Phys.* **104** (2014) 465 [[arXiv:1305.0533](#)] [[INSPIRE](#)].
- [61] F. Benini, R. Eager, K. Hori and Y. Tachikawa, *Elliptic genera of 2d $N = 2$ gauge theories*, *Commun. Math. Phys.* **333** (2015) 1241 [[arXiv:1308.4896](#)] [[INSPIRE](#)].
- [62] A. Kapustin and M.J. Strassler, *On mirror symmetry in three-dimensional Abelian gauge theories*, *JHEP* **04** (1999) 021 [[hep-th/9902033](#)] [[INSPIRE](#)].
- [63] D. Tong, *The holographic dual of $AdS_3 \times S^3 \times S^3 \times S^1$* , *JHEP* **04** (2014) 193 [[arXiv:1402.5135](#)] [[INSPIRE](#)].