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Chiral algebras from Ω-deformation

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Abstract: In the presence of an Ω-deformation, local operators generate a chiral algebra in the topological-holomorphic twist of a four-dimensional $\mathcal{N}=2$ supersymmetric field theory. We show that for a unitary $\mathcal{N}=2$ superconformal field theory, the chiral algebra thus defined is isomorphic to the one introduced by Beem et al. Our definition of the chiral algebra covers nonconformal theories with insertions of suitable surface defects.

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Contents

1 Introduction

A remarkable property of any $\mathcal{N} = 2$ superconformal field theory (SCFT) in four dimensions is that it possesses an infinite-dimensional symmetry in a certain protected sector. This symmetry is generated by local operators which depend holomorphically on a plane, just as the chiral algebra of a two-dimensional conformal field theory is generated by holomorphic currents. Since their discovery by Beem et al. [1], the chiral algebras of $\mathcal{N} = 2$ SCFTs have been studied intensely and shed new light on the physics of these theories, such as unitarity bounds on central charges [1–3].

Prior to the introduction of the chiral algebras in [1], Kapustin [4] had come up with another way to extract the structures of two-dimensional holomorphic field theories from $\mathcal{N} = 2$ supersymmetric field theories. Known as the topological-holomorphic twist, it is a twist [5, 6] of theories placed on a product $\Sigma \times C$ of two surfaces. As the name suggests, it renders the theories topological on Σ and holomorphic on C.

It has been suggested by Kevin Costello (and mentioned in [7]) that an Ω-deformation [8, 9] of Kapustin's construction should give rise to a chiral algebra. Moreover, this chiral algebra has been suspected to be isomorphic to the one introduced in [1]; for standard gauge theories, it has been shown by Dylan Butson [10] that this is indeed the case. One of the primary purposes of this paper is to establish this isomorphism for all unitary $\mathcal{N} = 2$ SCFTs.

To this end we start in two dimensions. In section 2, we formulate Ω-deformations of the topological twists of two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric field theories. It turns out that an Ω-deformation does not really deform a twisted theory on $\mathbb{R}^2$ if the theory before the twist is not only supersymmetric but also conformal: the only effect is that the supercharge Q that one uses to define cohomology is replaced with a linear combination $Q^\hbar$ of Poincaré and conformal supercharges. Furthermore, for a unitary $\mathcal{N} = (2, 2)$ SCFT, we show that the $Q^\hbar$-cohomology of local operators is isomorphic to the cohomology of local operators with respect to another supercharge $\mathcal{Q}^\hbar$ which squares to zero. We also recall a localization formula for Ω-deformed B-twisted gauge theories constructed from vector and chiral multiplets.

All of these results can be applied more or less directly to $\mathcal{N} = 2$ supersymmetric field theories in four dimensions. This is what we do in section 3. The point is that the $\mathcal{N} = 2$ superconformal algebra contains the global $\mathcal{N} = (2, 2)$ superconformal algebra as a subalgebra. Correspondingly, Kapustin's topological-holomorphic twist of an $\mathcal{N} = 2$ supersymmetric field theory on $\Sigma \times C$ may be regarded as the B-twist of an $\mathcal{N} = (2, 2)$ supersymmetric field theory on Σ. This B-twisted theory can be subjected to an Ω-deformation, and for $\Sigma = \mathbb{R}^2$, the $Q^\hbar$-cohomology of local operators is naturally a chiral algebra on C.

If the theory is a unitary $\mathcal{N} = 2$ SCFT on $\mathbb{R}^2 \times \mathbb{C}$, this chiral algebra may be described alternatively as the $\mathcal{Q}^\hbar$-cohomology of local operators. This alternative description turns out to coincide with the definition given in [1]. The isomorphism in question is thus established.

For an $\mathcal{N} = 2$ superconformal gauge theory consisting of vector multiplets and hypermultiplets, the localization formula immediately tells us that the associated chiral algebra is a gauged $\beta\gamma$ system. In fact, the path integral for the four-dimensional theory localizes to that for the gauged $\beta\gamma$ system, placed directly on the holomorphic surface C. This is an advantage of our approach.

Another advantage is that we can define chiral algebras for *nonconformal* theories, provided that suitable surface defects are inserted on C. For our definition of the chiral algebra to work, we only need to ensure that the $\mathcal{N} = 2$ supersymmetric field theory under consideration has a well-defined twisted rotation symmetry on $\mathbb{R}^2$ and an Ω-deformation can be turned on with respect to it. This is guaranteed if the R-symmetry used in the twist along Σ is nonanomalous — a condition which is tied to conformal invariance by $\mathcal{N} = 2$ supersymmetry. However, even if the R-symmetry is anomalous, we can add an unusual θ-term and make the theory invariant under the twisted rotations. The added term breaks the gauge symmetry, but it can be restored by a gauge anomaly from a surface defect which, in general, preserves only $\mathcal{N} = (0, 2)$ supersymmetry on C. In this way we can obtain, for example, all gauged $\beta\gamma$–bc systems from gauge theories with surface defects.

We emphasize that although for specific examples we consider theories with Lagrangian descriptions, the existence of a Lagrangian is not essential to our construction. What is essential is realization of a sensible Ω-deformation on $\mathbb{R}^2$. For $\mathcal{N}=2$ SCFTs, there is a canonical definition whether or not a Lagrangian formulation exists, and this definition, combined with unitarity, yields the chiral algebras of [1].

Lastly, our approach applies equally well to $\mathcal{N}=4$ supersymmetric field theories in three dimensions. As we show in section 4, considerations similar to those described above lead to the conclusion that an Ω-deformation of the Rozansky-Witten twist [11] of an $\mathcal{N}=4$ supersymmetric gauge theory on $\mathbb{R}^2 \times \mathbb{R}$, constructed from vector multiplets and hypermultiplets, is equivalent to a topological gauged quantum mechanics. For $\mathcal{N}=4$ SCFTs, we reproduce in this manner some of the results obtained in [12, 13]. Our approach clarifies the relation between the $\mathcal{Q}^\hbar$-cohomology, studied in [12], and deformation quantizations of the chiral rings.

In appendix A, we describe the supersymmetry transformation laws for vector and chiral multiplets in Ω-deformed topologically twisted $\mathcal{N}=(2,2)$ supersymmetric field theories.

2 Ω-deformations of $\mathcal{N}=(2,2)$ supersymmetric field theories

As explained in the introduction, our approach to the chiral algebras of four-dimensional $\mathcal{N}=2$ supersymmetric field theories is based on Ω-deformations in two dimensions. In this section, we formulate Ω-deformations of topologically twisted $\mathcal{N}=(2,2)$ supersymmetric field theories and SCFTs, and discuss some of their properties that will be important in the application to four-dimensional theories.

2.1 $\mathcal{N}=(2,2)$ supersymmetry and superconformal symmetry

An $\mathcal{N}=(2,2)$ supersymmetric field theory is a two-dimensional quantum field theory that is invariant under a symmetry generated by conserved charges forming the $\mathcal{N}=(2,2)$ supersymmetry algebra. Likewise, an $\mathcal{N}=(2,2)$ SCFT is a theory whose symmetry algebra contains the global $\mathcal{N}=(2,2)$ superconformal algebra.

The global $\mathcal{N}=(2,2)$ superconformal algebra is generated by the generators L_0, $L_{\pm 1}$, $\overline{L}_0$, $\overline{L}_{\pm 1}$ of global conformal transformations and J_0, $\overline{J}_0$ of U(1) R-symmetries, Poincaré supercharges $G^\pm_{-\frac{1}{2}}$, $\overline{G}^\pm_{-\frac{1}{2}}$, and conformal supercharges $G^\pm_{\frac{1}{2}}$, $\overline{G}^\pm_{\frac{1}{2}}$. They satisfy the commutation relations

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n}\,, & [\overline{L}_m, \overline{L}_n] &= (m-n)\overline{L}_{m+n}\,, \\
[L_m, G^\pm_r] &= \Big(\frac{m}{2} - r\Big)G^\pm_{m+r}\,, & [\overline{L}_m, \overline{G}^\pm_r] &= \Big(\frac{m}{2} - r\Big)\overline{G}^\pm_{m+r}\,, \\
[J_0, G^\pm_r] &= \pm G^\pm_r\,, & [\overline{J}_0, \overline{G}^\pm_r] &= \pm \overline{G}^\pm_r\,, \\
\{G^+_r, G^-_s\} &= L_{r+s} + \frac{r-s}{2}J_{r+s}\,, & \{\overline{G}^+_r, \overline{G}^-_s\} &= \overline{L}_{r+s} + \frac{r-s}{2}\overline{J}_{r+s}\,,
\end{aligned} \tag{2.1}$$

where m, $n = 0$, ± 1 and r, $s = \pm\frac{1}{2}$. The other combinations of the generators commute. If we let m, n run over all integers and r, s over all half-integers, then we obtain the local $\mathcal{N}=(2,2)$ superconformal algebra (with vanishing central charges).

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	Q_+	Q_-	$\overline{Q}_+$	$\overline{Q}_-$
M	-1	$+1$	-1	$+1$
F_V	-1	-1	$+1$	$+1$
F_A	-1	$+1$	$+1$	-1
M_A	-2	0	0	$+2$
M_B	-2	$+2$	0	0

Table 1. The spin, R-charges and twisted spins of the supercharges in the $\mathcal{N} = (2,2)$ supersymmetry algebra.

The supercharges are also denoted as

$$Q_+ = \overline{G}^-_{-\frac{1}{2}}\,, \qquad Q_- = G^-_{-\frac{1}{2}}\,, \qquad \overline{Q}_+ = \overline{G}^+_{-\frac{1}{2}}\,, \qquad \overline{Q}_- = G^+_{-\frac{1}{2}}\,, \tag{2.2}$$

$$S_+ = \overline{G}^-_{\frac{1}{2}}\,, \qquad S_- = G^-_{\frac{1}{2}}\,, \qquad \overline{S}_+ = \overline{G}^+_{\frac{1}{2}}\,, \qquad \overline{S}_- = G^+_{\frac{1}{2}}\,. \tag{2.3}$$

In this notation, the subscripts $+$ and $-$ indicate that the supercharges are from the left- and right-moving (or holomorphic and antiholomorphic) sectors, respectively. The supercharges are characterized by their spin M, vector R-charge F_V and axial R-charge F_A, defined by

$$M = 2(L_0 - \overline{L}_0)\,, \qquad F_V = J_0 + \overline{J}_0\,, \qquad F_A = -J_0 + \overline{J}_0\,. \tag{2.4}$$

See table 1 for a summary of the spin and R-charges of the supercharges.

The $\mathcal{N} = (2,2)$ supersymmetry algebra is the subalgebra of the $\mathcal{N} = (2,2)$ superconformal algebra generated by L_{-1}, $\overline{L}_{-1}$, M, $Q_\pm$, $\overline{Q}_\pm$, F_V and F_A. The fact that the $\mathrm{U}(1)_V \times \mathrm{U}(1)_A$ R-symmetry, generated by F_V and F_A, is unbroken implies that central charges do not appear in the algebra, although it is straightforward to incorporate them.

The commutation relations (2.1) are consistent with the hermiticity condition

$$\begin{aligned} (L_n)^\dagger &= L_{-n}\,, & (\overline{L}_n)^\dagger &= \overline{L}_{-n}\,, \\ (G^\pm_r)^\dagger &= G^\mp_{-r}\,, & (\overline{G}^\pm_r)^\dagger &= \overline{G}^\mp_{-r}\,, \\ (J_0)^\dagger &= J_0\,, & (\overline{J}_0)^\dagger &= \overline{J}_0\,. \end{aligned} \tag{2.5}$$

This is the action of hermitian conjugation in radial quantization. The map

$$G^\pm_r \mapsto G^\mp_r\,, \qquad J_0 \mapsto -J_0 \tag{2.6}$$

is an involution on the algebra. On the $\mathcal{N} = (2,2)$ supersymmetry algebra it acts by the exchange

$$Q_- \leftrightarrow \overline{Q}_-\,, \qquad F_V \leftrightarrow F_A\,. \tag{2.7}$$

2.2 Topological twists

A twist of a quantum field theory [5, 6] is an operation which changes the action of the rotation symmetry in such a way that its generator M is shifted to M', differing by a generator F of a global symmetry:

$$M \to M' = M + F\,. \tag{2.8}$$

One way to think about this operation is that it promotes the global symmetry to a local symmetry and identifies the associated gauge field with a multiple of the spin connection. Therefore, a twisted theory is equivalent to the original theory if the spacetime is flat — the twist merely relabels some symmetry generators and changes what we should call the energy-momentum tensor — but the difference becomes important otherwise.

For supersymmetric field theories, twists are particularly interesting when the twisted theories have scalar supercharges. The corresponding fermionic symmetries are parametrized by constants rather than covariantly constant spinors. As such, typically these symmetries can be realized on arbitrary manifolds.

Furthermore, in many cases one finds that a twisted theory has a scalar supercharge Q such that $Q^2 = 0$. Then, one can consider Q-cohomology in the space of operators and the space of states. By a standard argument, Q-invariant quantities in the twisted theory, such as correlation functions of Q-closed operators, depend on the objects involved only through their Q-cohomology classes. Often, the energy-momentum tensor is Q-exact and the diffeomorphisms act trivially in the Q-cohomology. In that situation, the Q-invariant sector of the twisted theory defines a topological quantum field theory.

In the case of $\mathcal{N} = (2, 2)$ supersymmetric field theories, there are essentially two such topological twists. In two dimensions, the rotation group $\mathrm{U}(1)_M$ has a single generator M. In the *A-twist* [6], the twisted rotation generator M' is given by

$$M_A = M + F_V \,, \tag{2.9}$$

and one considers the Q-cohomology with Q being the supercharge

$$Q_A = \overline{Q}_+ + Q_- \,. \tag{2.10}$$

(By these equations we mean that the generators on the left-hand sides are identified with those on the right-hand sides when the spacetime is flat.) In the *B-twist* [14, 15], M' is given by

$$M_B = M + F_A \,, \tag{2.11}$$

and Q is taken to be

$$Q_B = \overline{Q}_+ + \overline{Q}_- \,. \tag{2.12}$$

For either twist, the generators L_{-1}, $\overline{L}_{-1}$ of translations are Q-exact and annihilate Q-cohomology classes. For theories with local $\mathcal{N} = (2, 2)$ superconformal symmetry, the full topological invariance, for an appropriate definition of the energy-momentum tensor, follows from the superconformal algebra.

The Q_A-cohomology is graded by F_A, whereas the Q_B-cohomology is graded by F_V. The two twists are exchanged under the involution (2.7):

$$M_A \leftrightarrow M_B \,, \qquad Q_A \leftrightarrow Q_B \,, \qquad F_V \leftrightarrow F_A \,. \tag{2.13}$$

2.3 Ω-deformations

Consider a twisted supersymmetric field theory with a scalar square-zero supercharge Q, and place it on a spacetime Σ that has a continuous isometry generated by a Killing

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vector field V. An Ω-*deformation* [8, 9] of the twisted theory on Σ with respect to V is a deformation whereby Q is replaced with a scalar supercharge Q_V such that

$$Q_V^2 = L_V \,. \tag{2.14}$$

Here L_V is the conserved charge corresponding to V; on fields, L_V acts as the Lie derivative with respect to V. Slightly more generally, we allow V to be a complex linear combination of Killing vector fields and define the Ω-deformation with respect to it by the same relation.

Since Q_V no longer squares to zero, in order to define its cohomology we must restrict Q_V to V-invariant operators and states. Generically, this means that local operators must be located at zeros of V. This constraint is not a disadvantage compared to the undeformed Q-cohomology if the generators of translations are Q-exact, as the Q-cohomology class of a local operator does not depend on the location then. Rather, the lack of freedom can lead to a richer algebraic structure, as we will see later.

For a topological twist of an $\mathcal{N} = (2,2)$ supersymmetric field theory, an Ω-deformation may be constructed as follows [16, 17].

If Σ is a flat surface, the twisted theory has a one-form supercharge $\mathbf{Q} = \mathbf{Q}_\mu \mathrm{d}x^\mu$ such that

$$\{Q, \mathbf{Q}_\mu\} = \mathrm{i} P_\mu \,, \qquad \{\mathbf{Q}_\mu, \mathbf{Q}_\nu\} = 0 \,, \tag{2.15}$$

where (x^1, x^2) are flat coordinates and P_μ are generators of translations. For $\Sigma = \mathbb{C}$, this supercharge equals

$$\mathbf{Q}_A = -\overline{Q}_- \mathrm{d}z - Q_+ \mathrm{d}\bar{z} \tag{2.16}$$

for the A-twist and

$$\mathbf{Q}_B = -Q_- \mathrm{d}z - Q_+ \mathrm{d}\bar{z} \tag{2.17}$$

for the B-twist. If $V = V^\mu \partial_\mu$ has constant components V^μ, then $\iota_V \mathbf{Q} = V^\mu \mathbf{Q}_\mu$ is a conserved charge and

$$Q_V = Q + \iota_V \mathbf{Q} \tag{2.18}$$

satisfies $Q_V^2 = \iota_V P = L_V$.

After understanding how Q_V transforms the fields on a flat spacetime, we rewrite the transformation law so that it makes sense even when V is a vector field on a surface that is not necessarily flat, and moreover it squares to L_V. (Note that $\mathbf{Q}_\mu$ generally do not exist, and even if they do, $L_V \neq \iota_V P_\mu$ unless V^μ are constants.) Concretely, for vector and chiral multiplets, we are then led to the formulas listed in appendix A.

If the transformation thus obtained is already a symmetry of the twisted theory, the associated conserved charge serves as Q_V. In that event, the Ω-deformation changes the cohomology but not the underlying twisted theory.

In general, this is not what happens unless V is covariantly constant. Yet, when the theory has a standard Lagrangian description and V is a Killing vector field, we can construct a new action functional that is invariant under the Ω-deformed transformation. Roughly speaking, in the new action the Q-exact terms from the original action are replaced by the corresponding Q_V-exact terms. The Q_V-invariance relies on the fact that the action of the twisted theory is constructed with the exterior derivative and the Hodge star operator, acting on fields which are differential forms. Both of these operators commute with the Lie derivative L_V if V generates an isometry, so Q_V annihilates the Q_V-exact terms thanks to the formula $L_V = \mathrm{d}\iota_V + \iota_V \mathrm{d}$ and integration over Σ.

2.4 Ω-deformation of $\mathcal{N} = (2,2)$ SCFTs on $\mathbb{R}^2$

Now suppose that the theory under consideration is an $\mathcal{N} = (2,2)$ SCFT and defined on $\mathbb{R}^2$. In this case, there is a canonical way to make sense of $\iota_V \mathbf{Q}$ itself as a conserved charge. Therefore, the Ω-deformation does not really deform twisted $\mathcal{N} = (2,2)$ SCFTs on $\mathbb{R}^2$; it only changes which cohomology we should take.

This claim is clearly true if V is a generator of translations, so let us consider the case when it generates rotations. Using the angular coordinate θ on $\mathbb{R}^2$ we can write

$$V = 2\mathrm{i}\hbar\partial_\theta \,, \tag{2.19}$$

where we think of $\hbar$ as a complex parameter. We denote the corresponding Ω-deformed supercharge Q_V by $Q^\hbar$. It squares to the twisted rotation generator:

$$(Q^\hbar)^2 = \hbar M' \,. \tag{2.20}$$

The definition of $\iota_V \mathbf{Q}$ we are going to give works for all $\mathcal{N} = (2,2)$ SCFTs. To motivate the definition, however, we first look at theories with local $\mathcal{N} = (2,2)$ superconformal symmetry, such as free vector and free chiral multiplets.

For those theories, we can use a local conformal transformation to map $\mathbb{R}^2$ to a cylinder $\mathbb{R}\times S^1$ plus a point at infinity at one end. Because $\mathbb{R}^2$ and $\mathbb{R}\times S^1$ are flat, on these geometries the twisted and untwisted theories are naturally identified.

On the cylinder V is just a generator of translations, so $\iota_V \mathbf{Q}$ can, and should, be defined as $V^\mu \mathbf{Q}_\mu = 2\mathrm{i}\hbar \mathbf{Q}_\theta$. By the inverse conformal transformation, $\mathbf{Q}_\theta$ is mapped to a linear combination of $G^\pm_r$ and $\overline{G}^\pm_r$, with $r \in \mathbb{Z} + \frac{1}{2}$, in the theory on $\mathbb{R}^2$. Noting that neither $\mathbf{Q}_\theta$ nor the conformal map depends on $\hbar$, we can determine this linear combination by comparing both sides of the relation (2.20) order by order in $\hbar$. We find

$$Q^\hbar = Q + 2\hbar S \,, \tag{2.21}$$

with S given by

$$S_A = \overline{S}_- - S_+ \,, \tag{2.22}$$

$$S_B = S_- - S_+ \tag{2.23}$$

for the A-twist and the B-twist, respectively.

A possibly more direct derivation of the above formulas goes as follows. Recall that the fermionic generators of local $\mathcal{N} = (2,2)$ superconformal symmetry are the modes of holomorphic supercurrents $G^\pm$ and antiholomorphic supercurrents $\overline{G}^\pm$:

$$G^\pm(z) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \frac{G^\pm_r}{z^{r+\frac{3}{2}}} \,, \qquad \overline{G}^\pm(\bar{z}) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \frac{\overline{G}^\pm_r}{\bar{z}^{r+\frac{3}{2}}} \,. \tag{2.24}$$

In terms of these supercurrents, the one-form supercharge, say in the B-twist, is given by

$$\mathbf{Q}_B = -\left(\oint \frac{\mathrm{d}z}{2\pi\mathrm{i}} G^-\right)\mathrm{d}z + \left(\oint \frac{\mathrm{d}\bar{z}}{2\pi\mathrm{i}} \overline{G}^-\right)\mathrm{d}\bar{z}\,, \tag{2.25}$$

where the integration contours are circles of positive orientation centered at the origin. Then, for a vector field V of the form $V(z,\bar{z}) = V^z(z)\partial_z + V^{\bar{z}}(\bar{z})\partial_{\bar{z}}$, it is natural to define

$$\iota_V \mathbf{Q}_B = -\oint \frac{\mathrm{d}z}{2\pi\mathrm{i}} V^z G^- + \oint \frac{\mathrm{d}\bar{z}}{2\pi\mathrm{i}} V^{\bar{z}} \overline{G}^-\,. \tag{2.26}$$

For $V = 2\mathrm{i}\hbar(\mathrm{i}z\partial_z - \mathrm{i}\bar{z}\partial_{\bar{z}})$, this reproduces the formula $\iota_V \mathbf{Q}_B = 2\hbar(S_- - S_+)$.

We see that the resulting formulas for $\iota_V \mathbf{Q}$ for theories on $\mathbb{R}^2$ only involve conformal supercharges in the global $\mathcal{N} = (2,2)$ superconformal algebra. Hence, these formulas also make sense for theories with global but not local $\mathcal{N} = (2,2)$ superconformal symmetry, and can be employed as the definition of $\iota_V \mathbf{Q}$ for all $\mathcal{N} = (2,2)$ SCFTs on $\mathbb{R}^2$.

An important observation about the Ω-deformation of $\mathcal{N} = (2,2)$ SCFTs on $\mathbb{R}^2$, which follows from the formula (2.21) for $Q^\hbar$, is that the $Q^\hbar$-cohomologies for different values of $\hbar$ are all isomorphic as long as $\hbar \neq 0$. This is because $\hbar$ is rescaled by a $\mathbb{C}^\times$-action, generated by F_A in the A-twist and by F_V in the B-twist. The $Q^\hbar$-cohomology is still graded by F_A or F_V if we assign degree 2 to $\hbar$.

2.5 $Q^\hbar$-cohomology of local operators for unitary $\mathcal{N} = (2,2)$ SCFTs

The main object of interest in this paper is the $Q^\hbar$-cohomology in the space of local operators that are located at the origin of $\mathbb{R}^2$ and annihilated by M'. We will refer to it simply as the $Q^\hbar$-cohomology of local operators.

For unitary $\mathcal{N} = (2,2)$ SCFTs, this $Q^\hbar$-cohomology has an alternative description. Let us write $Q^\hbar$ as

$$Q^\hbar = \mathcal{Q}^\hbar + \widetilde{\mathcal{Q}}^\hbar\,, \tag{2.27}$$

where $\mathcal{Q}^\hbar$ and $\widetilde{\mathcal{Q}}^\hbar$ satisfy $(\mathcal{Q}^\hbar)^2 = (\widetilde{\mathcal{Q}}^\hbar)^2 = 0$. Specifically,

$$\mathcal{Q}^\hbar_A = \overline{Q}_+ + 2\hbar\overline{S}_-\,, \qquad \widetilde{\mathcal{Q}}^\hbar_A = Q_- - 2\hbar S_+ \tag{2.28}$$

for the A-twist and

$$\mathcal{Q}^\hbar_B = \overline{Q}_+ + 2\hbar S_-\,, \qquad \widetilde{\mathcal{Q}}^\hbar_B = \overline{Q}_- - 2\hbar S_+ \tag{2.29}$$

for the B-twist. Then, the $Q^\hbar$-cohomology of local operators is isomorphic to the cohomology of local operators at the origin with respect to either $\mathcal{Q}^\hbar$ or $\widetilde{\mathcal{Q}}^\hbar$.

Here we prove the isomorphism for $|\hbar| = 1/4$. Since the $Q^\hbar$-cohomology is independent of $\hbar$, we have no loss of generality.

First of all, note that we can restrict the action of the relevant supercharges to the space of local operators that are located at the origin and have $M' = 0$. For $Q^\hbar$, this restriction is part of the definition of the $Q^\hbar$-cohomology of local operators. For $\mathcal{Q}^\hbar$ and $\widetilde{\mathcal{Q}}^\hbar$, we lose nothing since M' is $\mathcal{Q}^\hbar$- and $\widetilde{\mathcal{Q}}^\hbar$-exact.

By conformal invariance, the space of local operators at the origin is isomorphic as a vector space to the space of states on a circle. A standard argument about the relation between cohomology classes and harmonic states then tells that the cohomology groups with respect to $Q^\hbar$, $\not{Q}^\hbar$ and $\widetilde{\not{Q}}^\hbar$ are isomorphic to the spaces of local operators at the origin with $\{Q^\hbar, (Q^\hbar)^\dagger\} = 0$, $\{\not{Q}^\hbar, (\not{Q}^\hbar)^\dagger\} = 0$ and $\{\widetilde{\not{Q}}^\hbar, (\widetilde{\not{Q}}^\hbar)^\dagger\} = 0$, respectively. The isomorphisms naturally extend to isomorphisms of rings.

The isomorphisms we wish to establish follow from the equalities

$$\{Q^\hbar, (Q^\hbar)^\dagger\} = 2\{\not{Q}^\hbar, (\not{Q}^\hbar)^\dagger\} = 2\{\widetilde{\not{Q}}^\hbar, (\widetilde{\not{Q}}^\hbar)^\dagger\}\,. \tag{2.30}$$

2.6 Localization of Ω-deformed B-twisted gauge theories

A salient feature of topologically twisted theories is localization of the path integral to a simpler, often finite-dimensional, integration over the space of supersymmetric configurations. In the well-known examples of the topological sigma models, the path integral localizes to the space of holomorphic maps in the A-twisted case [6] and the space of constant maps in the B-twisted case [14, 15].

Similar localization takes place for Ω-deformed twisted theories. Here we describe the localization formula for Ω-deformed B-twisted gauge theories [16–19]. See [19] for a derivation and detailed discussions.

Consider an $\mathcal{N} = (2, 2)$ supersymmetric gauge theory, constructed from a vector multiplet for gauge group $\mathcal{G}$ and a chiral multiplet in a representation $\mathcal{R}\colon \mathcal{G} \to \mathrm{GL}(\mathcal{V})$ of $\mathcal{G}$. The superpotential W of the theory is a gauge-invariant holomorphic function on the representation space $\mathcal{V}$, in which the complex scalar field φ of the chiral multiplet takes values. We perform the B-twist, place the twisted theory on $\mathbb{R}^2$, and turn on the Ω-deformation.

The path integral depends on the boundary condition at infinity. For the purpose of localization, a good boundary condition is specified as follows. Let $\mathcal{M}$ be the quotient of $\mathcal{V}$ by the action of the complexified gauge group $\mathcal{G}_\mathbb{C}$. (More precisely, we take $\mathcal{M}$ to be the set of semistable $\mathcal{G}_\mathbb{C}$-orbits.) Since W is $\mathcal{G}$-invariant and holomorphic, it is invariant under the $\mathcal{G}_\mathbb{C}$-action and defines a holomorphic function on $\mathcal{M}$. The moduli space $\mathcal{M}$ is Kähler, and so is the critical locus of W in $\mathcal{M}$. Choose a Lagrangian submanifold $\mathcal{L}_\infty$ of the critical locus. On the boundary, we require φ to lie in $\mathcal{L}_\infty$.

The localized path integral is then

$$\int_\mathcal{L} \mathrm{vol}_\mathcal{L} \exp\left(-\frac{\mathrm{i}\pi}{\hbar} W\right), \tag{2.31}$$

where $\mathrm{vol}_\mathcal{L}$ is the volume form on $\mathcal{L}$. The integration domain $\mathcal{L}$ is the union of all gradient flows generated by $\mathrm{Re}(W/\hbar)$ in $\mathcal{M}$, terminating on $\mathcal{L}_\infty$. This is a Lagrangian submanifold of $\mathcal{M}$.

The above integral may be viewed as the path integral for a zero-dimensional gauged sigma model with gauge group $\mathcal{G}_\mathbb{C}$ and target the preimage of $\mathcal{L}$ under the projection $\mathcal{V} \to \mathcal{M}$. The action function of this zero-dimensional theory is given by the superpotential, and the Ω-deformation parameter plays the role of the Planck constant.

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3 Chiral algebras from Ω-deformation

Now we apply what we have learned in the previous section to $\mathcal{N} = 2$ supersymmetric field theories in four dimensions. The goal of this section is to understand how chiral algebras arise from these theories via the combination of the topological-holomorphic twist and Ω-deformation, and how they are related to the chiral algebras introduced in [1].

3.1 $\mathcal{N} = 2$ supersymmetry and superconformal symmetry

To begin with, we review basic facts about $\mathcal{N} = 2$ supersymmetry and superconformal symmetry in four dimensions. We refer the reader to [1] for more details.

The $\mathcal{N} = 2$ superconformal algebra is generated by the generators $\mathcal{P}_{\alpha\dot{\alpha}}$ of translations, ${\mathcal{M}_\alpha}^\beta$, ${\mathcal{M}^{\dot{\alpha}}}_{\dot{\beta}}$ of rotations, $\mathcal{H}$ of dilatations, $\mathcal{K}^{\dot{\alpha}\alpha}$ of special conformal transformations, ${\mathcal{R}^\mathcal{I}}_\mathcal{J}$ of the R-symmetry group $\mathrm{U}(2)_\mathcal{R} \cong \mathrm{SU}(2)_R \times \mathrm{U}(1)_r$, as well as eight Poincaré supercharges $\mathcal{Q}^\mathcal{I}_\alpha$, $\widetilde{\mathcal{Q}}_{\mathcal{I}\dot{\alpha}}$ and eight conformal supercharges $\mathcal{S}^\alpha_\mathcal{I}$, $\widetilde{\mathcal{S}}^{\mathcal{I}\dot{\alpha}}$. Here $\mathcal{I}$, $\mathcal{J} = 1$, 2 are indices for $\mathrm{U}(2)_\mathcal{R}$ doublets, and α, $\beta = \pm$ and $\dot{\alpha}$, $\dot{\beta} = \pm$ are indices for Weyl spinors.

The $\mathcal{N} = 2$ supersymmetry algebra is the subalgebra of the $\mathcal{N} = 2$ superconformal algebra generated by $\mathcal{P}_{\alpha\dot{\alpha}}$, ${\mathcal{M}_\alpha}^\beta$, ${\mathcal{M}^{\dot{\alpha}}}_{\dot{\beta}}$, ${\mathcal{R}^\mathcal{I}}_\mathcal{J}$, $\mathcal{Q}^\mathcal{I}_\alpha$, $\widetilde{\mathcal{Q}}_{\mathcal{I}\dot{\alpha}}$. The unbroken $\mathrm{U}(1)_r$ symmetry implies that the central charge vanishes.

We list some of the commutation relations in the $\mathcal{N} = 2$ superconformal algebra which will be of particular importance in the subsequent discussions. The nonvanishing commutators between the supercharges are

$$
\begin{aligned}
\{\mathcal{Q}^\mathcal{I}_\alpha, \widetilde{\mathcal{Q}}_{\mathcal{J}\dot{\alpha}}\} &= \delta^\mathcal{I}_\mathcal{J} \mathcal{P}_{\alpha\dot{\alpha}} \,, \\
\{\widetilde{\mathcal{S}}^{\mathcal{I}\dot{\alpha}}, \mathcal{S}^\alpha_\mathcal{J}\} &= \delta^\mathcal{I}_\mathcal{J} \mathcal{K}^{\dot{\alpha}\alpha} \,, \\
\{\mathcal{Q}^\mathcal{I}_\alpha, \mathcal{S}^\beta_\mathcal{J}\} &= \frac{1}{2}\delta^\mathcal{I}_\mathcal{J}\delta^\beta_\alpha \mathcal{H} + \delta^\mathcal{I}_\mathcal{J} {\mathcal{M}_\alpha}^\beta - \delta^\beta_\alpha {\mathcal{R}^\mathcal{I}}_\mathcal{J} \,, \\
\{\widetilde{\mathcal{S}}^{\mathcal{I}\dot{\alpha}}, \widetilde{\mathcal{Q}}_{\mathcal{J}\dot{\beta}}\} &= \frac{1}{2}\delta^\mathcal{I}_\mathcal{J} {\delta^{\dot{\alpha}}}_{\dot{\beta}} \mathcal{H} + \delta^\mathcal{I}_\mathcal{J} {\mathcal{M}^{\dot{\alpha}}}_{\dot{\beta}} + \delta^{\dot{\alpha}}_{\dot{\beta}} {\mathcal{R}^\mathcal{I}}_\mathcal{J} \,.
\end{aligned}
\tag{3.1}
$$

Rotations act on the Poincaré supercharges as

$$
\begin{aligned}
[{\mathcal{M}_\alpha}^\beta, \mathcal{Q}^\mathcal{I}_\gamma] &= \delta^\beta_\gamma \mathcal{Q}^\mathcal{I}_\alpha - \frac{1}{2}\delta^\beta_\alpha \mathcal{Q}^\mathcal{I}_\gamma \,, \\
[{\mathcal{M}^{\dot{\alpha}}}_{\dot{\beta}}, \widetilde{\mathcal{Q}}_{\mathcal{I}\dot{\gamma}}] &= \delta^{\dot{\alpha}}_{\dot{\gamma}} \widetilde{\mathcal{Q}}_{\mathcal{I}\dot{\beta}} - \frac{1}{2}\delta^{\dot{\alpha}}_{\dot{\beta}} \widetilde{\mathcal{Q}}_{\mathcal{I}\dot{\gamma}} \,,
\end{aligned}
\tag{3.2}
$$

whereas R-symmetry transformations act on them as

$$
\begin{aligned}
[{\mathcal{R}^\mathcal{I}}_\mathcal{J}, \mathcal{Q}^\mathcal{K}_\alpha] &= \delta^\mathcal{K}_\mathcal{J} \mathcal{Q}^\mathcal{I}_\alpha - \frac{1}{4}\delta^\mathcal{I}_\mathcal{J} \mathcal{Q}^\mathcal{K}_\alpha \,, \\
[{\mathcal{R}^\mathcal{I}}_\mathcal{J}, \widetilde{\mathcal{Q}}_{\mathcal{K}\dot{\alpha}}] &= -\delta^\mathcal{I}_\mathcal{K} \widetilde{\mathcal{Q}}_{\mathcal{J}\dot{\alpha}} + \frac{1}{4}\delta^\mathcal{I}_\mathcal{J} \widetilde{\mathcal{Q}}_{\mathcal{K}\dot{\alpha}} \,.
\end{aligned}
\tag{3.3}
$$

Their action on the conformal supercharges may be found from the hermiticity condition in radial quantization:

$$
\begin{aligned}
\mathcal{H}^\dagger = \mathcal{H} \,, \qquad (\mathcal{P}_{\alpha\dot{\alpha}})^\dagger &= \mathcal{K}^{\dot{\alpha}\alpha} \,, \qquad ({\mathcal{M}_\alpha}^\beta)^\dagger = {\mathcal{M}_\beta}^\alpha \,, \qquad ({\mathcal{M}^{\dot{\alpha}}}_{\dot{\beta}})^\dagger = {\mathcal{M}^{\dot{\beta}}}_{\dot{\alpha}} \,, \\
({\mathcal{R}^\mathcal{I}}_\mathcal{J})^\dagger &= {\mathcal{R}^\mathcal{J}}_\mathcal{I} \,, \qquad (\mathcal{Q}^\mathcal{I}_\alpha)^\dagger = \mathcal{S}^\alpha_\mathcal{I} \,, \qquad (\widetilde{\mathcal{Q}}_{\mathcal{I}\dot{\alpha}})^\dagger = \widetilde{\mathcal{S}}^{\mathcal{I}\dot{\alpha}} \,.
\end{aligned}
\tag{3.4}
$$

	$\mathcal{Q}^1_+$	$\mathcal{Q}^1_-$	$\widetilde{\mathcal{Q}}_{1\dot{+}}$	$\widetilde{\mathcal{Q}}_{1\dot{-}}$	$\mathcal{Q}^2_+$	$\mathcal{Q}^2_-$	$\widetilde{\mathcal{Q}}_{2\dot{+}}$	$\widetilde{\mathcal{Q}}_{2\dot{-}}$
M_Σ	$+1$	-1	-1	$+1$	$+1$	-1	-1	$+1$
M_C	$+1$	-1	$+1$	-1	$+1$	-1	$+1$	-1
$2r$	$+1$	$+1$	-1	-1	$+1$	$+1$	-1	-1
$2R$	$+1$	$+1$	-1	-1	-1	-1	$+1$	$+1$
M'_Σ	$+2$	0	-2	0	$+2$	0	-2	0
M'_C	$+2$	0	0	-2	0	-2	$+2$	0

Table 2. The spins, R-charges and twisted spins of the Poincaré supercharges. Those of the conformal supercharges are opposite of their hermitian conjugates'.

3.2 Kapustin's topological-holomorphic twist

In [4], Kapustin introduced a twist of an $\mathcal{N} = 2$ supersymmetric field theory which is applicable when the theory is placed on a product $\Sigma \times C$ of two surfaces. Upon this twist the theory becomes topological on Σ and holomorphic on C. For this reason, Kapustin's twist is called a topological-holomorphic twist.

Let us choose a local frame on the spinor bundle on $\Sigma \times C$ in such a way that

$$M_\Sigma = 2(\mathcal{M}_+{}^+ - \mathcal{M}^{\dot{+}}{}_{\dot{+}}) \tag{3.5}$$

is a generator of rotations on Σ, while

$$M_C = 2(\mathcal{M}_+{}^+ + \mathcal{M}^{\dot{+}}{}_{\dot{+}}) \tag{3.6}$$

generates rotations on C. The topological-holomorphic twist replaces these generators with

$$M'_\Sigma = M_\Sigma + 2r\,, \tag{3.7}$$

$$M'_C = M_C + 2R\,, \tag{3.8}$$

where we have introduced r and R by

$$\mathcal{R}^1{}_1 = \frac{r}{2} + R\,, \qquad \mathcal{R}^2{}_2 = \frac{r}{2} - R\,. \tag{3.9}$$

The R-charges r and R are generators of $\mathrm{U}(1)_r$ and the diagonal subgroup $\mathrm{U}(1)_R$ of $\mathrm{SU}(2)_R$, respectively. Table 2 summarizes how the supercharges transform under the action of these U(1) charges.

The twisted theory has two scalar supercharges $\mathcal{Q}^1_-$ and $\widetilde{\mathcal{Q}}_{2\dot{-}}$. Both of these square to zero and they commute with each other, so we can consider the cohomology with respect to the linear combination

$$Q = \mathcal{Q}^1_- + t\widetilde{\mathcal{Q}}_{2\dot{-}} \tag{3.10}$$

for any $t \in \mathbb{C} \cup \{\infty\}$. However, the cohomologies for all values of t other than 0 or ∞ are isomorphic via the $\mathbb{C}^\times$-action generated by r. It is convenient to set

$$t = 1\,. \tag{3.11}$$

The momenta $\mathcal{P}_{+\dot{-}}$, $\mathcal{P}_{-\dot{+}}$ and $\mathcal{P}_{-\dot{-}}$ are Q-exact. From their commutators with M'_Σ and M'_C, one finds that the first two are generators of translations on Σ, whereas the last one is a generator of antiholomorphic translations on C. Therefore, Q-cohomology classes of local operators are independent of their positions on Σ and depend holomorphically on C. For a gauge theory with a standard Lagrangian formulation, one can moreover show that the action is independent of the metric on Σ, up to Q-exact terms. In this sense, the twisted theory is topological on Σ and holomorphic on C. We will call it a topological-holomorphic theory on $\Sigma \times C$.

3.3 Ω-deformed topological-holomorphic theories and chiral algebras

As it is, the algebra of local operators of the topological-holomorphic theory is not very interesting because the product of two local operators does not contain any singularities. This is a consequence of Hartogs's extension theorem, which states that for $n \geq 2$, a holomorphic function on $U \setminus V$ can be extended to a holomorphic function on U, where U is an open subset of $\mathbb{C}^n$ and V is a compact subset of U such that $U \setminus V$ is connected.

The problem is that the topological-holomorphic theory has the topological surface Σ on which local operators can move freely. In order to obtain an operator product structure that allows singularities, we must make a modification to the topological-holomorphic theory that eliminates this freedom and turns local operators into holomorphic functions of a single variable.

For $\Sigma = \mathbb{R}^2$, this is precisely what an Ω-deformation does: it requires local operators to stay at the origin. Thus, via the Ω-deformation, the topological-holomorphic theory on $\mathbb{R}^2 \times C$ produces a *chiral algebra* on C, that is, the algebra of local operators in a holomorphic field theory on C.

More precisely, this Ω-deformation of the topological-holomorphic theory for $\Sigma = \mathbb{R}^2$ is parametrized by $\hbar \in \mathbb{C}$, and deforms Q to $Q^\hbar$ satisfying

$$(Q^\hbar)^2 = \hbar M'_\Sigma \,. \tag{3.12}$$

We assume that the generator of holomorphic translations on C remains $Q^\hbar$-closed and that of antiholomorphic translations $Q^\hbar$-exact. (The generators, however, may be shifted from $\mathcal{P}_{+\dot{+}}$ and $\mathcal{P}_{-\dot{-}}$.) Under this assumption, the $Q^\hbar$-cohomology, taken in the space of rotation invariant local operators placed at the origin of $\mathbb{R}^2$, defines the chiral algebra of the theory.

To understand this construction better, let us describe the topological-holomorphic twist in two steps. In the first step we perform a twist along C, replacing M_C with M'_C. After that, we twist along Σ.

The first step turns the supercharges $\mathcal{Q}^1_-$ $\widetilde{\mathcal{Q}}_{1\dot{+}}$, $\mathcal{Q}^2_+$, $\widetilde{\mathcal{Q}}_{2\dot{-}}$ into scalars on C, thereby enabling them to be unbroken even when C is curved. Two of them have $M_\Sigma = +1$ and the other two have $M_\Sigma = -1$. Together with $\mathcal{P}_{+\dot{-}}$, $\mathcal{P}_{-\dot{+}}$, M_Σ, r and R, these supercharges generate two-dimensional supersymmetry, namely the $\mathcal{N} = (2,2)$ supersymmetry algebra on Σ.

The relation between the four-dimensional generators and the two-dimensional ones can be determined as follows. Suppose that the theory contains a vector multiplet. If we

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twist and reduce the theory on C, we obtain an $\mathcal{N}=(2,2)$ supersymmetric theory on Σ. In this process, the two real scalars in the vector multiplet become the two real scalars in an $\mathcal{N}=(2,2)$ vector multiplet.[1] Since complex linear combinations of the former scalars have $(R,r)=(0,\pm 1)$, while complex linear combinations of the latter have $(F_V,F_A)=(0,\pm 2)$, we learn that $F_A=\pm 2r$ up to generators that act trivially on these scalars. The choice of the sign is a matter of convention, so let us pick $+$. Requiring that two supercharges have $F_A=+1$ and the other two have $F_A=-1$, we find $F_A=2r+\alpha M'_C$ for some constant α. Looking at the values of M_Σ and F_A for the supercharges, we deduce the identification

$$Q_+=\widetilde{\mathcal{Q}}_{1\dot{+}}\,,\qquad Q_-=\mathcal{Q}^2_+\,.\qquad \overline{Q}_+=\mathcal{Q}^1_-\,,\qquad \overline{Q}_-=\widetilde{\mathcal{Q}}_{2\dot{-}}\,,\tag{3.13}$$

from which it also follows that $F_V=2R+\beta M'_C$ for some β. Finally, from the commutation relations of the supercharges we get

$$L_{-1}=\mathcal{P}_{+\dot{-}}\,,\qquad \overline{L}_{-1}=\mathcal{P}_{-\dot{+}}\,.\tag{3.14}$$

As far as the $\mathcal{N}=(2,2)$ supersymmetry algebra is concerned, we can always shift U(1) R-charges by generators of global U(1) symmetries commuting with the supercharges. Thus, we simply postulate $F_A=2r$; as we will see in section 3.4, this relation is consistent with what we find in the superconformal case. Then, the second step in the topological-holomorphic twist replaces M_Σ with $M'_\Sigma=M_\Sigma+F_A$. This is the B-twist of the $\mathcal{N}=(2,2)$ supersymmetry on Σ. With our choice $t=1$, the supercharge used for the cohomology is

$$Q_{\mathrm{K}}=\mathcal{Q}^1_-+\widetilde{\mathcal{Q}}_{2\dot{-}}\tag{3.15}$$

and coincides with the B-twist supercharge Q_B.

The above consideration shows that the relevant Ω-deformation is a four-dimensional counterpart of the one for B-twisted theories in two dimensions. For theories constructed from vector multiplets and hypermultiplets, one may obtain formulas for the Ω-deformation by lifting the two-dimensional formulas listed in appendix A to four dimensions. A similar lift was considered in [19] in the context of a topological-holomorphic twist of the six-dimensional maximally supersymmetric Yang-Mills theory.

3.4 Chiral algebras of $\mathcal{N}=2$ SCFTs

If the theory is superconformal and placed on $\Sigma\times C=\mathbb{R}^2\times\mathbb{C}$, we do not only get $\mathcal{N}=(2,2)$ supersymmetry on Σ. In this situation, we actually get global $\mathcal{N}=(2,2)$ superconformal symmetry. Indeed, the conformal supercharges $\mathcal{S}^1_-$ $\widetilde{\mathcal{S}}_{1\dot{+}}$, $\mathcal{S}^2_+$, $\widetilde{\mathcal{S}}_{2\dot{-}}$, which are scalars with respect to M'_C, can be identified with the two-dimensional conformal supercharges as

$$S_+=\mathcal{S}^-_1\,,\qquad S_-=\widetilde{\mathcal{S}}^{2\dot{-}}\,.\qquad \overline{S}_+=\widetilde{\mathcal{S}}^{1\dot{+}}\,,\qquad \overline{S}_-=\mathcal{S}^+_2\,.\tag{3.16}$$

[1]Here we are choosing to describe the two-dimensional theory using a vector multiplet as opposed to a twisted vector multiplet. This is different from the choice made in [20, 21]. As a result, our identification of the two- and four-dimensional supercharges differs from theirs by the involution (2.7).

The identification of the remaining generators are as follows:

$$\begin{aligned} L_0 &= \frac{1}{2}\mathcal{H} + \frac{1}{4}M_\Sigma\,, & \overline{L}_0 &= \frac{1}{2}\mathcal{H} - \frac{1}{4}M_\Sigma\,, \\ L_1 &= \mathcal{K}^{\dot{-}+}\,, & \overline{L}_1 &= \mathcal{K}^{\dot{+}-}\,, \\ J_0 &= \frac{1}{2}M'_C - r + R\,, & \overline{J}_0 &= \frac{1}{2}M'_C + r + R\,. \end{aligned} \tag{3.17}$$

In particular, we have

$$F_V = 2R + M'_C\,, \qquad F_A = 2r\,. \tag{3.18}$$

The hermiticity conditions are consistent in two and four dimensions.

Since the theory has $\mathcal{N} = (2,2)$ superconformal symmetry on $\mathbb{R}^2$, the Ω-deformation of the topological-holomorphic theory can be achieved by the procedure described in section 2.4. As explained there, in the superconformal case the Ω-deformation just changes the supercharge with respect to which the cohomology is defined. According to the formula (2.23), the Ω-deformed supercharge is

$$Q^\hbar_{\mathrm{K}} = \mathcal{Q}^1_- + \widetilde{\mathcal{Q}}_{2\dot{-}} + 2\hbar(\widetilde{\mathcal{S}}^{2\dot{-}} - \mathcal{S}_1^-)\,, \tag{3.19}$$

which indeed satisfies the relation (3.12).

For a unitary SCFT, the chiral algebra defined here is isomorphic to the one introduced in [1]. As we have seen in section 2.5, in the unitary case the $Q^\hbar_{\mathrm{K}}$-cohomology of local operators is isomorphic to the cohomology of local operators at the origin with respect to either $\mathscr{Q}^\hbar_{\mathrm{K}}$ or $\widetilde{\mathscr{Q}}^\hbar_{\mathrm{K}}$. In the present case we have

$$\mathscr{Q}^\hbar_{\mathrm{K}} = \mathcal{Q}^1_- + 2\hbar\widetilde{\mathcal{S}}^{2\dot{-}}\,, \qquad \widetilde{\mathscr{Q}}^\hbar_{\mathrm{K}} = \widetilde{\mathcal{Q}}_{2\dot{-}} - 2\hbar\mathcal{S}_1^-\,. \tag{3.20}$$

These are the supercharges used in the construction of the chiral algebra in [1].

The generator $\mathcal{P}_{+\dot{+}}$ of holomorphic translations on C is $Q^\hbar_{\mathrm{K}}$-closed, which is part of the assumption we have made. The generator of antiholomorphic translations is $Q^\hbar_{\mathrm{K}}$-exact, provided that it is shifted from $\mathcal{P}_{-\dot{-}}$ to $\mathcal{P}_{-\dot{-}} + 2\hbar\mathcal{R}^2{}_1$:

$$\{Q^\hbar_{\mathrm{K}}, \widetilde{\mathcal{Q}}^{1\dot{-}}\} = \{Q^\hbar_{\mathrm{K}}, \mathcal{Q}^2_-\} = \mathcal{P}_{-\dot{-}} + 2\hbar\mathcal{R}^2{}_1\,. \tag{3.21}$$

Since $\widetilde{\mathcal{Q}}^{1\dot{-}}$ and $\mathcal{Q}^2_-$ are the only fermionic generators with $(M'_\Sigma, M'_C) = (0,-2)$, there is no other candidate for a $Q^\hbar_{\mathrm{K}}$-exact generator of antiholomorphic translations.

In fact, $\mathcal{P}_{-\dot{-}} + 2\hbar\mathcal{R}^2{}_1$ is the only bosonic generator of the $\mathcal{N} = 2$ superconformal algebra such that it commutes with $Q^\hbar_{\mathrm{K}}$, reduces to $\mathcal{P}_{-\dot{-}}$ for $\hbar = 0$, and has the correct values $(M'_\Sigma, M'_C) = (0,-2)$ and $(F_V, F_A) = (-2,0)$ to represent $\partial_{\bar{w}}$ up to an overall factor. At first order in $\hbar$, the correction to $\mathcal{P}_{-\dot{-}}$ is given by a bosonic generator of the $\mathcal{N} = 2$ supersymmetry algebra that has $(M'_\Sigma, M'_C) = (0,-2)$ and $(F_V, F_A) = (-4,0)$. (Recall that $\hbar$ has $F_V = 2$ in the B-twist.) The only such generator is $\mathcal{R}^2{}_1$, and the commutativity with $Q^\hbar_{\mathrm{K}}$ fixes the coefficient of the first order correction. Higher order corrections are absent as there are no generators with $F_V \leq -6$. By the same token, $\mathcal{P}_{+\dot{+}}$ is the only $Q^\hbar_{\mathrm{K}}$-closed bosonic generator that can represent ∂_w. This argument shows that if we use a set of field variables on which the action of ∂_w and $\partial_{\bar{w}}$ commute with $Q^\hbar_{\mathrm{K}}$, then generically, $\mathcal{P}_{+\dot{+}}$ and $\mathcal{P}_{-\dot{-}} + 2\hbar\mathcal{R}^2{}_1$ generate translations along C.

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3.5 Vector multiplets and hypermultiplets

As an example, let us consider an $\mathcal{N}=2$ supersymmetric gauge theory constructed from a vector multiplet for a gauge group G and a hypermultiplet in a representation $R_H \colon G \to \mathrm{GL}(V)$ of G. For this theory, the chiral algebra arising from the Ω-deformation of the topological-holomorphic twist has been found by Butson [10] to be isomorphic to the one defined in [1].

Here we derive the same result by localization. The idea is to view the topological-holomorphic theory on $\mathbb{R}^2 \times C$ as a B-twisted gauge theory on $\mathbb{R}^2$, and apply the localization formula explained in section 2.6.

The gauge group $\mathcal{G}$ of this two-dimensional theory is the space of maps from C to G; its elements are four-dimensional gauge transformations that are constant on $\mathbb{R}^2$. The $\mathcal{N}=2$ vector multiplet splits into an $\mathcal{N}=(2,2)$ vector multiplet and an $\mathcal{N}=(2,2)$ chiral multiplet in the adjoint representation of $\mathcal{G}$. The complex scalar in the adjoint chiral multiplet is $A_{\bar{w}}$, which is annihilated by Q_{K} and has $(F_V, F_A) = (-2, 0)$.

The $\mathcal{N}=2$ hypermultiplet consists of a pair of $\mathcal{N}=1$ chiral multiplets valued in R_H and its dual $R_H^\vee$. From the two-dimensional point of view, it is a pair of $\mathcal{N}=(2,2)$ chiral multiplets valued in $\mathcal{R}_H$ and $\mathcal{R}_H^\vee$, where $\mathcal{R}_H \colon \mathcal{G} \to \mathrm{GL}(\mathcal{V})$ is the representation of $\mathcal{G}$ induced from R_H on the space $\mathcal{V}$ of maps from C to V. The complex scalars q and $\tilde{q}$ of these multiplets have $(F_V, F_A) = (2, 0)$. Under the topological-holomorphic twist, they become spinors on C with $M'_C = 1$, that is, sections of $K_C^{1/2}$.

If we wish to place the hypermultiplet on any choice of C, it is necessary to further twist the theory along C so that all fields become differential forms rather than spinors. For this twist we can use the $\mathrm{U}(1)$ symmetry which commutes with the supercharges and assigns q and $\tilde{q}$ opposite charges. We choose to make q and $\tilde{q}$ have $M'_C = 0$ and 2, respectively. To emphasize this change we rename them as

$$\beta = \tilde{q}\,, \qquad \gamma = q\,. \tag{3.22}$$

Thus, β is a $(1,0)$-form valued in $R_H^\vee$ and γ is a scalar valued in R_H on C.

In order to apply the localization formula, we need to determine the superpotential W of the two-dimensional theory. In the present setup, W is a gauge-invariant holomorphic functional of $(A_{\bar{w}}, q, \tilde{q})$ with $(F_V, F_A) = (2, 0)$. Furthermore, it must be the integral over C of a local density which is of first order in derivatives on C so that the Lagrangian is local and of second order. The only such functional, up to an overall factor, is

$$W = \int_C \beta \bar{\partial}_A \gamma\,, \tag{3.23}$$

where $\bar{\partial}_A = d\bar{w}(\partial_{\bar{w}} + A_{\bar{w}})$ is the gauge covariant Dolbeault operator on C. The prefactor is irrelevant as it can be absorbed by a rescaling of β and γ.

Now we turn on the Ω-deformation. According to the localization formula, the $Q_{\mathrm{K}}^\hbar$-invariant sector of the Ω-deformed B-twisted gauge theory on $\mathbb{R}^2$ is equivalent to a zero-dimensional gauge theory whose gauge group is $\mathcal{G}_{\mathbb{C}}$. The action of the theory (including the Planck constant) is $W/\hbar$. In the case at hand, this is a functional of fields that are maps from C to V, $V^\vee$ or the Lie algebra $\mathfrak{g}_\mathbb{C}$ of $G_\mathbb{C}$. Hence, the localized theory is really a quantum field theory on C. The algebra of local operators of this two-dimensional theory is the chiral algebra in question.

The chiral algebra does not depend on the global structures of C, so let us take $C = \mathbb{C}$. Then, by a gauge transformation we can set[2]

$$A_{\bar{w}} = 0\,. \tag{3.24}$$

The corresponding ghost action is

$$\frac{1}{\hbar}\int_C \mathrm{Tr}\bigl(b \wedge \bar{\partial}_A c + \mathsf{B} \wedge A\bigr)\,, \tag{3.25}$$

where c is a scalar and b, B are $(1,0)$-forms on C, all valued in the adjoint representation of G. Integrating out the auxiliary field B produces a delta function in $A_{\bar{w}}$ which imposes the gauge fixing condition, while the integration over b and c gives the associated Faddeev-Popov determinant.

Instead of integrating B out, let us first integrate over $A_{\bar{w}}$ to set

$$\mathsf{B} = -\eta^{ab}\beta_i (T_b)^i{}_j \gamma^j T_a + \{b, c\}\,. \tag{3.26}$$

Here we have chosen a basis $\{T_a\}_{a=1}^{\dim\mathfrak{g}}$ for $\mathfrak{g}$ and denoted by η^{ab} the (a,b) component of the inverse of the Killing form; also, we have used i, j for indices for the representations R_H and $R_H^\vee$. The action of the localized theory becomes

$$\frac{1}{\hbar}\int_C (\beta\bar{\partial}\gamma + b\bar{\partial}c)\,. \tag{3.27}$$

As explained in section 2.4, $\hbar$ can be rescaled by the $\mathbb{C}^\times$-action generated by F_V, and we see this property reflected in the fact that the action is quadratic. We made use of this rescaling when we identified the superpotential.

The chiral algebra described by the above action is the gauged $\beta\gamma$ system, which has the operator product expansions (OPEs)

$$\beta_j(w)\gamma^i(w') \sim -\frac{\hbar\delta^i_j \mathrm{d}w}{w - w'}\,, \tag{3.28}$$

$$b^a(w)c^b(w') \sim \frac{\hbar\eta^{ab}\,\mathrm{d}w}{w - w'} \tag{3.29}$$

and the BRST charge

$$Q_{\mathrm{BRST}} = \frac{1}{2\pi\mathrm{i}\hbar}\oint \bigl(-\beta c\gamma + \mathrm{Tr}(bcc)\bigr)\,. \tag{3.30}$$

[2]At first sight it may appear that holomorphic gauge transformations leave the gauge fixing condition intact. This is not the case because A is the gauge field for the compact gauge group G and obeys the hermiticity condition $A_w = -(A_{\bar{w}})^\dagger$. The residual gauge symmetry is therefore given by the constant gauge transformations. This is a global symmetry, with respect to which we do not quotient out the field space. Accordingly, we remove the constant gauge transformations from the BRST procedure, which means that the ghost c does not contain the zero mode. The ghosts without the zero mode of c form the so-called small bc system [22].

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Essentially the same result was derived by localization computations on S^4 [21] and $S^3 \times S^1$ [23, 24].

Computing the double contractions in the OPE of two BRST currents, we find an anomaly in the nilpotency of $Q_{\rm BRST}$:

$$Q_{\rm BRST}^2 = \bigl(\mathrm{Tr}_{\rm adj}(T_a T_b) - \mathrm{Tr}_{R_H}(T_a T_b)\bigr)\partial_w c^a c^b\,, \tag{3.31}$$

where $\mathrm{Tr}_{\rm adj}$ and Tr_{R_H} denote trace taken in the adjoint representation and in R_H, respectively. The factor in the parentheses on the right-hand side vanishes precisely when $\mathrm{U}(1)_r$ is nonanomalous. Indeed, under the action of $e^{2\mathrm{i}\alpha r} \in \mathrm{U}(1)_r$, the fermionic path integral measure changes by the phase factor

$$\exp\biggl(-\frac{\mathrm{i}\alpha}{2\pi^2}\int_{\mathbb{R}^2\times C}\bigl(\mathrm{Tr}_{\rm adj}(T_a T_b) - \mathrm{Tr}_{R_H}(T_a T_b)\bigr)F^a \wedge F^b\biggr)\,. \tag{3.32}$$

The appearance of the $\mathrm{U}(1)_r$ anomaly is natural from the perspective of the topological-holomorphic twist; as $\mathrm{U}(1)_r$ is used in the twist, it had better be well defined. The same condition also dictates the vanishing of the one-loop beta function of the theory. From the point of view of the untwisted theory on $\mathbb{R}^4$, this is necessary because the construction of [1] fails if the conformal symmetry is broken.

We remark that a care must be taken when we identify the fields in the chiral algebra and those in the underlying $\mathcal{N} = 2$ SCFT. In fact, the gauged $\beta\gamma$ system is written in terms of field variables on which the antiholomorphic derivative $\partial_{\bar w}$ on C is represented by the $Q_{\rm K}^\hbar$-exact operator $\mathcal{P}_{\dot{-}\dot{-}} + 2\hbar\mathcal{R}^2{}_1$, not $\mathcal{P}_{\dot{-}\dot{-}}$. The reason is that in describing the $\mathcal{N} = 2$ SCFT as a two-dimensional theory, we have treated the coordinates on C as "flavor indices" so that the theory is formulated as an Ω-deformed $\mathcal{N} = (2, 2)$ supersymmetric gauge theory in the standard manner, with integration over C taking the place of summation over the flavor indices. In particular, we have demanded that $\partial_{\bar w}$ commutes with $Q_{\rm K}^\hbar$, which requires the relation between $\partial_{\bar w}$ and $\mathcal{P}_{\dot{-}\dot{-}}$ to be corrected, as explained at the end of section 3.4. As a result, the identification (3.22) is modified in the Ω-deformed setting by conjugation by $\exp(2\hbar\bar w\mathcal{R}^2{}_1)$:

$$\beta = \tilde q - 2\hbar\bar w q^\dagger\,, \qquad \gamma = q + 2\hbar\bar w\tilde q^\dagger\,. \tag{3.33}$$

These are called twisted-translated operators in [1]. The $\hbar$-corrections with explicit position dependence are crucial for reproducing the correct OPEs within the $\mathcal{N} = 2$ SCFT.

3.6 Adding surface defects

The above story can be further enriched by introduction of surface defects. Adding a surface defect that covers the holomorphic surface C modifies the chiral algebra of the theory.

As we did in section 3.3, let us think of the topological-holomorphic twist as two successive two-dimensional twists. This time, however, we reverse the order and first perform the twist along $\Sigma = \mathbb{R}^2$. From the values of M'_Σ for the supercharges, we see that after this first twist we are left with $\mathcal{N} = (0, 4)$ supersymmetry on C. The second twist then gives a twisted version of $\mathcal{N} = (0, 4)$ supersymmetry.

The supercharge $Q_{\rm K}$ relevant for the chiral algebra actually belongs to the smaller subalgebra commuting with M'_Σ, generated by $Q_{\rm K}$, $\mathcal{Q}^2_- + \widetilde{\mathcal{Q}}_{1\dot{-}}$, $\mathcal{P}_{-\dot{-}}$, M_C and R. This subalgebra is isomorphic to the $\mathcal{N} = (0, 2)$ supersymmetry algebra. In the case of $\mathcal{N} = 2$ SCFTs, it is further contained in the global $\mathcal{N} = (0, 2)$ superconformal algebra, with the identification

$$\begin{aligned} &\overline{L}_0 = \frac{1}{2}\mathcal{H} - \frac{1}{4}M_C\,, &&\overline{L}_{-1} = (\overline{L}_1)^\dagger = \mathcal{P}_{-\dot{-}}\,, &&\overline{J}_0 = 2R\,, \\ &\overline{G}^+_{-\frac{1}{2}} = (\overline{G}^-_{\frac{1}{2}})^\dagger = \frac{1}{\sqrt{2}}Q_{\rm K}\,, &&\overline{G}^-_{-\frac{1}{2}} = (\overline{G}^+_{\frac{1}{2}})^\dagger = \frac{1}{\sqrt{2}}(\mathcal{Q}^2_- + \widetilde{\mathcal{Q}}_{1\dot{-}})\,. \end{aligned} \tag{3.34}$$

Hence, we may take an $\mathcal{N} = (0, 2)$ supersymmetric field theory in two dimensions, twist it, and couple it to the topological-holomorphic theory along $\{0\} \times C \subset \mathbb{R}^2 \times C$. From the four-dimensional viewpoint, the two-dimensional theory creates a surface defect at the origin of $\mathbb{R}^2$. Note that in general it preserves M'_Σ but not M_Σ and r separately. We let M'_Σ act trivially in the two-dimensional theory

The Ω-deformation replaces $Q_{\rm K}$ with $Q^\hbar_{\rm K}$ and $\mathcal{P}_{-\dot{-}}$ with $\mathcal{P}_{-\dot{-}} + 2\hbar\mathcal{R}^2{}_1$, but together with other generators they still generate a twisted $\mathcal{N} = (0, 2)$ supersymmetry subalgebra inside the two-dimensional theory where $M'_\Sigma = 0$. The $Q^\hbar_{\rm K}$-cohomology of local operators of the twisted $\mathcal{N} = (0, 2)$ supersymmetric field theory is a chiral algebra of its own. Therefore, the $Q^\hbar_{\rm K}$-cohomology of local operators of the coupled system is a chiral algebra constructed from those of the four-dimensional theory and the two-dimensional theory.

As an example, take the gauge theory considered in section 3.5 as the four-dimensional theory, and couple to it an $\mathcal{N} = (0, 2)$ chiral multiplet in a representation R_C and a Fermi multiplet in a representation R_F of the gauge group G. The coupling is done via gauging. The chiral algebras associated to the $\mathcal{N} = (0, 2)$ multiplets are readily identified [25, 26]. The chiral multiplet yields the $\beta\gamma$ system in the representation R_C. In terms of the complex scalar ϕ of the multiplet, the fields of this system are

$$\mathsf{\beta} = \partial\bar{\phi}\,, \qquad \mathsf{\gamma} = \phi\,. \tag{3.35}$$

The Fermi multiplet gives the bc system in the representation R_F, whose fields b and c are identified with the left-moving fermions of the multiplet. The chiral algebra of the coupled system is therefore the BRST reduction of the $\beta\gamma$ system in the representation $R_H \oplus R_C$ plus the bc system in the representation $\mathrm{adj} \oplus R_F$, governed by the action

$$\frac{1}{\hbar}\int_C (\beta\bar{\partial}\gamma + b\bar{\partial}c + \mathsf{\beta}\bar{\partial}\mathsf{\gamma} + \mathsf{b}\bar{\partial}\mathsf{c})\,, \tag{3.36}$$

with the BRST charge given by

$$Q_{\rm BRST} = \frac{1}{2\pi i\hbar}\oint \bigl(-\beta c\gamma + \mathrm{Tr}(bcc) - \mathsf{\beta} c\mathsf{\gamma} + \mathsf{b}c\mathsf{c}\bigr)\,. \tag{3.37}$$

The surface defect has its own contribution to the anomaly in the BRST symmetry. The BRST charge of the coupled system satisfies the relation

$$Q^2_{\rm BRST} = \bigl(\mathrm{Tr}_{\mathrm{adj}\oplus R_F}(T_aT_b) - \mathrm{Tr}_{R_H\oplus R_C}(T_aT_b)\bigr)\partial_w c^a c^b\,. \tag{3.38}$$

Assuming that the four-dimensional theory has no $\mathrm{U}(1)_r$ anomaly, the vanishing of Q^2_{BRST} requires

$$\mathrm{Tr}_{R_C}(T_a T_b) - \mathrm{Tr}_{R_F}(T_a T_b) = 0\,. \tag{3.39}$$

This is the condition for the absence of gauge anomaly in the two-dimensional theory.

While the absence of the BRST anomaly is indispensable for the construction of the chiral algebra, $\mathrm{U}(1)_R$ does not have to be anomaly free. Even though M'_C fails to be conserved if $\mathrm{U}(1)_R$ is anomalous, the chiral algebra can be defined without it. This is to be contrasted with M'_Σ which makes an appearance in $(Q^\hbar_{\mathrm{K}})^2$. Anomalies in $\mathrm{U}(1)_R$ may originate from nonperturbative effects in the two-dimensional theory, and break $\mathrm{U}(1)_R$ down to a finite subgroup. Such a situation was studied in [25, 27–29] in the context of $\mathcal{N} = (0,2)$ supersymmetric sigma models, for which $\mathrm{U}(1)_R$ anomalies are linked to conformal anomalies due to the curvature of the target space. It was shown in [29] that the chiral algebras of models with $\mathrm{U}(1)_R$ anomalies lack an energy-momentum tensor. In extreme cases, the entire chiral algebra vanish because worldsheet instantons make the identity operator Q-exact [27–29].

3.7 Nonconformal theories with surface defects

In view of the relation (3.38), the condition $Q^2_{\mathrm{BRST}} = 0$ does not require the $\mathrm{U}(1)_r$ anomaly of the four-dimensional theory and the gauge anomaly of the two-dimensional theory to vanish separately. All it asks is that the contributions to Q^2_{BRST} from these anomalies cancel out. This observation suggests that it may be possible to define chiral algebras even for theories with nonvanishing one-loop beta function if suitable surface defects are inserted.[3] As we now show, this is in fact true. In particular, we can obtain any nonanomalous gauged $\beta\gamma$–bc system from an $\mathcal{N} = 2$ supersymmetric gauge theory.

The key point is that $\mathrm{U}(1)_r$ anomalies by themselves are not fundamental obstructions to the existence of the chiral algebra. Rather, the real problem is that the twisted rotation symmetry on $\Sigma = \mathbb{R}^2$ is anomalous, which means that $Q^\hbar_{\mathrm{K}}$ is not conserved and the Ω-deformation cannot be turned on. *Assuming* that the rotation symmetry on Σ is unbroken, $\mathrm{U}(1)_r$ anomalies imply that the twisted rotation symmetry is also anomalous. In principle, however, $M'_\Sigma = M_\Sigma + 2r$ may be preserved while neither M_Σ nor r is.

We claim that this can be achieved by addition of the following term to the action of the four-dimensional theory:

$$-\frac{\mathrm{i}}{4\pi^2}\int_{\mathbb{R}^2\times C} \mathrm{d}\theta\big(\mathrm{CS}_{\mathrm{adj}}(A) - \mathrm{CS}_{R_H}(A)\big)\,, \tag{3.40}$$

where

$$\mathrm{CS}_R(A) = \mathrm{Tr}_R\left(A\wedge \mathrm{d}A + \frac{2}{3}A\wedge A\wedge A\right) \tag{3.41}$$

[3] This possibility was suggested to us by Kevin Costello. A three-dimensional analog of this mechanism has been considered in [7]. A similar construction has also appeared in [30] in the context of BPS strings in six-dimensional SCFTs.

is the Chern-Simons three-form with the trace taken in the representation R. Formally, we can rewrite this term as

$$\frac{\mathrm{i}}{4\pi^2}\int_{\mathbb{R}^2\times C}\theta\big(\mathrm{Tr}_{\mathrm{adj}}(F\wedge F)-\mathrm{Tr}_{R_H}(F\wedge F)\big)\,. \tag{3.42}$$

This is much like the θ-term, but here θ is not a parameter: it is literally the angle coordinate on $\mathbb{R}^2$. The second expression makes it clear that although this term is multivalued, its exponential is not.

Because of the explicit dependence on θ, the above term breaks the rotation symmetry on $\mathbb{R}^2$, unless the theory is superconformal in which case this term simply vanishes. The variation of the term under the action of $e^{\mathrm{i}\alpha M_\Sigma}\in \mathrm{U}(1)_\Sigma$ cancels the anomalous phase factor (3.32) of the fermionic path integral measure, thereby restoring the twisted rotation symmetry as desired. We expect that the anomaly in $Q^\hbar_{\mathrm{K}}$ is also canceled by the $Q^\hbar_{\mathrm{K}}$-variation of this term; although we are not able to prove it here, this is very plausible given the cancellation of the anomaly in M'_Σ.

The added term, however, is not gauge invariant. Under the gauge transformation $A\mapsto A+\mathrm{d}_A\chi$, it changes by

$$-\frac{\mathrm{i}}{4\pi^2}\int_{\mathbb{R}^2\times C}\mathrm{d}\theta\,\mathrm{d}\big(\mathrm{Tr}_{\mathrm{adj}}(\chi\,\mathrm{d}A)-\mathrm{Tr}_{R_H}(\chi\,\mathrm{d}A)\big)\,. \tag{3.43}$$

The integrand is not a total derivative since θ is not single valued; it satisfies $\mathrm{d}^2\theta=2\pi\delta_{0\in\mathbb{R}^2}$, with $\delta_{0\in\mathbb{R}^2}$ being the delta two-form on $\mathbb{R}^2$ supported at 0. Therefore, the gauge variation of the added term is localized at the origin of $\mathbb{R}^2$:

$$-\frac{\mathrm{i}}{2\pi}\int_{\{0\}\times C}\big(\mathrm{Tr}_{\mathrm{adj}}(\chi\,\mathrm{d}A)-\mathrm{Tr}_{R_H}(\chi\,\mathrm{d}A)\big)\,. \tag{3.44}$$

Since the integral is now performed over $\{0\}\times C$, it can be canceled by an anomaly inflow [31] from a two-dimensional object that produces an equal and opposite gauge anomaly.

The last integral is precisely of the form that the gauge anomalies of two-dimensional chiral fermions take. A left-moving fermion in a representation R induces the anomalous phase factor

$$\exp\left(-\frac{\mathrm{i}}{4\pi}\int_{\{0\}\times C}\mathrm{Tr}_R(\chi\,\mathrm{d}A)\right) \tag{3.45}$$

in the path integral measure. An $\mathcal{N}=(0,2)$ chiral multiplet contains a pair of right-moving fermions, while an $\mathcal{N}=(0,2)$ Fermi multiplet has a pair of left-moving fermions. Thus, the gauge variation is canceled by the gauge anomaly of a surface defect constructed from a chiral multiplet valued in R_C and a Fermi multiplet valued in R_F if and only if

$$\exp\left(-\frac{\mathrm{i}}{2\pi}\int_{\{0\}\times C}\big(\mathrm{Tr}_{\mathrm{adj}\oplus R_F}(T_aT_b)-\mathrm{Tr}_{R_H\oplus R_C}(T_aT_b)\big)(\chi^a\mathrm{d}A^b)\right)=1 \tag{3.46}$$

for any gauge transformation parameter χ. This is equivalent to the condition for Q_{BRST} to square to zero.

4 Topological quantum mechanics from three-dimensional $\mathcal{N} = 4$ supersymmetric field theories

As dimensional reduction of the $\mathcal{N} = 2$ superconformal algebra in four dimensions yields the $\mathcal{N} = 4$ superconformal algebra in three dimensions, much of the analysis from the previous section carries over to three-dimensional theories. We briefly discuss how some known results about $\mathcal{N} = 4$ SCFTs can be understood in our approach. We follow the conventions of [12], except that we rescale the supercharges by a factor of 1/2.

The R-symmetry group of the $\mathcal{N} = 4$ superconformal algebra is $\mathrm{SU}(2)_H \times \mathrm{SU}(2)_C$. In the dimensional reduction picture, $\mathrm{SU}(2)_H$ comes from $\mathrm{SU}(2)_R$ in four dimensions, whereas the diagonal subgroup of $\mathrm{SU}(2)_C$ can be identified with $\mathrm{U}(1)_r$. We denote the generators of $\mathrm{SU}(2)_H$ by $R^a{}_b$ and $\mathrm{SU}(2)_C$ by $\widetilde{R}^{\tilde{a}}{}_{\tilde{b}}$, and those of the rotation group $\mathrm{SU}(2)_M$ by $M^\alpha{}_\beta$. The eight Poincaré supercharges $Q^{\alpha a\tilde{a}}$ and the eight conformal supercharges $S^{\alpha a\tilde{a}}$ of the $\mathcal{N} = 4$ superconformal algebra transform in the trifundamental representation of $\mathrm{SU}(2)_M \times \mathrm{SU}(2)_H \times \mathrm{SU}(2)_C$. The remaining generators are $P_{\alpha\beta}$ of translations, D of dilatations and $K^{\alpha\beta}$ of special conformal transformations.

The supercharges satisfy the commutation relations

$$\begin{aligned}
\{Q^{a\tilde{a}}_\alpha, Q^{b\tilde{b}}_\beta\} &= \frac{1}{2}\epsilon^{ab}\epsilon^{\tilde{a}\tilde{b}}P_{\alpha\beta}\,,\\
\{S^\alpha_{a\tilde{a}}, S^\beta_{b\tilde{b}}\} &= \frac{1}{2}\epsilon_{ab}\epsilon_{\tilde{a}\tilde{b}}K^{\alpha\beta}\,,\\
\{Q^{a\tilde{a}}_\alpha, S^\beta_{b\tilde{b}}\} &= \frac{1}{2}\delta^a_b\delta^{\tilde{a}}_{\tilde{b}}(M_\alpha{}^\beta + \delta^\beta_\alpha D) - \frac{1}{2}\delta^\beta_\alpha(R^a{}_b\delta^{\tilde{a}}_{\tilde{b}} + \delta^a{}_b\widetilde{R}^{\tilde{a}}{}_{\tilde{b}})\,,
\end{aligned} \tag{4.1}$$

and in radial quantization obey the hermiticity condition

$$(Q^{a\tilde{a}}_\alpha)^\dagger = S^\alpha_{a\tilde{a}}\,. \tag{4.2}$$

To locate the $\mathcal{N} = (2,2)$ superconformal algebra inside the $\mathcal{N} = 4$ superconformal algebra, we embed $\mathrm{U}(1)_M \times \mathrm{U}(1)_V \times \mathrm{U}(1)_A$ into $\mathrm{SU}(2)_M \times \mathrm{SU}(2)_H \times \mathrm{SU}(2)_C$. As $\mathrm{U}(1)_V$ and $\mathrm{U}(1)_A$ are identified with the diagonal subgroup of $\mathrm{SU}(2)_R$ and $\mathrm{U}(1)_r$ in four dimensions, we take

$$M = 2M^+{}_+\,, \qquad F_V = 2R^1{}_1\,, \qquad F_A = 2\widetilde{R}^{\tilde{1}}{}_{\tilde{1}}\,. \tag{4.3}$$

Then we have

$$Q_+ = Q^{2\tilde{2}}_+\,, \qquad Q_- = Q^{2\tilde{1}}_-\,, \qquad \overline{Q}_+ = Q^{1\tilde{1}}_+\,, \qquad \overline{Q}_- = Q^{1\tilde{2}}_-\,, \tag{4.4}$$

$$S_+ = S^+_{1\tilde{1}}\,, \qquad S_- = S^-_{1\tilde{2}}\,, \qquad \overline{S}_+ = S^+_{2\tilde{2}}\,, \qquad \overline{S}_- = S^-_{2\tilde{1}}\,. \tag{4.5}$$

The commutation relations between the supercharges fix the identification for the bosonic generators:

$$\begin{aligned}
L_{-1} &= -\frac{1}{2}P_{--}\,, & \overline{L}_{-1} &= \frac{1}{2}P_{++}\,,\\
L_0 &= \frac{1}{2}(M^+{}_+ + D)\,, & \overline{L}_0 &= \frac{1}{2}(-M^+{}_+ + D)\,,\\
L_1 &= -\frac{1}{2}K^{--}\,, & \overline{L}_1 &= \frac{1}{2}K^{++}\,,\\
J_0 &= R^1{}_1 - \widetilde{R}^{\tilde{1}}{}_{\tilde{1}}\,, & \overline{J}_0 &= R^1{}_1 + \widetilde{R}^{\tilde{1}}{}_{\tilde{1}}\,.
\end{aligned} \tag{4.6}$$

Now, consider the topological-holomorphic twist of an $\mathcal{N} = 2$ supersymmetric field theory on $\Sigma \times C$. If we take $C = \mathbb{R} \times S^1$ and perform dimensional reduction on S^1, we obtain a topological twist of an $\mathcal{N} = 4$ supersymmetric field theory on $\Sigma \times \mathbb{R}$. This sets the derivative along S^1 to zero, hence turns the antiholomorphic derivative $\partial_{\bar{w}}$ on C into the derivative ∂_t along $\mathbb{R}$. The relation $\partial_{\bar{w}} = 0$ in the topological-holomorphic theory on $\Sigma \times C$ then implies that we have $\partial_t = 0$ after the reduction. Therefore, the resulting twisted theory on $\Sigma \times \mathbb{R}$ is fully topological.

There are two topological twists of a three-dimensional $\mathcal{N} = 4$ supersymmetric field theory. In the *Rozansky-Witten twist* [11], the rotation group $\mathrm{SU}(2)_M$ is replaced with the diagonal subgroup of $\mathrm{SU}(2)_M \times \mathrm{SU}(2)_C$ and Q is taken to be a linear combination of two supercharges that are singlets under the twisted rotation group:

$$(M_{\mathrm{RW}})^{\alpha}{}_{\beta} = M^{\alpha}{}_{\beta} + \widetilde{R}^{\tilde{\alpha}}{}_{\tilde{\beta}}\,, \tag{4.7}$$

$$Q_{\mathrm{RW}} = Q_+^{1\tilde{1}} + Q_-^{1\tilde{2}}\,. \tag{4.8}$$

(Here we are using the indices $(+,-)$ and $(1,2)$ interchangeably.) In the *mirror Rozansky-Witten twist* [32, 33], we use $\mathrm{SU}(2)_H$ to twist the rotation group:

$$(M_{\mathrm{mRW}})^{\alpha}{}_{\beta} = M^{\alpha}{}_{\beta} + R^{\alpha}{}_{\beta}\,, \tag{4.9}$$

$$Q_{\mathrm{mRW}} = Q_+^{1\tilde{1}} + Q_-^{2\tilde{1}}\,. \tag{4.10}$$

Under the identification (4.4) we have

$$Q_A = Q_{\mathrm{mRW}}\,, \tag{4.11}$$

$$Q_B = Q_{\mathrm{RW}}\,. \tag{4.12}$$

Thus, from the point of view of the $\mathcal{N} = (2,2)$ supersymmetry on Σ, the Rozansky-Witten twist is the B-twist and the mirror Rozansky-Witten twist is the A-twist. The topological-holomorphic twist reduces to the Rozansky-Witten twist.

The Q-cohomology of local operators in a topological twist of an $\mathcal{N} = 4$ supersymmetric field theory is called the *chiral ring* of the twisted theory. It is necessarily commutative since there is no invariant notion of operator ordering in three dimensions (or for that matter, in any dimension greater than one). For $\Sigma = \mathbb{R}^2$, an Ω-deformation introduces just such an ordering by forcing local operators to line up on $\{0\} \times \mathbb{R}$. Therefore, the $Q^{\hbar}$-cohomology of local operators is a quantization of the chiral ring. Some aspects of the Ω-deformed chiral rings of $\mathcal{N} = 4$ supersymmetric field theories were studied in [33–40].

For the Rozansky-Witten twist of an $\mathcal{N} = 4$ supersymmetric gauge theory constructed from vector multiplets and hypermultiplets, we can easily identify the Ω-deformed chiral ring. The dimensional reduction of the localization formula discussed in section 3.5 shows that the Ω-deformed chiral ring is the algebra of local operators in the $G_{\mathbb{C}}$-gauged quantum mechanics, described by the action

$$\frac{1}{\hbar}\int_{\mathbb{R}} (\tilde{q}\mathrm{d}_A q)\,. \tag{4.13}$$

The Hamiltonian for this quantum mechanical system vanishes as a consequence of the topological invariance on $\mathbb{R}$. This result was also obtained in [13] via localization on S^3. The case of sigma models has been studied by localization in [16], and from the point of view of secondary products and equivariant homology in [39].

For a unitary $\mathcal{N} = 4$ SCFT on $\mathbb{R}^2 \times \mathbb{R}$, the $Q^{\hbar}_{\rm RW}$-cohomology of local operators coincides with the $\mathcal{Q}^{\hbar}_{\rm RW}$-cohomology of local operators at the origin of $\mathbb{R}^2$. According to the formula (2.29), we have

$$\mathcal{Q}^{\hbar}_{\rm RW} = Q^{1\tilde{1}}_{+} + 2\hbar S^{-}_{1\tilde{2}}\,. \tag{4.14}$$

This is the supercharge used in [12] to define a noncommutative operator algebra. There, it was found that this algebra is a deformation quantization of the chiral ring for the Rozansky-Witten twist. The above localization result, and the results obtained in [39], explain why this is true.

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A Ω-deformations for vector and chiral multiplets

In this appendix, we describe Ω-deformations of the topological twists of $\mathcal{N} = (2,2)$ supersymmetric gauge theories constructed from vector multiplets and charged chiral multiplets.

In the A-twist, a vector multiplet consists of a gauge field A, a complex scalar σ, an auxiliary scalar D, fermionic scalars α, β and a fermionic one-form λ:

$$\sigma\,, \mathsf{D} \in \Omega^0\,, \quad A \in \Omega^1\,, \quad \alpha\,, \beta \in \Pi\Omega^0\,, \quad \lambda \in \Pi\Omega^1\,. \tag{A.1}$$

Here Ω^p is the space of p-forms on the spacetime surface and Π denotes parity reversal. A chiral multiplet (with $F_V = 0$) consists of a complex scalar φ, an auxiliary one-form F, fermionic scalars η, $\bar{\eta}$ and a fermionic one-form ψ:

$$\varphi \in \Omega^0\,, \quad \mathsf{F} \in \Omega^1\,, \quad \eta\,, \bar{\eta} \in \Pi\Omega^0\,, \quad \psi \in \Pi\Omega^1\,. \tag{A.2}$$

The fields in the vector multiplet are valued in the adjoint representation of the gauge group, whereas those in the chiral multiplet are valued in some representation.

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The Ω-deformed supercharge Q_V acts on the vector multiplet by the transformation

$$\begin{aligned} &\delta_V A = \lambda\,, && \delta_V \lambda = \iota_V F - \mathrm{d}_A \sigma\,, \\ &\delta_V \sigma = -\iota_V \lambda\,, && \delta_V \alpha = \iota_V \mathrm{d}_A \bar{\sigma} + [\sigma, \bar{\sigma}] \\ &\delta_V \bar{\sigma} = \alpha\,, && \delta_V \beta = \mathsf{D}\,, \\ &\delta_V \mathsf{D} = \iota_V \mathrm{d}_A \beta + [\sigma, \beta]\,, && \end{aligned} \tag{A.3}$$

where $F = \mathrm{d}A + A \wedge A$ is the curvature of A. On the chiral multiplet, Q_V acts by

$$\begin{aligned} &\delta_V \varphi = \eta\,, && \delta_V \bar{\varphi} = \bar{\eta}\,, \\ &\delta_V \eta = \iota_V \mathrm{d}_A \varphi + \sigma \phi\,, && \delta_V \bar{\eta} = \iota_V \mathrm{d}_A \bar{\phi} + \sigma \bar{\phi}\,, \\ &\delta_V \psi = \mathsf{F}\,, && \delta_V \mathsf{F} = (\mathrm{d}_A \iota_V + \iota_V \mathrm{d}_A)\psi + \sigma \psi\,. \end{aligned} \tag{A.4}$$

The square of this transformation is the gauge covariant Lie derivative with respect to V plus the infinitesimal gauge transformation generated by σ:

$$\delta_V^2 = \mathrm{d}_A \iota_V + \iota_V \mathrm{d}_A + \delta_\sigma\,. \tag{A.5}$$

For the B-twist, a vector multiplet contains a one-form σ rather than a complex scalar, and a two-form fermion ζ instead of a scalar fermion:

$$\mathsf{D} \in \Omega^0\,, \quad A\,, \sigma \in \Omega^1\,, \quad \alpha \in \Pi\Omega^0\,, \quad \lambda \in \Pi\Omega^1\,, \quad \zeta \in \Pi\Omega^2\,, \tag{A.6}$$

A chiral multiplet (with $F_A = 0$) consists of

$$\varphi \in \Omega^0\,, \quad \mathsf{F} \in \Omega^2\,, \quad \bar{\eta} \in \Pi\Omega^0\,, \quad \rho \in \Pi\Omega^1\,, \quad \bar{\mu} \in \Pi\Omega^2\,, \tag{A.7}$$

where F is a complex two-form.

The action of Q_V is given by

$$\begin{aligned} &\delta_V \mathcal{A} = \iota_V \zeta\,, && \delta_V \overline{\mathcal{A}} = \lambda - \iota_V \zeta\,, \\ &\delta_V \lambda = 2\iota_V F - 2\mathrm{i} D \iota_V \sigma\,, && \delta_V \zeta = \mathcal{F}\,, \\ &\delta_V \alpha = \mathsf{D}\,, && \delta_V \mathsf{D} = \iota_V \mathcal{D} \alpha \end{aligned} \tag{A.8}$$

for the vector multiplet and

$$\begin{aligned} &\delta_V \varphi = \iota_V \rho\,, && \delta_V \bar{\varphi} = \bar{\eta}\,, \\ &\delta_V \rho = \mathcal{D}\varphi + \iota_V \mathsf{F}\,, && \delta_V \bar{\eta} = \iota_V \mathcal{D} \bar{\varphi}\,, \\ &\delta_V \mathsf{F} = \mathcal{D}\rho - \zeta\varphi\,, && \delta_V \overline{\mathsf{F}} = \mathcal{D} \iota_V \bar{\mu}\,, \\ & && \delta_V \bar{\mu} = \overline{\mathsf{F}}\,. \end{aligned} \tag{A.9}$$

for the chiral multiplet. Its square is the covariant Lie derivative with respect to the complexified gauge field $\mathcal{A} = A + \mathrm{i}\sigma$:

$$\delta_V^2 = \mathrm{d}_{\mathcal{A}} \iota_V + \iota_V \mathrm{d}_{\mathcal{A}}\,. \tag{A.10}$$

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