Linear Chern-Simons-matter theories in the planar limit

Takao Suyama

*KEK Theory Center, High Energy Accelerator Research Organization (KEK),
Oho 1-1, Tsukuba, Ibaraki 305-0801, Japan*

E-mail: tsuyama@post.kek.jp

ABSTRACT: We study $\mathcal{N} = 3$ linear Chern-Simons-matter theories in the planar limit. The matter content of the theory is depicted by a linear-shape diagram with n nodes and $n - 1$ links for any n . The free energy and the vevs of BPS Wilson loops are given in terms of a single 1-form on $\mathbb{CP}^1$ which can be determined explicitly for all linear theories. The analytic structure of the vevs of the Wilson loops is investigated in detail for $n = 1$ and $n = 2$. The addition of fundamental matters is also discussed.

KEYWORDS: Matrix Models, AdS-CFT Correspondence, Chern-Simons Theories

ARXIV EPRINT: [1904.11728](https://arxiv.org/abs/1904.11728)

Contents

1	Introduction	1
2	$\mathcal{N} = 3$ linear theories in the planar limit	3
2.1	Localization formulas	4
2.2	Planar limit	5
3	Planar solution	7
3.1	A function on $\mathbb{CP}^1$	7
3.2	Fundamental 1-form	9
4	Examples	11
4.1	Pure Chern-Simons theory	11
4.2	Theories for $n = 2$	12
5	Adding flavors	13
5.1	$n = 1$	14
6	Discussion	15
A	Equations for resolvents	17
B	Planar formulas for observables	19
C	Constraints on parameters in $\Omega(s)$ and $z(s)$	20

1 Introduction

A Chern-Simons-matter theory appears as the worldvolume theory on a D-brane system. For example, one may first construct a three-dimensional gauge theory, without Chern-Simons terms, by a brane construction using D3-branes intersecting with NS5-branes [1]. The gauge group of the worldvolume theory is of the form $G = \prod_{a=1}^n \mathrm{U}(N_a)$. Chern-Simons terms are then induced in the theory by replacing the i -th NS5-brane to a $(1, \tilde{k}_i)$ 5-brane [2, 3]. Some of the integers $\tilde{k}_i$ can be zero. The Chern-Simons level k_a for each $\mathrm{U}(N_a)$ gauge factor is determined by these $\tilde{k}_i$.

There are particular kind of Chern-Simons-matter theories studied intensively so far. The theories are constructed by considering the above brane system in $\mathbb{R}^{1,8} \times S^1$ in which the D3-branes wrap on S^1 . This is depicted in figure 1. Each $\mathrm{U}(N_a)$ factor in the gauge group G correspond to a set of N_a D3-brane segments suspended between successive $(1, \tilde{k}_i)$ 5-branes. In addition, there are bi-fundamental matter fields for each $(1, \tilde{k}_i)$ 5-branes. In

the following, we refer to these theories as circular theories because of the shape of the brane configuration. These theories are interesting since, via a chain of dualities, the brane system can be related to multiple M2-branes in some background. This relation enables us to determine the worldvolume theory on the M2-branes. ABJM theory [4] corresponds to the case $n = 2$. The theories with $n > 2$ were also discussed in [5, 6]. Their gravity duals are M-theory on backgrounds of the form $\text{AdS}_4 \times M_7$ where M_7 is a seven-dimensional 3-Sasakian manifold.

In this paper, we consider another family of Chern-Simons-matter theories. We require that the theory has $\mathcal{N} = 3$ supersymmetry. They can be obtained from the circular theories by eliminating one set of D3-brane segments, or in other words, by taking a limit $N_n \rightarrow 0$. Such a theory is usually depicted by a diagram like the one in figure 2. Similar theories can be obtained by taking a limit $k_n \rightarrow 0$, by which the gauge symmetry $U(N_n)$ becomes a global symmetry, and then some of the bi-fundamental matters become fundamental matters. In this paper, we refer to a theory of both kinds as a linear theory, due to the shape of the corresponding diagram. More details on this theory are given in section 2. Note that, when parameters of a linear theory are chosen appropriately, it is known that the supersymmetry is enhanced to $\mathcal{N} = 4$, resulting in a theory discussed in [7, 8].

It would be natural to expect that the gravity dual of a linear theory could be obtained by taking a suitable limit in the gravity dual [4–6] of the circular theory. However, the limit in the gravity dual seems to involve a large flux or a singular geometry, and it is not clear whether such a dual theory is useful for further analysis. Interestingly, as we will show in this paper, the field theory analysis of the linear theories turns out to be quite simple and explicit, even in the strong coupling region. Therefore, it might be possible to use the strongly coupled field theory to study a possibly singular gravity theory.

In this paper, we investigate the linear theories based on the localization formula for the partition function [9]. We take a limit, called the planar limit, in which the ranks N_a of the $U(N_a)$ factors and the Chern-Simons levels k_a become large, keeping their ratios finite. This limit enables us to analyze some observables of the theories, the free energy and the vevs of BPS Wilson loops, for arbitrary values of the 't Hooft couplings for all the linear theories. We obtain the formulas (3.20)(3.26) for the vevs of BPS Wilson loops as functions of the 't Hooft couplings. These are given as parametric representations using simple functions, whose analytic structure can be studied easily. We also obtain an integral formula for the free energy of the theories. Their detailed analysis will be postponed to a future publication.

This paper is organized as follows. In section 2, we describe the $\mathcal{N} = 3$ linear Chern-Simons-matter theories in more detail. We also recall the localization formulas for the observables studied in this paper. Then, we define the planar limit for the theories. In section 3, we show that all the information necessary in this paper is encoded in a 1-form Λ on $\mathbb{CP}^1$, which we call the fundamental 1-form of the theory. The explicit form of Λ gives simple formulas for the vevs of BPS Wilson loops. The integral formula for the free energy is also given in terms of Λ . The results obtained in section 3 are examined in detail in section 4. We show that our formulas reproduce known results for $n = 1$ theory, that is, $\mathcal{N} = 3$ pure Chern-Simons theory. For $n = 2$ theory, which was studied recently [10, 11],

Figure 1. The brane configuration which gives a Chern-Simons-matter theory with $n = 4$.

our formulas give new results. In section 5, we briefly explain how to introduce fundamental matters to the theories discussed so far. Section 6 is devoted to discussion. Some technical details are relegated to appendices.

2 $\mathcal{N} = 3$ linear theories in the planar limit

We consider a family of Chern-Simons theories coupled to matters in three dimensions. We require that the theories have $\mathcal{N} = 3$ supersymmetry, which is the largest supersymmetry allowed for arbitrary gauge groups and their representations of the matter fields. In such theories, the matter fields form $\mathcal{N} = 4$ hypermultiplets. The general structure of the Lagrangian of those theories can be found, for example, in [12].

In this paper, we mainly focus our attention on theories of the following type:

- The gauge group is of the form $\prod_{a=1}^n \text{U}(N_a)_{k_a}$ where $k_a \in \mathbb{Z}$ are the Chern-Simons levels.
- For each factor $\text{U}(N_a) \times \text{U}(N_{a+1})$ ($a = 1, 2, \dots, n-1$) of the gauge group, there is a single $\mathcal{N} = 4$ hypermultiplet belonging to the bi-fundamental representation.

These data can be depicted by, for example, a diagram in figure 2. In section 5, we will also briefly discuss similar theories with additional $\mathcal{N} = 4$ hypermultiplets belonging to the fundamental representation of $\text{U}(N_a)$. Note that an $\mathcal{N} = 4$ hypermultiplet consists of two $\mathcal{N} = 2$ chiral multiplets belonging to a representation and its complex conjugate. Therefore, each link in figure 2 may be replaced with two arrows with opposite directions.

We call these theories (both with and without flavors) the linear theories because of the shape of the corresponding diagram. The linear theory with $n = 2$ were recently studied in [10, 11]. In our investigation, the value of n can be arbitrary. We will take the parameters N_a, k_a to be large, while their ratios N_a/k_b can be arbitrary finite values.

As observables of interest, we consider the free energy of the theory, and the vevs of BPS Wilson loops [12] defined as

$$W_a(C) := \frac{1}{N_a} \text{Tr} P \exp \left[\int_C d\tau \left(i A_\mu^{(a)} \dot{x}^\mu + \sigma^{(a)} |\dot{x}| \right) \right] \quad (2.1)$$

Figure 2. The diagram depicting the matter contents of the linear theory with $n = 4$.

for each $U(N_a)$, where $A_\mu^{(a)}$ and $\sigma^{(a)}$ are the gauge field and the real scalar field, respectively, in the $\mathcal{N} = 2$ vector multiplet for $U(N_a)$, and the trace is taken for the fundamental representation.¹ For a suitable choice of the contour C , they preserve two supercharges.

Note that there is another kind of BPS Wilson loops, associated to two nodes, first constructed for ABJM theory [13], and later generalized to other theories [14]. It was shown in [13, 14] that the calculations for these Wilson loops are reduced to the calculations of (2.1). Therefore, the analysis in this paper can be also applied to these BPS Wilson loops for two nodes.

2.1 Localization formulas

Due to the supersymmetric localization [9], the free energy F and the vevs W_a of the Wilson loops (2.1) can be given in terms of finite-dimensional integrals.

Let n_{ab} be the number of bi-fundamental hypermultiplets for a factor $U(N_a) \times U(N_b)$ of the gauge group. We define $n_{aa} = 0$. The localization formula for the partition function of an $\mathcal{N} = 3$ Chern-Simons matter theory whose matters consists of bi-fundamental fields is given as

$$Z = \int du \exp \left[\frac{i}{4\pi} \sum_{a=1}^n k_a \sum_{i=1}^{N_a} (u_i^a)^2 \right] \frac{\prod_{a=1}^n \prod_{i < j}^{N_a} \sinh^2 \frac{u_i^a - u_j^a}{2}}{\prod_{a,b=1}^n \prod_{i=1}^{N_a} \prod_{j=1}^{N_b} \left(\cosh \frac{u_i^a - u_j^b}{2} \right)^{\frac{1}{2} n_{ab}}}, \quad (2.2)$$

where $du := \prod_{a,i} du_i^a$ is the integration measure. We have omitted the overall numerical factor which are irrelevant below. It is convenient to rewrite the integrand in (2.2) as $\exp(-S_{\text{eff}}[u])$ where

$$\begin{aligned} S_{\text{eff}}[u] = & \frac{1}{4\pi i} \sum_{a=1}^n k_a \sum_{i=1}^{N_a} (u_i^a)^2 - \sum_{a=1}^n \sum_{i < j}^{N_a} \log \sinh^2 \frac{u_i^a - u_j^a}{2} \\ & + \frac{1}{2} \sum_{a,b=1}^n n_{ab} \sum_{i=1}^{N_a} \sum_{j=1}^{N_b} \log \cosh \frac{u_i^a - u_j^b}{2}. \end{aligned} \quad (2.3)$$

For a linear theory, n_{ab} are given as

$$n_{ab} = \delta_{a,b+1} + \delta_{a,b-1}. \quad (2.4)$$

Note that the indices are *not* considered modulo n . The free energy F is then given as

$$F := -\log Z. \quad (2.5)$$

¹BPS Wilson loops in other representations in ABJM theory are discussed, for example, in [15] in relation to D-branes.

The vevs W_a of the Wilson loops (2.1) are given as

$$W_a := \frac{1}{Z} \int du e^{-S_{\text{eff}}[u]} \cdot \frac{1}{N_a} \sum_{i=1}^{N_a} e^{u_i^a}. \quad (2.6)$$

2.2 Planar limit

The integrals (2.2) and (2.6) are too complicated to analyze exactly for generic values of the parameters. To proceed further, we take a limit of the parameters described below.

We observe that $S_{\text{eff}}[u]$ scales as k^2 for large k when the parameters k_a and N_a scale as

$$k_a = \kappa_a k, \quad 2\pi i N_a = t_a k, \quad (2.7)$$

where κ_a and t_a are fixed constants. Therefore, the saddle point approximation for the integrals (2.2) and (2.6) becomes exact in the large k limit. We call this the planar limit.

In the following, we call t_a the 't Hooft couplings. The physical values of κ_a and t_a are real and pure imaginary, respectively. However, in the following sections, we will regard them as complex parameters. Since the overall scale of (κ_a, t_a) can be absorbed into k , the parameter space becomes $\mathbb{CP}^{2n-1}$. The physical parameter region is a $2n - 1$ real-dimensional subspace.

In the planar limit, the observables F and W_a are determined in terms of the solution of the saddle point equations

$$\frac{\partial S_{\text{eff}}}{\partial u_i^a} = \frac{k_a}{2\pi i} u_i^a - \sum_{j \neq i}^{N_a} \coth \frac{u_i^a - u_j^a}{2} + \sum_{b \neq a}^n \frac{n_{ab}}{2} \sum_{j=1}^{N_b} \tanh \frac{u_i^a - u_j^b}{2} = 0. \quad (2.8)$$

Since these equations have complex coefficients, the solution $\{\bar{u}_i^a\}$ consists of complex values in general. It is known that there are infinite number of saddle points for equations of this kind [16], which is argued to be related to D2-brane instantons. In the following, we select one of the saddle points which is assumed to give a dominant contribution to the integrals. The validity of this choice is then justified by showing that the selected saddle point reproduces results of perturbative calculations in the planar and weak 't Hooft coupling limit. In section 4, we will show this explicitly for the cases $n = 1, 2$. Other saddle points are expected to produce non-perturbative corrections to it.

In the context of the matrix model, it is well-known that the complex analysis is a very powerful tool for dealing with the information of the saddle point given by (2.8). We introduce the resolvents

$$v_a(z) := \frac{t_a}{N_a} \sum_{i=1}^{N_a} \frac{z + z_i^a}{z - z_i^a}, \quad z_i^a := -(-1)^a \exp(\bar{u}_i^a), \quad (2.9)$$

in terms of the solution $\{\bar{u}_i^a\}$ of (2.8). The resolvents encode the information of the saddle point as poles at $z = z_i^a$.

In the planar limit, the poles $z = z_i^a$ for each a are expected to be accumulated, forming a segment I_a of a curve in $\mathbb{C}$. Then, $v_a(z)$ has a square-root branch cut on I_a . The segments I_a are not necessarily lying on the real axis of $\mathbb{C}$ since $\bar{u}_i^a$ are complex in general.

The saddle point equations (2.8) can be rewritten in terms of $v_a(z)$. In fact, it was observed in [17] that their derivatives $v'_a(z)$ are more suitable quantities than $v_a(z)$ for solving the equations (2.8). The derivatives $v'_a(z)$ satisfy

$$2\kappa_a = xv'_a(x_+) + xv'_a(x_-) - xv'_{a-1}(x) - xv'_{a+1}(x), \quad x \in I_a, \quad (2.10)$$

where we defined $v_0(z) = v_{n+1}(z) = 0$, and $x_{\pm}$ are points in the vicinity of x above or below I_a . When I_a is a segment on the real axis, then $x_{\pm} = x \pm i0$. A derivation of these equations can be found in appendix A.

Once $v'_a(z)$ are obtained, we can calculate the observables F and W_a . The vevs W_a are given rather simply as

$$W_a = \frac{(-1)^a}{2t_a} \lim_{z \rightarrow \infty} z \cdot zv'_a(z). \quad (2.11)$$

The free energy F can be obtained as follows. Recalling that $S_{\text{eff}}[u]$ scales as k^2 in the planar limit, we notice that the free energy F is a homogeneous function of k_a and N_a of degree two. Therefore, F satisfies

$$F = \frac{1}{2} \sum_{a=1}^n \left(k_a \frac{\partial F}{\partial k_a} + N_a \frac{\partial F}{\partial N_a} \right). \quad (2.12)$$

Note that the finite quantity in the planar limit is F/k^2 which can be written in terms of κ_a and t_a . The derivatives of F can be obtained from $v'_a(z)$ as

$$\frac{\partial F}{\partial k_a} = \frac{iN_a}{24\pi t_a} \int_{C_a} \frac{dz}{2\pi i} \frac{zv'_a(z)}{z} (\log(\epsilon_a z))^3, \quad (2.13)$$

$$\frac{\partial F}{\partial N_a} = \int_{p_a}^{\infty} dz \left[-\frac{N_a}{t_a} \frac{zv'_a(z)}{z} \log(\epsilon_a z) + \sum_{b \neq a}^n \frac{n_{ab}}{2} \frac{N_b}{t_b} \frac{zv'_b(z)}{z} \log(\epsilon_a z) \right], \quad (2.14)$$

where C_a is a closed contour encircling I_a , and p_a is an endpoint of I_a . The details of these formulas can be found in appendix B.

Note that there is no dependence of the equations (2.10) on the 't Hooft couplings t_a . As will be shown in the next section, the equations (2.10) for fixed κ_a determine $v'_a(z)$ up to n complex parameters, say ξ_a . Then, the observables F and W_a are given as functions of κ_a and ξ_a . They can be converted to functions of κ_a and t_a by using

$$v_a(\infty) = -v_a(0) = t_a \quad (2.15)$$

which relate ξ_a to t_a . In terms of $v'_a(z)$, we can use instead

$$t_a = \frac{1}{2} \int_0^{\infty} \frac{dz}{z} \cdot zv'_a(z). \quad (2.16)$$

In the next section, we will find that t_a and W_a are simple functions of κ_a and ξ_a . On the other hand, F is given by an integral of elementary functions. We will mainly discuss the properties of W_a in section 4, while the investigation of F will appear in a future publication.

3 Planar solution

In this section, we solve the equations

$$2\kappa_a = xv'_a(x_+) + xv'_a(x_-) - xv'_{a-1}(x) - xv'_{a+1}(x), \quad x \in I_a. \quad (3.1)$$

Recall that we set $v_0(z) = v_{n+1}(z) = 0$.

Our first step is to eliminate the constants in the left-hand side. We define $\omega_a(z)$ to be constant shifts of $zv'_a(z)$, that is, $\omega_a(z)$ satisfy

$$zv'_a(z) = c_a + \omega_a(z). \quad (3.2)$$

We choose c_a to be a solution of

$$2\kappa_a = 2c_a - c_{a-1} - c_{a+1}, \quad (3.3)$$

where we set $c_0 = c_{n+1} = 0$. These equations always have a solution. It is interesting to notice that these equations can be written as

$$(2\kappa_1, \dots, 2\kappa_n) = (c_1, \dots, c_n)A, \quad (3.4)$$

where A is the Cartan matrix of $su(n+1)$ which is non-degenerate [18]. For this choice of c_a , we find that $\omega_a(z)$ satisfy

$$\omega_a(x_+) + \omega_a(x_-) - \omega_{a-1}(x) - \omega_{a+1}(x) = 0, \quad x \in I_a, \quad (3.5)$$

where $\omega_0(z) = \omega_{n+1}(z) = 0$.

The equations (3.5) can be further simplified by introducing $\Omega_\alpha(z)$ ($\alpha = 0, 1, \dots, n$) defined as

$$\Omega_\alpha(z) := \omega_{\alpha+1}(z) - \omega_\alpha(z). \quad (3.6)$$

We find that (3.5) can be written as

$$\Omega_a(x_\pm) = \Omega_{a-1}(x_\mp), \quad x \in I_a, \quad (3.7)$$

for $a = 1, 2, \dots, n$.

3.1 A function on $\mathbb{CP}^1$

The following geometric consideration is useful for solving the equations (3.7).

Let $\Sigma_\alpha := \mathbb{C} \setminus (I_\alpha \cup I_{\alpha+1})$ be a domain of $\mathbb{C}$ on which $\Omega_\alpha(z)$ is defined, where we set $I_0 = I_{n+1} = \emptyset$. By adding the point $z = \infty$ to each Σ_α , we obtain two disks from Σ_0 and Σ_n , and $n-1$ cylinders from the other Σ_α . From these pieces, we obtain $\mathbb{CP}^1$ by gluing them along the cuts I_α . This procedure is depicted in figure 3. The equations (3.7) then imply that $\Omega_\alpha(z)$ consistently define a function $\Omega(s)$ on $\mathbb{CP}^1$, where s is a coordinate of $\mathbb{CP}^1$.

Suppose that $\Omega(s)$ had been determined. Then, $\Omega_\alpha(z)$ can be recovered by composing $\Omega(s)$ with the map $s_\alpha : \Sigma_\alpha \rightarrow \mathbb{CP}^1$ which was used for the gluing. To obtain $zv'_a(z)$ from

Figure 3. The gluing of three complex planes. The branch cuts are represented by red lines.

$\Omega_\alpha(z)$ is then a trivial task. A difficulty in this strategy for obtaining $v'_a(z)$ is that the maps s_α seem to be quite complicated. Indeed, each of them would be given by a multi-valued function satisfying a polynomial equations of degree $n + 1$ with function coefficients.

Fortunately, it turns out that the inverse of s_α can be described quite explicitly. Let $U_\alpha \subset \mathbb{CP}^1$ be the image of $\Sigma_\alpha \cup \{\infty\}$ by the map s_α . The inverse of s_α defines a function $z|_{U_\alpha}(s)$ on U_α . This function has a simple pole at $s = s_\alpha(\infty)$. Since the closures $\overline{U}_\alpha$ of U_α cover $\mathbb{CP}^1$, we can define a meromorphic function $z(s)$ on $\mathbb{CP}^1$ from $z|_{U_\alpha}(s)$. Since any meromorphic function on $\mathbb{CP}^1$ is a rational function, we conclude that $z(s)$ is of the form

$$z(s) = D \prod_{\alpha=0}^n \frac{s - \eta_\alpha}{s - \xi_\alpha}. \quad (3.8)$$

The analytic structure of $\Omega(s)$ is determined from those of $v_a(z)$ as follows. Recall that $v_a(z)$ are holomorphic except at their branch points. This implies that $\Omega(s)$ is holomorphic on $\mathbb{CP}^1$ except at $2n$ points corresponding to the branch points of $v_a(z)$. Let $z = p$ be one of the branch point. At this point, $v_a(z)$ is expected to behave as

$$v_a(z) \sim v_a(p) + c\sqrt{z - p}. \quad (3.9)$$

Then, the derivative behaves as

$$zv'_a(z) \sim \frac{cp}{2\sqrt{z - p}}. \quad (3.10)$$

The same behavior is shared by $\Omega_a(z)$.

Let $s = \sigma$ be a point on $\mathbb{CP}^1$ such that $z(\sigma) = p$ is satisfied. At this point, $z(s)$ should behave as

$$z(s) \sim p + c'(s - \sigma)^2 \quad (3.11)$$

since $z = p$ is a branch point of $v_a(z)$. This implies that the pull-back by $z(s)$ of the functions $\Omega_\alpha(z)$ behaving like (3.10) has a simple pole at $s = \sigma$.

We have found that $\Omega(s)$ is a meromorphic function on $\mathbb{CP}^1$ with $2n$ simple poles. This is a rational function of the form

$$\Omega(s) = A + \sum_{a=1}^n \left[\frac{B_a}{s - \sigma_a} + \frac{C_a}{s - \tau_a} \right], \quad (3.12)$$

where $z(\sigma_a)$ and $z(\tau_a)$ are the branch points of $v_a(z)$.

The observables discussed in section 2 can be given directly in terms of $\Omega(s)$ and $z(s)$ defined above. For example, the vevs W_a of the Wilson loops are given as

$$t_{\alpha+1}W_{\alpha+1} - t_\alpha W_\alpha = -\frac{1}{2}D(\xi_\alpha - \eta_\alpha)\Omega'(\xi_\alpha) \prod_{\beta \neq \alpha} \frac{\xi_\alpha - \eta_\beta}{\xi_\alpha - \xi_\beta}, \quad (3.13)$$

and the 't Hooft couplings t_a are given as

$$t_{\alpha+1} - t_\alpha = \frac{1}{2} \int_{\eta_\alpha}^{\xi_\alpha} ds \frac{z'(s)}{z(s)} (\Omega(s) - \Omega_\alpha), \quad \Omega_\alpha := \Omega(\xi_\alpha). \quad (3.14)$$

Note that we set $t_0 = t_{n+1} = W_0 = W_{n+1} = 0$, so that t_a and W_a are completely determined by (3.13) and (3.14). By eliminating some of the parameters, we obtain W_a as functions of t_a . The integrands for the 't Hooft couplings are in fact rational functions, and therefore the integration can be done readily. The results are given in terms of the parameters in $\Omega(s)$ and $z(s)$ which are to be determined.

The free energy F can be also rewritten in terms of $\Omega(s)$ and $z(s)$. Note that in any case we do not need to deal with the inverse of $z(s)$ for the calculations of the observables.

3.2 Fundamental 1-form

We need to determine $\Omega(s)$ and $z(s)$. It turns out that there are constraints on them deduced from the properties of $v_a(z)$, by which enough number of the parameters in $\Omega(s)$ and $z(s)$ can be determined. This is shown in appendix C. The undetermined parameters are related to t_a , as explained at the end of section 2. Therefore, we conclude that we can solve the equations (3.1) completely, in principle. In practice, the values of the parameters are given as solutions of a set of algebraic equations. The existence of the solution can be shown, but it is almost impossible to understand the analytic behaviors of them as functions of t_a for general values of n .

Fortunately, there is a shortcut. It turns out that all the information we need to know is encoded in the following 1-form

$$\Lambda := \frac{z'(s)}{z(s)} \Omega(s) ds. \quad (3.15)$$

It is curious to notice that Λ is obtained from the pull-back of a 1-form $dv_{\alpha+1} - dv_\alpha$ on Σ_α to $\mathbb{CP}^1$ by $z(s)$ as

$$z^*(dv_{\alpha+1} - dv_\alpha) = \Lambda - \Omega_\alpha d \log z(s). \quad (3.16)$$

The advantage of considering Λ instead of $\Omega(s)$ and $z(s)$ separately is that Λ has a very simple expression given below.

Recall that $\Omega(s)$ has simple poles at $s = \sigma_a, \tau_a$. Since $z(\sigma_a)$ and $z(\tau_a)$ correspond to the branch points of $v_a(z)$, the values of σ_a and τ_a are determined as the solutions of $z'(s) = 0$. Therefore, the poles of $\Omega(s)$ are canceled by the zeros of $z'(s)$ in Λ . There are other poles coming from

$$\frac{z'(s)}{z(s)} = \sum_{\alpha=0}^n \left(\frac{1}{s - \eta_\alpha} - \frac{1}{s - \xi_\alpha} \right). \quad (3.17)$$

Therefore, Λ is given simply as the sum of these poles with appropriate residues. We obtain

$$\Lambda = \sum_{\alpha=0}^n \left(\frac{\Omega_\alpha}{s - \eta_\alpha} - \frac{\Omega_\alpha}{s - \xi_\alpha} \right) ds, \quad (3.18)$$

where we have used $\Omega(\eta_\alpha) = \Omega_\alpha$, which is shown in appendix C. The residues Ω_α are given as

$$\Omega_\alpha = c_\alpha - c_{\alpha+1}, \quad (3.19)$$

and c_α are given in terms of κ_a by (3.3).

Now, the 't Hooft couplings t_a can be obtained very explicitly as

$$t_{\alpha+1} - t_\alpha = \frac{1}{2} \sum_{\beta \neq \alpha} (\Omega_\beta - \Omega_\alpha) \log r_{\alpha\beta}, \quad (3.20)$$

where $r_{\alpha\beta}$ are the cross ratios

$$r_{\alpha\beta} := \frac{\xi_\alpha - \eta_\beta}{\eta_\alpha - \eta_\beta} \frac{\eta_\alpha - \xi_\beta}{\xi_\alpha - \xi_\beta}, \quad (3.21)$$

and the coefficients $\Omega_\beta - \Omega_\alpha$ can be given simply as

$$\Omega_\beta - \Omega_\alpha = 2 \sum_{\gamma=\alpha+1}^{\beta} \kappa_\gamma \quad (3.22)$$

for $\beta > \alpha$.

We need to know $\Omega'(\xi_\alpha)$ which is necessary in (3.13) for the vevs W_a . It turns out that these quantities can be also obtained from Λ . By expanding (3.15) at $s = \xi_\alpha$, we find

$$\Lambda = \left[-\frac{\Omega_\alpha}{s - \xi_\alpha} - \Omega'(\xi_\alpha) + \frac{\Omega_\alpha}{\xi_\alpha - \eta_\alpha} + \sum_{\beta \neq \alpha} \left(\frac{\Omega_\alpha}{\xi_\alpha - \eta_\beta} - \frac{\Omega_\alpha}{\xi_\alpha - \xi_\beta} \right) + \mathcal{O}(s - \xi_\alpha) \right] ds. \quad (3.23)$$

On the other hand, by expanding (3.18), we find

$$\Lambda = \left[-\frac{\Omega_\alpha}{s - \xi_\alpha} + \frac{\Omega_\alpha}{\xi_\alpha - \eta_\alpha} + \sum_{\beta \neq \alpha} \left(\frac{\Omega_\beta}{\xi_\alpha - \eta_\beta} - \frac{\Omega_\beta}{\xi_\alpha - \xi_\beta} \right) + \mathcal{O}(s - \xi_\alpha) \right] ds. \quad (3.24)$$

Comparing these two expressions, we obtain

$$\Omega'(\xi_\alpha) = \sum_{\beta \neq \alpha} \left(\frac{\Omega_\alpha - \Omega_\beta}{\xi_\alpha - \eta_\beta} - \frac{\Omega_\alpha - \Omega_\beta}{\xi_\alpha - \xi_\beta} \right). \quad (3.25)$$

Therefore, the vevs W_a can be given as

$$t_{\alpha+1} W_{\alpha+1} - t_\alpha W_\alpha = -\frac{1}{2} D(\xi_\alpha - \eta_\alpha) \sum_{\beta \neq \alpha} \left(\frac{\Omega_\alpha - \Omega_\beta}{\xi_\alpha - \eta_\beta} - \frac{\Omega_\alpha - \Omega_\beta}{\xi_\alpha - \xi_\beta} \right) \prod_{\gamma \neq \alpha} \frac{\xi_\alpha - \eta_\gamma}{\xi_\alpha - \xi_\gamma}. \quad (3.26)$$

The free energy F can be also written in terms of Λ . As a result, we obtain F as a sum of integrals whose integrands are combinations of rational functions and logarithms.

4 Examples

In this section, we examine the implications of the formulas (3.20)(3.26) obtained in the last section. We will consider the cases $n = 1$ and $n = 2$. In the following, we employ

$$z(s) = - \prod_{\alpha=0}^n \frac{\xi_\alpha s - 1}{s - \xi_\alpha}. \quad (4.1)$$

Note that this corresponds to taking $\eta_\alpha = \xi_\alpha^{-1}$. In appendix C, we show that this choice gives a rather simple solution to the equations (3.1). Note that the choice of the coordinate on $\mathbb{CP}^1$ is not physically meaningful, and they can be changed by an $SL(2, \mathbb{C})$ transformation. After fixing $\eta_\alpha = \xi_\alpha^{-1}$ as above, there is still a freedom to make an $SL(2, \mathbb{C})$ transformation which brings one of ξ_α to infinity. In appendix C, we choose $\xi_0 = \infty$.

4.1 Pure Chern-Simons theory

This theory corresponds to $n = 1$. In this case, we can choose $k = k_1$, and then we have $\kappa_1 = 1$. For the simplicity of notation, we denote $t = t_1$. The equations (3.1) become

$$2 = xv'(x_+) + xv'(x_-), \quad x \in I. \quad (4.2)$$

The formula (3.20) implies

$$t = \log \frac{\xi_0 - \eta_1}{\eta_0 - \eta_1} \frac{\eta_0 - \xi_1}{\xi_0 - \xi_1}, \quad \eta_\alpha = \xi_\alpha^{-1}. \quad (4.3)$$

As explained above, we can choose $\xi_0 = \infty$. We denote $\xi = \xi_1$. Then, we obtain

$$t = \log \xi^2. \quad (4.4)$$

The weak coupling limit corresponds to $\xi^2 = 1$. Since we defined that $t = 2\pi i N_1 / k_1$ is pure imaginary for the physical parameter region, the corresponding ξ is a pure phase. The 't Hooft coupling t grows as ξ rotates around the unit circle in $\mathbb{C}$.

In fact, it is not possible to decide whether $\xi = +1$ or $\xi = -1$ should correspond to the weak coupling limit, based only on the results on pure Chern-Simons theory. In the next section, we will find that $\xi = -1$ is preferable by examining theories with flavors added.

The vev $W = W_1$ of the Wilson loop is given as

$$W = \frac{\xi^2 - 1}{t}. \quad (4.5)$$

By rewriting this as

$$W = e^{\frac{1}{2}t} \frac{e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}}{t}, \quad (4.6)$$

we find that this reproduces the planar limit of the exact result obtained in [19] with a framing factor $e^{\frac{1}{2}t}$ [9]. This approaches 1 in the weak coupling limit, as it should be.

It is curious to observe that W vanishes at $t = 2\pi i$, or in other words, $N = k$. This is known to be the boundary of the region in the parameter space in which the supersymmetry is broken [2, 3, 20]. In the next subsection, we will show that a similar phenomenon is observed also for $n = 2$. However, this is not a general property of the vevs of the Wilson loop. We will see a counter-example in section 5

4.2 Theories for $n = 2$

The 't Hooft couplings are given as

$$t_1 = \kappa_1 \log r_{01} + (\kappa_1 + \kappa_2) \log r_{02}, \quad (4.7)$$

$$t_2 = (\kappa_1 + \kappa_2) \log r_{20} + \kappa_2 \log r_{21}. \quad (4.8)$$

The analysis for generic parameters looks to be involved. For simplicity, we consider the case $k_1 + k_2 = 0$, for which the supersymmetry is enhanced to $\mathcal{N} = 4$ [7]. For this case, we choose $k = k_1$ so that we have $\kappa_1 = -\kappa_2 = 1$.

We find that the choice $\xi_1 = \infty$ is the most convenient. For this choice, we obtain

$$t_1 = \log(\xi_0)^2, \quad t_2 = -\log(\xi_2)^2. \quad (4.9)$$

As explained in appendix C, the parameters ξ_0 and ξ_2 are unconstrained. The physical parameter space then corresponds to $S^1 \times S^1$ given by $|\xi_0| = |\xi_2| = 1$.

The vevs of the Wilson loops are given as

$$W_1 = \frac{1}{t_1} \frac{(\xi_0^2 - 1)(\xi_0 \xi_2 - 1)}{\xi_0 - \xi_2}, \quad (4.10)$$

$$W_2 = -\frac{1}{t_2} \frac{(\xi_2^2 - 1)(\xi_0 \xi_2 - 1)}{\xi_0 - \xi_2}. \quad (4.11)$$

To express them in terms of t_1 and t_2 , we need to fix the sign ambiguity in the relation between ξ_0 (ξ_2) and t_1 (t_2), respectively. This can be done by checking whether both W_1 and W_2 approach 1 as t_1 and t_2 go to zero. We find that the correct choice of the signs is

$$\xi_0 = -e^{\frac{1}{2}t_1}, \quad \xi_2 = e^{-\frac{1}{2}t_2}. \quad (4.12)$$

Then, the vevs are given as

$$W_1 = \frac{e^{t_1} - 1}{t_1} \frac{e^{\frac{1}{2}(t_1 - t_2)} + 1}{e^{\frac{1}{2}t_1} + e^{-\frac{1}{2}t_2}}, \quad (4.13)$$

$$W_2 = -\frac{e^{-t_2} - 1}{t_2} \frac{e^{\frac{1}{2}(t_1 - t_2)} + 1}{e^{\frac{1}{2}t_1} + e^{-\frac{1}{2}t_2}}. \quad (4.14)$$

The perturbative expansions of these expressions are given as

$$W_1 = 1 + \frac{t_1}{2} + \frac{t_1^2}{6} - \frac{t_1 t_2}{8} + \mathcal{O}(t^3), \quad (4.15)$$

$$W_2 = 1 - \frac{t_2}{2} + \frac{t_2^2}{6} - \frac{t_1 t_2}{8} + \mathcal{O}(t^3). \quad (4.16)$$

They reproduce the perturbative results found in [17] which were obtained directly from the localization formula (2.6). Note that the coefficients of the linear terms in t_1 or t_2 above are $\pm \frac{1}{2}$, in analogy with the one in pure Chern-Simons case. This seems to suggest that the framings implicitly chosen for the above Wilson loops in $n = 2$ case would be $+1$ or -1 .

The vevs W_1 and W_2 have interesting non-perturbative behaviors. Since the structures of W_1 and W_2 are almost identical, we focus on W_1 .

W_1 has a factor which also appeared in W for pure Chern-Simons theory. Therefore, W_1 vanishes whenever $N_1 = k_1$. There is another factor in the numerator which vanishes when $\frac{1}{2}(t_1 - t_2) = \pi i \bmod 2\pi i$, or in other words,

$$N_1 - N_2 \in (2\mathbb{Z} + 1)k. \quad (4.17)$$

It can be shown by the s-rule [1] that the supersymmetry of the theory is broken when $|N_1 - N_2| = k$. This condition for the supersymmetry breaking coincides with the vanishing condition (4.17) of W_1 .

Curiously, the denominator of W_1 vanishes when $\frac{1}{2}(t_1 + t_2) = \pi i \bmod 2\pi i$, or

$$N_1 + N_2 \in (2\mathbb{Z} + 1)k. \quad (4.18)$$

Note that one of them, $N_1 + N_2 = k$, was discussed in [10, 11] as the boundary between the good theories and bad theories [21]. It would be interesting to investigate this singularity in more detail. It would be also interesting to investigate the parameter region beyond the singularity by the analytic continuation of the parameters. This is indeed possible since the singularity is a pole for $t_1 + t_2$ which can be avoided by taking complex values of t_1 and t_2 .

5 Adding flavors

It is rather straightforward to analyze theories which are obtained by adding an arbitrary number of fundamental hypermultiplets to the theories considered so far. If we add a hypermultiplet belonging to the fundamental representation of $U(N_a)$, the localization formula for the partition function (2.2) includes the factor

$$\prod_{i=1}^{N_a} \left(\cosh \frac{u_i^a}{2} \right)^{-1}. \quad (5.1)$$

Let n_a be the number of such fundamental hypermultiplets. Then, the resolvents $v_a(z)$ defined by (2.9) satisfy

$$2\kappa_a + 2\epsilon_a \nu_a \frac{x}{(x + \epsilon_a)^2} = x v_a'(x_+) + x v_a'(x_-) - x v_{a-1}'(x) - x v_{a+1}'(x), \quad (5.2)$$

where we have defined ϵ_a and ν_a as

$$\epsilon_a := -(-1)^a, \quad \nu_a := \frac{2\pi i n_a}{k}. \quad (5.3)$$

To solve these equations, we decompose $z v_a'(z)$ as

$$z v_a'(z) = c_a + d_a(z) + \omega_a(z), \quad (5.4)$$

where c_a are defined by (3.3) as before, and $d_a(z)$ are rational functions satisfying

$$2\epsilon_a \nu_a \frac{z}{(z + \epsilon_a)^2} = 2d_a(z) - d_{a-1}(z) - d_{a+1}(z). \quad (5.5)$$

The solution of these equations has of the form

$$d_a(z) = \frac{d_{a,-2}}{(z-1)^2} + \frac{d_{a,-1}}{z-1} + \frac{\tilde{d}_{a,-2}}{(z+1)^2} + \frac{\tilde{d}_{a,-1}}{z+1} + d_{a,0}. \quad (5.6)$$

Since $zv'_a(z)$ are expected to be holomorphic at $z = \pm 1$, $\omega_a(z)$ are required to have poles of the form

$$\omega_a(z) = -\frac{d_{a,-2}}{(z-1)^2} - \frac{d_{a,-1}}{z-1} - \frac{\tilde{d}_{a,-2}}{(z+1)^2} - \frac{\tilde{d}_{a,-1}}{z+1} + (\text{holomorphic}). \quad (5.7)$$

The 't Hooft couplings are given as

$$t_a = \frac{1}{2} \int_0^\infty \frac{dz}{z} (c_a + \omega_a(z)) + t_a^{(\nu)}, \quad (5.8)$$

where $t_a^{(\nu)}$ are the solutions of the equations

$$\nu_a = 2t_a^{(\nu)} - t_{a-1}^{(\nu)} - t_{a+1}^{(\nu)}. \quad (5.9)$$

The vevs W_a of the Wilson loops are given as

$$W_a = -\frac{\epsilon_a}{2t_a} \lim_{z \rightarrow \infty} z \cdot (c_a + \omega_a(z)) - \frac{1}{2t_a} w_a, \quad (5.10)$$

where w_a satisfy

$$2\nu_a = 2w_a + w_{a-1} + w_{a+1}. \quad (5.11)$$

5.1 $n = 1$

Let us illustrate the above procedure for the case $n = 1$, that is, for $\mathcal{N} = 3$ U(N) Chern-Simons theory coupled to n_1 fundamental hypermultiplets. The resolvent $v(z)$ satisfy

$$2 + 2\nu \frac{x}{(x+1)^2} = xv'(x_+) + xv'(x_-), \quad x \in I_1, \quad (5.12)$$

where $\nu = \nu_1$. We decompose $zv'(z)$ into $c + d(z) + \omega(z)$ where

$$d(z) = \nu \frac{z}{(z+1)^2} = \frac{-\nu}{(z+1)^2} + \frac{\nu}{z+1}. \quad (5.13)$$

The part $c + \omega(z)$ satisfies (4.2) for pure Chern-Simons theory. Therefore, the determination of $\omega(z)$ is almost the same as before, except for the requirement that $\omega(z)$ must have poles at $z = -1$ of the form

$$\omega(z) = \frac{\nu}{(z+1)^2} + \frac{-\nu}{z+1} + (\text{holomorphic}). \quad (5.14)$$

We use the same formula (4.1) for $z(s)$. Recall that the necessary object for determining the observables is the fundamental 1-form Λ , not $\Omega(s)$ itself. We find that Λ is given as

$$\begin{aligned} \Lambda = & \left[-\frac{1}{s-\eta_0} + \frac{1}{s-\xi_0} + \frac{1}{s-\eta_1} - \frac{1}{s-\xi_1} \right. \\ & \left. + \frac{(\xi_0-1)(\xi_1-1)}{2(\xi_0\xi_1-1)} \frac{\mu}{(s-1)^2} + \frac{(\xi_0+1)(\xi_1+1)}{2(\xi_0\xi_1-1)} \frac{\mu}{(s+1)^2} \right] ds, \end{aligned} \quad (5.15)$$

where the second line comes from poles of $\omega(z)$ at $z = 1$. As for pure Chern-Simons theory, we choose $\xi_0 = \infty$. Then, all the observables are functions of $\xi = \xi_1$. The 't Hooft coupling t is given as

$$t = \log \xi^2 + \frac{\nu}{2} \left(1 + \frac{1}{\xi} \right), \quad (5.16)$$

which is the same as the expression obtained in [22]. The weak coupling limit corresponds to the limit $\xi \rightarrow -1$. The vev W of the Wilson loop is given as

$$W = \frac{1}{t} \left(\xi^2 - 1 - \frac{\nu}{2} (1 + \xi) \right). \quad (5.17)$$

To write W explicitly as a function of t , we need to solve (5.16) for ξ . Since this equation is complicated, we assume that ν is small, and solve it perturbatively. As a result, we obtain

$$\xi = -\exp \left[\frac{1}{2} \left(t - \frac{1}{2} (1 - e^{-\frac{1}{2}t}) \nu + \frac{1}{8} e^{-\frac{1}{2}t} (1 - e^{-\frac{1}{2}t}) \nu^2 \right) \right] + \mathcal{O}(\nu^3). \quad (5.18)$$

Substituting this into (5.17), we obtain

$$W = \frac{e^t - 1}{t} - \frac{1}{2} \frac{(e^{\frac{1}{2}t} - 1)^2}{t} \nu + \frac{1}{8} \frac{(e^{\frac{1}{2}t} - 1)^2}{t} \nu^2 + \mathcal{O}(\nu^3). \quad (5.19)$$

We find that W approaches 1 in the limit $t \rightarrow 0$, as expected.

We have observed so far that the vevs of the Wilson loops vanish for values of t_a corresponding to the supersymmetry breaking. For the theory under consideration, the condition for the supersymmetry breaking is $N > k + m$ [22]. Then, one might expect that W above vanishes when $t = 2\pi i + \nu$. However, we find

$$W \Big|_{t=2\pi i + \nu} = -\nu + \mathcal{O}(\nu^3). \quad (5.20)$$

This indicates that the condition for the vanishing of Wilson loop vevs does not always coincide with the condition for the supersymmetry breaking. Note that W vanishes $\xi = 1 + \frac{\nu}{2}$ instead.

6 Discussion

We have investigated the linear Chern-Simons-matter theories in the planar limit, based on the localization formula (2.2) of the partition function. We have found that all the information necessary for obtaining the free energy F and the vevs W_a of Wilson loops in the planar limit is encoded into a single 1-form Λ on an auxiliary $\mathbb{CP}^1$ which we called the fundamental 1-form of the theory. The form of Λ was determined explicitly. As a result, we obtained the formulas for W_a as functions of the 't Hooft couplings. The analytic behavior of W_a was investigated in detail for $n = 1$ and $n = 2$. For the latter case, W_a may vanish or diverge for finite values of t_a . This seems to be related to some non-perturbative physics, for example, the one discussed in [10, 11]. For F , we obtained an integral formula in which the integrand is given by an elementary function.

It would be interesting to investigate the linear theories for $n > 2$ in more detail. For those theories, the formulas for the observables were already obtained in this paper. However, even though they are given in terms of elementary functions, their analysis does not look straightforward. For example, the physical parameter subspace in $\mathbb{CP}^{2n-1}$ would be quite involved for general n , although for $n = 1$ and $n = 2$ it is simply S^1 and T^2 , respectively. It would be interesting to understand what kinds of zeros and poles for the observables exist in the physical parameter subspace. If a non-trivial analytic structure exists, it is important to identify physical properties underlying it. In particular, the investigation of the free energy would be important since it contains important information such as the conditions for the supersymmetry breaking [23].

It would be interesting to extend the analysis in this paper to more general Chern-Simons-matter theories in the planar limit. The extension to the circular theories mentioned in the Introduction would be rather straightforward. It would be accomplished by replacing the auxiliary $\mathbb{CP}^1$ to T^2 , and therefore the roles played by rational functions would be taken by elliptic functions. The analysis would be almost parallel at least when $\sum_a k_a = 0$ holds. For a general case, we probably need the logarithm of a theta function as a part of the resolvents.

There could be further generalization. In [17], we studied a family of Chern-Simons-matter theories which is another generalization of the linear theory with $n = 1$. In the analysis, not functions on T^2 but sections of a non-trivial line bundle on T^2 played a crucial role. The use of such non-trivial line bundles would enable us to analyze more general Chern-Simons-matter theories. It is interesting if the structure of the Kac-Moody algebra found in [18] plays some role in the generalizations.

In this paper, we have investigated the planar limit of linear theories using resolvents. There are other methods for analyzing Chern-Simons-matter theories, some of which can be applied beyond the planar limit. For example, the Fermi gas formalism [26] has been shown to be quite powerful in analyzing many theories. In [26], circular theories are analyzed using this formalism, where the ranks of the $U(N_a)$ factors were chosen to be equal. Since the Chern-Simons levels can be arbitrary in this analysis, one could study linear theories with the same ranks by taking a limit $k_n \rightarrow 0$. An application of the Fermi gas formalism to linear theories are also argued in [10]. A successful application will give us much more information on linear theories. There is another method [27] which is restricted to a large N limit but the analysis is much simpler. Up to now, the application of this method is limited to theories in which a “long-range force” is absent. Such theories are systematically investigated in [28], in which linear theories are not included. If this limitation can be overcome, the method in [27] could be applied to linear theories. In addition, it is also very interesting if a relation to a topological string theory can be found, like in the case of ABJM theory [29]. Our geometric viewpoint on the resolvent method might give some hints for such a possibility.

Finally, it would be interesting if we could obtain some implications on AdS/CFT correspondence based on our planar analysis on Chern-Simons-matter theories.

Acknowledgments

This work was supported in part by Grants-in-Aid for Scientific Research (No.16K05329).

A Equations for resolvents

In this appendix, we derive the equations (2.10) satisfied by the resolvents $v_a(z)$.

We start with the localization formula [9] for the partition function

$$Z = \int du \exp \left[\frac{i}{4\pi} \sum_{a=1}^n k_a \sum_{i=1}^{N_a} (u_i^a)^2 \right] \frac{\prod_{a=1}^n \prod_{i < j}^{N_a} \sinh^2 \frac{u_i^a - u_j^a}{2}}{\prod_{a,b=1}^n \prod_{i=1}^{N_a} \prod_{j=1}^{N_b} \left(\cosh \frac{u_i^a - u_j^b}{2} \right)^{\frac{1}{2} n_{ab}}}, \quad (\text{A.1})$$

for an $\mathcal{N} = 3$ Chern-Simons-matter theory whose matters consist of bi-fundamental fields. Our analysis here is not restricted to the linear theories. In the following, we allow n_{ab} to be more general non-negative integers, provided that they satisfy (i) $n_{ab} = n_{ba}$, (ii) $n_{aa} = 0$, and (iii) the condition (A.5) below.

The saddle point equations for this integral are

$$\frac{k_a}{2\pi i} u_i^a = \sum_{j \neq i}^{N_a} \coth \frac{u_i^a - u_j^a}{2} - \sum_{b \neq a}^n \frac{n_{ab}}{2} \sum_{j=1}^{N_b} \tanh \frac{u_i^a - u_j^b}{2}. \quad (\text{A.2})$$

We rewrite these equations into more manageable ones. Introduce new variables

$$z_i^a := \epsilon_a \exp(u_i^a), \quad \epsilon_a := -(-1)^a. \quad (\text{A.3})$$

In terms of z_i^a , (A.2) can be written as

$$\frac{k_a}{2\pi i} \log(\epsilon_a z_i^a) = \sum_{j \neq i}^{N_a} \frac{z_i^a + z_j^a}{z_i^a - z_j^a} - \sum_{b \neq a}^n \frac{n_{ab}}{2} \sum_{j=1}^{N_b} \frac{z_i^a + z_j^b}{z_i^a - z_j^b}. \quad (\text{A.4})$$

Here we have assumed that n_{ab} satisfy the following condition

$$a \equiv b \pmod{2} \quad \Rightarrow \quad n_{ab} = 0. \quad (\text{A.5})$$

This is satisfied by the linear theories. For a theory with n_{ab} satisfying this condition, a diagram corresponding to figure 2 is bipartite. Non-bipartite theories were discussed in [25].

We introduce the resolvents

$$v_a(z) := \frac{t_a}{N_a} \sum_{i=1}^{N_a} \frac{z + z_i^a}{z - z_i^a}, \quad (\text{A.6})$$

where t_a are the 't Hooft couplings defined in (2.7). For finite values of the parameters k_a and N_a , these are simple rational functions. In the planar limit defined in subsection 2.2, it is usually expected that the poles of $v_a(z)$ at $z = z_i^a$ accumulate to form a segment I_a

of a curve in $\mathbb{C}$. As a result, $v_a(z)$ are expected to become non-trivial functions in the planar limit.

We assume the following analytic structure of $v_a(z)$ in the planar limit. Each $v_a(z)$ is an analytic function on $\mathbb{C} \setminus I_a$. On the segment I_a , $v_a(z)$ has a branch cut. At a branch point, say $z = p_a$, the non-analytic part of $v_a(z)$ behaves as $\sqrt{z - p_a}$, due to the Wigner law. In addition, $v_a(z)$ is analytic at infinity which is anticipated from the definition (A.6).

In terms of the resolvents, the saddle point equations (A.4) can be written as

$$2\kappa_a \log(\epsilon_a x) = v_a(x_+) + v_a(x_-) - \sum_{b \neq a}^n n_{ab} v_b(x), \quad x \in I_a, \quad (\text{A.7})$$

where $x_{\pm}$ are points in the vicinity of x above or below I_a , and κ_a were defined in (2.7). This rewriting uses the fact that the sum $\frac{1}{N_a} \sum_{j \neq i}$ becomes the principal value of an integral in the planar limit.

We have assumed that the segment I_a does not have any intersections with other segments, and therefore $v_b(x_+) = v_b(x_-)$ hold for $x \in I_a$ with $b \neq a$. The resolvents for a more general configuration of the segments are obtained via the analytic continuation of the positions of the branch points.

The equations (A.7) are not convenient for determining $v_a(z)$ since $\log x$ in the left-hand side requires us to handle the log branch cuts. To avoid this difficulty, we take the derivative of the equations [17]. As a result, we obtain

$$2\kappa_a = x v'_a(x_+) + x v'_a(x_-) - \sum_{b \neq a}^n n_{ab} x v'_b(x), \quad x \in I_a. \quad (\text{A.8})$$

These are the equations analyzed in section 3.

A generalization for adding fundamental hypermultiplets is rather straightforward. Let n_a be the number of $\mathcal{N} = 4$ hypermultiplets belonging to the fundamental representation of $U(N_a)$. The localization formula for such a theory is obtained by inserting the factor

$$\prod_{a=1}^n \prod_{i=1}^{N_a} \left(\cosh \frac{u_i^a}{2} \right)^{-n_a} \quad (\text{A.9})$$

to the integral (A.1). Formally, this corresponds to adding an extra node $n = 0$. For the extra node, the level and the rank are $k_0 = 0$ and $N_0 = 1$. The variable u_1^0 is fixed to zero by hand. The number n_a is regarded as n_{0a} .

The saddle point equations then become

$$\frac{k_a}{2\pi i} u_i^a = \sum_{j \neq i}^{N_a} \coth \frac{u_i^a - u_j^a}{2} - \sum_{b \neq a}^n \frac{n_{ab}}{2} \sum_{j=1}^{N_b} \tanh \frac{u_i^a - u_j^b}{2} - \frac{n_a}{2} \tanh \frac{u_i^a}{2}, \quad (\text{A.10})$$

where $a = 1, 2, \dots, n$.

The resolvents $v_a(z)$ are defined as in (A.6). In terms of them, the saddle point equations (A.10) can be written as

$$2\kappa_a + 2\epsilon_a \mu_a \frac{x}{(x + \epsilon_a)^2} = x v'_a(x_+) + x v'_a(x_-) - \sum_{b \neq a}^n n_{ab} x v'_b(x), \quad x \in I_a, \quad (\text{A.11})$$

where $\nu_a := 2\pi i n_a / k$. These equations are analyzed in section 5.

B Planar formulas for observables

The derivatives $v'_a(z)$ contain the information on the observables F and W_a .

The expansion of $v_a(z)$ at infinity is given as

$$v_a(z) = t_a \left(1 + \frac{2}{N_a} \sum_{i=1}^{N_a} z_i^a \cdot z^{-1} \right) + \mathcal{O}(z^{-2}). \quad (\text{B.1})$$

Recall that we defined $z_i^a = -(-1)^a \exp(\bar{u}_i^a)$. Therefore, W_a can be read off from the coefficient of z^{-1} in $v_a(z)$. In terms of $v'_a(z)$, W_a is given as

$$W_a = \frac{(-1)^a}{2t_a} \lim_{z \rightarrow \infty} z \cdot v'_a(z). \quad (\text{B.2})$$

The free energy F is given by $S_{\text{eff}}[\bar{u}]$, in principle. However, the formula for $S_{\text{eff}}[\bar{u}]$ in the planar limit is complicated since the double sums in $S_{\text{eff}}[\bar{u}]$ become double integrals in the limit. There is a relatively simple formula for F using $zv'_a(z)$.

Recall that F scales as k^2 in the planar limit. Since k_a and N_a are proportional to k , we find that F is a homogeneous function of k_a and N_a of degree 2. This implies that F satisfies

$$F = \frac{1}{2} \sum_{a=1}^n \left(k_a \frac{\partial F}{\partial k_a} + N_a \frac{\partial F}{\partial N_a} \right). \quad (\text{B.3})$$

The k_a -derivatives of F can be written as

$$\frac{\partial F}{\partial k_a} = \frac{1}{Z} \int du e^{-S_{\text{eff}}[u]} \frac{-i}{4\pi} \sum_{i=1}^{N_a} (u_i^a)^2. \quad (\text{B.4})$$

Since $v_a(z)$ can be written as

$$v_a(z) = 2t_a z \cdot \frac{1}{N_a} \sum_{i=1}^{N_a} \frac{1}{z - z_i^a} - t_a, \quad (\text{B.5})$$

the expectation values like (B.4) can be written as

$$\frac{1}{Z} \int du e^{-S_{\text{eff}}[u]} \frac{1}{N_a} \sum_{i=1}^{N_a} f(e^{u_i^a}) = \int_{C_a} \frac{dz}{2\pi i} \frac{v_a(z)}{2t_a z} f(\epsilon_a z), \quad (\text{B.6})$$

where the contour C_a encircles the poles at $z = z_i^a$ but excludes the origin. Therefore, $\partial_{k_a} F$ is given as

$$\begin{aligned} \frac{\partial F}{\partial k_a} &= -\frac{iN_a}{8\pi t_a} \int_{C_a} \frac{dz}{2\pi i} \frac{v_a(z)}{z} (\log(\epsilon_a z))^2 \\ [2mm] &= \frac{iN_a}{24\pi t_a} \int_{C_a} \frac{dz}{2\pi i} \frac{zv'_a(z)}{z} (\log(\epsilon_a z))^3. \end{aligned} \quad (\text{B.7})$$

The N_a -derivatives of F can be written as follows:

$$\begin{aligned}\frac{\partial F}{\partial N_a} &= - \int_{u_a}^{\infty} du \frac{\partial S_{\text{eff}}}{\partial u} \\ &= - \int_{u_a}^{\infty} du \left[\frac{k_a}{2\pi i} u - \sum_{i=1}^{N_a} \coth \frac{u - u_i^a}{2} + \sum_{b \neq a}^n \frac{n_{ab}}{2} \sum_{i=1}^{N_b} \tanh \frac{u - u_i^b}{2} \right],\end{aligned}\quad (\text{B.8})$$

where u_a is an endpoint of I_a . Note that this integral must be regularized in some manner. The quantity $\partial_{N_a} F$ is the change of the free energy F when a single eigenvalue is brought from infinity to an edge of I_a . The value u_a can be changed to any value on I_a as long as u_i^a are replaced with the saddle point values $\bar{u}_i^a$.

This can be written in terms of the resolvents as

$$\begin{aligned}\frac{\partial F}{\partial N_a} &= - \int_{p_a}^{\infty} \frac{dz}{z} \left[\frac{k_a}{2\pi i} \log(\epsilon_a z) - \frac{N_a}{t_a} v_a(z) + \sum_{b \neq a}^n \frac{n_{ab}}{2} \frac{N_b}{t_b} v_b(z) \right] \\ &= - \frac{k_a}{4\pi i} (\log(\epsilon_a z))^2 + \log(\epsilon_a z) \left(\frac{N_a}{t_a} v_a(z) - \sum_{b \neq a}^n \frac{n_{ab}}{2} \frac{N_b}{t_b} v_b(z) \right) \Big|_{p_a}^{\infty} \\ &\quad + \int_{p_a}^{\infty} dz \left[- \frac{N_a}{t_a} \frac{z v'_a(z)}{z} \log(\epsilon_a z) + \sum_{b \neq a}^n \frac{n_{ab}}{2} \frac{N_b}{t_b} \frac{z v'_b(z)}{z} \log(\epsilon_a z) \right].\end{aligned}\quad (\text{B.9})$$

Since $\partial_u S_{\text{eff}} = 0$ is satisfied on the branch cut, most of the surface terms at $z = p_a$ cancels. The surface term at $z = \infty$ should be subtracted for the regularization. Therefore, the relevant part of the free energy is given by the last integral in (B.9).

C Constraints on parameters in $\Omega(s)$ and $z(s)$

We have shown in subsection 3.1 that the information we need is encoded into two rational functions $\Omega(s)$ and $z(s)$ on $\mathbb{CP}^1$. The general form of $\Omega(s)$ is given in (3.12). There are $4n + 1$ parameters:

$$A, \quad B_a, \quad C_a, \quad \sigma_a, \quad \tau_a, \quad a = 1, 2, \dots, n. \quad (\text{C.1})$$

The general form of $z(s)$ is given in (3.8). There are $2n + 3$ parameters:

$$D, \quad \xi_\alpha, \quad \eta_\alpha, \quad \alpha = 0, 1, 2, \dots, n. \quad (\text{C.2})$$

We should not fix all the above parameters since n of them will be related to n 't Hooft couplings t_a by (2.16). Therefore, we need $5n + 4$ constraints for fixing the other parameters. They are given as follows:

- The choice of the coordinate s on $\mathbb{CP}^1$ is arbitrary. We can make an $\text{SL}(2, \mathbb{C})$ transformation to fix 3 parameters in $z(s)$.

- The values of σ_a and τ_a , corresponding to the branch points of $v_a(z)$, are determined by

$$z'(\sigma_a) = z'(\tau_a) = 0. \quad (\text{C.3})$$

Indeed, the equation $z'(s) = 0$ has $2n$ solutions since the numerator of $z'(s)$ is a polynomial of degree $2n$. The relations between the coefficients of the polynomial and its solutions give $2n$ constraints on $\sigma_a, \tau_a, \xi_\alpha$ and η_α .

- The saddle point equations (A.2) are invariant under the sign flip $u_i^a \rightarrow -u_i^a$. Since the variables z_i^a are given by $\exp(u_i^a)$ up to sign, this invariance implies that the endpoints $z = p_a, q_a$ of the segment I_a must satisfy $p_a q_a = 1$. These relations give n constraints

$$z(\sigma_a)z(\tau_a) = 1. \quad (\text{C.4})$$

- The resolvents $v_a(z)$ should be holomorphic at $z = 0, \infty$. This requirement implies that $z v'_a(z)$ vanish at $z = 0, \infty$. These conditions then require $\Omega(s)$ to satisfy

$$\Omega(\eta_\alpha) = \Omega(\xi_\alpha) = c_\alpha - c_{\alpha+1}. \quad (\text{C.5})$$

Recall that c_a are determined by (3.3). These give $2n + 2$ constraints. We will see shortly that only $2n + 1$ of them are independent constraints.

In total, we have $3 + 2n + n + 2n + 1 = 5n + 4$ constraints, as expected.

In the following, we show that the constraints listed above really fix the $5n + 4$ parameters.

First, we show that only $2n + 1$ of the constraints (C.5) are independent. The general form (3.12) of $\Omega(s)$ implies

$$\sum_{\alpha=0}^n (\Omega(\eta_\alpha) - \Omega(\xi_\alpha)) = \sum_{\alpha=0}^n \sum_{a=1}^n \left[\frac{B_a}{\eta_\alpha - \sigma_a} + \frac{C_a}{\eta_\alpha - \tau_a} - \frac{B_a}{\xi_\alpha - \sigma_a} - \frac{C_a}{\xi_\alpha - \tau_a} \right]. \quad (\text{C.6})$$

The conditions (C.3) can be written as

$$\sum_{\alpha=0}^n \left[\frac{1}{\sigma_a - \eta_\alpha} - \frac{1}{\sigma_a - \xi_\alpha} \right] = 0, \quad (\text{C.7})$$

$$\sum_{\alpha=0}^n \left[\frac{1}{\tau_a - \eta_\alpha} - \frac{1}{\tau_a - \xi_\alpha} \right] = 0. \quad (\text{C.8})$$

Using these relations, we find

$$\sum_{\alpha=0}^{\infty} (\Omega(\eta_\alpha) - \Omega(\xi_\alpha)) = 0. \quad (\text{C.9})$$

Therefore, at least one equation in (C.5) is redundant.

To show that the remaining equations are independent, we choose $\eta_0 = 0$ and $\xi_0 = \infty$ by using an $\text{SL}(2, \mathbb{C})$ transformation. Then, we obtain $A = \Omega(\eta_0) = \Omega(\xi_0)$. The rest of the equations can be written as

$$\sum_{a=1}^n \left[\frac{B_a}{\eta_b - \sigma_a} + \frac{C_a}{\eta_b - \tau_a} \right] = \Omega(\eta_b) - \Omega(\eta_0), \quad (\text{C.10})$$

$$\sum_{a=1}^n \left[\frac{B_a}{\xi_b - \sigma_a} + \frac{C_a}{\xi_b - \tau_a} \right] = \Omega(\xi_b) - \Omega(\xi_0). \quad (\text{C.11})$$

These equations determine B_a and C_a uniquely if and only if

$$\frac{\prod_{a < b}^n (\eta_a - \eta_b)(\xi_a - \xi_b)(\sigma_a - \sigma_b)(\tau_a - \tau_b) \prod_{a,b=1}^n (\eta_a - \xi_b)(\sigma_a - \tau_b)}{\prod_{a,b=1}^n (\eta_a - \sigma_b)(\xi_a - \sigma_b)(\eta_a - \tau_b)(\xi_a - \tau_b)} \quad (\text{C.12})$$

is non-vanishing. Therefore, generically the $2n + 1$ constraints are independent.

In fact, a large part of the constraints can be solved explicitly. To show this, we choose

$$z(s) = s \prod_{a=1}^n \frac{\xi_a s - 1}{s - \xi_a}. \quad (\text{C.13})$$

The freedom to perform the $\text{SL}(2, \mathbb{C})$ transformations is now fixed. This satisfies $z(s^{-1}) = z(s)^{-1}$. Differentiating this relation, we obtain

$$-\frac{1}{s^2} z'(s^{-1}) = -\frac{1}{z(s)^2} z'(s). \quad (\text{C.14})$$

This implies that, if σ satisfies $z'(\sigma) = 0$, then $\tau := \sigma^{-1}$ also satisfies $z'(\tau) = 0$. Therefore, by choosing $\tau_a = \sigma_a^{-1}$, the half of (C.3) are solved. Also, the constraints (C.4) are automatically satisfied. Then, σ_a are determined by solving the equations

$$\frac{1}{s} + \sum_{a=1}^n \left[\frac{1}{s - \xi_a^{-1}} - \frac{1}{s - \xi_a} \right] = 0. \quad (\text{C.15})$$

To determine $\Omega(s)$, we choose

$$B_a = \sigma_a E_a, \quad C_a = -\tau_a E_a, \quad (\text{C.16})$$

using new parameters E_a . For this choice, $\Omega(s)$ satisfies

$$\Omega(s^{-1}) = \Omega(s). \quad (\text{C.17})$$

Since we chose $\eta_a = \xi_a^{-1}$ in (C.13), the half of the conditions (C.5) are automatically solved. The remaining $n + 1$ equations determine A and E_a uniquely.

We have shown that the $5n + 4$ constraints determine $z(s)$ and $\Omega(s)$. All the parameters are determined by ξ_a . Since there is no further constraint on ξ_a , the parameter space is $\mathbb{C}^n$ corresponding to a subspace of $\mathbb{CP}^{2n-1}$ specified by κ_a not all of them vanishing. Note that there could be a set of discrete choices for the solutions of the constraints. We will fix this ambiguity by examining the weak coupling results, as in section 4, since they can be also obtained directly from the localized partition function (A.1).

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] A. Hanany and E. Witten, *Type IIB superstrings, BPS monopoles and three-dimensional gauge dynamics*, *Nucl. Phys. B* **492** (1997) 152 [[hep-th/9611230](#)] [[INSPIRE](#)].
- [2] T. Kitao, K. Ohta and N. Ohta, *Three-dimensional gauge dynamics from brane configurations with (p,q) -fivebrane*, *Nucl. Phys. B* **539** (1999) 79 [[hep-th/9808111](#)] [[INSPIRE](#)].
- [3] O. Bergman, A. Hanany, A. Karch and B. Kol, *Branes and supersymmetry breaking in three-dimensional gauge theories*, *JHEP* **10** (1999) 036 [[hep-th/9908075](#)] [[INSPIRE](#)].
- [4] O. Aharony, O. Bergman, D.L. Jafferis and J. Maldacena, *$N = 6$ superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*, *JHEP* **10** (2008) 091 [[arXiv:0806.1218](#)] [[INSPIRE](#)].
- [5] Y. Imamura and K. Kimura, *On the moduli space of elliptic Maxwell-Chern-Simons theories*, *Prog. Theor. Phys.* **120** (2008) 509 [[arXiv:0806.3727](#)] [[INSPIRE](#)].
- [6] D.L. Jafferis and A. Tomasiello, *A simple class of $N = 3$ gauge/gravity duals*, *JHEP* **10** (2008) 101 [[arXiv:0808.0864](#)] [[INSPIRE](#)].
- [7] D. Gaiotto and E. Witten, *Janus Configurations, Chern-Simons Couplings, And The θ -Angle in $N = 4$ Super Yang-Mills Theory*, *JHEP* **06** (2010) 097 [[arXiv:0804.2907](#)] [[INSPIRE](#)].
- [8] K. Hosomichi, K.-M. Lee, S. Lee, S. Lee and J. Park, *$N = 4$ Superconformal Chern-Simons Theories with Hyper and Twisted Hyper Multiplets*, *JHEP* **07** (2008) 091 [[arXiv:0805.3662](#)] [[INSPIRE](#)].
- [9] A. Kapustin, B. Willett and I. Yaakov, *Exact Results for Wilson Loops in Superconformal Chern-Simons Theories with Matter*, *JHEP* **03** (2010) 089 [[arXiv:0909.4559](#)] [[INSPIRE](#)].
- [10] T. Nosaka and S. Yokoyama, *Complete factorization in minimal $\mathcal{N} = 4$ Chern-Simons-matter theory*, *JHEP* **01** (2018) 001 [[arXiv:1706.07234](#)] [[INSPIRE](#)].
- [11] T. Nosaka and S. Yokoyama, *Index and duality of minimal $\mathcal{N} = 4$ Chern-Simons-matter theories*, *JHEP* **06** (2018) 028 [[arXiv:1804.04639](#)] [[INSPIRE](#)].
- [12] D. Gaiotto and X. Yin, *Notes on superconformal Chern-Simons-Matter theories*, *JHEP* **08** (2007) 056 [[arXiv:0704.3740](#)] [[INSPIRE](#)].
- [13] N. Drukker and D. Trancanelli, *A supermatrix model for $N = 6$ super Chern-Simons-matter theory*, *JHEP* **02** (2010) 058 [[arXiv:0912.3006](#)] [[INSPIRE](#)].
- [14] H. Ouyang, J.-B. Wu and J.-j. Zhang, *Construction and classification of novel BPS Wilson loops in quiver Chern-Simons-matter theories*, *Nucl. Phys. B* **910** (2016) 496 [[arXiv:1511.02967](#)] [[INSPIRE](#)].
- [15] J. Cookmeyer, J.T. Liu and L.A. Pando Zayas, *Higher Rank ABJM Wilson Loops from Matrix Models*, *JHEP* **11** (2016) 121 [[arXiv:1609.08165](#)] [[INSPIRE](#)].
- [16] T. Morita and K. Sugiyama, *Multi-cut solutions in Chern-Simons matrix models*, *Nucl. Phys. B* **929** (2018) 1 [[arXiv:1704.08675](#)] [[INSPIRE](#)].

- [17] T. Suyama, *Notes on Planar Resolvents of Chern-Simons-matter Matrix Models*, *JHEP* **11** (2016) 049 [[arXiv:1605.09110](#)] [[INSPIRE](#)].
- [18] T. Suyama, *Strong Coupling Limit of A Family of Chern-Simons-matter Theories*, *JHEP* **01** (2018) 125 [[arXiv:1706.08204](#)] [[INSPIRE](#)].
- [19] E. Witten, *Quantum Field Theory and the Jones Polynomial*, *Commun. Math. Phys.* **121** (1989) 351 [[INSPIRE](#)].
- [20] K. Ohta, *Supersymmetric index and s rule for type IIB branes*, *JHEP* **10** (1999) 006 [[hep-th/9908120](#)] [[INSPIRE](#)].
- [21] D. Gaiotto and E. Witten, *S-duality of Boundary Conditions In $N = 4$ Super Yang-Mills Theory*, *Adv. Theor. Math. Phys.* **13** (2009) 721 [[arXiv:0807.3720](#)] [[INSPIRE](#)].
- [22] T. Suyama, *Supersymmetry Breaking and Planar Free Energy in Chern-Simons-matter Theories*, [arXiv:1405.7469](#) [[INSPIRE](#)].
- [23] T. Morita and V. Niarchos, *F-theorem, duality and SUSY breaking in one-adjoint Chern-Simons-Matter theories*, *Nucl. Phys. B* **858** (2012) 84 [[arXiv:1108.4963](#)] [[INSPIRE](#)].
- [24] N. Drukker, M. Mariño and P. Putrov, *From weak to strong coupling in ABJM theory*, *Commun. Math. Phys.* **306** (2011) 511 [[arXiv:1007.3837](#)] [[INSPIRE](#)].
- [25] T. Suyama, *A Systematic Study on Matrix Models for Chern-Simons-matter Theories*, *Nucl. Phys. B* **874** (2013) 528 [[arXiv:1304.7831](#)] [[INSPIRE](#)].
- [26] M. Mariño and P. Putrov, *ABJM theory as a Fermi gas*, *J. Stat. Mech.* **1203** (2012) P03001 [[arXiv:1110.4066](#)] [[INSPIRE](#)].
- [27] C.P. Herzog, I.R. Klebanov, S.S. Pufu and T. Tesileanu, *Multi-Matrix Models and Tri-Sasaki Einstein Spaces*, *Phys. Rev. D* **83** (2011) 046001 [[arXiv:1011.5487](#)] [[INSPIRE](#)].
- [28] D.R. Gulotta, C.P. Herzog and T. Nishioka, *The ABCDEF's of Matrix Models for Supersymmetric Chern-Simons Theories*, *JHEP* **04** (2012) 138 [[arXiv:1201.6360](#)] [[INSPIRE](#)].
- [29] M. Mariño and P. Putrov, *Exact Results in ABJM Theory from Topological Strings*, *JHEP* **06** (2010) 011 [[arXiv:0912.3074](#)] [[INSPIRE](#)].