Boundary conditions for General Relativity in three-dimensional spacetimes, integrable systems and the KdV/mKdV hierarchies

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ABSTRACT: We present a new set of boundary conditions for General Relativity on AdS_3 , where the dynamics of the boundary degrees of freedom are described by two independent left and right members of the Gardner hierarchy of integrable equations, also known as the “mixed KdV-mKdV” hierarchy. This integrable system has the very special property that simultaneously combines both, the Korteweg-de Vries (KdV) and modified Korteweg-de Vries (mKdV) hierarchies in a single integrable structure. This relationship between gravitation in three-dimensional spacetimes and two-dimensional integrable systems is based on an extension of the recently introduced “soft hairy boundary conditions” on AdS_3 , where the chemical potentials are now allowed to depend locally on the dynamical fields and their spatial derivatives. The complete integrable structure of the Gardner system, i.e., the phase space, the Poisson brackets and the infinite number of commuting conserved charges, are directly obtained from the asymptotic analysis and the conserved surface integrals in the gravitational theory. These boundary conditions have the particular property that they can also be interpreted as being defined in the near horizon region of spacetimes with event horizons. Black hole solutions are then naturally accommodated within our boundary conditions, and are described by static configurations associated to the corresponding member of the Gardner hierarchy. The thermodynamic properties of the black holes in the ensembles defined by our boundary conditions are also discussed. Finally, we show that our results can be naturally extended to the case of a vanishing cosmological constant, and the integrable system turns out to be precisely the same as in the case of AdS_3 .

KEYWORDS: Black Holes, Classical Theories of Gravity, Gauge-gravity correspondence

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1 Introduction

The asymptotic structure of General Relativity in 2+1 spacetime dimensions has been recently shown to be deeply linked with certain special classes of integrable systems in 1+1 dimensions. For example, a new set of boundary conditions for Einstein gravity on AdS_3 , labeled by a nonnegative integer k , was proposed in ref. [1]. The dynamics of the gravitational excitations at the boundary are then described by two independent copies of the k -th element of the Korteweg-de Vries (KdV) hierarchy of nonlinear partial differential equations, where the standard Brown-Henneaux boundary conditions [2] are contained as a particular case ($k = 0$). For a vanishing cosmological constant, the associated integrable system corresponds to a generalization of the Hirota-Satsuma coupled KdV system [3], which possesses a BMS_3 Poisson structure [4].

The key of this relationship between three-dimensional gravity and two-dimensional integrable systems is the precise way in which the length and time scales are fixed in the

asymptotic region of spacetime in the gravitational theory. Indeed, the asymptotic values of the lapse and shift functions in an ADM decomposition of the metric, can be chosen to depend explicitly on the dynamical fields in a very tight form, compatible with the action principle.

In the case of the KdV hierarchy analyzed in ref. [1], the fall-off of the metric coincides with the one of Brown and Henneaux when one includes the most general form of the chemical potentials [5, 6]. However, the lapse and shift functions, and consequently the boundary metric of AdS_3 , are no longer kept fixed at infinity, because they are now allowed to explicitly depend on the Virasoro currents. As a consequence of this particular choice of chemical potentials, Einstein equations in the asymptotic region precisely reduce to two independent left and right copies of the k -th element of the KdV hierarchy. Furthermore, the complete integrable structure of the KdV system, i.e., the phase space, the Poisson brackets and the infinite number of commuting conserved charges, are directly obtained from the asymptotic analysis and the conserved surface integrals in the gravitational theory. It is worth pointing out that there exists a deep relation between the KdV hierarchy and two-dimensional conformal field theories (CFT_2), indeed, the infinite (quantum) commuting KdV charges can be expressed as composite operators in terms of the stress tensor of a CFT_2 [7–9]. This fact has been recently used to describe Generalized Gibbs ensembles in these theories [10–16], as well as their holographic description [1].

In this paper, we explore the consequences of extending the “soft hairy” boundary conditions on AdS_3 introduced in refs. [17, 18] by choosing the chemical potentials as appropriate *local* functions of the fields, and its relation with integrable systems. For this class of boundary conditions, the fall-off of the metric near infinity differs from the one of Brown and Henneaux, and they have the particular property that, by virtue of the topological nature of three-dimensional General Relativity, they can also be interpreted as being defined in the near horizon region of spacetimes with event horizons.

The corresponding integrable system is shown to be the Gardner hierarchy of non-linear partial differential equations, also known as the “mixed KdV-mKdV” hierarchy (see e.g. [19]), whose first member is given by

$$\dot{\mathcal{J}} = 3a\mathcal{J}\mathcal{J}' + 3b\mathcal{J}^2\mathcal{J}' - 2\mathcal{J}'''. \quad (1.1)$$

Here, dots and primes denote derivatives with respect to the temporal and spatial coordinates respectively, while a, b are arbitrary constants. Equation (1.1) has the very special property that simultaneously combines both, KdV and modified KdV (mKdV) equations. Remarkably, in spite of the deformations generated by the arbitrary parameters a and b , the integrability of this equation and of the complete hierarchy associated to it, is maintained. Indeed, when $b = 0$ and $a \neq 0$, eq. (1.1) precisely reduces to the KdV equation, while when $a = 0$ and $b \neq 0$ it coincides with the mKdV equation.¹ From the point of view of the gravitational theory, there exists a precise choice of chemical potentials, extending

¹There is a very deep link between mKdV and KdV equations. If the field $\mathcal{J}$ obeys the mKdV equation, then the field $\mathcal{L} = b\mathcal{J}^2 + 2\sqrt{b}\mathcal{J}'$ will precisely obey the KdV equation with $a = 1$. This relationship between $\mathcal{L}$ and $\mathcal{J}$ is known as the Miura transformation, and turns out to be fundamental in order to prove the integrability of the KdV and mKdV equations [20–23] (see also [24]).

the boundary conditions in [17, 18], such that Einstein equations reduce to two (left and right) copies of eq. (1.1). Moreover, the infinite number of abelian conserved quantities of the Gardner hierarchy are obtained from the asymptotic symmetries and the canonical generators in General Relativity using the Regge-Teitelboim approach [25]. These results are generalized to the whole hierarchy, and can also be applied to the case with a vanishing cosmological constant. We also show that if the hierarchy is “extended backwards” one recovers the soft hairy boundary conditions of refs. [17, 18] as a particular case.

For a negative cosmological constant, black hole solutions are naturally accommodated within our boundary conditions, and are described by static configurations associated to the corresponding member of the Gardner hierarchy. In particular, BTZ black holes [26, 27] are described by constant values of the left and right fields $\mathcal{J}_\pm$. Static solutions with nonconstants $\mathcal{J}_\pm$ can also be found, and describe black holes carrying nontrivial conserved charges. The thermodynamic properties of the black holes in the ensembles defined by our boundary conditions will be also discussed.

Note added. After this work was completed we received the preprint [28], where the authors also considered the KdV hierarchy in the context of generalizations of the soft hairy boundary conditions introduced in refs. [17, 18]. In our notation this corresponds to the particular case $b = 0$. However, they mostly focus on the description of the boundary actions associated with this integrable system, and its interpretation from a near horizon perspective.

2 Asymptotic behavior of the gravitational field

In this section we describe the asymptotic behavior (fall-off) of the gravitational field without specifying yet what is fixed at the boundary of spacetime, i.e., without imposing at this step of the analysis a precise boundary condition. For the purpose of simplicity and clarity in the presentation, we mostly work in the Chern-Simons formulation of three-dimensional Einstein gravity with a negative cosmological constant [29, 30].

The action for General Relativity on AdS_3 can be written as a Chern-Simons action for the gauge group $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$

$$I = I_{CS} [A^+] - I_{CS} [A^-],$$

where

$$I_{CS} [A] = \frac{\kappa}{4\pi} \int \left\langle AdA + \frac{2}{3} A^3 \right\rangle. \quad (2.1)$$

Here, the gauge connections $A^\pm$ are 1-forms valued on the $sl(2, \mathbb{R})$ algebra, and are related to the vielbein e and spin connection ω through $A^\pm = \omega \pm e\ell^{-1}$. The level in (2.1) is given by $\kappa = \ell/4G$, where ℓ is the AdS radius and G is the Newton constant, while the bilinear form $\langle, \rangle$ is defined by the trace in the fundamental representation of $sl(2, \mathbb{R})$. The $sl(2, \mathbb{R})$ generators L_n with $n = 0, \pm 1$, obey the commutation relations $[L_n, L_m] = (n - m) L_{n+m}$, with non-vanishing components of the bilinear form given by $\langle L_1 L_{-1} \rangle = -1$, and $\langle L_0^2 \rangle = 1/2$.

2.1 Asymptotic form of the gauge field

In order to describe the fall-off of the gauge connection, we closely follow the analysis in refs. [17, 18]. We will assume that the gauge fields in the asymptotic region take the form

$$A^\pm = b_\pm^{-1} (d + \mathfrak{a}^\pm) b_\pm, \quad (2.2)$$

where $b_\pm$ are gauge group elements depending only on the radial coordinate, and $\mathfrak{a}^\pm = \mathfrak{a}_t^\pm dt + \mathfrak{a}_\phi^\pm d\phi$ correspond to auxiliary connections that only depend on the temporal and angular coordinates t and ϕ , respectively [31]. It is worth pointing out that, by virtue of the lack of local propagating degrees of freedom of three-dimensional Einstein gravity, all the relevant physical information necessary for the asymptotic analysis is completely encoded in the auxiliary connections $\mathfrak{a}^\pm$, independently of the precise choice of $b_\pm$. We will not specify any particular $b_\pm$ until section 4, where the metric formulation will be discussed.

The auxiliary connections $\mathfrak{a}^\pm$ are chosen to be given by

$$\mathfrak{a}^\pm = L_0 (\zeta^\pm dt \pm \mathcal{J}^\pm d\phi). \quad (2.3)$$

Here, $\mathfrak{a}^\pm$ are diagonal matrices in the fundamental representation of $sl(2, \mathbb{R})$, and by that reason we say that the connections $\mathfrak{a}^\pm$ in eq. (2.3) are written in the “diagonal gauge”. The fields $\zeta^\pm$ are defined along the temporal components of $\mathfrak{a}^\pm$, and consequently they correspond to Lagrange multipliers (chemical potentials). On the other hand, $\mathcal{J}^\pm$ belong to the spatial components of the gauge connections and therefore they are identified as the dynamical fields.

The field equations are determined by the vanishing of the field strength $F^\pm = dA^\pm + A^{\pm 2} = 0$, and take the form

$$\dot{\mathcal{J}}_\pm = \pm \zeta'_\pm. \quad (2.4)$$

Note that there are no time derivatives associated to the fields $\zeta_\pm$, which is consistent with the fact that they are chemical potentials.

It is important to emphasize that, as was pointed out in refs. [17, 18], one can also interpret the connections in (2.2) and (2.3) as describing the behavior of the fields in a region near a horizon. Indeed, the reconstructed metric can always be expressed (in a co-rotating frame) as the direct product of the two-dimensional Rindler metric times S^1 , together with appropriate deviations from it. Furthermore, the charges are independent of the gauge group element $b_\pm(r)$, and consequently they do not depend on the precise value of the radial coordinate where they are evaluated. In this sense, one can also interpret the present analysis as describing “near horizon boundary conditions”. This near horizon interpretation can also be extended to higher spacetime dimensions [32].

2.2 Consistency with the action principle and canonical generators

A fundamental requirement that must be guaranteed is the consistency of the asymptotic form of the gauge connections in eqs. (2.2) and (2.3) with the action principle. In order to analyze the physical consequences of this requirement, we will use the Regge-Teitelboim approach [25] in the canonical formulation of Chern-Simons theory.

The canonical action can be written as

$$I_{can} [A^\pm] = -\frac{\kappa}{4\pi} \int dt d^2x \varepsilon^{ij} \left\langle A_i^\pm A_j^\pm - A_t^\pm F_{ij}^\pm \right\rangle + B_\infty^\pm,$$

where $B_\infty^\pm$ are boundary terms needed to ensure that the action principle attains an extremum. Their variations are then given by

$$\delta B_\infty^\pm = -\frac{\kappa}{2\pi} \int dt d\phi \left\langle A_t^\pm \delta A_\phi^\pm \right\rangle,$$

so that, when one evaluates them for the asymptotic form of the connections (2.2) and (2.3) yields

$$\delta B_\infty^\pm = \mp \frac{\kappa}{4\pi} \int dt d\phi \zeta_\pm \delta \mathcal{J}_\pm. \quad (2.5)$$

Consistency of the analysis requires that one must be able to “take the delta outside” in the variation of the boundary terms (2.5), i.e., they must be integrable in the functional space.² This can only be achieved provided one specifies a precise boundary condition, which turns out to be equivalent to specify what fields are fixed, i.e., without functional variation, at the boundary of spacetime. Following ref. [1], a generic possible choice of boundary conditions is to assume that the chemical potentials $\zeta_\pm$ depend on the dynamical fields $\mathcal{J}_\pm$ through

$$\zeta_\pm = \frac{4\pi}{\kappa} \frac{\delta H^\pm}{\delta \mathcal{J}_\pm}, \quad (2.6)$$

where $H^\pm = \int d\phi \mathcal{H}^\pm[\mathcal{J}_\pm, \mathcal{J}'_\pm, \mathcal{J}''_\pm, \dots]$ are functionals depending *locally* on the fields $\mathcal{J}_\pm$ and their spatial derivatives. Here we also assume that the left and right sectors are decoupled. With the particular choice of boundary conditions given by (2.6), the delta in (2.5) can be immediately taken outside, and consequently the boundary terms necessary to improve the canonical action take the form

$$B^\pm = \mp \int dt H^\pm.$$

An immediate consequence of this choice is that the total energy of the system, defined as the on-shell value of the generator of translations in time, can be directly written in terms of the “Hamiltonians” $H^\pm$ as

$$E = H^+ + H^-. \quad (2.7)$$

The conserved charges associated to the asymptotic symmetries are also sensitive to the choice of boundary conditions, but in a more subtle way. The form of the connections $\mathfrak{a}^\pm$ in eq. (2.3) is preserved under gauge transformations $\delta \mathfrak{a}^\pm = d\lambda^\pm + [\mathfrak{a}^\pm, \lambda^\pm]$, with gauge parameters $\lambda^\pm$ given by³

$$\lambda^\pm = \eta_\pm L_0,$$

²The requirement of integrability of the boundary terms can be relaxed in the presence of ingoing or outgoing radiation, where their lack of integrability precisely gives the rate of change of the charges in time [33–35]. In the present case this possibility is not at hand, because General Relativity in 2+1-dimensions does not have local propagating degrees of freedom.

³Extra terms along L_1 and L_{-1} might also be added, however they are pure gauge.

provided the fields $\mathcal{J}^\pm$ and the chemical potentials $\zeta^\pm$ transform as

$$\delta\mathcal{J}_\pm = \pm\eta'_\pm, \quad (2.8)$$

$$\delta\zeta_\pm = \dot{\eta}_\pm. \quad (2.9)$$

Following the Regge-Teitelboim approach [25], the variation of the conserved charges reads

$$\delta Q^\pm[\eta_\pm] = \pm \frac{\kappa}{4\pi} \oint d\phi \eta_\pm \delta\mathcal{J}_\pm. \quad (2.10)$$

Here, the parameters $\eta^\pm$ are not arbitrary, because now the chemical potentials $\zeta^\pm$ depend on the fields $\mathcal{J}^\pm$ and their spatial derivatives. Hence, one must use the chain rule in the variation at the left hand-side of eq. (2.9), giving the following first order differential equations in time for $\eta^\pm$

$$\dot{\eta}_\pm = \pm \frac{4\pi}{\kappa} \frac{\delta}{\delta\mathcal{J}_\pm} \int d\phi \frac{\delta H^\pm}{\delta\mathcal{J}_\pm} \partial_\phi \eta_\pm. \quad (2.11)$$

Generically, these differential equations will depend explicitly on $\mathcal{J}_\pm$, and consequently finding explicit solutions to them becomes a very hard task. However, this can be achieved for certain special choices of boundary conditions, which are related to the integrable hierarchy associated to the Gardner equation. This will be discussed in detail in the next section.

3 Different choices of boundary conditions in the diagonal gauge

3.1 Soft hairy boundary conditions

A simple choice of boundary conditions corresponds to fix the chemical potentials $\zeta_\pm$ at the boundary, such that they are arbitrary functions without variation, i.e., $\delta\zeta_\pm = 0$. This possibility was analyzed in detail in refs. [17, 18], and it was termed “soft hairy boundary conditions”. The asymptotic symmetry algebra is then spanned by the generators

$$Q_{\text{soft hairy}}^\pm[\eta_\pm] = \pm \frac{\kappa}{4\pi} \oint d\phi \eta_\pm \mathcal{J}_\pm,$$

where, by the consistency with time evolution, the parameters $\eta_\pm$ are arbitrary functions without variation ($\delta\eta_\pm = 0$). The global charges are then characterized by $\mathcal{J}_\pm$, which obey the following Poisson brackets

$$\{\mathcal{J}_\pm(\phi), \mathcal{J}_\pm(\bar{\phi})\} = \pm \frac{4\pi}{\kappa} \partial_\phi \delta(\phi - \bar{\phi}). \quad (3.1)$$

The asymptotic symmetry algebra is then given by two copies of $\hat{u}(1)$ current algebras.

In a co-rotating frame, $\zeta_+ = \zeta_- = \text{const.}$, the generator of time evolution is identified with the sum of the left and right zero modes of $\mathcal{J}_\pm$, that commute with all the members of the algebra. In this sense, one can say that the higher modes describe “soft hair excitations” in the sense of Hawking, Perry and Strominger [36, 37], because they do not change the energy of the gravitational configuration. Besides, in this frame, it is possible to find solutions with non-extremal horizons which are diffeomorphic to BTZ black holes, and

that are endowed with not trivial soft hair charges. These solutions were called “black flowers”⁴ [17, 18].

3.2 Gardner equation

A different choice of boundary conditions that makes contact with the Gardner (mixed KdV-mKdV) equation is $\zeta^\pm = \frac{3}{2}a\mathcal{J}_\pm^2 + b\mathcal{J}_\pm^3 - 2\mathcal{J}_\pm''$, which according to eq. (2.6) corresponds to use the following Hamiltonians

$$H_{(1)}^\pm = \frac{\kappa}{4\pi} \oint d\phi \left(\frac{1}{2}a\mathcal{J}_\pm^3 + \frac{1}{4}b\mathcal{J}_\pm^4 + \mathcal{J}_\pm'^2 \right). \quad (3.2)$$

With this particular choice of chemical potentials, Einstein equations (2.4) precisely reduce to two independent left and right copies of the Gardner equation (1.1), that read

$$\dot{\mathcal{J}}_\pm = \pm (3a\mathcal{J}_\pm\mathcal{J}_\pm' + 3b\mathcal{J}_\pm^2\mathcal{J}_\pm' - 2\mathcal{J}_\pm'''). \quad (3.3)$$

Equations (2.11), describing the time evolution of the gauge parameters $\eta^\pm$, now take the form

$$\dot{\eta}_\pm = \pm (3a\mathcal{J}_\pm\eta_\pm' + 3b\mathcal{J}_\pm^2\eta_\pm' - 2\eta_\pm'''), \quad (3.4)$$

where the explicit dependence on $\mathcal{J}_\pm$ is manifest. These equations are linear in $\eta_\pm$, and by virtue of the integrability of the system, it is possible to find their general solutions under the assumption that they depend locally on $\mathcal{J}_\pm$ and their spatial derivatives. Indeed, the gauge parameters $\eta_\pm$ obeying eq. (3.4), are expressed as a linear combination of “generalized Gelfand-Dikii polynomials” $R_{(n)}^\pm$, i.e.,

$$\eta_\pm = \frac{4\pi}{\kappa} \sum_{n \geq 0} \alpha_{(n)}^\pm R_{(n)}^\pm, \quad (3.5)$$

where $\alpha_{(n)}^\pm$ are arbitrary constants, and the polynomials $R_{(n)}^\pm$ are defined through the following recursion relation

$$\partial_\phi R_{(n+1)}^\pm = \mathcal{D}_\phi R_{(n)}^\pm, \quad (3.6)$$

where $\mathcal{D}_\phi$ is a non-local operator given by

$$\mathcal{D}_\phi := a(\partial_\phi \mathcal{J}_\pm + 2\mathcal{J}_\pm \partial_\phi) + 2b\partial_\phi \left(\mathcal{J}_\pm \partial_\phi^{-1} (\mathcal{J}_\pm \partial_\phi) \right) - 2\partial_\phi^3. \quad (3.7)$$

The polynomials $R_{(n)}^\pm$ can be expressed as the “gradient” of the Hamiltonians $H_{(n)}^\pm$ of the integrable system, i.e.,

$$R_{(n)}^\pm = \frac{\delta H_{(n)}^\pm}{\delta \mathcal{J}_\pm}. \quad (3.8)$$

The first generalized Gelfand-Dikii polynomials together with their corresponding Hamiltonians are explicitly displayed in appendix A.

⁴A different type of black flower solution was found in ref. [38] in the context of new massive gravity [39, 40].

If one replaces the general solution of eqs. (3.4), given by (3.5), into the expression for the variation of the canonical generators (2.10), one can take immediately the delta outside, and write the conserved charges as

$$Q^\pm = \pm \sum_{n \geq 0} \alpha_{(n)}^\pm H_{(n)}^\pm, \quad (3.9)$$

where the Hamiltonians $H_{(n)}^\pm$ are in involution with respect to the Poisson brackets (3.1),

$$\{H_{(n)}^\pm, H_{(m)}^\pm\} = 0.$$

The presence of the operator $\mathcal{D}_\phi$ in the recurrence relation for the generalized Gelfand-Dikii polynomials (3.6), is related to the fact that this integrable system is bi-Hamiltonian. Consequently, it is possible to define a second Poisson bracket between the dynamical fields

$$\{\mathcal{J}_\pm(\phi), \mathcal{J}_\pm(\bar{\phi})\}_2 := \pm \frac{4\pi}{\kappa} \mathcal{D}_\phi \delta(\phi - \bar{\phi}), \quad (3.10)$$

so that the Gardner equations can be written as

$$\dot{\mathcal{J}}_\pm = \{\mathcal{J}_\pm, H_{(1)}^\pm\} = \{\mathcal{J}_\pm, H_{(0)}^\pm\}_2,$$

where the Hamiltonians $H_{(1)}^\pm$ are given by eq. (3.2), while the Hamiltonians $H_{(0)}^\pm$ take the form

$$H_{(0)}^\pm = \frac{\kappa}{4\pi} \oint d\phi \left(\frac{1}{2} \mathcal{J}_\pm^2 \right). \quad (3.11)$$

It is worth pointing out that only the first Poisson bracket structure, given by eq. (3.1), is obtained from the gravitational theory using the Dirac method for constrained systems. It is not clear how to obtain the second Poisson structure (3.10) directly from the canonical structure of General Relativity.

As it was discussed in the introduction, the (left/right) Gardner equation in (3.3) has the special property that when $a = 0$ and $b \neq 0$, the equation and its whole integrable structure, precisely reduce to those of the mKdV integrable system. Conversely, when $b = 0$ and $a \neq 0$, the integrable system reduces to KdV. In this case, the operator $\mathcal{D}_\phi$ in (3.7) becomes a local differential operator, and the second Poisson structure associated to it through eq. (3.10) corresponds to two copies of the Virasoro algebra with left and right central charges given by $c^\pm = 3\ell / (Ga^2)$. Note that these central charges do not coincide with the ones of Brown and Henneaux. This is not surprising because, as we say before, this Poisson structure does not come directly from the canonical structure of the gravitational theory.

The Gardner equation does not have additional symmetries besides the ones generated by the infinite number of commuting Hamiltonians described above. In particular, it is not invariant under Galilean boosts unless $b = 0$ (KdV case). Nevertheless, it is possible to show that there exists a particular Galilean boost with parameter $\omega = 3a^2 / (4b)$, together with a shift in $\mathcal{J}_\pm$ given by $\mathcal{J}_\pm \rightarrow \mathcal{J}_\pm - a / (2b)$, such that the Gardner equation reduces to the mKdV equation. However, the conserved charges are not mapped into each other, which is a direct consequence of the fact that this transformation does not map the higher members of the Gardner and mKdV hierarchies.

In the next section we will show how to extend the previous results in order to incorporate an arbitrary member of the Gardner hierarchy.

3.3 Extension to the Gardner hierarchy

It is possible to find precise boundary conditions for General Relativity on AdS_3 , such that the dynamics of the boundary degrees of freedom are described by the k -th member of the Gardner hierarchy. This is achieved by choosing the chemical potentials $\zeta_{\pm}$ as follows

$$\zeta_{\pm} = \frac{4\pi}{\kappa} R_{(k)}^{\pm}, \quad (3.12)$$

where $R_{(k)}^{\pm}$ are the k -th generalized Gelfand-Dikii polynomials. The functionals $H^{\pm}$ in eq. (2.6), are identified with the k -th Hamiltonian of the hierarchy defined by eq. (3.8). The particular case $k = 1$ corresponds to the one developed in the previous section.

With the boundary condition specified by eq. (3.12), Einstein equations in (2.4) take the form

$$\dot{\mathcal{J}}_{\pm} = \pm \frac{4\pi}{\kappa} \partial_{\phi} R_{(k)}^{\pm}, \quad (3.13)$$

which coincide with two (left/right) copies of the k -th element of the Gardner hierarchy. By virtue of the bi-Hamiltonian character of the system, eq. (3.13) can also be written as

$$\dot{\mathcal{J}}_{\pm} = \left\{ \mathcal{J}_{\pm}, H_{(k)}^{\pm} \right\} = \left\{ \mathcal{J}_{\pm}, H_{(k-1)}^{\pm} \right\}_2. \quad (3.14)$$

The equations that describe the time evolution of the gauge parameters $\eta_{\pm}$ take the form of eq. (2.11), but with $H^{\pm} \rightarrow H_{(k)}^{\pm}$, whose general solution that depends locally on $\mathcal{J}_{\pm}$ and their spatial derivatives coincides with eq. (3.5). Consequently, the global charges integrate as (3.9), which as it was discussed in the previous section, commute among them with both Poisson structures.

Lifshitz scaling. The members of the Gardner hierarchy are not invariant under Lifshitz scaling. However, in the particular cases when they belong to KdV or mKdV hierarchies, the anisotropic scaling, with dynamical exponent $z = 2k + 1$, is restored.

Case 1: $a = 0$ (mKdV).

When $a = 0$, the k -th member of the hierarchy is invariant under

$$t \rightarrow \lambda^z t, \quad \phi \rightarrow \lambda \phi, \quad \mathcal{J}_{\pm} \rightarrow \lambda^{-1} \mathcal{J}_{\pm}. \quad (3.15)$$

Case 2: $b = 0$ (KdV).

When $b = 0$, the k -th member of the hierarchy is invariant under

$$t \rightarrow \lambda^z t, \quad \phi \rightarrow \lambda \phi, \quad \mathcal{J}_{\pm} \rightarrow \lambda^{-2} \mathcal{J}_{\pm}. \quad (3.16)$$

Note that the dynamical exponent is the same in both cases, but the fields $\mathcal{J}_{\pm}$ scale in different ways.

Extending the hierarchy backwards. The first nonlinear members of the Gardner hierarchy correspond to the case $k = 1$, and are given by eq. (3.3). However, if one uses the Hamiltonians $H_{(0)}^{\pm}$ defined in eq. (3.11) together with the first Poisson bracket structure (3.1), one obtains a linear equation describing left and right chiral movers

$$\dot{\mathcal{J}}_{\pm} = \pm \mathcal{J}'_{\pm},$$

which can be considered as the member with $k = 0$ of the hierarchy.

From the point of view of the gravitational theory, it is useful to extend the hierarchy an additional step backwards by using the recursion relation (3.6) in the opposite direction. Hence, one obtains a new Hamiltonian for each copy of the form

$$H_{(-1)}^{\pm} = \frac{\kappa}{4\pi} \oint d\phi (a^{-1} \mathcal{J}_{\pm}). \quad (3.17)$$

These Hamiltonians can only be defined when $a \neq 0$, and their corresponding generalized Gelfand-Dikii polynomials $R_{(-1)}^{\pm} = \kappa/(4\pi a)$ can be used as a seed that generates the whole hierarchy through the recursion relation.

Using the Hamiltonians $H_{(-1)}^{\pm}$ in (3.17), together with the first Poisson bracket structure (3.1), the soft hairy boundary conditions of refs. [17, 18], reviewed in section 3.1, are recovered in a co-rotating frame. In this case, the chemical potentials $\zeta_{\pm}$ are constants and given by

$$\zeta_{\pm} = \frac{4\pi}{\kappa} R_{(-1)}^{\pm} = a^{-1},$$

while the equations of motion become $\dot{\mathcal{J}}_{\pm} = 0$. In this sense, one can consider the soft hairy boundary conditions as being part of the hierarchy, and besides as the first member of it.

4 Metric formulation

In this section, we provide a metric description of the results previously obtained in the context of the Chern-Simons formulation of General Relativity on AdS_3 . As was discussed in section 2.1, the boundary conditions that describe the Gardner hierarchy can be interpreted as being defined either at infinity or in the near horizon region. We will analyze these two possible interpretations in the metric formalism following the lines of ref. [18].

4.1 Asymptotic behavior

The spacetime metric can be directly reconstructed from the Chern-Simons fields (2.2), (2.3), provided a particular gauge group element $b_{\pm}(r)$ is specified. In order to describe the metric in the asymptotic region, it is useful to choose

$$b_{\pm}(r) = \exp \left[\pm \frac{1}{2} \log \left(\frac{2r}{\ell} \right) (L_1 - L_{-1}) \right].$$

The expansion of the metric for $r \rightarrow \infty$ then reads

$$\begin{aligned} g_{tt} &= -\zeta_+ \zeta_- r^2 + \frac{\ell^2}{4} (\zeta_+^2 + \zeta_-^2) + \mathcal{O}(r^{-1}), \\ g_{tr} &= \mathcal{O}(r^{-2}), \\ g_{t\phi} &= (\zeta_+ \mathcal{J}_- - \zeta_- \mathcal{J}_+) \frac{r^2}{2} + \frac{\ell^2}{4} (\zeta_+ \mathcal{J}_+ - \zeta_- \mathcal{J}_-) + \mathcal{O}(r^{-1}), \\ g_{rr} &= \frac{\ell^2}{r^2} + \mathcal{O}(r^{-5}), \\ g_{r\phi} &= \mathcal{O}(r^{-2}), \\ g_{\phi\phi} &= \mathcal{J}_+ \mathcal{J}_- r^2 + \frac{\ell^2}{4} (\mathcal{J}_+^2 + \mathcal{J}_-^2) + \mathcal{O}(r^{-1}). \end{aligned} \quad (4.1)$$

As it was explained in section 3.3, in order to implement the boundary conditions associated to the Gardner hierarchy we must choose the chemical potentials $\zeta_{\pm}$ according to eq. (3.12), i.e.,

$$\zeta_{\pm} = \frac{4\pi}{\kappa} R_{(k)}^{\pm}.$$

The differential equations associated to the (left/right) k -th element of the hierarchy are precisely recovered if one imposes that the metric in eq. (4.1) obeys Einstein equations with a negative cosmological constant in the asymptotic region of spacetime.

The fall-off in (4.1) is preserved under the asymptotic symmetries generated by the following asymptotic Killing vectors

$$\begin{aligned}\xi^t &= \frac{\eta^+ \mathcal{J}_- + \eta^- \mathcal{J}_+}{\zeta_+ \mathcal{J}_- + \zeta_- \mathcal{J}_+} + \mathcal{O}\left(\frac{1}{r^3}\right), \\ \xi^r &= \mathcal{O}\left(\frac{1}{r^2}\right), \\ \xi^\phi &= \frac{\eta^+ \zeta_- - \eta^- \zeta_+}{\zeta_+ \mathcal{J}_- + \zeta_- \mathcal{J}_+} + \mathcal{O}\left(\frac{1}{r^3}\right).\end{aligned}\tag{4.2}$$

The conserved charges can be directly computed using the Regge-Teitelboim approach [25], and as expected coincide with the expression in eq. (2.10) obtained using the Chern-Simons formulation.

Note that the boundary metric

$$d\bar{s}^2 = r^2 \left(-\zeta_+ \zeta_- dt^2 + (\zeta_+ \mathcal{J}_- - \zeta_- \mathcal{J}_+) dt d\phi + \mathcal{J}_+ \mathcal{J}_- d\phi^2 \right),$$

explicitly depends on the dynamical fields $\mathcal{J}_{\pm}$ and consequently *it is not fixed* at the boundary of spacetime, i.e., it has a nontrivial functional variation.

4.2 Near horizon behavior

Following ref. [18], the metric in the near horizon region can be reconstructed using

$$b_{\pm}(r) = \exp\left(\pm \frac{r}{2\ell} (L_1 - L_{-1})\right),$$

and considering an expansion around $r = 0$. The metric then reads

$$\begin{aligned}g_{tt} &= \frac{\ell^2}{4} (\zeta_+ - \zeta_-)^2 - \zeta_+ \zeta_- r^2 + \mathcal{O}(r^3), \\ g_{tr} &= \mathcal{O}(r^2), \\ g_{t\phi} &= \frac{\ell^2}{4} (\mathcal{J}_+ + \mathcal{J}_-) (\zeta_+ - \zeta_-) + (\zeta_+ \mathcal{J}_- - \zeta_- \mathcal{J}_+) \frac{r^2}{2} + \mathcal{O}(r^3), \\ g_{rr} &= 1 + \mathcal{O}(r^2), \\ g_{r\phi} &= \mathcal{O}(r^2), \\ g_{\phi\phi} &= \frac{\ell^2}{4} (\mathcal{J}_+ + \mathcal{J}_-)^2 + \mathcal{J}_+ \mathcal{J}_- r^2 + \mathcal{O}(r^3).\end{aligned}\tag{4.3}$$

Again, the chemical potentials $\zeta_{\pm}$ are expressed in terms of the generalized Gelfand-Dikii polynomials according to eq. (3.12).⁵

The behavior of the metric near the horizon is preserved under the action of the following near horizon Killing vectors

$$\begin{aligned}\xi^t &= \frac{\eta^+ \mathcal{J}_- + \eta^- \mathcal{J}_+}{\zeta_+ \mathcal{J}_- + \zeta_- \mathcal{J}_+} + \mathcal{O}(r^3), \\ \xi^r &= \mathcal{O}(r^3), \\ \xi^\phi &= \frac{\eta^+ \zeta_- - \eta^- \zeta_+}{\zeta_+ \mathcal{J}_- + \zeta_- \mathcal{J}_+} + \mathcal{O}(r^3).\end{aligned}\tag{4.4}$$

The conserved charges can be obtained using the Regge-Teitelboim approach, and evaluating them at $r = 0$. The results coincide with eq. (2.10), as expected.

4.3 General solution

In refs. [17, 18], it was shown that it is possible to construct the general solution of Einstein equations obeying the fall-off described in (4.1). It is given by

$$\begin{aligned}ds^2 &= dr^2 + \frac{\ell^2}{4} \cosh^2(r/\ell) [(\zeta_+ - \zeta_-) dt + (\mathcal{J}_+ + \mathcal{J}_-) d\phi]^2 \\ &\quad - \frac{\ell^2}{4} \sinh^2(r/\ell) [(\zeta_+ + \zeta_-) dt + (\mathcal{J}_+ - \mathcal{J}_-) d\phi]^2,\end{aligned}\tag{4.5}$$

and satisfies Einstein equations provided that $\mathcal{J}_{\pm}$ obey the differential equations associated to the k -th member of the hierarchy when $\zeta_{\pm}$ is fixed according to eq. (3.12). In the near horizon region, this solution also obeys the fall-off in (4.3). Note that the metric (4.5) is diffeomorphic to a BTZ geometry, but as we will show in the next section, it carries non-trivial charges associated to improper (large) gauge transformations [41], and consequently describes a different physical state.

It is worth emphasizing that there is a one-to-one map between three-dimensional geometries described by eq. (4.5), and solutions of the members of the Gardner hierarchy. In this sense, we can say that this integrable system was “fully geometrized” in terms of certain three-dimensional spacetimes which are locally of constant curvature.

5 Black holes

5.1 Regularity conditions and thermodynamics

Euclidean black holes solutions are obtained by requiring regularity of the Euclidean geometries associated to the family of metrics in (4.5). This fixes the inverse of left and right temperatures $\beta_{\pm} = T_{\pm}^{-1}$ in terms of the fields $\zeta_{\pm}$ according to

$$\beta_{\pm} = \frac{2\pi}{\zeta_{\pm}}.\tag{5.1}$$

⁵Note that with our choice of boundary conditions it is not possible to write the metric (4.3) in a co-rotating frame ($\zeta_+ = \zeta_- = \text{const.}$), because $\zeta_{\pm}$ have a very precise dependence on the fields $\mathcal{J}_{\pm}$, and generically cannot be set to be equal to constants.

These conditions can also be obtained by requiring that the holonomy around the thermal cycle for the gauge connections (2.3) be trivial.

A direct consequence of eq. (5.1) is that the chemical potentials $\zeta_{\pm}$ are now constants, and hence from the field equations (2.4), the regular Euclidean solutions are characterized by $\tilde{\mathcal{J}}_{\pm} = 0$, i.e., by static solutions of the members of the Gardner hierarchy. In sum, in order to obtain an explicit black hole solution, the following equations must be solved

$$\partial_{\phi} R_{(k)}^{\pm} = 0, \quad (5.2)$$

restricted to the conditions

$$T_{\pm} = \frac{2}{\kappa} R_{(k)}^{\pm}. \quad (5.3)$$

The Bekenstein-Hawking entropy can be directly obtained from the near horizon expansion (4.3), and gives

$$S = \frac{A}{4G} = \frac{\kappa}{2} \oint d\phi (\mathcal{J}_+ + \mathcal{J}_-). \quad (5.4)$$

As expected, the first law is automatically fulfilled. Indeed, using (5.3) one obtains

$$\beta_+ \delta H_{(k)}^+ + \beta_- \delta H_{(k)}^- = \oint d\phi \left(\beta_+ R_{(k)}^+ \delta \mathcal{J}_+ + \beta_- R_{(k)}^- \delta \mathcal{J}_- \right) = \delta \left[\frac{\kappa}{2} \oint d\phi (\mathcal{J}_+ + \mathcal{J}_-) \right] = \delta S.$$

Here, $\beta_{\pm}$ turn out to be the conjugates to the left and right energies $H_{(k)}^{\pm}$. The inverse temperature, conjugate to the energy E in eq. (2.7), is expressed in terms of the left and right temperatures according to $T^{-1} = \frac{1}{2} (T_+^{-1} + T_-^{-1})$.

The previous analysis was performed in a rather abstract form without using an explicit solution to eq. (5.2), which in general are very hard to find. A simple solution corresponds to $\mathcal{J}_{\pm} = \text{const.}$, which describes a BTZ configuration. In this case the Hamiltonians take the form

$$H_{(k)}^{\pm} = \sum_{n=k+2}^{2k+2} \alpha_n^{\pm} \mathcal{J}_{\pm}^n,$$

where $\alpha_n^{\pm}$ are constant coefficients which are not specified in general, but whose values can be determined once the corresponding Hamiltonians are explicitly computed through the recursion relation for the generalized Gelfand-Dikii polynomials.

Some simplifications occur when we turn off either a or b (mKdV and KdV cases), that we discuss next.

5.2 mKdV case ($a = 0$)

When $a = 0$, the metric associated to the black hole solution with $\mathcal{J}_{\pm} = \text{const.}$ can be written as

$$ds^2 = dr^2 + \frac{\ell^2}{4} \cosh^2(r/\ell) \left[4\pi^2 \left(\frac{\pi\kappa}{2\sigma_{(k)}(k+1)} \right)^{2k+1} \left(\mathcal{J}_+^{2k+1} - \mathcal{J}_-^{2k+1} \right) dt + (\mathcal{J}_+ + \mathcal{J}_-) d\phi \right]^2 - \frac{\ell^2}{4} \sinh^2(r/\ell) \left[4\pi^2 \left(\frac{\pi\kappa}{2\sigma_{(k)}(k+1)} \right)^{2k+1} \left(\mathcal{J}_+^{2k+1} + \mathcal{J}_-^{2k+1} \right) dt + (\mathcal{J}_+ - \mathcal{J}_-) d\phi \right]^2, \quad (5.5)$$

where the constant $\sigma_{(k)}$, given by

$$\sigma_{(k)} := \left(\frac{\pi \kappa}{2k+2} \right)^{\frac{k+1}{k+\frac{1}{2}}} \left(\frac{\sqrt{\pi}}{\kappa 2^{k-2} b^k} \frac{\Gamma(k+2)}{\Gamma(k+\frac{1}{2})} \right)^{\frac{1}{2k+1}},$$

will play the role of the anisotropic Stefan-Boltzmann constant of the system. The metric can be written in Schwarzschild-like coordinates using the following coordinate transformation

$$r = \frac{\ell}{2} \log \left(\frac{4 \sqrt{\left(\bar{r}^2 - \frac{\ell^2}{4} (\mathcal{J}_+ - \mathcal{J}_-)^2 \right) \left(\bar{r}^2 - \frac{\ell^2}{4} (\mathcal{J}_+ + \mathcal{J}_-)^2 \right) - \ell^2 (\mathcal{J}_+^2 + \mathcal{J}_-^2) + 4 \bar{r}^2}}{2 \ell^2 \mathcal{J}_+ \mathcal{J}_-} \right). \quad (5.6)$$

It coincides with the metric of a BTZ black hole in a rotating frame, with outer and inner horizons located at $\bar{r}_{\pm} = \frac{\ell}{2} (\mathcal{J}_+ \pm \mathcal{J}_-)$.

With the choice $a = 0$, the expression for the left and right energies written in terms of the constants $\mathcal{J}_{\pm}$ becomes simpler than the one in the general case. Indeed, it can be written in a closed form as

$$H_{(k)}^{\pm} = \sigma_{(k)}^{-z} \left(\frac{\pi \kappa}{(z+1)} \mathcal{J}_{\pm} \right)^{z+1}, \quad (5.7)$$

where $z = 2k + 1$ is the dynamical exponent of the Lifshitz scale symmetry (3.15) of the k -th element of the mKdV hierarchy.

Using eqs. (5.3) and (5.7), the left and right energies $H_{(k)}^{\pm}$ can be expressed in terms of the left and right temperatures $T_{\pm}$, acquiring the form dictated by the Stefan-Boltzmann law for a two-dimensional system with anisotropic Lifshitz scaling [42]

$$H_{(k)}^{\pm} = \sigma_{(k)} T_{\pm}^{1+\frac{1}{z}}.$$

The Bekenstein-Hawking entropy, given by eq. (5.4), can be expressed in terms of the left and right energies $H_{(k)}^{\pm}$ as follows

$$S = (1+z) \sigma_{(k)}^{\frac{z}{1+z}} \left(\left(H_{(k)}^+ \right)^{\frac{1}{z+1}} + \left(H_{(k)}^- \right)^{\frac{1}{z+1}} \right). \quad (5.8)$$

Note that the dependence of the entropy in terms of the left/right energies is consistent with the Lifshitz scaling of the k -th element of the mKdV hierarchy.

Power partitions and microstate counting. The dependence of the entropy in terms of the left/right energies in eq. (5.8) might be understood from a microscopic point of view if, following [43], we assume that there exists a two-dimensional field theory with Lifshitz scaling, defined on a circle, whose dispersion relation for very high energies takes the form

$$E_n^{\pm} = \varepsilon_{(z)}^{\pm} n^z, \quad (5.9)$$

where n is a non-negative integer, and $\varepsilon_{(z)}^\pm$ denote the characteristic energy of the left/right modes. The problem of computing the entropy in the microcanonical ensemble is then equivalent to compute the power partitions of given integers $N_\pm = E^\pm / \varepsilon_{(z)}^\pm$. Here $E^\pm$ are the left/right energies given by

$$E^\pm = \varepsilon_{(z)}^\pm \sum_i n_i^z.$$

This problem was solved long ago by Hardy and Ramanujan in [44], where at the end of their paper they conjecture that the asymptotic growth of power partitions is given by

$$p_z(N_\pm) \approx \exp \left[(1+z) \left(\frac{\Gamma(1+\frac{1}{z}) \zeta(1+\frac{1}{z})}{z} \right)^{\frac{z}{1+z}} N_\pm^{\frac{1}{1+z}} \right],$$

result that was proven later by Wright in 1934 [45].

The (left/right) entropies then reads

$$S^\pm = \log[p_z(N_\pm)] = (1+z) \left(\frac{\Gamma(1+\frac{1}{z}) \zeta(1+\frac{1}{z})}{z} \right)^{\frac{z}{1+z}} \left(\frac{E^\pm}{\varepsilon_{(z)}^\pm} \right)^{\frac{1}{1+z}}. \quad (5.10)$$

This expression precisely coincides with the entropy of the black hole in eq. (5.8), provided $E^\pm = H_{(k)}^\pm$, and

$$\varepsilon_{(z)}^\pm = \left(\frac{\Gamma(1+\frac{1}{z}) \zeta(1+\frac{1}{z})}{\sigma_{(k)} z} \right)^z.$$

Note that AdS spacetime is not contained within the spectrum of our boundary conditions, and consequently one can naively think that the anisotropic extension of Cardy formula of refs. [1, 42] cannot be used to reproduce the entropy of the black hole (5.8). However, there is a known case [42], where the anisotropic extension of Cardy formula can still be used in spite of the fact that ground state, given by a gravitational soliton, does not fit within the boundary conditions that accommodate the Lifshitz black hole. This approach is based on the use of an anisotropic extension of modular invariance that relates the Euclidean black hole and its corresponding soliton, which turn out to be diffeomorphic. It would be interesting to explore in the future whether this approach could be applied to the BTZ black holes in the context of our boundary conditions.

5.3 KdV case ($b = 0$)

When $b = 0$, the metric associated to the black hole solution with $\mathcal{J}_\pm = \text{const.}$ takes the form

$$ds^2 = dr^2 + \frac{\ell^2}{4} \cosh^2(r/\ell) \left[4\pi^2 \left(\frac{\pi\kappa}{\bar{\sigma}_{(k)}(k+2)} \right)^{k+1} \left(\mathcal{J}_+^{k+1} - \mathcal{J}_-^{k+1} \right) dt + (\mathcal{J}_+ + \mathcal{J}_-) d\phi \right]^2 - \frac{\ell^2}{4} \sinh^2(r/\ell) \left[4\pi^2 \left(\frac{\pi\kappa}{\bar{\sigma}_{(k)}(k+2)} \right)^{k+1} \left(\mathcal{J}_+^{k+1} + \mathcal{J}_-^{k+1} \right) dt + (\mathcal{J}_+ - \mathcal{J}_-) d\phi \right]^2, \quad (5.11)$$

where

$$\bar{\sigma}_{(k)} = \left(\frac{\pi \kappa}{k+2} \right)^{\frac{k+2}{k+1}} \left(\frac{\sqrt{\pi}}{\kappa (2a)^k} \frac{\Gamma(k+3)}{\Gamma(k+\frac{3}{2})} \right)^{\frac{1}{k+1}}.$$

Using the change of coordinates (5.6), the metric (5.11) can be written in Schwarzschild-like form, and coincides with the one of a BTZ black hole with outer and inner horizons located at $\bar{r}_{\pm} = \frac{\ell}{2} (\mathcal{J}_+ \pm \mathcal{J}_-)$.

The left and right energies $H_{(k)}^{\pm}$ can then be expressed in terms of the constants $\mathcal{J}_{\pm}$ according to

$$H_{(k)}^{\pm} = \bar{\sigma}_{(k)}^{-\frac{z+1}{2}} \left(\frac{2\pi\kappa}{z+3} \mathcal{J}_{\pm} \right)^{\frac{z+3}{2}}, \quad (5.12)$$

where $z = 2k + 1$ is the dynamical exponent associated to the Lifshitz symmetry (3.16) of the k -th member of the KdV hierarchy.

The expression for the left/right energies $H_{(k)}^{\pm}$ in terms of the left/right temperatures $T_{\pm}$, is then given by

$$H_{(k)}^{\pm} = \bar{\sigma}_{(k)} T_{\pm}^{\frac{z+3}{z+1}}.$$

In spite of the fact that the k -th equation of the KdV hierarchy is invariant under Lifshitz scaling with dynamical exponent z , the power in the temperature is not the one expected for a two-dimensional theory with this symmetry. Furthermore, this is inherited to the expression for the entropy written in terms of the left/right energies $H_{(k)}^{\pm}$

$$S = \left(\frac{z+3}{2} \right) \bar{\sigma}_{(k)}^{\frac{z+1}{z+3}} \left(\left(H_{(k)}^+ \right)^{\frac{2}{z+3}} + \left(H_{(k)}^- \right)^{\frac{2}{z+3}} \right),$$

which is not of the expected form (5.10).

Remarkably, if instead of the Hamiltonians $H_{(k)}^{\pm}$, one uses the Hamiltonians $H_{(2k)}^{\pm}$, this naive incompatibility with the Lifshitz symmetry disappears. Indeed, the relation between $H_{(2k)}^{\pm}$ and the left/right temperatures takes the form

$$H_{(2k)}^{\pm} = \bar{\sigma}_{(2k)} T_{\pm}^{1+\frac{1}{z}},$$

while the expression for the entropy in terms of the extensive quantities $H_{(2k)}^{\pm}$ reads

$$S = (1+z) \bar{\sigma}_{(2k)}^{\frac{z}{1+z}} \left(\left(H_{(2k)}^+ \right)^{\frac{1}{z+1}} + \left(H_{(2k)}^- \right)^{\frac{1}{z+1}} \right).$$

The entropy then takes the expected Hardy-Ramanujan form (5.10), with the characteristic energy of the dispersion relation given by

$$\varepsilon_{(z)}^{\pm} = \left(\frac{\Gamma(1+\frac{1}{z}) \zeta(1+\frac{1}{z})}{\bar{\sigma}_{(2k)} z} \right)^z.$$

Black hole with nonconstants $\mathcal{J}_\pm$. In the particular case when $b = 0$ and $k = 1$, it is possible to find explicit nonconstants solutions to eq. (5.2) that characterize a regular Euclidean black hole. These are static solutions of the left/right KdV equations, which take the form of periodic cnoidal waves. The solutions are then given by

$$\mathcal{J}_\pm = -\frac{8K^2(m_\pm)}{3a\pi^2} \left(1 - 2m_\pm + 3m_\pm \text{cn}^2 \left(\frac{K(m_\pm)}{\pi} \phi | m^\pm \right) \right), \quad (5.13)$$

where $m_\pm$ are constants in the range $0 \leq m_\pm < 1$, cn denotes the Jacobi elliptic cosine function, and $K(m)$ is the complete elliptic integral of the first kind defined as

$$K(m) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - m \sin^2 \theta}}. \quad (5.14)$$

The regularity conditions (5.3) fix the left and right temperatures $T_\pm$ in terms of the constant $m_\pm$ according to

$$T_\pm = \frac{16(m_\pm^2 - m_\pm + 1)}{3\pi^5 a} K(m_\pm)^4.$$

6 Final remarks

We have shown that by imposing appropriate boundary conditions to General Relativity on AdS_3 , one can describe the boundary dynamics of the gravitational field by the integrable system corresponding to the Gardner (mixed KdV-mKdV) hierarchy. These results can be naturally extended to the case with a vanishing cosmological constant. Indeed, following ref. [18], if one chooses an auxiliary gauge connection of the form

$$\mathfrak{a} = \frac{1}{2} ((\zeta_+ + \zeta_-) L_0 + (\zeta_+ - \zeta_-) P_0) dt + \frac{1}{2} ((\mathcal{J}_+ - \mathcal{J}_-) L_0 + (\mathcal{J}_+ + \mathcal{J}_-) P_0) d\phi, \quad (6.1)$$

the field equations precisely reduce to eq. (2.4), i.e.,

$$\dot{\mathcal{J}}_\pm = \pm \zeta'_\pm.$$

Here P_0 and L_0 are the zero modes of the $isl(2)$ generators, whose non-vanishing brackets are $[L_n, L_m] = (n - m) L_{n+m}$ and $[L_n, P_m] = (n - m) P_{n+m}$, with $n, m = 0, \pm 1$. Furthermore, the expression for the variation of the charges acquire precisely the form (2.10), with $\kappa = 1/(4G)$. Consequently, the complete integrable structure associated to the Gardner hierarchy is recovered provided we fix the chemical potentials $\zeta_\pm$ according to eq. (2.6), i.e.,

$$\zeta_\pm = \frac{4\pi}{\kappa} R_{(k)}^\pm.$$

Remarkably, the boundary dynamics in the absence of a cosmological constant, described by the connection (6.1), is associated to the same integrable system than in the case of a negative cosmological constant. In this sense, there is a certain universality in our results.

The integrable systems discussed here possess an infinite number of conserved charges that commute among them. This opens the possibility of studying more general thermodynamic ensembles containing not only the left/right Hamiltonians, but also the additional

conserved charges. This kind of ensembles are called Generalized Gibbs Ensembles and have been recently studied in the context of the KdV charges [1, 10–16]. In our analysis, the conserved charges are more general than the ones of KdV, which are obtained as a particular case ($b = 0$). One could then investigate the behavior of the Generalized Gibbs Ensembles in this more general setup, as well as their consequences for the corresponding dual gravitational theory.

One can also investigate the possibility of introducing suitable “potentials” for the fields $\mathcal{J}_\pm$ to define an appropriate action principle, for which the conserved charges $H_{(k)}^\pm$ are obtained from the Noether theorem. This could be achieved by performing a Hamiltonian reduction along the lines of [31, 46], in order to find the corresponding action describing the “boundary dynamics”.

It would be also interesting to explore the generalization to the case of three-dimensional higher spin gravity [47–51], where one can use the “diagonal gauge” introduced in ref. [52] and choose the chemical potentials as appropriate functions of the dynamical fields. One would expect that, for the particular case of spin $s = 3$, the corresponding integrable system would be related to the Boussinesq hierarchy and its corresponding modified version. This might be also extended to the case of higher spin gravity with vanishing cosmological constant, using the fall-off described in [53].

A new class of deformations of two-dimensional field theories preserving integrability was introduced in ref. [54]. In particular, deformations of the form $T\bar{T}$ [55–60], and $J\bar{T}$ [61–64], as well as their holographic descriptions [65–68], have recently received a great deal of attention. It would be interesting to apply this class of deformations to the integrable systems studied in this work, and explore their consequences in the context of the dual gravitational theory through the boundary conditions proposed here.

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A Hamiltonians and generalized Gelfand-Dikii polynomials

In this appendix, we explicitly display the first Hamiltonians $H_{(k)}^\pm$ associated to the Gardner integrable system, as well as their corresponding generalized Gelfand-Dikii polynomials $R_{(k)}^\pm$.

The first Hamiltonians $H_{(k)}^\pm$ are given by

$$\begin{aligned}
H_{(-1)}^\pm &= \frac{\kappa}{4\pi} \oint d\phi (a^{-1} \mathcal{J}_\pm), \\
H_{(0)}^\pm &= \frac{\kappa}{4\pi} \oint d\phi \left(\frac{1}{2} \mathcal{J}_\pm^2 \right), \\
H_{(1)}^\pm &= \frac{\kappa}{4\pi} \oint d\phi \left(\frac{1}{2} a \mathcal{J}_\pm^3 + \frac{1}{4} b \mathcal{J}_\pm^4 + \mathcal{J}_\pm'^2 \right), \\
H_{(2)}^\pm &= \frac{\kappa}{4\pi} \oint d\phi \left(\frac{5}{8} a^2 \mathcal{J}_\pm^4 - \frac{5}{2} a \mathcal{J}_\pm^2 \mathcal{J}_\pm'' + \frac{1}{4} b^2 \mathcal{J}_\pm^6 - \frac{5}{3} b \mathcal{J}_\pm^3 \mathcal{J}_\pm'' + \frac{3}{4} ab \mathcal{J}_\pm^5 + 2 \mathcal{J}_\pm'^2 \right), \\
H_{(3)}^\pm &= \frac{\kappa}{4\pi} \oint d\phi \left(\frac{7}{8} a^3 \mathcal{J}_\pm^5 - \frac{35}{6} a^2 \mathcal{J}_\pm^3 \mathcal{J}_\pm'' + 7a \mathcal{J}_\pm^2 \mathcal{J}_\pm''' + \frac{5}{16} b^3 \mathcal{J}_\pm^8 + \frac{35}{2} b^2 \mathcal{J}_\pm'^2 \mathcal{J}_\pm^4 \right. \\
&\quad \left. + \frac{7}{3} b (2 \mathcal{J}_\pm^3 \mathcal{J}_\pm''' + \mathcal{J}_\pm'^4) + \frac{7}{4} a^2 b \mathcal{J}_\pm^6 + \frac{5}{4} ab^2 \mathcal{J}_\pm^7 + 35ab \mathcal{J}_\pm'^2 \mathcal{J}_\pm^3 - 4 \mathcal{J}_\pm \mathcal{J}_\pm'''' \right).
\end{aligned} \tag{A.1}$$

Note that the Hamiltonians of the Gardner hierarchy cannot be written as the sum of the Hamiltonians of KdV ($b = 0$) and mKdV ($a = 0$), because there are cross terms. The case $H_{(-1)}^\pm$, that it is obtained by extending the hierarchy backwards, is special because contains a^{-1} , and consequently it cannot be defined in the mKdV hierarchy.

The generalized Gelfand-Dikii polynomials $R_{(k)}^\pm$ are obtained using eq. (3.8), and take the form

$$\begin{aligned}
R_{(-1)}^\pm &= \frac{\kappa}{4\pi a}, \\
R_{(0)}^\pm &= \frac{\kappa}{4\pi} \mathcal{J}_\pm, \\
R_{(1)}^\pm &= \frac{\kappa}{4\pi} \left(\frac{3}{2} a \mathcal{J}_\pm^2 + b \mathcal{J}_\pm^3 - 2 \mathcal{J}_\pm'' \right), \\
R_{(2)}^\pm &= \frac{\kappa}{4\pi} \left(\frac{5}{2} a^2 \mathcal{J}_\pm^3 - 5a \left(\mathcal{J}_\pm'^2 + 2 \mathcal{J}_\pm \mathcal{J}_\pm'' \right) + \frac{3}{2} b^2 \mathcal{J}_\pm^5 \right. \\
&\quad \left. - 10b \left(\mathcal{J}_\pm \mathcal{J}_\pm'^2 + \mathcal{J}_\pm^2 \mathcal{J}_\pm'' \right) + \frac{15}{4} ab \mathcal{J}_\pm^4 + 4 \mathcal{J}_\pm'''' \right), \\
R_{(3)}^\pm &= \frac{\kappa}{4\pi} \left(\frac{35}{8} a^3 \mathcal{J}_\pm^4 - 35a^2 \left(\mathcal{J}_\pm \mathcal{J}_\pm'^2 + \mathcal{J}_\pm^2 \mathcal{J}_\pm'' \right) + 7a \left(4 \mathcal{J}_\pm \mathcal{J}_\pm'''' + 6 \mathcal{J}_\pm'^2 + 8 \mathcal{J}_\pm' \mathcal{J}_\pm''' \right) \right. \\
&\quad \left. + \frac{5}{2} b^3 \mathcal{J}_\pm^7 + 7b \left(4 \mathcal{J}_\pm^2 \mathcal{J}_\pm'''' + 12 \mathcal{J}_\pm \mathcal{J}_\pm'^2 + 16 \mathcal{J}_\pm \mathcal{J}_\pm' \mathcal{J}_\pm''' + 20 \mathcal{J}_\pm'^2 \mathcal{J}_\pm'' \right) - 8 \mathcal{J}_\pm'''''' \right. \\
&\quad \left. - 35b^2 \left(\mathcal{J}_\pm^4 \mathcal{J}_\pm'' + 2 \mathcal{J}_\pm^3 \mathcal{J}_\pm'^2 \right) + \frac{21}{2} a^2 b \mathcal{J}_\pm^5 + \frac{35}{4} ab^2 \mathcal{J}_\pm^6 - 35ab \left(2 \mathcal{J}_\pm^3 \mathcal{J}_\pm'' + 3 \mathcal{J}_\pm^2 \mathcal{J}_\pm'^2 \right) \right).
\end{aligned} \tag{A.2}$$

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