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Superstrings on AdS_3 at $k = 1$

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ABSTRACT: We study superstring theory in three dimensional Anti-de Sitter spacetime with NS-NS flux, focusing on the case where the radius of curvature is equal to the string length. This corresponds to the critical level $k = 1$ in the formulation as a Wess-Zumino-Witten model. Previously, it was argued that a transition takes place at this special radius, from a phase dominated by black holes at larger radius to one dominated by long strings at smaller radius. We argue that the infinite tower of modes that become massless at $k = 1$ is a signal of this transition. We propose a simple two-dimensional conformal field theory as the holographic dual to superstring theory at $k = 1$. As evidence for our conjecture, we demonstrate that our putative dual exactly reproduces the full spectrum of the long strings of the weakly coupled string theory, including states unprotected by supersymmetry.

KEYWORDS: AdS-CFT Correspondence, Long strings

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Contents

1	Introduction	1
2	The phase structure of string theory in AdS_3	2
3	String theory in AdS_3	5
3.1	The bosonic string on AdS_3	5
3.2	Superstrings on AdS_3	7
4	Holographic dual	10
4.1	The permutation orbifold $(\mathbb{R} \times \mathcal{N})^N/S_N$	12
4.2	Fermionic boundary conditions	13
5	Comparison of spectra	14
5.1	Multiplicities	14
5.2	Invariant states of the symmetric orbifold	15
5.2.1	Shift in the conformal weight	16
5.3	An equivalent construction	17
5.3.1	Odd n	18
5.3.2	Even n	18
6	Observations on interactions	19
7	Conclusions	20

1 Introduction

String theory in three-dimensional anti-de Sitter spacetime AdS_3 has a dimensionless coupling $k = R_{AdS}^2/l_s^2$, where R_{AdS} is the radius of curvature of AdS_3 and l_s is the string length. The theory is well-understood in the semi-classical regime of large k . Our purpose here is to explore the interesting phenomena that occur in the stringy regime at a specific small value of k where $R_{AdS} \sim l_s$. It was argued in [1] that there is a critical value $k = k_c$ (given by $k_c = 3$ for the bosonic string and $k_c = 1$ for the superstring with NS-NS flux) at which the theory undergoes a transition from a phase $k > k_c$ dominated by black holes to a phase $k < k_c$ dominated by long strings. This was based on both UV and IR considerations. At this same value $k = k_c$ an infinite tower of higher spin states become massless [2]. This suggests that the appearance of the massless higher spin states is the IR manifestation of a stringy phase transition. The linear dilaton background of [1] has a similar phase transition, and also appears to have an infinite tower of massless states at $k = k_c$.

A phase of string theory in which there are an infinite number of massless higher spin states would be a highly symmetric phase in which there are unbroken stringy symmetries and which could reveal much about the fundamental structure of string theory. It has been argued that such phases could be probed in high-energy large-angle scattering [3, 4] or for strings in anti-de Sitter space in a limit where the radius of curvature is much smaller than the string length [5–7], or of order of the string length [8]. In both these cases, the massless states can be associated with fundamental strings becoming tensionless. However for strings in AdS_3 at $k = k_c$ the situation is rather different: it is long strings near the boundary that become tensionless.

Some states in the short-string spectrum, including the L_2 -normalizable vacuum and the graviton, are absent from the spectrum when $k \leq k_c$. Precisely at $k = k_c$, the short-string vacuum and graviton are replaced by long-string states with the same spacetime quantum numbers, along with an infinite tower of massless higher spin states. As long strings, all these states are plane-wave rather than strictly normalizable, and lie at the bottom of a continuum of states parameterised by a continuous variable σ . (For large k , this variable can be interpreted as the radial momentum of long strings.) It was argued in [1] that for $k < k_c$ the spectrum does not contain either black holes or other non-perturbative excitations, so that the phase with $k < k_c$ would be a novel theory dominated by the long-string degrees of freedom.

The special features of the spectrum at $k = k_c$ provide clues as to the nature of the holographic dual conformal field theory (CFT). In this paper we first discuss the physics of the theory near $k = k_c$, and then identify a CFT that exactly reproduces the *full* spectrum (not just the BPS spectrum) of the long strings of the weakly-coupled string theory in AdS_3 at this critical level.

This paper is organized as follows: section 2 describes the phase structure of string theory on a WZW AdS_3 space. The phase transition at $k = k_c$ is discussed in some detail. Section 3 studies in greater detail the spectrum of the $N = 1$ superstring at $k = k_c = 1$. Section 4 proposes a 2-d CFT dual to the $k = 1$ AdS_3 superstring. It is a symmetric-product orbifold, whose complete spectrum we describe exactly. Section 5 compares the spectrum of the long strings on AdS_3 with the spectrum of the proposed holographic dual CFT and finds perfect agreement. Section 6 presents a preliminary comparison of interactions in the two theories. A potential disagreement is shown to exist and possible resolutions are discussed. In section 7 we give some concluding remarks.

2 The phase structure of string theory in AdS_3

A useful key to unravelling the phase structure and high energy behavior of a system is to determine which sets of degrees of freedom control the dynamics at different scales. A given system will typically have many kinds of degrees of freedom — e.g. particles, solitons, radiation, black holes, strings etc — and the contribution to the entropy associated with each will depend on the temperature or energy, coupling constants etc. In a given regime, there will typically be one set of degrees of freedom that dominates the entropy. There

will then be various regimes in each of which a different degree of freedom dominates, corresponding to different phases of the theory.

In gravity, the key players are light particles, radiation, strings, branes and various black objects. Each player indeed dominates for some energy range and this realization has led to interesting phase diagrams for gravitational systems. A particularly interesting transition is that between phases dominated by black holes and phases dominated by strings. For dimensions $D > 3$, the intuitive arguments involved in describing the transition relate to the change in the curvature at the black hole horizon as a function of the energy/mass of the configuration: the weaker the horizon curvature, the more the black hole description is justified. The transition between the black hole picture, which is valid for small horizon curvature, and the perturbative string picture, which is valid for large curvature, occurs when the curvature at the horizon is of order the string scale and is smooth.

For $D > 3$, the horizon curvature of a black hole depends on its mass, and goes to zero for large mass. This does not apply for black holes in AdS_3 for which the curvature depends only on the cosmological constant — it is the same everywhere in the spacetime and is independent of the black hole mass. Accordingly, a somewhat different approach was proposed in [1] for the case of string theory on an AdS_3 background. The focus was on the curvature scalar, which is a constant given by the cosmological constant

$$\Lambda = -\frac{1}{R_{\text{AdS}}^2} = -\frac{1}{kl_s^2}.$$

For $D = 3$, one can then use k as a tuneable parameter to probe the string/black hole transition.

In AdS_3 , there is an asymptotic space-time Virasoro algebra with central charge c , given for large R_{AdS} by

$$c = \frac{3R_{\text{AdS}}}{2l_p}, \quad (2.1)$$

where l_p is the three dimensional Planck length. The entropy behaves at high energies as

$$S = 2\pi\sqrt{\frac{c_{\text{eff}}L_0}{6}} + 2\pi\sqrt{\frac{\bar{c}_{\text{eff}}\bar{L}_0}{6}}, \quad (2.2)$$

for some effective central charges $c_{\text{eff}}, \bar{c}_{\text{eff}}$, where L_0 and $\bar{L}_0$ are the asymptotic Virasoro generators related to the energy E and angular momentum s in AdS_3 by

$$ER_{\text{AdS}} = L_0 + \bar{L}_0 - \frac{c}{12}; \quad s = L_0 - \bar{L}_0. \quad (2.3)$$

Unitarity implies that the effective central charge c_{eff} satisfies

$$c_{\text{eff}} \leq c \quad (2.4)$$

and $c_{\text{eff}} = c$ if and only if the $\text{SL}(2, \mathbb{C})$ invariant vacuum is normalizable.

In general, the spectrum of strings in AdS_3 includes excited short strings and long strings near the boundary. In addition there are BTZ black hole states and possible further states such as branes wrapping additional dimensions that depend on the details of the string background $\text{AdS}_3 \times \mathcal{N}$ [9].

Figure 1. The behavior of the entropy S at high energies, as a function of the WZW level k . The black solid line is the entropy of black holes, which do not exist for $k < 1$. The dashed line is the entropy of long strings, that dominate for $k < 1$ (see [1] for details).

In [1], several pieces of evidence were discovered that pointed to the value of $k = k_c$ as the string/black hole transition point. The first is related to the proper identification of the effective degrees of freedom. For $k < k_c$ the dominant degrees of freedom at all energy scales are the long strings. There is no graviton and no states corresponding to BTZ black holes. In this regime, $c_{\text{eff}} < c$ and the $\text{SL}(2, \mathbb{C})$ invariant vacuum is not normalizable, and so perturbative string states are also not normalizable. In particular, there is not a normalizable graviton state. For $k > k_c$, $c_{\text{eff}} = c$ and the $\text{SL}(2, \mathbb{C})$ vacuum and perturbative string states are normalizable. The BTZ black hole states are also normalizable and dominate the entropy at high energy [1, 9]. The k dependence of the entropy for long strings and black holes is shown in figure 1.

The entropy of the black holes for $k > k_c$ and of the long strings for $k < k_c$ matches exactly at $k = k_c$, but the first derivative of the entropy with respect to k is not continuous. The fact that $c_{\text{eff}} = c$ for $k > k_c$, $c_{\text{eff}} < c$ for $k < k_c$, and c_{eff} is continuous through the critical value suggests that c_{eff} can be considered an order parameter for this transition. Finally, the effective coupling of the long strings near the boundary switches from being weak if $k < k_c$ to becoming strong if $k > k_c$. The entropy matching is a UV property, and the behavior of c_{eff} can be considered to be an IR property.

Thus there is some evidence for a transition at $k = k_c$ from a black hole phase to a phase dominated by long strings. Note that this is different from the usual string/black hole transition in which the string phase is dominated by highly excited short strings that for large excitation number begin to resemble a black hole [10, 11]. Here the strings are of a different nature, long and localized near the boundary of AdS_3 [12]. For further confirmation of the presence of a string/black hole phase transition, one would want to identify the IR signature of a phase transition, for example the emergence of massless excitations at the transition point. Such massless modes were discovered in [2] and we argue that these are indeed the signal of a phase transition.

Before discussing these modes in more detail, it will be useful to discuss the differences between second order phase transitions in QFT and those in string theory. In a QFT

one typically varies a coupling to a relevant operator to go from one phase to another, passing through a point or surface in parameter space in which massless degrees of freedom play a role (in some cases they play a role in a whole region of parameter space). By contrast, perturbative string theory is heavily constrained — the worldsheet theory must be a quantum conformal field theory with specific features, among them a specific value for the central charge. The worldsheet theory of strings on AdS_3 must be accompanied by a unitary CFT, so that as one varies the value of the level k one must also vary the attendant CFT so as to keep the total central charge unchanged at the critical value. In the QFT case one can have Goldstone modes or a Brout-Englert-Higgs mechanism at and around the fixed point with a finite number of massless particles involved. It is not a priori clear exactly what to expect for a transition in a string theory. The results obtained in [2] for the bosonic case and in [2, 13] for the supersymmetric case give two chiral trajectories of a discrete infinity of massless states with an ever growing spin at the same transition point $k = k_c$. This is the desired IR manifestation of the transition.

3 String theory in AdS_3

The bosonic string in AdS_3 is formulated in terms of a WZW model with target space AdS_3 (the universal covering space of the group manifold $\text{SL}(2, \mathbb{R})$) with level k . The complete perturbative spectrum (at weak string coupling but to all orders in the string tension α') was found in [12]. The superstring in AdS_3 with NS-NS flux is given by a supersymmetric WZW model, and has been analyzed in [13–19]. The bosonic string and the superstring with NS-NS flux in AdS_3 are formally very similar, but the bosonic string has a tachyon [12] which leads to an exponential divergence in the partition function and presumably renders the theory unstable. In both cases, the spectrum consists of both discrete and continuous representations of $\text{SL}(2, \mathbb{R})$, and a “spectral flow” acts on the representations. The excitations of the fundamental strings fit into discrete representations, but there are also long strings near the boundary of AdS_3 that are associated with spectrally-flowed continuous representations [12]. We shall first review the bosonic string, which introduces many of the key features, and then turn to the superstring.

3.1 The bosonic string on AdS_3

The Virasoro central charge for the $\text{SL}(2, \mathbb{R})$ WZW model at level k is

$$c_k = \frac{3k}{k-2}. \quad (3.1)$$

For a critical string theory, we must consider the tensor product of this with a CFT $\mathcal{N}$ with central charge $c_N = 26 - c_k$. For a unitary theory $c_N \geq 0$, which implies

$$k \geq 52/23. \quad (3.2)$$

This in particular excludes $k = 2$, so that the $k = 2$ $\text{SL}(2, \mathbb{R})$ WZW model cannot be part of a conventional critical string theory. As mentioned previously, if we are to consider the behavior of the theory as k changes, then we have to change the CFT $\mathcal{N}$ as we vary k ,

so that the central charge of $\mathcal{N}$ remains $c_N = 26 - c_k$. For $k = 3$, we note that $c_k = 9$ and $c_N = 17$.

States are labelled by the eigenvalue C of the quadratic Casimir of $SL(2, \mathbb{R})$ and by the eigenvalue m of J_0^3 for the left-handed $SL(2, \mathbb{R})$, together with the corresponding quantities $\bar{C}$ and $\bar{m}$ for the right-handed $SL(2, \mathbb{R})$. They are also labelled by an integer w characterising the spectral flow; w is associated with the winding number for long strings [12]. The spacetime energy E and spin s are

$$E = m + \bar{m}, \quad s = m - \bar{m}. \quad (3.3)$$

The AdS_3 mass is given by:

$$m_{\text{AdS}_3}^2 = (E - |s|)(E + |s| - 2). \quad (3.4)$$

States for which $E = |s|$ are then massless higher spin fields.

For $w = 0$, there are two kinds of representations that appear in the spectrum. First, there are the discrete lowest weight representations $\mathcal{D}_j^+$ for which $m = j, j+1, j+2, \dots$ and $C = -j(j-1)$ (with j a positive real number). These correspond to the so-called short string states. Secondly, there are the continuous representations $\mathcal{C}(\sigma, \alpha)$ labelled by $\sigma \in \mathbb{R}$, $\sigma > 0$, $\alpha \in \mathbb{R}$ for which m takes the values $m \in \alpha + \mathbb{Z}$ and $C = \frac{1}{4} + \sigma^2$. These correspond to the long string states. Without loss of generality, the real parameter α can be taken to be in the range $0 \leq \alpha < 1$. Spectral flow then generates versions of these representations with $w \neq 0$. The limit $\sigma \rightarrow 0$ gives a reducible representation that can be decomposed into the direct sum of highest- and lowest-weight discrete representations. All string states are in these two classes of representation, and a unitary string theory with a modular invariant partition function is obtained by imposing the Virasoro constraints.

The mass-shell condition for a state of given C, m, w and excitation number N is [2, 12]

$$\frac{C}{k-2} - w\tilde{m} - \frac{k}{4}w^2 + h_{\text{int}} + N = 1, \quad (3.5)$$

where h_{int} is the eigenvalue of the L_0 operator for the internal CFT $\mathcal{N}$ and $\tilde{m}$ is the eigenvalue of the spectrally flowed J_0 operator, related to m by

$$m = \tilde{m} + \frac{k}{2}w \quad (3.6)$$

A similar equation arises for the right-moving operators.

Reference [12] shows that the constraints from requiring unitarity, normalizability of physical states, and symmetry under spectral flow imply that short string physical states exist only for

$$1/2 < j < (k-1)/2. \quad (3.7)$$

The analysis of the partition function in [36] supports this conclusion. For $k \leq 3$ this excludes many short string states, including the vacuum and the graviton with $j = 1$. For $k = 3$, these are replaced by plane-wave normalizable long string states with the same spacetime quantum numbers.

The massless higher spin states identified in [2] arise from the continuous representation with

$$C = \bar{C} = \frac{1}{4} + \sigma^2, \quad w = 1, \quad h_{\text{int}} = \bar{h}_{\text{int}} = 0 \quad (3.8)$$

Then the physical state conditions give

$$m = N + \sigma^2, \quad \bar{m} = \bar{N} + \sigma^2. \quad (3.9)$$

The continuous parameter σ is real and positive; massless states then arise if $m = 0$ or $\bar{m} = 0$, i.e. if $\sigma^2 = 0$ and either N or $\bar{N} = 0$. If $\bar{N} = 0, \sigma^2 = 0$, these have $E = s = N$ and there is a massless higher spin state of spin N for each positive integer N . These arise as part of a continuum of states labelled by σ with $E = N + 2\sigma^2$ and spin $s = N$ which are massive if $\sigma \neq 0$. Similarly, if $N = 0$, there is a continuum of states with $E = \bar{N} + 2\sigma^2$ and spin $s = -\bar{N}$ which are massless for $\sigma = 0$.

It was suggested in [2] that the stringy tower of higher spin states that become massless in the limit $k \rightarrow 3$ might correspond to a symmetry structure similar to that of the Higher Spin Square [23, 24].

This fits with the picture of Seiberg and Witten [25], who argued that a long fundamental string in AdS_3 is effectively described, for large k , by a Liouville theory with background charge Q and central charge $c = 1 + 3Q^2$, where

$$Q = (k - 3)\sqrt{\frac{2}{k - 2}}. \quad (3.10)$$

The Liouville field corresponds to the distance of the long string from the boundary of AdS_3 . It has a spectrum that is continuous above a threshold energy

$$\Delta_0 = \frac{Q^2}{8} = \frac{(k - 3)^2}{4(k - 2)}. \quad (3.11)$$

In the limit $k \rightarrow k_c = 3$ this threshold energy goes to zero, giving massless states as part of a continuum, so that the effective theory reproduces some of the features of the exact analysis. In this limit, a pure Liouville description becomes problematic, as the Liouville theory is likely no longer a unitary CFT for $Q < 2\sqrt{2}$. Unitarity may be recovered when the Liouville theory is a sector of a larger CFT which contains other marginal operators besides the Liouville potential [26]. Moreover, the central charge and the threshold energy are robust properties that hold beyond the large- k limit [25]; therefore, we expect that the effective CFT for the long string should be a $c = 1$ CFT in the limit $k \rightarrow k_c = 3$.¹

3.2 Superstrings on AdS_3

The situation for the superstring is similar [13–22]. The $SL(2, \mathbb{R})$ $N = 1$ supersymmetric WZW model at level k has super-Virasoro central charge

$$c_k = \frac{9}{2} + \frac{6}{k}. \quad (3.12)$$

¹Early work that highlighted the interesting phenomena in the WZW model at $k = 3$ and the resemblance to Liouville coupled to $c = 1$ matter is [27].

For a critical string theory this must be tensored with a SCFT $\mathcal{N}$ with central charge $c_N = 15 - c_k$. For a unitary theory with $c_N \geq 0$, we need

$$k \geq \frac{4}{7}. \quad (3.13)$$

For $k = 1$, $c_k = 21/2$ so that $c_N = 9/2$. For instance, $\mathcal{N}$ could be three free $N = 1$ supermultiplets, e.g. the supersymmetric sigma model with target space T^3 . Here we shall focus on the particular case in which the SCFT $\mathcal{N}$ is the SCFT on S^1 with $c = 3/2$ tensored with a $c = 3$ model of six free fermions, which can be thought of as the fermionization of the $N = 1$ SCFT on T^2 .² The number of fermionic fields will play a significant role later.

The constraints from requiring unitarity, normalizability of physical states, and symmetry under spectral flow imply that short string physical states now exist only for [13]

$$1/2 < j < (k+1)/2. \quad (3.14)$$

For $k \leq 1$ this excludes many short string states, including the vacuum and graviton with $j = 1$. As in the bosonic string, for $k = 1, w = 1$, there is a stringy tower of massless states [2, 13]. The results again are in agreement with the effective description of Seiberg and Witten [25] for the supersymmetric case.

The current algebra is a supersymmetrization of the affine $SL(2, \mathbb{R})$ algebra. Shifting the bosonic currents by fermion bilinears gives a direct product of two commuting subalgebras, a free fermion algebra with generators $\psi^\pm, \psi^3$ and an affine $SL(2, \mathbb{R})$ algebra of level $\kappa = k + 2$ with generators $j_n^\pm, j_n^3$. The nonzero commutation and anticommutation relations are

$$\begin{aligned} [j_m^3, j_n^3] &= -\frac{k+2}{2} m \delta_{n+m,0}, & [j_m^3, j_n^\pm] &= \pm j_{m+n}^\pm, & [j_m^+, j_n^-] &= (k+2) m \delta_{m+n,0} - 2 j_{m+n}^3, \\ \{\psi_r^+, \psi_s^-\} &= k \delta_{r+s,0}, & \{\psi_r^3, \psi_s^3\} &= -\frac{k}{2} \delta_{r+s,0}. \end{aligned} \quad (3.15)$$

The central charge (3.12) can be written in a form similar to (3.1) as

$$c_k = \frac{3\kappa}{\kappa-2} + \frac{3}{2} \quad (3.16)$$

and $k = 1$ corresponds to $\kappa = 3$.

The quantum states fit into representations of the bosonic $SL(2, \mathbb{R})$ algebra of level $\kappa = k + 2$ and of the free fermion algebra, where the fermions can have Ramond or Neveu-Schwarz boundary conditions. The spectrum has been analyzed in [13–19]. We will now summarize the results for the spectrum of one-particle states for the superstring in AdS_3 that we need here, following [13].

The physical-state condition in the NS sector with w units of spectral flow is

$$\tilde{L}_0 - 1/2 = 0, \quad \tilde{L}_0 = L_0 - w \tilde{J}_0 + k w^2 / 4. \quad (3.17)$$

²Note that these six fermions generate an $su(2) \times su(2)$ Kač-Moody algebra, which allows one to interpret this sigma model as $S^3 \times S^3$. In this sense the full theory can also be thought of as $AdS_3 \times S^3 \times S^3 \times S^1$.

The target space energy E and spin s are given by

$$\frac{E+s}{2} = \tilde{J}_0, \quad \frac{E-s}{2} = \tilde{\tilde{J}}_0, \quad (3.18)$$

while

$$L_0 = -j(j-1)/k + N + h \quad (3.19)$$

where N is the level of the state in the $\text{SL}(2, \mathbb{R})$ current algebra and is given by

$$N = \frac{1}{k} \left[\sum_{m=1}^{\infty} (j_{-m}^+ j_m^- + j_{-m}^- j_m^+ - 2j_{-m}^3 j_m^3) + \sum_{r>0} r(\psi_{-r}^+ \psi_r^- + \psi_{-r}^- \psi_r^+ - 2\psi_{-r}^3 \psi_r^3) \right]. \quad (3.20)$$

The spin of the continuous representation is $j = 1/2 + i\sigma$, $\sigma \in \mathbb{R}$, $\sigma > 0$. Here h is the (worldsheet) conformal weight for the internal SCFT $\mathcal{N}$, which for the free field theory we are considering here is $h = N_{\mathcal{N}} + h_0$ where $N_{\mathcal{N}}$ is the excitation number for the internal SCFT. All excitation number are either integer or half integer.

The GSO projection in the NS sector is

$$N + N_{\mathcal{N}} + (w+1)/2 \in \mathbb{N}. \quad (3.21)$$

To understand the shift $(w+1)/2$, we recall that the three fermionic coordinates $\psi^\pm, \psi^3$ transform under spectral flow according to

$$\psi_r^\pm \rightarrow \tilde{\psi}_r^\pm = \psi_{r \pm w}^\pm, \quad \psi_r^3 \rightarrow \tilde{\psi}_r^3 = \psi_r^3. \quad (3.22)$$

Consider next the fermionic number operator in the NS sector,

$$N_F \equiv \sum_{r \in \mathbb{N}+1/2} r(\psi_{-r}^+ \psi_r^- + \psi_{-r}^- \psi_r^+ - 2\psi_{-r}^3 \psi_r^3). \quad (3.23)$$

Under spectral flow it transforms to

$$\begin{aligned} \tilde{N}_F &\equiv \sum_{r \in \mathbb{N}+1/2} r(\tilde{\psi}_{-r}^+ \tilde{\psi}_r^- + \tilde{\psi}_{-r}^- \tilde{\psi}_r^+ - 2\tilde{\psi}_{-r}^3 \tilde{\psi}_r^3) \\ &= \sum_{r \in \mathbb{N}+1/2} r(\psi_{-r+w}^+ \psi_{r-w}^- + \psi_{-r-w}^- \psi_{r+w}^+ - 2\psi_{-r}^3 \psi_r^3). \end{aligned} \quad (3.24)$$

Some of the fermionic products in this equation are not normal ordered and the shift in (3.21) comes from reordering them. To be concrete, consider the flow with $w = 1$. Then

$$\tilde{N}_F = \frac{1}{2}(\psi_{+1/2}^+ \psi_{-1/2}^- - \psi_{-1/2}^- \psi_{+1/2}^+) + N_F \quad (3.25)$$

We follow ref. [13] and define the GSO projection in the same way in all flowed sectors *in terms of the flowed fermionic oscillators*: $\tilde{N}_F + N_{\mathcal{N}} \in \mathbb{N} + 1/2$, so the shift $(w+1)/2$ follows trivially from eq. (3.25).

The R sector physical-state condition is

$$\tilde{L}_0 = L_0 - w\tilde{J}_0 + kw^2/4 = 0, \quad (3.26)$$

while the GSO projection does not eliminate any conformal weight from the spectrum because of the existence of fermionic zero-modes. Also, all the excitation numbers in the R sector are always integers, so that $N + N_{\mathcal{N}} \in \mathbb{N}$.

Note that the vacuum energy has been normalized so that the conformal weights of the ground states of both the NS and R sector are zero.

The weight $C = -j(j-1) = \sigma^2 + 1/4$ is the same for both the left and right sectors of the superstring, so the normalization of the ground state energy tells us that combining the left and right sectors we obtain integer spin $s = \tilde{J}_0 - \tilde{\bar{J}}_0$ from the NS-NS and R-R sectors and half-integer spins from the NS-R and R-NS sectors, because for our SCFT we can check explicitly that $h_0 - \bar{h}_0 \in \mathbb{Z}$. Of course we would have arrived at the same result by noticing that the ground state of the R sector is in the spinorial representation of $\text{SL}(2, \mathbb{R})$.

Notice that a consistent GSO projection requires 10 world-sheet fermions (eight plus two that cancel the world-sheet supersymmetry ghosts). This is necessary because otherwise the one-loop string integral could not be interpreted as a vacuum energy. Technically, the reason is that we need theta-functions with characteristics to appear as $\theta^4 \bmod 8$. If we had chosen $\mathcal{N}$ to be the SCFT on T^3 , we would have only six fermions: three from the supersymmetric $\text{SL}(2, \mathbb{R})$, three from T^3 . Instead we take $\mathcal{N}$ to be the SCFT on S^1 (with one fermion) tensored with the SCFT with 6 free fermions, which together with $\psi^\pm, \psi^3$ gives 10 fermions in total, as required.

4 Holographic dual

Superstring theory in AdS_3 is expected to have a 2-d CFT dual. While the supersymmetric case with RR flux has been much discussed, the case with NS-NS flux has been studied less. In [18], it was conjectured that the dual CFT for the superstrings on $\text{AdS}_3 \times \mathcal{N}$ should be a SCFT in the moduli space of the symmetric product $\mathcal{M}^N/S_N$. The integer N is associated with the number of long fundamental strings required to construct the AdS_3 background and parameterizes the string coupling constant, and $\mathcal{M}$ is the spacetime SCFT with $c = 6k$ corresponding to transverse excitations of a single long string. There have been many attempts in the literature to identify pieces of this exact SCFT, some of which suggest a boundary theory that describes the BPS part of the spectrum for larger values of k . Clearly, a key issue is identifying the correct $\mathcal{M}$.

To avoid confusion, note that there are four 2-d CFTs which play a role here. The first is the worldsheet CFT on $\text{AdS}_3 \times \mathcal{N}$ with critical central charge of 26 or 15. The second is the worldsheet theory describing the high energy transverse excitations of one long string; an effective theory at large k is given by Liouville theory for the bosonic string and super-Liouville theory for the superstring. The third is a class of CFTs capturing only the BPS part of the spectrum. The fourth is the holographic dual CFT, the exact description of the space-time theory at the transition point.

We now attempt to identify the holographic dual for superstrings with $k = k_c = 1$. First, note that at this point the cosmological constant is of the order of the string scale (though not necessarily of the order of the Plank scale), implying that there are large string

corrections. We may expect to be in the familiar territory of a system which is rather simple in CFT language but which deviates strongly from the semi-classical geometric picture.

This of course occurs for duals of AdS_5 as well as for the candidates for the duals for higher spin theories on AdS_4 , the so-called Vasiliev models. The difference is that in our case we can compare the exact spectrum of our proposed CFT to the exact bulk spectrum using the results of [12], at least in the limit of large N .

The excitations of a long string are described by an effective CFT, which for large k is a (super-)Liouville theory, that has a continuous spectrum above a gap. For $k \leq k_c$ long strings become weakly coupled close to the boundary [1]. Since the strings are weakly coupled, we may hope that this description applies at $k = k_c$. For the bosonic string, the central charge $1 + 3Q^2$ of the Liouville theory tends to $c = 1$ as k approaches $k_c = 3$ (3.10), and the gap (3.11) vanishes so that the spectrum is bounded from below by zero. For the superstring, the situation is similar. The SCFT governing the long string, which for large k is effectively given by a super-Liouville theory, is expected to exist at any k and have a central charge tending to $c = 3/2$ as k approaches $k_c = 1$, with the spectrum again bounded from below by zero.

Based on these hints, we propose a dual boundary *SCFT* theory for the $k = 1$ point of the supersymmetric theory. The excitations of a long string at $k = 1$ are expected to be described by a $c = 3/2$ SCFT which has a continuous spectrum bounded from below by zero. The simplest possible theory with these properties is a free boson plus a free fermion. Therefore *we propose that the $d = 2$ SCFT on the symmetric orbifold $(\mathbb{R} \times \mathcal{N})^N / S_N$ is the holographic dual to string theory on $\text{AdS}_3 \times \mathcal{N}$ at $k = 1$.* If $\mathcal{N} = T^3$ for the critical $k = 1$ theory, then the dual would be the SCFT on $(\mathbb{R} \times T^3)^N / S_N$, but here we will focus on the case in which $\mathcal{N}$ is the S^1 SCFT tensored with the free fermion theory.

As we will see in the next section, in the large N limit, the spectrum of the SCFT on $(\mathbb{R} \times T^3)^N / S_N$ coincides *precisely* with that of the long strings of the $SL(2, \mathbb{R})$ WZW model found in [12]. This provides non-trivial evidence for the correctness of our duality conjecture. However, there are caveats. It could be that the $c = 3/2$ SCFT that arises in the limit is not the free SCFT on $\mathbb{R}$, but a different theory with the same spectrum (or nearly the same spectrum).³ Another possibility is that $\mathbb{R}$ could be replaced by $\mathbb{R}/\mathbb{Z}_2$. This theory has additional normalizable states in addition to those that arise for $\mathbb{R}$. Careful consideration of the interactions is needed to determine which $c = 3/2$ SCFT is correct here, and what role the remaining short string states play. We leave this investigation to future work.

The presence of the $\mathbb{R}$ factor in the boundary CFT suggests considering a similar factor in the bulk theory. Indeed, we shall show in the next section that the parameter σ in the bulk spectrum should be identified with $p/\sqrt{2}$ where p is the momentum in the $\mathbb{R}$ factor in the boundary CFT. Note that if the target space of the seed theory in our CFT orbifold is $\mathbb{R} \times \mathcal{N}$, the momentum p ranges over the entire real line. On the other hand if it is

³For the bosonic case, the corresponding limit would give a $c = 1$ CFT. One candidate is a free boson, another is the theory discussed by Runkel and Watts in [28–30]. According to [28], this theory also has a continuous spectrum bounded from below by zero, but missing a discrete set of points.

$(\mathbb{R}/\mathbb{Z}_2) \times \mathcal{N}$, there is only one state for each $|p|$ (and there are additional discrete states in the twisted sector). Certain observables are insensitive to this distinction (cf. section 6).

This system of higher-spin massless states may enjoy the conformal invariance that might be expected at a phase transition. However, there remain massive states in the spectrum at this point (e.g. from continuous representations with $w \neq 1$, or from states with both N and $\bar{N}$ non-zero), so the full theory is presumably not conformally invariant, but may have spontaneously broken conformal symmetry.

4.1 The permutation orbifold $(\mathbb{R} \times \mathcal{N})^N/S_N$

The Hilbert space of a permutation orbifold sigma model M^N/S_N decomposes into twisted sectors labeled by the conjugacy classes of S_N (see e.g. [31]). The sigma-model fields are maps $S^1 \times \mathbb{R} \rightarrow M^N/S_N$, and the S^1 coordinate σ has periodicity π . The conjugacy classes are associated with partitions of N into integers: $N = \sum_{n=1}^N nM_n$. Each class $[g]$ consists of all permutations of the form

$$[g] = (1)^{M_1} (2)^{M_2} \dots (N)^{M_N} = \prod_{n=1}^N (n)^{M_n}. \quad (4.1)$$

In the above we denote a cyclic permutation of n elements by (n) , with M_n its multiplicity (for instance, a conjugacy class of S_9 is $(1)^2(2)^2(3)$, with $M_1 = 2, M_2 = 2, M_3 = 1$, and all other $M_i = 0$). Each conjugacy class gives rise to a twisted sector of the symmetric product, where the coordinate X_i in the i -th copy of M obeys the boundary condition

$$X_i(\sigma + \pi) = X_{[g](i)}(\sigma). \quad (4.2)$$

The untwisted sector is given by $[g] = (1)^N$, for which $[g](i) = i \ \forall i$. The Hilbert space obtained by quantizing the fields X_i with boundary condition (4.2) must be projected to the invariant subspace of the commutant C_g of $[g]$ [31]

$$C_g = S_{M_1} \times (S_{M_2} \rtimes \mathbb{Z}_2) \times \dots (S_{M_N} \rtimes \mathbb{Z}_N). \quad (4.3)$$

Explicitly, the Hilbert space of the twisted sector $[g]$ is

$$\mathcal{H}_g^{C_g} = \otimes_{n=1}^N S^{M_n} \mathcal{H}_{(n)}^{\mathbb{Z}_n}. \quad (4.4)$$

The various symbols appearing here are defined as in [31]:

1. $S^k(\mathcal{H}) = (\mathcal{H} \otimes \dots \mathcal{H})^{Sym}$ is the k -fold product of the vector space $\mathcal{H}$ symmetrized according to the grading of $\mathcal{H}$.
2. $\mathcal{H}_{(n)}^{\mathbb{Z}_n}$ is the $\mathbb{Z}_n$ -invariant sector of $\mathcal{H}_{(n)}$.
3. $\mathcal{H}_{(n)}$ is the Hilbert space of the sigma model M^n with boundary condition

$$X_i(\sigma + \pi) = X_{i+1 \bmod n}(\sigma). \quad (4.5)$$

The full Hilbert space of the theory is the direct sum of (4.4) over all conjugacy classes. Our proposal is that the single-particle states of the bulk with winding w correspond to states where $M_1 = N - w$ and $M_w = 1$ (and all other $M_n = 0$). Multi-particle states are those with more than one sector $M_{n>1} \neq 0$. Notice that at finite N , the maximum possible winding w is $w = N$. As there is no limit on w in the bulk according to classical string theory, it is only in the $N \rightarrow \infty$ limit that our boundary theory could reproduce the bulk spectrum of long strings found in [12]. This is as expected, since the limit $N \rightarrow \infty$ should correspond to classical physics in the bulk.

The sector (4.5) can be thought of as a map from an S^1 with periodic coordinate $\sigma \sim \sigma + n\pi$ to M . Therefore all energies are divided by n and the vacuum energy is also shifted to $\frac{c}{24}(n - 1/n)$ (see e.g. [32, 33]). Then, a state with right conformal weight h and left conformal weight $\bar{h}$ in the conformal sigma model on M maps to a state in $\mathcal{H}_{(n)}$ with conformal weights

$$h_n = \frac{h}{n} + \frac{c}{24} \left(n - \frac{1}{n} \right), \quad \bar{h}_n = \frac{\bar{h}}{n} + \frac{c}{24} \left(n - \frac{1}{n} \right). \quad (4.6)$$

In our case $M = \mathbb{R} \times \mathcal{N} = \mathbb{R} \times S^1 \times \mathcal{F}$ where $\mathcal{F}$ is the SCFT of 6 free fermions and $c = 6$. To find the specific form of $h, \bar{h}$, we can write the conformal weight as a sum of excitation numbers $N + N_{\mathcal{N}}$ (with $N_{\mathcal{N}} = N_{S^1} + N_{\mathcal{F}}$) plus the primary weight h_0 . The excitation numbers are each the sum of a bosonic excitation number, which is always integer, plus a fermionic excitation number, that can be either integer or half-integer. We will need only the explicit form of the excitation number in the SCFT on $\mathbb{R}$, consisting of a free boson with modes a_n and a free fermion with modes ψ_m

$$N = \sum_{m=1}^{\infty} (a_{-m}a_m + m\psi_{-m}\psi_m), \quad [a_m, a_n] = m\delta_{m+n,0}, \quad \{\psi_m, \psi_n\} = \delta_{m+n,0}. \quad (4.7)$$

The primary weights $h_0, \bar{h}_0$ are

$$h_0 = \frac{1}{2}p^2 + h_0^{\mathcal{N}}, \quad \bar{h}_0 = \frac{1}{2}p^2 + \bar{h}_0^{\mathcal{N}} \quad (4.8)$$

where p is the momentum in the $\mathbb{R}$ factor.

4.2 Fermionic boundary conditions

The sigma model on M is defined only once we specify its spectrum. To do so, we have to decide whether fermions have periodic (R) boundary conditions, antiperiodic (NS) boundary conditions, or both. In the first case the Hilbert space of the sigma model is only the R sector, in the second it is only the NS sector while in the third it is a sum of R plus NS. Likewise, we can decide to project out states with odd fermion number in either sector. We want to reproduce the spectrum of superstrings on $\text{AdS}_3 \times \mathcal{N}$ with $\mathcal{N} = S^1 \times \mathcal{F}$, where fermions can be both NS and R and in the sector with w units of flow we project over states with fermion parity $(-1)^F = (-1)^{1+w}$ (see eq. (3.21)). To mimic the string construction as closely as possible, we include both periodic and antiperiodic fermions in the CFT make a projection on fermion number *before* the symmetrization S_N . By considering both NS

and R sectors and imposing the projection before the symmetrization, we obtain a model that describes transverse oscillations of long strings on $\text{AdS}_3 \times S_1 \times \mathcal{F}$ instead of a generic symmetric-orbifold CFT. In section 5.3 we relate this to an equivalent construction where all fermions have NS boundary conditions.

Since the fermion parity of the i -th copy in M^N and $(-1)_i^F$ and S_N do not commute, we have to carefully examine the possible boundary conditions of fermions. We will do so in the next subsection.

5 Comparison of spectra

The spectrum of conformal dimensions given by eq. (3.17) at $k = 1$ can be written in a form valid for both the NS and R sectors. In the sector with w units of flow it is

$$\begin{aligned} h &= \frac{E+s}{2} = \frac{1}{w} \left(\sigma^2 + N + N_{\mathcal{N}} + h_0 + \frac{a}{2} \right) + \frac{1}{4} \left(w - \frac{1}{w} \right), \\ \bar{h} &= \frac{E-s}{2} = \frac{1}{w} \left(\sigma^2 + \bar{N} + \bar{N}_{\mathcal{N}} + \bar{h}_0 + \frac{b}{2} \right) + \frac{1}{4} \left(w - \frac{1}{w} \right). \end{aligned} \quad (5.1)$$

The constants a, b are zero in the NS sector and one in the R sector. This is the spectrum of single-particle states. We want to identify it with the spectrum of states in $\mathcal{H}_{(w)}^{\mathbb{Z}_w}$, because single-particle states map to states where only one copy of the Hilbert space of M in (4.4) is not in its ground state. Using (4.6), the definition of the conformal weight given below it, and (4.8) we find, on changing $n \rightarrow w$,

$$\begin{aligned} h_w &= \frac{1}{w} \left(\frac{1}{2} p^2 + N + N_{\mathcal{N}} + h_0^{\mathcal{N}} \right) + \frac{1}{4} \left(w - \frac{1}{w} \right), \\ \bar{h}_w &= \frac{1}{w} \left(\frac{1}{2} p^2 + \bar{N} + \bar{N}_{\mathcal{N}} + \bar{h}_0^{\mathcal{N}} \right) + \frac{1}{4} \left(w - \frac{1}{w} \right). \end{aligned} \quad (5.2)$$

Eqs. (5.1) and (5.2) are very similar. To prove that they are in fact identical we have to take care of a few details.

5.1 Multiplicities

The excitation number operators $N_{\mathcal{N}}$ in (5.1), (5.2) are evidently the same, as are the barred excitation numbers $\bar{N}_{\mathcal{N}}$. The operator N in (5.1) is given in eq. (3.20) while the operator N in (5.2) is given in (4.7). Despite appearances they give the same spectrum with the same multiplicities thanks to the no-ghost theorem of [12, 19]. The theorem says that up to null states, the physical states of the fermionic string on AdS_3 obey $J_n^3 |Phys\rangle = 0$, $\psi_n^3 |Phys\rangle = 0$ for all $n > 0$. If we denote by L_n^3, G_r^3 the Virasoro and world-sheet supersymmetry generators of the sigma model associated to J^3 in $\text{SL}(2, \mathbb{R})$ then $\hat{L}_n \equiv L_n - L_n^3$ and $\hat{G}_r \equiv G_r - G_r^3$ commute with J_n^3 , so that every physical state is a null state plus a linear combination of vectors

$$\hat{L}_{n_1} \dots \hat{L}_{n_I} \hat{G}_{r_1} \dots \hat{G}_{r_J} |j, m, \alpha\rangle, \quad (5.3)$$

where the state $|j, m, \alpha\rangle$ denotes a primary of the current algebra of $\text{SL}(2, \mathbb{R})$. The Hilbert space generated by these vectors is isomorphic to the Fock space of one fermionic oscillator

plus one bosonic oscillator. The natural isomorphism of basis vectors maps the operator N given in (3.20) into the N given in (4.7).

Invariant states of the superstring on $\text{AdS}_3 \times \mathcal{N}$. Not all values of N , $N_{\mathcal{N}}$ are allowed. The single-string states in $\text{AdS}_3 \times \mathcal{N}$ are constrained by the condition $h - \bar{h} \in \mathbb{Z} + a'/2$, with $a' = 0$ in the NS-NS and R-R sectors and $a' = 1$ in the NS-R and R-NS sectors. Thanks to eq. (5.1), this condition becomes, in all sectors,

$$N + N_{\mathcal{N}} + h_0 + \frac{a}{2} - \bar{N} - \bar{N}_{\mathcal{N}} - \bar{h}_0 - \frac{b}{2} = w \left(m + \frac{a-b}{2} \right), \quad m \in \mathbb{Z}. \quad (5.4)$$

As we mentioned earlier, the GSO projection imposes the further constraint

$$(-1)^F = (-1)^{w+1}, \quad (-1)^{\bar{F}} = (-1)^{w+1}. \quad (5.5)$$

5.2 Invariant states of the symmetric orbifold

Projecting over invariant states is straightforward for bosonic coordinates but less so for fermions. Fermions can be twisted in two ways: by the permutation and by the fermionic parity. Consider the cyclic permutation (n) inside $[g] = (1)^{M_1}(2)^{M_2} \dots (N)^{M_N}$. Up to trivial field redefinitions the boundary conditions for the n cyclically-permuted fermions are

$$R: \psi_j(\sigma + \pi) = \psi_{j+1}(\sigma) \mod n, \quad NS: \psi_j(\sigma + \pi) = (-1)^{\delta_{nj}} \psi_{j+1}(\sigma) \mod n. \quad (5.6)$$

Call g the generator of $\mathbb{Z}_n$, the cyclic permutation of n elements;⁴ $(-1)^{F_i}$ are the generator of the fermionic parities $\mathbb{Z}_2$. Then the R boundary conditions in eq. (5.6) is a twist by the element g in $\mathbb{Z}_n \rtimes (\mathbb{Z}_2)^n$ while the NS boundary condition is a twist by the element $g(-1)^{F_n}$. *To find invariant states we must project over the commutant of the twist element.* For R boundary condition the projection is obviously

$$P = \frac{1}{2} [1 + (-1)^F] \frac{1}{n} \left(\sum_{j=1}^n g^j \right), \quad F \equiv \sum_{j=1}^n F_j \quad (5.7)$$

For NS, the commutant is found as follows. A product of fermion parities, $P_A = \prod_{j \in A} (-1)^{F_j}$, acts on fermions as $P\psi_j = \eta_j \psi_j P$ with $\eta_j = -1$ if $j \in A$ and $\eta_j = +1$ if $j \notin A$. The action of g is of course $g\psi_j = \psi_{j+1} \mod n$. Call $A = g(-1)^{F_n}$; it acts on fermions as $A\psi_j = \eta_j \psi_{j+1} A$, $\eta_j = (-1)^{\delta_{jn}}$. Call $B = g^k P_A$ then

$$AB\psi_j = \eta_{j+k} \epsilon_j \psi_{j+k+1} AB, \quad BA\psi_j = \eta_j \epsilon_{j+1} \psi_{j+k+1} BA. \quad (5.8)$$

It is easy to see that the commutant of A is the subgroup generated by 1, $(-1)^F$, and the elements $B_k \equiv g^k \prod_{j=n-k-1}^n (-1)^{F_j}$, $k = 1, \dots, n-1$. Next, expand the fermionic coordinate in oscillators

$$\psi_j(\sigma) = \sum_{r \in \mathbb{Z} + 1/2} d_r e^{2ir[\sigma + \pi(j-1)]/n}, \quad j = 1, \dots, n, \quad \psi_{j+n}(\sigma) \equiv \psi_j(\sigma). \quad (5.9)$$

On the oscillators d_r , $B_k d_r = \exp[2irk\pi/n] d_r B_k$.

⁴The winding number w of the bulk theory corresponds to the cycle length n in the holographic dual discussed in section 4. Hereafter we will write n .

There is one last point to consider. The sigma model on M has left- and right-moving fermions. The GSO projection acts independently on the left and right movers. On the other hand, the twist acts simultaneously on left movers and right movers. So, in the NS-NS sector, the commutant is generated by 1 , $(-1)^F$, $(-1)^{\bar{F}}$, and $g^k \prod_{j=n-k-1}^n (-1)^{F_j + \bar{F}_j}$, $k = 1, \dots, n-1$. In the NS-R sector the commutant is generated by 1 , $(-1)^F$, $(-1)^{\bar{F}}$, $g^k \prod_{j=n-k-1}^n (-1)^{F_j}$, $k = 1, \dots, n-1$; in the R-NS sector, barred and unbarred operators exchange their role.

Finally, we can use the identity $(-1)^{F_j} g = g(-1)^{F_{j-1}}$ to write the projectors over invariant states as

$$\begin{aligned} NS - NS: \quad P &= \frac{1}{4} [1 + (-1)^F] [1 + (-1)^{\bar{F}}] \sum_{j=0}^{n-1} [g(-1)^{F_n + \bar{F}_n}]^j, \\ NS - R: \quad P &= \frac{1}{4} [1 + (-1)^F] [1 + (-1)^{\bar{F}}] \sum_{j=0}^{n-1} [g(-1)^{F_n}]^j. \end{aligned} \quad (5.10)$$

Invariant states can be written in a more transparent way, valid for all sectors, by using the action of the commutant on the oscillators d_r , $\bar{d}_{\bar{r}}$. The expansion (5.9) holds in the R sector too if the index $r \in \mathbb{Z}$. The projections (5.7) and (5.10) keep states invariant under three distinct transformations:

$$d_r \rightarrow -d_r, \quad \bar{d}_{\bar{r}} \rightarrow -\bar{d}_{\bar{r}}, \quad (d_r, \bar{d}_{\bar{r}}) \rightarrow (e^{2\pi i r/n} d_r, e^{-2\pi i \bar{r}/n} \bar{d}_{\bar{r}}). \quad (5.11)$$

The first two fix the left and right fermion parity. To find states invariant under the last transformation we must define its action on the ground states of the fermionic oscillators, $|h_0^{\mathcal{N}}\rangle |\bar{h}_0^{\mathcal{N}}\rangle$. The correct definition turns out to be

$$B_1 |h_0^{\mathcal{N}}\rangle |\bar{h}_0^{\mathcal{N}}\rangle = (-1)^{a+b} e^{2i\pi(h_0^{\mathcal{N}} - \bar{h}_0^{\mathcal{N}})/n} |h_0^{\mathcal{N}}\rangle |\bar{h}_0^{\mathcal{N}}\rangle, \quad (5.12)$$

with $a, b = 0$ in the NS sectors and $a, b = 1$ in the R sectors. The resulting constraint is

$$N + N_{\mathcal{N}} + h_0^{\mathcal{N}} - \bar{N} - \bar{N}_{\mathcal{N}} - \bar{h}_0^{\mathcal{N}} \in n \left(\mathbb{Z} + \frac{a-b}{2} \right). \quad (5.13)$$

This equation matches eq. (5.4) perfectly, once the shift in conformal weights described in the next subsection is taken into account, so it is natural to define the fermion parity of the vacuum states in the (n) sector as $(-1)^F |0\rangle = (-1)^{\bar{F}} |0\rangle = (-1)^{1+n} |0\rangle$ to match eq. (5.5).

5.2.1 Shift in the conformal weight

There is one last point to clarify before comparing the spectra in eqs. (5.1) and (5.2). Let us examine the unbarred sector, since the story will be the same for the barred sector. There seems to be a discrepancy between the conformal weight $\sigma^2 + h_0 + a/2$ in (5.1) and the conformal weight $p^2/2 + h_0^{\mathcal{N}}$ in (5.2). In reality there is no discrepancy. In the NS sector the weights agree once we identify $\sigma = p/\sqrt{2}$. When comparing R sectors we must recall that we defined $\sigma^2 + h_0$ by subtracting the energy of the R vacuum, so that the relation between $\sigma^2 + h_0$ and $p^2/2 + h_0^{\mathcal{N}}$ is

$$\sigma^2 + h_0 = \frac{p^2}{2} + h_0^{\mathcal{N}} - \frac{n_F}{16}, \quad (5.14)$$

where n_F is the number of fermions in the SCFT $\mathbb{R} \times \mathcal{N}$. To match weights between (5.1) and (5.2) we need $n_F/16 = 1/2$ i.e. $n_F = 8$. Taking $\mathcal{N} = S^1 \times \mathcal{F}$, we have one fermion from the $\mathbb{R}$ sigma model, one from S^1 and six from the free fermion factor $\mathcal{F}$, giving a total of $n_F = 8$, as required. If we had chosen $\mathcal{N} = T^3$, we would have had one fermion from the $\mathbb{R}$ sigma model, one from S^1 , and two from T^2 , which would be too few; we would have needed to fermionize the two coordinates of T^2 to get another four fermions, so that the T^2 sigma model would be replaced by the free fermion model $\mathcal{F}$. The same replacement of the T^2 SCFT with $\mathcal{F}$ was needed to make sense of the superstring on $\text{AdS}_3 \times S^1 \times T^2$ (see the last paragraph of section 3.2).⁵

5.3 An equivalent construction

The dual CFT can be defined in an equivalent way by choosing the following fermion boundary conditions in the cyclic-permutation sector (n),⁶

$$\psi_j(\sigma + \pi) = -\psi_{j+1}(\sigma) \pmod{n}. \quad (5.15)$$

Mapping the cylinder coordinates τ, σ into the complex plane coordinate z by the exponential mapping $z = \exp(\tau + 2i\sigma)$ these become the “pure NS” conditions

$$\psi_j(e^{2i\pi}z) = \psi_{j+1}(z) \pmod{n} \quad (5.16)$$

usually considered in the literature on supersymmetric permutation orbifolds beginning with [35]. Eq. (5.15) implies that (after a simple field redefinition) the first of eqs. (5.6) holds for n even while the second holds for n odd.

The complete spectrum of the bulk theory given in eqs. (5.1) is obtained by *not* imposing a GSO projection on the symmetric-orbifold states. The only constraint is the projection over Z_n invariant states. The action on bosons is well known [31]; to find the action on fermions it is convenient to change eq. (5.9) and expand the eight fermions as

$$\psi_j^I(\sigma) = \sum_{r \in \mathbb{Z} + n/2} d_r^I e^{i2r[\sigma + \pi(j-1)]/n + i\pi(j-1)}, \quad \forall j, \quad I = 1, \dots, 8. \quad (5.17)$$

An identical expansion holds for $\bar{\psi}_j^I$ in terms of oscillators $\bar{d}_r^I$. The cyclic permutation g that sends j into $j + 1$ now acts on the oscillators as

$$(d_r^I, \bar{d}_r^I) \rightarrow (e^{2ir\pi/n + i\pi} d_r^I, e^{2i\bar{r}\pi/n + i\pi} \bar{d}_r^I), \quad (5.18)$$

while it acts on the fermionic oscillator ground state as

$$g|h_0^{\mathcal{N}}\rangle|\bar{h}_0^{\mathcal{N}}\rangle = e^{2i\pi(h_0^{\mathcal{N}} - \bar{h}_0^{\mathcal{N}})/n} |h_0^{\mathcal{N}}\rangle|\bar{h}_0^{\mathcal{N}}\rangle. \quad (5.19)$$

The projection over invariant states therefore changes from (5.13) to

$$N + N_{\mathcal{N}} + h_0^{\mathcal{N}} - \bar{N} - \bar{N}_{\mathcal{N}} - \bar{h}_0^{\mathcal{N}} \in n(\mathbb{Z} + F/2 - \bar{F}/2). \quad (5.20)$$

⁵We can of course bosonize the four the fermions to recover a torus, but the radii of the torus must be fixed to correspond to the free fermion theory.

⁶We are indebted to M. Gaberdiel, who raised a number of questions regarding the relationship of our construction to that of [34]. In this subsection, we address the concerns he raised and would like to thank him for his input.

5.3.1 Odd n

Specifically, when the cyclic permutation (n) has odd length, the symmetric-orbifold fermions are in the NS sector (as defined in (5.6)) so that there is no shift in the conformal weight and eq. (5.14) gives $h_0 = h_0^{\mathcal{N}}$.

Target space bosons. Symmetric-orbifold states with an even number of fermions are in one-to-one correspondence with long string states with $w = n$ units of flow in the NS sector of the WZW model. The latter are subject to constraint (5.4), which agrees with eq. (5.20) once we recall that the WZW model fermions have to be projected according to eq. (5.5) $(-1)^F = (-1)^{n+1} = 1$ and therefore they too have even fermion number.

Target space fermions. Symmetric-orbifold states with an odd number of fermions have $N + N_{\mathcal{N}} \in \mathbb{N} + 1/2$; this matches the WZW fermionic states in the R sector, whose spectrum contains $N + N_{\mathcal{N}} + 1/2$ with $N + N_{\mathcal{N}} \in \mathbb{N}$. The shift by $1/2$ also makes eq. (5.4) agree with (5.20). To complete the matching of WZW states with orbifold states multiplicities must also agree. To see that they do, we recall that the symmetric-orbifold states corresponding to the R vacuum of the WZW model are

$$d_{-1/2}^I |0\rangle, \quad I = 1, \dots, 8. \quad (5.21)$$

These eight states are in the vector representation of the $SO(8)$ rotating the fermion indices I . The R vacuum states of the WZW model are instead in one of the spinor representations of $SO(8)$. The two representations are mapped into each other by $SO(8)$ triality.

5.3.2 Even n

For even cycle length the symmetric-orbifold fermions are in the R sector (as defined in (5.6)), so their fermion number $N + N_{\mathcal{N}}$ is always integer valued. When comparing conformal weights eq. (5.14) gives

$$h_0 = h_0^{\mathcal{N}} - 1/2. \quad (5.22)$$

Target space bosons. The spectrum of the WZW NS sector is projected on states with odd fermion number $(-1)^F = (-1)^{n+1} = -1$ so conformal dimensions and multiplicities match thanks to the shift (5.22), which also makes eq. (5.4) agree with (5.20). Here the $SO(8)$ triality works “in reverse”: the ground states of the symmetric orbifold, which are in one of the spinor representations of $SO(8)$, are mapped to the ground states of the WZW model, which are in the vectorial representation.

Target space fermions. The spectrum of WZW R-sector states is given by eq. (5.1) with $a = 1$ (or equivalently $b = 1$). Thanks to the shift (5.22) and to the fermion number operators in right hand side of eq. (5.20) we find again an exact match between these states and the states of the symmetric orbifold. The ground states in both descriptions belong to a spinorial representation of $SO(8)$.

What we have described in this subsection coincides at $p = 0$ and zero winding number on S^1 with the states found in the construction proposed in [34] when $\mathcal{N}$ is interpreted as $S^3 \times S^3 \times S^1$ as in footnote 5. The equivalence of the two constructions of $(\mathbb{R} \times \mathcal{N})^N/S_N$ at $p = 0$ and zero S^1 winding number can be seen for the “seed theory” of the symmetric orbifold, i.e. for the case $N = 1$, also from the fact that the eight fermions in $\mathbb{R} \times \mathcal{N}$ enter the partition function of our construction –which has an R sector and is GSO-projected– in the combination $\frac{1}{2}(\vartheta_{00}^4/\eta^4 + \vartheta_{01}^4/\eta^4 + \vartheta_{02}^4/\eta^4)$ while they enter in the construction of [34] as ϑ_{00}^4/η^4 . The two expressions are identical thanks to the famous Jacobi *æquatio identica satis abstrusa*.

6 Observations on interactions

The continuous spectrum of the symmetric orbifold has a density of single-particle states of the form $[L + \rho(p)]dp/2\pi$, with L an infrared regulator. The physical limit is $L \rightarrow \infty$. The corresponding density of states in the WZW theory is also of the form $[L + \rho(p)]dp/2\pi$ [36]. Without a detailed computation of the two-point function of primary operators in WZW at $k = k_c$ we cannot tell whether the range of integration in p is over the positive real numbers or all real numbers (this is the source of the uncertainty regarding whether the seed theory of the symmetric orbifold should be a supersymmetric version of the CFT on $\mathbb{R}$ or $\mathbb{R}/\mathbb{Z}_2$, or the CFT of Runkel and Watts [28–30], or something else). We can only compare the divergent parts of the spectra, which in fact agree.

Certain physical observables do not depend on either the discrete part of the density of states or the range of p . The reason is that long string states in AdS_3 propagate as asymptotically free particles in an infinite radial direction; in other words, they behave as particles moving in a potential $V(\rho)$ such that $\lim_{\rho \rightarrow +\infty} V(\rho) = 0$. A toy model of such a system is a particle moving in a potential with a wall located at $\rho = -L$. At finite ρ in the limit $L \rightarrow \infty$, the finite-time dynamics of a localized wave packet is the same as that for a free theory, where the momentum ranges from $p = -\infty$ to $p = +\infty$. To detect the existence of the wall one has to wait a time $O(2L/v)$, where v is the speed of the wave packet. This would remain true even if there are normalizable bound states attached to the wall. We conjecture that the short strings that remain as $k \rightarrow k_c$ are of this nature.

To check this conjecture and match the discrete part (and answer the question about the range of integration) we need to perform a careful re-evaluation of the two point functions at $k = k_c$. These comparisons are particular cases of a more general problem: comparing interactions between the bulk theory and the boundary CFT. We plan to do that in detail in a forthcoming paper; here we will only mention some obvious problems arising from the attempt to match bulk string amplitudes to boundary CFT correlators.

To simplify the discussion, consider first the interactions of the bosonic string.⁷ String amplitudes in AdS_3 are given in terms of correlation functions of vertex operators integrated

⁷Amplitudes of the supersymmetric theory are obtained from those of the bosonic theory by implementing three changes: first, one takes into account the shifting $k \rightarrow \kappa = k + 2$ in the level; second, one dresses the worldsheet operators with the free fermion contributions; third, one includes in the correlators $n - 2$ picture changing operators; see [37, 38].

over the inserting points z_i ($i = 1, 2, \dots, n$):

$$\mathcal{A}_{p_1, p_2, \dots, p_N}^n = \int \frac{\prod_{i=1}^n d^2 z_i}{\text{Vol}(\text{PSL}(2, \mathbb{C}))} \left\langle \prod_{i=1}^n : V_{j_i}^{w_i}(x_i; z_i) : \right\rangle_{\text{WZW}}, \quad (6.1)$$

where $\text{Vol}(\text{PSL}(2, \mathbb{C}))$ stands for the volume of the conformal Killing group and where in the left hand side we omitted for brevity the labels $w_1, w_2, \dots, w_n$ on which the amplitude also depends. We are also omitting ghost contributions and picture labels. The complex variables $(x_i, \bar{x}_i)$ are auxiliary variables — in some sense conjugate to $(m_i, \bar{m}_i)$ — that serve to organize the $\text{SL}(2, \mathbb{R})$ representations. From the boundary theory point of view, these variables have a clear interpretation as the coordinates of the local dual operator $\mathcal{O}_{h_i}(x_i)$ [39]. Each worldsheet operator $V_j^w(x; z)$ in (6.1) creates a physical state with quantum numbers $j = 1/2 + i\sigma$ and $w \geq 0$.

Problems appear when trying to match the bulk observables (6.1) with expectations for the boundary theory because the selection rules for the scattering amplitudes seem to differ from those one expects for the boundary observables. For instance, one can compute three point functions with arbitrary values of the winding numbers w_i [40].⁸ If $w_i - \sum_{j \neq i}^3 w_j > 1$ for any $i = 1, 2, 3$ they vanish [42]; more generally, WZW correlation functions (6.1) vanish unless the winding numbers w_i satisfy the bounds $w_i - \sum_{j \neq i}^n w_j \leq n - 2$ [42]. On the other hand, we can compute the three point function of twist operators that create cycles of length w_i as a function of position on the boundary using the techniques of [43]. A result of their analysis is that three point functions of operators that create the twisted states $g_1, g_2, g_3 \in S_N$ can be nonzero only when there exist three elements of S_N , h_i, h_2, h_3 , such that $\prod_{i=1}^3 h_i^{-1} g_i h_i = 1$. It is not obvious that these two conditions are consistent. Finally, if the CFT dual is in fact $(\mathbb{R} \times S^1 \times \mathcal{F})^N / S_N$, there will be a selection rule associated with a conserved momentum $p = \sqrt{2}\sigma$ in $\mathbb{R}$. Conservation of momentum should make amplitudes with $\sum_i \sigma_i \neq 0$ vanish, yet (6.1) does not seem to obey this rule. We plan to address these puzzles in a separate paper.

7 Conclusions

Among the various candidate phases of gravity is a phase in which the string has no tension. The transition point we address here, instead, contains two chiral trajectories of massless excitations of all spins, but it also contains many more massive states whose mass scale derives from the original theory on $\text{AdS}_3 \times \mathcal{N}$. While the tensionless phase of gravity is still mysterious, the $k = k_c$ theory can be described very concretely. Its long string spectrum is completely described by a symmetric orbifold of a free CFT. We do not understand yet fully if the interactions of the WZW superstring at $k = 1$ pinpoint the undeformed orbifold CFT as *the* holographic dual to the full theory, but we do have encouraging indications that this may be indeed true. In any event, the fact that the *complete* spectrum of the long strings, not just the BPS sector, can be matched to such a simple CFT is remarkable and novel. Finally, it would be interesting to better understand the long string phase that arises for $k < k_c$ and which has no graviton.

⁸Superstring amplitudes in AdS_3 with winding states have also been studied in [14, 41].

Note added. Related work has independently been done in [34]. We thank the authors of [34] for sharing their draft with us prior to publication.

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