Deformed $\mathcal{N} = 8$ mechanics of $(8, 8, 0)$ multiplets

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ABSTRACT: We construct new models of “curved” $SU(4|1)$ supersymmetric mechanics based on two versions of the off-shell multiplet $(8, 8, 0)$ which are “mirror” to each other. The worldline realizations of the supergroup $SU(4|1)$ are treated as a deformation of flat $\mathcal{N} = 8$, $d = 1$ supersymmetry. Using $SU(4|1)$ chiral superfields, we derive invariant actions for the first-type $(8, 8, 0)$ multiplet, which parametrizes special Kähler manifolds. Since we are not aware of a manifestly $SU(4|1)$ covariant superfield formalism for the second-type $(8, 8, 0)$ multiplet, we perform a general construction of $SU(4|1)$ invariant actions for both multiplet types in terms of $SU(2|1)$ superfields. An important class of such actions enjoys superconformal $OSp(8|2)$ invariance. We also build off-shell actions for the $SU(4|1)$ multiplets $(6, 8, 2)$ and $(7, 8, 1)$ through appropriate substitutions for the component fields in the $(8, 8, 0)$ actions. The $(6, 8, 2)$ actions are shown to respect superconformal $SU(4|1, 1)$ invariance.

KEYWORDS: Extended Supersymmetry, Superspaces, Space-Time Symmetries

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1 Introduction

In recent years, mainly motivated by the study of higher-dimensional models with “curved” rigid supersymmetries (see e.g. [1]), there was a growth of activity in supersymmetric mechanics (SM) models underlain by some semi-simple superalgebras treated as deformations of flat one-dimensional supersymmetries with the same number of supercharges. The simplest superalgebra of this kind is $su(2|1)$ (and its central-charge extension $\widehat{su}(2|1)$), which is a deformation of rigid $\mathcal{N}=4, d=1$ supersymmetry by a mass-dimension parameter m . The first examples with a worldline realization of $su(2|1)$ supersymmetry were considered more than 10 years ago (prior to [1] and related works) in [2, 3] and in [4] (where it was named “weak $d=1$ supersymmetry”). The corresponding worldline $su(2|1)$ multiplets had $d=1$ field contents $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ and $(\mathbf{1}, \mathbf{4}, \mathbf{3})$.¹

A systematic superfield approach to $su(2|1)$ supersymmetry was worked out in [6–8] and [9]. The models built on the multiplets $(\mathbf{1}, \mathbf{4}, \mathbf{3})$, $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ and $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ were studied at the classical and quantum level. Recently, $su(2|1)$ invariant versions of super Calogero-Moser systems were constructed and quantized [10–12]. The common notable features of all these models are:

- Oscillator-type Lagrangians for the bosonic fields, with m^2 as the oscillator strength,
- Wess-Zumino type terms for the bosonic fields, of the type $\sim im(\dot{z}\bar{z} - z\dot{\bar{z}})$,
- At the lowest energy levels, wave functions form *atypical* $su(2|1)$ multiplets, with unequal numbers of the bosonic and fermionic states.

It was of obvious interest to move one step further and to consider mechanics models with analogous deformations of $\mathcal{N}=8, d=1$ supersymmetry. In contrast to $\mathcal{N}=4$ supersymmetry, in the $\mathcal{N}=8$ case there exist two different possibilities for deformation due to the existence of two different superalgebras with eight supercharges: $su(2|2)$ and $su(4|1)$, with R -symmetry algebra $su(2) \oplus su(2)$ or $su(4) \oplus u(1)$, respectively.² The $su(2|2)$ models have been considered in [13] by analogy with the $su(2|1)$ case, on the basis of the appropriate superfield worldline formalism, as deformations of flat $\mathcal{N}=8$ SM models [14–18]. They were built on the off-shell multiplets $(\mathbf{3}, \mathbf{8}, \mathbf{5})$, $(\mathbf{4}, \mathbf{8}, \mathbf{4})$ and $(\mathbf{5}, \mathbf{8}, \mathbf{3})$. One class of $(\mathbf{5}, \mathbf{8}, \mathbf{3})$ actions represents a massive deformation for the same multiplet in the flat case [19, 20]. Another class enjoys superconformal $OSp(4^*|4)$ invariance. Remarkably, the superconformal group $OSp(4^*|4)$ is a closure of its two different $SU(2|2)$ subgroups, with deformation parameters m and $-m$. So any $SU(2|2)$ invariant action involving only even powers of m is automatically superconformal. Based on this observation, the general $SU(2|2)$ action of the multiplet $(\mathbf{3}, \mathbf{8}, \mathbf{5})$ was shown to be superconformal.

¹Our notation follows ref. [5]: bold numerals denote, respectively, the number of physical bosonic, physical fermionic and auxiliary bosonic degrees of freedom in the given supermultiplet.

²In the $su(2|2)$ case one can also add two central charges.

It turns out that some admissible multiplets of flat $\mathcal{N}=8$ supersymmetry do not have $SU(2|2)$ analogs, most importantly the so called “root” $\mathcal{N}=8$ multiplet $(\mathbf{8}, \mathbf{8}, \mathbf{0})$. The significance of this root multiplet derives from the fact that all other flat $\mathcal{N}=8$ multiplets and their invariant actions can be obtained from the root one and its general actions through appropriate covariant substitution of the auxiliary fields (or Hamiltonian reductions, in the Hamiltonian formalism) [17] as a generalization of the phenomenon found in [5] at the linearized level.³ Deforming the flat $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ multiplet has remained an open problem.

In the present paper we show that the latter becomes possible within the alternative $SU(4|1)$ deformation. Interestingly, there exist two such root $SU(4|1)$ multiplets, which are complementary to each other in the sense that the $SU(4)$ assignments of their fermionic and bosonic components are interchanged. Namely, in one multiplet, the bosonic $d=1$ fields are in $\underline{\mathbf{1}} \oplus \underline{\mathbf{1}}^* \oplus \underline{\mathbf{6}}$ of $SU(4)$ (eight real fields) and the fermionic fields in $\underline{\mathbf{4}} \oplus \underline{\mathbf{4}}^*$ (4 complex fields), while in the other multiplet the bosonic fields are in $\underline{\mathbf{4}} \oplus \underline{\mathbf{4}}^*$ and the fermionic fields in $\underline{\mathbf{1}} \oplus \underline{\mathbf{1}}^* \oplus \underline{\mathbf{6}}$. In the “flat” $\mathcal{N}=8, d=1$ limit they go over to two different 8-dimensional multiplets of the $SO(8)$ R -symmetry related by triality (see, e.g., [22, 23]). These two multiplets are analogs of the mutually “mirror” $\mathcal{N}=4$ multiplets $(\mathbf{4}, \mathbf{4}, \mathbf{0})$, for which bosonic and fermionic components form doublets with respect to different $SU(2)$ factors of the $SO(4)$ R -symmetry. For this reason it is natural to treat the two root $SU(4|1)$ $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ multiplets as “mirror” to each other.

The main incentive of our paper is constructing invariant actions for both types of the $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ multiplets. To this end, we will use a manifestly $SU(4|1)$ covariant superspace formalism along with the $SU(2|1)$ superfield approach, in which the extra $SU(4|1)/SU(2|1)$ transformations are realized in a hidden way. In some cases, it is simplest to use the component approach. The point is that $SU(4|1)$ possesses many non-equivalent worldline supercosets, including the harmonic ones [24], and it is not easy to decide which superfield formalism is most adequate for one or another $SU(4|1)$ multiplet. We utilize several versions of such an extended superfield approach for constructing invariant actions.

The paper is organized as follows. In section 2 we present the superalgebra $su(2|1)$ and describe the relevant worldline supercosets. In section 3, on the example of flat $\mathcal{N}=8, d=1$ supersymmetry, we discuss three possible $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ multiplets, which are not equivalent if the $SO(8)$ R -symmetry is broken, and argue that only two of them can be extended to the deformed $SU(4|1)$ case. The various superfield and component descriptions of the first version of the $SU(4|1)$ $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ multiplet are the subject of section 4. We find three different classes of invariant actions for this multiplet, including an $OSp(8|2)$ invariant one, with an R -symmetry enhanced to $SO(8)$. The analogous treatment of the second version of the multiplet $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ is given in section 5. We show that its general invariant action is superconformal and equivalent to the superconformal action of the first version. Summary and outlook are given in section 6. An appendix A contains details of calculating the invariant actions in the appropriate harmonic $SU(4|1)$ superspaces, and in appendices B and C the off-shell actions for the $SU(4|1)$ multiplets $(\mathbf{6}, \mathbf{8}, \mathbf{2})$ and $(\mathbf{7}, \mathbf{8}, \mathbf{1})$ are presented. The full set of (anti)commutation relations of the conformal superalgebra $osp(8|2)$ is given in appendix D.

³As an aside, the multiplet $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ has a puzzling relationship with the octonion algebra [21].

2 Supergroup SU(4|1) and its worldline realizations

We consider SU(4|1) supersymmetry as a deformation of the standard $\mathcal{N}=8$, $d=1$ supersymmetry [14–17]. The superalgebra $su(4|1)$ is given by the following non-vanishing (anti)commutators:

$$\begin{aligned} \{Q^I, \bar{Q}_J\} &= 2m L_J^I + 2\delta_J^I \mathcal{H}, & [L_J^I, L_L^K] &= \delta_J^K L_L^I - \delta_L^I L_J^K, \\ [L_J^I, Q^K] &= \delta_J^K Q^I - \frac{1}{4} \delta_J^I Q^K, & [L_J^I, \bar{Q}_L] &= \frac{1}{4} \delta_J^I \bar{Q}_L - \delta_L^I \bar{Q}_J, \\ [\mathcal{H}, Q^K] &= -\frac{3m}{4} Q^K, & [\mathcal{H}, \bar{Q}_L] &= \frac{3m}{4} \bar{Q}_L. \end{aligned} \quad (2.1)$$

Here, L_J^I are the generators of the R -symmetry group SU(4), and the capital indices I, J, K, L ($I=1, 2, 3, 4$) refer to the SU(4) fundamental (“quark”) representation and its conjugate. $\mathcal{H}$ is the U(1) generator. In the contraction limit $m=0$ the above superalgebra goes over to the SU(4) covariant form of the flat $\mathcal{N}=8$, $d=1$ superalgebra. This limiting superalgebra actually possesses an enhanced R -symmetry group SO(8) which mixes Q^I with $\bar{Q}_J$ (they are joined into SO(8) spinor). In what follows we will not need the explicit form of these enhanced SO(8)/SU(4) transformations, except for their realizations on the covariant “flat” spinor derivatives.

The basic real SU(4|1), $d=1$ superspace is defined as the coset superspace

$$\frac{\text{SU}(4|1)}{\text{SU}(4)} \sim \frac{\{Q^I, \bar{Q}_J, L_J^I, \mathcal{H}\}}{\{L_J^I\}}, \quad (2.2)$$

with the coset parameters being the superspace coordinates:

$$\zeta = \{t, \theta_I, \bar{\theta}^J\}, \quad \overline{(\theta_I)} = \bar{\theta}^I. \quad (2.3)$$

One could define these coordinates within the standard exponential parametrization of the supercoset. However, it will be more convenient to use another parametrization, the one associated with the purely fermionic coset SU($n|1$)/U(n) defined in [30] (see also [31]). We uplift the U(1) group from the stability subgroup U(4) into the numerator and consider an extension of the SU(4|1)/U(4) coordinate set by a time coordinate t . Thus this U(1) generator is associated with the Hamiltonian. Following to [30], one can then write generators of (2.1) acting on the extended coset (2.2) as

$$\begin{aligned} Q^I &= \frac{\partial}{\partial \theta_I} - 2m \bar{\theta}^I \bar{\theta}^K \frac{\partial}{\partial \bar{\theta}^K} + i \bar{\theta}^I \partial_t, & \bar{Q}_J &= \frac{\partial}{\partial \bar{\theta}^J} + 2m \theta_J \theta_K \frac{\partial}{\partial \theta_K} + i \theta_J \partial_t, \\ L_J^I &= \left(\bar{\theta}^I \frac{\partial}{\partial \bar{\theta}^J} - \theta_J \frac{\partial}{\partial \theta_I} \right) - \frac{\delta_J^I}{4} \left(\bar{\theta}^K \frac{\partial}{\partial \bar{\theta}^K} - \theta_K \frac{\partial}{\partial \theta_K} \right), \\ \mathcal{H} &= i \partial_t - \frac{3m}{4} \left(\bar{\theta}^K \frac{\partial}{\partial \bar{\theta}^K} - \theta_K \frac{\partial}{\partial \theta_K} \right). \end{aligned} \quad (2.4)$$

Then, odd transformations corresponding to these supercharges are given by

$$\delta \theta_I = \epsilon_I + 2m \bar{\epsilon}^K \theta_K \theta_I, \quad \delta \bar{\theta}^J = \bar{\epsilon}^J - 2m \epsilon_K \bar{\theta}^K \bar{\theta}^J, \quad \delta t = i (\bar{\epsilon}^K \theta_K + \epsilon_K \bar{\theta}^K). \quad (2.5)$$

According to [30], one can define the integration measure as

$$d\zeta := dt d^4\theta d^4\bar{\theta} (1 + 2m \bar{\theta}^K \theta_K)^3. \quad (2.6)$$

It is easily checked to be invariant under the transformations (2.5).

Note that the Hamiltonian in (2.4) is not a pure time derivative. One could pass to the new parametrization of superspace as

$$\tilde{\theta}_I = \theta_I e^{3imt/4}, \quad \bar{\tilde{\theta}}^I = \bar{\theta}^I e^{-3imt/4}, \quad t = t, \quad (2.7)$$

in which the Hamiltonian takes the standard form $\mathcal{H} = i\partial_t$. The advantage of the parametrization (2.3) is the simplest form of the transformations (2.5). So, in what follows it will be convenient to deal with such a simple parametrization. Due to the non-standard form of the Hamiltonian in this parametrization, all transformations and θ -expansions of the SU(4|1) superfields will be accompanied by the factors like $e^{\pm 3imt/4}$.

2.1 Chiral superspaces

The supergroup SU(4|1) admits two mutually conjugated complex supercosets which can be identified with the left and right chiral subspaces:

$$\zeta_L = (t_L, \theta_I), \quad \zeta_R = (t_R, \bar{\theta}^J). \quad (2.8)$$

The left coordinate t_L is related to the real time coordinate t via

$$t_L = t + \frac{i}{2m} \log(1 + 2m \bar{\theta}^K \theta_K). \quad (2.9)$$

Then we check that the left chiral space ζ_L is closed under the supersymmetry transformations

$$\delta\theta_I = \epsilon_I + 2m \bar{\epsilon}^K \theta_K \theta_I, \quad \delta t_L = 2i \bar{\epsilon}^K \theta_K. \quad (2.10)$$

The invariant left chiral measure is defined as

$$d\zeta_L := dt_L d^4\theta e^{-3imt_L}, \quad \delta(d\zeta_L) = 0, \quad \int d\zeta_L \theta_I \theta_J \theta_K \theta_L e^{3imt_L} = \varepsilon_{IJKL}. \quad (2.11)$$

2.2 Reduction to SU(2|1), $d=1$ superspace

One can consider reduction of the superspace (2.2) to the SU(2|1) superspace. It is performed on the superspace coordinates (2.3) as

$$\{t, \theta_i, \bar{\theta}^i\}, \quad \overline{(\theta_i)} = \bar{\theta}^i, \quad i = 1, 2. \quad (2.12)$$

Limiting to the ϵ_1 and ϵ_2 transformations in (2.5), we obtain the reduced SU(2|1) supersymmetric transformations which coincide with those found in [6]:

$$\delta\theta_i = \epsilon_i + 2m \bar{\epsilon}^k \theta_k \theta_i, \quad \delta\bar{\theta}^j = \bar{\epsilon}^j - 2m \epsilon_k \bar{\theta}^k \bar{\theta}^j, \quad \delta t = i \left(\bar{\epsilon}^k \theta_k + \epsilon_k \bar{\theta}^k \right). \quad (2.13)$$

Respectively, the superalgebra (2.1) contains as a subalgebra the extended $su(2|1) \oplus u(1)$ superalgebra:

$$\begin{aligned}
 \{Q^i, \bar{Q}_j\} &= 2m I_j^i + m \delta_j^i F + 2\delta_j^i \mathcal{H}, & [I_j^i, I_l^k] &= \delta_j^k I_l^i - \delta_l^i I_j^k, \\
 [I_j^i, Q^k] &= \delta_k^i Q^j - \frac{1}{2} \delta_j^i Q^k, & [I_j^i, \bar{Q}_l] &= \frac{1}{2} \delta_j^i \bar{Q}_l - \delta_l^i \bar{Q}_j, \\
 [\mathcal{H}, Q^k] &= -\frac{3m}{4} Q^k, & [\mathcal{H}, \bar{Q}_l] &= \frac{3m}{4} \bar{Q}_l, \\
 [F, Q^k] &= \frac{1}{2} Q^k, & [F, \bar{Q}_l] &= -\frac{1}{2} \bar{Q}_l.
 \end{aligned} \tag{2.14}$$

Here, $SU(2)$ generators of $SU(2|1)$ are defined as

$$I_j^i = L_j^i - \frac{1}{2} \delta_j^i F. \tag{2.15}$$

The combination $\mathcal{H} + \frac{m}{2} F$ can be identified with the internal $U(1)$ generator of $SU(2|1)$, while F becomes an external R -symmetry $U(1)$ generator.

The explicit expressions for the covariant spinor derivatives $\mathcal{D}^k, \bar{\mathcal{D}}^k$ corresponding to the basic real coset of $SU(2|1)$ defined in [8] and parametrized by the coordinates (2.12) with the transformation properties (2.13) are given by

$$\begin{aligned}
 \mathcal{D}^i &= e^{-3imt/4} \left\{ \left[1 + m \bar{\theta}^k \theta_k - \frac{3m^2}{8} (\theta)^2 (\bar{\theta})^2 \right] \frac{\partial}{\partial \theta^i} - m \bar{\theta}^i \theta_j \frac{\partial}{\partial \theta_j} - i \bar{\theta}^i \partial_t \right. \\
 &\quad \left. - \frac{m}{2} \bar{\theta}^i \tilde{F} - m \bar{\theta}^j \left(1 - m \bar{\theta}^k \theta_k \right) \tilde{I}_j^i \right\}, \\
 \bar{\mathcal{D}}_j &= e^{3imt/4} \left\{ - \left[1 + m \bar{\theta}^k \theta_k - \frac{3m^2}{8} (\theta)^2 (\bar{\theta})^2 \right] \frac{\partial}{\partial \bar{\theta}^j} + m \bar{\theta}^k \theta_j \frac{\partial}{\partial \bar{\theta}^k} + i \theta_j \partial_t \right. \\
 &\quad \left. + \frac{m}{2} \theta_j \tilde{F} + m \theta_k \left(1 - m \bar{\theta}^l \theta_l \right) \tilde{I}_j^k \right\},
 \end{aligned} \tag{2.16}$$

where

$$\begin{aligned}
 \tilde{I}_j^i \bar{\mathcal{D}}_l &= \delta_l^i \bar{\mathcal{D}}_j - \frac{1}{2} \delta_j^i \bar{\mathcal{D}}_l, & \tilde{I}_j^i \mathcal{D}^k &= \frac{1}{2} \delta_j^i \mathcal{D}^k - \delta_j^k \mathcal{D}^i, \\
 \tilde{F} \bar{\mathcal{D}}_l &= \frac{1}{2} \bar{\mathcal{D}}_l, & \tilde{F} \mathcal{D}^k &= -\frac{1}{2} \mathcal{D}^k.
 \end{aligned} \tag{2.17}$$

In what follows we will avoid using the explicit form of the $SU(4|1)$ counterparts of these derivatives, though they can be straightforwardly constructed by applying the standard coset (super)space machinery.

3 $SU(4)$ covariant formulations of $(8, 8, 0)$ multiplet in flat $\mathcal{N} = 8$ supersymmetry

Prior to the discussion of the superfield description of the root $(8, 8, 0)$ multiplets in $SU(4|1)$ supersymmetry, we will consider $SU(4)$ covariant form of its defining constraints in the standard flat $\mathcal{N} = 8$ superspace, bearing in mind that the deformation to $SU(4|1)$ mechanics must respect R -symmetry $SU(4)$.

Such constraints can be written in the two superfield forms, both preserving not only SU(4) but also a non-manifest SO(8) R -symmetry.⁴

In the first formulation one deals with a chiral superfield Φ and an antisymmetric tensor superfield Y^{IJ} satisfying the constraints⁵

$$\begin{aligned} \bar{D}_J \Phi &= 0, & D^I \bar{\Phi} &= 0, & \bar{D}_I \bar{D}_J \bar{\Phi} &= \frac{1}{2} \varepsilon_{IJKL} D^K D^L \Phi, \\ \sqrt{2} D^I Y^{JK} &= -\varepsilon^{IJKL} \bar{D}_L \bar{\Phi}, & \sqrt{2} \bar{D}_J Y_{KL} &= \varepsilon_{IJKL} D^I \Phi, \\ \overline{(Y^{IJ})} &= Y_{IJ} = \frac{1}{2} \varepsilon_{IJKL} Y^{KL}, & \overline{(\Phi)} &= \bar{\Phi}, \end{aligned} \quad (3.1)$$

where the flat covariant derivatives are defined as

$$D^I = \frac{\partial}{\partial \theta_I} - i \bar{\theta}^I \partial_t, \quad \bar{D}_J = -\frac{\partial}{\partial \bar{\theta}^J} + i \theta_J \partial_t. \quad (3.2)$$

It is straightforward to check that (3.1) is covariant under the non-manifest SO(8)/SU(4) symmetry transformations realized as

$$\delta D^I = -\sqrt{2} \Lambda^{IJ} \bar{D}_J + i \lambda D^I, \quad \delta \bar{D}_J = \sqrt{2} \bar{\Lambda}_{IJ} D^I - i \lambda \bar{D}_J, \quad (3.3)$$

$$\begin{aligned} \delta \Phi &= -\bar{\Lambda}^{IJ} Y_{IJ} - 2i \lambda \Phi, & \delta \bar{\Phi} &= -\Lambda^{IJ} Y_{IJ} + 2i \lambda \bar{\Phi}, \\ \delta Y_{IJ} &= \Lambda_{IJ} \Phi + \bar{\Lambda}_{IJ} \bar{\Phi}, \end{aligned} \quad (3.4)$$

where the antisymmetric complex 4×4 matrix

$$\Lambda_{IJ} = \frac{1}{2} \varepsilon_{IJKL} \Lambda^{KL}, \quad \bar{\Lambda}_{IJ} = \frac{1}{2} \varepsilon_{IJKL} \bar{\Lambda}^{KL}, \quad (3.5)$$

accommodates just 12 real parameters of the coset SO(8)/U(4) and λ is the real U(1) $\sim$ SO(2) parameter. One can check that indeed

$$\Phi \bar{\Phi} + \frac{1}{2} Y^{IJ} Y_{IJ} = \text{inv}. \quad (3.6)$$

Another form of the SU(4) covariant superfield description of the multiplet $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ involves the general superfield V^I which is subject to the constraints

$$\begin{aligned} D^I V^J &= \frac{1}{2} \varepsilon^{IJKL} \bar{D}_K \bar{V}_L, & D^{(I} V^{J)} &= 0, & \bar{D}_{(K} \bar{V}_{L)} &= 0, \\ D^I \bar{V}_J &= \frac{1}{4} \delta_J^I D^K \bar{V}_K, & \bar{D}_J V^I &= \frac{1}{4} \delta_J^I \bar{D}_K V^K & \overline{(V^I)} &= \bar{V}_I. \end{aligned} \quad (3.7)$$

The non-manifest SO(8)/SU(4) transformations of V^I leaving covariant the system (3.7) are written this time as

$$\delta V^I = \sqrt{2} \bar{\Lambda}^{IJ} \bar{V}_J - i \lambda V^I, \quad \delta \bar{V}_J = -\sqrt{2} \Lambda_{IJ} V^I + i \lambda \bar{V}_J. \quad (3.8)$$

⁴In general, flat constraints defining the multiplet $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ can be given many equivalent forms. For instance, in [15], they were written in SU(2) $\times$ SU(2) $\times$ SU(2) $\times$ SU(2) covariant form. The common feature of all these formulations is the hidden covariance of the constraints under the full R -symmetry group of $\mathcal{N}=8$ superalgebra, the group SO(8).

⁵For further use, we introduce the antisymmetric tensor $\varepsilon^{IJKL} \equiv \varepsilon^{[IJKL]}$, such that

$$\varepsilon^{1234} = \varepsilon_{1234} = 1, \quad \varepsilon^{IJKL} \varepsilon_{IJKL} = 24.$$

These transformations, together with the transformations of the covariant derivatives (3.3), preserve the constraints (3.7). One can also see that

$$V^I \bar{V}_I = \text{inv.} \quad (3.9)$$

It is rather easy to check that the constraints (3.1) leave in the bosonic sector of Φ, Y^{IJ} just the complex bosonic field $\phi(t)$ and tensorial field $y^{IJ}(t)$ which are first components of these superfields and transform as $\underline{1}$ and $\underline{6}$ of $\text{SU}(4)$. The physical fermions are defined as $D^I \Phi|_{\theta=0}$ and transform as $\underline{4}$ of $\text{SU}(4)$. In the case of the constraints (3.9) the $\text{SU}(4)$ assignment of the physical fields changes to the opposite: the physical bosons are the first components of V^J and transform as $\underline{4}$, while fermions are defined as $\bar{D}_K V^K|_{\theta=0}, D^K \bar{V}_K|_{\theta=0}, D^{[I} V^{J]}|_{\theta=0}$ and transform as $\underline{1} \oplus \underline{1}^* \oplus \underline{6}$. Thus, two $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ multiplets have “inverted” $\text{SU}(4)$ contents: the contents of bosons and fermions of the first version coincide with those of fermions and bosons in the second one.

In order to better understand the interplay between the two forms of the $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ multiplet, we note that the fermionic superfield $D^I \Phi$ transforms precisely as V^I . It is easy to check that it satisfies the constraints (3.7) as a consequence of (3.1). Analogously, the fermionic superfields $-2\sqrt{2} D^I V^J$ and $D^K \bar{V}_K$ possess the same transformation properties as Y^{IJ} and $\bar{\Phi}$, respectively. It is also straightforward to check that such fermionic superfields satisfy (3.1) as a consequence of (3.7). In other words, by the first multiplet one can construct the “derivative” fermionic multiplet satisfying the Grassmann-odd version of the second multiplet constraints (3.7). After establishing this correspondence, we could consider (3.7) for some new independent Grassmann-even superfield V^I and so come to the system (3.7) as an alternative description of the $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ multiplet with the same Grassmann parities for the component fields as in the first version, but with “inverted” $\text{SU}(4)$ assignments of these components. Its fermionic “derivative” satisfies the constraints (3.1).

This interplay between two $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ multiplets resembles a similar feature of “mirroring” of $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ multiplets in the standard (flat) $\mathcal{N}=4$ mechanics [25, 26]. The bosonic and fermionic components of the mutually mirror $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ multiplets form doublets with respect to different $\text{SU}(2)$ factors of the full $\text{SO}(4)$ R -symmetry group and are equivalent up to switching the roles of these two commuting $\text{SU}(2)$ groups. However, there is an essential difference. In the $\mathcal{N}=4$ case the bosonic fields of the mutually mirror $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ multiplets are doublets of *different* R -symmetry $\text{SU}(2)$ groups (the same is true for fermionic fields). As is seen from (3.6) and (3.9), in the $\mathcal{N}=8$ case the relevant fields form 8-dimensional irreps of the *same* full R -symmetry $\text{SO}(8)$ and differ only in their assignments with respect to the fixed $\text{U}(4) \subset \text{SO}(8)$. So these two descriptions are associated with different embeddings of $\text{U}(4)$ into $\text{SO}(8)$. The first version corresponds to splitting $\text{SO}(8) \rightarrow \text{SO}(2) \times \text{SO}(6)$ and representing the $\text{SO}(8)$ -multiplet of superfields as a sum of $\text{SO}(2)$ and $\text{SO}(6)$ vectors. Then $\text{SU}(4)$ is identified with $\text{SO}(6)$, the additional R -symmetry $\text{U}(1)$ with $\text{SO}(2)$, while Φ and Y^{IK} with the corresponding $\text{SO}(2)$ and $\text{SO}(6)$ vectors. The second version corresponds to splitting $\text{SO}(8) \rightarrow \text{SO}(4) \times \text{SO}(4)'$ and representing the relevant $\text{SO}(8)$ superfield set as a sum of two 4-vectors. The diagonal $\text{SO}(4)$ is identified with the “minimally embedded” $\text{SO}(4) \subset \text{SU}(4)$, and two 4-vectors are joined into a complex fundamental spinor V^I of $\text{SU}(4)$.

Actually, the hidden $\text{SO}(8)$ symmetry reveals the triality [22] between bosonic fields, fermionic fields and covariant derivatives. This triality interrelates the three irreducible fundamental representations of $\text{SO}(8)$, *viz.* the vector representation and two spinorial ones.⁶ All three representations can be written in the $\text{SU}(4) \times \text{U}(1) \sim \text{SO}(6) \times \text{SO}(2)$ notation [23] as

$$\begin{aligned} \text{vector} & \quad \underline{\mathbf{1}}_1 \oplus \underline{\mathbf{1}}_{-1}^* \oplus \underline{\mathbf{6}}_0, \\ \text{spinor} & \quad \underline{\mathbf{4}}_{1/2} \oplus \underline{\mathbf{4}}_{-1/2}^*, \\ \text{spinor} & \quad \underline{\mathbf{4}}_{-1/2} \oplus \underline{\mathbf{4}}_{1/2}^*, \end{aligned} \tag{3.10}$$

where the subscript index refers to the $\text{U}(1)$ charge. Comparing this with the $\text{U}(4)$ assignments of the bosonic and fermionic fields of the $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ multiplets, as well as of the covariant derivatives, we observe that just these $\text{SO}(8)$ representations are realized on the quantities in question.

Supposing that the roles of two spinor representations can be switched, in flat $\mathcal{N}=8$, $d=1$ supersymmetry we can introduce yet a *third* multiplet $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ living on a different superspace, with the covariant derivatives defined as

$$\tilde{D}_{IJ} = \frac{1}{2} \varepsilon_{IJKL} \tilde{D}^{KL}, \quad \overline{(\tilde{D}_{IJ})} = \tilde{D}^{IJ}, \quad \tilde{D}, \quad \tilde{\bar{D}} = \overline{(\tilde{D})}, \tag{3.11}$$

and so belonging to the vector representation. However, an $\text{SU}(4)$ covariant formulation of this third $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ multiplet is beyond our purpose because the $\text{SU}(4|1)$ covariant derivatives are $\text{SU}(4)$ spinors by definition. So, this third option does not admit a generalization to $\text{SU}(4|1)$ supersymmetry, in contrast to the first two.

In the case of the constraints (3.1), the bosonic fields belong to the $\text{SO}(8)$ vector representation, and the fermionic fields form $\text{SO}(8)$ spinor. For the multiplet given by (3.7) the picture is reversed, that is, the bosonic fields form an $\text{SO}(8)$ spinor and the fermionic fields are combined into $\text{SO}(8)$ vector. So, from the standpoint of $\text{SO}(8)$ R -symmetry, due to the triality property, both $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ multiplets can be considered as equivalent, once the spinorial representation of the covariant spinor derivatives has been fixed and one deals with $\text{SO}(8)$ invariant actions for these multiplets (for more detail, see section 5.4).

The crucial point of the equivalence just discussed is the hidden $\text{SO}(8)$ covariance of both sets of constraints (3.1) and (3.7). In the case of $\text{SU}(4|1)$ -deformed mechanics there is no $\text{SO}(8)$ R -symmetry, only $\text{U}(4)$ remains. For this reason one cannot expect the corresponding counterparts of the two “flat” $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ multiplets to be equivalent to one another.⁷

⁶To be more exact, the triality property is inherent to the group $\text{Spin}(8)$.

⁷In the flat case the $\mathcal{N}=8$ supersymmetric Lagrangians are not obliged to simultaneously respect the full $\text{SO}(8)$ symmetry. So for $\text{SO}(8)$ non-invariant Lagrangians the equivalency of different $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ multiplets may be broken in the flat case too.

4 The SU(4|1) multiplet (8, 8, 0): first version

The first version of the multiplet (8, 8, 0) is defined by the SU(4|1) covariant constraints

$$\begin{aligned}
 \bar{\mathcal{D}}_J \Phi &= 0, & \mathcal{D}^I \bar{\Phi} &= 0, & \bar{\mathcal{D}}_I \bar{\mathcal{D}}_J \bar{\Phi} &= \frac{1}{2} \varepsilon_{IJKL} \mathcal{D}^K \mathcal{D}^L \Phi, \\
 \sqrt{2} \mathcal{D}^I Y^{JK} &= -\varepsilon^{IJKL} \bar{\mathcal{D}}_L \bar{\Phi}, & \sqrt{2} \bar{\mathcal{D}}_J Y_{KL} &= \varepsilon_{IJKL} \mathcal{D}^I \Phi, \\
 \overline{(Y^{IJ})} &= Y_{IJ} = \frac{1}{2} \varepsilon_{IJKL} Y^{KL}, & \overline{(\Phi)} &= \bar{\Phi},
 \end{aligned} \tag{4.1}$$

where Φ is a chiral superfield and Y^{IJ} is an antisymmetric tensor superfield. In the flat limit, when $m \rightarrow 0$, $\bar{\mathcal{D}}_J \rightarrow \bar{D}_J$, $\mathcal{D}^I \rightarrow D^I$, this set of constraints becomes the set of superfield constraints (3.1) defining the standard $\mathcal{N}=8$, $d=1$ multiplet (8, 8, 0) [15], such that only $\text{SU}(4) \subset \text{SO}(8)$ is manifest.

In what follows, we avoid calculation of the deformed covariant derivatives $\mathcal{D}^I$, $\bar{\mathcal{D}}_J$ (they in general involve complicated U(4) connection terms) and consider the multiplet (8, 8, 0) in the chiral superspace description, harmonic superspace description and SU(2|1) superfield approach.

4.1 Chiral superfield

We consider the chiral superfield Φ given by the general θ -expansion

$$\begin{aligned}
 \Phi(t_L, \theta_I) &= \phi + \sqrt{2} \theta_K \chi^K e^{3imt_L/4} + \theta_I \theta_J A^{IJ} e^{3imt_L/2} + \frac{\sqrt{2}}{3} \theta_I \theta_J \theta_K \xi^{IJK} e^{9imt_L/4} \\
 &+ \frac{1}{4} \varepsilon^{IJKL} \theta_I \theta_J \theta_K \theta_L B e^{3imt_L}, \quad A^{IJ} \equiv A^{[IJ]}, \quad \xi^{IJK} \equiv \xi^{[IJK]}.
 \end{aligned} \tag{4.2}$$

The superfield Φ transforms as a singlet of the stability subgroup SU(4), i.e. $\delta_{su(4)} \Phi = 0$. Taking into account (2.10), we can find the transformations of its components under the odd generators:

$$\begin{aligned}
 \delta\phi &= -\sqrt{2} \epsilon_K \chi^K e^{3imt/4}, \\
 \delta\chi^I &= \sqrt{2} \bar{\epsilon}^I \left(i\dot{\phi} \right) e^{-3imt/4} - \sqrt{2} \epsilon_K A^{IK} e^{3imt/4}, \\
 \delta A^{IJ} &= 2\sqrt{2} \bar{\epsilon}^{[I} \left(i\dot{\chi}^{J]} + \frac{m}{4} \chi^{J]} \right) e^{-3imt/4} - \sqrt{2} \epsilon_K \xi^{IJK} e^{3imt/4}, \\
 \frac{\sqrt{2}}{3} \delta\xi^{IJK} &= 2\bar{\epsilon}^{[K} \left(i\dot{A}^{IJ]} + \frac{m}{2} A^{IJ]} \right) e^{-3imt/4} - \varepsilon^{IJKL} \epsilon_L B e^{3imt/4}, \\
 \varepsilon^{IJKL} \delta B &= \frac{8\sqrt{2}}{3} \bar{\epsilon}^{[L} \left(i\dot{\xi}^{IJK]} + \frac{3m}{4} \xi^{IJK]} \right) e^{-3imt/4}.
 \end{aligned} \tag{4.3}$$

The general supersymmetric action can be written as a sum of integrals over chiral subspaces [13, 18] as

$$S_{\text{chiral}} = \int dt \mathcal{L}_{\text{chiral}} = -\frac{1}{4} \left[\int d\zeta_L K(\Phi) + \int d\zeta_R \bar{K}(\bar{\Phi}) \right], \tag{4.4}$$

where the overall coefficient $-1/4$ is chosen for further convenience. The component form of this $SU(4|1)$ invariant is found to be

$$S_{\text{chiral}} = -\frac{1}{4} \int dt \left\{ 6B \partial_\phi K + \varepsilon_{IJKL} \left[\frac{2}{3} \chi^L \xi^{IJK} + \frac{1}{2} A^{IJ} A^{KL} \right] (\partial_\phi)^2 K \right. \\ \left. - \varepsilon_{IJKL} A^{IJ} \chi^K \chi^L (\partial_\phi)^3 K + \frac{1}{6} \varepsilon_{IJKL} \chi^I \chi^J \chi^K \chi^L (\partial_\phi)^4 K + \text{c.c.} \right\}. \quad (4.5)$$

This invariant does not display the kinetic term of the fields in (4.2) and so must be treated as a kind of “pre-action” for the $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ multiplet. The genuine action appears after imposing some extra $SU(4|1)$ covariant conditions on the components in (4.2). Of course they should follow from the rest of the superfield constraints (4.1), but it is easier to guess their form directly at the component level, requiring the final field content to be $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ and resorting to the $SU(4|1)$ covariance reasonings.

In this way we find that the components of the chiral superfield (4.2) must be subjected to the following additional constraints

$$A^{IJ} = \sqrt{2} \left(i\dot{y}^{IJ} - \frac{m}{2} y^{IJ} \right), \quad \overline{(y^{IJ})} = y_{IJ} = \frac{1}{2} \varepsilon_{IJKL} y^{KL}, \\ \xi^{IJK} = -\varepsilon^{IJKL} \left(i\dot{\chi}_L - \frac{5m}{4} \bar{\chi}_L \right), \quad \overline{(\chi^I)} = \bar{\chi}_I, \\ B = \frac{2}{3} \left(\ddot{\phi} + 2im\dot{\phi} \right). \quad (4.6)$$

The odd $SU(2|1)$ transformations are realized on this minimal set of fields as:

$$\delta\phi = -\sqrt{2} \epsilon_I \chi^I e^{3imt/4}, \quad \delta\bar{\phi} = \sqrt{2} \bar{\epsilon}^I \bar{\chi}_I e^{-3imt/4}, \\ \delta y^{IJ} = -2 \bar{\epsilon}^{[I} \chi^{J]} e^{-3imt/4} + \varepsilon^{IJKL} \epsilon_K \bar{\chi}_L e^{3imt/4}, \\ \delta\chi^I = \sqrt{2} \bar{\epsilon}^I \left(i\dot{\phi} \right) e^{-3imt/4} - 2 \epsilon_J \left(i\dot{y}^{IJ} - \frac{m}{2} y^{IJ} \right) e^{3imt/4}, \\ \delta\bar{\chi}_I = -\sqrt{2} \epsilon_I \left(i\dot{\phi} \right) e^{3imt/4} + 2 \bar{\epsilon}^J \left(i\dot{y}_{IJ} + \frac{m}{2} y_{IJ} \right) e^{-3imt/4}. \quad (4.7)$$

They are consistent with the transformations (4.3) and leave invariant the constraints (4.6).

4.2 The final action

Substituting the constraints (4.6) into the pre-action (4.5), we find the correct component Lagrangian in the form

$$\mathcal{L}_{\text{SK}} = g_1 \left[\dot{\phi}\dot{\phi} + \frac{1}{2} \dot{y}^{IJ} \dot{y}_{IJ} + \frac{i}{2} (\chi^K \dot{\chi}_K - \dot{\chi}^K \bar{\chi}_K) - \frac{5m}{4} \chi^K \bar{\chi}_K - \frac{m^2}{8} y^{IJ} y_{IJ} \right] \\ - \frac{im}{4} \left(\dot{\phi} \partial_\phi g_1 - \dot{\phi} \partial_{\bar{\phi}} g_1 \right) y^{IJ} y_{IJ} + 2im \left(\dot{\phi} \partial_{\bar{\phi}} \bar{K} - \dot{\phi} \partial_\phi K \right) \\ + \frac{1}{\sqrt{2}} \left(i\dot{y}_{IJ} - \frac{m}{2} y_{IJ} \right) \chi^I \chi^J \partial_\phi g_1 + \frac{1}{\sqrt{2}} \left(i\dot{y}^{IJ} + \frac{m}{2} y^{IJ} \right) \bar{\chi}_I \bar{\chi}_J \partial_{\bar{\phi}} g_1 \\ - \frac{i}{2} \left(\dot{\phi} \partial_\phi g_1 - \dot{\phi} \partial_{\bar{\phi}} g_1 \right) \chi^K \bar{\chi}_K - \frac{1}{24} \varepsilon^{IJKL} \bar{\chi}_I \bar{\chi}_J \bar{\chi}_K \bar{\chi}_L \partial_{\bar{\phi}} \partial_{\bar{\phi}} g_1 \\ - \frac{1}{24} \varepsilon_{IJKL} \chi^I \chi^J \chi^K \chi^L \partial_\phi \partial_\phi g_1. \quad (4.8)$$

We observe that the complex fields ϕ parametrizes a special Kähler (SK) manifold with the metric

$$g_1(\phi, \bar{\phi}) = \partial_\phi \partial_{\bar{\phi}} K(\phi) + \partial_{\bar{\phi}} \partial_{\bar{\phi}} \bar{K}(\bar{\phi}). \quad (4.9)$$

4.3 Supercharges

The matrix models based on the multiplet under consideration, in the case of the simplest target space metric $g = 1$ (i.e for the free model), were studied in [28]. Here, we consider a one-particle model generalized to the general SK metric (4.9) and find the relevant classical SU(4|1) supercharges. Poisson (Dirac) brackets are written as

$$\begin{aligned} \{\phi, p_\phi\} &= 1, & \{\bar{\phi}, p_{\bar{\phi}}\} &= 1, & \{y^{KL}, p_{IJ}\} &= \frac{1}{2} (\delta_I^K \delta_J^L - \delta_I^L \delta_J^K), \\ \{\chi^I, \bar{\chi}_J\} &= -i \delta_J^I (g_1)^{-1}, \\ \{p_\phi, \chi^I\} &= \frac{1}{2} (g_1)^{-1} \partial_\phi g_1 \chi^I, & \{p_{\bar{\phi}}, \chi^I\} &= \frac{1}{2} (g_1)^{-1} \partial_{\bar{\phi}} g_1 \chi^I, \\ \{p_\phi, \bar{\chi}_J\} &= \frac{1}{2} (g_1)^{-1} \partial_\phi g_1 \bar{\chi}_J, & \{p_{\bar{\phi}}, \bar{\chi}_J\} &= \frac{1}{2} (g_1)^{-1} \partial_{\bar{\phi}} g_1 \bar{\chi}_J. \end{aligned} \quad (4.10)$$

Then the Noether supercharges are given by

$$\begin{aligned} Q^I &= e^{3imt/4} \left\{ 2\bar{\chi}_K \left[p^{IK} + \frac{i}{2} m g_1 y^{IK} - \frac{i}{6\sqrt{2}} \varepsilon^{IKLM} \bar{\chi}_L \bar{\chi}_M \partial_{\bar{\phi}} g_1 \right] \right. \\ &\quad \left. - \sqrt{2} \chi^I \left(p_{\bar{\phi}} - 2im \partial_{\bar{\phi}} \bar{K} + \frac{i}{4} m \partial_\phi g_1 y^{KL} y_{KL} + \frac{i}{2} \partial_\phi g_1 \chi^K \bar{\chi}_K \right) \right\}, \\ \bar{Q}_J &= e^{-3imt/4} \left\{ 2\chi^K \left[p_{JK} - \frac{i}{2} m g_1 y_{JK} - \frac{i}{6\sqrt{2}} \varepsilon_{JKLM} \chi^L \chi^M \partial_\phi g_1 \right] \right. \\ &\quad \left. - \sqrt{2} \bar{\chi}_J \left(p_{\bar{\phi}} + 2im \partial_\phi K - \frac{i}{4} m \partial_{\bar{\phi}} g_1 y^{KL} y_{KL} - \frac{i}{2} \partial_{\bar{\phi}} g_1 \chi^K \bar{\chi}_K \right) \right\}. \end{aligned} \quad (4.11)$$

Taking into account the brackets (4.10), these supercharges close on the following bosonic generators

$$\begin{aligned} \mathcal{H}_{\text{SK}} &= (g_1)^{-1} \left(p_\phi - 2im \partial_{\bar{\phi}} \bar{K} + \frac{i}{4} m \partial_\phi g_1 y^{IJ} y_{IJ} + \frac{i}{2} \partial_\phi g_1 \chi^K \bar{\chi}_K \right) \\ &\quad \times \left(p_{\bar{\phi}} + 2im \partial_\phi K - \frac{i}{4} m \partial_{\bar{\phi}} g_1 y^{IJ} y_{IJ} - \frac{i}{2} \partial_{\bar{\phi}} g_1 \chi^K \bar{\chi}_K \right) \\ &\quad + \frac{1}{2g_1} \left[p^{IJ} - \frac{i}{\sqrt{2}} \left(\chi^I \chi^J \partial_\phi g_1 + \frac{1}{2} \varepsilon^{IJKL} \bar{\chi}_K \bar{\chi}_L \partial_{\bar{\phi}} g_1 \right) \right] \\ &\quad \times \left[p_{IJ} - \frac{i}{\sqrt{2}} \left(\bar{\chi}_I \bar{\chi}_J \partial_{\bar{\phi}} g_1 + \frac{1}{2} \varepsilon_{IJMN} \chi^M \chi^N \partial_\phi g_1 \right) \right] \\ &\quad + g_1 \left[\frac{5m}{4} \chi^K \bar{\chi}_K + \frac{m^2}{8} y^{IJ} y_{IJ} \right] + \frac{m}{2\sqrt{2}} (y_{IJ} \chi^I \chi^J \partial_\phi g_1 - y^{IJ} \bar{\chi}_I \bar{\chi}_J \partial_{\bar{\phi}} g_1) \\ &\quad + \frac{1}{24} \varepsilon^{IJKL} \bar{\chi}_I \bar{\chi}_J \bar{\chi}_K \bar{\chi}_L \partial_{\bar{\phi}} \partial_{\bar{\phi}} g_1 + \frac{1}{24} \varepsilon_{IJKL} \chi^I \chi^J \chi^K \chi^L \partial_\phi \partial_\phi g_1, \end{aligned} \quad (4.12)$$

$$L_J^I = 2i \left(y^{IK} p_{JK} - \frac{\delta_J^I}{4} y^{KL} p_{KL} \right) + g_1 \left(\chi^I \bar{\chi}_J - \frac{\delta_J^I}{4} \chi^K \bar{\chi}_K \right), \quad (4.13)$$

in full agreement with the superalgebra (2.1). The quantum version of these SU(4|1) (super)charges can be straightforwardly constructed and will be presented elsewhere.

4.4 Harmonic superspace description

We consider the harmonic coset of SU(4|1) with the harmonic part $\frac{\text{SU}(4)}{[\text{SU}(2) \times \text{SU}(2) \times \text{U}(1)]}$ [32]. The relevant harmonic variables are $u_a^{(+I)}$, $u_I^{(+i)}$, $u_I^{(-a)}$, $u_i^{(-I)}$ where $i = 1, 2$ and $a = 1, 2$ are the indices of the fundamental representations of the subgroup SU(2) × SU(2). The unitarity and unimodularity conditions are written as

$$\begin{aligned} u_K^{(+i)} u_j^{(-)K} &= \delta_j^i, & u_K^{(-a)} u_b^{(+K)} &= \delta_b^a, & u_J^{(-a)} u_a^{(+I)} + u_J^{(+i)} u_i^{(-)I} &= \delta_J^I, \\ u_K^{(-a)} u_j^{(-)K} &= u_K^{(+i)} u_b^{(+K)} = 0, & \varepsilon^{IJKL} \varepsilon_{ij} u_K^{(+i)} u_L^{(+j)} + 2\varepsilon^{ab} u_a^{(+I)} u_b^{(+J)} &= 0. \end{aligned} \quad (4.14)$$

Defining the harmonic projections of the SU(4|1) Grassmann coordinates as

$$\begin{aligned} \theta_a^{(+)} &= \theta_I \left(u_a^{(+I)} + m \bar{\theta}^{(+k)} \theta_a^{(+)} u_k^{(-)I} \right), & \theta_i^{(-)} &= \theta_I u_i^{(-)I}, \\ \bar{\theta}^{(+i)} &= \bar{\theta}^J \left(u_J^{(+i)} + m \bar{\theta}^{(+i)} \theta_c^{(+)} u_J^{(-)c} \right), & \bar{\theta}^{(-a)} &= \bar{\theta}^J u_J^{(-)a}, \\ t_A &= t + i \left(\bar{\theta}^{(-a)} \theta_a^{(+)} - \bar{\theta}^{(+i)} \theta_i^{(-)} \right) \left[1 - m \left(\bar{\theta}^{(-a)} \theta_a^{(+)} + \bar{\theta}^{(+i)} \theta_i^{(-)} \right) \right], \end{aligned} \quad (4.15)$$

one can find that they transform as

$$\begin{aligned} \delta \theta_i^{(-)} &= \epsilon_i^{(-)} + 2m \left[\bar{\epsilon}^{(-)c} \theta_c^{(+)} \left(1 + m \bar{\theta}^{(+k)} \theta_k^{(-)} \right) + \bar{\epsilon}^{(+k)} \theta_k^{(-)} \right] \theta_i^{(-)}, \\ \delta \bar{\theta}^{(-a)} &= \bar{\epsilon}^{(-)a} - 2m \left[\epsilon_c^{(+)} \bar{\theta}^{(-)c} + \epsilon_k^{(-)} \bar{\theta}^{(+k)} \left(1 + m \bar{\theta}^{(-)c} \theta_c^{(+)} \right) \right] \bar{\theta}^{(-a)}, \\ \delta \theta_a^{(+)} &= \epsilon_a^{(+)} + m \epsilon_k^{(-)} \bar{\theta}^{(+k)} \theta_a^{(+)} + 2m \bar{\epsilon}^{(-)c} \theta_c^{(+)} \theta_a^{(+)}, \\ \delta \bar{\theta}^{(+i)} &= \bar{\epsilon}^{(+i)} - m \bar{\epsilon}^{(-)c} \theta_c^{(+)} \bar{\theta}^{(+i)} - 2m \epsilon_k^{(-)} \bar{\theta}^{(+k)} \bar{\theta}^{(+i)}, \\ \delta u_I^{(+i)} &= -\Lambda_b^{(+2)i} u_I^{(-)b}, & \delta u_i^{(-)I} &= 0, \\ \delta u_b^{(+I)} &= \Lambda_b^{(+2)i} u_i^{(-)I}, & \delta u_I^{(-)b} &= 0, \\ \delta t_A &= 2i \left(\epsilon_k^{(-)} \bar{\theta}^{(+k)} + \bar{\epsilon}^{(-)c} \theta_c^{(+)} \right), \end{aligned} \quad (4.16)$$

where

$$\begin{aligned} \Lambda_a^{(+2)i} &= m \left(\epsilon_a^{(+)} \bar{\theta}^{(+i)} + \bar{\epsilon}^{(+i)} \theta_a^{(+)} \right) + m^2 \left(\epsilon_k^{(-)} \bar{\theta}^{(+k)} + \bar{\epsilon}^{(-)c} \theta_c^{(+)} \right) \bar{\theta}^{(+i)} \theta_a^{(+)}, \\ \epsilon_i^{(-)} &= \epsilon_I u_i^{(-)I}, & \epsilon_a^{(+)} &= \epsilon_I u_a^{(+I)}, & \bar{\epsilon}^{(+i)} &= \bar{\epsilon}^J u_J^{(+i)}, & \bar{\epsilon}^{(-a)} &= \bar{\epsilon}^J u_J^{(-)a}. \end{aligned} \quad (4.17)$$

We observe the existence of the analytic subspace closed under the SU(4|1) supersymmetry

$$\zeta_A = \left\{ t_A, \theta_a^{(+)}, \bar{\theta}^{(+i)}, u_a^{(+I)}, u_I^{(+i)}, u_I^{(-a)}, u_i^{(-)I} \right\}. \quad (4.18)$$

Its integration measure is given by

$$\begin{aligned} d\zeta_A^{(-4)} &= dt_A du d^2\theta^{(+)} d^2\bar{\theta}^{(+)} \Rightarrow \\ \Rightarrow \delta \left(d\zeta_A^{(-4)} \right) &= 2 d\zeta_A^{(-4)} \Lambda^{(0)}, & \Lambda^{(0)} &= m \left(\epsilon_k^{(-)} \bar{\theta}^{(+k)} - \bar{\epsilon}^{(-)c} \theta_c^{(+)} \right). \end{aligned} \quad (4.19)$$

The only harmonic derivative $\mathcal{D}_a^{(+2)i}$ preserving the analytic subspace reads

$$\begin{aligned}\mathcal{D}_a^{(+2)i} = & u_a^{(+K)} \frac{\partial}{\partial u_i^{(-)K}} - u_K^{(+i)} \frac{\partial}{\partial u_K^{(-)a}} - 2i\bar{\theta}^{(+i)}\theta_a^{(+)}\partial_A \\ & + m\bar{\theta}^{(+i)}\theta_a^{(+)} \left(\bar{\theta}^{(+k)} \frac{\partial}{\partial \bar{\theta}^{(+k)}} - \theta_c^{(+)} \frac{\partial}{\partial \theta_c^{(+)}} \right) \\ & + \frac{m^2}{4} \varepsilon^{ij} \varepsilon_{ab} \left(\theta^{(+)} \right)^4 \left(u_j^{(-)K} \frac{\partial}{\partial u_b^{(+K)}} - u_K^{(-b)} \frac{\partial}{\partial u_K^{(+)j}} \right),\end{aligned}\quad (4.20)$$

where

$$\left(\theta^{(+)} \right)^4 = \left(\bar{\theta}^{(+)} \right)^2 \left(\theta^{(+)} \right)^2 = \bar{\theta}^{(+k)} \bar{\theta}_k^{(+)} \theta_c^{(+)} \theta^{(+c)}.\quad (4.21)$$

The remaining harmonic covariant derivatives prove undeformed:

$$\begin{aligned}\mathcal{D}^{(0)} = & u_K^{(+k)} \frac{\partial}{\partial u_K^{(+k)}} + u_c^{(+K)} \frac{\partial}{\partial u_c^{(+K)}} - u_k^{(-)K} \frac{\partial}{\partial u_k^{(-)K}} - u_K^{(-c)} \frac{\partial}{\partial u_K^{(-c)}} \\ & + \theta_c^{(+)} \frac{\partial}{\partial \theta_c^{(+)}} + \bar{\theta}^{(+k)} \frac{\partial}{\partial \bar{\theta}^{(+k)}}, \\ \mathcal{D}_j^i = & u_K^{(+i)} \frac{\partial}{\partial u_K^{(+j)}} - u_j^{(-)K} \frac{\partial}{\partial u_i^{(-)K}} + \bar{\theta}^{(+i)} \frac{\partial}{\partial \bar{\theta}^{(+j)}} \\ & - \frac{\delta_j^i}{2} \left(u_K^{(+k)} \frac{\partial}{\partial u_K^{(+k)}} - u_k^{(-)K} \frac{\partial}{\partial u_k^{(-)K}} + \bar{\theta}^{(+k)} \frac{\partial}{\partial \bar{\theta}^{(+k)}} \right), \\ \mathcal{D}_b^a = & u_K^{(-a)} \frac{\partial}{\partial u_K^{(-b)}} - u_b^{(+K)} \frac{\partial}{\partial u_a^{(+K)}} - \theta_b^{(+)} \frac{\partial}{\partial \theta_a^{(+)}} \\ & - \frac{\delta_b^a}{2} \left(u_K^{(-c)} \frac{\partial}{\partial u_K^{(-c)}} - u_c^{(+K)} \frac{\partial}{\partial u_c^{(+K)}} - \theta_c^{(+)} \frac{\partial}{\partial \theta_c^{(+)}} \right).\end{aligned}\quad (4.22)$$

One can check that

$$\begin{aligned}\Lambda_a^{(+2)i} = & \mathcal{D}_a^{(+2)i} \Lambda^{(0)}, \quad \varepsilon^{ab} \varepsilon_{ij} \mathcal{D}_b^{(+2)j} \Lambda_a^{(+2)i} = m^2 \delta \left(\theta^{(+)} \right)^4, \\ \delta \left(\theta^{(+)} \right)^4 = & 2 \left(\delta \bar{\theta}^{(+k)} \right) \bar{\theta}_k^{(+)} \theta_c^{(+)} \theta^{(+c)} + 2 \bar{\theta}^{(+k)} \bar{\theta}_k^{(+)} \theta_c^{(+)} \left(\delta \theta^{(+c)} \right),\end{aligned}\quad (4.23)$$

and

$$\delta \mathcal{D}_a^{(+2)i} = \Lambda_c^{(+2)i} \mathcal{D}_a^c - \Lambda_a^{(+2)k} \mathcal{D}_k^i - \frac{\Lambda_a^{(+2)i}}{2} \mathcal{D}^{(0)}.\quad (4.24)$$

4.4.1 Analytic harmonic superfield

The relevant analytic harmonic superfield is defined by the conditions

$$\mathcal{D}_a^{(+2)i} Y^{(+2)} = 0, \quad \mathcal{D}_j^i Y^{(+2)} = \mathcal{D}_b^a Y^{(+2)} = 0,\quad (4.25)$$

and it transforms as

$$\delta Y^{(+2)} = \Lambda^{(0)} Y^{(+2)}.\quad (4.26)$$

It can be obtained by the “harmonization” of the superfield Y^{IJ} satisfying the constraints

$$\mathcal{D}^{(K} Y^{I)J} = 0, \quad \bar{\mathcal{D}}_{(K} Y_{I)J} = 0. \quad (4.27)$$

These constraints in fact define the multiplet $(\mathbf{6}, \mathbf{8}, \mathbf{2})$. On the other hand, they are part of the full set of the constraints (4.1) defining the multiplet $(\mathbf{8}, \mathbf{8}, \mathbf{0})$. Indeed, the solution of (4.25)

$$\begin{aligned} Y^{(+2)} = & y^{(+2)} + 2i\bar{\theta}^{(+i}\theta^{(+a)}\dot{y}_{ia} - \bar{\theta}^{(+k)}\bar{\theta}_k^{(+)}\theta_a^{(+)}\theta^{(+a)}\ddot{y}^{(-2)} \\ & + \bar{\theta}_i^{(+)}u_K^{(+i)}\chi^K e^{-3imt_{\Lambda}/4} + \bar{\theta}^{(+i)}\bar{\theta}_i^{(+)}\theta_a^{(+)}u_K^{(-a)}\left(i\dot{\chi}^K + \frac{m}{4}\chi^K\right)e^{-3imt_{\Lambda}/4} \\ & + \theta^{(+a)}u_a^{(+K)}\bar{\chi}_K e^{3imt_{\Lambda}/4} + \theta_a^{(+)}\theta^{(+a)}\bar{\theta}^{(+i)}u_i^{(-K)}\left(i\dot{\bar{\chi}}_K - \frac{m}{4}\bar{\chi}_K\right)e^{3imt_{\Lambda}/4} \\ & + \frac{1}{\sqrt{2}}\left(\bar{\theta}^{(+i)}\bar{\theta}_i^{(+)}D e^{-3imt_{\Lambda}/2} + \theta_a^{(+)}\theta^{(+a)}\bar{D} e^{3imt_{\Lambda}/2}\right), \end{aligned} \quad (4.28)$$

reveals the field content $(\mathbf{6}, \mathbf{8}, \mathbf{2})$, where

$$\begin{aligned} y^{(+2)} = & \frac{1}{2}\varepsilon^{ab}u_a^{(+I)}u_b^{(+J)}y_{IJ} + \frac{m^2}{4}\left(\theta^{(+)}\right)^4 y^{(-2)}, \\ y_{ia} = & u_a^{(+I)}u_i^{(-J)}y_{IJ}, \quad y^{(-2)} = \frac{1}{2}\varepsilon^{ij}u_i^{(-I)}u_j^{(-J)}y_{IJ}. \end{aligned} \quad (4.29)$$

The component fields transform as

$$\begin{aligned} \delta D = & -\sqrt{2}\epsilon_I\left(i\dot{\chi}^I - \frac{3m}{4}\chi^I\right)e^{3imt/4}, \quad \delta\bar{D} = -\sqrt{2}\bar{\epsilon}^I\left(i\dot{\bar{\chi}}_I + \frac{3m}{4}\bar{\chi}_I\right)e^{-3imt/4}, \\ \delta y^{IJ} = & -2\bar{\epsilon}^{[I}\chi^{J]}e^{-3imt/4} + \varepsilon^{IJKL}\epsilon_K\bar{\chi}_L e^{3imt/4}, \\ \delta\chi^I = & \sqrt{2}\bar{\epsilon}^I D e^{-3imt/4} - 2\epsilon_J\left(i\dot{y}^{IJ} - \frac{m}{2}y^{IJ}\right)e^{3imt/4}, \\ \delta\bar{\chi}_I = & \sqrt{2}\epsilon_I\bar{D} e^{3imt/4} + 2\bar{\epsilon}^J\left(i\dot{y}_{IJ} + \frac{m}{2}y_{IJ}\right)e^{-3imt/4}. \end{aligned} \quad (4.30)$$

The substitution $D = i\dot{\phi}$ in these transformations gives just the transformations (4.7) of the multiplet $(\mathbf{8}, \mathbf{8}, \mathbf{0})$. Thus this substitution ensures the validity of the additional constraints imposed on the superfield Y^{IJ} . We conclude that the $SU(4|1)$ multiplet $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ admits an alternative description within harmonic $SU(4|1)$ superspace.

4.4.2 Invariant action via harmonic superspace

Introducing the shifted superfield

$$\begin{aligned} Y^{(+2)} = & \hat{Y}^{(+2)} + c^{(+2)}, \\ c^{(+2)} = & \frac{1}{2}\varepsilon^{ab}u_a^{(+I)}u_b^{(+J)}c_{IJ} + \frac{m^2}{4}\left(\theta^{(+)}\right)^4 c^{(-2)}, \quad c^{(-2)} = \frac{1}{2}\varepsilon^{ij}u_i^{(-I)}u_j^{(-J)}c_{IJ}, \\ \delta\hat{Y}^{(+2)} = & \Lambda^{(0)}\left(\hat{Y}^{(+2)} + c^{(+2)}\right) + \varepsilon^{ab}\varepsilon_{ij}\Lambda_a^{(+2)i}\left(\mathcal{D}_b^{(+2)j}c^{(-2)}\right) \\ & - \frac{1}{4}\varepsilon^{ab}\varepsilon_{ij}c^{(-2)}\left(\mathcal{D}_b^{(+2)j}\Lambda_a^{(+2)i}\right), \end{aligned} \quad (4.31)$$

we calculate the invariant action (see appendix A) as

$$S_{(\mathbf{6},\mathbf{8},\mathbf{2})} = \frac{1}{16} \int d\zeta_A^{(-4)} L^{(+4)},$$

$$L^{(+4)} = \frac{\hat{Y}^{(+2)} \hat{Y}^{(+2)}}{\left(1 + c^{(-2)} \hat{Y}^{(+2)}\right)^4} + \frac{m^2}{2} \left(\theta^{(+)}\right)^4 \left[1 - \frac{1 - c^{(-2)} \hat{Y}^{(+2)}}{\left(1 + c^{(-2)} \hat{Y}^{(+2)}\right)^5}\right]. \quad (4.32)$$

This action of the multiplet $(\mathbf{6}, \mathbf{8}, \mathbf{2})$ is in fact superconformal with respect to the supergroup $SU(4|1, 1)$ (see appendix B). The relevant metric is $SO(6)$ invariant and given by

$$g_2 = \left[\frac{1}{2} y^{IJ} y_{IJ} \right]^{-2}. \quad (4.33)$$

Substituting $D = i\dot{\phi}$, one can finally find the bosonic truncation of the component Lagrangian for the multiplet $(\mathbf{8}, \mathbf{8}, \mathbf{0})$:

$$\mathcal{L}_{\text{bos.}} = g_2 \left(\dot{\phi} \dot{\phi} + \frac{1}{2} \dot{y}^{IJ} \dot{y}_{IJ} - \frac{m^2}{8} y^{IJ} y_{IJ} \right). \quad (4.34)$$

Calculation of all terms in harmonic superspace is rather complicated. We skip all these calculations and write the full component Lagrangian (4.50) in the next subsection by employing $SU(2|1)$ superfields.

4.5 $SU(2|1)$ superfield approach

To simplify the construction of $SU(4|1)$ invariant actions, it will be convenient to employ $SU(2|1)$ superfield approach elaborated in [6–9]. We split the multiplet $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ into $SU(2|1)$ multiplets as a sum of the conventional multiplet $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ and the “mirror” multiplet $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ [9]. To obtain such a decomposition, we need to single out the ϵ_1 and ϵ_2 subvariety of the transformations of (4.7) corresponding to the $SU(2|1)$ superspace transformations (2.13). The $SU(2|1)$ covariant constraints given below involve the covariant derivatives (2.16).

4.5.1 The standard multiplet $(\mathbf{4}, \mathbf{4}, \mathbf{0})$

Introducing the new notations

$$\begin{aligned} x^{11} &:= y^{14}, & x^{12} &:= y^{13}, & x^{21} &:= y^{24}, & x^{22} &:= y^{23}, \\ \xi^1 &:= \bar{\chi}_3, & \xi^2 &:= -\bar{\chi}_4, & \bar{\xi}_1 &:= \chi^3, & \bar{\xi}_2 &:= -\chi^4, \\ \overline{(x^{ia})} &= \varepsilon_{ab} \varepsilon_{ij} x^{jb}, & \overline{(\xi^a)} &= \bar{\xi}_a, & \overline{(\bar{\xi}^a)} &= \bar{\xi}_a, \end{aligned} \quad (4.35)$$

we obtain the same deformed transformations as in [9]:

$$\begin{aligned} \delta x^{ia} &= - \left(\epsilon^i \xi^a e^{3imt/4} + \bar{\epsilon}^i \bar{\xi}^a e^{-3imt/4} \right), \\ \delta \xi^a &= \bar{\epsilon}^k (2i\dot{x}_k^a + m x_k^a) e^{-3imt/4}, & \delta \bar{\xi}^a &= \epsilon_k (2i\dot{x}^{ka} - m x^{ka}) e^{3imt/4}. \end{aligned} \quad (4.36)$$

The indices $i = 1, 2$ and $a = 1, 2$ correspond to the fundamental representations of the sub-group $SU(2) \times SU(2) \subset SU(4)$.

The corresponding superfield q^{ia} obeys the $SU(2|1)$ covariant constraints

$$\mathcal{D}^{(k} q^{i)a} = \bar{\mathcal{D}}^{(k} q^{i)a} = 0, \quad \tilde{F} q^{ia} = 0, \quad \overline{(q^{ia})} = q_{ia}. \quad (4.37)$$

These constraints are solved by

$$\begin{aligned} q^{ia} = & \left[1 + \frac{m}{2} \bar{\theta}^k \theta_k - \frac{5m^2}{16} (\bar{\theta})^2 (\theta)^2 \right] x^{ia} + \left(1 + \frac{m}{4} \bar{\theta}^k \theta_k \right) \left(\theta^i \xi^a e^{3imt/4} + \bar{\theta}^i \bar{\xi}^a e^{-3imt/4} \right) \\ & + i \left(\bar{\theta}^k \theta^i \dot{x}_k^a - \bar{\theta}^i \theta_k \dot{x}^{ka} \right) - i \bar{\theta}^k \theta_k \left(\theta^i \dot{\xi}^a e^{3imt/4} - \bar{\theta}^i \dot{\bar{\xi}}^a e^{-3imt/4} \right) + \frac{1}{4} (\bar{\theta})^2 (\theta)^2 \ddot{x}^{ia}, \end{aligned} \quad (4.38)$$

where the following conventions for the Grassmann monomials were employed: $(\theta)^2 = \theta_i \theta^i$, $(\bar{\theta})^2 = \bar{\theta}^i \bar{\theta}_i$.

4.5.2 The mirror multiplet (4, 4, 0)

The “mirror” (4, 4, 0) multiplet is defined by the transformations

$$\begin{aligned} \delta z &= -\epsilon_k \psi^k e^{3imt/4}, & \delta \bar{z} &= \bar{\epsilon}^k \bar{\psi}_k e^{-3imt/4}, \\ \delta y &= -\epsilon_k \bar{\psi}^k e^{3imt/4}, & \delta \bar{y} &= -\bar{\epsilon}^k \psi_k e^{-3imt/4}, \\ \delta \psi^i &= \bar{\epsilon}^i (2i\dot{z}) e^{-3imt/4} + \epsilon^i (2i\dot{y} - m\bar{y}) e^{3imt/4}, \\ \delta \bar{\psi}_i &= -\epsilon_i (2i\dot{\bar{z}}) e^{3imt/4} + \bar{\epsilon}_i (2i\dot{y} + m\bar{y}) e^{-3imt/4}, \end{aligned} \quad (4.39)$$

where

$$\begin{aligned} \sqrt{2} z &:= \phi, & \sqrt{2} \bar{z} &:= \bar{\phi}, & y &:= y^{34}, & \bar{y} &:= y^{12}, \\ \psi^1 &:= \chi^1, & \psi^2 &:= \chi^2, & \bar{\psi}_1 &:= \bar{\chi}_1, & \bar{\psi}_2 &:= \bar{\chi}_2. \end{aligned} \quad (4.40)$$

These transformations differ from those given in [9]. In the present case, Pauli-Gürsey $SU(2)$ symmetry is broken. For this case the $SU(2|1)$ superfield constraints defining the mirror (4, 4, 0) multiplet are written as

$$\begin{aligned} \bar{\mathcal{D}}^i Z &= \bar{\mathcal{D}}^i Y = 0, & \mathcal{D}^i \bar{Z} &= \mathcal{D}^i \bar{Y} = 0, \\ \mathcal{D}^i Z &= -\bar{\mathcal{D}}^i \bar{Y}, & \mathcal{D}^i Y &= \bar{\mathcal{D}}^i \bar{Z}, \\ \tilde{F} Z &= 0, & \tilde{F} Y &= Y. \end{aligned} \quad (4.41)$$

Their solution reads

$$\begin{aligned} Z &= z + \theta_i \psi^i e^{3imt/4} + i \bar{\theta}^j \theta_j \dot{z} - (\theta)^2 \left(i\dot{y} - \frac{m}{2} \bar{y} \right) e^{3imt/2} + \bar{\theta}^j \theta_j \theta_i \left(i\dot{\psi}^i - \frac{3m}{4} \psi^i \right) e^{3imt/4} \\ &\quad - \frac{1}{4} (\bar{\theta})^2 (\theta)^2 (\ddot{z} + 2im\dot{z}), \\ Y &= y + \theta_i \bar{\psi}^i e^{3imt/4} + \bar{\theta}^j \theta_j \left(i\dot{y} + \frac{m}{2} y \right) + i (\theta)^2 \dot{\bar{z}} e^{3imt/2} + \bar{\theta}^j \theta_j \theta_i \left(i\dot{\bar{\psi}}^i - \frac{m}{4} \bar{\psi}^i \right) e^{3imt/4} \\ &\quad - \frac{1}{4} (\bar{\theta})^2 (\theta)^2 \left(\ddot{y} + im\dot{y} + \frac{3m^2}{4} y \right). \end{aligned} \quad (4.42)$$

4.5.3 SU(2|1) superfield action

The construction of SU(4|1) invariant actions in terms of the SU(2|1) superfields (4.38), (4.42) goes as follows. The general SU(2|1) superfield action can be written as

$$S = \int dt d^2\theta d^2\bar{\theta} \left(1 + 2m \bar{\theta}^k \theta_k\right) f(Z, \bar{Z}, Y\bar{Y}, q^{ia} q_{ia}). \quad (4.43)$$

The target space metric g is defined according to [17] as

$$\begin{aligned} g &= \Delta_2 f = -\Delta_1 f, & f &= f(z, \bar{z}, y\bar{y}, x^{ia} x_{ia}), & g &= g(z, \bar{z}, y\bar{y}, x^{ia} x_{ia}), \\ \Delta_1 f + \Delta_2 f &= 0 & \Rightarrow & \Delta_1 g + \Delta_2 g = 0, \end{aligned} \quad (4.44)$$

where

$$\begin{aligned} \partial_{ia} &= \partial/\partial x^{ia}, & \Delta_1 &= \varepsilon^{ik} \varepsilon^{ab} \partial_{ia} \partial_{kb}, \\ \partial_z &= \frac{\partial}{\partial z}, & \partial_{\bar{z}} &= \frac{\partial}{\partial \bar{z}}, & \partial_y &= \frac{\partial}{\partial y}, & \partial_{\bar{y}} &= \frac{\partial}{\partial \bar{y}}, & \Delta_2 &= 2(\partial_z \partial_{\bar{z}} + \partial_y \partial_{\bar{y}}). \end{aligned} \quad (4.45)$$

Since SU(2|1) supersymmetry implies SU(2) $\times$ U(1) symmetry, the function f and g are functions of the following coordinate monomials: $z, \bar{z}, y\bar{y}, x^{ia} x_{ia}$.

Requiring SU(4) invariance of the corresponding component action amounts to the constraints:

$$\begin{aligned} m(\bar{y}g + 2\partial_y f + x^{ia} \partial_{ia} \partial_y f) &= 0 & \Rightarrow & m(x_{ia} \partial_y - \bar{y} \partial_{ia}) g = 0, \\ m(yg + 2\partial_{\bar{y}} f + x^{ia} \partial_{ia} \partial_{\bar{y}} f) &= 0 & \Rightarrow & m(x_{ia} \partial_{\bar{y}} - y \partial_{ia}) g = 0. \end{aligned} \quad (4.46)$$

These constraints admit three different solutions:

- 1) Special Kähler manifold metric (4.9)

$$\begin{aligned} f_1 &= \frac{1}{2} [\bar{z} \partial_z K(z) + z \partial_{\bar{z}} \bar{K}(\bar{z})] - \frac{1}{16} (x^{ia} x_{ia} + 4y\bar{y}) [\partial_z \partial_z K(z) + \partial_{\bar{z}} \partial_{\bar{z}} \bar{K}(\bar{z})], \\ g_1 &= \frac{1}{2} [\partial_z \partial_z K(z) + \partial_{\bar{z}} \partial_{\bar{z}} \bar{K}(\bar{z})] \implies g_1 = \partial_\phi \partial_\phi K(\phi) + \partial_{\bar{\phi}} \partial_{\bar{\phi}} \bar{K}(\bar{\phi}). \end{aligned} \quad (4.47)$$

- 2) SO(6)-invariant metric (4.33)

$$\begin{aligned} f_2 &= \frac{1}{4} (x^{ia} x_{ia})^{-1} \log(2y\bar{y} + x^{ia} x_{ia}), \\ g_2 &= (2y\bar{y} + x^{ia} x_{ia})^{-2} \implies g_2 = \left[\frac{1}{2} y^{IJ} y_{IJ} \right]^{-2}. \end{aligned} \quad (4.48)$$

- 3) SO(8)-invariant metric

$$\begin{aligned} f_3 &= -\frac{1}{8} (x^{ia} x_{ia})^{-1} (2z\bar{z} + 2y\bar{y} + x^{ia} x_{ia})^{-1}, \\ g_3 &= (2z\bar{z} + 2y\bar{y} + x^{ia} x_{ia})^{-3} \implies g_3 = \left[\phi \bar{\phi} + \frac{1}{2} y^{IJ} y_{IJ} \right]^{-3}. \end{aligned} \quad (4.49)$$

The first solution (4.47) reproduces the Lagrangian (4.8) with the metric (4.9). Other solutions correspond to new $SU(4|1)$ invariant actions.

The second solution (4.48) gives the Lagrangian

$$\begin{aligned}\mathcal{L}_{SO(6)} = g_2 & \left[\dot{\phi}\dot{\bar{\phi}} + \frac{1}{2}\dot{y}^{IJ}\dot{y}_{IJ} + \frac{i}{2}(\chi^K\dot{\bar{\chi}}_K - \dot{\chi}^K\bar{\chi}_K) - \frac{m}{4}\chi^K\bar{\chi}_K - \frac{m^2}{8}y^{IJ}y_{IJ} \right] \\ & - \frac{i}{\sqrt{2}}\dot{\phi}\partial_{IJ}g_2\chi^I\chi^J - \frac{i}{\sqrt{2}}\dot{\bar{\phi}}\partial^{IJ}g_2\bar{\chi}_I\bar{\chi}_J + i(\dot{y}_{IK}\partial^{JK}g_2 - \dot{y}^{JK}\partial_{IK}g_2)\chi^I\bar{\chi}_J \\ & - \frac{1}{2}\partial_{IJ}\partial^{KL}g_2\chi^I\chi^J\bar{\chi}_K\bar{\chi}_L, \end{aligned} \quad (4.50)$$

where

$$\partial_{IJ} = \frac{\partial}{\partial y^{IJ}}, \quad \partial_{IJ}y^{KL} = \frac{1}{2}(\delta_I^K\delta_J^L - \delta_I^L\delta_J^K), \quad \partial_{IJ}(y_{KL}) = \frac{1}{2}\varepsilon_{IJKL}. \quad (4.51)$$

Substitution $i\dot{\phi} = D$ gives $SU(4|1)$ invariant Lagrangian for the multiplet $(\mathbf{6}, \mathbf{8}, \mathbf{2})$, which is in fact superconformal, with the relevant group $SU(4|1, 1)$ (see appendix B).

The third solution (4.49) exhibits an invariance under the maximal R -symmetry group $SO(8)$ and produces the component Lagrangian

$$\begin{aligned}\mathcal{L}_{SO(8)} = g_3 & \left[\dot{\phi}\dot{\bar{\phi}} + \frac{1}{2}\dot{y}^{IJ}\dot{y}_{IJ} + \frac{i}{2}(\chi^K\dot{\bar{\chi}}_K - \dot{\chi}^K\bar{\chi}_K) + \frac{m}{4}\chi^K\bar{\chi}_K - \frac{m^2}{8}y^{IJ}y_{IJ} \right] \\ & - \frac{i}{\sqrt{2}}\dot{\phi}\partial_{IJ}g_3\chi^I\chi^J - \frac{i}{\sqrt{2}}\dot{\bar{\phi}}\partial^{IJ}g_3\bar{\chi}_I\bar{\chi}_J + i(\dot{y}_{IK}\partial^{JK}g_3 - \dot{y}^{JK}\partial_{IK}g_3)\chi^I\bar{\chi}_J \\ & + \frac{1}{\sqrt{2}}\left(i\dot{y}_{IJ} - \frac{m}{2}y_{IJ}\right)\partial_\phi g_3\chi^I\chi^J + \frac{1}{\sqrt{2}}\left(i\dot{y}^{IJ} + \frac{m}{2}y^{IJ}\right)\partial_{\bar{\phi}}g_3\bar{\chi}_I\bar{\chi}_J \\ & - \frac{i}{2}\left(\dot{\phi}\partial_\phi g_3 - \dot{\bar{\phi}}\partial_{\bar{\phi}}g_3\right)\chi^K\bar{\chi}_K - \frac{im}{2}\left(\dot{\phi}\bar{\phi} - \dot{\bar{\phi}}\phi\right)g_3 \\ & + \frac{m}{4}\left(\dot{\phi}\partial_\phi g_3 + \dot{\bar{\phi}}\partial_{\bar{\phi}}g_3\right)\chi^K\bar{\chi}_K - \frac{1}{\sqrt{2}}\left(\chi^I\chi^J\partial_{IJ}\partial_\phi g_3 + \bar{\chi}_I\bar{\chi}_J\partial^{IJ}\partial_{\bar{\phi}}g_3\right)\chi^K\bar{\chi}_K \\ & - \frac{1}{24}\left(\varepsilon_{IJKL}\chi^I\chi^J\chi^K\chi^L\partial_\phi\partial_\phi g_3 + \varepsilon^{IJKL}\bar{\chi}_I\bar{\chi}_J\bar{\chi}_K\bar{\chi}_L\partial_{\bar{\phi}}\partial_{\bar{\phi}}g_3\right) \\ & - \frac{1}{2}\partial_{IJ}\partial^{KL}g_3\chi^I\chi^J\bar{\chi}_K\bar{\chi}_L + \frac{1}{2}\partial_\phi\partial_{\bar{\phi}}g_3\chi^I\bar{\chi}_I\chi^J\bar{\chi}_J. \end{aligned} \quad (4.52)$$

4.6 Superconformal symmetry

Redefining the component fields in (4.52) as

$$\begin{aligned}\phi & \rightarrow \phi e^{-imt/2}, & \chi^I & \rightarrow \chi^I e^{-imt/4}, \\ \bar{\phi} & \rightarrow \bar{\phi} e^{imt/2}, & \bar{\chi}_I & \rightarrow \bar{\chi}_I e^{imt/4}, \end{aligned} \quad (4.53)$$

we eliminate all the deformed terms proportional to m and write the Lagrangian in SO(8) invariant formulation:

$$\begin{aligned}
 \mathcal{L}_{\text{conf}} = g_3 \bigg[& \dot{\phi}\dot{\bar{\phi}} + \frac{1}{2} \dot{y}^{IJ} \dot{y}_{IJ} + \frac{i}{2} (\chi^K \dot{\chi}_K - \dot{\chi}^K \bar{\chi}_K) - \frac{m^2}{4} \left(\phi\bar{\phi} + \frac{1}{2} y^{IJ} y_{IJ} \right) \bigg] \\
 & - \frac{i}{\sqrt{2}} \dot{\phi} \partial_{IJ} g_3 \chi^I \chi^J - \frac{i}{\sqrt{2}} \dot{\bar{\phi}} \partial^{IJ} g_3 \bar{\chi}_I \bar{\chi}_J + i (\dot{y}_{IK} \partial^{JK} g_3 - \dot{y}^{JK} \partial_{IK} g_3) \chi^I \bar{\chi}_J \\
 & + \frac{i}{\sqrt{2}} (\dot{y}_{IJ} \chi^I \chi^J \partial_{\phi} g_3 + \dot{y}^{IJ} \bar{\chi}_I \bar{\chi}_J \partial_{\bar{\phi}} g_3) - \frac{i}{2} (\dot{\phi} \partial_{\phi} g_3 - \dot{\bar{\phi}} \partial_{\bar{\phi}} g_3) \chi^K \bar{\chi}_K \\
 & - \frac{1}{24} (\varepsilon_{IJKL} \chi^I \chi^J \chi^K \chi^L \partial_{\phi} \partial_{\phi} g_3 + \varepsilon^{IJKL} \bar{\chi}_I \bar{\chi}_J \bar{\chi}_K \bar{\chi}_L \partial_{\bar{\phi}} \partial_{\bar{\phi}} g_3) \\
 & - \frac{1}{\sqrt{2}} (\chi^I \chi^J \partial_{IJ} \partial_{\phi} g_3 + \bar{\chi}_I \bar{\chi}_J \partial^{IJ} \partial_{\bar{\phi}} g_3) \chi^K \bar{\chi}_K \\
 & - \frac{1}{2} \partial_{IJ} \partial^{KL} g_3 \chi^I \chi^J \bar{\chi}_K \bar{\chi}_L + \frac{1}{2} \partial_{\phi} \partial_{\bar{\phi}} g_3 \chi^I \bar{\chi}_I \chi^J \bar{\chi}_J. \tag{4.54}
 \end{aligned}$$

As a result, we obtain OSp(8|2) superconformal Lagrangian of the trigonometric type⁸ that contains only m^2 terms. Since the new Lagrangian (4.54) is an even function of m , it is invariant under two types of SU(4|1) transformations, with the deformation parameters m and $-m$:

$$\begin{aligned}
 \delta\phi &= -\sqrt{2} \epsilon_I \chi^I e^{imt}, & \delta\bar{\phi} &= \sqrt{2} \bar{\epsilon}^I \bar{\chi}_I e^{-imt}, \\
 \delta y^{IJ} &= -2 \bar{\epsilon}^{[I} \chi^{J]} e^{-imt} + \varepsilon^{IJKL} \epsilon_K \bar{\chi}_L e^{imt}, \\
 \delta\chi^I &= \sqrt{2} \bar{\epsilon}^I \left(i\dot{\phi} + \frac{m}{2} \phi \right) e^{-imt} - 2 \epsilon_J \left(i\dot{y}^{IJ} - \frac{m}{2} y^{IJ} \right) e^{imt}, \\
 \delta\bar{\chi}_I &= -\sqrt{2} \epsilon_I \left(i\dot{\bar{\phi}} - \frac{m}{2} \bar{\phi} \right) e^{imt} + 2 \bar{\epsilon}^J \left(i\dot{y}_{IJ} + \frac{m}{2} y_{IJ} \right) e^{-imt}, \tag{4.55}
 \end{aligned}$$

$$\begin{aligned}
 \delta\phi &= -\sqrt{2} \eta_I \chi^I e^{-imt}, & \delta\bar{\phi} &= \sqrt{2} \bar{\eta}^I \bar{\chi}_I e^{imt}, \\
 \delta y^{IJ} &= -2 \bar{\eta}^{[I} \chi^{J]} e^{imt} + \varepsilon^{IJKL} \eta_K \bar{\chi}_L e^{-imt}, \\
 \delta\chi^I &= \sqrt{2} \bar{\eta}^I \left(i\dot{\phi} - \frac{m}{2} \phi \right) e^{imt} - 2 \eta_J \left(i\dot{y}^{IJ} + \frac{m}{2} y^{IJ} \right) e^{-imt}, \\
 \delta\bar{\chi}_I &= -\sqrt{2} \eta_I \left(i\dot{\bar{\phi}} + \frac{m}{2} \bar{\phi} \right) e^{-imt} + 2 \bar{\eta}^J \left(i\dot{y}_{IJ} - \frac{m}{2} y_{IJ} \right) e^{imt}. \tag{4.56}
 \end{aligned}$$

In the closure of these transformations, we obtain superconformal algebra $osp(8|2)$ spanned by 16 supercharges and 31 bosonic generators (see appendix D),⁹ where the conformal Hamiltonian $\mathcal{H}_{\text{conf}}$ is defined as

$$\mathcal{H}_{\text{conf}} = \mathcal{H} - \frac{m}{2} F. \tag{4.57}$$

The generators F^{IJ} and $\bar{F}_{IJ}$ produce SO(8)/U(4) transformations realized as

$$\begin{aligned}
 \delta\chi^I &= \sqrt{2} \bar{\Lambda}^{IJ} \bar{\chi}_J, & \delta\bar{\chi}_I &= \sqrt{2} \Lambda_{IJ} \chi^J, \\
 \delta\phi &= -\bar{\Lambda}^{IJ} y_{IJ}, & \delta\bar{\phi} &= -\Lambda^{IJ} y_{IJ}, & \delta y_{IJ} &= \Lambda_{IJ} \phi + \bar{\Lambda}_{IJ} \bar{\phi}. \tag{4.58}
 \end{aligned}$$

⁸Here we follow the terminology suggested in [33].

⁹In the limit $m=0$ the Lagrangian (4.54) goes into the one invariant under the “parabolic” realization of OSp(8|2), as it was given in [34].

5 The SU(4|1) multiplet (8, 8, 0): second version

The second version of the multiplet (8, 8, 0) is described by a complex bosonic superfield V^I satisfying

$$\begin{aligned}\mathcal{D}^I V^J &= \frac{1}{2} \varepsilon^{IJKL} \bar{\mathcal{D}}_K \bar{V}_L, & \mathcal{D}^{(I} V^{J)} &= 0, & \bar{\mathcal{D}}_{(K} \bar{V}_{L)} &= 0, \\ \mathcal{D}^I \bar{V}_J &= \frac{1}{4} \delta_J^I \mathcal{D}^K \bar{V}_K & \bar{\mathcal{D}}_J V^I &= \frac{1}{4} \delta_J^I \bar{\mathcal{D}}_K V^K & \overline{(V^I)} &= \bar{V}_I.\end{aligned}\quad (5.1)$$

In the flat superspace limit $m \rightarrow 0$, these constraints go over to the SU(4) covariant constraints (3.7) specifying another form of the flat $\mathcal{N}=8$, $d=1$ multiplet (8, 8, 0).

To avoid calculation of the deformed covariant derivatives $\mathcal{D}^I$ and $\bar{\mathcal{D}}_J$, we instead consider harmonization of part of these constraints, *viz.*

$$\bar{\mathcal{D}}_{(K} \bar{V}_{L)} = 0, \quad \mathcal{D}^I \bar{V}_J = \frac{1}{4} \delta_J^I \mathcal{D}^K \bar{V}_K, \quad (5.2)$$

with the rest of constraints being solved at the component level.

5.1 Harmonic superspace

The option for harmonic superspace relevant to the given case uses the harmonic variables on $SU(4)/[SU(3) \times U(1)]$ [32]. The set of these harmonic variables is given by $u_I^{(+)\alpha}$, $u^{(+3)I}$, $u_\beta^{(-)I}$, $u_I^{(-3)}$, where the index $\alpha = 1, 2, 3$ refers to the SU(3) fundamental representation. The harmonics satisfy the following unitarity and unimodularity conditions:

$$\begin{aligned}u_I^{(-3)} u^{(+3)I} &= 1, & u_I^{(+)\alpha} u_\beta^{(-)I} &= \delta_\beta^\alpha, & u_J^{(+)\alpha} u_\alpha^{(-)I} + u_J^{(-3)} u^{(+3)I} &= \delta_J^I, \\ u_I^{(+)\alpha} u^{(+3)I} &= u_I^{(-3)} u_\alpha^{(-)I} = 0, & \varepsilon^{IJKL} u_I^{(+)\alpha} u_J^{(+)\beta} u_K^{(+)\gamma} u_L^{(-3)} &= \varepsilon^{\alpha\beta\gamma}.\end{aligned}\quad (5.3)$$

As in the previous case, we define the new coordinates

$$\begin{aligned}\theta^{(+3)} &= \theta_I \left(u^{(+3)I} + m \bar{\theta}^{(+)\alpha} \theta^{(+3)} u_\alpha^{(-)I} \right), & \theta_\alpha^{(-)} &= \theta_I u_\alpha^{(-)I}, \\ \bar{\theta}^{(+)\alpha} &= \bar{\theta}^J \left(u_J^{(+)\alpha} + m \bar{\theta}^{(+)\alpha} \theta^{(+3)} u_J^{(-3)} \right), & \bar{\theta}^{(-3)} &= \bar{\theta}^J u_J^{(-3)}, \\ t_A &= t + i \bar{\theta}^{(-3)} \theta^{(+3)} - i \bar{\theta}^{(+)\alpha} \theta_\alpha^{(-)} \left[1 - m \bar{\theta}^{(+)\beta} \theta_\beta^{(-)} + \frac{4m^2}{3} \left(\bar{\theta}^{(+)\beta} \theta_\beta^{(-)} \right)^2 \right].\end{aligned}\quad (5.4)$$

They transform as

$$\begin{aligned}\delta \theta_\alpha^{(-)} &= \epsilon_\alpha^{(-)} + 2m \left[\bar{\epsilon}^{(+)\beta} \theta_\beta^{(-)} + \bar{\epsilon}^{(-3)} \theta^{(+3)} \left(1 + m \bar{\theta}^{(+)\beta} \theta_\beta^{(-)} \right) \right] \theta_\alpha^{(-)}, \\ \delta \bar{\theta}^{(-3)} &= \bar{\epsilon}^{(-3)} - 2m \epsilon_\beta^{(-)} \bar{\theta}^{(+)\beta} \bar{\theta}^{(-3)}, \\ \delta \theta^{(+3)} &= \epsilon^{(+3)} + m \epsilon_\alpha^{(-)} \bar{\theta}^{(+)\alpha} \theta^{(+3)}, \\ \delta \bar{\theta}^{(+)\alpha} &= \bar{\epsilon}^{(+)\alpha} + m \bar{\epsilon}^{(-3)} \bar{\theta}^{(+)\alpha} \theta^{(+3)} - 2m \epsilon_\beta^{(-)} \bar{\theta}^{(+)\beta} \bar{\theta}^{(+)\alpha}, \\ \delta u^{(+3)I} &= \Lambda^{(+4)\alpha} u_\alpha^{(-)I} & \delta u_I^{(-3)} &= 0, \\ \delta u_I^{(+)\alpha} &= -\Lambda^{(+4)\alpha} u_I^{(-3)} & \delta u_\beta^{(-)I} &= 0, \\ \delta t_A &= 2i \left(\epsilon_\alpha^{(-)} \bar{\theta}^{(+)\alpha} + \bar{\epsilon}^{(-3)} \theta^{(+3)} \right),\end{aligned}\quad (5.5)$$

where

$$\begin{aligned}\Lambda^{(+4)\alpha} &= m \left(\epsilon^{(+3)} \bar{\theta}^{(+)\alpha} + \bar{\epsilon}^{(+)\alpha} \theta^{(+3)} \right) + m^2 \epsilon_\beta^{(-)} \bar{\theta}^{(+)\beta} \bar{\theta}^{(+)\alpha} \theta^{(+3)}, \\ \epsilon_\alpha^{(-)} &= \epsilon_I u_\alpha^{(-)I}, \quad \epsilon^{(+3)} = \epsilon_I u^{(+3)I}, \quad \bar{\epsilon}^{(+)\alpha} = \bar{\epsilon}^J u_J^{(+)\alpha}, \quad \bar{\epsilon}^{(-3)} = \bar{\epsilon}^J u_J^{(-3)}.\end{aligned}\quad (5.6)$$

It is straightforward to see that the analytic subspace

$$\zeta_A = \left\{ t_A, \theta^{(+3)}, \bar{\theta}^{(+)\alpha}, u_I^{(+)\alpha}, u^{(+3)I}, u_\beta^{(-)I}, u_I^{(-3)} \right\}, \quad (5.7)$$

is closed under the transformations (5.5). Its integration measure

$$d\zeta_A^{(-6)} = dt_A du d\theta^{(+3)} d^3 \bar{\theta}^{(+)} e^{3imt_A/2} \quad (5.8)$$

transforms as

$$\delta \left(d\zeta_A^{(-6)} \right) = m d\zeta_A^{(-6)} \left(\epsilon_\alpha^{(-)} \bar{\theta}^{(+)\alpha} - 3 \bar{\epsilon}^{(-3)} \theta^{(+3)} \right). \quad (5.9)$$

The harmonic derivatives are found to be

$$\begin{aligned}\mathcal{D}^{(+4)\alpha} &= \partial^{(+4)\alpha} - 2i \bar{\theta}^{(+)\alpha} \theta^{(+3)} \partial_A - \frac{m}{6} \bar{\theta}^{(+)\alpha} \theta^{(+3)} \mathcal{D}^0 + m \bar{\theta}^{(+)\alpha} \theta^{(+3)} \bar{\theta}^{(+)\beta} \frac{\partial}{\partial \bar{\theta}^{(+)\beta}}, \\ \mathcal{D}_\beta^\alpha &= \partial_\beta^\alpha + \bar{\theta}^{(+)\alpha} \frac{\partial}{\partial \bar{\theta}^{(+)\beta}} - \frac{\delta_\beta^\alpha}{3} \bar{\theta}^{(+)\gamma} \frac{\partial}{\partial \bar{\theta}^{(+)\gamma}}, \\ \mathcal{D}^0 &= \partial^0 + \bar{\theta}^{(+)\alpha} \frac{\partial}{\partial \bar{\theta}^{(+)\alpha}} + 3 \theta^{(+3)} \frac{\partial}{\partial \theta^{(+3)}},\end{aligned}\quad (5.10)$$

where

$$\begin{aligned}\partial^{(+4)\alpha} &= u^{(+3)K} \frac{\partial}{\partial u_\alpha^{(-)K}} - u_K^{(+)\alpha} \frac{\partial}{\partial u_K^{(-3)}}, \\ \partial_\beta^\alpha &= u_K^{(+)\alpha} \frac{\partial}{\partial u_K^{(+)\beta}} - u_\beta^{(-)K} \frac{\partial}{\partial u_\alpha^{(-)K}} - \frac{\delta_\beta^\alpha}{3} \left(u_K^{(+)\gamma} \frac{\partial}{\partial u_K^{(+)\gamma}} - u_\gamma^{(-)K} \frac{\partial}{\partial u_\gamma^{(-)K}} \right), \\ \partial^0 &= u_K^{(+)\alpha} \frac{\partial}{\partial u_K^{(+)\alpha}} - u_\alpha^{(-)K} \frac{\partial}{\partial u_\alpha^{(-)K}} + 3 \left(u^{(+3)K} \frac{\partial}{\partial u^{(+3)K}} - u_K^{(-3)} \frac{\partial}{\partial u_K^{(-3)}} \right).\end{aligned}\quad (5.11)$$

Note that

$$\mathcal{D}^{(+4)\alpha} \Lambda = \Lambda^{(+4)\alpha}, \quad \Lambda = m \left(\epsilon_\alpha^{(-)} \bar{\theta}^{(+)\alpha} - \bar{\epsilon}^{(-3)} \theta^{(+3)} \right). \quad (5.12)$$

The harmonic analytic superfield $\bar{V}^{(+3)}$ defined on (5.7) satisfies the harmonic constraints

$$\mathcal{D}^{(+4)\alpha} \bar{V}^{(+3)} = 0, \quad \mathcal{D}_\beta^\alpha \bar{V}^{(+3)} = 0, \quad \mathcal{D}^0 \bar{V}^{(+3)} = 3 \bar{V}^{(+3)}. \quad (5.13)$$

It can be treated as a harmonization of the superfield $\bar{V}_I$ defined by (5.2), where Grassmann analyticity constraints are provided by

$$u^{(+3)K} u^{(+3)L} \bar{\mathcal{D}}_{(K} \bar{V}_{L)} = \bar{\mathcal{D}}^{(+3)} \bar{V}^{(+3)} = 0, \quad u_I^{(+)\alpha} u^{(+3)J} \mathcal{D}^I \bar{V}_J = \mathcal{D}^{(+)\alpha} \bar{V}^{(+3)} = 0. \quad (5.14)$$

The full set of the constraints (5.1) operates with the set of superfields $V^{(+)\alpha}$, $V^{(-)}$, $\bar{V}^{(+3)}$ and $\bar{V}_\alpha^{(-)}$ living on the full harmonic superspace (5.4). Here we consider just the superfield $\bar{V}^{(+3)}$ treated as an unconstrained deformed harmonic superfield satisfying the analyticity conditions (5.13). The rest of constraints on $\bar{V}^{(+3)}$ will be imposed below “by hand” at the component level, like in the previous cases.

The general expansion of $\bar{V}^{(+3)}$ reads

$$\begin{aligned}\bar{V}^{(+3)} = & \bar{z}_I u^{(+3)I} + \sqrt{2} \theta^{(+3)} \chi e^{3imt/4} + 2 \bar{\theta}^{(+)\alpha} \chi_{JI} u^{(+3)J} u_\alpha^{(-)I} e^{-3imt/4} \\ & + 2 \bar{\theta}^{(+)\alpha} \theta^{(+3)} \left(i \dot{z}_K + \frac{m}{4} \bar{z}_K \right) u_\alpha^{(-)K} \\ & + \varepsilon_{IJKL} \bar{\theta}^{(+)\alpha} \bar{\theta}^{(+)\beta} C^K u^{(+3)L} u_\alpha^{(-)I} u_\beta^{(-)J} e^{-3imt/2} \\ & + \varepsilon_{\alpha\beta\gamma} \bar{\theta}^{(+)\alpha} \bar{\theta}^{(+)\beta} \bar{\theta}^{(+)\gamma} \pi e^{-9imt/4} \\ & - 2 \bar{\theta}^{(+)\alpha} \bar{\theta}^{(+)\beta} \theta^{(+3)} \left(i \dot{\chi}_{IJ} + \frac{m}{2} \chi_{IJ} \right) u_\alpha^{(-)I} u_\beta^{(-)J} e^{-3imt/4} \\ & - \frac{2}{3} \varepsilon_{IJKL} \bar{\theta}^{(+)\alpha} \bar{\theta}^{(+)\beta} \bar{\theta}^{(+)\gamma} \theta^{(+3)} \left(i \dot{C}^L + \frac{3m}{4} C^L \right) u_\alpha^{(-)I} u_\beta^{(-)J} u_\gamma^{(-)K} e^{-3imt/2},\end{aligned}\tag{5.15}$$

where

$$\chi_{IJ} \equiv \chi_{[IJ]}.\tag{5.16}$$

Taking into account the transformation rule

$$\begin{aligned}\delta \mathcal{D}^{(+4)\alpha} = & - \left(\frac{1}{3} \Lambda^{(+4)\alpha} \mathcal{D}^0 + \Lambda^{(+4)\beta} \mathcal{D}_\beta^\alpha \right) - \frac{m}{6} \left(\bar{\epsilon}^{(+)\alpha} \theta^{(+3)} - \epsilon^{(+3)} \bar{\theta}^{(+)\alpha} \right) \mathcal{D}^0 \\ & + \frac{m^2}{6} \epsilon_\beta^{(-)} \bar{\theta}^{(+)\beta} \bar{\theta}^{(+)\alpha} \theta^{(+3)} \mathcal{D}^0,\end{aligned}\tag{5.17}$$

the superfield $\bar{V}^{(+3)}$ transforms as

$$\delta \bar{V}^{(+3)} = \Lambda \bar{V}^{(+3)} - \frac{m}{2} \left(\epsilon_\alpha^{(-)} \bar{\theta}^{(+)\alpha} + \bar{\epsilon}^{(-3)} \theta^{(+3)} \right) \bar{V}^{(+3)}.\tag{5.18}$$

This superfield transformation law amounts to the following component transformations

$$\begin{aligned}\delta \bar{z}_J = & -2 \bar{\epsilon}^K \chi_{JK} e^{-3imt/4} - \sqrt{2} \epsilon_J \chi e^{3imt/4}, \\ \delta \chi = & \sqrt{2} \bar{\epsilon}^K \left(i \dot{z}_K + \frac{3m}{4} \bar{z}_K \right) e^{-3imt/4}, \\ \delta \chi_{IJ} = & \varepsilon_{IJKL} \bar{\epsilon}^K C^L e^{-3imt/4} - 2 \epsilon_{[I} \left(i \dot{z}_{J]} + \frac{m}{4} \bar{z}_{J]} \right) e^{3imt/4}, \\ \delta C^I = & \varepsilon^{IJKL} \epsilon_J \left(i \dot{\chi}_{KL} - \frac{m}{2} \chi_{KL} \right) e^{3imt/4} - 3 \bar{\epsilon}^I \pi e^{-3imt/4}, \\ \delta \pi = & \frac{2}{3} \epsilon_K \left(i \dot{C}^K - \frac{3m}{4} C^K \right) e^{3imt/4}.\end{aligned}\tag{5.19}$$

From the transformation properties of $\bar{V}^{(+3)}$ one can draw the conclusion that the construction of a “pre-action” similar to (4.5) cannot be performed within the analytic harmonic superspace. We conjecture that such a construction could become possible after taking account of the additional set of constraints defining the multiplet $(\mathbf{8}, \mathbf{8}, \mathbf{0})$. Then the action can probably be constructed in the full harmonic superspace approach (see [19]).

At the component level, the rest of the constraints (5.1) impose the relations

$$\begin{aligned}\overline{(\bar{z}_I)} &= z^I, \quad \overline{(\chi)} = \bar{\chi}, \quad \overline{(\chi_{IJ})} = \frac{1}{2} \varepsilon^{IJKL} \chi_{KL} = \chi^{IJ}, \\ C^I &= i\dot{z}^I + \frac{m}{4} z^I, \quad \pi = -\frac{\sqrt{2}}{3} (i\dot{\bar{\chi}} + m\bar{\chi}).\end{aligned}\quad (5.20)$$

The final form of the deformed transformations is

$$\begin{aligned}\delta z^I &= 2\epsilon_K \chi^{IK} e^{3imt/4} + \sqrt{2} \bar{\epsilon}^I \bar{\chi} e^{-3imt/4}, \\ \delta \bar{z}_J &= -2\bar{\epsilon}^K \chi_{JK} e^{-3imt/4} - \sqrt{2} \epsilon_J \chi e^{3imt/4}, \\ \delta \chi &= \sqrt{2} \bar{\epsilon}^K \left(i\dot{z}_K + \frac{3m}{4} \bar{z}_K \right) e^{-3imt/4}, \\ \delta \bar{\chi} &= -\sqrt{2} \epsilon_K \left(i\dot{z}^K - \frac{3m}{4} z^K \right) e^{3imt/4}, \\ \delta \chi^{IJ} &= 2\bar{\epsilon}^{[I} \left(i\dot{z}^{J]} + \frac{m}{4} z^{J]} \right) e^{-3imt/4} - \varepsilon^{IJKL} \epsilon_K \left(i\dot{\bar{z}}_L - \frac{m}{4} \bar{z}_L \right) e^{3imt/4},\end{aligned}\quad (5.21)$$

where

$$\overline{(z^I)} = \bar{z}_I, \quad \overline{(\chi)} = \bar{\chi}, \quad \overline{(\chi^{IJ})} = \chi_{IJ} = \frac{1}{2} \varepsilon_{IJKL} \chi^{KL}. \quad (5.22)$$

5.2 SU(2|1) superfield formulation

Once again, we split the given multiplet into SU(2|1) multiplets as $(\mathbf{4}, \mathbf{4}, \mathbf{0}) \oplus (\mathbf{4}, \mathbf{4}, \mathbf{0})$.

The first multiplet is associated with the fields

$$x^{i1} := z^i, \quad x_i^2 := \bar{z}_i, \quad \xi^1 := 2\chi^{12}, \quad \bar{\xi}_1 := 2\chi_{12}, \quad \xi^2 := \sqrt{2}\chi, \quad \bar{\xi}_2 := \sqrt{2}\bar{\chi}, \quad (5.23)$$

such that

$$\begin{aligned}\delta x^{iA} &= -\epsilon^i \xi^A e^{3imt/4} - \bar{\epsilon}^i \bar{\xi}^A e^{-3imt/4}, \\ \delta \xi^1 &= 2\bar{\epsilon}^k \left(i\dot{x}_k^1 + \frac{m}{4} x_k^1 \right) e^{-3imt/4}, \quad \delta \bar{\xi}_1 = 2\epsilon_k \left(i\dot{x}_1^k - \frac{m}{4} x_1^k \right) e^{3imt/4}, \\ \delta \xi^2 &= 2\bar{\epsilon}^k \left(i\dot{x}_k^2 - \frac{3m}{4} x_k^2 \right) e^{-3imt/4}, \quad \delta \bar{\xi}_2 = 2\epsilon_k \left(i\dot{x}_2^k + \frac{3m}{4} x_2^k \right) e^{3imt/4}.\end{aligned}\quad (5.24)$$

This first multiplet $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ is accommodated by a superfield q^{iA} obeying the SU(2|1) covariant constraints

$$\mathcal{D}^{(k} q^{i)A} = 0, \quad \bar{\mathcal{D}}^{(k} q^{i)A} = 0, \quad \tilde{F} q^{iA} = -\frac{1}{2} (\sigma_3)_B^A q^{iB}, \quad \overline{(q^{iA})} = q_{iA}. \quad (5.25)$$

As distinct from (4.37), Pauli-Gürsey SU(2) symmetry is broken. Taking into account (2.16), we solve these constraints as

$$\begin{aligned}q^{iA} &= \left[1 + \frac{m}{2} \bar{\theta}^k \theta_k - \frac{5m^2}{16} (\bar{\theta})^2 (\theta)^2 \right] x^{iA} - i\varepsilon_{kl} \left(\bar{\theta}^i \theta^l + \bar{\theta}^l \theta^i \right) \left(\dot{x}^{kA} + \frac{im}{4} (\sigma_3)_B^A x^{kB} \right) \\ &\quad - i\bar{\theta}^k \theta_k \left(\theta^i \dot{\xi}^A e^{3imt/4} - \bar{\theta}^i \dot{\bar{\xi}}^A e^{-3imt/4} \right) \\ &\quad + \left(1 + \frac{m}{4} \bar{\theta}^k \theta_k \right) \left(\theta^i \xi^A e^{3imt/4} + \bar{\theta}^i \bar{\xi}^A e^{-3imt/4} \right) \\ &\quad + \frac{m}{4} \bar{\theta}^k \theta_k \left(\theta^i \xi^B e^{3imt/4} - \bar{\theta}^i \bar{\xi}^B e^{-3imt/4} \right) (\sigma_3)_B^A \\ &\quad + \frac{1}{4} (\bar{\theta})^2 (\theta)^2 \left(\dot{x}^{iA} + \frac{im}{2} (\sigma_3)_B^A \dot{x}^{iB} - \frac{m^2}{16} x^{iA} \right).\end{aligned}\quad (5.26)$$

The second (mirror) multiplet $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ is formed by the fields

$$y^1 := z^4, \quad y^2 := z^3, \quad \bar{y}_1 := \bar{z}_4, \quad \bar{y}_2 := \bar{z}_3, \quad \psi^{i1} := 2\chi^{i4}, \quad \psi^{i2} := 2\chi^{i3}, \quad (5.27)$$

with the $SU(2|1)$ transformations

$$\begin{aligned} \delta y^a &= -\epsilon_i \psi^{ia} e^{3imt/4}, & \delta \bar{y}^a &= -\bar{\epsilon}_i \psi^{ia} e^{-3imt/4}, \\ \delta \psi^{ia} &= 2\bar{\epsilon}^i \left(i\dot{y}^a + \frac{m}{4} y^a \right) e^{-3imt/4} - 2\epsilon^i \left(i\dot{\bar{y}}^a - \frac{m}{4} \bar{y}^a \right) e^{3imt/4}. \end{aligned} \quad (5.28)$$

The superfield $SU(2|1)$ constraints defining the mirror $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ multiplet are written as

$$\begin{aligned} \bar{\mathcal{D}}^i Y^a &= \mathcal{D}^i \bar{Y}^a = 0, & \mathcal{D}^i Y^a &= \bar{\mathcal{D}}^i \bar{Y}^a, \\ \tilde{F} Y^a &= \frac{1}{2} Y^a, & \tilde{F} \bar{Y}^a &= -\frac{1}{2} \bar{Y}^a, & \overline{(Y^a)} &= \bar{Y}_a. \end{aligned} \quad (5.29)$$

They are solved by

$$\begin{aligned} Y^a &= \left[1 + \frac{m}{4} \bar{\theta}^k \theta_k - \frac{7m^2}{64} (\bar{\theta})^2 (\theta)^2 \right] y^a + i\dot{y}^a \left[\bar{\theta}^k \theta_k - \frac{3m}{8} (\bar{\theta})^2 (\theta)^2 \right] - \frac{1}{4} (\bar{\theta})^2 (\theta)^2 \ddot{y}^a \\ &\quad + \theta_k \theta^k \left(i\dot{\bar{y}}^a - \frac{m}{4} \bar{y}^a \right) e^{3imt/2} + \left[\left(1 - \frac{m}{2} \bar{\theta}^k \theta_k \right) \theta_i \psi^{ia} + i\bar{\theta}^k \theta_k \bar{\theta}_i \psi^{ia} \right] e^{3imt/4}, \\ \bar{Y}^a &= \left[1 + \frac{m}{4} \bar{\theta}^k \theta_k - \frac{7m^2}{64} (\bar{\theta})^2 (\theta)^2 \right] \bar{y}^a - i\dot{\bar{y}}^a \left[\bar{\theta}^k \theta_k - \frac{3m}{8} (\bar{\theta})^2 (\theta)^2 \right] - \frac{1}{4} (\bar{\theta})^2 (\theta)^2 \ddot{\bar{y}}^a \\ &\quad + \bar{\theta}^k \bar{\theta}_k \left(i\dot{y}^a + \frac{m}{4} y^a \right) e^{-3imt/2} + \left[\left(1 - \frac{m}{2} \bar{\theta}^k \theta_k \right) \bar{\theta}_i \psi^{ia} - i\bar{\theta}^k \theta_k \bar{\theta}_i \psi^{ia} \right] e^{-3imt/4}. \end{aligned} \quad (5.30)$$

5.3 Invariant Lagrangian

The general $SU(2|1)$ invariant action is written as

$$S = \int dt \mathcal{L} = \frac{1}{2} \int dt d^2\theta d^2\bar{\theta} \left(1 + 2m \bar{\theta}^k \theta_k \right) f(Y^a \bar{Y}_a, q^{iA} q_{iA}). \quad (5.31)$$

Requiring it to be $SU(4)$ invariant produces the following conditions:

$$\begin{aligned} \Delta_y &= -2\epsilon^{ab} \partial_a \bar{\partial}_b, & \partial_a &= \partial/\partial y^a, & \bar{\partial}_b &= \partial/\partial \bar{y}^b, \\ \Delta_x &= \epsilon^{ij} \epsilon^{AB} \partial_{iA} \partial_{jB}, & \partial_{iA} &= \partial/\partial x^{iA}, \\ G := \Delta_y f &= -\Delta_x f \quad \Rightarrow \quad (\Delta_y + \Delta_x) G = 0, \end{aligned} \quad (5.32)$$

$$\begin{aligned} m(2\partial_a f + \bar{y}_a G + x^{iA} \partial_{iA} \partial_a f) &= 0 \quad \Rightarrow \quad m(\bar{y}_a \partial_{iA} - x_{iA} \partial_a) G = 0, \\ m(2\bar{\partial}_a f - y_a G + x^{iA} \partial_{iA} \bar{\partial}_a f) &= 0 \quad \Rightarrow \quad m(y_a \partial_{iA} + x_{iA} \bar{\partial}_a) G = 0. \end{aligned} \quad (5.33)$$

The unique solution of these equations is given by

$$\begin{aligned} f &= \frac{1}{4} (y^a \bar{y}_a)^{-1} \left(y^a \bar{y}_a + \frac{1}{2} x^{iA} x_{iA} \right)^{-1} + c_1 (y^a \bar{y}_a)^{-1} (x^{iA} x_{iA})^{-1} + c_2 (x^{iA} x_{iA})^{-1} \Rightarrow \\ \Rightarrow G &= \left(y^a \bar{y}_a + \frac{1}{2} x^{iA} x_{iA} \right)^{-3}. \end{aligned} \quad (5.34)$$

Here, the terms with the constants c_1 and c_2 do not affect the metric G , since it is a harmonic function. One can check that these terms drop out of the component Lagrangian, which is finally written as

$$\begin{aligned} \mathcal{L} = & \left[\dot{z}^I \dot{\bar{z}}_I + \frac{i}{2} \chi^{IJ} \dot{\chi}_{IJ} + \frac{i}{2} (\chi \dot{\bar{\chi}} - \dot{\chi} \bar{\chi}) - \frac{im}{4} (\dot{z}^I \bar{z}_I - z^I \dot{\bar{z}}_I) + \frac{m}{4} \chi \bar{\chi} - \frac{3m^2}{16} z^I \bar{z}_I \right] G \\ & + i (\dot{z}^I \partial_J G - \dot{\bar{z}}_J \bar{\partial}^I G) \chi_{IK} \chi^{JK} - \frac{m}{4} (z^I \partial_J G + \bar{z}_J \bar{\partial}^I G) \chi_{IK} \chi^{JK} \\ & + \frac{i}{2} (\dot{z}^I \partial_I G - \dot{\bar{z}}_I \bar{\partial}^I G) \chi \bar{\chi} + \partial_J \bar{\partial}^I G \chi_{IK} \chi^{JK} \chi \bar{\chi} + \frac{1}{3} \partial_J \bar{\partial}^I G \chi_{IK} \chi^{LK} \chi_{LM} \chi^{JM} \\ & - \frac{\sqrt{2}}{3} \partial_I \partial_J G \bar{\chi} \chi^{IK} \chi^{JL} \chi_{KL} - \frac{\sqrt{2}}{3} \bar{\partial}^I \bar{\partial}^J G \chi \chi_{IK} \chi_{JL} \chi^{KL}, \end{aligned} \quad (5.35)$$

where

$$G = (z^I \bar{z}_I)^{-3}. \quad (5.36)$$

5.4 Superconformal symmetry

By analogy with the section 4.6, one can redefine the component fields as

$$z^I \rightarrow z^I e^{-imt/4}, \quad \bar{z}_I \rightarrow \bar{z}_I e^{imt/4}, \quad \chi \rightarrow \chi e^{imt/2}, \quad \bar{\chi} \rightarrow \bar{\chi} e^{-imt/2}, \quad (5.37)$$

after which the Lagrangian (5.35) becomes an even function of m . As a result, we obtain $\text{OSp}(8|2)$ superconformal Lagrangian that is equivalent to (4.54):

$$\begin{aligned} \mathcal{L}_{\text{conf}} = & \left[\dot{z}^I \dot{\bar{z}}_I + \frac{i}{2} \chi^{IJ} \dot{\chi}_{IJ} + \frac{i}{2} (\chi \dot{\bar{\chi}} - \dot{\chi} \bar{\chi}) - \frac{m^2}{4} z^I \bar{z}_I \right] G + i (\dot{z}^I \partial_J G - \dot{\bar{z}}_J \bar{\partial}^I G) \chi_{IK} \chi^{JK} \\ & + \frac{i}{2} (\dot{z}^I \partial_I G - \dot{\bar{z}}_I \bar{\partial}^I G) \chi \bar{\chi} + \partial_J \bar{\partial}^I G \chi_{IK} \chi^{JK} \chi \bar{\chi} + \frac{1}{3} \partial_J \bar{\partial}^I G \chi_{IK} \chi^{LK} \chi_{LM} \chi^{JM} \\ & - \frac{\sqrt{2}}{3} \partial_I \partial_J G \bar{\chi} \chi^{IK} \chi^{JL} \chi_{KL} - \frac{\sqrt{2}}{3} \bar{\partial}^I \bar{\partial}^J G \chi \chi_{IK} \chi_{JL} \chi^{KL}. \end{aligned} \quad (5.38)$$

In the same way, this Lagrangian is invariant under two types of ϵ_I and η_I transformations which close on the superalgebra $\text{osp}(8|2)$ (D.1)–(D.3):

$$\begin{aligned} \delta z^I &= 2 \epsilon_K \chi^{IK} e^{imt} + \sqrt{2} \bar{\epsilon}^I \bar{\chi} e^{-imt}, & \delta \bar{z}_J &= -2 \bar{\epsilon}^K \chi_{JK} e^{-imt} - \sqrt{2} \epsilon_J \chi e^{imt}, \\ \delta \chi &= \sqrt{2} \bar{\epsilon}^K \left(i \dot{\bar{z}}_K + \frac{m}{2} \bar{z}_K \right) e^{-imt}, & \delta \bar{\chi} &= -\sqrt{2} \epsilon_K \left(i \dot{z}^K - \frac{m}{2} z^K \right) e^{imt}, \\ \delta \chi^{IJ} &= 2 \bar{\epsilon}^{[I} \left(i \dot{z}^{J]} + \frac{m}{2} z^{J]} \right) e^{-imt} - \epsilon^{IJKL} \epsilon_K \left(i \dot{\bar{z}}_L - \frac{m}{2} \bar{z}_L \right) e^{imt}, \end{aligned} \quad (5.39)$$

$$\begin{aligned} \delta z^I &= 2 \eta_K \chi^{IK} e^{-imt} + \sqrt{2} \bar{\eta}^I \bar{\chi} e^{imt}, & \delta \bar{z}_J &= -2 \bar{\eta}^K \chi_{JK} e^{imt} - \sqrt{2} \eta_J \chi e^{-imt}, \\ \delta \chi &= \sqrt{2} \bar{\eta}^K \left(i \dot{\bar{z}}_K - \frac{m}{2} \bar{z}_K \right) e^{imt}, & \delta \bar{\chi} &= -\sqrt{2} \eta_K \left(i \dot{z}^K + \frac{m}{2} z^K \right) e^{-imt}, \\ \delta \chi^{IJ} &= 2 \bar{\eta}^{[I} \left(i \dot{\bar{z}}^{J]} - \frac{m}{2} \bar{z}^{J]} \right) e^{imt} - \epsilon^{IJKL} \eta_K \left(i \dot{z}_L + \frac{m}{2} z_L \right) e^{-imt}. \end{aligned} \quad (5.40)$$

We see that the Lagrangians (4.54) and (5.38) have conformally flat metrics g_3 and G which both depend on the quadratic $\text{SO}(8)$ invariants of the same power -3 . The fields z^I and $\bar{z}_J$ can be reexpressed, by a linear transformation, through the bosonic fields $y^{I'J'}$, ϕ and $\bar{\phi}$ of the first multiplet $(\mathbf{8}, \mathbf{8}, \mathbf{0})$, where I' and J' label the fundamental representation of a different $\text{SU}(4)'$ subgroup of the $\text{SO}(8)$ symmetry, such that it intersects with the first $\text{SU}(4)$ in a common $\text{SU}(3)$ subgroup. After an analogous linear transformation of the fermionic fields, the Lagrangian (5.38) will coincide with (4.54). So both superconformal Lagrangians are indeed equivalent. This feature of equivalence of $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ multiplets in the presence of exact $\text{SO}(8)$ symmetry was already noted in the end of section 3.

6 Summary and outlook

We have shown the existence of two non-equivalent “root” multiplets $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ of deformed $\mathcal{N}=8$ supersymmetry associated with the supergroup $\text{SU}(4|1)$. We described them in multiple ways with worldline superfields and in components and derived invariant actions for them. Some of these actions are superconformally $\text{OSp}(8|2)$ invariant. For a non-trivially interacting example we gave the explicit form of the (classical) $\text{SU}(4|1)$ supercharges. We also obtained the $\text{SU}(4|1)$ invariant actions for the off-shell multiplets $(\mathbf{6}, \mathbf{8}, \mathbf{2})$ and $(\mathbf{7}, \mathbf{8}, \mathbf{1})$ (in appendices B and C) from the $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ actions, reconfirming the root interpretation of the $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ multiplets for $\mathcal{N}=8$ mechanics. The $(\mathbf{6}, \mathbf{8}, \mathbf{2})$ action was shown to exhibit superconformal $\text{SU}(4|1, 1)$ invariance.

As for further applications of these results, the most appropriate arena might be provided by supersymmetric matrix models (see, e.g., [35–37]). These possess $\text{SU}(4|2)$ invariance, hence multi-particle mechanics based on $\text{SU}(2|2) \subset \text{SU}(4|2)$ or $\text{SU}(4|1) \subset \text{SU}(4|2)$ may appear as some truncation of such matrix models. The matrix models studied so far lead to free worldline multiplets and actions. Our approach allows one to generate non-trivial interactions, which hopefully may be interpreted as effective actions with quantum corrections taken into account. An important ingredient of matrix models is a gauging of appropriate isometries by non-propagating gauge multiplets. To promote this to the $\text{SU}(4|1)$ superfield language, one needs to define suitable gauge superfields generalizing those used in [38, 39] or [10].

Another problem for the future is finding an action including both types of deformed $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ multiplets and inquiring the ensuing target-space geometry.

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A Some calculations

Here we collect the necessary identities for calculation of the function $L^{(+4)}$ in the invariant action (4.32). We represent it as an infinite series:

$$L^{(+4)} = \sum_{n=0}^{\infty} a_n \left(c^{(-2)}\right)^n \left(\hat{Y}^{(+2)}\right)^{n+2} + m^2 \left(\theta^{(+)}\right)^4 \sum_{n=1}^{\infty} b_n \left(c^{(-2)} \hat{Y}^{(+2)}\right)^n. \quad (\text{A.1})$$

All the identities below are given up to terms with a total harmonic derivative $\mathcal{D}_a^{(+2)i}$. In addition, one must take into account the definitions (4.31). Each term in the variation of the series (A.1) also contains transformations compensating the measure transformations (4.19), i.e.

$$\begin{aligned} \delta \left(\hat{Y}^{(+2)}\right)^2 &= 4\Lambda^{(0)} \left(\hat{Y}^{(+2)}\right)^2 + 2\Lambda^{(0)} c^{(+2)} \hat{Y}^{(+2)} + 2\varepsilon^{ab} \varepsilon_{ij} \Lambda_a^{(+2)i} \mathcal{D}_b^{(+2)j} c^{(-2)} \hat{Y}^{(+2)} \\ &\quad - \frac{1}{2} \varepsilon^{ab} \varepsilon_{ij} c^{(-2)} \hat{Y}^{(+2)} \left(\mathcal{D}_b^{(+2)j} \Lambda_a^{(+2)i}\right), \end{aligned} \quad (\text{A.2})$$

$$\begin{aligned} \delta \left(c^{(-2)}\right)^n \left(\hat{Y}^{(+2)}\right)^{n+2} &= (n+4) \Lambda^{(0)} \left(c^{(-2)}\right)^n \left(\hat{Y}^{(+2)}\right)^{n+2} \\ &\quad + (n+2) \Lambda^{(0)} c^{(+2)} \left(c^{(-2)}\right)^n \left(\hat{Y}^{(+2)}\right)^{n+1} \\ &\quad + \frac{(n+2)}{(n+1)} \varepsilon^{ab} \varepsilon_{ij} \Lambda_a^{(+2)i} \mathcal{D}_b^{(+2)j} \left(c^{(-2)}\right)^{n+1} \left(\hat{Y}^{(+2)}\right)^{n+1} \\ &\quad - \frac{(n+2)}{4} \varepsilon^{ab} \varepsilon_{ij} \left(\mathcal{D}_b^{(+2)j} \Lambda_a^{(+2)i}\right) \left(c^{(-2)}\right)^{n+1} \left(\hat{Y}^{(+2)}\right)^{n+1} \\ &= (n+4) \Lambda^{(0)} \left(c^{(-2)}\right)^n \left(\hat{Y}^{(+2)}\right)^{n+2} \\ &\quad + n \Lambda^{(0)} \left(c^{(-2)}\right)^{n-1} \left(\hat{Y}^{(+2)}\right)^{n+1} \\ &\quad - \frac{(n+3)(n+4)}{4(n+1)} \varepsilon^{ab} \varepsilon_{ij} \left(\mathcal{D}_b^{(+2)j} \Lambda_a^{(+2)i}\right) \left(c^{(-2)} \hat{Y}^{(+2)}\right)^{n+1} \end{aligned} \quad (\text{A.3})$$

where

$$\begin{aligned} \Lambda^{(0)} \left(c^{(-2)}\right)^n c^{(+2)} &= -\frac{1}{4} \Lambda^{(0)} \left(c^{(-2)}\right)^n \left(\varepsilon^{ab} \varepsilon_{ij} \mathcal{D}_a^{(+2)i} \mathcal{D}_b^{(+2)j} c^{(-2)}\right) \\ &= \frac{1}{4} \Lambda_a^{(+2)i} \left(c^{(-2)}\right)^n \left(\varepsilon^{ab} \varepsilon_{ij} \mathcal{D}_b^{(+2)j} c^{(-2)}\right) \\ &\quad + \frac{n}{4} \Lambda^{(0)} \left(c^{(-2)}\right)^{n-1} \left(\varepsilon^{ab} \varepsilon_{ij} \mathcal{D}_b^{(+2)j} c^{(-2)}\right) \left(\mathcal{D}_a^{(+2)i} c^{(-2)}\right) \\ &= \frac{1}{4(n+1)} \Lambda_a^{(+2)i} \varepsilon^{ab} \varepsilon_{ij} \mathcal{D}_b^{(+2)j} \left(c^{(-2)}\right)^{n+1} \\ &\quad + \frac{n}{4} \Lambda^{(0)} \left(c^{(-2)}\right)^{n-1} \left(\varepsilon^{ab} \varepsilon_{ij} \mathcal{D}_b^{(+2)j} c^{(-2)}\right) \left(\mathcal{D}_a^{(+2)i} c^{(-2)}\right). \end{aligned} \quad (\text{A.4})$$

We also use the identity $(c^{IJ} c_{IJ} = 4)$

$$\Lambda^{(0)} \left[c^{(-2)} c^{(+2)} + \frac{1}{2} \varepsilon^{ab} \varepsilon_{ij} \left(\mathcal{D}_b^{(+2)j} c^{(-2)}\right) \left(\mathcal{D}_a^{(+2)i} c^{(-2)}\right) \right] = \frac{1}{4} c^{IJ} c_{IJ} \Lambda^{(0)} = \Lambda^{(0)}. \quad (\text{A.5})$$

Thus, the series is invariant when

$$\begin{aligned} a_0 &= 1, & a_1 &= -4a_0, & a_2 &= \frac{a_0 5!}{2! 3!}, & a_n &= \frac{(-1)^n (n+3)!}{n! 3!}, \\ b_n &= \frac{(n+2)(n+3)a_{n-1}}{4n} = -\frac{(-1)^n (n+2)(n+3)!}{n! 4!}. \end{aligned} \quad (\text{A.6})$$

Using these relations, the above series can be summed up to the expression

$$L^{(+4)} = \frac{\hat{Y}^{(+2)} \hat{Y}^{(+2)}}{\left(1 + c^{(-2)} \hat{Y}^{(+2)}\right)^4} + \frac{m^2}{2} \left(\theta^{(+)}\right)^4 \left[1 - \frac{1 - c^{(-2)} \hat{Y}^{(+2)}}{\left(1 + c^{(-2)} \hat{Y}^{(+2)}\right)^5}\right]. \quad (\text{A.7})$$

For calculating the component Lagrangians, we use the following identities (up to total harmonic derivatives):

$$\begin{aligned} c^{(-2)} \hat{y}^{(+2)} &= \frac{1}{12} c^{IJ} \hat{y}_{IJ}, \\ \left(c^{(-2)} \hat{y}^{(+2)}\right)^2 &= \frac{1}{80} \left[\left(c^{IJ} \hat{y}_{IJ}\right)^2 - \frac{2}{3} \hat{y}^{IJ} \hat{y}_{IJ} \right], \\ \left(c^{(-2)} \hat{y}^{(+2)}\right)^3 &= \frac{1}{400} \left[\left(c^{IJ} \hat{y}_{IJ}\right)^3 - \frac{3}{2} \hat{y}^{KL} \hat{y}_{KL} c^{IJ} y_{IJ} \right], \\ \left(c^{(-2)} \hat{y}^{(+2)}\right)^{2n} &= \frac{12 (2n)! (2n)!}{2^{2n} (2n+2)! (2n+3)!} \\ &\quad \times \sum_{k=0}^n \frac{(-1)^k (2n-k+1)!}{k! (2n-2k)!} \left(\hat{y}^{KL} \hat{y}_{KL}\right)^k \left(c^{IJ} \hat{y}_{IJ}\right)^{2n-2k}, \\ \left(c^{(-2)} \hat{y}^{(+2)}\right)^{2n+1} &= \frac{12 (2n+1)! (2n+1)!}{2^{2n+1} (2n+3)! (2n+4)!} \\ &\quad \times \sum_{k=0}^n \frac{(-1)^k (2n-k+2)!}{k! (2n-2k+1)!} \left(\hat{y}^{KL} \hat{y}_{KL}\right)^k \left(c^{IJ} \hat{y}_{IJ}\right)^{2n-2k+1}. \end{aligned} \quad (\text{A.8})$$

Using them, one obtains

$$\begin{aligned} \frac{\partial^2}{\partial \hat{y}^{(+2)} \partial \hat{y}^{(+2)}} \left[\frac{\hat{y}^{(+2)} \hat{y}^{(+2)}}{\left(1 + c^{(-2)} \hat{y}^{(+2)}\right)^4} \right] &= \sum_{n=0}^{\infty} (n+1)(n+2) a_n \left(c^{(-2)} \hat{y}^{(+2)}\right)^n \\ &= 2 \left(1 + \frac{c^{IJ} \hat{y}_{IJ}}{2} + \frac{\hat{y}^{IJ} \hat{y}_{IJ}}{4}\right)^{-2} \\ &= 8 \left[\frac{1}{2} y^{IJ} y_{IJ}\right]^{-2}, \\ \sum_{n=1}^{\infty} b_n \left(c^{(-2)} \hat{Y}^{(+2)}\right)^n &= -\frac{1}{2} \left(1 + \frac{c^{IJ} \hat{y}_{IJ}}{2} + \frac{\hat{y}^{IJ} \hat{y}_{IJ}}{4}\right)^{-1} \\ &= -\left[\frac{1}{2} y^{IJ} y_{IJ}\right]^{-1}. \end{aligned} \quad (\text{A.9})$$

B The multiplet (6, 8, 2)

As was already mentioned, the multiplets (6, 8, 2) and (8, 8, 0) are related to each other by the substitution $i\dot{\phi} = D$. The same substitution is admissible in the Lagrangian (4.50).

Then, the $SU(4|1)$ invariant action (4.32) is given by

$$\begin{aligned}\mathcal{L}_{(6,8,2)} = g_2 \left[\frac{1}{2} \dot{y}^{IJ} \dot{y}_{IJ} + \frac{i}{2} (\chi^K \dot{\bar{\chi}}_K - \dot{\chi}^K \bar{\chi}_K) + D\bar{D} - \frac{m}{4} \chi^K \bar{\chi}_K - \frac{m^2}{8} y^{IJ} y_{IJ} \right] \\ + \frac{1}{\sqrt{2}} \bar{D} \partial_{IJ} g_2 \chi^I \chi^J - \frac{1}{\sqrt{2}} D \partial^{IJ} g_2 \bar{\chi}_I \bar{\chi}_J + i (\dot{y}_{IK} \partial^{JK} g_2 - \dot{y}^{JK} \partial_{IK} g_2) \chi^I \bar{\chi}_J \\ - \frac{1}{2} \partial_{IJ} \partial^{KL} g_2 \chi^I \chi^J \bar{\chi}_K \bar{\chi}_L, \quad g_2 = \left[\frac{1}{2} y^{IJ} y_{IJ} \right]^{-2}.\end{aligned}\quad (\text{B.1})$$

One can obtain a superconformal Lagrangian by making the following substitutions of the component fields in (B.1)

$$D \rightarrow D e^{imt/2}, \quad \bar{D} \rightarrow \bar{D} e^{-imt/2}, \quad \chi^I \rightarrow \chi^I e^{imt/4}, \quad \bar{\chi}_I \rightarrow \bar{\chi}_I e^{-imt/4}. \quad (\text{B.2})$$

The resulting Lagrangian

$$\begin{aligned}\mathcal{L}_{\text{conf}} = g_2 \left[\frac{1}{2} \dot{y}^{IJ} \dot{y}_{IJ} + \frac{i}{2} (\chi^K \dot{\bar{\chi}}_K - \dot{\chi}^K \bar{\chi}_K) + D\bar{D} - \frac{m^2}{8} y^{IJ} y_{IJ} \right] \\ + \frac{1}{\sqrt{2}} \bar{D} \partial_{IJ} g_2 \chi^I \chi^J - \frac{1}{\sqrt{2}} D \partial^{IJ} g_2 \bar{\chi}_I \bar{\chi}_J + i (\dot{y}_{IK} \partial^{JK} g_2 - \dot{y}^{JK} \partial_{IK} g_2) \chi^I \bar{\chi}_J \\ - \frac{1}{2} \partial_{IJ} \partial^{KL} g_2 \chi^I \chi^J \bar{\chi}_K \bar{\chi}_L\end{aligned}\quad (\text{B.3})$$

is an even function of m and so is superconformal. Its form is typical for Lagrangians with the trigonometric realizations of superconformal groups.

It can be shown that the superconformal group of (B.3) is $SU(4|1,1)$. On the one hand, the transformations (4.30) become

$$\begin{aligned}\delta D &= -\sqrt{2} \epsilon_I (i \dot{\chi}^I - m \chi^I) e^{imt/2}, & \delta \bar{D} &= -\sqrt{2} \bar{\epsilon}^I (i \dot{\bar{\chi}}_I + m \bar{\chi}_I) e^{-imt/2}, \\ \delta y^{IJ} &= -2 \bar{\epsilon}^I \chi^J e^{-imt/2} + \epsilon^{IJKL} \epsilon_K \bar{\chi}_L e^{imt/2}, \\ \delta \chi^I &= \sqrt{2} \bar{\epsilon}^I B e^{-imt/2} - 2 \epsilon_J \left(i \dot{y}^{IJ} - \frac{m}{2} y^{IJ} \right) e^{imt/2}, \\ \delta \bar{\chi}_I &= \sqrt{2} \epsilon_I \bar{B} e^{imt/2} + 2 \bar{\epsilon}^J \left(i \dot{y}_{IJ} + \frac{m}{2} y_{IJ} \right) e^{-imt/2},\end{aligned}\quad (\text{B.4})$$

and leave the Lagrangian (B.3) invariant. On the other hand, new $SU(4|1)$ transformations leaving invariant (B.3) are defined by replacing $m \rightarrow -m$ in (B.4):

$$\begin{aligned}\delta D &= -\sqrt{2} \eta_I (i \dot{\chi}^I + m \chi^I) e^{-imt/2}, & \delta \bar{D} &= -\sqrt{2} \bar{\eta}^I (i \dot{\bar{\chi}}_I - m \bar{\chi}_I) e^{imt/2}, \\ \delta y^{IJ} &= -2 \bar{\eta}^I \chi^J e^{imt/2} + \epsilon^{IJKL} \eta_K \bar{\chi}_L e^{-imt/2}, \\ \delta \chi^I &= \sqrt{2} \bar{\eta}^I B e^{imt/2} - 2 \eta_J \left(i \dot{y}^{IJ} + \frac{m}{2} y^{IJ} \right) e^{-imt/2}, \\ \delta \bar{\chi}_I &= \sqrt{2} \eta_I \bar{B} e^{-imt/2} + 2 \bar{\eta}^J \left(i \dot{y}_{IJ} - \frac{m}{2} y_{IJ} \right) e^{imt/2}.\end{aligned}\quad (\text{B.5})$$

Introducing the conformal Hamiltonian as

$$\mathcal{H}'_{\text{conf}} = \mathcal{H} + \frac{m}{2} F, \quad (\text{B.6})$$

the superconformal algebra $su(4|1,1)$ amounts to the following set of (anti)commutators

$$\begin{aligned} \{Q^I, \bar{Q}_J\} &= 2\delta_J^I \mathcal{H}'_{\text{conf}} + 2m L_J^I - m \delta_J^I F, \\ \{S^I, \bar{S}_J\} &= 2\delta_J^I \mathcal{H}'_{\text{conf}} - 2m L_J^I + m \delta_J^I F, \\ \{Q^I, \bar{S}_J\} &= 2\delta_J^I T', \quad \{S^I, \bar{Q}_J\} = 2\delta_J^I \bar{T}', \end{aligned} \quad (\text{B.7})$$

$$\begin{aligned} [L_J^I, L_L^K] &= \delta_J^K L_L^I - \delta_L^K L_J^I, \\ [T', \bar{T}'] &= 2m \mathcal{H}'_{\text{conf}}, \quad [\mathcal{H}'_{\text{conf}}, T'] = -m T', \quad [\mathcal{H}'_{\text{conf}}, \bar{T}'] = m \bar{T}', \end{aligned} \quad (\text{B.8})$$

$$\begin{aligned} [L_J^I, Q^K] &= \delta_J^K Q^I - \frac{1}{4} \delta_J^I Q^K, \quad [L_J^I, \bar{Q}_L] = \frac{1}{4} \delta_J^I \bar{Q}_L - \delta_L^I \bar{Q}_J, \\ [L_J^I, S^K] &= \delta_J^K S^I - \frac{1}{4} \delta_J^I S^K, \quad [L_J^I, \bar{S}_L] = \frac{1}{4} \delta_J^I \bar{S}_L - \delta_L^I \bar{S}_J, \\ [F, Q^I] &= \frac{1}{2} Q^I, \quad [F, \bar{Q}_J] = -\frac{1}{2} \bar{Q}_J, \quad [F, S^I] = \frac{1}{2} S^I, \quad [F, \bar{S}_J] = -\frac{1}{2} \bar{S}_J, \\ [\mathcal{H}'_{\text{conf}}, Q^I] &= -\frac{m}{2} Q^I, \quad [\mathcal{H}'_{\text{conf}}, \bar{Q}_J] = \frac{m}{2} \bar{Q}_J, \\ [\mathcal{H}'_{\text{conf}}, S^I] &= \frac{m}{2} S^I, \quad [\mathcal{H}'_{\text{conf}}, \bar{S}_J] = -\frac{m}{2} \bar{S}_J, \\ [T', \bar{Q}_J] &= m \bar{S}_J, \quad [T', S^I] = m Q^I, \\ [\bar{T}', \bar{S}_J] &= -m \bar{Q}_J, \quad [\bar{T}', Q^I] = -m S^I. \end{aligned} \quad (\text{B.9})$$

Note that the “parabolic” realization of the superconformal group $SU(4|1,1)$ on the undeformed multiplet $(\mathbf{6}, \mathbf{8}, \mathbf{2})$ was given in [40]. The corresponding superconformal Lagrangian can be obtained as the $m=0$ limit of (B.3).

C The multiplet $(\mathbf{7}, \mathbf{8}, \mathbf{1})$

The substitution $\sqrt{2} i \dot{\phi} = C - i \dot{x}$ in (4.7) gives $SU(4|1)$ transformations of the multiplet $(\mathbf{7}, \mathbf{8}, \mathbf{1})$:

$$\begin{aligned} \delta C &= -\epsilon_I \left(i \dot{\chi}^I - \frac{3m}{4} \chi^I \right) e^{3imt/4} - \bar{\epsilon}^I \left(i \dot{\bar{\chi}}_I + \frac{3m}{4} \bar{\chi}_I \right) e^{-3imt/4}, \\ \delta x &= \epsilon_I \chi^I e^{3imt/4} - \bar{\epsilon}^I \bar{\chi}_I e^{-3imt/4}, \\ \delta y^{IJ} &= -2 \bar{\epsilon}^{[I} \chi^{J]} e^{-3imt/4} + \epsilon^{IJKL} \epsilon_K \bar{\chi}_L e^{3imt/4}, \\ \delta \chi^I &= \bar{\epsilon}^I (C - i \dot{x}) e^{-3imt/4} - 2 \epsilon_J \left(i \dot{y}^{IJ} - \frac{m}{2} y^{IJ} \right) e^{3imt/4}, \\ \delta \bar{\chi}_I &= \epsilon_I (C + i \dot{x}) e^{3imt/4} + 2 \bar{\epsilon}^J \left(i \dot{y}_{IJ} + \frac{m}{2} y_{IJ} \right) e^{-3imt/4}. \end{aligned} \quad (\text{C.1})$$

The same substitution is admissible in the Lagrangian (4.50). In this way we obtain the $SU(4|1)$ invariant Lagrangian for the multiplet $(\mathbf{7}, \mathbf{8}, \mathbf{1})$:

$$\begin{aligned} \mathcal{L}_{(\mathbf{7}, \mathbf{8}, \mathbf{1})} &= g_2 \left[\frac{\dot{x}^2}{2} + \frac{1}{2} \dot{y}^{IJ} \dot{y}_{IJ} + \frac{i}{2} (\chi^K \dot{\bar{\chi}}_K - \dot{\chi}^K \bar{\chi}_K) + \frac{C^2}{2} - \frac{m}{4} \chi^K \bar{\chi}_K - \frac{m^2}{8} y^{IJ} y_{IJ} \right] \\ &\quad + \frac{C}{2} (\partial_{IJ} g_2 \chi^I \chi^J - \partial^{IJ} g_2 \bar{\chi}_I \bar{\chi}_J) + \frac{i \dot{x}}{2} (\partial_{IJ} g_2 \chi^I \chi^J + \partial^{IJ} g_2 \bar{\chi}_I \bar{\chi}_J) \\ &\quad + i (\dot{y}_{IK} \partial^{JK} g_2 - \dot{y}^{JK} \partial_{IK} g_2) \chi^I \bar{\chi}_J - \frac{1}{2} \partial_{IJ} \partial^{KL} g_2 \chi^I \chi^J \bar{\chi}_K \bar{\chi}_L, \\ g_2 &= \left[\frac{1}{2} y^{IJ} y_{IJ} \right]^{-2}. \end{aligned} \quad (\text{C.2})$$

The corresponding action is not invariant with respect to the superconformal group $F(4)$ inherent to the multiplet $(\mathbf{7}, \mathbf{8}, \mathbf{1})$ [41] because (C.2) cannot be brought to a form in which it depends only on m^2 . On the other hand, since $F(4)$ includes $SU(4|1)$ as a subgroup, we expect the existence of an alternative, $F(4)$ superconformal, action for the $SU(4|1)$ multiplet $(\mathbf{7}, \mathbf{8}, \mathbf{1})$. At the component level, such a system has recently been constructed, without giving an action however [42].

D Superconformal algebra $osp(8|2)$

Superconformal algebra $osp(8|2)$ is given by the following non-vanishing (anti)commutators:

$$\begin{aligned}
 \{Q^I, \bar{Q}_J\} &= 2\delta_J^I \mathcal{H}_{\text{conf}} + 2m L_J^I + m \delta_J^I F, \\
 \{S^I, \bar{S}_J\} &= 2\delta_J^I \mathcal{H}_{\text{conf}} - 2m L_J^I - m \delta_J^I F, \\
 \{Q^I, S^J\} &= 2m F^{IJ}, & \{\bar{Q}_I, \bar{S}_J\} &= 2m \bar{F}_{IJ}, \\
 \{Q^I, \bar{S}_J\} &= 2\delta_J^I T, & \{S^I, \bar{Q}_J\} &= 2\delta_J^I \bar{T}, \\
 [L_J^I, L_L^K] &= \delta_J^K L_L^I - \delta_L^K L_J^I, \\
 [L_J^I, F^{KL}] &= \delta_J^K F^{IL} + \delta_J^L F^{KI} - \frac{1}{2} \delta_J^I F^{KL}, \\
 [L_J^I, \bar{F}_{KL}] &= \frac{1}{2} \delta_J^K \bar{F}_{KL} - \delta_K^I \bar{F}_{JL} - \delta_L^I \bar{F}_{KJ}, \\
 [F^{IJ}, \bar{F}_{KL}] &= \delta_K^I L_L^J - \delta_L^I L_K^J + \delta_L^J L_K^I - \delta_K^J L_L^I + (\delta_K^I \delta_L^J - \delta_L^I \delta_K^J) F, \\
 [F, F^{IJ}] &= F^{IJ}, & [F, \bar{F}_{IJ}] &= -\bar{F}_{IJ}, \\
 [T, \bar{T}] &= 4m \mathcal{H}_{\text{conf}}, & [\mathcal{H}_{\text{conf}}, T] &= -2m T, & [\mathcal{H}_{\text{conf}}, \bar{T}] &= 2m \bar{T}, \\
 [L_J^I, Q^K] &= \delta_J^K Q^I - \frac{1}{4} \delta_J^I Q^K, & [L_J^I, \bar{Q}_L] &= \frac{1}{4} \delta_J^I \bar{Q}_L - \delta_L^I \bar{Q}_J, \\
 [L_J^I, S^K] &= \delta_J^K S^I - \frac{1}{4} \delta_J^I S^K, & [L_J^I, \bar{S}_L] &= \frac{1}{4} \delta_J^I \bar{S}_L - \delta_L^I \bar{S}_J, \\
 [\bar{F}_{IJ}, Q^K] &= \delta_I^K \bar{S}_J - \delta_J^K \bar{S}_I, & [F^{IJ}, \bar{Q}_L] &= \delta_L^J S^I - \delta_L^I S^J, \\
 [\bar{F}_{IJ}, S^K] &= \delta_I^K \bar{Q}_J - \delta_J^K \bar{Q}_I, & [F^{IJ}, \bar{S}_L] &= \delta_L^J Q^I - \delta_L^I Q^J, \\
 [F, Q^I] &= \frac{1}{2} Q^I, & [F, \bar{Q}_J] &= -\frac{1}{2} \bar{Q}_J, \\
 [F, S^I] &= \frac{1}{2} S^I, & [F, \bar{S}_J] &= -\frac{1}{2} \bar{S}_J, \\
 [\mathcal{H}_{\text{conf}}, Q^I] &= -m Q^I, & [\mathcal{H}_{\text{conf}}, \bar{Q}_J] &= m \bar{Q}_J, \\
 [\mathcal{H}_{\text{conf}}, S^I] &= m S^I, & [\mathcal{H}_{\text{conf}}, \bar{S}_J] &= -m \bar{S}_J, \\
 [T, \bar{Q}_J] &= 2m \bar{S}_J, & [T, S^I] &= 2m Q^I, \\
 [\bar{T}, \bar{S}_J] &= -2m \bar{Q}_J, & [\bar{T}, Q^I] &= -2m S^I.
 \end{aligned} \tag{D.1}$$

$$\begin{aligned}
 [L_J^I, Q^K] &= \delta_J^K Q^I - \frac{1}{4} \delta_J^I Q^K, & [L_J^I, \bar{Q}_L] &= \frac{1}{4} \delta_J^I \bar{Q}_L - \delta_L^I \bar{Q}_J, \\
 [L_J^I, S^K] &= \delta_J^K S^I - \frac{1}{4} \delta_J^I S^K, & [L_J^I, \bar{S}_L] &= \frac{1}{4} \delta_J^I \bar{S}_L - \delta_L^I \bar{S}_J, \\
 [\bar{F}_{IJ}, Q^K] &= \delta_I^K \bar{S}_J - \delta_J^K \bar{S}_I, & [F^{IJ}, \bar{Q}_L] &= \delta_L^J S^I - \delta_L^I S^J, \\
 [\bar{F}_{IJ}, S^K] &= \delta_I^K \bar{Q}_J - \delta_J^K \bar{Q}_I, & [F^{IJ}, \bar{S}_L] &= \delta_L^J Q^I - \delta_L^I Q^J, \\
 [F, Q^I] &= \frac{1}{2} Q^I, & [F, \bar{Q}_J] &= -\frac{1}{2} \bar{Q}_J, \\
 [F, S^I] &= \frac{1}{2} S^I, & [F, \bar{S}_J] &= -\frac{1}{2} \bar{S}_J, \\
 [\mathcal{H}_{\text{conf}}, Q^I] &= -m Q^I, & [\mathcal{H}_{\text{conf}}, \bar{Q}_J] &= m \bar{Q}_J, \\
 [\mathcal{H}_{\text{conf}}, S^I] &= m S^I, & [\mathcal{H}_{\text{conf}}, \bar{S}_J] &= -m \bar{S}_J, \\
 [T, \bar{Q}_J] &= 2m \bar{S}_J, & [T, S^I] &= 2m Q^I, \\
 [\bar{T}, \bar{S}_J] &= -2m \bar{Q}_J, & [\bar{T}, Q^I] &= -2m S^I.
 \end{aligned} \tag{D.2}$$

$$\begin{aligned}
 [L_J^I, Q^K] &= \delta_J^K Q^I - \frac{1}{4} \delta_J^I Q^K, & [L_J^I, \bar{Q}_L] &= \frac{1}{4} \delta_J^I \bar{Q}_L - \delta_L^I \bar{Q}_J, \\
 [L_J^I, S^K] &= \delta_J^K S^I - \frac{1}{4} \delta_J^I S^K, & [L_J^I, \bar{S}_L] &= \frac{1}{4} \delta_J^I \bar{S}_L - \delta_L^I \bar{S}_J, \\
 [\bar{F}_{IJ}, Q^K] &= \delta_I^K \bar{S}_J - \delta_J^K \bar{S}_I, & [F^{IJ}, \bar{Q}_L] &= \delta_L^J S^I - \delta_L^I S^J, \\
 [\bar{F}_{IJ}, S^K] &= \delta_I^K \bar{Q}_J - \delta_J^K \bar{Q}_I, & [F^{IJ}, \bar{S}_L] &= \delta_L^J Q^I - \delta_L^I Q^J, \\
 [F, Q^I] &= \frac{1}{2} Q^I, & [F, \bar{Q}_J] &= -\frac{1}{2} \bar{Q}_J, \\
 [F, S^I] &= \frac{1}{2} S^I, & [F, \bar{S}_J] &= -\frac{1}{2} \bar{S}_J, \\
 [\mathcal{H}_{\text{conf}}, Q^I] &= -m Q^I, & [\mathcal{H}_{\text{conf}}, \bar{Q}_J] &= m \bar{Q}_J, \\
 [\mathcal{H}_{\text{conf}}, S^I] &= m S^I, & [\mathcal{H}_{\text{conf}}, \bar{S}_J] &= -m \bar{S}_J, \\
 [T, \bar{Q}_J] &= 2m \bar{S}_J, & [T, S^I] &= 2m Q^I, \\
 [\bar{T}, \bar{S}_J] &= -2m \bar{Q}_J, & [\bar{T}, Q^I] &= -2m S^I.
 \end{aligned} \tag{D.3}$$

The supercharges $Q^I, \bar{Q}_J$ together with the generators L_J^I and $H = \mathcal{H}_{\text{conf}} + \frac{m}{2} F$ form the subalgebra $su(4|1) \oplus u(1)$ in $osp(8|2)$, with F being an additional external R -symmetry $U(1)$ generator. The second set of $SU(4|1)$ supercharges $S^I, \bar{S}_J$ extends this subalgebra to the full superconformal algebra $osp(8|2)$. The latter involves twelve additional R -symmetry

generators $F^{IJ} \equiv F^{[IJ]}$, $\bar{F}_{IJ} \equiv \bar{F}_{[IJ]}$ which, together with the $U(4)$ generators L_J^I, F , form the full R -symmetry algebra $o(8)$. Additional conformal generators are $\bar{T}, T$, such that three bosonic generators $\mathcal{H}_{\text{conf}}, \bar{T}$ and T constitute the conformal $d=1$ subalgebra $o(2,1)$.

Actually, the parameter m drops out from the superconformal algebra after performing redefinitions similar to those made in [8] for the case of the $\mathcal{N}=4, d=1$ superconformal algebra $D(2,1;\alpha)$.

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