RECEIVED: February 23, 2018

REVISED: July 20, 2018

ACCEPTED: August 17, 2018

PUBLISHED: August 27, 2018

Yukawa couplings from magnetized D-brane models on non-factorisable tori

Stefan Förste and Christoph Liyanage

*Bethe Center for Theoretical Physics and Physikalisches Institut der Universität Bonn,
Nussallee 12, 53115 Bonn, Germany*

E-mail: forste@th.physik.uni-bonn.de, liyanage@th.physik.uni-bonn.de

ABSTRACT: We compute Yukawa couplings in type IIB string theory compactified on a non factorisable six-torus in the presence of D9 branes and fluxes. The setting studied in detail, is obtained by T-dualising an intersecting brane configuration of type IIA theory compactified on a torus generated by the $SO(12)$ root lattice. Particular deformations of such torus are taken into account and provide moduli dependent couplings. Agreement with the type IIA result is found in a non trivial way. The classical type IIB calculation gives also information on a factor accessible only by quantum computations on the type IIA side.

KEYWORDS: D-branes, Field Theories in Higher Dimensions, String Duality

ARXIV EPRINT: [1802.05136](https://arxiv.org/abs/1802.05136)

Contents

1	Introduction	1
2	<i>D9</i> branes as T-dualised <i>D6</i> branes	3
2.1	T-dual of $T_{\text{SO}(12)}^6$: closed string sector	3
2.2	T-dual of $T_{\text{SO}(12)}^6$: open string sector	6
3	Chiral matter	7
3.1	Labelling gauge indices	7
3.1.1	Example	10
3.2	Bifundamentals	10
3.3	Massless Dirac zero modes	11
4	Computing wavefunctions for bifundamentals	12
4.1	Warm-up: boundary conditions on T^2	12
4.1.1	Example	14
4.2	Boundary conditions on T^6	14
4.2.1	Example	16
4.3	Solving the boundary conditions	17
4.4	Chiral wavefunctions	19
5	Yukawa couplings	20
6	Conclusions	21
A	Quotient lattices, divisors and multiples of integral matrices	22
B	Normalisation	25
C	Overlap of three wavefunctions on T^6	28
C.1	Two extra dimensions	28
C.2	Yukawa couplings for the T-dual of $T_{\text{SO}(12)}^6$	30

1 Introduction

One possible extension of the Standard Model of particle physics is to assume the existence of extra dimensions as motivated by string theory. The appeal of such extensions lies in their capability to explain patterns in the Standard Model which are adjusted by hand to match observations. One such example is the hierarchy in the size of Yukawa couplings. In [1] super Yang-Mills theory with extra dimensions was studied in this context. Standard

Model fields correspond to zero modes of the extra dimensional Dirac equation. Different fields have different localisations within the extra dimensions. Yukawa couplings arise as overlap integrals of these zero modes; they are large if they are localised near to each other and small otherwise. The authors of [1] mainly focused on the extra dimensions being compactified on a torus which factorises into a product of two-tori. An initial $U(N)$ gauge group is broken by fluxes to $U(N_a) \times U(N_b) \times U(N_c)$ which can be further broken by Wilson lines. (The unbroken gauge group could e.g. be the Standard Model gauge group.) Computations in [1] are restricted to the case that N_a, N_b, N_c are mutually coprime. In the present paper, the discussion will be extended to particular non-factorisable tori. This will also make it necessary to abandon the restriction of N_a, N_b, N_c being mutually coprime, and hence the generalisation considered in [2] neither applies.

Restricting considerations to type II string model building, the above setting corresponds to type IIB theory, whereas most of type II string model building has been carried out on the type IIA side in the geometrically intuitive intersecting brane picture, see e.g. [3–29]. Some constructions have, however, been directly performed on the type IIB side [30–39]. Computing Yukawa couplings in the type IIB setting is useful also from an intersecting brane model builder’s perspective. Type IIA Yukawa couplings have been computed in [40]. There, they are given by sums over exponentials of classical worldsheet instanton actions. A factor in front of this sum cannot be fixed by classical calculations. In [1] also T-duality of intersecting brane models to type IIB flux compactifications is discussed. Couplings do match and further the type IIB calculation fixes the leading (in the small angle limit) contribution to the overall factor. Further discussions on the computation of interactions in type II models, including also quantum corrections, can be found in [41–59].

Usually toroidal constructions are performed on so called factorisable six-tori consisting of three mutually orthogonal two-tori. Generalisations to non factorisable tori are studied in [60–69]. In particular in [67] Yukawa couplings for intersecting branes on non factorisable six-tori have been computed. The calculations are restricted to cases where the torus is generated by a sublattice of a lattice belonging to a factorisable torus; as a representative example the $SO(12)$ root lattice is considered. In the present paper T-duality of this setup will be carried out. Yukawa couplings are found to match and the leading contribution to the overall factor can be computed in type IIB theory. Some technical details of the calculation are quite appealing. For instance, the $SO(12)$ structure of the type IIA compactification is scrambled in the process of T-duality along some of the lattice vectors. It resurfaces at a later stage when zero modes of the Dirac equation are labelled. As an aside, the methods developed for N_a, N_b, N_c not all being mutually coprime can easily be applied to factorisable compactifications. In phenomenological model building such situations are not unlikely to arise; for instance an initial $U(N)$ gauge symmetry can be broken by fluxes to Pati-Salam which in turn could be broken by Wilson lines to the Standard Model gauge group.

The paper is organised as follows. In the next section T-duality on the configuration of [67] is performed. In section three, equations for wavefunctions multiplying 4d chiral fields are derived. In section four, these wave equations are solved. The non trivial labelling of independent solutions is discussed in detail. In section five, Yukawa couplings are computed by integrating the product of three wavefunctions over compact space. Section six

contains some concluding remarks. In appendix A generalisations of the concept of greatest common divisors and lowest common multiples of lattices are reviewed and some examples given. Appendix B provides details on how wavefunctions are normalised. Computational details for Yukawa couplings (overlap integrals) have been moved to appendix C.

2 *D9* branes as T-dualised *D6* branes

In this section the T-dual of the setups considered in [67] will be constructed. The dual geometry will be a six-torus whose complex structure matrix has off-diagonal components. D-branes at angles give rise to magnetic flux, whereas multiple intersections with the T-dualised cycle result in constant Wilson lines.

2.1 T-dual of $T_{\text{SO}(12)}^6$: closed string sector

Before performing the T-duality, taking one from type IIA to type IIB, the (deformed) six-torus on the type IIA side will be described [67]. The compactification space is chosen to be a six dimensional flat torus T^6 . It is given by the quotient space $\mathbb{R}^6/\Lambda^6$, where Λ^6 is a six dimensional lattice

$$\Lambda^6 = \left\{ \sum_{i=1}^6 n_i \vec{\alpha}_i \mid n_i \in \mathbb{Z} \right\},$$

with $\{\vec{\alpha}_i\}_{i=1,\dots,6}$ generating the lattice. Hence, locally the torus looks like $\mathbb{R}^6$, but points differing by lattice vectors are identified

$$\vec{x} \sim \vec{x} + \vec{\lambda}, \quad x \in \mathbb{R}^6, \quad \vec{\lambda} \in \Lambda^6.$$

In the following, the canonical basis of $\mathbb{R}^6$ will be denoted by $\{\vec{e}_i\}_{i=1,\dots,6}$ with components

$$e_{i\mu} = \delta_{i\mu}. \quad (2.1)$$

The metric on flat $\mathbb{R}^6$ is given by

$$ds^2 = \sum_{h=1}^3 |du_h|^2,$$

where the six canonical coordinates have been combined into three complex coordinates according to

$$\vec{x} = \sum_{i=1}^6 x_i \vec{e}_i = \sum_{h=1}^3 \text{Re}(u_h) \vec{e}_{2h-1} + \text{Im}(u_h) \vec{e}_{2h}. \quad (2.2)$$

At the moment, this choice of pairs is arbitrary. Later D6 branes projecting onto straight lines in each of the complex planes and thus automatically wrapping Lagrangian cycles will be introduced.

A torus is called factorisable if its generators $\{\vec{\alpha}_i\}$ can be split into three mutually orthogonal pairs of vectors. In this case, one would arrange the choice of complex coordinates such that each of the mutually orthogonal pairs lies within one complex plane. For

non factorisable tori this is not possible. As a typical example the root lattice of $SO(12)$,

$$\begin{aligned}\vec{\alpha}_1 &= (1, -1, 0, 0, 0, 0)^T, & \vec{\alpha}_2 &= (0, 1, -1, 0, 0, 0)^T, & \vec{\alpha}_3 &= (0, 0, 1, -1, 0, 0)^T, \\ \vec{\alpha}_4 &= (0, 0, 0, 1, -1, 0)^T, & \vec{\alpha}_5 &= (0, 0, 0, 0, 1, -1)^T, & \vec{\alpha}_6 &= (0, 0, 0, 0, 1, 1)^T,\end{aligned}$$

will be considered. Here, vector components are given w.r.t. the canonical basis $\{\vec{e}_i\}$. If one was discussing just T^6 compactifications without any further ingredients (such as D branes or envisaged orientifolds) one could change metric and B field components by arbitrary constants. In particular, this allows deforming non factorisable into factorisable tori. Here, additional ingredients allowing deformations only within each of the complex planes will be assumed. This leads to the general metric

$$ds^2 = \sum_{h=1}^3 \frac{\text{Im}K_h}{\text{Im}\tau_h} |du_h|^2,$$

where K_h and τ_h are complex parameters with positive imaginary parts. The definition of the complex coordinates in (2.2) is also deformed

$$u_h = x_{2h-1} + \tau_h x_{2h}.$$

In addition, a constant B field of the form

$$B = \sum_{h=1}^3 2 \text{Re}K_h dx_{2h-1} \wedge dx_{2h} = i \sum_{h=1}^3 \frac{\text{Re}K_h}{\text{Im}\tau_h} du_h \wedge d\bar{u}_h$$

will be allowed. For compactifications on a factorisable T^6 the K_h 's would be the complexified Kähler moduli of the three T^2 's whereas the τ_h 's would form the complex structure moduli. In [67] it was observed that Yukawa couplings of type IIA intersecting branes exponentially depend on these ‘would be’ complex Kähler moduli even for non factorisable T^6 .

Before performing T-duality, it is useful to change coordinates to the lattice basis

$$\sum_{i=1}^6 x_i \vec{e}_i = \sum_{i=1}^6 y_i \vec{\alpha}_i, \quad (2.3)$$

such that integer shifts in any of the y_i coordinates correspond to lattice shifts. Again, expressions for metric and B field can be compressed by means of complex coordinates

$$w_1 = y_1 + \frac{\tau_1 y_2}{1 - \tau_1}, \quad w_2 = y_3 - \frac{y_2}{1 - \tau_2} + \frac{\tau_2 y_4}{1 - \tau_2}, \quad w_3 = y_5 - \frac{y_4}{1 - \tau_3} + \frac{1 + \tau_3}{1 - \tau_3} y_6, \quad (2.4)$$

for which one obtains,

$$ds^2 = \sum_{h=1}^3 \frac{\text{Im}K_h |1 - \tau_h|^2}{\text{Im}\tau_h} |dw_h|^2, \quad B = i \sum_{h=1}^3 \frac{\text{Re}K_h |1 - \tau_h|^2}{\text{Im}\tau_h} dw_h \wedge d\bar{w}_h. \quad (2.5)$$

Since the y_i coordinates are compactified on circles they are particularly useful for performing T-duality. The radii of these circles are taken to be at their selfdual value, $R = \sqrt{\alpha'}$. In the following

$$\alpha' = 1/(4\pi^2) \quad (2.6)$$

will be chosen such that $2\pi R = 1$. The Buscher rules [70, 71] for T-duality along the θ direction read

$$\begin{aligned}\tilde{G}_{\theta\theta} &= \frac{1}{G_{\theta\theta}}, & \tilde{G}_{ij} &= G_{ij} - \frac{G_{\theta i} G_{\theta j} - B_{\theta i} B_{\theta j}}{G_{\theta\theta}}, & \tilde{G}_{\theta i} &= \frac{B_{\theta i}}{G_{\theta\theta}}, \\ \tilde{B}_{\theta i} &= \frac{G_{\theta i}}{G_{\theta\theta}}, & \tilde{B}_{ij} &= B_{ij} - \frac{G_{\theta i} B_{\theta j} - B_{\theta i} G_{\theta j}}{G_{\theta\theta}},\end{aligned}\quad (2.7)$$

where i, j label directions other than θ . In addition, there is a shift in the dilaton

$$\Phi_b = \Phi_a - \frac{1}{2} \log G_{\theta\theta}. \quad (2.8)$$

The T-dual coordinate is again compactified on a circle of selfdual radius. Successively performing T-duality along the y_1 , y_3 and y_5 direction yields type IIB theory. To write the T-dual background the following complex coordinates are introduced (omitting tildes at dual coordinates)

$$z_1 = y_1 + K_1 y_2, \quad z_2 = y_3 - K_2 y_2 + K_2 y_4, \quad z_3 = y_5 - K_3 y_4 + 2K_3 y_6. \quad (2.9)$$

T-dual metric and B field can be written as

$$ds^2 = \sum_{h=1}^3 \frac{\text{Im}\tau_h}{\text{Im}K_h |1 - \tau_h|^2} |dz_h|^2, \quad (2.10)$$

$$\begin{aligned}B + dy_3 \wedge dy_4 - 2dy_5 \wedge dy_6 &= \frac{i}{\text{Im}K_1} \text{Re} \frac{\tau_1}{1 - \tau_1} dz_1 \wedge d\bar{z}_1 + \frac{i}{\text{Im}K_2} \text{Re} \frac{1}{1 - \tau_2} dz_2 \wedge d\bar{z}_2 \\ &+ \frac{i}{\text{Im}K_3} \text{Re} \frac{1}{1 - \tau_3} dz_3 \wedge d\bar{z}_3.\end{aligned}\quad (2.11)$$

Here, moduli have been suggestively split into complex structure appearing in (2.9) and the rest. This is not unique. The 6d metric has 21 independent real components whereas complex structure moduli and imaginary part of the Kähler moduli have 18 plus 9 real components. Uniqueness is achieved by imposing the six additional conditions that the B field should have components only along $(1, 1)$ forms [72].¹ To achieve that, the complex structure will not be modified but instead T-duality will be combined with the gauge transformation

$$B \rightarrow B - 2dy_3 \wedge dy_4 - 2dy_5 \wedge dy_6, \quad (2.12)$$

which has to be kept in mind when performing T-duality in the open string sector. Notice, that the previously ‘would be’ Kähler moduli become actual complex structure moduli in the T-dual type IIB theory. Finally, the relation between type IIB and type IIA dilaton is

$$\Phi_b = \Phi_a - \frac{1}{2} \sum_{h=1}^3 \log \frac{\text{Im}K_h |1 - \tau_h|^2}{\text{Im}\tau_h}. \quad (2.13)$$

¹The same argument can be also applied to the type IIA side. From (2.4) and (2.5) one learns that actual complex structure moduli are given purely in terms of ‘would be’ complex structure moduli, independent of ‘would be’ Kähler moduli.

2.2 T-dual of $T_{\text{SO}(12)}^6$: open string sector

As discussed in [63, 67] a D6 brane of type IIA theory spans the following three dimensional subspace of the six dimensional compact space

$$x_{2h} = \frac{m^h}{n^h} x_{2h-1}, \quad \text{for } h \in \{1, 2, 3\}. \quad (2.14)$$

For factorisable tori the wrapping numbers n^h and m^h should be coprime for each h . In the non factorisable case these conditions are modified. For instance if the pairs are still all coprime, $n^h + m^h$ has to be even for all h 's. Other possibilities are listed in [63, 67]. For simplicity, the case that branes pass through the origin will be considered. If one of the wrapping numbers n^h is zero the corresponding equation has to be replaced by $x_{2h-1} = 0$. Expressed in y_i coordinates (2.3), equations (2.14) take the form

$$y_1 = \frac{n^1 y_2}{N^{(1)}}, \quad y_3 = \frac{m^2 y_2 + n^2 y_4}{N^{(2)}}, \quad y_5 = \frac{m^3 y_4 + (n^3 - m^3) y_6}{N^{(3)}},$$

with $N^{(h)} = n^h + m^h$, $h \in \{1, 2, 3\}$ (2.15)

In the following, the case that any of the $N^{(h)}$ vanishes will be excluded, i.e. T-duality along a D-brane will not be performed. This case has to be treated separately and leads to D7, D5 or D3 branes in the T-dual picture. T duality for open strings has been discussed in e.g. in [73–79]. Eq. (2.15) represents Dirichlet conditions on the coordinates with respect to which T-duality will be performed. Dirichlet conditions turn into Neumann conditions, which are obtained by varying the worldsheet action with no boundary conditions on the variation and a gauge field coupling to the boundary. This gauge field is given by minus the right hand sides of (2.15),

$$\frac{\tilde{A}_{y_1}}{2\pi} = -\frac{n^1 y_2}{N^{(1)}}, \quad \frac{\tilde{A}_{y_3}}{2\pi} = -\frac{m^2 y_2 + n^2 y_4}{N^{(2)}}, \quad \frac{\tilde{A}_{y_5}}{2\pi} = -\frac{m^3 y_4 + (n^3 - m^3) y_6}{N^{(3)}}.$$

As to be discussed shortly, these gauge fields are multiplied by identity matrices whose appearance has been suppressed so far. The T-dual fieldstrength is finally computed as (recall (2.12))

$$\begin{aligned} F &= \frac{1}{2} F_{ij} dy_i \wedge dy_j = d\tilde{A} + \tilde{A} \wedge \tilde{A} - 2\pi (dy_3 \wedge dy_4 + dy_5 \wedge dy_6) \\ &= \frac{i\pi}{\text{Im}K_1} \frac{n^1}{N^{(1)}} dz_1 \wedge d\bar{z}_1 - \frac{i\pi}{\text{Im}K_2} \frac{m^2}{N^{(2)}} dz_2 \wedge d\bar{z}_2 - \frac{i\pi}{\text{Im}K_3} \frac{m^3}{N^{(3)}} dz_3 \wedge d\bar{z}_3, \end{aligned} \quad (2.16)$$

where T-dual complex coordinates are defined in (2.9). It is consistent that starting with D branes wrapping Lagrangian cycles in type IIA theory the T-dual D9 branes of type IIB carry flux only along (1,1) forms. So far, multiple wrappings of the D9 brane have not been taken into account. The D9 brane wrapping number, $N = N_{\text{D9}}$, is given by

$$N = N_{\text{D9}} = \frac{N^{(1)} N^{(2)} N^{(3)}}{2} N_{\text{D6}},$$

where N_{D6} is the wrapping number of the D6 brane and the additional multiplicity originates from the intersection number with the cycle along which T-duality has been performed. (Intersection numbers for the type IIA setting are taken from [63].) In the following

$$N_{D6} = 1$$

will be considered since for the calculation of Yukawa couplings this number is not relevant. (Given a gauge group $U(A) \times U(B) \times U(C)$ the Yukawa coupling of $(A\bar{B})(B\bar{C})(C\bar{A})$ does not depend on A, B, C .) For later convenience, the gauge transformation (2.12) will be included in a redefinition of the T-dual gauge field. Taking into account multiple wrappings,

$$F = dA + A \wedge A, \quad A = \left(\frac{\pi n^1 \text{Im} \bar{z}_1 dz_1}{N^{(1)} \text{Im} K_1} + \frac{\pi m^2 \text{Im} z_2 d\bar{z}_2}{N^{(2)} \text{Im} K_2} + \frac{\pi m^3 \text{Im} z_3 d\bar{z}_3}{N^{(3)} \text{Im} K_3} \right) \mathbb{1}_N + W. \quad (2.17)$$

is chosen, where W is a Wilson line originating from the finite separation of $N^{(1)}N^{(2)}N^{(3)}/2$ stacks of branes along the T-dualised direction. Although W can be written as $g^{-1}dg$ with $g \in \text{SU}(N)$ it cannot be removed by a globally single valued gauge transformation. On the type IIB side, it breaks the gauge group from $U(N)$ to $U(N_{D6} = 1)$. The Wilson line will be discussed more explicitly in the next section.

3 Chiral matter

This section follows closely the strategy of [1] in identifying chiral matter of the effective four dimensional theory. First, Wilson lines are specified. They are viewed as gauge transformations induced by lattice shifts. In the factorisable case these gauge transformations are associated to the direct product of three matrices, or in other words, each of the two group indices on the gauge transformation matrix is conveniently replaced by a triplet of indices. It will be argued that in the non factorisable case the gauge index should be expressed in terms of a vector in a quotient lattice. To really discuss the T-dual of intersecting branes, more than one unitary gauge group factor has to be considered. Zero modes of the Dirac equation in the bifundamental representation will give rise to chiral matter.

3.1 Labelling gauge indices

Consider a field ϕ as a function of torus coordinates transforming in the fundamental representation of $U(N)$. Dependence on uncompactified spacetime is also assumed but suppressed in the notation. It is imposed that this field is periodic under lattice shifts up to gauge transformations, i.e.

$$\phi(y_1, \dots, y_i + 1, \dots, y_6) = e^{i\chi_i(\vec{z})} \omega_i \phi(y_1, \dots, y_i, \dots, y_6), \quad (3.1)$$

where χ_i contains effects due to magnetic flux (2.17),

$$\chi_i(\vec{z}) = \oint_{(y_1, \dots, y_i, \dots, y_6)}^{(y_1, \dots, y_i+1, \dots, y_6)} (A - W). \quad (3.2)$$

The Wilsonline W has been encoded in a constant matrix $\omega_i \in \text{SU}(N)$. The phases χ_i are explicitly given by

$$\begin{aligned}\chi_1(\vec{z}) &= -\frac{\pi n^1}{N^{(1)}} \frac{\text{Im}(z_1)}{\text{Im}(K_1)}, & \chi_2(\vec{z}) &= -\frac{\pi n^1}{N^{(1)}} \frac{\text{Im}(\overline{K_1} z_1)}{\text{Im}(K_1)} + \frac{\pi m^2}{N^{(2)}} \frac{\text{Im}(\overline{K_2} z_2)}{\text{Im}(K_2)}, \\ \chi_3(\vec{z}) &= -\frac{\pi m^2}{N^{(2)}} \frac{\text{Im}(z_2)}{\text{Im}(K_2)}, & \chi_4(\vec{z}) &= -\frac{\pi m^2}{N^{(2)}} \frac{\text{Im}(\overline{K_2} z_2)}{\text{Im}(K_2)} + \frac{\pi m^3}{N^{(3)}} \frac{\text{Im}(\overline{K_3} z_3)}{\text{Im}(K_3)}, \\ \chi_5(\vec{z}) &= -\frac{\pi m^3}{N^{(3)}} \frac{\text{Im}(z_3)}{\text{Im}(K_3)}, & \chi_6(\vec{z}) &= -\frac{2\pi m^3}{N^{(3)}} \frac{\text{Im}(\overline{K_3} z_3)}{\text{Im}(K_3)}.\end{aligned}\quad (3.3)$$

The $\text{SU}(N)$ factors ω_i will be fixed by consistency. Taking the argument once through a closed loop should leave a field transforming in the fundamental representation invariant, i.e.

$$\omega_j^{-1} \omega_i^{-1} \omega_j \omega_i \phi(\vec{z}) = e^{2\pi i k_{ij}/N} \cdot \mathbb{1}_N \cdot \phi(\vec{z}). \quad (3.4)$$

The phases are fixed such that a phase originating from $A - W$ is cancelled, e.g.

$$k_{12} = -\frac{N}{2\pi} (\chi_1(0,0,0) + \chi_2(1,0,0) - \chi_1(1+K_1, -K_2, 0) - \chi_2(K_1, -K_2, 0)) \mod N$$

Non vanishing phases are

$$\begin{aligned}k_{12} &= \frac{n^1}{2} N^{(2)} N^{(3)} \mod N, & k_{32} &= \frac{m^2}{2} N^{(1)} N^{(3)} \mod N, \\ k_{34} &= -\frac{m^2}{2} N^{(1)} N^{(3)} \mod N, & k_{54} &= \frac{m^3}{2} N^{(1)} N^{(2)} \mod N, \\ k_{56} &= -m^3 N^{(1)} N^{(2)} \mod N.\end{aligned}\quad (3.5)$$

Notice, that all k_{ij} 's are integers. This is related to conditions that D6 branes should wrap closed cycles in the type IIA geometry [63, 67]. One may try to construct the ω_i 's by means of two dimensional solutions given in [1]: consider two matrices $w_1, w_2 \in \text{SU}(n)$, where $n \in \mathbb{Z}_+$. Impose the condition

$$w_2^{-1} w_1^{-1} w_2 w_1 = e^{2\pi i k/n} \cdot \mathbb{1}_n.$$

A solution for w_1 and w_2 is

$$w_1 = Q^m, \quad w_2 = P$$

where $m = k \mod n$ and

$$Q = \begin{pmatrix} 1 & & & \\ & e^{2\pi i/n} & & \\ & & \ddots & \\ & & & e^{2\pi i(n-1)/n} \end{pmatrix}, \quad \text{and} \quad P = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ 1 & & & \end{pmatrix}.$$

For the factorisable torus solutions of the form

$$\begin{aligned}\omega_1 &= Q_{(1)}^{n^1} \otimes \mathbb{1}_{(2)} \otimes \mathbb{1}_{(3)}, & \omega_3 &= \mathbb{1}_{(1)} \otimes Q^{-m^2} \otimes \mathbb{1}_{(3)}, & \omega_5 &= \mathbb{1}_{(1)} \otimes \mathbb{1}_{(2)} \otimes Q_{(3)}^{-m^3}, \\ \omega_2 &= P_{(1)} \otimes P_{(2)}^{-1} \otimes \mathbb{1}_{(3)}, & \omega_4 &= \mathbb{1}_{(1)} \otimes P_{(2)} \otimes P_{(3)}^{-1}, & \omega_6 &= \mathbb{1}_{(1)} \otimes \mathbb{1}_{(2)} \otimes P_{(3)}^2,\end{aligned}\quad (3.6)$$

where matrices with subscript (h) are $N^{(h)} \times N^{(h)}$ matrices, solve conditions corresponding to (3.4) [1]. The resulting ω_i are $2N \times 2N$ matrices. A similar overcounting arises on the type IIA side if one just multiplied the intersections numbers in each complex plane [67]. There the overcounting would happen due to an erroneous labelling of intersection points as $j^{(h)} \in \mathbb{Z}_{N^{(h)}}$. The resolution advocated in [67] is that the triplet of $j^{(l)}$'s takes values in a sublattice of $\prod_{h=1}^3 \mathbb{Z}_{N^{(h)}}$. With triple indices i, j , (3.6) reads

$$\begin{aligned} (\omega_1)_{ij} &= Q_{i(1),j(1)}^{n^1} \delta_{i(2),j(2)} \delta_{i(3),j(3)}, & (\omega_2)_{ij} &= P_{i(1),j(1)} P_{i(2),j(2)}^{-1} \delta_{i(3),j(3)}, \\ (\omega_3)_{ij} &= \delta_{i(1),j(1)} Q_{i(2),j(2)}^{-m^2} \delta_{i(3),j(3)}, & (\omega_4)_{ij} &= \delta_{i(1),j(1)} P_{i(2),j(2)} P_{i(3),j(3)}^{-1}, \\ (\omega_5)_{ij} &= \delta_{i(1),j(1)} \delta_{i(2),j(2)} Q_{i(3),j(3)}^{-m^3}, & (\omega_6)_{ij} &= \delta_{i(1),j(1)} \delta_{i(2),j(2)} P_{i(3),j(3)}^2. \end{aligned} \quad (3.7)$$

It remains to identify the lattice Λ^3 within which triple indices take values. Wrapping numbers on the type IIA side describe closed cycles if one of the following four cases applies:

- (i) all three $N^{(h)}$'s are even and all (n^h, m^h) are coprime,
- (ii) all three $N^{(h)}$'s are even but for exactly one i : $\text{g.c.d.}(n^i, m^i) = 2$, remaining are coprime pairs,
- (iii) two $N^{(h)}$'s are even, for exactly one i : $\text{g.c.d.}(n^i, m^i) = 2$, remaining are coprime pairs,
- (iv) one $N^{(h)}$ is even and for the corresponding pair $\text{g.c.d.}(n^h, m^h) = 2$, remaining are coprime pairs.

Whenever $\text{g.c.d.}(n^i, m^i) = 2$, $N^{(i)}/2$ has to be odd. Otherwise the corresponding wrapping numbers have to be divided by two and another case applies. The following quotient lattices, Λ^3 , turn out to yield useful sets for labels:

- (i), (ii) $\Lambda_3 = \Lambda_{\text{SO}(6)} / \bigotimes_{l=1}^3 N^{(l)} \mathbb{Z}$
- (iii) $\Lambda_3 = \Lambda_{\text{SO}(6)} / \Gamma$. If e.g. $\text{g.c.d.}(n^1, m^1) = 2$ and $N^{(2)}$ odd then Γ is generated by $(N^{(1)}/2, N^{(2)}, 0)$, $(N^{(1)}/2, -N^{(2)}, 0)$, $(0, 0, N^{(3)})$
- (iv) $\Lambda_3 = \Lambda_{\text{SO}(6)} / \Gamma$. If e.g. $\text{g.c.d.}(n^1, m^1) = 2$ then Γ is generated by $(N^{(1)}/2, N^{(2)}, 0)$, $(N^{(1)}/2, -N^{(2)}, 0)$, $(0, N^{(2)}, N^{(3)})$.

Here, $\Lambda_{\text{SO}(6)}$ denotes the $\text{SO}(6)$ root lattice generated by $(1, -1, 0)$, $(0, 1, 1)$, $(0, 1, -1)$. In cases where $\text{g.c.d.}(n^h, m^h) = 2$ the corresponding Q matrix has to be replaced by

$$Q_{N^{(h)}} \rightarrow Q_{N^{(h)}/2} \otimes \mathbb{1}_2 \text{ if } \text{g.c.d.}(n^h, m^h) = 2.$$

3.1.1 Example

Consider a D6-brane wrapping the cycle $\Pi = (2, 0) \times (1, 2) \times (1, -2)$ on the Type IIA side. The T-dual D9-brane on the Type IIB side contains a $U(3)$ gauge symmetry with a flux given by

$$F = \frac{i\pi}{\text{Im}K_1} dz_1 \wedge d\bar{z}_1 - \frac{2}{3} \frac{i\pi}{\text{Im}K_2} dz_2 \wedge d\bar{z}_2 - 2 \frac{i\pi}{\text{Im}K_3} dz_3 \wedge d\bar{z}_3.$$

The $SU(3)$ factors of the corresponding Wilson lines need to satisfy (3.4) with $N = 3$ and

$$k_{12} = -3 \pmod{3}, \quad k_{32} = -k_{34} = -2 \pmod{3}, \quad k_{54} = -\frac{1}{2}k_{56} = -1 \pmod{3}.$$

Using the above prescription, the gauge indices belong to the lattice $\Lambda_{SO(6)}/\Gamma$, with Γ generated by $(1, 3, 0)^T$, $(1, -3, 0)^T$ and $(0, 3, -1)^T$. That means the only inequivalent gauge indices are the three labels $(0, 0, 0)^T$, $(1, 1, 0)^T$, $(0, 2, 0)^T$. Hence the representation space on which the $SU(N)$ factors of the Wilson lines act is three dimensional instead of six dimensional as it would be when taking the standard tensorproduct.

3.2 Bifundamentals

Consider two D6 branes wrapping cycles labelled by a and b on the type IIA side. Assume that neither cycle has zero intersection number with the T-dualised cycle. On the type IIB side this corresponds to $N_a + N_b$ D9 branes wrapping the T-dual six torus where N_a and N_b are the respective intersection numbers with the T-dualised cycle. These D9 branes accommodate a $U(N_a + N_b)$ gauge symmetry which is broken to $U(N_a) \times U(N_b)$ by magnetic fluxes and finally to $U(1) \times U(1)$ by Wilson lines. The magnetic flux is given by the following non vanishing fieldstrength components

$$\begin{aligned} F_{z_1 \bar{z}_1} &= \frac{\pi i}{\text{Im}(K_1)} \begin{pmatrix} \frac{n_a^1}{N_a^{(1)}} \mathbb{1}_{N_a} & \\ & \frac{n_b^1}{N_b^{(1)}} \mathbb{1}_{N_b} \end{pmatrix}, & F_{z_2 \bar{z}_2} &= -\frac{\pi i}{\text{Im}(K_2)} \begin{pmatrix} \frac{m_a^2}{N_a^{(2)}} \mathbb{1}_{N_a} & \\ & \frac{m_b^2}{N_b^{(2)}} \mathbb{1}_{N_b} \end{pmatrix}, \\ F_{z_3 \bar{z}_3} &= -\frac{\pi i}{\text{Im}(K_3)} \begin{pmatrix} \frac{m_a^3}{N_a^{(3)}} \mathbb{1}_{N_a} & \\ & \frac{m_b^3}{N_b^{(3)}} \mathbb{1}_{N_b} \end{pmatrix}. \end{aligned} \quad (3.8)$$

Let ϕ be a field transforming in the $(N_a, \bar{N}_b)$ representation of $U(N_a) \times U(N_b)$. Formula (3.1) is modified to

$$\phi(y_1, \dots, y_i + 1, \dots, y_6) = e^{i\chi_i^{ab}(z)} \omega_i^a \phi(y_1, \dots, y_i, \dots, y_6) \omega_i^{b\dagger}, \quad (3.9)$$

where $\chi_i^{ab} = \chi_i^a - \chi_i^b$ denotes the difference between the two phases (3.3). Defining

$$I_{ab}^{(h)} = n_a^h m_b^h - m_a^h n_b^h, \quad \tilde{I}_{ab}^{(h)} = I_{ab}^{(h)} / N_a^{(h)} N_b^{(h)},$$

the phase differences can be written as

$$\begin{aligned}
 \chi_1^{ab}(\vec{z}) &= \frac{\pi \text{Im}(z_1) \tilde{I}_{ab}^{(1)}}{\text{Im}(K_1)}, & \chi_2^{ab}(\vec{z}) &= \frac{\pi \text{Im}(\overline{K_1} z_1) \tilde{I}_{ab}^{(1)}}{\text{Im}(K_1)} - \frac{\pi \text{Im}(\overline{K_2} z_2) \tilde{I}_{ab}^{(2)}}{\text{Im}(K_2)}, \\
 \chi_3^{ab}(\vec{z}) &= \frac{\pi \text{Im}(z_2) \tilde{I}_{ab}^{(2)}}{\text{Im}(K_2)}, & \chi_4^{ab}(\vec{z}) &= \frac{\pi \text{Im}(\overline{K_2} z_2) \tilde{I}_{ab}^{(2)}}{\text{Im}(K_2)} - \frac{\pi \text{Im}(\overline{K_3} z_3) \tilde{I}_{ab}^{(3)}}{\text{Im}(K_3)}, \\
 \chi_5^{ab}(\vec{z}) &= \frac{\pi \text{Im}(z_3) \tilde{I}_{ab}^{(3)}}{\text{Im}(K_3)}, & \chi_6^{ab}(\vec{z}) &= \frac{2\pi \text{Im}(\overline{K_3} z_3) \tilde{I}_{ab}^{(3)}}{\text{Im}(K_3)}.
 \end{aligned} \tag{3.10}$$

Inserting the explicit representations for the Wilson lines (3.7) one finds

$$\begin{aligned}
 \left(\omega_1^a \phi(\vec{z}) \omega_1^{b\dagger} \right)_{k_a k_b} &= e^{2\pi i \left(k_a^{(1)} n_a^1 / N_a^{(1)} - k_b^{(1)} n_b^1 / N_b^{(1)} \right)} \phi_{k_a k_b}(\vec{z}), \\
 \left(\omega_2^a \phi(\vec{z}) \omega_2^{b\dagger} \right)_{k_a k_b} &= \phi_{k_a + (1, -1, 0), k_b + (1, -1, 0)}(\vec{z}), \\
 \left(\omega_3^a \phi(\vec{z}) \omega_3^{b\dagger} \right)_{k_a k_b} &= e^{-2\pi i \left(k_a^{(2)} m_a^2 / N_a^{(2)} - k_b^{(2)} m_b^2 / N_b^{(2)} \right)} \phi_{k_a k_b}(\vec{z}), \\
 \left(\omega_4^a \phi(\vec{z}) \omega_4^{b\dagger} \right)_{k_a k_b} &= \phi_{k_a + (0, 1, -1), k_b + (0, 1, -1)}(\vec{z}), \\
 \left(\omega_5^a \phi(\vec{z}) \omega_5^{b\dagger} \right)_{k_a k_b} &= e^{-2\pi i \left(k_a^{(3)} m_a^3 / N_a^{(3)} - k_b^{(3)} m_b^3 / N_b^{(3)} \right)} \phi_{k_a k_b}(\vec{z}), \\
 \left(\omega_6^a \phi(\vec{z}) \omega_6^{b\dagger} \right)_{k_a k_b} &= \phi_{k_a + (0, 0, 2), k_b + (0, 0, 2)}(\vec{z}).
 \end{aligned} \tag{3.11}$$

Notice that ω_2 , ω_4 and ω_6 act as shifts by $\text{SO}(6)$ roots on the gauge indices k_a and k_b . Therefore the convention to label the gauge group elements by a subset of $\text{SO}(6)$ roots is consistent with gauge transformations.

3.3 Massless Dirac zero modes

First, consider open strings on a stack of branes with gauge symmetry $\text{U}(N_a + N_b)$. The fields Ψ corresponding to the open string states transform in the adjoint representation of the gauge group. Massless fermions in four dimensions arise from massless fermionic states in ten dimensions satisfying the Dirac equation

$$i \sum_{l=1}^3 \left(\Gamma^{\bar{l}} D_l - \Gamma^l D_l^\dagger \right) \Psi(\vec{z}) = 0, \tag{3.12}$$

with Γ^l and $\Gamma^{\bar{l}}$ being elements of the six dimensional Clifford algebra and

$$D_l \psi_{\epsilon^1 \epsilon^2 \epsilon^3}(\vec{z}) = \bar{\partial}_l \psi_{\epsilon^1 \epsilon^2 \epsilon^3}(\vec{z}) + [A_{\bar{l}}, \psi_{\epsilon^1 \epsilon^2 \epsilon^3}(\vec{z})].$$

Here, $\psi_{\epsilon^1 \epsilon^2 \epsilon^3}$ are the eight components of the Dirac fermion Ψ and $\epsilon^l \in \{\pm\}$ denotes the spin under the three Cartan generators of $\text{SO}(6)$ (i.e. the components of $\text{SO}(6)$ fundamental weights). For a given $\text{SO}(6)$ weight there are $(N_a + N_b)^2$ components forming the adjoint representation of the gauge group. Eq. (3.12) leads to three equations for each fundamental weight,

$$D_l^\alpha \psi_{\vec{\epsilon}} = 0 \quad \text{with} \quad \alpha = \begin{cases} 1 & \text{if } \epsilon^l = + \\ \dagger & \text{if } \epsilon^l = - \end{cases} \quad \text{and} \quad l \in \{1, 2, 3\}. \tag{3.13}$$

After turning on magnetic flux as in (3.8), the states in ψ_{ϵ^l} decompose into the adjoint representation of $U(N_a)$ and $U(N_b)$ and bifundamentals of $U(N_a) \times U(N_b)$ [1]. The bifundamentals will be denoted by $\phi_{\vec{\epsilon}}$. For e.g. $\epsilon_l = +$ the corresponding equation in (3.13) reads

$$\bar{\partial}_{\vec{l}} \phi_{\vec{\epsilon}} + \frac{\pi \tilde{I}_{ab}^{(l)}}{2\text{Im}(K_l)} z_l \phi_{\vec{\epsilon}} = 0, \quad (3.14)$$

As in [1], normalisable solutions to (3.14) for fermions in the $(\mathbf{N}_a, \bar{\mathbf{N}}_b)$ will be considered. Normalisability leads to the condition

$$\epsilon_i = \text{sign} \left(I_{ab}^{(i)} \right), \quad i \in \{1, 2, 3\}. \quad (3.15)$$

The chirality of the resulting four dimensional massless fermion is fixed by the sign of $\epsilon^1 \epsilon^2 \epsilon^3$, i.e. by $\text{sign} \left(\tilde{I}_{ab}^{(1)} \tilde{I}_{ab}^{(2)} \tilde{I}_{ab}^{(3)} \right)$. Apart from that the solutions depend only on the absolute values $|I_{ab}^{(i)}|$. In the following vertical bars will be dropped and positive $I_{ab}^{(i)}$'s will be assumed since negative values can be accomodated easily by changing the chirality. The following ansatz solves (3.14)

$$(\phi(\vec{z}))_{k_a k_b} = e^{i\pi \sum_{l=1}^3 \frac{\tilde{I}_{ab}^{(l)}}{\text{Im}(K_l)} z_l \text{Im}(z_l)} \xi_{k_a k_b}(\vec{z}), \quad (3.16)$$

where the $\vec{\epsilon}$ label has been dropped in accordance with the above discussion. The $\xi_{k_a k_b}$ are holomorphic functions of the z^i . In the next section the computation of $\xi_{k_a k_b}$ is presented. Readers who are interested mainly in the results should go straight to section 4.4 where solutions to the wave equations will be given.

4 Computing wavefunctions for bifundamentals

In order for the solutions of the massless Dirac equation (3.16) to transform in the bifundamental representation they need to satisfy the boundary conditions (3.11). Before dealing with the non factorisable T^6 it proves useful to recall and generalise the case with two extra dimensions on T^2 .

4.1 Warm-up: boundary conditions on T^2

In order to find solutions for (3.16), which satisfy (3.11), it is demonstrated in [1] that replacing the double index at matrix components by a single index is very useful. The details can be summarised as follows. Focusing on just one T^2 factor the expression corresponding to the first line in (3.11) reads (φ replaces ϕ for the case of two extra dimensions)

$$\left(\omega^a \varphi(z) \omega^{b\dagger} \right)_{k_a k_b} = e^{2\pi i \left(\frac{k_a n_a}{N_a} - \frac{k_b n_b}{N_b} \right)} \varphi_{k_a k_b}(z) = e^{2\pi i \tilde{I}_{ab} \ell} \varphi_{\ell \ell}, \quad (4.1)$$

where $\omega^a, \omega^b \in \text{SU}(N)$ are Wilson lines on the T^2 . Further in the last step the double index has been replaced by a single index

$$\ell \in \{0, \dots, N_a N_b - 1\}, \quad (4.2)$$

from which it is obtained by

$$k_a = \ell \pmod{N_a}, \quad k_b = \ell \pmod{N_b}. \quad (4.3)$$

This means that there is a pair of integers (s, t) such that

$$\ell = k_a + sN_a = k_b + tN_b \pmod{N_a N_b},$$

implying that the difference $k_a - k_b$ has to be an integer multiple of

$$d = g.c.d.(N_a, N_b).$$

For this reason the discussion in [1] is restricted to the case $d = 1$. For general d , the intersection number $I_{ab} = n_a N_a - n_b N_b$ is a multiple of d . Hence, (4.2) should be replaced by

$$\ell \in \left\{ 0, \dots, \frac{N_a N_b}{d} - 1 \right\}, \quad (4.4)$$

providing not enough labels. In addition, one should introduce another label

$$\delta \in \{0, \dots, d-1\},$$

with

$$k_a - k_b = 0 \pmod{d} \longrightarrow k_a - k_b = \delta \pmod{d}. \quad (4.5)$$

The distribution of δ among individual shifts of k_a and k_b is carried out as follows. First, one chooses a solution (p, q) of the linear Diophantine equation

$$d = N_a p - N_b q.$$

Then (4.5) is compatible with

$$k_a \rightarrow k_a + \frac{N_a p \delta}{d} \pmod{N_a}, \quad k_b \rightarrow k_b + \frac{N_b q \delta}{d} \pmod{N_b}.$$

Summarising, the correspondence between (k_a, k_b) and (ℓ, δ) is

$$k_a = \ell + \frac{p N_a \delta}{d} \pmod{N_a}, \quad k_b = \ell + \frac{q N_b \delta}{d} \pmod{N_b}. \quad (4.6)$$

Then, the second identity in (4.1) generalises to

$$e^{2\pi i \left(\frac{k_a n_a}{N_a} - \frac{k_b n_b}{N_b} \right)} \varphi_{k_a k_b}(z) = e^{2\pi i \left(\tilde{I}_{ab} \ell + \frac{\delta}{d} (p n_a + q n_b) \right)} \varphi_{\ell + \frac{p N_a \delta}{d} \ell + \frac{q N_b \delta}{d}}, \quad (4.7)$$

i.e. there is an additional phase taking values in $\mathbb{Z}_d$.

k_a	1	2	3	1	2	3	1	2	3	1	2	3
k_b	1	2	3	4	1	2	3	4	1	2	2	4
ℓ	1	2	3	4	5	6	7	8	9	10	11	12

Table 1. Index ℓ for bifundamental of $U(3) \times U(4)$.

k_a	1	2	1	2	2	1	2	1
k_b	1	2	3	4	1	2	3	4
ℓ	1	2	3	4	1	2	3	4
δ	0	0	0	0	1	1	1	1

Table 2. Assignment of (ℓ, δ) to (k_a, k_b) for $U(2) \times U(4)$.

4.1.1 Example

To illustrate the above discussion, take a look at the following setup: let $\varphi \in \mathbb{C}^3 \times \mathbb{C}^4$ transform in the bifundamental representation of $U(3) \times U(4)$ such that the Wilson lines act according to

$$\left(\omega_1^a \varphi(z) \omega_1^{b\dagger}\right)_{k_a, k_b} = e^{2\pi i \left(\frac{k_a}{3} - \frac{k_b}{4}\right)} \varphi_{k_a, k_b}(z), \quad \left(\omega_2^a \varphi(z) \omega_2^{b\dagger}\right)_{k_a, k_b} = \varphi_{k_a+1, k_b+1}(z).$$

Hence $(n_a, N_a) = (1, 3)$ and $(n_b, N_b) = (1, 4)$ for this example. Under the mapping (4.3) the corresponding index ℓ for the state φ_{k_a, k_b} is assigned as in table 1. Each component of φ is labelled unambiguously by the index $\ell \in \{1, \dots, 12\}$, since $\text{g.c.d.}(3, 4) = 1$.

Next, consider a field $\varphi \in \mathbb{C}^2 \times \mathbb{C}^4$, which transforms under the bifundamental representation of $U(2) \times U(4)$ and the Wilson line action is taken as

$$\left(\omega_1^a \varphi(z) \omega_1^{b\dagger}\right)_{k_a, k_b} = e^{2\pi i \left(\frac{k_a}{2} - \frac{k_b}{4}\right)} \varphi_{k_a, k_b}(z), \quad \left(\omega_2^a \varphi(z) \omega_2^{b\dagger}\right)_{k_a, k_b} = \varphi_{k_a+1, k_b+1}(z).$$

This time the mapping (4.3) leads to the assignment given in the $\delta = 0$ columns of table 2. Entries with different δ 's are not related by Wilson line actions. The mapping (4.6) yields an unambiguous correspondence between the matrix components of φ and states labelled by ℓ and δ as can be seen in table 2.

4.2 Boundary conditions on T^6

Returning to the non factorisable torus, discussed previously, it was noticed that

$$k_a \in \frac{\Lambda_{\text{SO}(6)}}{\Gamma_a}, \quad k_b \in \frac{\Lambda_{\text{SO}(6)}}{\Gamma_b}, \quad (4.8)$$

where Γ_a, Γ_b are sublattices of $\Lambda_{\text{SO}(6)}$.

The lattice Γ_d is defined as follows: Γ_a and Γ_b are sublattices of Γ_d and there is no proper sublattice of Γ_d containing Γ_a and Γ_b as sublattices. In other words, Γ_d is the coarsest lattice containing Γ_a and Γ_b .

The number of inequivalent index combinations (k_a, k_b) is given in terms of indices of quotient lattices²

$$\#(k_a, k_b) = \left| \frac{\Lambda_{\text{SO}(6)}}{\Gamma_a} \right| \left| \frac{\Lambda_{\text{SO}(6)}}{\Gamma_b} \right| = \left| \frac{\Lambda_{\text{SO}(6)}}{\Gamma_a \cap \Gamma_b} \right| \left| \frac{\Lambda_{\text{SO}(6)}}{\Gamma_d} \right|. \quad (4.9)$$

The second equality with a reference to its proof is discussed further in appendix A. These observations suggest replacing the index pair (k_a, k_b) by two lattice valued labels

$$l \in \frac{\Lambda_{\text{SO}(6)}}{\Gamma_a \cap \Gamma_b}, \quad \delta \in \frac{\Lambda_{\text{SO}(6)}}{\Gamma_d}.$$

The pair (k_a, k_b) can be again obtained by shifting values of l and modding out by lattices Γ_a , respectively Γ_b . The details are as follows. There are classes of differences $k_a - k_b$ labelled by different δ 's,

$$k_a - k_b = \delta. \quad (4.10)$$

Throughout the paper three dimensional lattice vectors are viewed as a column with three entries corresponding to the components with respect to a given basis (mostly (2.1)). Let a_i, b_i, d_i ($i \in \{1, 2, 3\}$) be the generators of the lattices $\Gamma_a, \Gamma_b, \Gamma_d$, respectively. It turns out to be convinient to combine these into three by three matrices

$$A = (a_1, a_2, a_3), \quad B = (b_1, b_2, b_3), \quad D = (d_1, d_2, d_3). \quad (4.11)$$

The requirement that Γ_a and Γ_b are sublattices of Γ_d is equivalent to the existence of three by three integral matrices³ M_a and M_b such that

$$A = DM_a, \quad B = DM_b. \quad (4.12)$$

Let (P, Q) be two three by three integral matrices satisfying⁴

$$D = AP - BQ. \quad (4.13)$$

A natural generalisation of (4.6) would be

$$k_a = l + APD^{-1}\delta \pmod{\Gamma_a}, \quad k_b = l + BQD^{-1}\delta \pmod{\Gamma_b}.$$

There is however a problem with that. The partitions $APD^{-1}\delta$ and $BQD^{-1}\delta$ are not always in $\Lambda_{\text{SO}(6)}$. A way out is to give up invariance under equivalence shifts of δ . So, in the following δ will be taken from a finite set consisting of one representative for each equivalence class. Then it makes sense to assign

$$k_a = l \pmod{\Gamma_a}, \quad k_b = l - \delta \pmod{\Gamma_b}, \quad (4.14)$$

since now shifts of k_b by elements of Γ_d cannot be absorbed by picking another δ from the same equivalence class. The non factorisable version of (4.7) reads

$$e^{2\pi i \left(\frac{k_a^{(1)} n_a^1}{N_a^{(1)}} - \frac{k_b^{(1)} n_b^1}{N_b^{(1)}} \right)} \phi_{k_a k_b} = e^{2\pi i \left(\tilde{I}_{ab} l^1 + \frac{n_b^1 \delta^1}{N_b^{(1)}} \right)} \phi_{l, l-\delta}.$$

²The index of a quotient lattice counts how often the fundamental cell of the lattice fits into the fundamental cell of the sublattice with respect to which the quotient is taken.

³Matrices with integer components are called integral matrices.

⁴These matrices exist, reference to a proof is given in appendix A.

4.2.1 Example

Consider a field φ_{k_a, k_b} , with its gauge indices belonging to

$$k_a \in \frac{\Lambda_{\text{SO}(6)}}{\Gamma_a}, \quad k_b \in \frac{\Lambda_{\text{SO}(6)}}{\Gamma_b},$$

with Γ_a generated by $a_1 = (3, 1, 0)^T$, $a_2 = (3, -1, 0)^T$, $a_3 = (0, 0, 2)^T$ and Γ_b generated by $b_1 = (2, 0, 0)^T$, $b_2 = (0, 2, 0)^T$, $b_3 = (0, 0, 2)^T$. Then ϕ has 24 componets with the gauge indices represented by e.g.

$$k_a \in \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\}, \quad k_b \in \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

The lattices Γ_a and Γ_b are contained in the lattice Γ_d spanned by $d_1 = (1, 1, 0)^T$, $d_2 = (1, -1, 0)^T$, $d_3 = (0, 0, 2)^T$. There are two inequivalent choices for $\Lambda_{\text{SO}(6)}/\Gamma_d$ vectors, e.g.

$$\delta^T \in \{(0, 0, 0), (0, 1, 1)\}.$$

Now, it is not possible to label the components ϕ_{k_a, k_b} by one triple index l . As in the second example of the T^2 discussion not all components of ϕ are related to each other by the action of Wilson lines and are distinguished by a label $\delta \in \Lambda_{\text{SO}(6)}/\Gamma_d$. One can check that $A = DM_a$ and $B = DM_b$ with

$$M_a = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_b = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The lattice $\Gamma_a \cap \Gamma_b$ is spanned by $(6, 0, 0)^T$, $(0, 2, 0)^T$, $(0, 0, 2)^T$, such that one can choose

$$\ell \in \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Prescription (4.8) allows to assign a label l to only 12 of the 24 inequivalent combinations (k_a, k_b) . For example the pair $k_a = (0, 1, 1)^T$ and $k_b = (1, 0, 1)^T$ is associated to $l = (3, 0, 1)^T$, but $k_a = (0, 1, 1)^T$ and $k_b = (0, 0, 0)^T$ cannot be represented by l as in (4.8). For the remaining 12 states one can shift the k_b index as in (4.14) by $\delta \in \Lambda_{\text{SO}(6)}/\Gamma_d$. For instance, after shifting with $\delta = (0, 1, 1)^T$, (4.8) maps the pair $k_a = (0, 1, 1)^T$, $k_b = (0, 0, 0)^T$ to $l = (0, 1, 1)^T$.

4.3 Solving the boundary conditions

Following the previous subsection the pair (k_a, k_b) is replaced by l and δ in (3.11). Plugging the ansatz (3.16) yields

$$\xi_{k_a, k_b}(z_1 + 1, z_2, z_3) = e^{2\pi i \left(\tilde{I}_{ab}^{(1)} l^1 + \frac{n_b^1 \delta^1}{N_b^{(1)}} \right)} \xi_{k_a, k_b}(\vec{z}), \quad (4.15)$$

$$\xi_{k_a, k_b}(z_1 + K_1, z_2 - K_2, z_3) = e^{-\pi i \tilde{I}_{ab}^{(1)} (2z_1 + K_1)} e^{\pi i \tilde{I}_{ab}^{(2)} (2z_2 - K_2)} \xi_{k_a + (1, -1, 0), k_b + (1, -1, 0)}(\vec{z}), \quad (4.16)$$

$$\xi_{k_a, k_b}(z_1, z_2 + 1, z_3) = e^{2\pi i \left(\tilde{I}_{ab}^{(2)} l^2 + \frac{m_b^2 \delta^2}{N_b^{(2)}} \right)} \xi_{k_a, k_b}(\vec{z}), \quad (4.17)$$

$$\xi_{k_a, k_b}(z_1, z_2 + K_2, z_3 - K_3) = e^{-\pi i \tilde{I}_{ab}^{(2)} (2z_2 + K_2)} e^{\pi i \tilde{I}_{ab}^{(3)} (2z_3 - K_3)} \xi_{k_a + (0, 1, -1), k_b + (0, 1, -1)}(\vec{z}), \quad (4.18)$$

$$\xi_{k_a, k_b}(z_1, z_2, z_3 + 1) = e^{2\pi i \left(\tilde{I}_{ab}^{(3)} l^3 + \frac{m_b^3 \delta^3}{N_b^{(3)}} \right)} \xi_{k_a, k_b}(\vec{z}), \quad (4.19)$$

$$\xi_{k_a, k_b}(z_1, z_2, z_3 + 2K_3) = e^{-4\pi i \tilde{I}_{ab}^{(3)} (z_3 + K_3)} \xi_{k_a + (0, 0, 2), k_b + (0, 0, 2)}(\vec{z}). \quad (4.20)$$

First, focus on boundary conditions (4.15), (4.17), (4.19), resulting in the general solution

$$\xi_{k_a, k_b}(\vec{z}) = \sum_{\vec{n} \in \mathbb{Z}^3} e^{2\pi i \sum_{k=1}^3 (n^k + \tilde{I}_{ab}^{(k)} l^k + \phi^{(k)}) z^k} \rho_{\vec{n}}(l), \quad (4.21)$$

where

$$\phi^{(1)} = -\frac{n_b^1 (M_b Q \delta)^1 + m_b^1 (M_a P \delta)^1}{N_b^{(1)}}, \quad (4.22)$$

$$\phi^{(i)} = \frac{m_b^i (M_b Q \delta)^i + n_b^i (M_a P \delta)^i}{N_b^{(i)}} \quad \text{for } i \in \{2, 3\}. \quad (4.23)$$

On the right-hand sides of (4.15), (4.17), (4.19) there will be additional, trivial phase factors of the form $\exp[2\pi i I]$ with the integer I given by $(M_a P \delta)^1$, $-(M_a P \delta)^2$, $-(M_a P \delta)^3$, respectively. The insertion of these factors of one will be helpful in mapping zero mode labels to intersection labels on the type IIA side, shortly. The last term, $\rho_{\vec{n}}(l)$ stands for $\vec{z}$ independent factors which will be further fixed by solving the remaining boundary conditions. Imposing conditions (4.16), (4.18), (4.20) and comparing coefficients at coinciding powers of $\exp z^i$ leads to

$$\frac{\rho_{\vec{n}}(l + (1, -1, 0)^T)}{\rho_{\vec{n}}(l)} = e^{2\pi i \left\{ \left(n^1 + \tilde{I}_{ab}^{(1)} (l^1 + \frac{1}{2}) + \phi^{(1)} \right) K_1 - \left(n^2 + \tilde{I}_{ab}^{(2)} (l^2 - \frac{1}{2}) + \phi^{(2)} \right) K_2 \right\}}, \quad (4.24)$$

$$\frac{\rho_{\vec{n}}(l + (0, 1, -1)^T)}{\rho_{\vec{n}}(l)} = e^{2\pi i \left\{ \left(n^2 + \tilde{I}_{ab}^{(2)} (l^2 + \frac{1}{2}) + \phi^{(2)} \right) K_2 - \left(n^3 + \tilde{I}_{ab}^{(3)} (l^3 - \frac{1}{2}) + \phi^{(3)} \right) K_3 \right\}}, \quad (4.25)$$

$$\frac{\rho_{\vec{n}}(l + (0, 0, 2)^T)}{\rho_{\vec{n}}(l)} = e^{4\pi i \left\{ \left(n^3 + \tilde{I}_{ab}^{(3)} (l^3 + 1) + \phi^{(3)} \right) K_3 \right\}}. \quad (4.26)$$

These conditions are solved by

$$\rho_{\vec{n}}(l) = \mathcal{N}_{\vec{n}} \prod_{l=1}^3 \exp \left[\frac{i\pi \left(n^h + \tilde{I}_{ab}^{(h)} l^h + \phi^{(h)} \right)^2 K_h}{\tilde{I}_{ab}^{(h)}} \right]. \quad (4.27)$$

Independent normalisation constants $\mathcal{N}_{\vec{n}}$ indicate independent zero modes. Imposing invariance under shifts of l by elements of $\Gamma_a \cap \Gamma_b$ identifies some constants. For $\lambda \in \Gamma_a \cap \Gamma_b$ this leads to

$$\mathcal{N}_{\vec{n}} = \mathcal{N}_{\vec{n}'} , \quad \text{for } n'^i = n^i + \tilde{I}_{ab}^{(i)} \lambda^i. \quad (4.28)$$

Possible lattices $\Gamma_a \cap \Gamma_b$ are listed in appendix A. For all cases one finds

$$\text{number of independent constants} = \frac{I_{ab}^{(1)} I_{ab}^{(2)} I_{ab}^{(3)}}{d^{(1)} d^{(2)} d^{(3)}}. \quad (4.29)$$

For the final counting of zero modes it is worthwhile noticing that boundary conditions relate different pairs k_a, k_b of identical $k_a - k_b$ (see (4.15)–(4.20)). In (4.24)–(4.26) this is reflected by relating different l 's but not different δ 's. In conclusion, the number of independent zero modes is given by multiplying the number of independent constants (4.29) times the number of inequivalent δ 's. Going through all examples in appendix A one finds

$$\text{number of independent zero modes} = \frac{I_{ab}^{(1)} I_{ab}^{(2)} I_{ab}^{(3)}}{2}.$$

As expected, this equals the intersection number in the T-dual type IIA configuration. It will be useful to detail the relation between intersections and zero modes by identifying their labellings. In [67] intersections in type IIA theory are labelled by a triplet $j^{(1)}, j^{(2)}, j^{(3)}$ of the following form

$$j = (t_1 m_b^1 - t_2 n_b^2, t_3 m_b^2 - t_4 n_b^2, t_5 m_b^3 - t_6 n_b^3), \quad (t_1, \dots, t_6) \in \Lambda_{\text{SO}(12)}. \quad (4.30)$$

This is subject to equivalence relations which will not be further discussed since matching of the overall numbers has already been established.

The relation (4.28) is taken into account by renaming the summation index

$$n^i = \tilde{I}_{ab} \lambda^i + k^i,$$

where k is a fixed label and $\lambda \in \Gamma_a \cap \Gamma_b$ is summed over. Combining (4.21) and (4.27) one obtains for one zero mode

$$\xi_{l,l-\delta}^{k,\delta} = \mathcal{N}_k \sum_{\lambda \in \Gamma_a \cap \Gamma_b} \prod_{h=1}^3 e^{2\pi i \tilde{I}_{ab}^{(h)} \left[\left(\lambda^h + l^h + \frac{k^h + \phi^{(h)}}{\tilde{I}_{ab}^{(h)}} \right) z^{h + \frac{K_h}{2}} \left(\lambda^h + l^h + \frac{k^h + \phi^{(h)}}{\tilde{I}_{ab}^{(h)}} \right)^2 \right]},$$

which is now labelled by the pair k, δ . To make contact with the type IIA labelling one notices that the solution depends only on the combination $k + \phi$ which can be brought into the form

$$k + \phi = \frac{j}{N_b}. \quad (4.31)$$

where j is the type IIA label (4.30) with

$$\begin{aligned} t_1 &= -(M_a P \delta)^1 + k^1, & t_2 &= (M_b Q \delta)^1 - k^1, & t_3 &= (M_b Q \delta)^2 + k^2, \\ t_4 &= -(M_a P \delta)^2 - k^2, & t_5 &= (M_b Q \delta)^3 + k^3, & t_6 &= -(M_a P \delta)^3 - k^3. \end{aligned} \quad (4.32)$$

That this is really in $\Lambda_{\text{SO}(12)}$ can be seen with (4.12), (4.13) and the fact that $\delta \in \Lambda_{\text{SO}(6)}$. Expressing the type IIB label in terms of the type IIA label via (4.31) leads finally to

$$\xi_{l,l-\delta}^j \equiv \xi_l^j = \mathcal{N}_j^{ab} \sum_{\lambda \in \Gamma_a \cap \Gamma_b} \prod_{h=1}^3 e^{2\pi i \tilde{I}_{ab}^{(h)} \left[\left(\lambda^h + l^h + \frac{N_a^{(h)} j^{(h)}}{I_{ab}^{(h)}} \right) z^h + \frac{K_h}{2} \left(\lambda^h + l^h + \frac{N_a^{(h)} j^{(h)}}{I_{ab}^{(h)}} \right)^2 \right]}, \quad (4.33)$$

where the notation has been changed to remove a redundancy in specifying the δ dependence of the zero mode. The notation for the original zero mode (3.16) will be changed accordingly $\phi_{k_a k_b} \rightarrow \phi_l^i$. Keep in mind that in (3.16) and (4.33) one should actually replace, $I_{ab}^h \rightarrow |I_{ab}^h|$ (see discussion after (3.15)). The normalisation factor $\mathcal{N}_j^{ab}$ has been computed in appendix B.

4.4 Chiral wavefunctions

The results of the above computation can be summarised as follows: the gauge group $U(N_a + N_b)$ in ten dimensions has been broken by the flux (3.8) to $U(N_a) \times U(N_b)$ and further by Wilson lines to $U(1)^2$. The wavefunctions multiplying 4d massless fermions transforming in the bifundamental representation take the form

$$\phi_{k_a, k_b}(\vec{z}) = \exp \left\{ i\pi \sum_{k=1}^3 \frac{\tilde{I}_{ab}^{(k)}}{\text{Im}(K_k)} z_k \text{Im}(z_k) \right\} \xi_{k_a, k_b}(\vec{z}). \quad (4.34)$$

As in the factorisable case the chirality of the 4d fermion is fixed by $\prod_{i=1}^3 \text{sign} \left(I_{ab}^{(i)} \right)$. In difference to the factorisable case, matrix elements are labelled by three dimensional representatives of quotient lattices

$$k_a \in \frac{\Lambda_{\text{SO}(6)}}{\Gamma_a}, \quad k_b \in \frac{\Lambda_{\text{SO}(6)}}{\Gamma_b}.$$

The final expression for the wavefunction can be obtained with

$$\begin{aligned} \xi_{k_a, k_b}^j &= \\ \mathcal{N}_j^{ab} \sum_{\lambda \in \Gamma_a \cap \Gamma_b} \prod_{h=1}^3 \exp \left\{ 2\pi i \tilde{I}_{ab}^{(h)} \left[\left(\lambda^h + l^h + \frac{N_a^{(h)} j^{(h)}}{I_{ab}^{(h)}} \right) z^h + \frac{K_h}{2} \left(\lambda^h + l^h + \frac{N_a^{(h)} j^{(h)}}{I_{ab}^{(h)}} \right)^2 \right] \right\}, \end{aligned} \quad (4.35)$$

where

$$k_a = l \mod \Gamma_a, \quad k_b = l - \delta \mod \Gamma_b, \quad \text{with } l \in \frac{\Lambda_{\text{SO}(6)}}{\Gamma_a \cap \Gamma_b}, \quad \delta \in \Gamma_d,$$

with Γ_d being the coarsest lattice containing Γ_a and Γ_b . The index $j = (j^{(1)}, j^{(2)}, j^{(3)})^T$ labels the independent zero modes and lives on a three dimensional lattice whose form depends on the flux quanta n_a^i, m_a^i and n_b^i, m_b^i and its computation is described in [67]. The normalisation constant $\mathcal{N}_j^{ab}$ is given in (B.11).

5 Yukawa couplings

The configuration considered in this section will be the T-dual of type IIA with three stacks of intersecting D6 branes. That is, (3.8) will be amended to⁵

$$\begin{aligned}
 F_{z_1 \bar{z}_1} &= \frac{\pi i}{\text{Im}(K_1)} \begin{pmatrix} \frac{n_a^1}{N_a^{(1)}} \mathbb{1}_{N_a} & & \\ & \frac{n_b^1}{N_b^{(1)}} \mathbb{1}_{N_b} & \\ & & \frac{n_c^1}{N_c^{(1)}} \mathbb{1}_{N_c} \end{pmatrix}, \\
 F_{z_2 \bar{z}_2} &= -\frac{\pi i}{\text{Im}(K_2)} \begin{pmatrix} \frac{m_a^2}{N_a^{(2)}} \mathbb{1}_{N_a} & & \\ & \frac{m_b^2}{N_b^{(2)}} \mathbb{1}_{N_b} & \\ & & \frac{m_c^2}{N_c^{(2)}} \mathbb{1}_{N_c} \end{pmatrix}, \\
 F_{z_3 \bar{z}_3} &= -\frac{\pi i}{\text{Im}(K_3)} \begin{pmatrix} \frac{m_a^3}{N_a^{(3)}} \mathbb{1}_{N_a} & & \\ & \frac{m_b^3}{N_b^{(3)}} \mathbb{1}_{N_b} & \\ & & \frac{m_c^3}{N_c^{(3)}} \mathbb{1}_{N_c} \end{pmatrix}.
 \end{aligned} \tag{5.1}$$

This breaks the original $U(N_a N_b N_c)$ gauge symmetry to $U(N_a) \times U(N_b) \times U(N_c)$ which is further broken by Wilson lines to $U(1)^3$. This gives rise to chiral matter in the bifundamental representations of $U(N_a) \times U(N_b)$, $U(N_b) \times U(N_c)$ and $U(N_c) \times U(N_a)$ and Yukawa couplings between those fields are given by the overlap of their wavefunctions on the T^6 [1]

$$|\lambda_{ijk}| = \int_{T^6} d^6 z \text{Tr} \left\{ \phi^{i, I_{ab}} \cdot \phi^{j, I_{ca}} \cdot \left(\phi^{k, I_{bc}} \right)^\dagger \right\}. \tag{5.2}$$

The details of the computation of the integral (5.2) can be found in appendix C. Inserting the normalisation factors (B.11) into (C.21), the Yukawa couplings take the form

$$|\lambda_{ijk}| = \alpha'^{3/2} e^{\Phi_b/2} \prod_{k=1}^3 \sqrt{\frac{|1 - \tau_k|^2}{\text{Im} \tau_k}} \left(2 \text{Im}(K_h) \frac{\tilde{I}_{ab}^{(h)} \tilde{I}_{ca}^{(h)}}{\tilde{I}_{cb}^{(h)}} \right)^{\frac{1}{4}} \sum_{\ell \in \Lambda_\ell^3} \exp \left\{ -\frac{A_{i,j,k}(\ell)}{2\pi\alpha'} \right\}, \tag{5.3}$$

with

$$A_{i,j,k}(\ell) = -2\pi^2 \alpha' i \sum_{h=1}^3 \left(\frac{i^{(h)}}{d_b^{(h)} I_{ab}^{(h)}} + \frac{j^{(h)}}{d_a^{(h)} I_{ca}^{(h)}} + \frac{k^{(h)}}{d_c^{(h)} I_{cb}^{(h)}} + \ell^{(h)} \right)^2 K_h |I_{ab}^{(h)} I_{bc}^{(h)} I_{ca}^{(h)}|$$

$\ell \in \Lambda_\ell^3$ (given in (C.20)) satisfying the following selection rules

$$\begin{aligned}
 I_{cb}^{(h)} N_a^{(h)} \ell^{(h)} + \frac{j^{(h)}}{d_a^{(h)}} N_b^{(h)} + \frac{k^{(h)}}{d_c^{(h)}} N_a^{(h)} &\in \mathbb{Z}, \\
 I_{ba}^{(h)} N_c^{(h)} \ell^{(h)} - \frac{i^{(h)}}{d_b^{(h)}} N_c^{(h)} - \frac{k^{(h)}}{d_c^{(h)}} N_a^{(h)} &\in \mathbb{Z}, \\
 I_{ac}^{(h)} N_b^{(h)} \ell^{(h)} - \frac{j^{(h)}}{d_a^{(h)}} N_b^{(h)} - \frac{i^{(h)}}{d_b^{(h)}} N_c^{(h)} &\in \mathbb{Z},
 \end{aligned} \tag{5.4}$$

⁵Now also $N_c = N_c^{(1)} N_c^{(2)} N_c^{(3)} / 2$.

and

$$\begin{aligned}
\sum_{h=1}^3 I_{cb}^{(h)} N_a^{(h)} \ell^{(h)} + \frac{j^{(h)}}{d_a^{(h)}} N_b^{(h)} + \frac{k^{(h)}}{d_c^{(h)}} N_a^{(h)} &= 0 \pmod{2}, \\
\sum_{h=1}^3 I_{ba}^{(h)} N_c^{(h)} \ell^{(h)} - \frac{i^{(h)}}{d_b^{(h)}} N_c^{(h)} - \frac{k^{(h)}}{d_c^{(h)}} N_a^{(h)} &= 0 \pmod{2}, \\
\sum_{h=1}^3 I_{ac}^{(h)} N_b^{(h)} \ell^{(h)} - \frac{j^{(h)}}{d_a^{(h)}} N_b^{(h)} - \frac{i^{(h)}}{d_b^{(h)}} N_c^{(h)} &= 0 \pmod{2},
\end{aligned} \tag{5.5}$$

where e.g. $d_a^{(h)} = g.c.d. \left(I_{ab}^{(h)}, I_{ac}^{(h)} \right)$. In the T-dual type IIA setting, the Yukawa couplings were computed in [67],

$$\left| \lambda_{ijk}^{(IIA)} \right| = h_{\text{qu}} \sum_{\ell \in \Lambda_\ell^3} \exp \left\{ -\frac{A_{i,j,k}(\ell)}{2\pi\alpha'} \right\},$$

where the computation of h_{qu} had not been performed. A direct calculation should be possible e.g. along the lines of [43]. Here, as in [1], its leading behaviour, in the small angle limit, will be deduced by T-dualising back the type IIB classical calculation. For easier comparison to the factorisable case the following abbreviations are useful

$$\mathcal{A}^{(h)} = 4\pi^2 \alpha' \frac{\text{Im}\tau_h}{|1 - \tau_h|^2} \text{ such that } \text{Volume}(T_{\text{IIB}}^6) = \prod_{h=1}^3 \mathcal{A}^{(h)}, \tag{5.6}$$

$$\theta_{ab}^{(h)} = 4\pi \frac{\tilde{I}_{ab}}{\mathcal{A}^{(h)}/\alpha'}. \tag{5.7}$$

Taking into account also the dilaton shift (2.13) and using (2.6) to obtain a manifestly dimensionless coupling, one finds

$$h_{\text{qu}} = \frac{e^{\Phi_a/2}}{(2\pi)^{9/4}} \prod_{h=1}^3 \left(\frac{\theta_{ab}^{(h)} \theta_{ca}^{(h)}}{\theta_{cb}^{(h)}} \right)^{\frac{1}{4}}.$$

This result looks exactly as the one reported in [1] for factorisable tori. Here, however the definition of $\theta_{\alpha\beta}^{(h)}$ has been modified through a modified $\mathcal{A}^{(h)}$ and $\tilde{I}_{\alpha\beta}^{(h)}$. The meaning is the same; in type IIB diluted flux implies small $\theta_{\alpha\beta}^{(h)}$'s which in type IIA yield the level spacing in the quantised open string stretching from brane α to brane β .

6 Conclusions

In the present paper, Yukawa couplings were computed along the lines of [1]. However, here a particular non factorisable six-torus was considered. This arose as a T-dual of a torus generated by the SO(12) root lattice. For cases in which the SO(12) root lattice is replaced by another sublattice of a factorisable lattice straightforward modifications of the presented calculations are expected. Compared to [1], however, some less straightforward

adjustments had to be performed. Gauge indices as well as zero mode labels take values in quotient lattices which appear as generalisations of products of finite sets of integers. On the type IIA side an $SO(12)$ lattice playing a role in labelling the intersection points was directly related to the compactification lattice. In the T-dual description, this $SO(12)$ lattice shows up in a rather indirect way when labelling zero modes. For non coprime flux ranks, not all components of a chiral multiplet are related by boundary conditions and hence expressed by the same set of zero modes.

T-dualising back to type IIA one can identify leading contributions to a factor which can be determined only by a quantum computation on the type IIA side. The result looks exactly as in the factorisable case [1], with some straightforward modifications in the definitions of variables. To confirm the presented result, one could in principle perform T-duality along other cycles. This is expected to be more complicated since the cycles of the presented calculation have been chosen such that they lie within complex planes.

It would be interesting to investigate to what extend the presented type IIB calculation can be generalised to cases being not T-dual to type IIA models of the considered kind. Abelian Wilson lines have not been turned on for simplicity. In the T-dual IIA setting they correspond to an offset from a brane passing through the origin. Their inclusion is expected to be straightforward. Finally, of course, applications to actual model building would be nice. The presented configuration generalises known cases and might help accommodating desirable phenomenological aspects.

Acknowledgments

We thank Josua Faller for collaboration at an early stage of the presented project. This work was supported by SFB-Transregio TR33 “The Dark Universe” (Deutsche Forschungsgemeinschaft) and “Bonn Cologne Graduate School for Physics and Astronomy” (BCGS).

A Quotient lattices, divisors and multiples of integral matrices

As discussed in (4.11) lattices will be associated to integral three by three matrices: Γ_a to A , Γ_b to B , Γ_d to D , and $\Gamma_a \cap \Gamma_b$ to M . With the following definitions one can establish relations among these matrices.

Definition: let A, B, D be integral matrices. Then D is a *left divisor* of A if there is an integral matrix M_a such that $A = DM_a$. Further, D is the *greatest common left divisor* of A and B if it is a left divisor of A and B and any other left divisor of A and B is a left divisor of D .

Clearly, the matrix D containing generators of Γ_d is a greatest common left divisor of A and B . The greatest common left divisor is unique up to multiplication by unimodular matrices which corresponds to choosing an equivalent set of lattice generators (see e.g. [80]). An explicit construction in terms of matrices taking the three by six matrix (A, B) to its Smith normal form can be found in [81] (proof of Proposition 3.4) where the existence of the matrices P and Q introduced in (4.13) is proven.

Similarly one can identify the matrix M with the lowest common right multiple of A and B . Its definition is:

Definition: the integral matrix M is a *right multiple* of the integral matrix A if there is an integral matrix N_a such that $M = AN_a$. M is the *lowest common right multiple* of the integral matrices A and B if it is a right multiple of A and B and any other right multiple of A and B is a right multiple of M .

The lowest common right multiple M and the greatest common left divisor D have been related in theorem 5 of [82],

$$M = AD^{-1}B. \quad (\text{A.1})$$

The index of a quotient lattice is related to the integral matrices of generators as follows. Let Λ_c be a sublattice of Λ_l . Further C and L denote integral matrices of the corresponding generators. Then the index of the quotient lattice is

$$\left| \frac{\Lambda_l}{\Lambda_c} \right| = \left| \frac{\det C}{\det L} \right|.$$

Hence, taking the determinat of (A.1) proves the second equality in (4.9).

In the following, examples, relevant for the present paper, will be listed. As discussed in section 3.1 there are three possible lattices for Γ_a or Γ_b

$$\begin{aligned} \Gamma_1 &= \bigotimes_{l=1}^3 N_x^{(l)} \mathbb{Z}, \quad \text{all } N_x^{(l)} \text{ even,} \\ \Gamma_2 &= \text{span} \left(\begin{pmatrix} \frac{N_x^{(1)}}{2} \\ N_x^{(2)} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{N_x^{(1)}}{2} \\ -N_x^{(2)} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ N_x^{(3)} \end{pmatrix} \right) \quad \frac{N_x^{(1)}}{2}, N_x^{(2)} \text{ odd and } N_x^{(3)} \text{ even,} \\ \Gamma_3 &= \text{span} \left(\begin{pmatrix} \frac{N_x^{(1)}}{2} \\ N_x^{(2)} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{N_x^{(1)}}{2} \\ -N_x^{(2)} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ N_x^{(2)} \\ N_x^{(3)} \end{pmatrix} \right) \quad \frac{N_x^{(1)}}{2}, N_x^{(2)} \text{ and } N_x^{(3)} \text{ odd,} \end{aligned}$$

where x stands for a or b , respectively. There are six inequivalent configurations corresponding to symmetric pairings of these lattices.

$$\Gamma_a = \Gamma_b = \Gamma_1.$$

$$\begin{aligned} \Gamma_d &= \bigotimes_{l=1}^3 d^{(l)} \mathbb{Z}, \quad \text{with } d^{(l)} = g.c.d. \left(N_a^{(l)}, N_b^{(l)} \right), & \left| \frac{\Lambda_{\text{SO}(6)}}{\Gamma_d} \right| &= \frac{d^{(1)} d^{(2)} d^{(3)}}{2}, \\ \Gamma_a \cap \Gamma_b &= \bigotimes_{l=1}^3 \frac{N_a^{(l)} N_b^{(l)}}{d^{(l)}} \mathbb{Z}, & \left| \frac{\Lambda_{\text{SO}(6)}}{\Gamma_a \cap \Gamma_b} \right| &= \frac{1}{2} \prod_{l=1}^3 \frac{N_a^{(l)} N_b^{(l)}}{d^{(l)}}, \\ \left| \frac{\Lambda_{\text{SO}(6)}}{\Gamma_a} \right| &= \frac{N_a^{(1)} N_a^{(2)} N_a^{(3)}}{2}, & \left| \frac{\Lambda_{\text{SO}(6)}}{\Gamma_b} \right| &= \frac{N_b^{(1)} N_b^{(2)} N_b^{(3)}}{2}. \end{aligned}$$

$$\Gamma_a = \Gamma_1, \Gamma_b = \Gamma_2.$$

$$\begin{aligned} \Gamma_d &= \text{span} \left(\begin{pmatrix} \frac{d^{(1)}}{2} \\ d^{(2)} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{d^{(1)}}{2} \\ -d^{(2)} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ d^{(3)} \end{pmatrix} \right), & \left| \frac{\Lambda_{\text{SO}(6)}}{\Gamma_d} \right| &= \frac{d^{(1)}d^{(2)}d^{(3)}}{2}, \\ \Gamma_a \cap \Gamma_b &= \bigotimes_{l=1}^3 \frac{N_a^{(l)} N_b^{(l)}}{d^{(l)}} \mathbb{Z}, & \left| \frac{\Lambda_{\text{SO}(6)}}{\Gamma_a \cap \Gamma_b} \right| &= \frac{1}{2} \prod_{l=1}^3 \frac{N_a^{(l)} N_b^{(l)}}{d^{(l)}}, \\ \left| \frac{\Lambda_{\text{SO}(6)}}{\Gamma_a} \right| &= \frac{N_a^{(1)} N_a^{(2)} N_a^{(3)}}{2}, & \left| \frac{\Lambda_{\text{SO}(6)}}{\Gamma_b} \right| &= \frac{N_b^{(1)} N_b^{(2)} N_b^{(3)}}{2}. \end{aligned}$$

$$\Gamma_a = \Gamma_1, \Gamma_b = \Gamma_3.$$

$$\begin{aligned} \Gamma_d &= \text{span} \left(\begin{pmatrix} \frac{d^{(1)}}{2} \\ d^{(2)} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{d^{(1)}}{2} \\ -d^{(2)} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ d^{(2)} \\ d^{(3)} \end{pmatrix} \right), & \left| \frac{\Lambda_{\text{SO}(6)}}{\Gamma_d} \right| &= \frac{d^{(1)}d^{(2)}d^{(3)}}{2}, \\ \Gamma_a \cap \Gamma_b &= \bigotimes_{l=1}^3 \frac{N_a^{(l)} N_b^{(l)}}{d^{(l)}} \mathbb{Z}, & \left| \frac{\Lambda_{\text{SO}(6)}}{\Gamma_a \cap \Gamma_b} \right| &= \frac{1}{2} \prod_{l=1}^3 \frac{N_a^{(l)} N_b^{(l)}}{d^{(l)}}, \\ \left| \frac{\Lambda_{\text{SO}(6)}}{\Gamma_a} \right| &= \frac{N_a^{(1)} N_a^{(2)} N_a^{(3)}}{2}, & \left| \frac{\Lambda_{\text{SO}(6)}}{\Gamma_b} \right| &= \frac{N_b^{(1)} N_b^{(2)} N_b^{(3)}}{2}. \end{aligned}$$

$$\Gamma_a = \Gamma_b = \Gamma_2.$$

$$\begin{aligned} \Gamma_d &= \text{span} \left(\begin{pmatrix} \frac{d^{(1)}}{2} \\ d^{(2)} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{d^{(1)}}{2} \\ -d^{(2)} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ d^{(3)} \end{pmatrix} \right), & \left| \frac{\Lambda_{\text{SO}(6)}}{\Gamma_d} \right| &= \frac{d^{(1)}d^{(2)}d^{(3)}}{2}, \\ \Gamma_a \cap \Gamma_b &= \text{span} \left(\begin{pmatrix} \frac{N_a^{(1)} N_b^{(1)}}{2d^{(1)}} \\ \frac{N_a^{(2)} N_b^{(2)}}{d^{(2)}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{N_a^{(1)} N_b^{(1)}}{2d^{(1)}} \\ -\frac{N_a^{(2)} N_b^{(2)}}{d^{(2)}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{N_a^{(3)} N_b^{(3)}}{d^{(3)}} \end{pmatrix} \right), & \left| \frac{\Lambda_{\text{SO}(6)}}{\Gamma_a \cap \Gamma_b} \right| &= \frac{1}{2} \prod_{l=1}^3 \frac{N_a^{(l)} N_b^{(l)}}{d^{(l)}}, \\ \left| \frac{\Lambda_{\text{SO}(6)}}{\Gamma_a} \right| &= \frac{N_a^{(1)} N_a^{(2)} N_a^{(3)}}{2}, & \left| \frac{\Lambda_{\text{SO}(6)}}{\Gamma_b} \right| &= \frac{N_b^{(1)} N_b^{(2)} N_b^{(3)}}{2}. \end{aligned}$$

$$\Gamma_a = \Gamma_2, \Gamma_b = \Gamma_3.$$

$$\begin{aligned} \Gamma_d &= \text{span} \left(\begin{pmatrix} \frac{d^{(1)}}{2} \\ d^{(2)} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{d^{(1)}}{2} \\ -d^{(2)} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ d^{(2)} \\ d^{(3)} \end{pmatrix} \right), & \left| \frac{\Lambda_{\text{SO}(6)}}{\Gamma_d} \right| &= \frac{d^{(1)}d^{(2)}d^{(3)}}{2}, \\ \Gamma_a \cap \Gamma_b &= \text{span} \left(\begin{pmatrix} \frac{N_a^{(1)} N_b^{(1)}}{2d^{(1)}} \\ \frac{N_a^{(2)} N_b^{(2)}}{d^{(2)}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{N_a^{(1)} N_b^{(1)}}{2d^{(1)}} \\ -\frac{N_a^{(2)} N_b^{(2)}}{d^{(2)}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{N_a^{(3)} N_b^{(3)}}{d^{(3)}} \end{pmatrix} \right), & \left| \frac{\Lambda_{\text{SO}(6)}}{\Gamma_a \cap \Gamma_b} \right| &= \frac{1}{2} \prod_{l=1}^3 \frac{N_a^{(l)} N_b^{(l)}}{d^{(l)}}, \\ \left| \frac{\Lambda_{\text{SO}(6)}}{\Gamma_a} \right| &= \frac{N_a^{(1)} N_a^{(2)} N_a^{(3)}}{2}, & \left| \frac{\Lambda_{\text{SO}(6)}}{\Gamma_b} \right| &= \frac{N_b^{(1)} N_b^{(2)} N_b^{(3)}}{2}. \end{aligned}$$

$$\Gamma_a = \Gamma_b = \Gamma_3.$$

$$\begin{aligned} \Gamma_d &= \text{span} \left(\begin{pmatrix} \frac{d^{(1)}}{2} \\ d^{(2)} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{d^{(1)}}{2} \\ -d^{(2)} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ d^{(2)} \\ d^{(3)} \end{pmatrix} \right), & \left| \frac{\Lambda_{\text{SO}(6)}}{\Gamma_d} \right| &= \frac{d^{(1)}d^{(2)}d^{(3)}}{2}, \\ \Gamma_a \cap \Gamma_b &= \text{span} \left(\begin{pmatrix} \frac{N_a^{(1)}N_b^{(1)}}{2d^{(1)}} \\ \frac{N_a^{(2)}N_b^{(2)}}{d^{(2)}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{N_a^{(1)}N_b^{(1)}}{2d^{(1)}} \\ -\frac{N_a^{(2)}N_b^{(2)}}{d^{(2)}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{N_a^{(2)}N_b^{(2)}}{d^{(2)}} \\ \frac{N_a^{(3)}N_b^{(3)}}{d^{(3)}} \end{pmatrix} \right), & \left| \frac{\Lambda_{\text{SO}(6)}}{\Gamma_a \cap \Gamma_b} \right| &= \frac{1}{2} \prod_{l=1}^3 \frac{N_a^{(l)}N_b^{(l)}}{d^{(l)}}, \\ \left| \frac{\Lambda_{\text{SO}(6)}}{\Gamma_a} \right| &= \frac{N_a^{(1)}N_a^{(2)}N_a^{(3)}}{2}, & \left| \frac{\Lambda_{\text{SO}(6)}}{\Gamma_b} \right| &= \frac{N_b^{(1)}N_b^{(2)}N_b^{(3)}}{2}. \end{aligned}$$

B Normalisation

In order to get canonically normalised kinetic terms in four dimensions the zero modes need to satisfy the orthogonality relation [1]⁶

$$\alpha'^{-3} e^{-\Phi_b} \prod_{l=1}^3 \frac{\text{Im} \tau_l}{\text{Im} K_l |1 - \tau_l|^2} \int_{T^6} d^6 z \text{Tr} \left\{ \phi^i \cdot (\phi^j)^\dagger \right\} = \delta_{i,j}, \quad (\text{B.1})$$

where the integration is over complex coordinates (2.9) and the metric is taken from (2.10). Further, Φ_b is the ten dimensional type IIB dilaton which is chosen to be constant. Its exponential in (B.1) is a universal factor at all open string tree level contributions to the effective action. The domain of integration in (B.1) is given by the fundamental domain of T^6 , which is the unit cell of the lattice spanned by

$$\begin{aligned} \vec{v}_1 &= (1, 0, 0)^T, & \vec{v}_2 &= (K_1, -K_2, 0)^T, & \vec{v}_3 &= (0, 1, 0)^T, \\ \vec{v}_4 &= (0, K_2, -K_3)^T, & \vec{v}_5 &= (0, 0, 1)^T, & \vec{v}_6 &= (0, 0, 2K_3)^T. \end{aligned} \quad (\text{B.2})$$

With that the integration in (B.1) can be expressed as

$$\prod_{l=1}^3 \frac{\text{Im} \tau_l}{\text{Im} K_l |1 - \tau_l|^2} \int_{T^6} d^6 z = 2 \prod_{l=1}^3 \frac{\text{Im}(\tau_l)}{|1 - \tau_l|^2} \int_0^1 dy_{2l-1} \int_0^1 dy_{2l}.$$

For each zero mode ϕ^i and ϕ^j the parameters $\delta_i, \delta_j \in \frac{\Lambda_{\text{SO}(6)}}{\Gamma_d}$ are fixed according to the definition of the labels in (4.31) (see also the discussion after (4.29)). Hence, the sum in the trace in (B.1) has to be taken only over $l \in \frac{\Lambda_{\text{SO}(6)}}{\Gamma_a \cap \Gamma_b}$

$$\text{Tr} \left\{ \phi^i \cdot (\phi^j)^\dagger \right\} = \sum_{l \in \frac{\Lambda_{\text{SO}(6)}}{\Gamma_a \cap \Gamma_b}} \delta_{\delta_i, \delta_j} \phi_{l, l - \delta_i}^i \cdot \left(\phi_{l, l - \delta_j}^j \right)^\dagger, \quad (\text{B.3})$$

⁶The factor α'^{-3} has been included to match finally the convention of [1] in which the gauge coupling is given by $e^{\Phi_b/2} \alpha'^{3/2}$. (At the moment α' is fixed as in (2.6)).

where the Kronecker delta $\delta_{\delta_i, \delta_j}$ ensures that the sum is indeed a trace. The product of wavefunctions $\phi_{l, l-\delta_i}^i \cdot \left(\phi_{l, l-\delta_j}^j\right)^\dagger$ satisfies the following boundary conditions

$$\begin{aligned} \phi_{l, l-\delta_i}^i \left(\phi_{l, l-\delta_j}^j\right)^\dagger (\dots, y_2+1, \dots) &= \phi_{l+(1, -1, 0), l+(1, -1, 0)-\delta_i}^i \left(\phi_{l+(1, -1, 0), l+(1, -1, 0)-\delta_j}^j\right)^\dagger (\dots, y_2, \dots), \\ \phi_{l, l-\delta_i}^i \left(\phi_{l, l-\delta_j}^j\right)^\dagger (\dots, y_4+1, \dots) &= \phi_{l+(0, 1, -1), l+(0, 1, -1)-\delta_i}^i \left(\phi_{l+(0, 1, -1), l+(0, 1, -1)-\delta_j}^j\right)^\dagger (\dots, y_4, \dots), \\ \phi_{l, l-\delta_i}^i \left(\phi_{l, l-\delta_j}^j\right)^\dagger (\dots, y_6+1) &= \phi_{l+(0, 0, 2), l+(0, 0, 2)-\delta_i}^i \left(\phi_{l+(0, 0, 2), l+(0, 0, 2)-\delta_j}^j\right)^\dagger (\dots, y_6), \end{aligned} \quad (\text{B.4})$$

and therefore integrals of $\phi_{l, l-\delta_i}^i \cdot \left(\phi_{l, l-\delta_j}^j\right)^\dagger$ over T^6 , with different values for l can be related to T^6 lattice shifts in the following way,

$$\begin{aligned} \int_0^1 dy_2 \phi_{l+(1, -1, 0), l+(1, -1, 0)-\delta_i}^i \left(\phi_{l+(1, -1, 0), l+(1, -1, 0)-\delta_j}^j\right)^\dagger &= \int_1^2 dy_2 \phi_{l, l-\delta_i}^i \left(\phi_{l, l-\delta_j}^j\right)^\dagger, \\ \int_0^1 dy_4 \phi_{l+(0, 1, -1), l+(0, 1, -1)-\delta_i}^i \left(\phi_{l+(0, 1, -1), l+(0, 1, -1)-\delta_j}^j\right)^\dagger &= \int_1^2 dy_4 \phi_{l, l-\delta_i}^i \left(\phi_{l, l-\delta_j}^j\right)^\dagger, \\ \int_0^1 dy_6 \left(\phi_{l+(0, 0, 2), l+(0, 0, 2)-\delta_j}^j\right)^\dagger &= \int_1^2 dy_2 \phi_{l, l-\delta_i}^i \left(\phi_{l, l-\delta_j}^j\right)^\dagger. \end{aligned} \quad (\text{B.5})$$

The relations (B.5) can be used to replace the sum over l by an enlarged domain of integration. That means instead of integrating all terms, belonging to the trace, over the fundamental domain of T^6 , we just need to integrate one term with a fixed l , for example $l = 0$, over the enlarged domain of integration $\tilde{C}$, where $\tilde{C}$ is given by the unit cell of the lattice spanned by $\vec{v}_1 = (1, 0, 0)^T$, $\vec{v}_3 = (0, 1, 0)^T$ and $\vec{v}_5 = (0, 0, 1)^T$ as before, but

$$\begin{aligned} \vec{v}_2 &= \begin{pmatrix} \frac{N_a^{(1)} N_b^{(1)}}{d^{(1)}} K_1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{v}_4 = \begin{pmatrix} 0 \\ \frac{N_a^{(2)} N_b^{(2)}}{d^{(2)}} K_2 \\ 0 \end{pmatrix}, \quad \vec{v}_6 = \begin{pmatrix} 0 \\ 0 \\ \frac{N_a^{(3)} N_b^{(3)}}{d^{(3)}} K_3 \end{pmatrix}, \quad \text{for } \Gamma_a \cap \Gamma_b = \Gamma_1, \\ \vec{v}_2 &= \begin{pmatrix} \frac{N_a^{(1)} N_b^{(1)}}{2d^{(1)}} K_1 \\ \frac{N_a^{(2)} N_b^{(2)}}{d^{(2)}} K_2 \\ 0 \end{pmatrix}, \quad \vec{v}_4 = \begin{pmatrix} \frac{N_a^{(1)} N_b^{(1)}}{2d^{(1)}} K_1 \\ -\frac{N_a^{(2)} N_b^{(2)}}{d^{(2)}} K_2 \\ 0 \end{pmatrix}, \quad \vec{v}_6 = \begin{pmatrix} 0 \\ 0 \\ \frac{N_a^{(3)} N_b^{(3)}}{d^{(3)}} K_3 \end{pmatrix}, \quad \text{for } \Gamma_a \cap \Gamma_b = \Gamma_2, \\ \vec{v}_2 &= \begin{pmatrix} \frac{N_a^{(1)} N_b^{(1)}}{2d^{(1)}} K_1 \\ \frac{N_a^{(2)} N_b^{(2)}}{d^{(2)}} K_2 \\ 0 \end{pmatrix}, \quad \vec{v}_4 = \begin{pmatrix} \frac{N_a^{(1)} N_b^{(1)}}{2d^{(1)}} K_1 \\ -\frac{N_a^{(2)} N_b^{(2)}}{d^{(2)}} K_2 \\ 0 \end{pmatrix}, \quad \vec{v}_6 = \begin{pmatrix} 0 \\ \frac{N_a^{(2)} N_b^{(2)}}{d^{(2)}} K_2 \\ \frac{N_a^{(3)} N_b^{(3)}}{d^{(3)}} K_3 \end{pmatrix}, \quad \text{for } \Gamma_a \cap \Gamma_b = \Gamma_3, \end{aligned} \quad (\text{B.6})$$

where e.g. $\Gamma_a \cap \Gamma_b = \Gamma_1$ means that it is of the form Γ_1 in appendix A.

The explicit expression for $\phi_{0,-\delta_i}^i \cdot \left(\phi_{0,-\delta_j}^j\right)^\dagger$ can be deduced by inserting (3.16) and (4.33),

$$\begin{aligned} \phi_{0,-\delta_i}^i \cdot \left(\phi_{0,-\delta_j}^j\right)^\dagger &= \mathcal{N}_i^{ab} \mathcal{N}_j^{\star ab} \exp \left\{ -2\pi \sum_{k=1}^3 \frac{\tilde{I}_{ab}^{(k)}}{\text{Im}(K_k)} (\text{Im}(z_k))^2 \right\} \\ &\times \sum_{\lambda \in \Gamma_a \cap \Gamma_b} \sum_{\rho \in \Gamma_a \cap \Gamma_b} \prod_{h=1}^3 \exp \left\{ 2\pi i \left[\left(\tilde{I}_{ab}^{(h)} \lambda_{ab}^{(h)} + \frac{i^{(h)}}{N_b^{(h)}} \right) z_h - \left(\tilde{I}_{ab}^{(h)} \rho_{ab}^{(h)} + \frac{j^{(h)}}{N_b^{(h)}} \right) \bar{z}_h \right] \right\} \\ &\cdot \exp \left\{ \pi i \left[\left(\tilde{I}_{ab}^{(h)} \lambda_{ab}^{(h)} + \frac{i^{(h)}}{N_b^{(h)}} \right)^2 \frac{K_h}{\tilde{I}_{ab}^{(h)}} - \left(\tilde{I}_{ab}^{(h)} \rho_{ab}^{(h)} + \frac{j^{(h)}}{N_b^{(h)}} \right)^2 \frac{\bar{K}_h}{\tilde{I}_{ab}^{(h)}} \right] \right\}. \end{aligned} \quad (\text{B.7})$$

The y_1, y_3, y_5 dependence of the integrand is contained in factors $\exp(2\pi i y_{2h-1} M_h)$ with

$$M_h = \tilde{I}_{ab}^{(h)} \lambda_{ab}^{(h)} + \frac{i^{(h)}}{N_b^{(h)}} - \tilde{I}_{ab}^{(h)} \rho_{ab}^{(h)} - \frac{j^{(h)}}{N_b^{(h)}}. \quad (\text{B.8})$$

A closer look at (B.8) reveals that, taking the trace condition $\delta_{\delta_i, \delta_j}$ into account, the terms are actually integer, because the potentially non integer part, which is according to (4.31) given by $\frac{\delta_i^{(h)}}{N_b^{(h)}} - \frac{\delta_j^{(h)}}{N_b^{(h)}}$, vanishes for $\delta_i = \delta_j$. Hence, the integration of (B.7) over y_1, y_3 and y_5 yields one if all M_h in (B.8) vanish and zero otherwise. This implies a non vanishing result only for $\rho_{ab} = \lambda_{ab}$ and $i = j$, establishing orthogonality of the zero modes. The final result of the y_1, y_3, y_5 integration is

$$\begin{aligned} \int dy_1 dy_3 dy_5 \phi_{0,-\delta_i}^i \cdot \left(\phi_{0,-\delta_j}^j\right)^\dagger &= \delta_{ij} \left| \mathcal{N}_i^{ab} \right|^2 \\ &\times \sum_{\lambda \in \Gamma_a \cap \Gamma_b} \prod_{h=1}^3 \exp \left\{ -2\pi \frac{\tilde{I}_{ab}^{(h)}}{\text{Im}(K_h)} \left(\text{Im}(z_h) + \lambda^{(h)} \text{Im}(K_h) + \frac{i^{(h)}}{N_a^{(h)}} \frac{\text{Im}(K_h)}{\tilde{I}_{ab}^{(h)}} \right)^2 \right\}. \end{aligned} \quad (\text{B.9})$$

Similar to (B.5), the sum over λ can be replaced by an enlarged domain of integration over y_2, y_4, y_6

$$\int_{\tilde{C}} dy_2 dy_4 dy_6 \sum_{\lambda \in \Gamma_a \cap \Gamma_b} \dots \rightarrow \int_{\mathbb{R}^3} dy_2 dy_4 dy_6 \dots = \frac{1}{2} \int_{\mathbb{R}^3} d \left(\frac{\text{Im}(z_h)}{\text{Im}(K_h)} \right) \dots$$

There are three remaining Gaussian integrals, solved by

$$\prod_{h=1}^3 \int_{\mathbb{R}^3} d(\text{Im}(z_h)) e^{-2\pi \frac{\tilde{I}_{ab}^{(h)}}{\text{Im}(K_h)} \left(\text{Im}(z_h) + \frac{i^{(h)}}{N_a^{(h)}} \frac{\text{Im}(K_h)}{\tilde{I}_{ab}^{(h)}} \right)^2} = \prod_{h=1}^3 \sqrt{\frac{\text{Im}(K_h)}{2\tilde{I}_{ab}^{(h)}}}. \quad (\text{B.10})$$

Plugging the results into (B.1), one finds the normalisation condition

$$\alpha'^{-3} e^{-\Phi_b} \left| \mathcal{N}_i^{ab} \right|^2 \prod_{h=1}^3 \frac{\text{Im}(\tau_h)}{|1 - \tau_h|^2} \left(2\tilde{I}_{ab}^{(h)} \text{Im}(K_h) \right)^{-\frac{1}{2}} = 1. \quad (\text{B.11})$$

The normalisation factors in the text are chosen to be real solutions of (B.11).

C Overlap of three wavefunctions on T^6

In the first part of this appendix generalisations of the case with two extra dimensions to non coprime flux ranks are discussed. This will make it easier to access the calculation on a non factorisable T^6 in the second part of the current appendix.

C.1 Two extra dimensions

It will be useful to recapitulate and to generalise the computation of Yukawa couplings in the case of two extra dimensions. This has been dealt with in [1] for the case that all pairs from $\{N_a, N_b, N_c\}$ are coprime. The computation of the Yukawa coupling boils down to evaluating integrals of the form⁷

$$|\lambda_{ijk}| = \int_{T^2} d^2z \sum_{k_a=0}^{N_a-1} \sum_{k_b=0}^{N_b-1} \sum_{k_c=0}^{N_c-1} \phi_{k_a k_b}^{i, I_{ab}} \phi_{k_c k_a}^{j, I_{ca}} \phi_{k_b k_c}^{\star k, I_{cb}}. \quad (\text{C.1})$$

The ϕ 's denote zero modes in bifundamentals as before. Now, the zero mode label has been supplemented by the intersection number. For all N_α 's being coprime the matrix elements are related by shifts by cycles of the T^2 (analogous to e.g. expression (4.16)). This enabled the authors of [1] to trade the sums (C.1) for an enlarged integration region $\tilde{T}^2$. Before outlining more details it will be useful to include also the discussion of non coprime pairs among the N_α 's. In this case there are subsets within all matrix elements invariant under shifting zero mode arguments by T^2 cycles. As discussed in (4.5) these sectors are characterised by differences in row and column number, e.g.

$$k_a - k_b = \delta_{ab} \pmod{d_{ab}} \quad (\text{C.2})$$

where

$$d_{\alpha\beta} = g.c.d.(N_\alpha, N_\beta), \text{ for } \alpha, \beta \in \{a, b, c\}. \quad (\text{C.3})$$

Different $\delta_{\alpha\beta}$'s belong to different zero modes. Expression (C.1) should be modified to

$$|\lambda_{ijk}| = \int_{T^2} d^2z \sum_{k_a=0}^{N_a-1} \sum_{k_b=0}^{N_b-1} \sum_{k_c=0}^{N_c-1} \delta_{k_a-k_b, \delta_{ab}} \delta_{k_c-k_a, \delta_{ca}} \delta_{k_b-k_c, \delta_{bc}} \delta_{\delta_{ab}+\delta_{bc}+\delta_{ca}} \phi_{k_a k_b}^{i, I_{ab}} \phi_{k_c k_a}^{j, I_{ca}} \phi_{k_b k_c}^{\star k, I_{cb}}. \quad (\text{C.4})$$

Here, the first three δ 's are usual Kronecker deltas on $\mathbb{Z}_{d_{ab}}$, e.g. the first is one if (C.2) holds and zero otherwise. The last δ ensures that the trace is taken and is defined as

$$\delta_\rho = \begin{cases} 1 & \text{for } \rho = 0 \pmod{g.c.d.(d_{ab}, d_{bc}, d_{ca})}, \\ 0 & \text{else.} \end{cases} \quad (\text{C.5})$$

The following abbreviations will be convenient. Similar to the greatest common divisor (C.3) the lowest common multiple will be denoted as

$$N_{\alpha\beta} = l.c.m.(N_\alpha, N_\beta) \text{ for } \alpha, \beta \in \{a, b, c\}.$$

⁷For simplicity, moduli dependence will be suppressed in the present discussion.

For d_{abc} given by

$$d_{abc} = g.c.d. (N_{ab}, N_c) = g.c.d. (N_{ca}, N_b) = g.c.d. (N_{bc}, N_a) \quad (\text{C.6})$$

one finds

$$d_{abc} = \begin{cases} d_{ab}d_{bc}d_{ca} & \text{for } d_{ab} \neq d_{bc}, d_{ab} \neq d_{ca}, d_{bc} \neq d_{ca}, \\ d_{ab}d_{bc} & \text{for } d_{ab} = d_{ac} \neq d_{bc}, \\ d_{ab}^2 & \text{for } d_{ab} = d_{bc} = d_{ca}, \end{cases}$$

where cases which can be obtained by permutations of (a, b, c) have not been explicitly written. With d_{abc} one can relate the product of three numbers to its lowest common multiple

$$N_a N_b N_c = d_{abc} l.c.m. (N_a, N_b, N_c).$$

The double index e.g. k_a, k_b can now be replaced by a single index l as in (4.7) where it proves useful to change notation slightly. For a fixed $\delta_{\alpha\beta}$ which is encoded in the label i one replaces

$$\phi_{k_a k_b}^{i, I_{\alpha\beta}} = \phi_l^{i, I_{\alpha\beta}}, \quad l \in \mathbb{Z}_{N_{\alpha\beta}}.$$

With that notation, the Yukawa coupling (C.4) reads

$$|\lambda_{ijk}| = \int_{T^2} d^2 z \sum_{l=0}^{N_{abc}-1} \phi_l^{i, I_{ab}} \phi_{l-\delta_{ab}}^{j, I_{ca}} \phi_{l+\delta_{bc}}^{\star k, I_{cb}}. \quad (\text{C.7})$$

Notice, that it has been possible to drop the first three Kronecker deltas of (C.4). However, the last δ function in (C.4) translates into a selection rule involving the labels i, j, k . Its explicit form depends on the so far unspecified way $\delta_{\alpha\beta}$ is encoded in the label. Therefore it has been left out in (C.7) but should be kept in mind. The gauge indices (summation labels) have been chosen such that the sum implies matrix multiplication, i.e. consecutive row and column indices match (see (4.6)). Analogous to e.g. (4.16) l can be shifted by one when replacing $z \rightarrow z + \tau$, where τ is the complex structure modulus of the compactification T^2 . Any factor induced by such shifts (cf (3.9)) drops out due to the identity

$$\tilde{I}_{ab} + \tilde{I}_{bc} + \tilde{I}_{ca} = 0. \quad (\text{C.8})$$

Therefore, one can replace the sum over l by an enlarged integration region leading to

$$|\lambda_{ijk}| = \int_{\tilde{T}^2} d^2 z \phi_0^{i, I_{ab}} \phi_{-\delta_{ab}}^{j, I_{ca}} \phi_{\delta_{bc}}^{\star k, I_{cb}}, \quad (\text{C.9})$$

where $\tilde{T}^2$ has complex structure $N_{abc}\tau$. From hereon one can use the techniques presented in [1] to complete the computation for the generalised configuration with two extra dimensions.

C.2 Yukawa couplings for the T-dual of $T_{\text{SO}(12)}^6$

Now the Yukawa coupling is determined via computing⁸

$$|\lambda_{ijk}| = \alpha'^{-3} e^{-\Phi_b} \prod_{l=1}^3 \frac{\text{Im}\tau_l}{\text{Im}K_l |1 - \tau_l|^2} \int_C d^6 z \sum_{l \in \frac{\Lambda_{\text{SO}(6)}}{\Gamma_a \cap \Gamma_b \cap \Gamma_c}} \phi_l^{j, I_{ca}} \phi_{l-\delta_{ca}}^{i, I_{ab}} \phi_{l+\delta_{cb}}^{\star k, I_{cb}},$$

where z has been introduced in (2.9), the prefactor comes from $\sqrt{G}$ with the metric taken from (2.10). The region of integration is a parallelepiped $C \subset \mathbb{C}^3$ whose edges are given by the following vectors

$$\begin{aligned} l_1^T &= (1, 0, 0), & l_3^T &= (0, 1, 0) & l_5^T &= (0, 0, 1), \\ l_2^T &= (K_1, -K_2, 0), & l_4^T &= (0, K_2, -K_3), & l_6^T &= (0, 0, 2K_3). \end{aligned} \quad (\text{C.10})$$

Again, the sum over gauge indices l can be replaced by an enlarged integration region since shifts by l_2 , l_4 or l_6 induce index shifts according to (4.16), (4.18), (4.20). To be more specific, one needs to identify $\Gamma_a \cap \Gamma_b \cap \Gamma_c$. Repeating the analysis given in appendix A one finds that $\Gamma_a \cap \Gamma_b \cap \Gamma_c$ is given by either Γ_1 , Γ_2 or Γ_3 with

$$N_x^{(l)} = N_{abc}^{(l)} \equiv \frac{N_a^{(l)} N_b^{(l)} N_c^{(l)}}{d_{abc}^{(l)}}.$$

Here, $d_{\alpha\beta}^{(l)}$, $d_{abc}^{(l)}$ are defined as in respectively (C.3), (C.6) for each $l \in \{1, 2, 3\}$. The summation over $l \in \frac{\Lambda_{\text{SO}(6)}}{\Gamma_a \cap \Gamma_b \cap \Gamma_c}$ can be traded for an integration over a larger parallelepiped $\tilde{C} \in \mathbb{C}^3$,

$$|\lambda_{ijk}| = \alpha'^{-3} e^{-\Phi_b} \prod_{l=1}^3 \frac{\text{Im}\tau_l}{\text{Im}K_l |1 - \tau_l|^2} \int_{\tilde{C}} d^6 z \phi_0^{j, I_{ca}} \phi_{-\delta_{ca}}^{i, I_{ab}} \phi_{\delta_{cb}}^{\star k, I_{cb}}.$$

The edges of $\tilde{C}$ are l_1 , l_3 and l_5 as in (C.10) but l_2 , l_4 and l_6 replaced by the generators of one of the lattices Γ_1 , Γ_2 or Γ_3 from appendix A with

$$N_x^{(l)} = N_{abc}^{(l)} K_l.$$

Next, the integration variables are replaced by $\{y_1, \dots, y_6\}$ as in (2.9),

$$|\lambda_{ijk}| = 2\alpha'^{-3} e^{-\Phi_b} \prod_{l=1}^3 \frac{\text{Im}\tau_l}{|1 - \tau_l|^2} \int_{\tilde{C}} d^6 y \phi_0^{j, I_{ca}} \phi_{-\delta_{ca}}^{i, I_{ab}} \phi_{\delta_{cb}}^{\star k, I_{cb}}.$$

The range for the $\{y_1, y_3, y_5\}$ integration is the cube spanned by l_1 , l_3 and l_5 . The range for $\{y_2, y_4, y_6\}$ is a parallelepiped whose form depends on whether $\Gamma_a \cap \Gamma_b \cap \Gamma_c$ is of the

⁸The sign is determined exactly as in the factorisable case [1] and not discussed here.

form Γ_1 , Γ_2 , or Γ_3 . One finds for the edges of the parallelepiped

$$\begin{aligned}
 l_2 &= \frac{N_{abc}^{(1)}}{2} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \quad l_4 = \frac{N_{abc}^{(2)}}{2} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \quad l_6 = \frac{N_{abc}^{(3)}}{2} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
 &\text{if form of } \Gamma_a \cap \Gamma_b \cap \Gamma_c \text{ is } \Gamma_1, \\
 l_2 &= \frac{N_{abc}^{(1)}}{4} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + \frac{N_{abc}^{(2)}}{2} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \quad l_4 = \frac{N_{abc}^{(1)}}{4} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} - \frac{N_{abc}^{(2)}}{2} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \\
 l_6 &= \frac{N_{abc}^{(3)}}{2} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{if form of } \Gamma_a \cap \Gamma_b \cap \Gamma_c \text{ is } \Gamma_2, \\
 l_2 &= \frac{N_{abc}^{(1)}}{4} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + \frac{N_{abc}^{(2)}}{2} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \quad l_4 = \frac{N_{abc}^{(1)}}{4} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} - \frac{N_{abc}^{(2)}}{2} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \\
 l_6 &= \frac{N_{abc}^{(2)}}{2} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \frac{N_{abc}^{(3)}}{2} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{if form of } \Gamma_a \cap \Gamma_b \cap \Gamma_c \text{ is } \Gamma_3.
 \end{aligned} \tag{C.11}$$

Inserting the expressions from (3.16) and (4.33) into $\phi_0^{j,I_{ca}} \phi_{-\delta_{ca}}^{i,I_{ab}} \phi_{\delta_{cb}}^{k,I_{cb}}$, and using the relation (C.8), with the choice

$$|\tilde{I}_{ab}^{(h)}| + |\tilde{I}_{ca}^{(h)}| = |\tilde{I}_{cb}^{(h)}|, \quad \text{and} \quad \tilde{I}_{ab}^{(h)}, I_{ca}^{(h)}, I_{cb}^{(h)} > 0,$$

the explicit expression for the product of wavefunctions is given by

$$\begin{aligned}
 \phi_0^{j,I_{ca}} \phi_{-\delta_{ca}}^{i,I_{ab}} \phi_{\delta_{cb}}^{k,I_{cb}} &= \mathcal{N}_j^{ca} \mathcal{N}_i^{ab} \mathcal{N}_k^{cb} e^{-2\pi \sum_{h=1}^3 \frac{\tilde{I}_{cb}^{(h)}}{\text{Im}(K_h)} (\text{Im}(z_h))^2} \\
 &\times \sum_{\lambda_{xy} \in \Gamma_x \cap \Gamma_y} \prod_{h=1}^3 e^{2\pi i \left\{ \left(\tilde{I}_{ab}^{(h)} (\lambda_{ab}^{(h)} - \delta_{ca}^{(h)}) + \tilde{I}_{ca}^{(h)} \lambda_{ca}^{(h)} + \frac{i^{(h)}}{N_b^{(h)}} + \frac{j^{(h)}}{N_a^{(h)}} \right) z_h - \left(\tilde{I}_{cb}^{(h)} (\lambda_{cb}^{(h)} - \delta_{ca}^{(h)} - \delta_{ab}^{(h)}) + \frac{k^{(h)}}{N_c^{(h)}} \right) \bar{z}_h \right\}} \\
 &\times e^{\pi i \left\{ \left(\tilde{I}_{ab}^{(h)} (\lambda_{ab}^{(h)} - \delta_{ca}^{(h)}) + \frac{i^{(h)}}{N_b^{(h)}} \right)^2 \frac{K_h}{\tilde{I}_{ab}^{(h)}} + \left(\tilde{I}_{ca}^{(h)} \lambda_{ca}^{(h)} + \frac{j^{(h)}}{N_a^{(h)}} \right)^2 \frac{\tau_h}{\tilde{I}_{ca}^{(h)}} - \left(\tilde{I}_{cb}^{(h)} (\lambda_{cb}^{(h)} - \delta_{ca}^{(h)} - \delta_{ab}^{(h)}) + \frac{k^{(h)}}{N_c^{(h)}} \right)^2 \frac{\bar{K}_h}{\tilde{I}_{cb}^{(h)}} \right\}},
 \end{aligned} \tag{C.12}$$

where $x, y \in \{a, b, c\}$ and the wavefunction $\phi_{k_b, k_c}^{k, I_{cb}}$ has been relabelled such that $k^{(h)}/N_b^{(h)}$ is replaced by $k^{(h)}/N_c^{(h)}$. (This corresponds to swapping the label of ξ_{k_c, k_b} with minus the label of ξ_{k_b, k_c}^* , see (4.33).)

Before performing the integration, a closer look at the terms

$$\tilde{I}_{ab}^{(h)} (\lambda_{ab}^{(h)} - \delta_{ca}^{(h)}) + \tilde{I}_{ca}^{(h)} \lambda_{ca}^{(h)} + \frac{i^{(h)}}{N_b^{(h)}} + \frac{j^{(h)}}{N_a^{(h)}} - \tilde{I}_{cb}^{(h)} (\lambda_{cb}^{(h)} - \delta_{ca}^{(h)} - \delta_{ab}^{(h)}) - \frac{k^{(h)}}{N_c^{(h)}}, \tag{C.13}$$

reveals them to be integers. From the way the labels i, j, k in (4.31) were introduced, it can be deduced that the potentially non integer part in (C.13) is

$$-\tilde{I}_{ab}^{(h)}\delta_{ca}^{(h)} + \frac{\delta_{ab}^{(h)}}{N_b^{(h)}} + \frac{\delta_{ca}^{(h)}}{N_a^{(h)}} + \tilde{I}_{cb}^{(h)}\left(\delta_{ca}^{(h)} + \delta_{ab}^{(h)}\right) - \frac{\delta_{cb}^{(h)}}{N_c^{(h)}}. \quad (\text{C.14})$$

However, when considering the trace of $\phi_l^{j,I_{ca}}\phi_{l-\delta_{ca}}^{i,I_{ab}}\phi_{l+\delta_{cb}}^{k,I_{cb}}$, only terms with $\delta_{ca} + \delta_{ab} = \delta_{cb}$ contribute and hence the terms in (C.14) vanish, the expression in (C.13) is indeed integer $\forall \{\lambda_{ab}, \lambda_{ca}, \lambda_{cb}\}$ and $\forall h \in \{1, 2, 3\}$. Therefore the integration over y_1, y_3 and y_5 leads to Kronecker deltas,

$$\begin{aligned} \int_0^1 dy_{2h-1} e^{2\pi i \left(\tilde{I}_{ab}^{(h)}(\lambda_{ab}^{(h)} - \delta_{ca}^{(h)}) + \tilde{I}_{ca}^{(h)}\lambda_{ca}^{(h)} + \frac{i^{(h)}}{N_b^{(h)}} + \frac{j^{(h)}}{N_a^{(h)}} - \tilde{I}_{cb}^{(h)}(\lambda_{cb}^{(h)} - \delta_{ca}^{(h)} - \delta_{ab}^{(h)}) - \frac{k^{(h)}}{N_c^{(h)}} \right) y_{2h-1}} &= (\text{C.15}) \\ &= \delta_{\tilde{I}_{ab}^{(h)}(\lambda_{ab}^{(h)} - \delta_{ca}^{(h)}) + \tilde{I}_{ca}^{(h)}\lambda_{ca}^{(h)} + \frac{i^{(h)}}{N_b^{(h)}} + \frac{j^{(h)}}{N_a^{(h)}} - \tilde{I}_{cb}^{(h)}(\lambda_{cb}^{(h)} - \delta_{ca}^{(h)} - \delta_{ab}^{(h)}) - \frac{k^{(h)}}{N_c^{(h)}}}, \end{aligned}$$

which imply the following Diophantine equations $h \in \{1, 2, 3\}$

$$\tilde{I}_{ab}^{(h)}\left(\lambda_{ab}^{(h)} - \delta_{ca}^{(h)}\right) + \tilde{I}_{ca}^{(h)}\lambda_{ca}^{(h)} + \frac{i^{(h)}}{N_b^{(h)}} + \frac{j^{(h)}}{N_a^{(h)}} - \tilde{I}_{cb}^{(h)}\left(\lambda_{cb}^{(h)} - \delta_{ca}^{(h)} - \delta_{ab}^{(h)}\right) - \frac{k^{(h)}}{N_c^{(h)}} = 0. \quad (\text{C.16})$$

In [67] Diophantine equations arose from the requirement that projections of the intersecting D6 branes form closed triangles in each plane. These equations could be solved in terms of wrapping numbers after performing a relabelling of the intersection points,

$$i^{(h)} \rightarrow \frac{i^{(h)}}{d_b^{(h)}} I_{cb}^{(h)}, \quad j^{(h)} \rightarrow \frac{j^{(h)}}{d_a^{(h)}} I_{ba}^{(h)}, \quad k^{(h)} \rightarrow \frac{k^{(h)}}{d_c^{(h)}} I_{ac}^{(h)}, \quad (\text{C.17})$$

where e.g. $d_a^{(h)} = g.c.d\left(I_{ab}^{(h)}, I_{ac}^{(h)}\right)$. It can happen that intersection points lose their label. The corresponding Yukawa couplings are equal to others for which no label is lost [67]. Performing the same relabelling (C.17) on the type IIB side, the solutions to (C.16) are given by

$$\begin{aligned} \lambda_{ab}^{(h)} &= N_{abc}^{(h)} p^{(h)} + N_a^{(h)} N_b^{(h)} M_c^{(h)} q^{(h)} + \frac{j^{(h)}}{d_a^{(h)}} N_b^{(h)} + \delta_{ca}^{(h)}, \\ \lambda_{ca}^{(h)} &= N_{abc}^{(h)} p^{(h)} + N_a^{(h)} M_b^{(h)} N_c^{(h)} q^{(h)} - \frac{k^{(h)}}{d_c^{(h)}} N_a^{(h)}, \\ \lambda_{cb}^{(h)} &= N_{abc}^{(h)} p^{(h)} + M_a^{(h)} N_b^{(h)} N_c^{(h)} q^{(h)} + \frac{i^{(h)}}{d_b^{(h)}} N_c^{(h)} + \delta_{ca}^{(h)} + \delta_{ab}, \end{aligned} \quad (\text{C.18})$$

where $p^{(h)}$ and $q^{(h)}$ are components of three dimensional lattice vectors to be specified shortly, and $M_\alpha^{(h)} = n_\alpha^{(h)} - m_\alpha^{(h)}$. After integrating $\phi^{j,I_{ca}}\phi^{i,I_{ab}}\phi^{*k,I_{cb}}$ over y_1, y_3 and y_5 and

evaluating the condition (C.18), one gets

$$\begin{aligned}
 & \int_0^1 \int_0^1 \int_0^1 dy_1 dy_3 dy_5 \phi_0^{j,I_{ca}} \phi_{\delta_{ca}}^{i,I_{ab}} \phi_{\delta_{cb}}^{*k,I_{cb}} = \mathcal{N}_j^{ca} \mathcal{N}_i^{ab} \mathcal{N}_k^{cb} \\
 & = \sum_{p \in \Lambda_p^3} \sum_{q \in \Lambda_q^3} \prod_{h=1}^3 e^{\pi i \left(\frac{i^{(h)}}{d_b^{(h)} I_{ab}^{(h)}} + \frac{j^{(h)}}{d_a^{(h)} I_{ca}^{(h)}} + \frac{k^{(h)}}{d_c^{(h)} I_{cb}^{(h)}} + 2q^{(h)} \right)^2} K_h \left| I_{ab}^{(h)} I_{bc}^{(h)} I_{ca}^{(h)} \right| \\
 & \quad \times e^{-2\pi \frac{\tilde{I}_{cb}^{(h)}}{\text{Im}(K_h)} \left[\text{Im}(z_h) + N_{abc}^{(h)} \text{Im}(K_h) p^{(h)} + \left(M_a^{(h)} N_b^{(h)} N_c^{(h)} q^{(h)} + \frac{i^{(h)}}{d_b^{(h)}} \frac{I_{cb}^{(h)}}{\tilde{I}_{cb}^{(h)}} - \frac{k^{(h)}}{d_c^{(h)}} \frac{I_{ca}^{(h)}}{\tilde{I}_{cb}^{(h)}} \right) \text{Im}(K_h) \right]^2},
 \end{aligned} \tag{C.19}$$

where Λ_p^3 and Λ_q^3 are three dimensional lattices, with a lattice structure such that λ_{ab} , λ_{ca} and λ_{cb} in (C.18) belong to the lattices $\Gamma_a \cap \Gamma_b$, $\Gamma_c \cap \Gamma_a$ and $\Gamma_c \cap \Gamma_b$, respectively. The components $p^{(h)}$ have to be chosen, such that the vectors $(N_{abc}^{(1)} p^{(1)}, N_{abc}^{(2)} p^{(2)}, N_{abc}^{(3)} p^{(3)})^T$ belong to the SO(6) lattice. Therefore Λ_p^3 takes the form

$$\begin{aligned}
 \Lambda_p^3 &= \text{span} \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) && \text{if } \Gamma_a \cap \Gamma_b \cap \Gamma_c \text{ is of the form } \Gamma_1, \\
 \Lambda_p^3 &= \text{span} \left(\begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) && \text{if } \Gamma_a \cap \Gamma_b \cap \Gamma_c \text{ is of the form } \Gamma_2, \\
 \Lambda_p^3 &= \text{span} \left(\begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right) && \text{if } \Gamma_a \cap \Gamma_b \cap \Gamma_c \text{ is of the form } \Gamma_3
 \end{aligned} \tag{C.20}$$

and the terms $N_{abc}^{(h)} \text{Im}(K_h) p^{(h)}$ in (C.19) are components of vectors in a lattice with fundamental cell $\tilde{C}$. That means a shift of p by one of the generators of Λ_p^3 in (C.19) can be absorbed into the integration by shifting the domain of integration to a neighbouring parallelepiped in the lattice with fundamental cell (C.11). That way, the sum over p can be absorbed into the integration by enlarging the domain of integration,

$$\int_{\tilde{C}} dy_2 dy_4 dy_6 \sum_{p \in \Lambda_p^3} \dots \rightarrow \int_{\mathbb{R}^3} dy_2 dy_4 dy_6 \dots$$

Now the remaining integration can be performed,

$$\begin{aligned}
 & \mathcal{N}_j^{ca} \mathcal{N}_k^{ab} \mathcal{N}_k^{cb} \prod_{k=1}^3 \frac{\text{Im} \tau_k}{\text{Im}(K_k) |1 - \tau_k|^2} \int_{\mathbb{R}} d(\text{Im}(z_k)) \\
 & \times \sum_{q \in \Lambda_q^3} \prod_{h=1}^3 e^{\pi i \left(\frac{i^{(h)}}{d_b^{(h)} I_{ab}^{(h)}} + \frac{j^{(h)}}{d_a^{(h)} I_{ca}^{(h)}} + \frac{k^{(h)}}{d_c^{(h)} I_{cb}^{(h)}} + 2q^{(h)} \right)^2} K_h \left| I_{ab}^{(h)} I_{bc}^{(h)} I_{ca}^{(h)} \right|
 \end{aligned} \tag{C.21}$$

$$\begin{aligned} & \times e^{-2\pi \frac{\tilde{f}_{cb}^{(h)}}{\text{Im}(K_h)} \left[\text{Im}(z_h) + \left(M_a^{(h)} N_b^{(h)} N_c^{(h)} q^{(h)} + \frac{i^{(h)}}{d_b^{(h)}} \frac{I_{cb}^{(h)}}{\tilde{f}_{cb}^{(h)}} - \frac{k^{(h)}}{d_c^{(h)}} \frac{I_{ca}^{(h)}}{\tilde{f}_{cb}^{(h)}} \right) \text{Im}(K_h) \right]^2} = \\ & = \mathcal{N}_j^{ca} \mathcal{N}_k^{ab} \mathcal{N}_l^{cb} \left[\prod_{l=1}^3 \frac{\text{Im}(\tau_l)}{|1 - \tau_l|^2} \left(2\tilde{f}_{cb}^{(l)} \text{Im}(K_l) \right)^{-\frac{1}{2}} \right] \sum_{q \in \Lambda_q^3} \exp \left\{ -\frac{A_{i,j,k}(q)}{2\pi\alpha'} \right\}, \end{aligned}$$

with

$$A_{i,j,k}(q) = -2\pi^2 \alpha' i \sum_{h=1}^3 \left(\frac{i^{(h)}}{d_b^{(h)} I_{ab}^{(h)}} + \frac{j^{(h)}}{d_a^{(h)} I_{ca}^{(h)}} + \frac{k^{(h)}}{d_c^{(h)} I_{cb}^{(h)}} + 2q^{(h)} \right)^2 K_h \left| I_{ab}^{(h)} I_{bc}^{(h)} I_{ca}^{(h)} \right|.$$

matching the definition of [67]. In [67] dimensionful Kähler moduli have been used. The explicit relation is (see (2.6))

$$4\pi^2 \alpha' \text{Im} K_h = g^{(h)}$$

where $g^{(h)}$ is the determinant of the metric in the h^{th} complex plane in the coordinates (2.1), as it was used in [67]. Up to now the lattice, to which the summation index q belongs, has not been specified. It can be deduced from (C.18): the p dependence in (C.18) is eliminated in linear combinations of the three Diophantine equations

$$\begin{aligned} \lambda_{ab}^{(h)} - \lambda_{ca}^{(h)} - \delta_{ca}^{(h)} &= 2I_{cb}^{(h)} N_a^{(h)} q^{(h)} + \frac{j^{(h)}}{d_a^{(h)}} N_b^{(h)} + \frac{k^{(h)}}{d_c^{(h)}} N_a^{(h)} \\ \lambda_{ca}^{(h)} - \lambda_{cb}^{(h)} - \delta_{ca}^{(h)} - \delta_{ab}^{(h)} &= 2I_{ba}^{(h)} N_c^{(h)} q^{(h)} - \frac{i^{(h)}}{d_b^{(h)}} N_c^{(h)} - \frac{k^{(h)}}{d_c^{(h)}} N_a^{(h)} \\ \lambda_{cb}^{(h)} - \lambda_{ab}^{(h)} - \delta_{ab}^{(h)} &= 2I_{ac}^{(h)} N_b^{(h)} q^{(h)} - \frac{j^{(h)}}{d_a^{(h)}} N_b^{(h)} - \frac{i^{(h)}}{d_b^{(h)}} N_c^{(h)}. \end{aligned} \quad (\text{C.22})$$

The left-hand sides of (C.22) are components of $\Lambda_{\text{SO}(6)}$ lattice vectors.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] D. Cremades, L.E. Ibáñez and F. Marchesano, *Computing Yukawa couplings from magnetized extra dimensions*, *JHEP* **05** (2004) 079 [[hep-th/0404229](#)] [[INSPIRE](#)].
- [2] I. Antoniadis, A. Kumar and B. Panda, *Fermion wavefunctions in magnetized branes: theta identities and Yukawa couplings*, *Nucl. Phys. B* **823** (2009) 116 [[arXiv:0904.0910](#)] [[INSPIRE](#)].
- [3] M. Berkooz, M.R. Douglas and R.G. Leigh, *Branes intersecting at angles*, *Nucl. Phys. B* **480** (1996) 265 [[hep-th/9606139](#)] [[INSPIRE](#)].
- [4] R. Blumenhagen, L. Görlich and B. Körs, *Supersymmetric 4D orientifolds of type IIA with D6-branes at angles*, *JHEP* **01** (2000) 040 [[hep-th/9912204](#)] [[INSPIRE](#)].
- [5] S. Förste, G. Honecker and R. Schreyer, *Supersymmetric $Z_N \times Z_M$ orientifolds in 4D with D-branes at angles*, *Nucl. Phys. B* **593** (2001) 127 [[hep-th/0008250](#)] [[INSPIRE](#)].

- [6] S. Förste, G. Honecker and R. Schreyer, *Orientifolds with branes at angles*, *JHEP* **06** (2001) 004 [[hep-th/0105208](#)] [[INSPIRE](#)].
- [7] M. Cvetič, G. Shiu and A.M. Uranga, *Three family supersymmetric standard-like models from intersecting brane worlds*, *Phys. Rev. Lett.* **87** (2001) 201801 [[hep-th/0107143](#)] [[INSPIRE](#)].
- [8] M. Cvetič, G. Shiu and A.M. Uranga, *Chiral four-dimensional $N = 1$ supersymmetric type IIA orientifolds from intersecting D6 branes*, *Nucl. Phys. B* **615** (2001) 3 [[hep-th/0107166](#)] [[INSPIRE](#)].
- [9] R. Blumenhagen, L. Görlich and T. Ott, *Supersymmetric intersecting branes on the type IIA T^6/Z_4 orientifold*, *JHEP* **01** (2003) 021 [[hep-th/0211059](#)] [[INSPIRE](#)].
- [10] M. Cvetič, I. Papadimitriou and G. Shiu, *Supersymmetric three family SU(5) grand unified models from type IIA orientifolds with intersecting D6-branes*, *Nucl. Phys. B* **659** (2003) 193 [Erratum *ibid.* **B 696** (2004) 298] [[hep-th/0212177](#)] [[INSPIRE](#)].
- [11] G. Honecker, *Chiral supersymmetric models on an orientifold of $Z_4 \times Z_2$ with intersecting D6-branes*, *Nucl. Phys. B* **666** (2003) 175 [[hep-th/0303015](#)] [[INSPIRE](#)].
- [12] M. Cvetič and I. Papadimitriou, *More supersymmetric standard-like models from intersecting D6-branes on type IIA orientifolds*, *Phys. Rev. D* **67** (2003) 126006 [[hep-th/0303197](#)] [[INSPIRE](#)].
- [13] R. Blumenhagen, D. Lüst and S. Stieberger, *Gauge unification in supersymmetric intersecting brane worlds*, *JHEP* **07** (2003) 036 [[hep-th/0305146](#)] [[INSPIRE](#)].
- [14] M. Cvetič, T. Li and T. Liu, *Supersymmetric Pati-Salam models from intersecting D6-branes: a road to the Standard Model*, *Nucl. Phys. B* **698** (2004) 163 [[hep-th/0403061](#)] [[INSPIRE](#)].
- [15] G. Honecker and T. Ott, *Getting just the supersymmetric Standard Model at intersecting branes on the Z_6 orientifold*, *Phys. Rev. D* **70** (2004) 126010 [Erratum *ibid.* **D 71** (2005) 069902] [[hep-th/0404055](#)] [[INSPIRE](#)].
- [16] G. Honecker, *Chiral $N = 1$ 4D orientifolds with D-branes at angles*, *Mod. Phys. Lett. A* **19** (2004) 1863 [[hep-th/0407181](#)] [[INSPIRE](#)].
- [17] R. Blumenhagen, F. Gmeiner, G. Honecker, D. Lüst and T. Weigand, *The statistics of supersymmetric D-brane models*, *Nucl. Phys. B* **713** (2005) 83 [[hep-th/0411173](#)] [[INSPIRE](#)].
- [18] R. Blumenhagen, M. Cvetič, F. Marchesano and G. Shiu, *Chiral D-brane models with frozen open string moduli*, *JHEP* **03** (2005) 050 [[hep-th/0502095](#)] [[INSPIRE](#)].
- [19] F. Gmeiner, R. Blumenhagen, G. Honecker, D. Lüst and T. Weigand, *One in a billion: MSSM-like D-brane statistics*, *JHEP* **01** (2006) 004 [[hep-th/0510170](#)] [[INSPIRE](#)].
- [20] D. Bailin and A. Love, *Towards the supersymmetric Standard Model from intersecting D6-branes on the Z'_6 orientifold*, *Nucl. Phys. B* **755** (2006) 79 [Erratum *ibid.* **B 783** (2007) 176] [[hep-th/0603172](#)] [[INSPIRE](#)].
- [21] D. Bailin and A. Love, *Almost the supersymmetric Standard Model from intersecting D6-branes on the Z'_6 orientifold*, *Phys. Lett. B* **651** (2007) 324 [Erratum *ibid.* **B 658** (2008) 292] [[arXiv:0705.0646](#)] [[INSPIRE](#)].
- [22] D. Bailin and A. Love, *Constructing the supersymmetric Standard Model from intersecting D6-branes on the Z'_6 orientifold*, *Nucl. Phys. B* **809** (2009) 64 [[arXiv:0801.3385](#)] [[INSPIRE](#)].
- [23] F. Gmeiner and G. Honecker, *Millions of standard models on Z'_6 ?*, *JHEP* **07** (2008) 052 [[arXiv:0806.3039](#)] [[INSPIRE](#)].

- [24] S. Förste and G. Honecker, *Rigid D6-branes on $T^6/(Z_2 \times Z_{2M} \times \Omega R)$ with discrete torsion*, *JHEP* **01** (2011) 091 [[arXiv:1010.6070](#)] [[INSPIRE](#)].
- [25] D. Bailin and A. Love, *Stabilising the supersymmetric Standard Model on the Z'_6 orientifold*, *Nucl. Phys. B* **854** (2012) 700 [[arXiv:1104.3522](#)] [[INSPIRE](#)].
- [26] G. Honecker, M. Ripka and W. Staessens, *The importance of being rigid: D6-brane model building on $T^6/Z_2 \times Z'_6$ with discrete torsion*, *Nucl. Phys. B* **868** (2013) 156 [[arXiv:1209.3010](#)] [[INSPIRE](#)].
- [27] G. Honecker and W. Staessens, *To tilt or not to tilt: discrete gauge symmetries in global intersecting D-brane models*, *JHEP* **10** (2013) 146 [[arXiv:1303.4415](#)] [[INSPIRE](#)].
- [28] J. Ecker, G. Honecker and W. Staessens, *Rigour and rigidity: systematics on particle physics D6-brane models on $Z_2 \times Z_6$* , *Fortsch. Phys.* **62** (2014) 981 [[arXiv:1409.1236](#)] [[INSPIRE](#)].
- [29] J. Ecker, G. Honecker and W. Staessens, *D6-brane model building on $Z_2 \times Z_6$: MSSM-like and left-right symmetric models*, *Nucl. Phys. B* **901** (2015) 139 [[arXiv:1509.00048](#)] [[INSPIRE](#)].
- [30] C. Bachas, *A way to break supersymmetry*, [hep-th/9503030](#) [[INSPIRE](#)].
- [31] R. Blumenhagen, L. Görlich, B. Kös and D. Lüst, *Noncommutative compactifications of type-I strings on tori with magnetic background flux*, *JHEP* **10** (2000) 006 [[hep-th/0007024](#)] [[INSPIRE](#)].
- [32] C. Angelantonj, I. Antoniadis, E. Dudas and A. Sagnotti, *Type I strings on magnetized orbifolds and brane transmutation*, *Phys. Lett. B* **489** (2000) 223 [[hep-th/0007090](#)] [[INSPIRE](#)].
- [33] C. Angelantonj and A. Sagnotti, *Type-I vacua and brane transmutation*, in *Quantization, gauge theory and strings. Proceedings, International Conference dedicated to the memory of professor Efim Fradkin*, Moscow, Russia, 5–10 June 2000, pg. 21 [[hep-th/0010279](#)] [[INSPIRE](#)].
- [34] G. Aldazabal, S. Franco, L.E. Ibáñez, R. Rabadán and A.M. Uranga, *$D = 4$ chiral string compactifications from intersecting branes*, *J. Math. Phys.* **42** (2001) 3103 [[hep-th/0011073](#)] [[INSPIRE](#)].
- [35] R. Blumenhagen, B. Kös and D. Lüst, *Type I strings with F flux and B flux*, *JHEP* **02** (2001) 030 [[hep-th/0012156](#)] [[INSPIRE](#)].
- [36] F. Marchesano and G. Shiu, *MSSM vacua from flux compactifications*, *Phys. Rev. D* **71** (2005) 011701 [[hep-th/0408059](#)] [[INSPIRE](#)].
- [37] F. Marchesano and G. Shiu, *Building MSSM flux vacua*, *JHEP* **11** (2004) 041 [[hep-th/0409132](#)] [[INSPIRE](#)].
- [38] E. Dudas and C. Timirgaziu, *Internal magnetic fields and supersymmetry in orientifolds*, *Nucl. Phys. B* **716** (2005) 65 [[hep-th/0502085](#)] [[INSPIRE](#)].
- [39] T. Kobayashi and S. Nagamoto, *Zero-modes on orbifolds: magnetized orbifold models by modular transformation*, *Phys. Rev. D* **96** (2017) 096011 [[arXiv:1709.09784](#)] [[INSPIRE](#)].
- [40] D. Cremades, L.E. Ibáñez and F. Marchesano, *Yukawa couplings in intersecting D-brane models*, *JHEP* **07** (2003) 038 [[hep-th/0302105](#)] [[INSPIRE](#)].
- [41] G. Aldazabal, S. Franco, L.E. Ibáñez, R. Rabadán and A.M. Uranga, *Intersecting brane worlds*, *JHEP* **02** (2001) 047 [[hep-ph/0011132](#)] [[INSPIRE](#)].
- [42] S.A. Abel and A.W. Owen, *CP violation and CKM predictions from discrete torsion*, *Nucl. Phys. B* **651** (2003) 191 [[hep-th/0205031](#)] [[INSPIRE](#)].

- [43] M. Cvetič and I. Papadimitriou, *Conformal field theory couplings for intersecting D-branes on orientifolds*, *Phys. Rev. D* **68** (2003) 046001 [*Erratum ibid.* **D 70** (2004) 029903] [[hep-th/0303083](#)] [[INSPIRE](#)].
- [44] S.A. Abel, M. Masip and J. Santiago, *Flavor changing neutral currents in intersecting brane models*, *JHEP* **04** (2003) 057 [[hep-ph/0303087](#)] [[INSPIRE](#)].
- [45] S.A. Abel and A.W. Owen, *Interactions in intersecting brane models*, *Nucl. Phys. B* **663** (2003) 197 [[hep-th/0303124](#)] [[INSPIRE](#)].
- [46] S.A. Abel and A.W. Owen, *N point amplitudes in intersecting brane models*, *Nucl. Phys. B* **682** (2004) 183 [[hep-th/0310257](#)] [[INSPIRE](#)].
- [47] D. Lüst, P. Mayr, R. Richter and S. Stieberger, *Scattering of gauge, matter and moduli fields from intersecting branes*, *Nucl. Phys. B* **696** (2004) 205 [[hep-th/0404134](#)] [[INSPIRE](#)].
- [48] S.A. Abel and B.W. Schofield, *One-loop Yukawas on intersecting branes*, *JHEP* **06** (2005) 072 [[hep-th/0412206](#)] [[INSPIRE](#)].
- [49] M. Bertolini, M. Billò, A. Lerda, J.F. Morales and R. Russo, *Brane world effective actions for D-branes with fluxes*, *Nucl. Phys. B* **743** (2006) 1 [[hep-th/0512067](#)] [[INSPIRE](#)].
- [50] S.A. Abel and M.D. Goodsell, *Intersecting brane worlds at one loop*, *JHEP* **02** (2006) 049 [[hep-th/0512072](#)] [[INSPIRE](#)].
- [51] S.A. Abel and M.D. Goodsell, *Realistic Yukawa couplings through instantons in intersecting brane worlds*, *JHEP* **10** (2007) 034 [[hep-th/0612110](#)] [[INSPIRE](#)].
- [52] D. Duo, R. Russo and S. Sciuto, *New twist field couplings from the partition function for multiply wrapped D-branes*, *JHEP* **12** (2007) 042 [[arXiv:0709.1805](#)] [[INSPIRE](#)].
- [53] R. Russo and S. Sciuto, *The twisted open string partition function and Yukawa couplings*, *JHEP* **04** (2007) 030 [[hep-th/0701292](#)] [[INSPIRE](#)].
- [54] M. Billò, M. Frau, I. Pesando, P. Di Vecchia, A. Lerda and R. Marotta, *Instantons in $N = 2$ magnetized D-brane worlds*, *JHEP* **10** (2007) 091 [[arXiv:0708.3806](#)] [[INSPIRE](#)].
- [55] M. Billò, M. Frau, I. Pesando, P. Di Vecchia, A. Lerda and R. Marotta, *Instanton effects in $N = 1$ brane models and the Kähler metric of twisted matter*, *JHEP* **12** (2007) 051 [[arXiv:0709.0245](#)] [[INSPIRE](#)].
- [56] I. Pesando, *Green functions and twist correlators for N branes at angles*, *Nucl. Phys. B* **866** (2013) 87 [[arXiv:1206.1431](#)] [[INSPIRE](#)].
- [57] I. Pesando, *Correlators of arbitrary untwisted operators and excited twist operators for N branes at angles*, *Nucl. Phys. B* **886** (2014) 243 [[arXiv:1401.6797](#)] [[INSPIRE](#)].
- [58] I. Pesando, *Canonical quantization of a string describing N branes at angles*, *Nucl. Phys. B* **889** (2014) 120 [[arXiv:1407.4627](#)] [[INSPIRE](#)].
- [59] I. Pesando, *Towards a fully stringy computation of Yukawa couplings on non factorized tori and non Abelian twist correlators (I): the classical solution and action*, *Nucl. Phys. B* **910** (2016) 618 [[arXiv:1512.07920](#)] [[INSPIRE](#)].
- [60] R. Blumenhagen, J.P. Conlon and K. Suruliz, *Type IIA orientifolds on general supersymmetric Z_N orbifolds*, *JHEP* **07** (2004) 022 [[hep-th/0404254](#)] [[INSPIRE](#)].
- [61] A.E. Faraggi, S. Förste and C. Timirgaziu, *$Z_2 \times Z_2$ heterotic orbifold models of non factorisable six dimensional toroidal manifolds*, *JHEP* **08** (2006) 057 [[hep-th/0605117](#)] [[INSPIRE](#)].

- [62] S. Förste, T. Kobayashi, H. Ohki and K.-J. Takahashi, *Non-factorisable $Z_2 \times Z_2$ heterotic orbifold models and Yukawa couplings*, *JHEP* **03** (2007) 011 [[hep-th/0612044](#)] [[INSPIRE](#)].
- [63] S. Förste, C. Timirgaziu and I. Zavala, *Orientifold's landscape: non-factorisable six-tori*, *JHEP* **10** (2007) 025 [[arXiv:0707.0747](#)] [[INSPIRE](#)].
- [64] T. Kimura, M. Ohta and K.-J. Takahashi, *Type IIA orientifolds and orbifolds on non-factorizable tori*, *Nucl. Phys. B* **798** (2008) 89 [[arXiv:0712.2281](#)] [[INSPIRE](#)].
- [65] S. Förste and I. Zavala, *Oddness from rigidity*, *JHEP* **07** (2008) 086 [[arXiv:0806.2328](#)] [[INSPIRE](#)].
- [66] D. Bailin and A. Love, *Intersecting D6-branes on the Z_{12} -II orientifold*, *JHEP* **01** (2014) 009 [[arXiv:1310.8215](#)] [[INSPIRE](#)].
- [67] S. Förste and C. Liyanage, *Yukawa couplings for intersecting D-branes on non-factorisable tori*, *JHEP* **03** (2015) 110 [[arXiv:1412.3645](#)] [[INSPIRE](#)].
- [68] M. Berasaluce-González, G. Honecker and A. Seifert, *Towards geometric D6-brane model building on non-factorisable toroidal Z_4 -orbifolds*, *JHEP* **08** (2016) 062 [[arXiv:1606.04926](#)] [[INSPIRE](#)].
- [69] M. Berasaluce-González, G. Honecker and A. Seifert, *Massless spectra and gauge couplings at one-loop on non-factorisable toroidal orientifolds*, *Nucl. Phys. B* **926** (2018) 112 [[arXiv:1709.07866](#)] [[INSPIRE](#)].
- [70] T.H. Buscher, *A symmetry of the string background field equations*, *Phys. Lett. B* **194** (1987) 59 [[INSPIRE](#)].
- [71] T.H. Buscher, *Path integral derivation of quantum duality in nonlinear σ -models*, *Phys. Lett. B* **201** (1988) 466 [[INSPIRE](#)].
- [72] G.W. Moore, *Arithmetic and attractors*, [hep-th/9807087](#) [[INSPIRE](#)].
- [73] J. Dai, R.G. Leigh and J. Polchinski, *New connections between string theories*, *Mod. Phys. Lett. A* **4** (1989) 2073 [[INSPIRE](#)].
- [74] P. Horava, *Background duality of open string models*, *Phys. Lett. B* **231** (1989) 251 [[INSPIRE](#)].
- [75] M.B. Green, *Space-time duality and Dirichlet string theory*, *Phys. Lett. B* **266** (1991) 325 [[INSPIRE](#)].
- [76] E. Alvarez, J.L.F. Barbon and J. Borlaf, *T duality for open strings*, *Nucl. Phys. B* **479** (1996) 218 [[hep-th/9603089](#)] [[INSPIRE](#)].
- [77] H. Dorn and H.J. Otto, *On T duality for open strings in general Abelian and non-Abelian gauge field backgrounds*, *Phys. Lett. B* **381** (1996) 81 [[hep-th/9603186](#)] [[INSPIRE](#)].
- [78] S. Förste, A.A. Kehagias and S. Schwager, *Non-Abelian duality for open strings*, *Nucl. Phys. B* **478** (1996) 141 [[hep-th/9604013](#)] [[INSPIRE](#)].
- [79] J. Borlaf and Y. Lozano, *Aspects of T duality in open strings*, *Nucl. Phys. B* **480** (1996) 239 [[hep-th/9607051](#)] [[INSPIRE](#)].
- [80] M. Newman, *Integral matrices*, Academic Press, U.S.A., (1972).
- [81] A. Bachem, *Beiträge zur Theorie der Corner Polyeder* (in German), Anton Hain Verlag, Meisenheim, Germany, (1976).
- [82] R.C. Thompson, *Left multiples and right divisors of integral matrices*, *Lin. Multilin. Alg.* **19** (1986) 287.