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Discrete quotients of 3-dimensional $\mathcal{N} = 4$ Coulomb branches via the cycle index

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Abstract: The study of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories via the associated Hilbert series, the so-called monopole formula, has been proven useful not only for 3-dimensional theories, but also for Higgs branches of 5 and 6-dimensional gauge theories with 8 supercharges. Recently, a conjecture connected different phases of 6-dimensional Higgs branches via *gauging of a discrete global S_n symmetry*. On the corresponding 3-dimensional Coulomb branch, this amounts to a geometric S_n-quotient. In this note, we prove the conjecture on Coulomb branches with unitary nodes and, moreover, extend it to Coulomb branches with other classical groups. The results promote discrete S_n-quotients to a versatile tool in the study of Coulomb branches.

Keywords: Discrete Symmetries, Extended Supersymmetry, Global Symmetries, Supersymmetric Gauge Theory

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Contents

1 Introduction and motivation

The study of Coulomb branches of 3-dimensional $\mathcal{N}=4$ gauge theories has been proven vital for the understanding of gauge theories with 8 supercharges in 5 and 6 dimensions. A powerful tool for analysing Coulomb branches as algebraic varieties is the Hilbert series — called monopole formula [1] — of the associated chiral ring. In the original 3-dimensional set-up, the monopole formula has provided a large number of interesting results and geometric insights [2–16], for instance in the geometry of nilpotent orbits.

The standard lore suggests that Higgs branches of theories with 8 supercharges in dimensions 3, 4, 5, and 6 are classically exact. In 5-dimensional $\mathcal{N}=1$ theories, the Higgs branch $\mathcal{M}_H^{5d}\big|_{g=\infty}$ at infinite gauge coupling, however, grows as new massless degrees of freedom appear in the form of instanton operators. As the Higgs branch is still a hyper-Kähler space, $\mathcal{M}_H^{5d}\big|_{g=\infty}$ has a 3-dimensional Coulomb branch counterpart, provided the global symmetry is large enough [17, 18]. To be precise, this means that a 3-dimensional $\mathcal{N}=4$ gauge theory exists such that its Coulomb branch agrees with $\mathcal{M}_H^{5d}\big|_{g=\infty}$. Such a quiver is further realised in the study of 5-brane webs and 7-branes.

Similarly, 6-dimensional $\mathcal{N}=(1,0)$ theories exhibit a previously unappreciated rich phase structure of the Higgs branch as recent developments have shown [19, 20]. As it turns out, many interesting effects on the 6-dimensional Higgs branches can be described neatly

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by associated 3-dimensional $\mathcal{N}=4$ theories, whose Coulomb branches $\mathcal{M}_C^{3d}$ agree with the 6-dimensional Higgs branches $\mathcal{M}_H^{6d}$ as algebraic varieties. In particular, the 3-dimensional quiver gauge theory is understood as a tool that captures the geometry of the moduli space. Besides the small E_8-instanton transition [19], another interesting phenomenon is *discrete gauging* [20]. For the latter, it is crucial to realise that the gauging of a discrete global symmetry Γ on $\mathcal{M}_H^{6d}$ corresponds to a quotient of $\mathcal{M}_C^{3d}$ by Γ. In other words, restriction to the Γ-invariant sector.

In this note we prove earlier conjectures and extend the concept of discrete quotients of 3-dimensional Coulomb branches to other scenarios. To begin with, we recall two examples.

Symmetric products of ALE spaces. Consider k D2 branes in the presence of n D6 branes in flat space. The worldvolume theory on the D2 branes is a 3-dimensional $\mathcal{N}=4$ $\mathrm{U}(k)$ gauge theory with one adjoint and n fundamental hypermultiplets. The corresponding quiver theory is the A-type ADHM quiver

$$T_{k,n}^{\text{A-type}} = \begin{array}{l} \bigcirc\ \text{Adj} \\ \bigcirc\ \mathrm{U}(k) \\ \ | \\ \square\ \mathrm{SU}(n) \end{array} \tag{1.1}$$

such that the Higgs branch is the moduli space $\mathcal{M}_{k,\mathrm{SU}(n),\mathbb{C}^2}$ of k $\mathrm{SU}(n)$-instantons on $\mathbb{C}^2$. Moreover, 3d mirror symmetry predicts that the Coulomb branch is the symmetric product of k copies of the A_{n-1} singularity [21, 22]. In detail,

$$\mathcal{M}_H\left(T_{k,n}^{\text{A-type}}\right) = \mathcal{M}_{k,\mathrm{SU}(n),\mathbb{C}^2} \qquad \text{and} \qquad \mathcal{M}_C\left(T_{k,n}^{\text{A-type}}\right) = \mathrm{Sym}^k\left(\mathbb{C}^2/\mathbb{Z}_n\right) . \tag{1.2}$$

We recall that a 3-dimensional $\mathcal{N}=4$ $\mathrm{U}(1)$ gauge theory with n fundamentals has $\mathbb{C}^2/\mathbb{Z}_n$ as Coulomb branch; hence, we may write

$$\mathcal{M}_C\left(\begin{array}{l} \bigcirc\ \text{Adj} \\ \bigcirc\ \mathrm{U}(k) \\ \ | \\ \square\ \mathrm{SU}(n) \end{array}\right) = \mathrm{Sym}^k\left(\mathcal{M}_C\left(\begin{array}{l} \bigcirc\ \mathrm{U}(1) \\ \ | \\ \square\ \mathrm{SU}(n) \end{array}\right)\right) . \tag{1.3}$$

These symmetry properties have been conjectured in two complementary studies: firstly, by computing the quantum corrections to the Coulomb branch metric in [21] and, secondly, by computing the Coulomb branch Hilbert series in [1].

Extending the above setting by an orientifold O6 plane changes the resulting 3-dimensional $\mathcal{N}=4$ worldvolume theory to an $\mathrm{USp}(2k)$ gauge theory with one antisymmetric and n fundamental hypermultiplets. The quiver theory is again an ADHM quiver

$$T_{k,n}^{\text{D-type}} = \begin{array}{l} \bigcirc\ \Lambda^2 \\ \bigcirc\ \mathrm{USp}(2k) \\ \ | \\ \square\ \mathrm{SO}(2n) \end{array} \tag{1.4}$$

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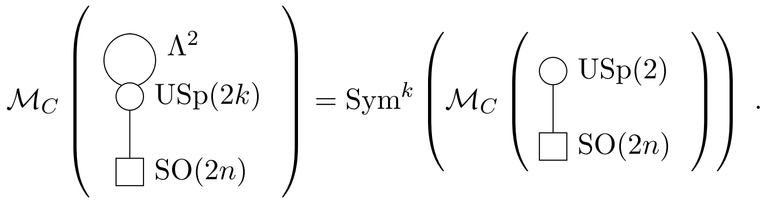
such that the Higgs branch is the moduli space of k SO$(2n)$-instantons on $\mathbb{C}^2$. Again, 3d mirror symmetry predicts that the Coulomb branch is the symmetric product of k copies of the D_n singularity [21, 22]. In other words,

$$\mathcal{M}_H\left(T^{\text{D-type}}_{k,n}\right) = \mathcal{M}_{k,\mathrm{SO}(2n),\mathbb{C}^2} \qquad \text{and} \qquad \mathcal{M}_C\left(T^{\text{D-type}}_{k,n}\right) = \mathrm{Sym}^k\left(\mathbb{C}^2/D_n\right) \ . \tag{1.5}$$

Recalling that the Coulomb branch of a SU(2) $\cong$ USp(2) gauge theory with n fundamentals is the D_n-singularity, one may conclude

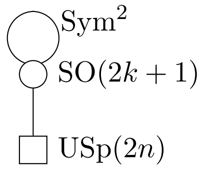


(1.6)

Again, this has been conjectured in [21] and [1] from different approaches.

Lastly, replacing the O6 plane by a hypothetical $\widetilde{\mathrm{O6}}^+$ plane [23, 24] implies that the 3-dimensional $\mathcal{N}=4$ worldvolume theory turns into a SO$(2k+1)$ gauge theory with one symmetric and n fundamental hypermultiplets. The quiver is given by

$T^{\text{D}'\text{-type}}_{k,n} =$



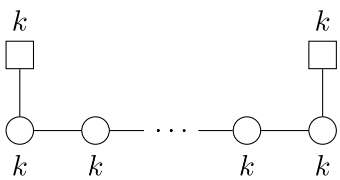
(1.7)

and it has been conjectured in [1] that the Coulomb branch is again the k-th symmetric product of a D-type singularity, i.e.

$$\mathcal{M}_C\left(T^{\text{D}'\text{-type}}_{k,n}\right) = \mathrm{Sym}^k\left(\mathbb{C}^2/D_{n+3}\right) \ . \tag{1.8}$$

6d Higgs branches. Following [20], consider n separated M5-branes on a $\mathbb{C}^2/\mathbb{Z}_k$ singularity. The 6-dimensional $\mathcal{N}=(1,0)$ worldvolume theory has $(n-1)$ tensor multiplets, a SU$(k)^{n-1}$ gauge group and bifundamental matter determined by a linear quiver

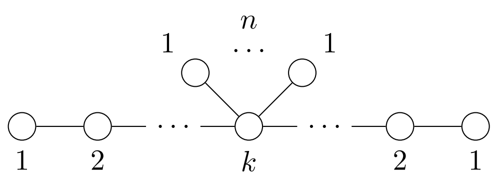
$Q^{\text{A}}_{n,k} =$



(1.9)

where all nodes are special unitary gauge or flavour nodes. The corresponding 3-dimensional $\mathcal{N}=4$ quiver gauge theory for n separated M5-branes is equipped with a *bouquet* of n nodes with 1, i.e.

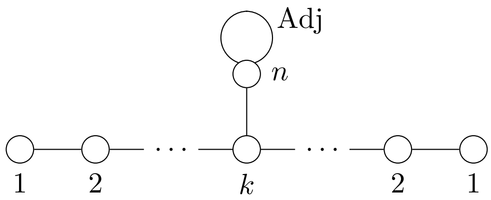
$F^{\text{A}}_{n,k} =$



(1.10)

with all nodes being unitary gauge groups. It is important to appreciate the global discrete S_n symmetry present in the problem of n identical objects. In particular, the Coulomb branch quiver has an apparent S_n symmetry. Following [20], the system admits different phases, in which n_i of the n separated M5-branes become coincident at positions x_i. These phases are then obtained by gauging a discrete $\prod_i S_{n_i} \subset S_n$ global symmetry in the 6-dimensional theory. Let us summarise the main conjectures of [20]:

(i) At infinite gauge coupling, the 3-dimensional quiver gauge theory for n coinciding M5-branes on a A_{k-1}-singularity is given by

$I^{\rm A}_{n,k} =$  (1.11)

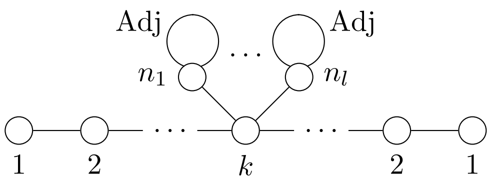
where all nodes are unitary gauge groups.

(ii) The 6-dimensional Higgs branches and 3-dimensional Coulomb branches then satisfy the following relations:

$$\mathcal{M}_H^{6d}(Q^{\rm A}_{n,k})\big|_{g<\infty} = \mathcal{M}_C^{3d}(F^{\rm A}_{n,k}) \,, \qquad \mathcal{M}_H^{6d}(Q^{\rm A}_{n,k})\big|_{g=\infty} = \mathcal{M}_C^{3d}(I^{\rm A}_{n,k}) \,, \tag{1.12}$$

$$\mathcal{M}_C^{3d}(I^{\rm A}_{n,k}) = \mathcal{M}_C^{3d}(F^{\rm A}_{n,k})/S_n \,. \tag{1.13}$$

(iii) Suppose a partition $\{n_i\}$ of n describes that the n M5-branes coincide in a pattern of n_i coinciding branes at different locations. The case of all branes separated corresponds to $\{1^n\}$, while all of them coinciding corresponds to $\{n\}$, i.e. infinite gauge coupling. Then the associated 3-dimensional quiver is conjectured to be

$F^{\rm A}_{\{n_i\},k} =$  (1.14)

with the relations

$$\mathcal{M}_H^{6d}(Q^{\rm A}_{n,k})\big|_{\{n_i\}} = \mathcal{M}_C^{3d}(F^{\rm A}_{\{n_i\},k}) \,, \quad \mathcal{M}_C^{3d}(F^{\rm A}_{\{n_i\},k}) = \mathcal{M}_C^{3d}(F^{\rm A}_{\{1^n\},k})/\prod_i S_{n_i} \,. \tag{1.15}$$

Here, $\prod_i S_{n_i}$ denotes the product of permutation groups which act on the sets of n_i coincident M5-branes.

Similarly, n M5-branes on a $\mathbb{C}^2/D_k$ singularity have been considered in [19]. The 6-dimensional $\mathcal{N}=(1,0)$ worldvolume theory is comprised of $(2n-1)$ tensor multiplets as

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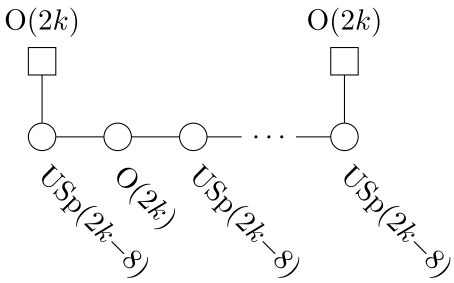
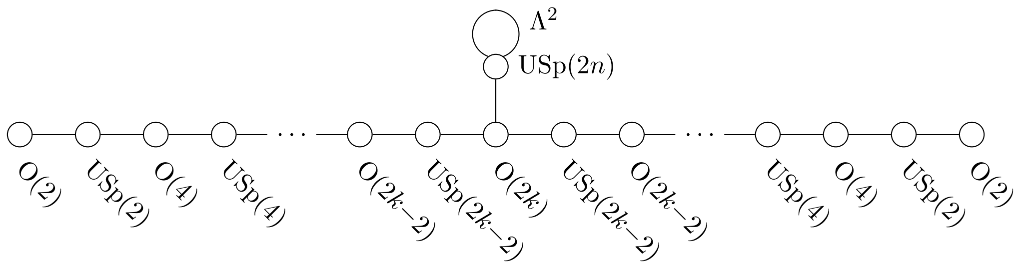
well as gauge groups and hypermultiplets determined by the quiver

$Q^{\mathrm{D}}_{n,k} =$ 

(1.16)

such that there are n $\mathrm{USp}(2k-8)$ and $(n-1)$ $\mathrm{O}(2k)$ gauge nodes in total. The associated 3-dimensional quiver theory for n coincident M5 branes, i.e. infinite gauge coupling in the 6-dimensional theory, has been conjectured to be

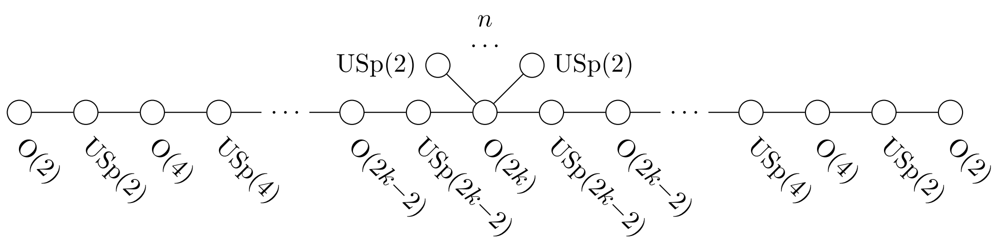
$I^{\mathrm{D}}_{n,k} =$ 

(1.17)

where Λ^2 denotes the traceless rank-2 antisymmetric representation of $\mathrm{USp}(2n)$. The theories are related via

$$\mathcal{M}^{6d}_{H}(Q^{\mathrm{D}}_{n,k})\big|_{g=\infty} = \mathcal{M}^{3d}_{C}(I^{\mathrm{D}}_{n,k}) \ . \tag{1.18}$$

One can argue, as shown below, that $\mathcal{M}^{3d}_{C}(I^{\mathrm{D}}_{n,k})$ is the S_n-quotient of the Coulomb branch of

$F^{\mathrm{D}}_{n,k} =$ 

(1.19)

which is a quiver with a *bouquet* of n nodes of $\mathrm{USp}(2)$. Physically, $F^{\mathrm{D}}_{n,k}$ captures the phase in which all n M5-branes are separated. The discrete gauging on the Higgs branch is reflected by the relation of the Coulomb branches

$$\mathcal{M}^{3d}_{C}(I^{\mathrm{D}}_{n,k}) = \mathcal{M}^{3d}_{C}(F^{\mathrm{D}}_{n,k})/S_n \ . \tag{1.20}$$

As shown below, one can generalise the setting to the analogue of (1.15).

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Outline. From the above examples, there is an apparent action of an S_n group on a Coulomb branch quiver, which appears to be a local operation on the quiver. These examples serve as guideline to prove exact statements on the Coulomb branch Hilbert series upon the action of an S_n group.

The remainder is organised as follows: the generalisations of the examples discussed in the introduction are the focus of section 2. In detail, the generalisation to an arbitrary quiver coupled to either a bouquet of U(1), USp(2), or SO(3) nodes is considered and the statements of discrete S_n-quotients are proven on the level of the monopole formula. In section 3, applications to other types of bouquets, composed of (different) USp(2), SO(3), and O(2) nodes, are considered and proven. Thereafter, section 4 summarises and concludes. Appendix A provides some background on the cycle index and a proof of an auxiliary identity.

As a remark, a complementary perspective of discrete gauging and its manifestation as discrete quotients on Coulomb branches is presented in [25].

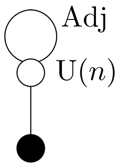
2 A and D-type

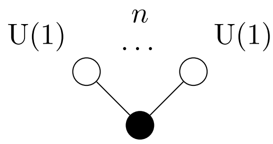
This section focuses on generic good 3-dimensional $\mathcal{N} = 4$ quiver gauge theories that are coupled to bouquets of either U(1), USp(2), or SO(3) nodes. Upon S_n-quotient, the bouquet is replaced by a single U(n), USp($2n$), or SO($2n+1$) node supplemented by an additional hypermultiplet that transforms as in the corresponding ADHM quiver.

2.1 A-type — U(1)-bouquet

Taking (1.3) as well as (1.13) and (1.15) as motivation, one can generalise the statement to a generic quiver with one (or many) bouquet(s) attached and provide a proof on the level of the monopole formula.

Consider an arbitrary quiver, denoted by $\bullet$, coupled to either a U(n) gauge node with one additional adjoint hypermultiplet or a bouquet of n U(1) nodes. I.e. define the two quiver theories

$T_{\{n\},\bullet} =$ 

and $T_{\{1^n\},\bullet} =$ 

. (2.1)

To be precise, the U(n) as well as all of the U(1) nodes couple to the same *single* node in $\bullet$ via bifundamental matter. Viewed from the U(n) or U(1) nodes, the quiver $\bullet$ is considered as providing background charges $\vec{k} = (k_1, \dots, k_s)$ for some $s \in \mathbb{N}$, i.e. the magnetic charges from the single node they couple to. To see this, consider this single node in $\bullet$ as flavour node with background charges $\vec{k}$ as, for example, in [4, 5, 16]. Thus, there are two Hilbert series to compute: (i) the monopole formula $H(t, \vec{k})$ of $\bullet$ with the single node turned into a flavour node with fluxes $\vec{k}$, and (ii) the monopole formula of $T_{\{n\},\square}$ (or $T_{\{1^n\},\square}$) where the flavour node $\square$ provides fluxes $\vec{k}$. The Hilbert series of $T_{\{n\},\bullet}$ (or $T_{\{1^n\},\bullet}$) can be obtained via gluing the Hilbert series $H(t, \vec{k})$ with that of $T_{\{n\},\square}$ (or $T_{\{1^n\},\square}$) along the common

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flavour node, which turns it into a gauge node. Since the Hilbert series with background charges for $\bullet$ is same in both cases and is not affected by the S_n-quotient, it will henceforth be ignored. Now, the above conjecture (1.13) is generalised by

Proposition 1. *Let the quiver gauge theories $T_{\{n\},\bullet}$ and $T_{\{1^n\},\bullet}$ be as defined in (2.1), then their Coulomb branches satisfy*

$$\mathcal{M}_C\left(T_{\{n\},\bullet}\right) = \mathcal{M}_C\left(T_{\{1^n\},\bullet}\right)/S_n \,. \tag{2.2}$$

Preliminaries. In order to prove Proposition 1, one defines

$$f(t,z) = \mathrm{HS}_{\mathcal{M}_C}\left(\begin{array}{c}\mathrm{U}(1)\\ \circ \\ | \\ \bullet\end{array}\right) \equiv \mathrm{HS}_{\mathcal{M}_C(T_{\{1\},\bullet})} \tag{2.3}$$

as Hilbert series of the 3-dimensional $\mathcal{N}=4$ U(1) gauge theory with background charges $\vec{k}$. In detail, the conformal dimension and dressing factor read

$$\Delta(q;\vec{k}) = \frac{1}{2}|q-\vec{k}| \,, \qquad P(t;q) = \frac{1}{1-t} \tag{2.4}$$

for $q\in\mathbb{Z}$. Then (2.2) can be expressed via the following generating series:

$$F[\nu;t,z] = \mathrm{PE}[\nu\cdot f(t,z)] = \sum_{n=0}^{\infty}\nu^n\,\mathrm{HS}_n(t,z) \tag{2.5}$$

such that Proposition 1 becomes

$$\mathrm{HS}_n(t,z) \equiv \mathrm{HS}_{\mathcal{M}_C(T_{\{n\},\bullet})} \overset{\text{Prop. 1}}{=} \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\nu^n}\mathrm{PE}\left[\nu\cdot\mathrm{HS}_{\mathcal{M}_C(T_{\{1\},\bullet})}\right]\bigg|_{\nu=0} \equiv \mathrm{HS}_{\mathcal{M}_C(T_{\{1^n\},\bullet})/S_n} \,. \tag{2.6}$$

In order to compute $\mathrm{HS}_n(t,z)$ explicitly from the symmetrisation of $f(t,z)$, one employs the cycle index (A.1).

To compare the result, recall the ingredients for the monopole formula of an U(n) gauge node with one adjoint hypermultiplet and background charges. The conformal dimension reads

$$\Delta(q_1,\ldots,q_n;\vec{k}) = \frac{1}{2}\sum_{i=1}^{n}|q_i-\vec{k}| = \sum_{i=1}^{n}\Delta(q_i;\vec{k}) \tag{2.7}$$

wherein the contributions from the adjoint hypermultiplet cancel the vector multiplet contributions entirely. The magnetic charges appearing in the monopole formula are ordered $q_1\geq q_2\geq\ldots\geq q_n$. The U($n$) dressing factors have been defined in [1]. The shorthand notation $|q_i-\vec{k}|\equiv\sum_{l=1}^{s}|q_i-k_l|$ summarises the contributions from the magnetic charges k_l of the single node in $\bullet$ the U(n) couples to via bifundamental matter.

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Proof. The recursive formula (A.2) suggests to prove (2.2) by induction. One explicitly verifies the claim for $n = 1, 2$; thereafter one proceeds to $\mathrm{HS}_n(t, z)$ with general n, i.e.

$$\mathrm{HS}_n(t,z) = \frac{1}{n}\sum_{k=1}^{n} a_k \cdot \mathrm{HS}_{n-k}(t,z)\,, \quad a_k = f(t^k, z^k)\,, \tag{2.8}$$

assuming the validity for all $\mathrm{HS}_k(t, z)$ with $k<n$. To begin with, one verifies the base case:

(i) $n = 1$: trivial. Returns the U(1) case.

(ii) $n = 2$: the two contributions read

$$\begin{aligned} a_1^2 &= \frac{1}{(1-t)^2}\sum_{q_1,q_2} z^{q_1+q_2} t^{\Delta(q_1)+\Delta(q_2)} \\ &= \frac{2}{(1-t)^2}\sum_{q_1>q_2} z^{q_1+q_2} t^{\Delta(q_1)+\Delta(q_2)} + \frac{1}{(1-t)^2}\sum_{q_1=q_2} z^{q_1+q_2} t^{\Delta(q_1)+\Delta(q_2)}\,, \end{aligned} \tag{2.9}$$

$$a_2 = \frac{1}{1-t^2}\sum_{q} z^{2q} t^{2\Delta(q)}\,. \tag{2.10}$$

Combining both, one obtains

$$\begin{aligned} \mathrm{HS}_2(t,z) &= \frac{1}{(1-t)^2}\sum_{q_1>q_2} z^{q_1+q_2} t^{\Delta(q_1)+\Delta(q_2)} \\ &\quad + \frac{1}{2}\left(\frac{1}{(1-t)^2} + \frac{1}{1-t^2}\right)\sum_{q_1=q_2} z^{q_1+q_2} t^{\Delta(q_1)+\Delta(q_2)} \\ &= \frac{1}{(1-t)^2}\sum_{q_1>q_2} z^{q_1+q_2} t^{\Delta(q_1)+\Delta(q_2)} \\ &\quad + \frac{1}{(1-t)(1-t^2)}\sum_{q_1=q_2} z^{q_1+q_2} t^{\Delta(q_1)+\Delta(q_2)}\,, \end{aligned} \tag{2.11}$$

which coincides with the monopole formula for the quiver $\bullet$ coupled to a U(2) gauge node with an adjoint hypermultiplet.

Thereafter, one proceeds with the inductive step $(n-1) \to n$ for (2.8). The strategy of the proof is to consider the different contributions for the distinct summation regions of the magnetic charges q_i in detail and show that these agree with the monopole formula of $T_{\{n\},\bullet}$.

(i) $q_1 > q_2 > \ldots > q_n$ can only originate from one term: $a_1\mathrm{HS}_{n-1}$, in which one denotes the magnetic charge in a_1 by q and those of HS_{n-1} by q_i, $i = 1, \ldots, n-1$. Then there are exactly n contributing cases:

$$\begin{aligned} q > q_1 > \ldots > q_{n-1}\,, \\ q_1 > q > q_2 > \ldots\,, \\ \ldots\,, \\ q_1 > \ldots > q > q_{n-1}\,, \\ q_1 > \ldots > q_{n-1} > q\,, \end{aligned} \tag{2.12}$$

but these can all be relabelled to a single case. Then one finds

$$\frac{1}{n} a_1 \mathrm{HS}_{n-1} \supset \frac{1}{(1-t)^n} \sum_{q_1 > \ldots > q_n} z^{\sum_{j=1}^n q_j} t^{\sum_{j=1}^n \Delta(q_j)} \tag{2.13}$$

and observes the dressing factor for a residual $\mathrm{U}(1)^n$ gauge symmetry, which is in fact the correct stabiliser of $q_1 > q_2 > \ldots > q_n$ in $\mathrm{U}(n)$.

(ii) $q_1 > q_2 > \ldots q_i > q_{i+1} = \ldots = q_{i+l} > \ldots > q_n$. The relevant contributions can only arise from $a_1 \mathrm{HS}_{n-1}$ to $a_l \mathrm{HS}_{n-l}$. Then $a_1 \mathrm{HS}_{n-1}$ has two different contributions: firstly,

$$a_1 \mathrm{HS}_{n-1} \supset \sum_q \frac{1}{1-t} z^q t^{\Delta(q)} \sum_{\substack{l \text{ equal } q_i \\ \text{out of } n-1}} \frac{1}{\prod_{a=1}^l (1-t^a)} \frac{1}{(1-t)^{n-1-l}} z^{\sum_{j=1}^{n-1} q_j} t^{\sum_{j=1}^{n-1} \Delta(q_j)}, \tag{2.14}$$

but recall that the q_i in HS_{n-1} are already ordered. Then there are exactly $(n-l)$ possible ways to arrange q in between the q_i. However, a simple relabelling makes them all identical and one obtains:

$$a_1 \mathrm{HS}_{n-1} \supset (n-l) \cdot \sum_{\substack{l \text{ equal } q_i \\ \text{out of } n}} \frac{1}{\prod_{a=1}^l (1-t^a)} \frac{1}{(1-t)^{n-l}} \cdot z^{\sum_{j=1}^n q_j} t^{\sum_{j=1}^n \Delta(q_j)} . \tag{2.15}$$

Secondly, there is the contribution where $(l-1)$ q_i are equal in HS_{n-1} and one has to align the q from a_1 with those $(l-1)$ equal magnetic charges. This yields precisely one case

$$a_1 \mathrm{HS}_{n-1} \supset \sum_{\substack{l \text{ equal } q_i \\ \text{out of } n}} \frac{1}{\prod_{a=1}^{l-1} (1-t^a)} \frac{1}{(1-t)^{n-l}} \frac{1}{(1-t)} \cdot z^{\sum_{j=1}^n q_j} t^{\sum_{j=1}^n \Delta(q_j)} . \tag{2.16}$$

Similarly, there exists exactly one matching contribution for $a_j \mathrm{HS}_{n-j}$, where the q from a_j has to match the $(l-j)$ equal q_i from HS_{n-j}. One obtains

$$a_j \mathrm{HS}_{n-j} \supset \sum_{\substack{l \text{ equal } q_i \\ \text{out of } n}} \frac{1}{\prod_{a=1}^{l-j} (1-t^a)} \frac{1}{(1-t)^{n-l}} \frac{1}{(1-t^j)} \cdot z^{\sum_{j=1}^n q_j} t^{\sum_{j=1}^n \Delta(q_j)} . \tag{2.17}$$

The total contribution becomes

$$\begin{aligned} \sum_{j=1}^l a_j \mathrm{HS}_{n_j}(t) \supset & \left(\frac{(n-l)}{\prod_{a=1}^l (1-t^a)} \frac{1}{(1-t)^{n-l}} + \sum_{j=1}^l \frac{1}{\prod_{a=1}^{l-j} (1-t^a)} \frac{1}{(1-t)^{n-l}} \frac{1}{(1-t^j)} \right) \\ & \cdot \sum_{\substack{l \text{ equal } q_i \\ \text{out of } n}} z^{\sum_{j=1}^n q_j} t^{\sum_{j=1}^n \Delta(q_j)} \\ = & \frac{1}{\prod_{a=1}^l (1-t^a)} \frac{1}{(1-t)^{n-l}} \left(n-l+Q_l(t)\right) \sum_{\substack{l \text{ equal } q_i \\ \text{out of } n}} z^{\sum_{j=1}^n q_j} t^{\sum_{j=1}^n \Delta(q_j)} \end{aligned} \tag{2.18}$$

$$\text{with} \quad Q_l(t) := \sum_{j=1}^l \frac{1}{(1-t^j)} \prod_{a=l-j+1}^{l} (1-t^a) . \tag{2.19}$$

As proven in appendix A.2, $Q_l(t)$ satisfies

$$Q_l(t) = l \qquad \forall t \,. \tag{2.20}$$

Such that one obtains the contribution:

$$\frac{1}{n}\sum_{j=1}^{l} a_j \mathrm{HS}_{n_j}(t) \supset \frac{1}{\prod_{a=1}^{l}(1-t^a)}\frac{1}{(1-t)^{n-l}} \sum_{\substack{l \text{ equal } q_i \\ \text{out of } n}} z^{\sum_{j=1}^n q_j} t^{\sum_{j=1}^n \Delta(q_j)} \tag{2.21}$$

and one recognises the correct dressing factor for the residual $S(\mathrm{U}(l)\times\mathrm{U}(1)^{n-l})$ gauge symmetry of l equal magnetic charges.

(iii) Now, one can easily generalise to any partition $(l_1,\ldots,l_p)$, $\sum_{j=1}^{i} l_i = n$ (not necessarily ordered) that describes the set-up of

$$q_1 = \ldots = q_{l_1} > q_{l_1+1} = \ldots = q_{l_1+l_2} > \ldots > q_{l_1+l_2+\ldots+l_{p-1}+1} = \ldots = q_{l_1+l_2+\ldots+l_p} \,. \tag{2.22}$$

The total contribution becomes

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^{\max(l_i)} a_j \mathrm{HS}_{n_j}(t) &\supset \frac{1}{n}\frac{1}{\prod_{j=1}^{p}\prod_{a_j=1}^{l_j}(1-t_j^a)}\left(n+\sum_{j=1}^{p}\underbrace{(Q_{l_j}(t)-l_j)}_{=0}\right) \\ &\quad\cdot \sum_{\substack{l \text{ equal } q_i \\ \text{out of } n}} z^{\sum_{j=1}^n q_j} t^{\sum_{j=1}^n \Delta(q_j)} \\ &= \frac{1}{\prod_{j=1}^{p}\prod_{a_j=1}^{l_j}(1-t_j^a)} \sum_{\substack{l \text{ equal } q_i \\ \text{out of } n}} z^{\sum_{j=1}^n q_j} t^{\sum_{j=1}^n \Delta(q_j)} \,,\end{aligned} \tag{2.23}$$

which is the correct contribution with a dressing factor reflecting the residual $S(\prod_{j=1}^{p}\mathrm{U}(l_j))$ gauge symmetry. Again, the factor n is cancelled by the $\frac{1}{n}$ pre-factor in the cycle index.

Consequently, one has addressed all possible $\{q_i\}$, $i = 1,\ldots,n$, summation regions that appear in (2.8) and, most importantly, one has proven that these correspond exactly to the definition of the fully refined monopole formula for a $\mathrm{U}(n)$ gauge node with one adjoint hypermultiplet and background charges. This concludes the proof of Proposition 1.

Comments. The proof shows that given the Coulomb branch of an arbitrary quiver $\bullet$ with a $\mathrm{U}(1)$-bouquet of size n, one may quotient by S_n. The result is the same as the Coulomb branch of $\bullet$ coupled to a $\mathrm{U}(n)$-node with one additional adjoint hypermultiplet. From the nature of the proof, i.e. the S_n-quotient is a local operation on the Coulomb branch, there exist two immediate corollaries:

(i) Similarly to (1.15), one can consider a generic partition $\{n_i\}$ of n which corresponds to the quotient by $\prod_i S_{n_i}$ on $T_{\{1^n\},\bullet}$. Since $\bullet$ has been arbitrary, one can repeat the proof by subdividing the size n bouquet, focusing on the sub-bouquet of size n_i, while

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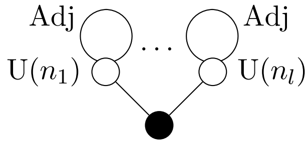
treating the remaining U(1)-nodes as part of the background quiver. In other word, the quiver

$$T_{\{n_i\},\bullet} = \quad (2.24)$$



has a Coulomb branch which satisfies

$$\mathcal{M}_C\left(T_{\{n_i\},\bullet}\right) = \mathcal{M}_C\left(T_{\{1^n\},\bullet}\right) / \prod_i S_{n_i} \,. \tag{2.25}$$

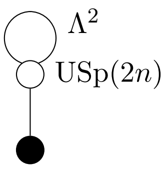
(ii) Furthermore, one may consider quivers where multiple bouquets are attached to different nodes. Then one can repeat the process of discrete quotients to any of the bouquets individually, as the operation is entirely local.

2.2 D-type — USp(2)-bouquet

Next, one can generalise (1.6) by considering an arbitrary quiver coupled to one (or many) bouquet(s) of USp(2) $\cong$ SU(2) gauge nodes and provide a proof at the level of the monopole formula.

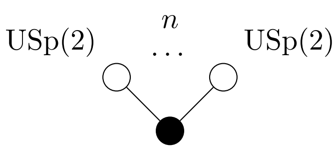
Again, consider an arbitrary quiver, labelled by $\bullet$, coupled to either an USp($2n$) gauge node with one additional anti-symmetric hypermultiplet or a USp(2)-bouquet of size n. Again, the following notation is employed:

$$T_{\{n\},\bullet} = $$



and

$$T_{\{1^n\},\bullet} = $$



. (2.26)

As above, the USp($2n$) as well as all of the USp(2) nodes couple to the same *single* node in $\bullet$ via bifundamental matter. From the USp($2n$) or USp(2) point of view, the quiver $\bullet$ contributes background charges $\vec{k} = (k_1, \ldots, k_s)$ for some $s \in \mathbb{N}$, i.e. the magnetic charges from the single node they couple to. All other contributions from $\bullet$ could be summarised in a function of the fugacity, which is not affected by the S_n-quotient and is henceforth ignored, cf. the discussion below (2.1).

Proposition 2. *Let the theories $T_{\{n\},\bullet}$ and $T_{\{1^n\},\bullet}$ be as defined in (2.26), then the Coulomb branches satisfy*

$$\mathcal{M}_C\left(T_{\{n\},\bullet}\right) = \mathcal{M}_C\left(T_{\{1^n\},\bullet}\right) / S_n \,. \tag{2.27}$$

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Preliminaries. To begin with, define the basic ingredient:

$$f(t) = \mathrm{HS}_{\mathcal{M}_C} \Bigg($$

USp(2)

$$\Bigg) \equiv \mathrm{HS}_{\mathcal{M}_C(T_{\{1\},\bullet})}\,, \tag{2.28}$$

which is the Coulomb branch Hilbert series in the presence of background charges. Note that there is no extra topological fugacity for USp(2). The relevant conformal dimension is

$$\Delta(q;\vec{k}) = \frac{1}{2}(|q-\vec{k}| + |q+\vec{k}|) - 2|q| \tag{2.29}$$

for the magnetic charge $q \in \mathbb{N}$ and background fluxes $\vec{k}$. The dressing factors associated to USp(2) are

$$P(t,q) = \begin{cases} \frac{1}{1-t} & , \; q>0\,, \\ \frac{1}{1-t^2} & , \; q=0\,. \end{cases} \tag{2.30}$$

The USp$(2n)$ gauge node with a hypermultiplet transforming in $\Lambda^2([1,0,\ldots,0])$ has the following conformal dimension:

$$\Delta(q_1,\ldots,q_n;\vec{k}) = \frac{1}{2}\sum_{i=1}^{n}\left(|q_i-\vec{k}| + |q_i+\vec{k}|\right) - 2\sum_{i=1}^{n}|q_i| = \sum_{i=1}^{n}\Delta(q_i;\vec{k}) \tag{2.31}$$

because $\Lambda^2([1,0,\ldots,0]) = [0,1,0,\ldots,0]\oplus[0,0,\ldots,0]$ has non-trivial weights $e_i \pm e_j$, $-(e_i \pm e_j)$ for $1 \leq i < j \leq n$ such that $[0,1,0,\ldots,0]$ cancels the vector multiplet contribution partially. In the monopole formula, the magnetic charges q_i are restricted to $q_1 \geq q_2 \geq \ldots \geq q_n \geq 0$. Moreover, the dressing factors for a USp$(2n)$ gauge node have been presented in [1]. The shorthand notation $|q_i \pm \vec{k}| \equiv \sum_{l=1}^{s}|q_i \pm k_l|$ summarises the contributions from the magnetic charges k_l of the single node in $\bullet$ the USp$(2n)$ couples to via bifundamental matter.

With this preliminaries at hand, the statement of Proposition 2 becomes

$$\mathrm{HS}_n \equiv \mathrm{HS}_{\mathcal{M}_C(T_{\{n\},\bullet})} \overset{\text{Prop. 2}}{=} \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\nu^n}\mathrm{PE}\left[\nu\cdot \mathrm{HS}_{\mathcal{M}_C(T_{\{1\},\bullet})}\right]\Bigg|_{\nu=0} \equiv \mathrm{HS}_{\mathcal{M}_C(T_{\{1^n\},\bullet})/S_n}\,. \tag{2.32}$$

Proof. As before, the cycle index (A.1) can be employed to realise the symmetrisation in (2.32) such that the proof proceeds by induction in n

$$\mathrm{HS}_n(t) = \frac{1}{n}\sum_{k=1}^{n} a_k \cdot \mathrm{HS}_{n-k}(t)\,, \quad a_k = f(t^k)\,. \tag{2.33}$$

To begin, one verifies the base case:

(i) $n=1$: trivial, as $\Lambda^2[1]=0$.

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(ii) $n=2$: the proposal reads

$$\mathrm{HS}_2(t) = \frac{1}{2}\left(a_2 + a_1^2\right) \tag{2.34}$$

where the two contributions are treated as follows:

$$\begin{aligned} a_1^2 &= \sum_{q_1,q_2\geq 0} P(t,q_1)P(t,q_2) t^{\Delta(q_1)+\Delta(q_2)} \\ &= 2\sum_{q_1>q_2>0} \frac{1}{(1-t)^2} t^{\Delta(q_1)+\Delta(q_2)} + 2\sum_{q_1>0=q_2} \frac{1}{(1-t)(1-t^2)} t^{\Delta(q_1)+\Delta(0)} \\ &\quad + \sum_{q_1=q_2>0} \frac{1}{(1-t)^2} t^{2\Delta(q_1)} + \frac{1}{(1-t^2)^2} t^{2\Delta(0)} \,, \end{aligned} \tag{2.35}$$

$$\begin{aligned} a_2 &= \sum_{q\geq 0} P(t^2,q) t^{2\Delta(q)} \\ &= \sum_{q>0} \frac{1}{1-t^2} t^{2\Delta(q)} + \frac{1}{1-t^4} t^{2\Delta(0)} \,. \end{aligned} \tag{2.36}$$

Adding them up yields

$$\begin{aligned} \mathrm{HS}_2(t) &= \sum_{q_1>q_2>0} \frac{1}{(1-t)^2} t^{\Delta(q_1)+\Delta(q_2)} + \sum_{q_1>0=q_2} \frac{1}{(1-t)(1-t^2)} t^{\Delta(q_1)+\Delta(0)} \\ &\quad + \frac{1}{2}\left(\frac{1}{(1-t)^2} + \frac{1}{1-t^2}\right) \sum_{q_1=q_2>0} t^{2\Delta(q_1)} + \frac{1}{2}\left(\frac{1}{(1-t^2)^2} + \frac{1}{1-t^4}\right) t^{2\Delta(0)} \\ &= \sum_{q_1>q_2>0} \frac{1}{(1-t)^2} t^{\Delta(q_1)+\Delta(q_2)} + \sum_{q_1>0=q_2} \frac{1}{(1-t)(1-t^2)} t^{\Delta(q_1)+\Delta(0)} \\ &\quad + \frac{1}{(1-t)(1-t^2)} \sum_{q_1=q_2>0} t^{2\Delta(q_1)} + \frac{1}{(1-t^2)(1-t^4)} t^{2\Delta(0)} \,. \end{aligned} \tag{2.37}$$

Comparing this to the Hilbert series of USp(4) with a $\Lambda^2[1,0]$ hypermultiplet and background charges, one has the conformal dimension (2.31) and the dressing factors [1]

$$P(t,q_1,q_2) = \begin{cases} \frac{1}{(1-t)^2} \,, & q_1>q_2>0 \,, \\ \frac{1}{(1-t)(1-t^2)} \,, & q_1=q_2>0 \,, \\ \frac{1}{(1-t)(1-t^2)} \,, & q_1>0=q_2 \,, \\ \frac{1}{(1-t^2)(1-t^4)} \,, & q_1=q_2=0 \,. \end{cases} \tag{2.38}$$

Consequently, Proposition 2 is true for $n=2$.

Next, one proceeds as in the A-type case of section 2.1, i.e. the inductive step $(n-1)\to n$. Since there is a slight complication when considering $q_1 \geq \ldots \geq q_n \geq 0$, the details of the proof need to be elaborated.

(i) $q_1 > \ldots > q_n > 0$ can only originate from $a_1 \mathrm{HS}_{n-1}$ via

$$\begin{aligned} \frac{1}{n} a_1 \mathrm{HS}_{n-1} &\supset \frac{1}{n} \sum_{q>0} \frac{1}{1-t} t^{\Delta(q)} \sum_{q_1 > \ldots > q_{n-1} > 0} \frac{1}{(1-t)^{n_1}} t^{\sum_{i=1}^{n-1} \Delta(q_i)} \\ &\supset \frac{1}{(1-t)^n} \sum_{q_1 > \ldots > q_n > 0} t^{\sum_{i=1}^{n} \Delta(q_i)} \,, \end{aligned} \tag{2.39}$$

where the n different possibilities to place q between the $(n-1)$ q_i eliminated the pre-factor $\frac{1}{n}$. Moreover, the dressing factor correctly reproduces the stabiliser of $q_1 > \ldots > q_n > 0$ inside $\mathrm{USp}(2n)$, i.e. $\mathrm{U}(1)^n$.

(ii) $q_1 > \ldots > q_{n-l} > 0 = q_{n-l+1} = \ldots = q_n$ for which contributions arise from $a_1 \mathrm{HS}_{n-1}$ to $a_l \mathrm{HS}_{n-l}$. To start with, $a_1 \mathrm{HS}_{n-1}$ provides two contributions

$$\begin{aligned} a_1 \mathrm{HS}_{n-1} &\supset \sum_{q>0} \frac{1}{1-t} t^{\Delta(q)} \sum_{\substack{l \text{ vanishing } q_i \\ \text{out of } n-1}} \frac{1}{(1-t)^{n-l-1} \prod_{a=1}^{l} (1-t^{2a})} t^{\sum_{i=1}^{n-1} \Delta(q_i)} \\ &\supset \frac{1}{(1-t)^{n-l}} \frac{n-l}{\prod_{a=1}^{l}(1-t^{2a})} \sum_{\substack{l \text{ vanishing } q_i \\ \text{out of } n}} t^{\sum_{i=1}^{n} \Delta(q_i)} \end{aligned} \tag{2.40}$$

and arranging q between the non-vanishing q_i gives a multiplicity of $(n-l)$. The other term is

$$\begin{aligned} a_1 \mathrm{HS}_{n-1} &\supset \sum_{q=0} \frac{1}{1-t^2} t^{\Delta(0)} \sum_{\substack{(l-1) \text{ vanishing } q_i \\ \text{out of } n-1}} \frac{1}{(1-t)^{n-l} \prod_{a=1}^{l-1} (1-t^{2a})} t^{\sum_{i=1}^{n-1} \Delta(q_i)} \\ &\supset \frac{1}{(1-t)^{n-l}} \frac{1}{(1-t^2) \prod_{a=1}^{l-1}(1-t^{2a})} \sum_{\substack{l \text{ vanishing } q_i \\ \text{out of } n}} t^{\sum_{i=1}^{n} \Delta(q_i)} \,, \end{aligned} \tag{2.41}$$

which has multiplicity one. Similarly, the contribution form $a_j \mathrm{HS}_{n-j}$ is

$$\begin{aligned} a_j \mathrm{HS}_{n-j} &\supset \sum_{q>0} \frac{1}{1-t^{2j}} t^{j\Delta(q)} \sum_{\substack{(l-j) \text{ vanishing } q_i \\ \text{out of } n-j}} \frac{1}{(1-t)^{n-l}} \frac{1}{\prod_{a=1}^{l-j} (1-t^{2a})} t^{\sum_{i=1}^{n-j} \Delta(q_i)} \\ &\supset \frac{1}{(1-t)^{n-l}} \frac{1}{(1-t^{2j}) \prod_{a=1}^{l-j}(1-t^{2a})} \sum_{\substack{l \text{ vanishing } q_i \\ \text{out of } n}} t^{\sum_{i=1}^{n} \Delta(q_i)} \,. \end{aligned} \tag{2.42}$$

Summing up all contributions, one obtains

$$
\begin{aligned}
\frac{1}{n}\sum_{j=1}^{l} a_j \mathrm{HS}_{n-j} \supset\ & \frac{1}{n}\frac{1}{(1-t)^{n-l}\prod_{a=1}^{l}(1-t^{2a})} \\
&\cdot \left(n-l+\sum_{m=1}^{l}\frac{1}{(1-t^{2m})}\prod_{b=l-m+1}^{l}(1-t^{2b})\right)\sum_{\substack{l \text{ vanishing } q_i\\ \text{out of } n}} t^{\sum_{i=1}^{n}\Delta(q_i)} \\
\supset\ & \frac{1}{n}\frac{1}{(1-t)^{n-l}\prod_{a=1}^{l}(1-t^{2a})}\left(n-l+Q_l(t^2)\right)\sum_{\substack{l \text{ vanishing } q_i\\ \text{out of } n}} t^{\sum_{i=1}^{n}\Delta(q_i)} \\
\supset\ & \frac{1}{(1-t)^{n-l}\prod_{a=1}^{l}(1-t^{2a})}\sum_{\substack{l \text{ vanishing } q_i\\ \text{out of } n}} t^{\sum_{i=1}^{n}\Delta(q_i)}
\end{aligned}
\tag{2.43}
$$

and one recognises the dressing factor of $\mathrm{U}(1)^{n-l}\times \mathrm{USp}(2l)$. Note in particular the use of the results of appendix A.2, but this time for $Q_l(t^2)=l$.

(iii) $q_1=\ldots=q_l>q_{l+1}>\ldots>q_n>0$ for which contributions arise from $a_1\mathrm{HS}_{n-1}$ to $a_l\mathrm{HS}_{n-l}$. To start with, $a_1\mathrm{HS}_{n-1}$ provides two contributions

$$
\begin{aligned}
a_1\mathrm{HS}_{n-1} \supset\ & \sum_{q>0}\frac{1}{1-t}t^{\Delta(q)}\sum_{\substack{l \text{ equal } q_i\\ \text{out of } n-1}}\frac{1}{(1-t)^{n-l-1}\prod_{a=1}^{l}(1-t^{a})}t^{\sum_{i=1}^{n-1}\Delta(q_i)} \\
\supset\ & \frac{1}{(1-t)^{n-l}}\frac{n-l}{\prod_{a=1}^{l}(1-t^{a})}\sum_{\substack{l \text{ equal } q_i\\ \text{out of } n}} t^{\sum_{i=1}^{n}\Delta(q_i)}
\end{aligned}
\tag{2.44}
$$

and arranging q between the non-equal q_i gives a multiplicity of $n-l$. The other term is

$$
\begin{aligned}
a_1\mathrm{HS}_{n-1} \supset\ & \sum_{q>0}\frac{1}{1-t}t^{\Delta(q)}\sum_{\substack{(l-1) \text{ equal } q_i\\ \text{out of } n-1}}\frac{1}{(1-t)^{n-l}\prod_{a=1}^{l-1}(1-t^{a})}t^{\sum_{i=1}^{n-1}\Delta(q_i)} \\
\supset\ & \frac{1}{(1-t)^{n-l}}\frac{1}{(1-t)\prod_{a=1}^{l-1}(1-t^{a})}\sum_{\substack{l \text{ equal } q_i\\ \text{out of } n}} t^{\sum_{i=1}^{n}\Delta(q_i)}
\end{aligned}
\tag{2.45}
$$

which has multiplicity one. Similarly, the contribution form $a_j\mathrm{HS}_{n-j}$ is

$$
\begin{aligned}
a_j\mathrm{HS}_{n-j} \supset\ & \sum_{q>0}\frac{1}{1-t^j}t^{j\Delta(q)}\sum_{\substack{(l-j) \text{ equal } q_i\\ \text{out of } n-j}}\frac{1}{(1-t)^{n-l}}\frac{1}{\prod_{a=1}^{l-j}(1-t^{a})}t^{\sum_{i=1}^{n-j}\Delta(q_i)} \\
\supset\ & \frac{1}{(1-t)^{n-l}}\frac{1}{(1-t^j)\prod_{a=1}^{l-j}(1-t^{a})}\sum_{\substack{l \text{ equal } q_i\\ \text{out of } n}} t^{\sum_{i=1}^{n}\Delta(q_i)}\,.
\end{aligned}
\tag{2.46}
$$

Summing up all contributions, one finds

$$\begin{aligned}
\frac{1}{n}\sum_{j=1}^{l} a_j \mathrm{HS}_{n-j} \supset\ & \frac{1}{n}\frac{1}{(1-t)^{n-l}\prod_{a=1}^{l}(1-t^a)} \\
&\cdot\left(n-l+\sum_{m=1}^{l}\frac{1}{(1-t^m)}\prod_{b=l-m+1}^{l}(1-t^b)\right)\sum_{\substack{l \text{ equal } q_i \\ \text{out of } n}} t^{\sum_{i=1}^{n}\Delta(q_i)} \\
\supset\ & \frac{1}{n}\frac{1}{(1-t)^{n-l}\prod_{a=1}^{l}(1-t^a)}\left(n-l+Q_l(t)\right)\sum_{\substack{l \text{ equal } q_i \\ \text{out of } n}} t^{\sum_{i=1}^{n}\Delta(q_i)} \\
\supset\ & \frac{1}{(1-t)^{n-l}\prod_{a=1}^{l}(1-t^a)}\sum_{\substack{l \text{ equal } q_i \\ \text{out of } n}} t^{\sum_{i=1}^{n}\Delta(q_i)}
\end{aligned} \tag{2.47}$$

and one recognises the dressing factor of $\mathrm{U}(1)^{n-l}\times\mathrm{U}(l)$.

(iv) In general, consider a (not necessarily ordered) partition $(l_1,\ldots,l_p;l_0)$ such that $\sum_{j=1}^{p} l_p + l_0 = n$. Here, l_0 counts the number of vanishing fluxes, i.e.

$$\begin{aligned}
q_1 = \ldots = q_{l_1} > q_{l_1+1} = \ldots = q_{l_1+l_2} > \ldots > q_{l_1+\ldots+l_{p-1}+1} = \ldots = q_{l_1+\ldots+l_p} > 0 \\
0 = q_{l_1+\ldots+l_p+1} = \ldots = q_{l_1+\ldots+l_p+l_0} \equiv q_n \,.
\end{aligned} \tag{2.48}$$

Then from the cases consider above, one obtains

$$\begin{aligned}
\frac{1}{n}\sum_{j=1}^{\max(\{l_j\},l_0)} a_j\mathrm{HS}_{n-j} \supset\ & \frac{1}{n}\frac{1}{\prod_{j=1}^{p}\prod_{a_j=1}^{l_j}(1-t^{a_j})\cdot\prod_{a_0=1}^{l_0}(1-t^{2a_0})} \\
&\cdot\left(n+\sum_{j=1}^{p}(Q_{l_j}(t)-l_j)+(Q_{l_0}(t^2)-l_0)\right)\sum_{q's} t^{\sum_{i=1}^{n}\Delta(q_i)} \\
\supset\ & \frac{1}{\prod_{j=1}^{p}\prod_{a_j=1}^{l_j}(1-t^{a_j})\cdot\prod_{a_0=1}^{l_0}(1-t^{2a_0})}\sum_{q's} t^{\sum_{i=1}^{n}\Delta(q_i)}
\end{aligned} \tag{2.49}$$

from which one recognises the dressing factor of $\left(\prod_{j=1}^{p}\mathrm{U}(l_j)\right)\times\mathrm{USp}(2l_0)$.

Therefore, the pieces together form exactly the Hilbert series for the n-th step. This concludes the proof of Proposition 2.

Comments. The proof establishes that the Coulomb branch of an arbitrary quiver $\bullet$ with a USp(2)-bouquet of size n coincides upon quotient by S_n with the Coulomb branch of the quiver $\bullet$ where the bouquet is replaced by a USp($2n$) gauge node with an additional anti-symmetric hypermultiplet.

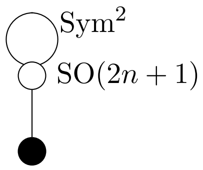
The nature of the proof allows to draw two immediate corollaries, as in the A-type case:

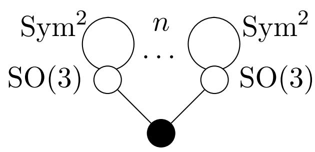
(i) One may consider an arbitrary partition $\{n_i\}$ of n such that one quotients $T_{\{1^n\},\bullet}$ by $\prod_i S_{n_i}$.

(ii) In addition, one may consider quivers with more multiple bouquets, as the operation is local on the Coulomb branch.

2.3 D-type — SO(3)-bouquet

Next, consider a SO(3)-bouquet in which each SO(3)-node is equipped with a loop corresponding to a hypermultiplet in the second symmetric representation. The reason for this will become clear below. The starting point is again an arbitrary quiver $\bullet$ coupled either to an $\mathrm{SO}(2n+1)$ gauge node with one additional hypermultiplet transforming in $\mathrm{Sym}^2([1,0,\ldots,0])$ or an SO(3)-bouquet of size n. Note that the SO(3) nodes on the bouquet also have one additional symmetric hypermultiplet. Define the following two sets of quivers:

$T_{\{n\},\bullet} =$


and $T_{\{1^n\},\bullet} =$


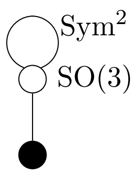
. (2.50)

To clarify, the $\mathrm{SO}(2n+1)$ as well as all of the SO(3) nodes couple to the same *single* node in $\bullet$ via bifundamental matter. From the view point of the $\mathrm{SO}(2n+1)$ or SO(3) nodes, the quiver $\bullet$ contributes background charges $\vec{k} = (k_1, \ldots, k_s)$ for some $s \in \mathbb{N}$, i.e. the magnetic charges from the single node they couple to. All other contributions from $\bullet$ could be summarised in a function of the fugacity, which is not affected by the S_n-quotient and is henceforth ignored, cf. the discussion below (2.1).

Proposition 3. *Let $T_{\{n\},\bullet}$ and $T_{\{1^n\},\bullet}$ be as defined in* (2.50) *then their Coulomb branches satisfy*

$$\mathcal{M}_C\left(T_{\{n\},\bullet}\right) = \mathcal{M}_C\left(T_{\{1^n\},\bullet}\right)/S_n\,. \tag{2.51}$$

Preliminaries. For the proof below, one defines the basic ingredient:

$f(t) = \mathrm{HS}_{\mathcal{M}_C}\Big($


$\Big) \equiv \mathrm{HS}_{\mathcal{M}_C(T_{\{1\},\bullet})}$ (2.52)

which is the Coulomb branch Hilbert series. Here, the conformal dimension reads

$$\Delta(q;\vec{k}) = \frac{1}{2}\left(|q-\vec{k}| + |q+\vec{k}|\right) + |q| \tag{2.53}$$

for the magnetic charge $q \in \mathbb{N}$ and background fluxes $\vec{k}$. The dressing factors associated to SO(3) are those of USp(2), i.e.

$$P(t,q) = \begin{cases} \frac{1}{1-t}\,, & q>0\,, \\ \frac{1}{1-t^2}\,, & q=0\,. \end{cases} \tag{2.54}$$

The SO$(2n+1)$ gauge node with one hypermultiplet transforming in $\mathrm{Sym}^2([1,0,\ldots,0])$ and background charges $\vec{k}$ has conformal dimension

$$\Delta(q_1,\ldots,q_n;\vec{k}) = \frac{1}{2}\sum_{i=1}^{n}\left(|q_i-\vec{k}|+|q_i+\vec{k}|\right)+\sum_{i=1}^{n}|q_i| = \sum_{i=1}^{n}\Delta(q_i;\vec{k})\,, \tag{2.55}$$

because $\mathrm{Sym}^2([1,0,\ldots,0]) = [2,0,\ldots,0]\oplus[0,\ldots,0]$ with non-trivial weights $e_i \pm e_j$, $-(e_i \pm e_j)$ for $1\leq i<j\leq n$ and $\pm 2e_i$ for $1\leq i\leq n$ such that $[2,0,\ldots,0]$ cancels the vector multiplet contribution. In the monopole formula, the magnetic charges q_i are restricted to $q_1 \geq q_2 \geq \ldots \geq q_n \geq 0$. Moreover, the dressing factors of SO$(2n+1)$ are those of USp$(2n)$, see [1]. The shorthand notation $|q_i \pm \vec{k}| \equiv \sum_{l=1}^{s}|q_i \pm k_l|$ summarises the contributions from the magnetic charges k_l of the single node in $\bullet$ the SO$(2n+1)$ couples to via bifundamental matter.

Proof. To prove Proposition 3, one needs to verify the Hilbert series relations (2.6) or (2.32) for the case of a SO(3)-bouquet. As before, the proof relies on the recursive formula (A.2) of the cycle index and proceeds by induction as in (2.33). As a first step, one considers the base case.

(i) $n=1$: trivial.

(ii) $n=2$: the proposal reads

$$\mathrm{HS}_2(t) = \frac{1}{2}\left(a_2+a_1^2\right) \qquad \text{with} \quad a_k := f(t^k)\,, \tag{2.56}$$

where the two contributions are treated as follows:

$$\begin{aligned}
a_1^2 &= \sum_{q_1,q_2\geq 0} P(t,q_1)P(t,q_2)t^{\Delta(q_1)+\Delta(q_2)} \\
&= 2\sum_{q_1>q_2>0}\frac{1}{(1-t)^2}t^{\Delta(q_1)+\Delta(q_2)} + 2\sum_{q_1>0=q_2}\frac{1}{(1-t)(1-t^2)}t^{\Delta(q_1)+\Delta(0)} \\
&\quad + \sum_{q_1=q_2>0}\frac{1}{(1-t)^2}t^{2\Delta(q_1)} + \frac{1}{(1-t^2)^2}t^{2\Delta(0)}\,,
\end{aligned} \tag{2.57}$$

$$\begin{aligned}
a_2 &= \sum_{q\geq 0} P(t^2,q)t^{2\Delta(q)} \\
&= \sum_{q>0}\frac{1}{1-t^2}t^{2\Delta(q)} + \frac{1}{1-t^4}t^{2\Delta(0)}\,.
\end{aligned} \tag{2.58}$$

Adding them up, yields

$$\begin{aligned}
\mathrm{HS}_2(t) &= \sum_{q_1>q_2>0}\frac{1}{(1-t)^2}t^{\Delta(q_1)+\Delta(q_2)} + \sum_{q_1>0=q_2}\frac{1}{(1-t)(1-t^2)}t^{\Delta(q_1)+\Delta(0)} \\
&\quad + \frac{1}{2}\left(\frac{1}{(1-t)^2}+\frac{1}{1-t^2}\right)\sum_{q_1=q_2>0}t^{2\Delta(q_1)} + \frac{1}{2}\left(\frac{1}{(1-t^2)^2}+\frac{1}{1-t^4}\right)t^{2\Delta(0)} \\
&= \sum_{q_1>q_2>0}\frac{1}{(1-t)^2}t^{\Delta(q_1)+\Delta(q_2)} + \sum_{q_1>0=q_2}\frac{1}{(1-t)(1-t^2)}t^{\Delta(q_1)+\Delta(0)} \\
&\quad + \frac{1}{(1-t)(1-t^2)}\sum_{q_1=q_2>0}t^{2\Delta(q_1)} + \frac{1}{(1-t^2)(1-t^4)}t^{2\Delta(0)}\,.
\end{aligned} \tag{2.59}$$

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Comparing this to the monopole formula of SO(5) with one $\mathrm{Sym}^2[1,0]$ hypermultiplet and background charges, the conformal dimension follows from (2.55) and the dressing factors read

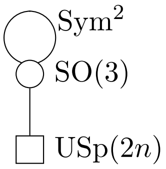
$$P(t,q_1,q_2) = \begin{cases} \frac{1}{(1-t)^2} & q_1 > q_2 > 0 \,, \\ \frac{1}{(1-t)(1-t^2)} & q_1 = q_2 > 0 \,, \\ \frac{1}{(1-t)(1-t^2)} & q_1 > 0 = q_2 \,, \\ \frac{1}{(1-t^2)(1-t^4)} & q_1 = q_2 = 0 \,. \end{cases} \tag{2.60}$$

Consequently, Proposition 3 is true for $n = 2$.

The argument proceeds as in the D-type case of section 2.2, one proves the inductive step $(n-1) \to n$. As the dressing factors as well as the lattice of magnetic charges are identical to the $\mathrm{USp}(2n)$ case, it is unnecessary to spell out the details of the proof. The only point to appreciate is that the conformal dimension is the sum of the individual SO(3) conformal dimensions.

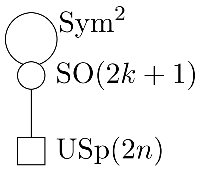
Comments. Again, the same corollaries are in order: (i) one can generalise to arbitrary partitions, and (ii) one can consider multiple bouquets.

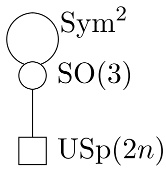
Moreover, note that the Coulomb branch of

$$T_{1,n}^{\mathrm{D'\text{-}type}} =$$


$$\text{with} \qquad \mathrm{HS}_{T_{1,n}^{\mathrm{D'\text{-}type}}} = \frac{1-t^{2n+4}}{(1-t^2)(1-t^{n+1})(1-t^{n+2})} \tag{2.61}$$

is the D_{n+3}-singularity. Hence, Proposition 3 provides the missing analogue of (1.3), (1.6) for (1.7), i.e.

$$\mathcal{M}_C \left(\right.$$


$$\left. \right) = \mathrm{Sym}^k \left(\mathcal{M}_C \left(\right. \right.$$


$$\left. \left. \right) \right) \,. \tag{2.62}$$

3 Other applications

After establishing the generalisations underlying the A and D-type singularities, one may wonder if there are other types of bouquets that can be considered. One could ask what are sufficient conditions such that a G_n gauge node, which may be supplemented by additional matter, can be obtained from an S_n-quotient of a certain G_1-bouquet. To be precise, the G_n node as well as all nodes of the G_1-bouquet are coupled to the same *single* node in a given quiver gauge theory via bifundamental matter. From the aforementioned cases one formulates three conditions:

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$\mathfrak{g}$	$\mathcal{W}$	adjoint of G
A_n	S_{n+1}	$[1,0,\ldots,0,1]$
B_n	$S_n \ltimes (\mathbb{Z}_2)^n$	$\Lambda^2([1,0,\ldots,0])$
C_n	$S_n \ltimes (\mathbb{Z}_2)^n$	$\mathrm{Sym}^2([1,0,\ldots,0])$
D_n	$S_n \ltimes (\mathbb{Z}_2)^{n-1}$	$\Lambda^2([1,0,\ldots,0])$

Table 1. Classical algebras and their Weyl groups.

(i) Additivity of the conformal dimension: i.e. Δ of G_n is the sum of the conformal dimensions of the G_1 nodes.

(ii) Compatibility of the GNO lattices.

(iii) Compatibility of the dressing factors.

While the first statement is concise, the second and third are less precise. However, by recalling [7, 8] the interpretation of the dressing factors $P(t, q_i)$ as Hilbert series of $\mathbb{C}[\mathfrak{g}]^G \cong \mathbb{C}[\mathfrak{t}]^{\mathcal{W}_G}$, with $\mathfrak{t}$ a Cartan sub-algebra of $\mathfrak{g}$ and $\mathcal{W}_G$ the Weyl group, one can identify all classical groups that allow for a S_n factor in $\mathcal{W}_G$.

Suppose $\mathcal{W}_{G_n} = S_n \ltimes (\Gamma)^n$ and denote the chosen Cartan sub-algebra as $\mathfrak{t} \cong V^n$, for some 1-dimensional vector space V, then

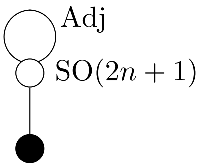
$$\mathbb{C}[\mathrm{Lie}(G_n)]^{G_n} \cong \mathbb{C}[V^n]^{S_n \ltimes (\Gamma)^n} \cong \mathrm{Sym}^n \left(\mathbb{C}[V]^\Gamma\right) \cong \mathrm{Sym}^n \left(\mathbb{C}[\mathrm{Lie}(G_1)]^{G_1}\right) \tag{3.1}$$

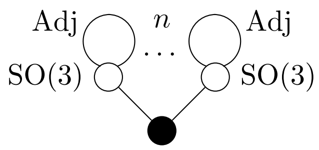
and a similar argument is valid for the summation ranges in the monopole formula of G_n and G_1. By inspecting classical Weyl groups in table 1 one concludes that choosing G_n to be either $\mathrm{U}(n)$, $\mathrm{SO}(2n+1)$, $\mathrm{USp}(2n)$, or $\mathrm{O}(2n)$ together with one adjoint hypermultiplet leads to possible S_n-quotients on the Coulomb branch. All the different cases are elaborated on in the subsequent sections. The only exception is $\mathrm{U}(n)$ as it agrees with Proposition 1.

3.1 SO(3)-bouquet

Specifying the above to a $\mathrm{SO}(2n+1)$ gauge node with one adjoint hypermultiplet coupled to an arbitrary quiver $\bullet$, one finds:

Corollary 1. *Let $T_{\{n\},\bullet}$ and $T_{\{1^n\},\bullet}$ be defined as*

$T_{\{n\},\bullet} =$ 

and $T_{\{1^n\},\bullet} =$ 

, (3.2)

then their Coulomb branches satisfy

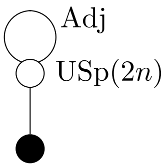
$$\mathcal{M}_C\left(T_{\{n\},\bullet}\right) = \mathcal{M}_C\left(T_{\{1^n\},\bullet}\right)/S_n\,. \tag{3.3}$$

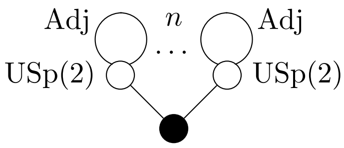
To prove Corollary 1 one follows all the steps of the proof of Proposition 3. The only point the take care of is that the conformal dimensions add up, which is not difficult to see.

3.2 USp(2)-bouquet

Next, consider a USp($2n$) gauge node with one adjoint hypermultiplet coupled to an arbitrary quiver $\bullet$ via bifundamental hypermultiplets.

Corollary 2. *Let $T_{\{n\},\bullet}$ and $T_{\{1^n\},\bullet}$ be defined as*

$T_{\{n\},\bullet} =$ 

and $T_{\{1^n\},\bullet} =$ 

, (3.4)

then their Coulomb branches satisfy

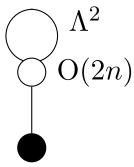
$$\mathcal{M}_C\left(T_{\{n\},\bullet}\right) = \mathcal{M}_C\left(T_{\{1^n\},\bullet}\right)/S_n\ . \tag{3.5}$$

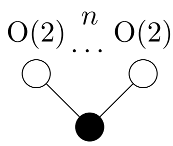
The proof of Corollary 2 follows from the proof of Proposition 2, by verifying that the conformal dimensions add up appropriately.

3.3 O(2)-bouquet

Finally, let an arbitrary quiver $\bullet$ be coupled either to an O($2n$) gauge node with one additional anti-symmetric hypermultiplet or to a O(2)-bouquet of size n.

Corollary 3. *For the quiver gauge theories $T_{\{n\},\bullet}$ and $T_{\{1^n\},\bullet}$, defined as*

$T_{\{n\},\bullet} =$ 

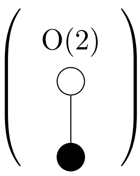
and $T_{\{1^n\},\bullet} =$ 

(3.6)

the Coulomb branches satisfy

$$\mathcal{M}_C\left(T_{\{n\},\bullet}\right) = \mathcal{M}_C\left(T_{\{1^n\},\bullet}\right)/S_n\ . \tag{3.7}$$

Since this is the first time $O(2n)$ gauge nodes appear, some remarks on the proof are in order. Analogous to section 2, define the basic ingredient:

$\mathrm{HS}_{\mathcal{M}_C}$ 

$\equiv \mathrm{HS}_{\mathcal{M}_C(T_{\{1\},\bullet})}$ (3.8)

which is the Coulomb branch Hilbert series. Here, the conformal dimension reads

$$\Delta(q;\vec{k}) = \frac{1}{2}(|q-\vec{k}| + |q+\vec{k}|) \tag{3.9}$$

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for the magnetic charge $q \in \mathbb{N}$ and background fluxes $\vec{k}$. Following [3], the dressing factors associated to O(2) are those of SO(3), i.e.

$$P(t,q) = \begin{cases} \frac{1}{1-t}\,, & q > 0\,, \\ \frac{1}{1-t^2}\,, & q = 0\,. \end{cases} \tag{3.10}$$

Note that there is no extra topological fugacity for O(2).

The $\mathrm{O}(2n)$ gauge node with one hypermultiplet transforming as $\Lambda^2([1,0,\dots,0])$ and background charges has conformal dimension

$$\Delta(q_1,\dots,q_n;\vec{k}) = \frac{1}{2}\sum_{i=1}^{n}\Big(|q_i - \vec{k}| + |q_i + \vec{k}|\Big) = \sum_{i=1}^{n} \Delta(q_i;\vec{k}) \tag{3.11}$$

because $\Lambda^2([1,0,\dots,0]) = [0,1,0,\dots,0]$ with non-trivial weights $e_i \pm e_j$, $-(e_i \pm e_j)$ for $1 \leq i < j \leq n$ such that $[0,1,0,\dots,0]$ cancels the vector multiplet contribution partially. Again, the magnetic charges q_i satisfy $q_1 \geq q_2 \geq \ldots \geq q_n \geq 0$ in the monopole formula. The dressing factors for $\mathrm{O}(2n)$ have been discussed in [3]. The shorthand notation $|q_i \pm \vec{k}| \equiv \sum_{l=1}^{s} |q_i \pm k_l|$ summarises the contributions from the magnetic charges k_l of the single node in $\bullet$ the $\mathrm{O}(2n)$ couples to via bifundamental matter.

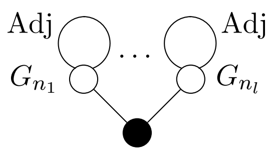
As the dressing factors and GNO lattice for $\mathrm{O}(2n)$ originate from $\mathrm{SO}(2n+1)$, which are the same as for $\mathrm{USp}(2n)$, the proof of Corollary 3 is consequence of the proofs of Propositions 2 and 3.

3.4 Remarks and example

With Corollaries 1–3 at ones disposal, one can immediately generalise to the following:

Corollary 4. *Let G_n be either $\mathrm{U}(n)$, $\mathrm{SO}(2n{+}1)$, $\mathrm{USp}(2n)$, or $\mathrm{O}(2n)$ and $\{n_i\}$ be a partition of n. The Coulomb branch of the quiver gauge theory*

$$T_{\{n_i\},\bullet} = \tag{3.12}$$



satisfies

$$\mathcal{M}_C\left(T_{\{n_i\},\bullet}\right) = \mathcal{M}_C\left(T_{\{1^n\},\bullet}\right) / \prod_i S_{n_i}\,. \tag{3.13}$$

Likewise, one could consider quiver gauge theories coupled to various bouquets at different nodes.

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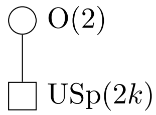
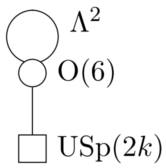
Example. Before closing, it is interesting to study an example of Corollary 3. This highlights the use of the monopole formula as very fortunate because the corresponding statements on rational functions would have been very cumbersome to prove. To begin with, one readily computes (3.8) for $\bullet$ being a flavour node and obtains

$$\mathrm{HS}_{\mathcal{M}_C(T_{\{1\},\square})}(t) = \frac{1-t^{2k+2}}{(1-t^2)(1-t^k)(1-t^{k+1})} \qquad \text{for} \quad T_{\{1\},\square} = \tag{3.14}$$



A similar computation can be performed for O(6) with one adjoint hypermultiplet and k flavours. In detail:

$$\mathrm{HS}_{\mathcal{M}_C(T_{\{3\},\square})}(t) = \frac{f(t)}{g(t)} \qquad \text{for} \qquad T_{\{3\},\square} = \tag{3.15}$$



$$\begin{aligned} f(t) = 1 &+ t^{k+1}(1+t+t^2+t^3+t^4) + t^{2k+1}(1+2t+2t^2+2t^3+2t^4+t^5+t^6) \\ &+ t^{3k+1}(1+t+3t^2+2t^3+2t^4+3t^5+t^6+t^7) \\ &+ t^{4k+2}(1+t+2t^2+2t^3+2t^4+2t^5+t^6) + t^{5k+4}(1+t+t^2+t^3+t^4) + t^{6k+9}\,, \\ g(t) = &\left(1-t^2\right)\left(1-t^4\right)\left(1-t^6\right)\left(1-t^k\right)\left(1-t^{2k}\right)\left(1-t^{3k}\right)\,. \end{aligned}$$

Then Corollary 3 becomes equivalent to the claim

$$\begin{aligned} \mathrm{HS}_{\mathcal{M}_C(T_{\{3\},\square})}(t) = \frac{1}{3!}\bigg(&\Big(\mathrm{HS}_{\mathcal{M}_C(T_{\{1\},\square})}(t)\Big)^3 + 3\cdot \mathrm{HS}_{\mathcal{M}_C(T_{\{1\},\square})}(t^2)\cdot \mathrm{HS}_{\mathcal{M}_C(T_{\{1\},\square})}(t) \\ &+ 2\cdot \mathrm{HS}_{\mathcal{M}_C(T_{\{1\},\square})}(t^3)\bigg)\,, \end{aligned} \tag{3.16}$$

which can be verified explicitly by inserting the rational functions.

4 Discussion and conclusions

In this note we have shown that discrete S_n-quotients on Coulomb branches of quivers with various bouquets are entirely local operations. By this we mean that the geometric S_n-quotient on $\mathcal{M}_C$ is realised on the quiver (and the monopole formula) as an operation on the bouquet alone; the remainder of the quiver is untouched by the S_n action.

We provided the A and D-type Propositions 1–3 in section 2 and proved them via the cycle index for S_n. Subsequently, we explore various other possibilities in section 3 and derived Corollaries 1–4. In comparison, the gauge nodes in section 2 are supplemented by loops corresponding to matter as in the ADHM quivers, whereas the gauge nodes in section 3 are equipped with one additional adjoint hypermultiplet. The A-type case of $\mathrm{U}(n)$ nodes is the only scenario for which both notions coincide.

The results are important for a number of reasons: firstly, it allows to deduce if certain 3-dimensional $\mathcal{N}=4$ Coulomb branches are S_n orbifolds of one another. For instance, the

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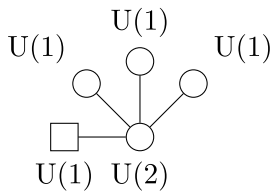
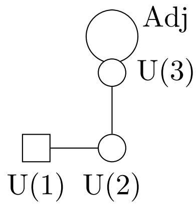
sub-regular nilpotent orbit of G_2 is an S_3 quotient of the minimal nilpotent orbit of SO(8), cf. [26]. Due to the discrete quotient proposition, the statement follows immediately by inspecting the 3-dimensional $\mathcal{N} = 4$ quivers

$$T_{\{1^3\},\Box\!-\!\circ} =$$



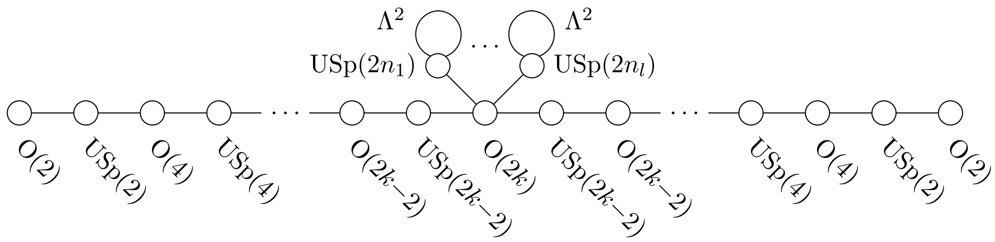
$$\xrightarrow[\text{quotient}]{S_3} \quad T_{\{3\},\Box\!-\!\circ} = \tag{4.1}$$



$$\overline{\mathcal{O}}^{G_2}_{\text{subreg}} = \mathcal{M}_C\left(T_{\{3\},\Box\!-\!\circ}\right) = \mathcal{M}_C\left(T_{\{1^3\},\Box\!-\!\circ}\right)/S_3 = \overline{\mathcal{O}}^{\text{SO}(8)}_{\text{min}}/S_3\,. \tag{4.2}$$

Secondly, the propositions allow to systematically study the different phases of 6-dimensional Higgs branches as put forward by [20]. For instance, Proposition 2 allows to conclude a similar statement to (1.15) on the different phases of the Higgs branches of multiple M5-branes on a $\mathbb{C}^2/D_k$ singularity [19]. The conjecture becomes that for a partition $\{n_i\}$ of n, such that the M5-branes coincide in a pattern of n_i, the 3-dimensional quiver reads

$$F^{\text{D}}_{n,k} = \tag{4.3}$$



and its Coulomb branch satisfies

$$\mathcal{M}^{6d}_H(Q^{\text{D}}_{n,k})|_{\{n_i\}} = \mathcal{M}^{3d}_C(F^{\text{D}}_{\{n_i\},k})\,, \qquad \mathcal{M}^{3d}_C(F^{\text{D}}_{\{n_i\},k}) = \mathcal{M}^{3d}_C(F^{\text{D}}_{\{1^n\},k})/\prod_i S_{n_i}\,. \tag{4.4}$$

Thirdly, the discrete quotient procedure establishes another operation on quiver gauge theories solely through their associated Hilbert series. This highlights the diverse applicability of the Hilbert series and adds to the catalogue of quiver operations such as the ideas of quiver subtraction [27] and Kraft-Procesi small instanton transition [19].

In view of other approaches to 3-dimensional $\mathcal{N} = 4$ Coulomb branches, like the abelianisation method [28, 29] or the attempt to define the Coulomb branch mathematically [30–32], it would be interesting to understand whether these can reproduce the discrete quotients.

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A Background material

A.1 Cycle index

The cycle index of a permutation group Γ of degree n is defined as average of the cycle index monomials of all permutations $g \in \Gamma$. Every $g \in \Gamma$ can be decomposed into disjoint cycles $c_1 c_2 c_3 \cdots$. Let $j_k(g)$ be the number of cycles in g of length k, then

$$Z(\Gamma) = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \prod_{k=1}^{n} a_k^{j_k(g)} \, . \tag{A.1}$$

If one considers the symmetric group S_n then cycle index can be cast into a recursive relation:

$$Z(S_n) = \frac{1}{n} \sum_{l=1}^{n} a_l Z(S_{n-l}) \tag{A.2}$$

where one defines $Z(S_0) = 1$. The first recursions yield:

$$Z(S_1) = a_1 \, , \qquad Z(S_2) = \frac{1}{2!}(a_2 + a_1^2) \, , \qquad Z(S_3) = \frac{1}{3!}(a_1^3 + 3a_1a_2 + 2a_3) \, . \tag{A.3}$$

A.2 q-theory

To prove the auxiliary identity

$$Q_l(t) := \sum_{j=1}^{l} \frac{1}{1-t^j} \prod_{a=0}^{j-1} (1 - t^{l-a}) = l \qquad \forall t \, , \tag{A.4}$$

one recalls the following definitions from q-theory:

q-bracket
$$[k]_q = \frac{1-q^k}{1-q} \, , \tag{A.5a}$$

q-factorial
$$[k]_q! = \begin{cases} 1 \, , & k = 0 \\ [k]_q \cdot [k-1]_q \cdot \ldots \cdot [1]_q \, , & k = 1, 2, \ldots \end{cases} \, , \tag{A.5b}$$

q-binomial coefficient
$$\begin{bmatrix} k \\ j \end{bmatrix}_q = \frac{[k]_q!}{[j]_q![k-j]_q!} \, . \tag{A.5c}$$

For the q-binomial coefficient exists a q-version of the Pascal identities; for instance

$$\begin{bmatrix} k \\ j \end{bmatrix}_q = q^j \begin{bmatrix} k-1 \\ j \end{bmatrix}_q + \begin{bmatrix} k-1 \\ j-1 \end{bmatrix}_q \, , \tag{A.5d}$$

for $j = 1, 2, \ldots, k-1$. Then, one can rewrite

$$\begin{aligned} Q_l(t) &= \sum_{j=1}^{l} \frac{1}{1-t^j}(1-t^l)(1-t^{l-1})\cdot\ldots\cdot(1-t^{l-(j-1)}) \\ &= \sum_{j=1}^{l} \frac{(1-t^l)(1-t^{l-1})\cdot\ldots\cdot(1-t^{l-(j-1)})}{(1-t^j)(1-t^{j-1})\cdot\ldots\cdot(1-t)} \cdot (1-t^{j-1})\cdot\ldots\cdot(1-t) \\ &= \sum_{j=1}^{l} \begin{bmatrix} l \\ j \end{bmatrix}_t \cdot (1-t)^{j-1} \cdot [j-1]_t! \,. \end{aligned} \tag{A.6}$$

Having expressed (A.4) as in (A.6) has the benefit that one can follow an argument of [33]. The proof proceeds by induction over l employing (A.5d). Firstly, the base case is verified easily

$$Q_1(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_t \cdot (1-t)^0 \cdot [0]_t! = 1 \,, \tag{A.7a}$$

$$Q_2(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_t + \begin{bmatrix} 2 \\ 2 \end{bmatrix}_t \cdot (1-t) \cdot [1]_t! = (1+t) + (1-t) = 2 \,. \tag{A.7b}$$

Secondly, the inductive step is shown via

$$\begin{aligned} Q_l(t) &= \sum_{j=1}^{l} \begin{bmatrix} l \\ j \end{bmatrix}_t \cdot (1-t)^{j-1} \cdot [j-1]_t! \\ &= \sum_{j=1}^{l-1} \begin{bmatrix} l-1 \\ j \end{bmatrix}_t t^j \cdot (1-t)^{j-1} \cdot [j-1]_t! + \sum_{j=0}^{l-2} \begin{bmatrix} l-1 \\ j \end{bmatrix}_t \cdot (1-t)^j \cdot [j]_t! \\ &= \sum_{j=1}^{l-1} \begin{bmatrix} l-1 \\ j \end{bmatrix}_t t^j \cdot (1-t)^{j-1} \cdot [j-1]_t! + 1 + \sum_{j=1}^{l-1} \begin{bmatrix} l-1 \\ j \end{bmatrix}_t \cdot (1-t)^{j-1} \cdot (1-t^j) \cdot [j-1]_t! \\ &= \sum_{j=1}^{l-1} \begin{bmatrix} l-1 \\ j \end{bmatrix}_t \cdot (1-t)^{j-1} \cdot [j-1]_t! + 1 \\ &= Q_{l-1}(t) + 1 = (l-1) + 1 \end{aligned} \tag{A.8}$$

where the induction hypothesis has been used in the last step.

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