

Supersymmetric localization in AdS_5 and the protected chiral algebra

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ABSTRACT: $\mathcal{N} = 4$ super Yang-Mills theory admits [1] a protected subsector isomorphic to a two-dimensional chiral algebra, obtained by passing to the cohomology of a certain supercharge. In the large N limit, we expect this chiral algebra to have a dual description as a subsector of IIB supergravity on $AdS_5 \times S^5$. This subsector can be carved out by a version of supersymmetric localization, using the bulk analog of the boundary supercharge. We illustrate this procedure in a simple model, the theory of an $\mathcal{N} = 4$ vector multiplet in AdS_5 , for which a convenient off-shell description is available. This model can be viewed as the simplest truncation of the full $AdS_5 \times S^5$ supergravity, in which case the vector multiplet should be taken in the adjoint representation of $\mathfrak{g}_F = \mathfrak{su}(2)_F$. Localization yields Chern-Simons theory on AdS_3 with gauge algebra \mathfrak{g}_F , whose boundary dual is the affine Lie algebra $\widehat{\mathfrak{g}}_F$. We comment on the generalization to the full bulk theory. We propose that the large N limit of the chiral algebra of $\mathcal{N} = 4$ SYM is again dual to Chern-Simons theory, with gauge algebra a suitable higher-spin superalgebra.

KEYWORDS: AdS-CFT Correspondence, Conformal Field Theory, Supersymmetric Gauge Theory

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1 Introduction

Any four-dimensional $\mathcal{N} = 2$ superconformal field theory (SCFT) admits a subsector of correlation functions that exhibits the structure of a two-dimensional chiral algebra [1]. This is in particular the case for $\mathcal{N} = 4$ super Yang-Mills (SYM) theories. The associated chiral algebra is labelled by the gauge algebra \mathfrak{g} and is independent of the complexified gauge coupling. It encodes an infinite amount of information about a very rich protected subsector of the SYM theory. In this paper we start addressing the question of finding a holographic description of this protected chiral algebra for $\mathfrak{g} = \mathfrak{su}(N)$, in the large N limit. Answering this question would provide us with a new solvable model of holography. Rather than a mere toy example, this would be an intricate yet tractable model carved out naturally from the standard holographic duality.

While in the general $\mathcal{N} = 2$ case the protected chiral algebra has no residual supersymmetry, the chiral algebra associated to an $\mathcal{N} = 4$ SCFT contains the *small* $\mathcal{N} = 4$ superconformal algebra (SCA) as a subalgebra. Conjecturally [1], the chiral algebra for $\mathcal{N} = 4$ SYM theory with gauge algebra \mathfrak{g} is a novel $\mathcal{N} = 4$ super chiral algebra, strongly generated by a finite number of currents. The super chiral algebra generators descend from the generators of the one-half BPS chiral ring of the SYM theory, and are thus in one-to-one correspondence with the Casimir invariants of \mathfrak{g} . For example, for $\mathfrak{g} = \mathfrak{su}(N)$, the super chiral algebra is conjectured to have $N - 1$ generators,¹ of holomorphic dimension $h = 1, \frac{3}{2}, \dots, \frac{N}{2}$, in correspondence with the familiar single-trace one-half BPS operators of the SYM theory, namely $\text{Tr } X^{2h}$ in the symmetric traceless representation of the $\mathfrak{so}(6)$ R-symmetry. As we will review in detail below, only an $\mathfrak{su}(2)_F$ subalgebra of $\mathfrak{so}(6)$ is visible in the chiral algebra, where it is in fact enhanced to the affine Kac-Moody algebra $\widehat{\mathfrak{su}(2)}_F$ that is part of the small $\mathcal{N} = 4$ SCA. The super chiral algebra generator of dimension h transforms in the spin h representation of $\mathfrak{su}(2)_F$. This is a BPS condition — the generators with $h > 1$ are the highest-weight states of *short* representations of the $\mathcal{N} = 4$ subalgebra.² The central charge of the chiral algebra is given by $c_{2d} = -3 \dim \mathfrak{g} = -3(N^2 - 1)$. It is not known whether the chiral algebra for fixed $N > 2$ admits a deformation³ to general values of the central charge. In fact, for this special value of c_{2d} one finds several null relations that might be essential to ensure associativity of the operator algebra.

Let us now consider the large N holographic description. As familiar, $\mathcal{N} = 4$ SYM theory is dual to IIB string theory on $AdS_5 \times S^5$, with the $1/N$ expansion on the field theory side corresponding to the genus expansion on the string theory side. It would be extremely interesting to construct a “topological” string theory whose genus expansion reproduces the $1/N$ expansion of the $\mathcal{N} = 4$ SYM chiral algebra. Here we will address the simpler question of finding a holographic description for the leading large N limit of the chiral algebra, in terms of a classical field theory in the bulk. There are two ways we can imagine to proceed: attempting to construct the bulk theory by bottom-up guesswork; or deriving it from the top-down as a subsector of $AdS_5 \times S^5$ string field theory.

From the bottom-up perspective, the natural conjecture is that the bulk theory is a Chern-Simons field theory in AdS_3 , with gauge algebra a suitable infinite-dimensional supersymmetric higher-spin algebra. Such a duality would mimic several examples of “higher-spin holography” that have been studied in recent years in the context of the AdS_3/CFT_2 correspondence. A duality has been proposed in [4]⁴ between higher-spin Vasiliev theory

¹We are counting the $\mathfrak{psu}(1, 1|2)$ primaries; the chiral algebra is generated by the $\mathfrak{psu}(1, 1|2)$ primaries and their supersymmetry descendants.

²Other examples of $\mathcal{N} = 4$ super chiral algebra have been considered in the literature, see e.g., [2, 3], where the generators are taken to be $\mathfrak{su}(2)_F$ singlets and are thus highest-weight states of *long* representations of the $\mathcal{N} = 4$ SCA. Long generators give a qualitatively different behavior: the higher-spin currents encountered in the CFT duals of Vasiliev theories are not protected and are in fact expected to be lifted when one moves away from the tensionless point.

³For $N = 2$, the chiral algebra coincides with the small $\mathcal{N} = 4$ SCA, which is of course consistent for any central charge.

⁴This proposal, reviewed in [5], has passed several tests [6, 7] and has been extended in various directions, both in a purely bosonic context [8, 9] and in models with supersymmetry, see for instance [10–16].

in AdS_3 [17] and a suitable 't Hooft limit of \mathcal{W}_N minimal models, i.e. the coset CFTs $\mathfrak{su}(N)_k \otimes \mathfrak{su}(N)_1 / \mathfrak{su}(N)_{k+1}$. In this example, the chiral algebra that controls the large N limit is the $\mathcal{W}_\infty[\mu]$ algebra, where the parameter μ is identified with the 't Hooft coupling $N/(N+k)$, kept fixed as $N \rightarrow \infty$. The bulk dual description involves Chern-Simons theory with gauge algebra the infinite-dimensional Lie algebra $\mathfrak{hs}[\mu]$. The (non-linear) $\mathcal{W}_\infty[\mu]$ algebra arises at the asymptotic symmetry of this Chern-Simons theory [18–20]. We find it likely that our example will work along similar lines, but we have not yet been able to identify the correct supersymmetric higher-spin algebra. An obvious feature of the sought after higher-spin algebra is that it must contain $\mathfrak{psu}(1,1|2)$ as a subalgebra. Indeed the asymptotic symmetry of AdS_3 Chern-Simons theory with algebra $\mathfrak{psu}(1,1|2)$ is the small $\mathcal{N} = 4$ SCA (see for instance [21, 22] and references therein), which, as reviewed above, is a consistent truncation of the full super W-algebra. Higher-spin holography with small $\mathcal{N} = 4$ superconformal symmetry has also been studied in [23, 24]. In our case, the construction of the complete higher-spin algebra is made more challenging by the absence of an obvious deformation parameter analogous to the 't Hooft coupling of the \mathcal{W}_N minimal models⁵ — as we have remarked, the chiral algebra for $\mathfrak{su}(N)$ SYM theory might be isolated, stuck at a specific value of the central charge.

The top-down approach is conceptually straightforward. The dual bulk theory must be a subsector of IIB *supergravity* on $AdS_5 \times S^5$. Indeed, the generators of the chiral algebra descend from the single-trace one-half BPS operators of $\mathcal{N} = 4$ SYM, which are dual to the infinite tower of Kaluza-Klein (KK) supergravity modes on S^5 . In principle, our task is clear. In the boundary SYM theory, the $2d$ chiral subsector is carved out by passing to the cohomology of either one of two nilpotent supercharges [1]. The bulk supergravity admits analogous nilpotent supercharges. We then expect to find the bulk dual to the large N limit of the chiral algebra by localization of the supergravity theory with respect to either supercharge. In practice however, this program is difficult to implement rigorously. The technique of supersymmetric localization requires an off-shell formalism, but we are not aware of such a formalism for $AdS_5 \times S^5$ supergravity, or even for its consistent truncation to $\mathcal{N} = 8$ AdS_5 supergravity.

In this paper, we give a proof of concept that this localization program works as expected, producing an AdS_3 Chern-Simons theory out of AdS_5 supergravity. We consider the simplest truncation of the supergravity theory for which a convenient off-shell formalism is readily available: the theory of an $\mathcal{N} = 4$ vector multiplet in AdS_5 , covariant under an $\mathfrak{su}(2,2|2)$ subalgebra of the full $\mathfrak{psu}(2,2|4)$ superalgebra. We obtain this model by a straightforward analytic continuation of the analogous model on S^5 [25]. When viewed as part of the $\mathcal{N} = 8$ supergravity multiplet, the $\mathcal{N} = 4$ vector multiplet transforms in the adjoint representation of $\mathfrak{su}(2)_F$ (the centralizer of the embedding $\mathfrak{su}(2,2|2) \subset \mathfrak{psu}(2,2|4)$), but it is no more difficult to consider a general simple Lie algebra \mathfrak{g}_F . We show by explicit calculation that supersymmetric localization with respect to the relevant supercharge yields Chern-Simons theory in AdS_3 , with gauge algebra \mathfrak{g}_F , and level k related to the Yang-Mills

⁵Recall that the usual 't Hooft coupling $g_{YM}^2 N$ is not visible in the chiral algebra, which describes a protected subsector of observables of the SYM theory.

coupling. As is well-known, its dual boundary theory is the affine Kac-Moody algebra $\widehat{\mathfrak{g}}_F$ at level k . Apart from confirming the general picture that we have outlined, we believe that the details of our calculations are interesting in their own right, and may find a broader range of applications. Localization computations involving non-compact AdS backgrounds have been considered in the literature, see for instance [26–34] and more recently [35, 36]. It is worth pointing out that the Killing spinor used in our localization computation satisfies somewhat unusual algebraic properties compared to those usually assumed in past work. This is a consequence of the fact that our choice of supercharge mimics the (somewhat unusual) cohomological construction on the field theory side.

Localization of the full maximally supersymmetric AdS_5 supergravity would be technically challenging, but it seems very plausible (by supersymmetrizing the above result) that it would yield AdS_3 Chern-Simons theory with gauge algebra $\mathfrak{psu}(1, 1|2)$, whose boundary dual is the small $\mathcal{N} = 4$ superconformal algebra. Inclusion of the KK modes is however much harder, and at present the quest for the full holographic dual seems best pursued by bottom-up guesswork of the higher-spin superalgebra.

The rest of the paper is organized as follows. In section 2 we review the construction and main features of the chiral algebra associated to an $\mathcal{N} = 2$ SCFT. Section 3 contains our main result, the localization of the $\mathcal{N} = 4$ super Yang-Mills action in AdS_5 to bosonic Chern-Simons theory in AdS_3 . In section 4 we collect some useful facts and offer some speculations for the construction of the full holographic dual of the $\mathcal{N} = 4$ SYM chiral algebra. We conclude in section 5 with a brief discussion. An appendix contains conventions and technical material.

2 Review of the chiral algebra construction

In an effort to make this paper self-contained, we briefly review in this section the construction of the two-dimensional chiral algebra associated to a four-dimensional $\mathcal{N} = 2$ superconformal field theory [1]. Our main focus is on $\mathcal{N} = 4$ SYM theory, but the calculations of section 3 will be relevant for any $\mathcal{N} = 2$ SCFT that admits a supergravity dual and enjoys a global symmetry. To this end, we review in section 2.2 the special properties of chiral algebras associated to SCFTs with additional (super)symmetries.

2.1 Cohomological construction

The spacetime signature and the reality properties of operators are largely inessential in the following. We thus consider the complexified theory on flat \mathbb{C}^4 . The complexified superconformal algebra is $\mathfrak{sl}(4|2)$, with maximal bosonic subalgebra $\mathfrak{sl}(4) \oplus \mathfrak{sl}(2)_R \oplus \mathbb{C}_r$. The first term corresponds to the action of the complexified conformal algebra on \mathbb{C}^4 , while the other terms constitute the complexification of the R-symmetry of the theory.

Let us consider a fixed (complexified) plane $\mathbb{C}^2 \subset \mathbb{C}^4$. The subalgebra of the conformal algebra preserving the plane is of the form

$$\mathfrak{sl}(2) \oplus \overline{\mathfrak{sl}(2)} \oplus \mathbb{C}_\perp \oplus \mathbb{C} \subset \mathfrak{sl}(4). \tag{2.1}$$

The first two summands comprise the complexified two-dimensional conformal algebra acting on the fixed plane, while \mathbb{C}_\perp corresponds to complexified rotations in the two directions orthogonal to the plane. We use the notation L_n , $n = -1, 0, 1$ for the generators of the first summand, and \bar{L}_n for the second summand, while the generator of \mathbb{C}_\perp will be denoted \mathcal{M}_\perp . It is natural to adopt coordinates $\zeta, \bar{\zeta}$ on the selected plane \mathbb{C}^2 , with $\mathfrak{sl}(2)$ acting on ζ via Möbius transformations, and $\overline{\mathfrak{sl}(2)}$ acting similarly on $\bar{\zeta}$.

The R-symmetry of the superconformal theory allows us to define a suitable diagonal subalgebra

$$\widetilde{\mathfrak{sl}(2)} \subset \overline{\mathfrak{sl}(2)} \oplus \mathfrak{sl}(2)_R, \tag{2.2}$$

given explicitly by

$$\tilde{L}_{-1} = \bar{L}_{-1} + \mathcal{R}^-, \quad \tilde{L}_0 = \bar{L}_0 - \mathcal{R}, \quad \tilde{L}_1 = \bar{L}_1 - \mathcal{R}^+, \tag{2.3}$$

where $\mathcal{R}, \mathcal{R}^\pm$ denote the generators of $\mathfrak{sl}(2)_R$ with commutators

$$[\mathcal{R}^+, \mathcal{R}^-] = 2\mathcal{R}, \quad [\mathcal{R}, \mathcal{R}^\pm] = \pm\mathcal{R}^\pm. \tag{2.4}$$

The relevance of the twisted subalgebra $\widetilde{\mathfrak{sl}(2)}$ stems from the following crucial fact. There exist two linear combinations $\mathbb{Q}_1, \mathbb{Q}_2$ of the supercharges of $\mathfrak{sl}(4|2)$, inequivalent under similarity transformations, that enjoy the following properties:

$$\begin{aligned} \{\mathbb{Q}_1, \mathbb{Q}_1\} &= 0, & \{\mathbb{Q}_2, \mathbb{Q}_2\} &= 0, \\ [L_n, \mathbb{Q}_1] &= 0, & [L_n, \mathbb{Q}_2] &= 0, \\ \tilde{L}_n = \{\mathbb{Q}_1, \mathcal{F}_{1,n}\} &= \{\mathbb{Q}_2, \mathcal{F}_{2,n}\}, & & \text{for suitable odd generators } \mathcal{F}_{1,n}, \mathcal{F}_{2,n} \text{ of } \mathfrak{sl}(4|2), \\ \{\mathbb{Q}_1, \mathbb{Q}_2\} &= -r - \mathcal{M}_\perp, \end{aligned} \tag{2.5}$$

where r denotes the generator of \mathbb{C}_r . In other words, the supercharges $\mathbb{Q}_1, \mathbb{Q}_2$ are nilpotent, are invariant under the action of the holomorphic factor $\mathfrak{sl}(2)$ of the conformal algebra of the plane, and are such that the twisted antiholomorphic factor $\widetilde{\mathfrak{sl}(2)}$ is both a \mathbb{Q}_1 - and a \mathbb{Q}_2 -commutator. Explicit expressions for $\mathbb{Q}_1, \mathbb{Q}_2$ in a convenient basis are found in [1], where it is also shown that

$$[\tilde{L}_0, \mathbb{Q}_i] = 0, \quad [r + \mathcal{M}_\perp, \mathbb{Q}_i] = 0, \quad i = 1, 2. \tag{2.6}$$

The chiral algebra associated to the four-dimensional superconformal field theory is then defined by considering cohomology classes of operators with respect to \mathbb{Q}_i ($i = 1$ or 2), i.e. the set of operators (anti)commuting with \mathbb{Q}_i modded out by addition of arbitrary \mathbb{Q}_i -commutators.

Let \mathcal{O} be a local operator of the four-dimensional theory such that its insertion at the origin $\mathcal{O}(0)$ defines a non-trivial \mathbb{Q}_i cohomology class ($i = 1$ or 2), i.e. $[\mathbb{Q}_i, \mathcal{O}(0)] = 0$, but $\mathcal{O}(0)$ is not itself a \mathbb{Q}_i -commutator. It follows that $\mathcal{O}(0)$ necessarily commutes with \tilde{L}_0 and $r + \mathcal{M}_\perp$. In terms of the four dimensional quantum numbers of \mathcal{O} , this amounts to

$$\frac{1}{2}(\Delta - j_1 - j_2) - R = 0, \quad r + j_1 - j_2 = 0, \tag{2.7}$$

where Δ is the conformal dimension of \mathcal{O} , j_1, j_2 are its Lorentz Cartan quantum numbers, R is the $\mathfrak{sl}(2)_R$ Cartan quantum number, and r is the \mathbb{C}_r quantum number. The restrictions (2.7) on the quantum numbers of a four dimensional operators define the so-called class of *Schur operators* of the theory. A Schur operator is always the highest-weight state of its Lorentz and R-symmetry multiplet. If the latter is denoted schematically as $\mathcal{O}_{\alpha_1 \dots \alpha_{2j_1} \dot{\beta}_1 \dots \dot{\beta}_{2j_2}}^{I_1 \dots I_{2R}}$, where $I = 1, 2$ are $\mathfrak{sl}(2)_R$ fundamental indices, and $\alpha = +, -, \dot{\beta} = \dot{+}, \dot{-}$ are spacetime Weyl indices, then the Schur operator is $\mathcal{O}_{+\dots+\dot{+}\dots\dot{+}}^{1\dots 1}$. Let us also point out that, if the theory is unitary, (2.7) are not only necessary but also sufficient conditions for $\mathcal{O}(0)$ to define a non-trivial \mathbb{Q}_i cohomology class. Furthermore, in that case \mathbb{Q}_1 and \mathbb{Q}_2 define the same cohomology. We refer the reader to [1] for an explanation of these points.

Suppose $\mathcal{O}(0)$ defines a non-trivial \mathbb{Q}_i -cohomology class. We cannot translate this operator away from the origin along the directions orthogonal to the $(\zeta, \bar{\zeta})$ plane without losing \mathbb{Q}_i -closure, since \mathbb{Q}_i is not invariant under translations in those directions. We can, however, construct the following twisted translated operator

$$\tilde{\mathcal{O}}(\zeta, \bar{\zeta}) = e^{\zeta L_{-1} + \bar{\zeta} \tilde{L}_{-1}} \mathcal{O}(0) e^{-\zeta L_{-1} - \bar{\zeta} \tilde{L}_{-1}}, \quad (2.8)$$

which is still annihilated by \mathbb{Q}_i . This object can also be written as a $\bar{\zeta}$ -dependent linear combination of the R-symmetry components of the multiplet to which the Schur operator belongs,

$$\tilde{\mathcal{O}}(\zeta, \bar{\zeta}) = u_{I_1}(\bar{\zeta}) \dots u_{I_{2R}}(\bar{\zeta}) \mathcal{O}_{+\dots+\dot{+}\dots\dot{+}}^{I_1 \dots I_{2R}}(\zeta, \bar{\zeta}), \quad u_1(\bar{\zeta}) = 1, \quad u_2(\bar{\zeta}) = \bar{\zeta}. \quad (2.9)$$

Crucially, thanks to the fact that the generators $\widetilde{\mathfrak{sl}(2)}$ are \mathbb{Q}_i -exact, the antiholomorphic dependence of $\tilde{\mathcal{O}}(\zeta, \bar{\zeta})$ is trivial in cohomology,

$$\partial_{\bar{\zeta}} \tilde{\mathcal{O}}(\zeta, \bar{\zeta}) = [\mathbb{Q}_i, \dots]. \quad (2.10)$$

This suggests the notation

$$\chi[\mathcal{O}](\zeta) = \mathbb{Q}_i\text{-cohomology class of } \tilde{\mathcal{O}}(\zeta, \bar{\zeta}). \quad (2.11)$$

Correlators of cohomology classes $\chi[\mathcal{O}](\zeta)$ are defined in terms of correlators of the representatives $\tilde{\mathcal{O}}(\zeta, \bar{\zeta})$,

$$\langle \chi[\mathcal{O}_1](\zeta_1) \dots \chi[\mathcal{O}_n](\zeta_n) \rangle = \langle \tilde{\mathcal{O}}_1(\zeta_1, \bar{\zeta}_1) \dots \tilde{\mathcal{O}}_n(\zeta_n, \bar{\zeta}_n) \rangle, \quad (2.12)$$

are independent of the representative chosen, and depend meromorphically on the insertion points. By the same token, the four-dimensional OPE of two twisted translated Schur operators $\tilde{\mathcal{O}}_1, \tilde{\mathcal{O}}_2$ induces a meromorphic OPE of cohomology classes $\chi[\mathcal{O}_1], \chi[\mathcal{O}_2]$. Let us remark that the holomorphic dimension h of $\chi[\mathcal{O}]$ is given in terms of the quantum numbers of the four-dimensional Schur operator \mathcal{O} by

$$h = \frac{1}{2}(\Delta + j_1 + j_2). \quad (2.13)$$

This quantity is generically a half-integer, but as a consequence of four-dimensional $\mathfrak{sl}(2)_R$ selection rules, the OPE of any two cohomology classes $\chi[\mathcal{O}_1], \chi[\mathcal{O}_2]$ is single-valued in the ζ -plane.⁶

2.2 Affine enhancement of symmetries

The stress tensor of a four-dimensional $\mathcal{N} = 2$ theory sits in a supersymmetry multiplet of type $\hat{\mathcal{C}}_{0(0,0)}$ in the notation of [37]. The same multiplet contains the $\mathfrak{sl}(2)_R$ symmetry current of the theory, $J_{\alpha\dot{\beta}}^{IJ}$. Its Lorentz and R-symmetry highest-weight component is a Schur operator and determines an element of the chiral algebra with holomorphic dimension two,

$$\mathbb{T} = \chi[J_{++}^{11}]. \tag{2.14}$$

This object is identified with the stress tensor of the chiral algebra.⁷ The meromorphic $\mathbb{T}\mathbb{T}$ OPE is determined by the OPE of R-symmetry currents in four dimensions, and has the expected form with a two-dimensional central charge

$$c_{2d} = -12 c_{4d}, \tag{2.15}$$

where c_{4d} is one of the two conformal anomaly coefficients of the four-dimensional theory [1]. Unitarity in four dimensions requires $c_{4d} > 0$, yielding a non-unitary chiral algebra in two dimensions.

If the four-dimensional theory is invariant under a continuous flavor symmetry group G_F , its spectrum contains a conserved current in the adjoint of the flavor symmetry algebra \mathfrak{g}_F . The latter is contained in a supersymmetry multiplet of type $\hat{\mathcal{B}}_1$, which also includes an $\mathfrak{sl}(2)_R$ triplet of scalars M^{IJ} in the adjoint of \mathfrak{g}_F with $\Delta = 2$. The R-symmetry highest weight component of M^{IJ} is a Schur operator, yielding an element of the chiral algebra with holomorphic dimension one,

$$\mathbb{J} = \chi[M^{11}]. \tag{2.16}$$

The $\mathbb{J}\mathbb{J}$ OPE reveals that this object can be identified with an affine current in two dimensions, satisfying a Kac-Moody algebra based on the Lie algebra \mathfrak{g}_F with level

$$k_{2d} = -\frac{1}{2}k_{4d}, \tag{2.17}$$

where k_{4d} is an anomaly coefficient entering the four-dimensional OPE of two flavor currents [1].

The cohomological construction of the previous section can also be performed in theories with $\mathcal{N} = 3$ or $\mathcal{N} = 4$ superconformal symmetry. The spectrum of such a theory, expressed in $\mathcal{N} = 2$ language, contains additional conserved spin 3/2 supersymmetry currents. The latter are contained in supermultiplets of type $\mathcal{D}_{\frac{1}{2}(0,0)}, \bar{\mathcal{D}}_{\frac{1}{2}(0,0)}$, which also include

⁶This implies that the NS sector is the relevant sector of the chiral algebra from the point of view of the parent four-dimensional SCFT.

⁷We refer the reader to [1] for a careful discussion of the relative normalization of \mathbb{T}, J_{++}^{11} in a standard set of conventions for four-dimensional and two-dimensional operators, and similarly for other pairs of four-dimensional and two-dimensional operators discussed below.

an $\mathfrak{sl}(2)_R$ triplet of spin 1/2 operators $\Psi_\alpha^{(IJ)}$, $\bar{\Psi}_\alpha^{(IJ)}$ with $\Delta = 5/2$. The highest-weight components of Ψ , $\bar{\Psi}$ are Schur operators and yield elements of the chiral algebra with holomorphic dimension 3/2,

$$G = \chi[\Psi_+^{11}], \quad \tilde{G} = \chi[\bar{\Psi}_+^{11}]. \tag{2.18}$$

The operators G , \tilde{G} are supersymmetry currents in two-dimensions. Both Ψ , $\bar{\Psi}$ and G , \tilde{G} carry implicitly flavor symmetry indices associated to the commutant of the $\mathcal{N} = 2$ R-symmetry $\mathfrak{sl}(2)_R \oplus \mathbb{C}_r$ inside the larger R-symmetry group of the $\mathcal{N} = 3$ or $\mathcal{N} = 4$ theory.

Focusing on the case of an $\mathcal{N} = 4$ theory, the larger (complexified) R-symmetry group is $\mathfrak{sl}(4)_R$ and the commutant of $\mathfrak{sl}(2)_R \oplus \mathbb{C}_r$ is $\mathfrak{sl}(2)_F$. The relevant branching rule is

$$\begin{aligned} \mathfrak{sl}(4)_R &\longrightarrow \mathfrak{sl}(2)_R \times \mathfrak{sl}(2)_F \times \mathbb{C}_r \\ [1, 0, 0] &= \left(\frac{1}{2}, 0\right)_{1/2} \oplus \left(0, \frac{1}{2}\right)_{-1/2}, \end{aligned} \tag{2.19}$$

where we denoted the fundamental representation of $\mathfrak{sl}(4)_R$ by its Dynkin indices, $\mathfrak{sl}(2)$ representations by their half-integral spin, and the subscript is the \mathbb{C}_r charge. Fundamental indices of $\mathfrak{sl}(2)_F$ will be denoted $\hat{I}, \hat{J} = 1, 2$. It follows that the chiral algebra always contains the two-dimensional small $\mathcal{N} = 4$ chiral algebra [38]. The latter is generated by the stress tensor T , two supersymmetry currents $G^{\hat{I}}, \tilde{G}^{\hat{I}}$ with holomorphic dimension 3/2 in the fundamental of $\mathfrak{sl}(2)_F$, and an $\mathfrak{sl}(2)_F$ current $J^{(\hat{I}\hat{J})}$. The Virasoro modes $L_{0,\pm 1}$, the supercurrent modes $G_{\pm 1/2}^{\hat{I}}, \tilde{G}_{\pm 1/2}^{\hat{I}}$ and the modes $J_0^{\hat{I}\hat{J}}$ of the affine current generate a global $\mathfrak{psl}(2|2)$ symmetry.

3 Localization argument

Given a four-dimensional $\mathcal{N} = 2$ theory admitting a holographic dual, it is natural to ask what is the bulk analog of the field-theoretic cohomological construction that we have just reviewed. In this section we address this problem in a simplified model.

The superconformal algebra on the field theory side is realized on the gravity side as the algebra of superisometries of the background. In particular, the background admits suitable Killing spinors that can be identified with the linear combinations $\mathbb{Q}_1, \mathbb{Q}_2$ of section 2. In light of the cohomological construction on the field theory side, we expect the following picture on the gravity side. If we only switch on sources dual to twisted translated Schur operators on the field theory side, the partition function on the gravity side should be subject to supersymmetric localization and should define an effective dynamics localized on an AdS_3 slice of the original AdS_5 spacetime. The boundary of the AdS_3 slice is identified to the preferred $(\zeta, \bar{\zeta})$ plane singled out by the cohomological construction on the field theory side. Implementing this program rigorously appears challenging in any realistic holographic duality, e.g., in the canonical duality between large N $\mathcal{N} = 4$ SYM theory and IIB string theory on $AdS_5 \times S^5$. We are not aware of the requisite off-shell formalism for IIB supergravity on $AdS_5 \times S^5$, or even for its consistent truncation to $\mathcal{N} = 8$ gauged supergravity on AdS_5 . We can, however, address explicitly a simplified version of the problem, along the following lines.

Consider an $\mathcal{N} = 2$ SCFT with a flavor symmetry algebra \mathfrak{g}_F . Our main target is $\mathcal{N} = 4$ SYM theory, for which $\mathfrak{g}_F = \mathfrak{su}(2)_F$ (the centralizer of the $4d$ $\mathcal{N} = 2$ superconformal algebra $\mathfrak{su}(2, 2|2)$ inside the $\mathcal{N} = 4$ superconformal algebra $\mathfrak{psu}(2, 2|4)$), but we may as well keep \mathfrak{g}_F general. According to the standard AdS/CFT dictionary, on the gravity side we find massless gauge fields with gauge algebra \mathfrak{g}_F , which must belong to an $\mathcal{N} = 4$ vector multiplet (half-maximal susy). The vector multiplet is part of the spectrum of a suitable half-maximal supergravity in five dimensions admitting an AdS_5 vacuum. We will consider the truncation of the full supergravity to the $\mathcal{N} = 4$ supersymmetric five-dimensional gauge theory with gauge algebra \mathfrak{g}_F on a non-dynamical AdS_5 background. This setup *can* be explicitly analyzed using available localization techniques.

We should point out from the outset that the restriction to the vector multiplet is not a *bona fide* consistent truncation of the full equations of motion of five-dimensional supergravity.⁸ It is however guaranteed to be a “twisted” consistent truncation, i.e., to hold in \mathbb{Q} -cohomology. Indeed, the corresponding sector of the chiral algebra is just the affine Kac-Moody algebra $\widehat{\mathfrak{g}}_F$, which is clearly a closed subalgebra.

3.1 Summary of the localization results

As the details of our calculations are somewhat technical, we begin with a summary of the main results. Our goal is to show that the five-dimensional super Yang-Mills action defined on AdS_5 localizes to an effective action defined on an AdS_3 slice inside AdS_5 , and determine this effective action. The relevant AdS_3 slice is specified as follows. We can write the Euclidean AdS_5 background in Poincaré coordinates as

$$ds_5^2 = \frac{R^2}{z^2} \left[d\zeta d\bar{\zeta} + d\rho^2 + \rho^2 d\varphi + dz^2 \right], \tag{3.1}$$

where R is the AdS_5 radius, z is the AdS_5 radial coordinate, $\zeta, \bar{\zeta}$ are complex coordinates on a selected plane on the boundary, ρ, φ are polar coordinates along the two other directions on the boundary. The coordinates $\zeta, \bar{\zeta}$ are identified with those used in section 2 in the discussion of the chiral algebra. In particular, the plane selected by the cohomological construction is the plane spanned by $\zeta, \bar{\zeta}$. With this notation, the relevant AdS_3 slice of AdS_5 is the one located at $\rho = 0$ and spanned by $\zeta, \bar{\zeta}, z$,

$$ds_3^2 = \frac{R^2}{z^2} \left[d\zeta d\bar{\zeta} + dz^2 \right]. \tag{3.2}$$

Let us remind the reader that the bosonic field content of maximal super Yang-Mills theory in five dimensions consists of a gauge connection A and five real adjoint scalars,

⁸This can be understood from the viewpoint of the boundary CFT. The $5d$ gauge field A_μ is dual to a conserved current J_m , of conformal dimension three. The *singular* OPE of two currents contains the stress tensor T_{mn} , of conformal dimension four. This is reflected in the bulk in the presence of a cubic coupling between two gauge fields and the fluctuation of the metric $h_{\mu\nu}$, which cannot be removed by a field redefinition. Setting $h_{\mu\nu} = 0$ is not a consistent truncation of the equations of motion, because the equation of motion for $h_{\mu\nu}$ would still induce a spurious constraint for the gauge fields. By contrast, in the chiral algebra the Sugawara stress tensor appears (by definition) as the leading *non-singular* term in the OPE of two affine currents.

denoted here $\phi_6, \phi_7, \phi_8, \phi_9, \phi_0$. (Our terminology is related to the ten-dimensional origin of these fields, described in the following subsection.) The realization of off-shell supersymmetry used in the localization computation induces a split of the five scalars into (ϕ_6, ϕ_7) and (ϕ_8, ϕ_9, ϕ_0) .

After these preliminaries, we can exhibit the value of the localized super Yang-Mills action. It can be written as the sum of two decoupled contributions,

$$S = S_{\text{free}} + S_{\text{CS}}, \quad (3.3)$$

where

$$S_{\text{free}} = \frac{i\pi R}{g_{\text{YM}}^2} \int_{AdS_3} d\zeta d\bar{\zeta} dz \frac{R^2}{z^2} \text{tr}(\phi_6^2 + \phi_7^2), \quad (3.4)$$

$$S_{\text{CS}} = -\frac{ik}{4\pi} \int_{AdS_3} \text{tr} \left(\mathbf{A} d\mathbf{A} + \frac{2}{3} \mathbf{A}^3 \right), \quad k = -\frac{8\pi^2 R}{g_{\text{YM}}^2}. \quad (3.5)$$

Here g_{YM}^2 denotes the Yang-Mills coupling of the five-dimensional super Yang-Mills theory, and the symbol tr stands for the trace in a reference representation of the gauge algebra (the fundamental for gauge algebra $\mathfrak{su}(N)$). The scalars ϕ_6, ϕ_7 are implicitly evaluated at $\rho = 0$, i.e. on the AdS_3 slice of AdS_5 . The object \mathbf{A} is an emergent complex gauge connection living on the AdS_3 slice of AdS_5 . Its expression in terms of the fields of the original Yang-Mills theory reads

$$\mathbf{A} = \mathbf{A}_\zeta d\zeta + \mathbf{A}_{\bar{\zeta}} d\bar{\zeta} + \mathbf{A}_z dz, \quad (3.6)$$

with

$$\mathbf{A}_\zeta = A_\zeta - \frac{iR^2}{2z^2} \left[(\phi_8 + i\phi_9) + \frac{2i\bar{\zeta}}{R} \phi_0 - \frac{\bar{\zeta}^2}{R^2} (\phi_8 - i\phi_9) \right], \quad (3.7)$$

$$\mathbf{A}_{\bar{\zeta}} = A_{\bar{\zeta}} - \frac{i}{2} (\phi_8 - i\phi_9), \quad (3.8)$$

$$\mathbf{A}_z = A_z + \frac{R}{z} \left[\phi_0 + \frac{i\bar{\zeta}}{R} (\phi_8 - i\phi_9) \right]. \quad (3.9)$$

The symbols $A_\zeta, A_{\bar{\zeta}}, A_z$ denote the components of the pullback of the original Yang-Mills connection A from AdS_5 to the AdS_3 slice. The scalars ϕ_8, ϕ_9, ϕ_0 are implicitly evaluated on the AdS_3 slice.

The quadratic action S_{free} for $\phi_{6,7}$ is expected to be completely decoupled from the rest of the dynamics on the AdS_3 slice, even if suitable supersymmetric insertions are considered in the path integral. As a result, $\phi_{6,7}$ are expected to provide only an inconsequential field-independent Gaussian factor in the computation of correlators, and can be effectively ignored. The emergent gauge field \mathbf{A} , on the other hand, has dynamics specified by the Chern-Simons action S_{CS} , which according to the classic results of [39–41] defines a WZWN theory on the boundary of AdS_3 based on the group G_F . This provides a realization of the two-dimensional affine current algebra of the Lie algebra \mathfrak{g}_F , as expected from the cohomological construction on the field theory side.

The outline of the rest of this section is as follows. The derivation of the above results is described in subsections 3.2, 3.3, 3.4. Further comments about our results are collected in subsection 3.5, in which we also test our findings against predictions from the chiral algebra based on the case of $\mathcal{N} = 4$ super Yang-Mills with gauge algebra $\mathfrak{su}(N)$.

3.2 Lagrangian with off-shell supersymmetry

We consider maximally supersymmetric Yang-Mills theory in five dimensions on a Euclidean AdS_5 background. Following [25, 42] this theory can be constructed in two steps. Firstly, the flat-space $10d$ maximally supersymmetric Yang-Mills theory with signature $(1, 9)$ is formally dimensionally reduced on a five-torus with signature $(1, 4)$. Secondly, the external flat metric is replaced with the curved Euclidean AdS_5 metric, minimal coupling to gravity is introduced, as well as extra non-minimal couplings needed for supersymmetry. In order to set up our notation, we review here the field content, Lagrangian, and off-shell supersymmetry variations, following closely [25]. Maximally supersymmetric Yang-Mills theory with off-shell supersymmetry has also been studied in [43] in a variety of backgrounds, including AdS_5 .

Curved $5d$ spacetime indices are denoted $\mu, \nu = 1, \dots, 5$. We adopt Poincaré coordinates for the background AdS_5 metric,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \frac{R^2}{z^2} [(dx^1)^2 + \dots + (dx^4)^2 + dz^2], \quad x^5 \equiv z. \quad (3.10)$$

A convenient choice of vielbein is

$$e^{\hat{\lambda}}{}_{\mu} = \frac{R}{z} \delta^{\lambda}_{\mu}, \quad \lambda, \mu = 1, \dots, 5, \quad (3.11)$$

where a hat is used to denote flat $5d$ spacetime indices.

All dynamical bosonic fields of the $5d$ theory originate from the $10d$ gauge connection A_M , $M = 0, 1, \dots, 9$, where 0 denotes the time direction. Upon dimensional reduction we obtain the $5d$ gauge connection A_{μ} , $\mu = 1, \dots, 5$, as well as five real scalars $\phi_{\mathcal{I}} \equiv A_{\mathcal{I}}$, $\mathcal{I} = 6, \dots, 9, 0$ in the adjoint representation of the gauge group. The index \mathcal{I} is a vector index of the R-symmetry $\mathfrak{so}(4, 1)_R$. The latter, however, is explicitly broken to $\mathfrak{so}(2)_R \oplus \mathfrak{so}(2, 1)_R$ by the way off-shell supersymmetry is realized below. Correspondingly, it is useful to introduce the notation

$$\phi_{\mathcal{I}} = (\phi_i, \phi_A), \quad i = 6, 7, \quad A = 8, 9, 0. \quad (3.12)$$

We use anti-Hermitian generators for the gauge algebra and the $10d$ field strength reads

$$F_{MN} = 2\partial_{[M} A_{N]} + [A_M, A_N]. \quad (3.13)$$

Its components after dimensional reduction are given by

$$F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]} + [A_{\mu}, A_{\nu}], \quad F_{\mu\mathcal{I}} = \partial_{\mu} \phi_{\mathcal{I}} + [A_{\mu}, \phi_{\mathcal{I}}] \equiv D_{\mu} \phi_{\mathcal{I}}, \quad F_{\mathcal{I}\mathcal{J}} = [\phi_{\mathcal{I}}, \phi_{\mathcal{J}}]. \quad (3.14)$$

In order to close supersymmetry off-shell we also need to introduce seven real auxiliary scalars K^m , $m = 1, \dots, 7$ in the adjoint representation of the gauge group. Their vector $\mathfrak{so}(7)$ index is raised and lowered with the flat invariant δ_{mn} .

All fermionic degrees of freedom are encoded in a 16-component Grassmann-odd $10d$ Majorana-Weyl gaugino Ψ_α , $\alpha = 1, \dots, 16$, in the adjoint representation of the gauge group. The chiral blocks of $10d$ gamma matrices are denoted $\Gamma^{M\alpha\beta}$, $\tilde{\Gamma}^M_{\alpha\beta}$, and we also use the notation

$$\Gamma^{MN} = \tilde{\Gamma}^{[M}\Gamma^{N]}, \quad \tilde{\Gamma}^{MN} = \Gamma^{[M}\tilde{\Gamma}^{N]}, \quad \Gamma^{MNP} = \Gamma^{[M}\tilde{\Gamma}^N\Gamma^{P]}, \quad \tilde{\Gamma}^{MNP} = \tilde{\Gamma}^{[M}\Gamma^N\tilde{\Gamma}^{P]}. \quad (3.15)$$

Weyl indices are henceforth suppressed. After dimensional reduction and coupling to the curved AdS_5 background, the $10d$ covariant derivative of the gaugino $D_M\Psi$ gives rise in five dimensions to

$$D_\mu\Psi = \partial_\mu\Psi + \frac{1}{4}\omega_{\mu\hat{\lambda}\hat{\tau}}\Gamma^{\hat{\lambda}\hat{\tau}}\Psi + [A_\mu, \Psi], \quad D_{\mathcal{I}}\Psi = [\phi_{\mathcal{I}}, \Psi], \quad (3.16)$$

where $\omega_{\mu\hat{\lambda}\hat{\tau}}$ is the spin connection associated to the background vielbein (3.11).

The off-shell supersymmetric Lagrangian reads

$$\begin{aligned} \mathcal{L} = \frac{1}{g_{\text{YM}}^2} \text{tr} \left[\frac{1}{2} F_{MN} F^{MN} - \Psi \Gamma^M D_M \Psi + \frac{i}{2R} \Psi \Lambda \Psi - \frac{3}{R^2} \phi^i \phi_i - \frac{4}{R^2} \phi^A \phi_A \right. \\ \left. - \frac{2i}{3R} \epsilon^{ABC} [\phi_A, \phi_B] \phi_C - K^m K_m \right], \end{aligned} \quad (3.17)$$

where tr denotes the trace in a reference representation (the fundamental for gauge algebra $\mathfrak{su}(N)$), and we defined

$$\Lambda = \Gamma^8 \tilde{\Gamma}^9 \Gamma^0, \quad \epsilon_{890} = +1, \quad \epsilon^{890} = -1. \quad (3.18)$$

Note that we adopted the customary compact notation

$$F_{MN} F^{MN} = F_{\mu\nu} F^{\mu\nu} + 2D_\mu \phi_{\mathcal{I}} D^\mu \phi^{\mathcal{I}} + [\phi_{\mathcal{I}}, \phi_{\mathcal{J}}] [\phi^{\mathcal{I}}, \phi^{\mathcal{J}}], \quad (3.19)$$

in which the spacetime indices μ, ν are curved and thus raised with the metric (3.10), while the indices \mathcal{I}, \mathcal{J} are flat and raised with the $\mathfrak{so}(4, 1)_R$ metric $\eta_{\mathcal{I}\mathcal{J}} = \text{diag}(1, 1, 1, 1, -1)$. In a similar way we have

$$\Gamma^M D_M \Psi = \Gamma^\mu D_\mu \Psi + \Gamma^{\mathcal{I}} [\phi_{\mathcal{I}}, \Psi] = \Gamma^{\hat{\lambda}} e_{\hat{\lambda}}^\mu D_\mu \Psi + \Gamma^{\mathcal{I}} [\phi_{\mathcal{I}}, \Psi], \quad (3.20)$$

where $e_{\hat{\lambda}}^\mu$ denotes the inverse of the $5d$ vielbein (3.11). All spinor bilinears in (3.17) and in the following are Majorana bilinears. Further details about our spinor conventions are collected in appendix A.

The Lagrangian (3.17) is invariant up to total derivatives under the off-shell supersymmetry transformations

$$\delta A_\mu = \epsilon \Gamma_\mu \Psi, \quad (3.21)$$

$$\delta \phi_{\mathcal{I}} = \epsilon \Gamma_{\mathcal{I}} \Psi, \quad (3.22)$$

$$\delta \Psi = \frac{1}{2} \Gamma^{MN} F_{MN} \epsilon + \frac{2}{5} \Gamma^{\mu i} \phi_i \nabla_\mu \epsilon + \frac{4}{5} \Gamma^{\mu A} \phi_A \nabla_\mu \epsilon + K^m \nu_m, \quad (3.23)$$

$$\delta K^m = -\nu^m \Gamma^M D_M \Psi + \frac{i}{2R} \nu^m \Lambda \Psi. \quad (3.24)$$

In these expressions ϵ is a Grassmann-even 16-component Majorana-Wey spinor with the same chirality as Ψ . It satisfies the AdS_5 Killing spinor equation

$$\nabla_\mu \epsilon = \frac{i}{2R} \tilde{\Gamma}_\mu \Lambda \epsilon. \quad (3.25)$$

Note that in this equation $\tilde{\Gamma}_\mu = \tilde{\Gamma}_{\hat{\lambda}} e^{\hat{\lambda}}{}_\mu$, $\nabla_\mu \epsilon = \partial_\mu \epsilon + \frac{1}{4} \omega_{\mu\hat{\lambda}\hat{\tau}} \Gamma^{\hat{\lambda}\hat{\tau}} \epsilon$. Let us also stress that the compact notation $\Gamma^{MN} F_{MN}$ in (3.23) is subject to remarks similar to those around (3.19) and (3.20) above. We have also introduced a set ν_m , $m = 1, \dots, 7$ of auxiliary Grassmann-even spinors with the same chirality as ϵ , determined up to an $\mathfrak{so}(7)$ rotation by the algebraic relations

$$\nu_m \Gamma^M \epsilon = 0, \quad \nu_m \Gamma^M \nu_n = \delta_{mn} \epsilon \Gamma^M \epsilon. \quad (3.26)$$

The $\mathfrak{so}(7)$ index m on ν_m is raised with δ^{mn} .

The square of the supersymmetry transformations (3.21)–(3.24) can be written as combinations of the bosonic symmetries of the theory without using the equations of motion. More precisely, one has

$$\delta^2 A_\mu = -v^\nu F_{\nu\mu} + D_\mu (v^\mathcal{I} \phi_\mathcal{I}), \quad (3.27)$$

$$\delta^2 \phi_i = -v^\nu D_\nu \phi_i - [v^\mathcal{I} \phi_\mathcal{I}, \phi_i] - \frac{i}{R} \epsilon \tilde{\Gamma}_{ij} \Lambda \epsilon \phi^j, \quad (3.28)$$

$$\delta^2 \phi_A = -v^\nu D_\nu \phi_A - [v^\mathcal{I} \phi_\mathcal{I}, \phi_A] - \frac{2i}{R} \epsilon \tilde{\Gamma}_{AB} \Lambda \epsilon \phi^B, \quad (3.29)$$

$$\delta^2 \Psi = -v^\nu D_\nu \Psi - \frac{1}{4} \nabla_\mu v_\nu \Gamma^{\mu\nu} \Psi - [v^\mathcal{I} \phi_\mathcal{I}, \Psi] - \frac{i}{2R} (\epsilon \tilde{\Gamma}^{AB} \Lambda \epsilon) \Gamma_{AB} \Psi - \frac{i}{4R} (\epsilon \tilde{\Gamma}^{ij} \Lambda \epsilon) \Gamma_{ij} \Psi, \quad (3.30)$$

$$\delta^2 K^m = -v^\nu D_\nu K^m - [v^\mathcal{I} \phi_\mathcal{I}, K^m] + \left(-\nu^{[m} \Gamma^\mu \nabla_\mu \nu^{n]} + \frac{i}{2R} \nu^{[m} \Lambda \nu^{n]} \right) K_n, \quad (3.31)$$

where we utilized the spinor bilinears

$$v^\mu = \epsilon \Gamma^\mu \epsilon = e_{\hat{\lambda}}{}^\mu \epsilon \Gamma^{\hat{\lambda}} \epsilon, \quad v^\mathcal{I} = \epsilon \Gamma^\mathcal{I} \epsilon. \quad (3.32)$$

The $5d$ vector v^μ is a Killing vector for the AdS_5 background metric, $\nabla_{(\mu} v_{\nu)} = 0$. Note that $\delta^2 K^m$ contains an $\mathfrak{so}(7)$ rotation, which is a symmetry of the Lagrangian.

All our formulae can be obtained as an analytic continuation of the formulae given in [25] for the case of the five-sphere. More precisely, the radius r of S^5 is related to the radius R of Euclidean AdS_5 as

$$R = ir. \quad (3.33)$$

Note, however, that the coordinate system utilized in [25] is different from the one adopted here, and would correspond in the case of Euclidean AdS_5 to the disk model of hyperbolic space, rather than the half-space model.

3.3 Identification of the relevant supercharge

Our first task in the implementation of the localization argument is the identification of the Killing spinor corresponding to the relevant supercharge on the field theory side. As

reviewed in section 2, for a unitary theory \mathbb{Q}_1 and \mathbb{Q}_2 define the same cohomology classes on the field theory side. From the point of view of localization it is most convenient to consider

$$\mathbb{Q} = \mathbb{Q}_1 + \mathbb{Q}_2. \quad (3.34)$$

The holomorphic $\mathfrak{sl}(2)$ factor on the fixed plane is \mathbb{Q} -closed, and the twisted antiholomorphic factor $\widehat{\mathfrak{sl}(2)}$ is \mathbb{Q} -exact. Note, however, that \mathbb{Q} is not nilpotent, but rather satisfies

$$\{\mathbb{Q}, \mathbb{Q}\} = 2\{\mathbb{Q}_1, \mathbb{Q}_2\} = -2(r + \mathcal{M}_\perp). \quad (3.35)$$

On the gravity side, if we select the Killing spinor ϵ corresponding to \mathbb{Q} , the associated Killing vector v contains the spacetime action \mathcal{M}_\perp , consisting of rotations in the directions orthogonal to the fixed plane. As a result, we localize on the fixed point set of \mathcal{M}_\perp , consisting of the fixed plane itself.

In order to identify the Killing spinor corresponding to \mathbb{Q} we have to analyze the space of solutions to the Killing spinor equation (3.25) with Λ given in (3.18). We refer the reader to section A.2 in the appendices for a thorough discussion and for the explicit expression for the Killing spinor ϵ . Let us summarize here some of its key properties. To this end, it is convenient to use complex coordinates $\zeta, \bar{\zeta}$ in the x^1x^2 plane and polar coordinates ρ, φ in the x^3x^4 plane,

$$\zeta = x^1 + ix^2, \quad \bar{\zeta} = x^1 - ix^2, \quad x^3 = \rho \cos \varphi, \quad x^4 = \rho \sin \varphi. \quad (3.36)$$

The Killing vector v^μ defined in (3.32) takes the form

$$v^\mu \partial_\mu = \partial_\phi, \quad (3.37)$$

while the field-dependent gauge parameter that enters the square of the supersymmetry transformations (3.27)–(3.31) is given by

$$v^{\mathcal{I}} \phi_{\mathcal{I}} = \frac{iR\rho}{z} (\cos \varphi \phi_6 - \sin \varphi \phi_7). \quad (3.38)$$

Our Killing spinor induces no $\mathfrak{so}(2, 1)_R$ R-symmetry rotation, but yields a non-zero $\mathfrak{so}(2)_R$ rotation,

$$\epsilon^{\tilde{\Gamma}AB} \Lambda \epsilon = 0, \quad \epsilon^{\tilde{\Gamma}ij} \Lambda \epsilon = iR \epsilon^{ij}, \quad \epsilon^{67} = +1. \quad (3.39)$$

Recall that the Lie derivative of a spinor in the direction of a Killing vector k^μ is given by

$$\mathcal{L}_k \epsilon = k^\mu \nabla_\mu \epsilon + \frac{1}{4} \nabla_\mu k_\nu \Gamma^{\mu\nu} \epsilon. \quad (3.40)$$

Using this expression one can check that our Killing spinor is invariant under the action of the Killing vectors associated to the holomorphic conformal generators in the $(\zeta, \bar{\zeta})$ plane. More precisely, if we consider the Killing vectors

$$k(L_{-1}) = -\partial_\zeta, \quad (3.41)$$

$$k(L_0) = -\zeta \partial_\zeta - \frac{1}{2} \rho \partial_\rho - \frac{1}{2} z \partial_z, \quad (3.42)$$

$$k(L_{+1}) = -\zeta^2 \partial_\zeta - \zeta \rho \partial_\rho - \zeta z \partial_z + (z^2 + \rho^2) \partial_{\bar{\zeta}}, \quad (3.43)$$

we have

$$\mathcal{L}_{k(L_m)}\epsilon = 0, \quad m = -1, 0, +1. \quad (3.44)$$

This corresponds to the fact that our supercharge commutes with the holomorphic conformal generators in the $(\zeta, \bar{\zeta})$ plane. Furthermore, we expect the anti-holomorphic generators to be exact. This expectation is confirmed by checking that each of the Killing vectors

$$k(\bar{L}_{-1}) = -\partial_{\bar{\zeta}}, \quad (3.45)$$

$$k(\bar{L}_0) = -\bar{\zeta}\partial_{\bar{\zeta}} - \frac{1}{2}\rho\partial_{\rho} - \frac{1}{2}z\partial_z, \quad (3.46)$$

$$k(\bar{L}_{+1}) = -\bar{\zeta}^2\partial_{\bar{\zeta}} - \bar{\zeta}\rho\partial_{\rho} - \bar{\zeta}z\partial_z + (z^2 + \rho^2)\partial_{\zeta}, \quad (3.47)$$

can be written in the form $\epsilon\Gamma^{\mu}\epsilon'$ for a suitable Killing spinor ϵ' .

Once the suitable Killing spinor ϵ is identified, we are left with the task of finding the associated auxiliary spinors ν_m satisfying (3.26). We refer the reader to section A.4 in the appendix for more details on this point.

3.4 BPS locus and classical action

The localization argument ensures that in the computation of \mathbb{Q} -closed observables the path integral localizes to the BPS locus⁹

$$\Psi = 0, \quad \delta\Psi = 0. \quad (3.48)$$

In particular this implies that δ^2 annihilates all fields on the BPS locus. Making use of the expression for δ^2 recorded in the previous section one can verify that, for our choice of supercharge, this implies

$$\begin{aligned} D_{\varphi}\phi_i &= -[v^{\mathcal{J}}\phi_{\mathcal{J}}, \phi_i] + \epsilon_{ij}\phi^j, & F_{\rho\varphi} &= -D_{\rho}(v^{\mathcal{I}}\phi_{\mathcal{I}}), \\ D_{\varphi}\phi_A &= -[v^{\mathcal{J}}\phi_{\mathcal{J}}, \phi_A], & F_{\tilde{\mu}\varphi} &= -D_{\tilde{\mu}}(v^{\mathcal{I}}\phi_{\mathcal{I}}), \end{aligned} \quad (3.49)$$

where we introduced a new $3d$ curved spacetime index $\tilde{\mu} = \zeta, \bar{\zeta}, z$ and $v^{\mathcal{I}}\phi_{\mathcal{I}}$ is given in (3.38). Let us point out that, in all the above equations, the covariant derivative acts on spacetime scalars and therefore contains the gauge field but no spacetime connection.

Once the constraints coming from $\delta^2 = 0$ are implemented, one can show that the 16 equations $\delta\Psi = 0$ are all solved by determining the seven auxiliary scalars K^m as a functional of all other bosonic fields. In summary,

$$\text{BPS locus: } \left\{ \begin{array}{l} \Psi = 0, \\ D_{\varphi}\phi_i = -[v^{\mathcal{J}}\phi_{\mathcal{J}}, \phi_i] + \epsilon_{ij}\phi^j, \quad D_{\varphi}\phi_A = -[v^{\mathcal{J}}\phi_{\mathcal{J}}, \phi_A], \\ F_{\rho\varphi} = -D_{\rho}(v^{\mathcal{I}}\phi_{\mathcal{I}}), \quad F_{\tilde{\mu}\varphi} = -D_{\tilde{\mu}}(v^{\mathcal{I}}\phi_{\mathcal{I}}), \\ K^m \text{ determined in terms of } A_{\mu}, \phi_{\mathcal{I}}. \end{array} \right. \quad (3.50)$$

⁹The fact that only field configurations invariant under the action of \mathbb{Q} can contribute to the path integral can be argued on very general grounds, independently of the choice of localizing terms. See, for instance, section 3.1 of [44]. We restrict our attention to the bosonic part of the BPS locus because we are interested in extracting the classical action. The integral over the fermionic component of the BPS locus is best thought of as part of the computation of the one-loop determinant, which lies beyond the scope of this work.

We refrain from recording here the explicit expressions for the auxiliary scalars K^m in terms of $A_\mu, \phi_{\mathcal{I}}$, which are lengthy and not particularly illuminating.

As a next step in the localization we evaluate the classical Lagrangian (3.17) on the BPS locus (3.50). A straightforward but tedious computation shows that the entire bosonic Lagrangian, including the appropriate volume form, collapses on the BPS locus to a sum of total derivatives. More precisely, one finds

$$\sqrt{g} \mathcal{L} d^4x dz = \left[\partial_\rho Y^\rho + \partial_\zeta Y^\zeta + \partial_{\bar{\zeta}} Y^{\bar{\zeta}} + \partial_\varphi Y^\varphi + \partial_z Y^z \right] d\zeta d\bar{\zeta} d\rho d\varphi dz, \quad (3.51)$$

where the quantities Y are suitable functionals of the gauge field and scalars whose explicit expressions are not recorded for the sake of brevity. On the l.h.s. the notation $d^4x dz$ is a shorthand for the five form $dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \wedge dz$, and by a similar token we have omitted wedge products on the r.h.s. In checking (3.51) it is essential to take into account the factors coming from the expression of the AdS_5 volume form in the $(\zeta, \bar{\zeta}, \rho, \varphi, z)$ coordinate system,

$$\sqrt{g} d^4x dz = \frac{iR^5 \rho}{2z^5} d\zeta d\bar{\zeta} d\rho d\varphi dz. \quad (3.52)$$

The classical action on the BPS locus is given by the integral of (3.51) over the factorized domain

$$-\infty < \text{Re}\zeta < \infty, \quad -\infty < \text{Im}\zeta < \infty, \quad 0 \leq \rho < \infty, \quad 0 \leq \varphi < 2\pi, \quad 0 \leq z < \infty. \quad (3.53)$$

Let us discuss the possible boundary contributions. Of course, since all fields are periodic in the angular variable φ no boundary term can be generated by integrating $\partial_\varphi Y^\varphi$. We assume that all fields fall off sufficiently rapidly at infinity in all directions orthogonal to the radial coordinate z of AdS_5 . As a result, we get no contributions from $\partial_\zeta Y^\zeta + \partial_{\bar{\zeta}} Y^{\bar{\zeta}}$, while $\partial_\rho Y^\rho$ contributes exclusively via the lower limit of integration $\rho = 0$. The asymptotic behavior of fields in the z direction is more subtle and is related to the implementation of the AdS/CFT prescription for the computation of correlators. The goal of our localization computation is the identification of an effective $3d$ bulk theory that could be then used to compute correlators of twisted-translated Schur operators according to the standard prescription. For the purpose of identifying the $3d$ theory we do not need to consider boundary terms coming from the z direction.

In conclusion, the relevant classical action on the BPS locus can be written as

$$S_{\text{cl}} = 2\pi \int d\zeta d\bar{\zeta} dz \left(-Y^\rho \Big|_{\rho=0} \right), \quad (3.54)$$

where we anticipated that Y^ρ is actually independent of φ and we performed the φ integration. The fact that all fields are evaluated at $\rho = 0$ shows manifestly the expected localization of the dynamics on the $\zeta\bar{\zeta}$ plane which is fixed under the action of the Killing vector (3.37) associated to our Killing spinor.

Let us now record the expression of the integrand in (3.54) in a convenient way. To this end, it is useful to trade the scalar fields ϕ_A , $A = 8, 9, 0$ of the original Yang-Mills theory with the components of a one-form $\Phi_{\bar{\mu}}$ living on the AdS_3 slice of AdS_5 identified

by $\rho = 0$ and parametrized by $\zeta, \bar{\zeta}, z$. This twist is achieved by means of a suitable object $\mathcal{V}^{A\tilde{\mu}}$ built from bilinears of the Killing spinor ϵ in the following way. To begin with, let us define the $5d$ three-vector

$$\mathcal{X}^{A\mu_1\mu_2\mu_3} = -\frac{i}{2R} \epsilon \Lambda \tilde{\Gamma}^A \Gamma^{\mu_1\mu_2\mu_3} \epsilon, \quad (3.55)$$

where μ_1, μ_2, μ_3 are curved $5d$ indices. Our choice of spinor breaks $5d$ covariance by selecting the plane spanned by ρ, φ . It is thus natural to consider the components $\mathcal{X}^{A\tilde{\mu}\rho\varphi}$ with $\tilde{\mu} = \zeta, \bar{\zeta}, z$. The sought-for intertwiner $\mathcal{V}^{A\tilde{\mu}}$ is then constructed as

$$\mathcal{V}^{A\tilde{\mu}} = \lim_{\rho \rightarrow 0^+} \rho \mathcal{X}^{A\tilde{\mu}\rho\varphi}, \quad (3.56)$$

where the prefactor ρ has been introduced to guarantee finiteness of the limit. The relation between the scalars ϕ_A and the twisted one-form Φ_μ is then

$$\phi_A = \frac{2R^2}{z^2} \Phi_{\tilde{\mu}} \eta_{AB} \mathcal{V}^{B\tilde{\mu}}, \quad \eta_{AB} = \text{diag}(1, 1, -1), \quad (3.57)$$

where the normalization factor has been chosen for later convenience, and all fields are implicitly evaluated at $\rho = 0$. More explicitly, in our conventions the components of $\Phi_{\tilde{\mu}}$ are given by

$$\Phi_\zeta = -\frac{iR^2}{2z^2} \left[(\phi_8 + i\phi_9) + \frac{2i\bar{\zeta}}{R} \phi_0 - \frac{\bar{\zeta}^2}{R^2} (\phi_8 - i\phi_9) \right], \quad (3.58)$$

$$\Phi_{\bar{\zeta}} = -\frac{i}{2} (\phi_8 - i\phi_9), \quad (3.59)$$

$$\Phi_z = \frac{R}{z} \left[\phi_0 + \frac{i\bar{\zeta}}{R} (\phi_8 - i\phi_9) \right]. \quad (3.60)$$

We can finally present the explicit expression for the classical action (3.54). It can be written as the sum of a non-topological and a topological term,

$$S_{\text{cl}} = S_{\text{free}} + S_{\text{CS}}, \quad (3.61)$$

where

$$S_{\text{free}} = \frac{i\pi R}{g_{\text{YM}}^2} \int_{AdS_3} d\zeta d\bar{\zeta} dz \frac{R^2}{z^2} \text{tr} \phi^i \phi_i, \quad (3.62)$$

$$S_{\text{CS}} = \frac{2\pi i R}{g_{\text{YM}}^2} \int_{AdS_3} \text{tr} \left(\Phi d_A \Phi + \frac{2}{3} \Phi^3 + 2F \Phi + A dA + \frac{2}{3} A^3 \right). \quad (3.63)$$

Let us remind the reader that all quantities are implicitly evaluated at $\rho = 0$. In the second line we have adopted a differential form notation suppressing wedge products and, by slight abuse of notation, A, F denote the restriction of the $5d$ gauge connection and field strength the AdS_3 slice spanned by coordinates $\zeta, \bar{\zeta}, z$. The symbol d_A denotes the exterior gauge-covariant derivative

$$d_A \Phi = d\Phi + A\Phi + \Phi A. \quad (3.64)$$

Let us point out that the appearance of the topological term $\text{tr} \Phi^3$ in (3.63) is a consequence of the cubic term $\epsilon^{ABC} \phi_A[\phi_B, \phi_C]$ in the scalar potential of the original Yang-Mills Lagrangian (3.17).

It is useful to construct the quantity

$$\mathbf{A} = A + \Phi, \tag{3.65}$$

which transforms as a connection since Φ is an adjoint-valued one-form. Thanks to the identity

$$\text{tr} \left(\mathbf{A} d\mathbf{A} + \frac{2}{3} \mathbf{A}^3 \right) = \text{tr} \left(AdA + \frac{2}{3} A^3 + \Phi d_A \Phi + \frac{2}{3} \Phi^3 + 2\Phi F \right) + d\text{tr}(\Phi A), \tag{3.66}$$

the topological part of the action S_{top} can be written compactly as a Chern-Simons action,

$$S_{\text{CS}} = -\frac{ik}{4\pi} \int_{AdS_3} \text{tr} \left(\mathbf{A} d\mathbf{A} + \frac{2}{3} \mathbf{A}^3 \right), \quad k = -\frac{8\pi^2 R}{g_{\text{YM}}^2}. \tag{3.67}$$

The minus sign in front of the Chern-Simons term is introduced because, in our conventions, the pairing $\text{tr}(ab)$ is negative definite, since we are using antihermitian generators. For instance, for gauge algebra $\mathfrak{su}(n)$,¹⁰ we adopt the standard normalization with tr denoting the trace in the fundamental representation,

$$\text{tr}(t_{\mathbf{a}} t_{\mathbf{b}}) = -\frac{1}{2} \delta_{\mathbf{ab}}, \quad [t_{\mathbf{a}}, t_{\mathbf{b}}] = f_{\mathbf{ab}^c} t_{\mathbf{c}}, \tag{3.68}$$

where $\mathbf{a}, \mathbf{b} = 1, \dots, n^2 - 1$ are adjoint indices of $\mathfrak{su}(n)$. As a result, we may also write

$$S_{\text{CS}} = \frac{ik}{8\pi} \delta_{\mathbf{ab}} \int_{AdS_3} \left(\mathbf{A}^{\mathbf{a}} d\mathbf{A}^{\mathbf{b}} + \frac{1}{3} f_{\mathbf{cd}}{}^{\mathbf{b}} \mathbf{A}^{\mathbf{a}} \mathbf{A}^{\mathbf{c}} \mathbf{A}^{\mathbf{d}} \right), \quad k = -\frac{8\pi^2 R}{g_{\text{YM}}^2}. \tag{3.69}$$

The localization technique can be applied with arbitrary insertions of \mathbb{Q} -closed functionals of the fields. The functionals should be well-defined on the AdS_3 slice and have a vanishing supersymmetry variation (3.21)–(3.22) on that slice. It is not hard to check that functionals built from \mathbf{A}_ζ , $\mathbf{A}_{\bar{\zeta}}$ and \mathbf{A}_z (and independent of $\phi_{6,7}$) satisfy these requirements,¹¹ in agreement with the conclusion that $\phi_{6,7}$ can be consistently decoupled.

¹⁰This $5d/3d$ gauge algebra should not be confused with the gauge algebra $\mathfrak{su}(N)$ of $4d \mathcal{N} = 4$ SYM. The case $n = 2$ will be relevant below, where we make contact to the chiral algebra dual to $4d \mathfrak{su}(N) \mathcal{N} = 4$ SYM by specializing the $5d$ gauge algebra to be $\mathfrak{su}(2)_F \supset \mathfrak{su}(4)_R$, the R-symmetry algebra of SYM.

¹¹It also appears that these are the *only* admissible functionals. At first sight, the linear combinations

$$A_\rho + \frac{R}{2z} \left[e^{i\varphi} (\phi_6 + i\phi_7) - e^{-i\varphi} (\phi_6 - i\phi_7) \right], \quad A_\varphi, \tag{3.70}$$

seem admissible, since they have a vanishing supersymmetry variation for $\rho \rightarrow 0$. However, assuming that the Cartesian components A_3, A_4 of A_μ are smooth near $\rho = 0$, the quantity $A_\rho = \cos \varphi A_3 + \sin \varphi A_4$ does not admit a unique limit for $\rho \rightarrow 0$, but rather depends on the φ angle with which we approach $\rho = 0$, while the quantity $A_\varphi = \rho(-\sin \varphi A_3 + \cos \varphi A_4)$ vanishes for $\rho \rightarrow 0$. These combinations must therefore be discarded.

3.5 Remarks

Our implementation of the localization recipe is different from the one usually applied to supersymmetric theories on Euclidean compact manifolds. In the latter case it is customary to supplement the classical Lagrangian by an explicit \mathbb{Q} -exact localizing term, $S_{\text{tot}} = S + t\mathbb{Q}V$. The functional V and the reality conditions on the fields are chosen in such a way as to guarantee that, as $t \rightarrow \infty$, the path integral converges and localizes on a suitable real slice of (a subspace of) the BPS locus. Different choices for V and for the reality conditions yield different localization schemes. The computation of the previous section shows that, in *any* localization scheme, the classical action must reduce to (3.61). This conclusion only relies on the form of BPS locus (3.50) without choosing a specific real slice in the space of field configurations. For example, the BPS locus allows for a non-zero profile for the scalars ϕ_i , but they enter the classical action via the quadratic, algebraic action $S_{\text{non-top}}$ only, and therefore decouple from the dynamics after localization.

We are ready to go back to our main physical goal — the determination of the bulk holographic dual to the $2d$ chiral algebra. The bulk action (3.61) must be supplemented with suitable boundary conditions for the fields \mathbf{A} , $\phi_{6,7}$ in order to implement the holographic recipe for the computation of correlators in the boundary theory. We should also contemplate the possibility of additional boundary terms to the AdS_3 action. The boundary conditions and boundary terms for the theory on the AdS_3 slice could be derived via localization of an appropriate set of boundary conditions and boundary terms in the original super Yang-Mills theory defined in AdS_5 . These $5d$ data are constrained by the requirement of compatibility with the action of the supercharge \mathbb{Q} selected for localization.¹² For the problem at hand, we can follow a simpler route without making reference to the parent $5d$ bulk theory. Since we have already argued that the scalars $\phi_{6,7}$ play no relevant role, we focus on \mathbf{A} only.

To begin with, we observe that supercharge \mathbb{Q} induces an asymmetry in the treatment of holomorphic and antiholomorphic components \mathbf{A}_ζ , $\mathbf{A}_{\bar{\zeta}}$. This is most easily detected by looking at the expressions (3.58)–(3.60) for the components of the twisted one-form Φ . Inspection of (3.58)–(3.60) reveals a hierarchy of the three components with respect to the radial coordinate of AdS_3 . In particular, if we prescribe the boundary conditions $\phi_{8,9,0} \sim z^2$ as $z \rightarrow 0$ in order to get a finite Φ_ζ , then $\Phi_{\bar{\zeta}}$ and Φ_z necessarily vanish at the boundary. We can regard \mathbf{A}_ζ , $\mathbf{A}_{\bar{\zeta}}$, \mathbf{A}_z as the supersymmetrizations of Φ_ζ , $\Phi_{\bar{\zeta}}$, Φ_z , and argue that the asymmetric pattern for the holomorphic and antiholomorphic components persists. This mechanism is the bulk dual of the emergence of a purely meromorphic dynamic on the field theory side, once we consider cohomology classes of the supercharges $\mathbb{Q}_1, \mathbb{Q}_2$.

From this observation we can deduce the correct boundary terms that must be added to the action. As explained for instance in [41, 46–48], meromorphic boundary conditions are selected by supplementing the Chern-Simons action with the boundary term

$$S_{\text{bdy}} = \frac{sk}{8\pi} \int_{\partial AdS_3} \text{tr}(\mathbf{A} * \mathbf{A}) = -\frac{sk}{16\pi} \delta_{\mathbf{ab}} \int_{\partial AdS_3} d^2x \sqrt{g} g^{\mu\nu} \mathbf{A}_\mu^{\mathbf{a}} \mathbf{A}_\nu^{\mathbf{b}}, \quad (3.71)$$

¹²We refer the reader to [36, 45] for a discussion of these points in similar contexts.

where s is a constant to be fixed momentarily and, by slight abuse of notation, we used μ, ν to denote two dimensional curved indices on ∂AdS_3 . The combined variation of the bulk action (3.67) and the boundary action (3.71) with respect to the gauge field takes the form

$$\delta(S_{CS} + S_{\text{bdy}}) = -\frac{ik}{2\pi} \int_{AdS_3} \text{tr}(\delta \mathbf{A} \mathbf{F}) - \frac{ik}{2\pi} \int_{\partial AdS_3} d\zeta d\bar{\zeta} \text{tr}(\delta \mathbf{A}_\zeta \mathbf{J}_{\bar{\zeta}} + \delta \mathbf{A}_{\bar{\zeta}} \mathbf{J}_\zeta), \quad (3.72)$$

where

$$\mathbf{J}_{\bar{\zeta}} = \frac{1}{2}(1-s)\mathbf{A}_{\bar{\zeta}}, \quad \mathbf{J}_\zeta = \frac{1}{2}(1+s)\mathbf{A}_\zeta. \quad (3.73)$$

The bulk term in the variation imposes that the connection be flat. The currents $\mathbf{J}_\zeta, \mathbf{J}_{\bar{\zeta}}$ entering the boundary terms of the variation are identified with the currents of the boundary CFT. Because of our choice of supercharge we know that the antiholomorphic component $\mathbf{J}_{\bar{\zeta}}$ of the boundary current should be zero. Thus, we must select $s = 1$. This implies that $k_{2d} = k$, which is negative in our case (recall (3.69)), in agreement with field theory expectations.

Our choice differs from the one in [48], where $s = \text{sgn } k$ is advocated on the basis of the following argument. The boundary action (3.71) contributes to the boundary stress tensor,

$$\delta S_{\text{bdy}} = \frac{1}{2} \int_{\partial AdS_3} d^2x \sqrt{g} T_{\text{bdy}}^{\mu\nu} \delta g_{\mu\nu}, \quad T_{\text{bdy}}^{\mu\nu} = \frac{sk}{8\pi} \delta_{\mathbf{ab}} \left[\mathbf{A}^{\mathbf{a}\mu} \mathbf{A}^{\mathbf{b}\nu} - \frac{1}{2} g^{\mu\nu} \mathbf{A}^{\mathbf{a}\tau} \mathbf{A}_\tau^{\mathbf{b}} \right], \quad (3.74)$$

which in complex components reads

$$T_{\zeta\zeta}^{\text{bdy}} = \frac{sk}{8\pi} \delta_{\mathbf{ab}} \mathbf{A}_\zeta^{\mathbf{a}} \mathbf{A}_\zeta^{\mathbf{b}}, \quad T_{\bar{\zeta}\bar{\zeta}}^{\text{bdy}} = \frac{sk}{8\pi} \delta_{\mathbf{ab}} \mathbf{A}_{\bar{\zeta}}^{\mathbf{a}} \mathbf{A}_{\bar{\zeta}}^{\mathbf{b}}, \quad T_{\zeta\bar{\zeta}}^{\text{bdy}} = 0. \quad (3.75)$$

As explained in [48], with these conventions a positive coefficient of $\mathbf{A}_\zeta^{\mathbf{a}} \mathbf{A}_\zeta^{\mathbf{b}}$ in $T_{\zeta\zeta}^{\text{bdy}}$ corresponds to a positive definite contribution to the boundary energy in a semi-classical picture, leading to the prescription $s = \text{sgn } k$ and $k_{2d} = |k|$. In other terms, in the standard case unitarity of the boundary CFT is enforced by hand. In our case, we must let supersymmetry dictate the correct boundary conditions, and we naturally land on a non-unitary chiral algebra.

We can now specialize (3.67) to the case in which the boundary theory is $4d \mathcal{N} = 4$ super Yang-Mills with gauge algebra $\mathfrak{su}(N)$. Regarded as a $4d \mathcal{N} = 2$ theory, this theory has a flavor symmetry $\mathfrak{su}(2)_F$, the commutant of $\mathfrak{su}(2)_R \times \mathfrak{u}(1)_r$ inside $\mathfrak{su}(4)_R$. The gravity dual of $\mathcal{N} = 4$ super Yang-Mills is maximal gauged supergravity with gauge group $\mathfrak{su}(4)_R$. The corresponding gauge coupling function, evaluated at the origin of the scalar potential corresponding to the AdS_5 vacuum, is [49]

$$\frac{g_{\mathfrak{su}(4)_R}^2}{R} = \frac{16\pi^2}{N^2}, \quad (3.76)$$

at leading order in $1/N$. We have to consider the branching $\mathfrak{su}(4)_R \rightarrow \mathfrak{su}(2)_R \times \mathfrak{u}(1)_r \times \mathfrak{su}(2)_F$ and restrict to the $\mathfrak{su}(2)_F$ factor. One can easily check that this does not affect the normalization of the Yang-Mills kinetic term, so $g_{\text{YM}} = g_{\mathfrak{su}(2)_F} = g_{\mathfrak{su}(4)_R}$. As a result, in this case the Chern-Simons level k in (3.67) is

$$k = -\frac{1}{2}N^2. \quad (3.77)$$

We can compare this result with the level of the affine current algebra of the $\mathfrak{su}(2)_F$ current $J^{\hat{I}\hat{J}}$ in the chiral algebra dual to $\mathcal{N} = 4$ super Yang-Mills with gauge algebra $\mathfrak{su}(N)$. One finds [1]

$$k_{2d} = -\frac{1}{2}k_{4d} = -\frac{1}{2}(N^2 - 1), \tag{3.78}$$

which agrees with the Chern-Simons level k (3.67) in the large N limit.

Finally, let us briefly comment about quantum corrections to the classical result (3.61). While it may be of some technical interest to compute the one-loop determinant factor associated to fluctuations of super Yang-Mills fields transversely to the localization locus, the physical relevance of such a calculation is *a priori* unclear. What would be physically relevant is a calculation of quantum fluctuations in a fully consistent holographic theory, e.g. in IIB string field theory on $AdS_5 \times S^5$, but this is clearly beyond the scope of this work — indeed even the classical problem seems prohibitively hard in the complete theory. Given the agreement of (3.61) with the expected large N result, we may speculate that the only effect of quantum fluctuations is an $O(1)$ shift of the Chern-Simons level.

4 Towards the complete holographic dual

In this section we propose a strategy to determine the complete holographic dual for the chiral algebra of $\mathcal{N} = 4$ SYM theory. A straightforward task is the identification of the linearized bulk modes that correspond to non-trivial boundary operators in \mathbb{Q} -cohomology. Their non-linear interactions are however extremely complicated, and extending the localization procedure to the full theory is presently beyond our technical abilities. Encouraged by the emergence of a Chern-Simons action in the simplified model discussed above, and drawing inspiration from minimal model holography, we will outline a bottom-up construction of the dual theory. We will argue that it is a Chern-Simons theory with gauge algebra a suitable higher-spin Lie superalgebra, defined implicitly by the large N OPE coefficients of $\mathcal{N} = 4$ SYM.

We begin in section 4.1 with a review of the super chiral algebra conjecture of [1]. We give a simple argument in favor of the conjecture in the large N limit. The generators¹³ of the chiral algebra are the single-trace Schur operators, which are in 1-1 correspondence with KK sugra modes obeying the Schur condition. In section 4.2 we give the details of this correspondence. Following the blueprint of minimal model holography, we propose in section 4.3 that the sought after holographic dual is AdS_3 Chern-Simons theory with gauge algebra given by the wedge algebra of the large N super W-algebra. While it is unclear whether such a wedge algebra exists for finite N , we outline its construction at infinite N .

4.1 The $\mathcal{N} = 4$ SYM chiral algebra at large N

As we have reviewed in section 2.2, the chiral algebra always admits the small $\mathcal{N} = 4$ superconformal algebra (SCA) as a subalgebra. The global part of the small $\mathcal{N} = 4$ SCA is $\mathfrak{psu}(1, 1|2)$, and we henceforth organize the operator content in terms of $\mathfrak{psu}(1, 1|2)$

¹³A *generator* of a chiral algebra is an operator that does *not* appear in the non-singular OPE of other operators. In the mathematics literature, what we call a generator is usually referred to as a *strong generator*.

primaries and their descendants. It was conjectured in [1] that the chiral algebra associated to $\mathcal{N} = 4$ SYM with gauge algebra $\mathfrak{su}(N)$ is generated by $N - 1$ $\mathfrak{psu}(1, 1|2)$ primaries and their supersymmetry descendants. The primaries obey the shortening condition $h = k$, where h is the holomorphic dimension (L_0 eigenvalue) and k the $\mathfrak{su}(2)_F$ spin. These supergenerators have $h = k = \frac{1}{2}(n + 2)$ where $n = 0, \dots, N - 2$, and will be denoted as $J^{(n)\hat{I}_1 \dots \hat{I}_{n+2}}$. The central charge is fixed by the general formula (2.15) to $c_{2d} = -3(N^2 - 1)$.

The $\mathfrak{psu}(1, 1|2)$ primary operator $J^{(n)\hat{I}_1 \dots \hat{I}_{n+2}}$ is clearly also an $\mathfrak{sl}(2)$ primary, and among its superdescendants we find additional $\mathfrak{sl}(2)$ primary operators, which we denote as $G^{(n)\hat{I}_1 \dots \hat{I}_{n+1}}$, $\tilde{G}^{(n)\hat{I}_1 \dots \hat{I}_{n+1}}$, $T^{(n)\hat{I}_1 \dots \hat{I}_n}$.¹⁴ The schematic form of the supermultiplet containing $J^{(n)\hat{I}_1 \dots \hat{I}_{n+2}}$, $G^{(n)\hat{I}_1 \dots \hat{I}_{n+1}}$, $\tilde{G}^{(n)\hat{I}_1 \dots \hat{I}_{n+1}}$, $T^{(n)\hat{I}_1 \dots \hat{I}_n}$ is

$$\begin{array}{l}
 h = 1 + \frac{1}{2}n \\
 h = \frac{3}{2} + \frac{1}{2}n \\
 h = 2 + \frac{1}{2}n
 \end{array}
 \begin{array}{ccc}
 & J^{(n)\hat{I}_1 \dots \hat{I}_{n+2}} & \\
 Q_{\hat{I}} \swarrow & & \searrow \tilde{Q}_{\hat{I}} \\
 G^{(n)\hat{I}_1 \dots \hat{I}_{n+1}} & & \tilde{G}^{(n)\hat{I}_1 \dots \hat{I}_{n+1}} \\
 \tilde{Q}_{\hat{I}} \swarrow & & \searrow Q_{\hat{I}} \\
 & T^{(n)\hat{I}_1 \dots \hat{I}_n &
 \end{array}
 , \quad (4.1)$$

where $Q_{\hat{I}}$, $\tilde{Q}_{\hat{I}}$ denote the Poincaré supercharges of $\mathfrak{psu}(1, 1|2)$. For gauge algebra $\mathfrak{su}(2)$, the conjecture asserts that there is a unique supergenerator, the $\mathfrak{su}(2)_F$ affine current $J^{(0)\hat{I}\hat{J}} \equiv J^{\hat{I}\hat{J}}$. The full $\mathfrak{psu}(1, 1|2)$ supermodule comprises $J^{\hat{I}\hat{J}}$, the supercurrents $G^{(0)\hat{I}} \equiv G^{\hat{I}}$ and $\tilde{G}^{(0)\hat{I}} \equiv \tilde{G}^{\hat{I}}$ and the stress tensor $T^{(0)} \equiv T$, which is of course the operator content of the small $\mathcal{N} = 4$ SCA.¹⁵ In this special case there is clearly no obstruction in deforming the central charge to arbitrary values.

The supergenerators $J^{(n)}$ in 1-to-1 correspondence with the generators of the 1/2 BPS chiral ring of the SYM theory, i.e. with the single-trace 1/2 BPS operators of the form $\text{tr} Z^{n+2}$, with $n = 0, \dots, N - 2$ (see (4.4) below for the definition of Z). In terms of $\mathfrak{psu}(2, 2|4)$ representation theory, these operators are the bottom components of multiplets of type $\mathcal{B}_{[0, n+2, 0](0, 0)}^{\frac{1}{2}, \frac{1}{2}}$ in the notations of [37]. An $\mathfrak{psu}(2, 2|4)$ multiplet of type $\mathcal{B}_{[0, n+2, 0](0, 0)}^{\frac{1}{2}, \frac{1}{2}}$ decomposes into $\mathfrak{su}(2, 2|2)$ as [37]

$$\begin{array}{cccc}
 \mathcal{B}_{[0, n+2, 0](0, 0)}^{\frac{1}{2}, \frac{1}{2}} = (n+3)\hat{\mathcal{B}}_{\frac{1}{2}(n+2)} \oplus (n+2) \left[\mathcal{D}_{\frac{1}{2}(n+1)(0, 0)} \oplus \bar{\mathcal{D}}_{\frac{1}{2}(n+1)(0, 0)} \right] \oplus (n+1)\hat{\mathcal{C}}_{\frac{1}{2}n(0, 0)} \oplus \dots & & & \\
 \downarrow & \downarrow & \downarrow & \downarrow \\
 J^{(n)\hat{I}_1 \dots \hat{I}_{n+2}} & G^{(n)\hat{I}_1 \dots \hat{I}_{n+1}} & \tilde{G}^{(n)\hat{I}_1 \dots \hat{I}_{n+1}} & T^{(n)\hat{I}_1 \dots \hat{I}_n
 \end{array}
 \quad (4.2)$$

Below each shown $\mathfrak{su}(2, 2|2)$ multiplet we have indicated the corresponding chiral algebra operator, which arises from as the \mathbb{Q} -cohomology class of the Schur operator in the multiplet.

¹⁴A fully explicit statement of the super W-algebra conjecture for $N \geq 3$ is then the following: (i) as a vector space, the chiral algebra is the linear span of derivatives of $J^{(n)}$, $G^{(n)}$, $\tilde{G}^{(n)}$, $T^{(n)}$ ($n = 0, \dots, N - 2$) and their conformally ordered products; (ii) the operators $J^{(n)}$, $G^{(n)}$, $\tilde{G}^{(n)}$, $T^{(n)}$ ($n = 0, \dots, N - 2$) cannot be written as conformally ordered products of derivatives of other operators.

¹⁵In fact, for $c_{2d} = -9$, the stress tensor T is not an independent generator, but is rather identified with the Sugawara stress tensor built from the affine current $J^{\hat{I}\hat{J}}$.

(The ellipsis in (4.2) represents additional multiplets that are not relevant for our discussion, since they do not contain Schur operators.) The degeneracies in (4.2) are accounted for by the dimensions of the $\mathfrak{su}(2)_F$ representations.

The fact that the operators $J^{(n)}$ are supergenerators of the chiral algebra is easily established. Indeed, they arise from $4d$ 1/2 BPS operators, which are absolutely protected against quantum corrections;¹⁶ as they correspond to generators of the 1/2 BPS chiral ring it is clear that they *cannot* appear in the non-singular OPE of other operators. The hard part of the conjecture of [1] is showing that these are *all* the supergenerators. The chiral algebra is specified by a non-trivial BRST procedure, which in physical terms amounts to selecting operators obeying the Schur shortening condition in the interacting theory.

The conjecture, however, can be proved at infinite N . All Schur operators are in particular 1/16 BPS operators of $\mathcal{N} = 4$ SYM, i.e., operators in the cohomology of a single Poincaré supercharge. This cohomology was studied in [50], where it was proved that at infinite N it is obtained by taking arbitrary products of 1/16 BPS single-trace operators,¹⁷ which were further shown to be in 1-1 correspondence with single 1/16 BPS gravitons in the dual $AdS_5 \times S^5$ supergravity. Schur operators are in the simultaneous cohomology of *two* Poincaré supercharges of opposite chirality, say Q_-^1 and \tilde{Q}_{2-} in the conventions of [1]. Specializing the results of [50] to this double cohomology, we find that at infinite N it is given by arbitrary products¹⁸ of the single-trace Schur operators corresponding to $J^{(n)}$, $G^{(n)}$, $\tilde{G}^{(n)}$, $T^{(n)}$, $n \in \mathbb{Z}_+$. This shows that these operators comprise the full set of generators for the chiral algebra at infinite N .

4.2 Single-trace Schur operators of $\mathcal{N} = 4$ SYM and supergravity

Having established the super chiral algebra conjecture of [1] for infinite N , we now proceed to give more details on the single-trace Schur operators in (4.2) and to map them to the Kaluza-Klein modes in type IIB supergravity on $AdS_5 \times S^5$ [51]. Our conclusions are summarized in table 4.2.

All Kaluza-Klein modes in the compactification of type IIB supergravity on $AdS_5 \times S^5$ are dual to operators that are organized in 1/2-BPS short $\mathcal{N} = 4$ multiplets of type $\mathcal{B}_{[0,n+2,0](0,0)}^{\frac{1}{2},\frac{1}{2}}$. In Lagrangian language, the $\mathcal{N} = 4$ superprimaries of these multiplets can be written as

$$\mathcal{O}^{(n)\mathcal{A}_1 \dots \mathcal{A}_{n+2}} = \text{tr}(X^{\{\mathcal{A}_1 \dots \mathcal{A}_{n+2}\}}), \quad n = 0, 1, 2, \dots, \quad (4.3)$$

where $\mathcal{A}_1, \dots, \mathcal{A}_{n+2}$ are vector indices of $\mathfrak{so}(6)_R$, $X^{\mathcal{A}}$ are the real scalars of $\mathcal{N} = 4$ super Yang-Mills, and curly brackets denote the traceless symmetric part.

¹⁶Recall that 1/2 BPS multiplets *cannot* recombine into long multiplets [37].

¹⁷The product operation relevant here is the commutative product induced by the ordinary OPE of the $4d$ theory, which is non-singular for 1/16 BPS operators.

¹⁸In this statement, the product operation is again the commutative product induced by the standard OPE of the $4d$ theory. The twisted translation prescription of [1] deforms this commutative algebra into the chiral algebra that we are interested in. There is an easy general argument that the generators of the chiral algebra are a subset of the generators of the commutative algebra. This is just what we need (the other inclusion is obvious in our case).

supermultiplet	$\hat{\mathcal{B}}_{\frac{1}{2}(n+2)}$	$\mathcal{D}_{\frac{1}{2}(n+1)(0,0)}$	$\bar{\mathcal{D}}_{\frac{1}{2}(n+1)(0,0)}$	$(n+1)\hat{\mathcal{C}}_{\frac{1}{2}n(0,0)}$
superprimary	$\text{tr}(Q^{n+2})$	$\text{tr}(ZQ^{n+1})$	$\text{tr}(\bar{Z}Q^{n+1})$	$\text{tr}(Z\bar{Z}Q^n)$
Schur operator	$\text{tr}(Q^{n+2})$	$\text{tr}(\lambda_+Q^{n+1})$	$\text{tr}(\tilde{\lambda}_+Q^{n+1})$	$\text{tr}(\lambda_+\tilde{\lambda}_+Q^n)$
Δ	$n+2$	$n+\frac{5}{2}$	$n+\frac{5}{2}$	$n+3$
chiral algebra operator	$\mathbf{J}^{(n)\hat{I}_1\dots\hat{I}_{n+2}}$	$\mathbf{G}^{(n)\hat{I}_1\dots\hat{I}_{n+1}}$	$\bar{\mathbf{G}}^{(n)\hat{I}_1\dots\hat{I}_{n+1}}$	$\mathbf{T}^{(n)\hat{I}_1\dots\hat{I}_n}$
h	$\frac{1}{2}(n+2)$	$\frac{1}{2}(n+3)$	$\frac{1}{2}(n+3)$	$\frac{1}{2}(n+4)$
J_F	$\frac{1}{2}(n+2)$	$\frac{1}{2}(n+1)$	$\frac{1}{2}(n+1)$	$\frac{1}{2}n$
KK mode	π^{I_1}	ψ^{I_L}	ψ^{I_L}	$B_\mu^{I_5}$
KK mass	$m^2R^2 = n^2 - 4$	$ m R = n + \frac{1}{2}$	$ m R = n + \frac{1}{2}$	$ m R = n^2 + 2n$

Table 1. Families of Schur operators of $\mathcal{N} = 4$ super Yang-Mills theory. For each family we give the $\mathcal{N} = 2$ supermultiplet in the notation of [37], the schematic form of the superprimary in the multiplet, and the schematic form of the Schur operator. The quantum numbers Δ , h , J_F are the $4d$ scaling dimension of the Schur operator, the $2d$ holomorphic dimension of the chiral algebra element, and the half-integer $\mathfrak{sl}(2)_F$ spin of both, respectively. The scalar fields Q , Z , \bar{Z} are defined in (4.4), while λ , $\tilde{\lambda}$ denote the gaugini in $\mathcal{N} = 2$ language, which are a subset of all gaugini in $\mathcal{N} = 4$ language. The KK modes are given in the notation of [51]. All families are labelled in such a way that the range of n is $n = 0, 1, 2, \dots$.

We have already reviewed the decomposition of an $\mathcal{N} = 4$ multiplet of type $\mathcal{B}_{[0,n+2,0](0,0)}^{\frac{1}{2},\frac{1}{2}}$ into superconformal multiplets of $\mathcal{N} = 2$, see (4.2). In order to elucidate the connection between the branching rule (4.2) and the Lagrangian presentation (4.3), it is convenient to reorganize the scalars X^A schematically as

$$Z = X^5 + iX^6, \quad \bar{Z} = X^5 - iX^6, \quad Q^{I\hat{J}} = X^a \sigma_a^{I\hat{J}}, \quad (4.4)$$

where $a = 1, \dots, 4$, $I = 1, 2$ is a fundamental index of $\mathfrak{su}(2)_R$, $\hat{J} = 1, 2$ is a fundamental index of $\mathfrak{su}(2)_F$, and $\sigma_a^{I\hat{J}}$ are chiral blocks of $\mathfrak{so}(4)$ gamma matrices. The scalar Z is the complex scalar in the $\mathcal{N} = 2$ vector multiplet, while $Q^{I\hat{J}}$ are the scalars in the $\mathcal{N} = 2$ hypermultiplet. We can now easily identify a Lagrangian realization of the $\mathcal{N} = 2$ superconformal primary for each of the multiplets on the r.h.s. of (4.2). Let us list them together with their R - and F -isospins and r charges,

$$\begin{aligned} \hat{\mathcal{B}}_{\frac{1}{2}(n+2)} &: \text{tr}(Q^{(I_1|\hat{J}_1} \dots Q^{I_{n+2})\hat{J}_{n+2}}), & J_R = \frac{1}{2}(n+2), & J_F = \frac{1}{2}(n+2), & r = 0, \\ \mathcal{D}_{\frac{1}{2}(n+1)(0,0)} &: \text{tr}(ZQ^{(I_1|\hat{J}_1} \dots Q^{I_{n+1})\hat{J}_{n+1}}), & J_R = \frac{1}{2}(n+1), & J_F = \frac{1}{2}(n+1), & r = 1, \\ \bar{\mathcal{D}}_{\frac{1}{2}(n+1)(0,0)} &: \text{tr}(\bar{Z}Q^{(I_1|\hat{J}_1} \dots Q^{I_{n+1})\hat{J}_{n+1}}), & J_R = \frac{1}{2}(n+1), & J_F = \frac{1}{2}(n+1), & r = -1, \\ \hat{\mathcal{C}}_{\frac{1}{2}n(0,0)} &: \text{tr}(Z\bar{Z}Q^{(I_1|\hat{J}_1} \dots Q^{I_n)\hat{J}_n}), & J_R = \frac{1}{2}n, & J_F = \frac{1}{2}n, & r = 0. \end{aligned}$$

Each of these $\mathcal{N} = 2$ supermultiplets yields a Schur operator. Let us discuss them in turn and relate them to the associated Kaluza-Klein mode in the spectrum of type IIB supergravity on $AdS_5 \times S^5$.

Multiplets of type $\hat{\mathcal{B}}_{\frac{1}{2}(n+2)}$. In this case the Schur operator is directly the $\mathfrak{su}(2)_R$ highest-weight component of the superconformal primary listed in (4.3). It follows that

the operator in the chiral algebra is simply

$$J^{(n)\hat{I}_1 \dots \hat{I}_{n+2}} = \chi \left[\text{tr}(Q^{1\hat{I}_1} \dots Q^{1\hat{I}_{n+2}}) \right], \quad h = \frac{1}{2}(n+2), \quad J_F = \frac{1}{2}(n+2), \quad n = 0, 1, \dots, \quad (4.5)$$

where we summarized its $2d$ quantum numbers. The gravity duals of the $\mathcal{N} = 4$ chiral primaries in (4.3) are given by the Kaluza-Klein modes named π^{I_1} in table III of [51]. It follows that the gravity duals of the operators (4.5) are given by the subset of the modes π^{I_1} corresponding to the $J_R = \frac{1}{2}(n+2)$, $J_F = \frac{1}{2}(n+2)$ representation of $\mathfrak{su}(2)_R \oplus \mathfrak{su}(2)_F$ inside the $[0, n+2, 0]$ of $\mathfrak{so}(6)_R$. The masses of the π^{I_1} Kaluza-Klein tower are

$$m^2 R^2 = (n-2)(n+2), \quad n = 0, 1, \dots \quad (4.6)$$

The case $n = 0$ deserves special attention. The superconformal primary of $\hat{\mathcal{B}}_1$ is the moment map for the $\mathfrak{su}(2)_F$ flavor symmetry. The associated operator in the chiral algebra $J^{(0)\hat{I}\hat{J}} \equiv J^{\hat{I}\hat{J}}$ is the affine $\mathfrak{su}(2)_F$ current of the small $\mathcal{N} = 4$ subalgebra. The dual scalar mode in supergravity has a negative mass-squared that saturates the Breitenlohner-Friedmann bound [52].

Multiplets of types $\mathcal{D}_{\frac{1}{2}(n+1)(0,0)}$ and $\bar{\mathcal{D}}_{\frac{1}{2}(n+1)(0,0)}$. In this case the Schur operator is a component of a super-descendant of the scalar operator listed in (4.2). More precisely, for $\mathcal{D}_{\frac{1}{2}(n+1)(0,0)}$ we need to act with \tilde{Q}_α^I , obtaining a right-handed spinor operator of the schematic form

$$\Psi^{(n)I_0 I_1 \dots I_{n+1} \hat{I}_1 \dots \hat{I}_{n+1}} = \text{tr}(\tilde{\lambda}_\alpha^{I_0} Q^{I_1 \hat{I}_1} \dots Q^{I_{n+1} \hat{I}_{n+1}}) + \dots, \quad n = 0, 1, \dots, \quad (4.7)$$

where we recorded explicitly the part coming from the action of \tilde{Q}_α^I on Z , which yields the $\mathcal{N} = 2$ gaugino $\tilde{\lambda}_\alpha^I$, but we omitted additional terms arising from the action of \tilde{Q}_α^I on the $\mathcal{N} = 2$ hypermultiplet scalars. The quantum numbers of the operator in (4.7) are

$$J_R = \frac{1}{2}(n+2), \quad J_F = \frac{1}{2}(n+1), \quad r = \frac{1}{2}, \quad \Delta = n + \frac{5}{2}, \quad (4.8)$$

and its $\mathfrak{su}(4)_R$ orbit is the one of the $\mathcal{N} = 4$ superdescendant of (4.3) in the $[0, n+1, 1]$ representation of $\mathfrak{su}(4)_R$.

The Schur operator is the highest-weight component of $\Psi^{(n)}$ and the associated chiral operator is then

$$\mathbf{G}^{(n)\hat{I}_1 \dots \hat{I}_{n+1}} = \chi \left[\Psi^{(n)\hat{I}_1 \dots \hat{I}_{n+1}} \right], \quad h = \frac{1}{2}(n+3), \quad J_F = \frac{1}{2}(n+1), \quad n = 0, 1, \dots \quad (4.9)$$

Completely analogous considerations hold for type $\bar{\mathcal{D}}_{\frac{1}{2}(n+1)(0,0)}$ multiplets. The analog of $\Psi^{(n)}$, denoted $\tilde{\Psi}^{(n)}$, is built using the supercharge Q_α^I and thus contains a λ_α^I insertion. It has the same quantum numbers as $\Psi^{(n)}$, except $r = -\frac{1}{2}$, and $\tilde{\mathbf{G}}^{(n)}$ is the associated operator in the chiral algebra.

The gravitational dual of the operators $\Psi^{(n)}$, $\tilde{\Psi}^{(n)}$ is encoded in the suitable R-symmetry components of $5d$ Dirac spinor modes denoted ψ^{I_L} in table III of [51]. Their masses are

$$mR = - \left(n + \frac{1}{2} \right), \quad n = 0, 1, 2, \dots \quad (4.10)$$

The minus sign is relative to the positive masses of the excited Kaluza-Klein modes in the tower of the $5d$ gravitino.

In the case $n = 0$ the multiplets $\mathcal{D}_{\frac{1}{2}(0,0)}$, $\bar{\mathcal{D}}_{\frac{1}{2}(0,0)}$ contain among their descendants the spin-3/2 supersymmetry currents of dimension $7/2$ associated to the supercharges of the $\mathcal{N} = 4$ superalgebra that are not in the selected $\mathcal{N} = 2$ subalgebra. The corresponding operators $\mathbf{G}^{(0)\hat{I}} \equiv \mathbf{G}^{\hat{I}}$ and $\tilde{\mathbf{G}}^{(0)\hat{I}} = \tilde{\mathbf{G}}^{\hat{I}}$ in the chiral algebra are supersymmetry currents.

Multiplets of type $\hat{\mathcal{C}}_{\frac{1}{2}n(0,0)}$. In this case the Schur operator is a component of the operator obtained acting with one \mathcal{Q} and one $\tilde{\mathcal{Q}}$ on the superprimary in (4.2). Schematically, we have

$$J_{\alpha\beta}^{(n) K_1 K_2 I_1 \dots I_n \hat{J}_1 \dots \hat{J}_n} = \text{tr}(\lambda_{\alpha}^{K_1} \tilde{\lambda}_{\beta}^{K_2} Q^{I_1 \hat{J}_1} \dots Q^{I_n \hat{J}_n}) + \dots, \quad n = 0, 1, \dots, \quad (4.11)$$

where we omitted several other terms for the sake of brevity. The quantum numbers of this $4d$ operator are

$$J_R = \frac{1}{2}(n+2), \quad J_F = \frac{1}{2}n, \quad r = 0, \quad \Delta = n+3, \quad (4.12)$$

and its $\mathfrak{su}(4)_R$ completion is the $\mathcal{N} = 4$ superdescendant of (4.3) in the $[1, n, 1]$ representation of $\mathfrak{su}(4)_R$.

The chiral algebra operator is therefore

$$\mathbb{T}^{(n)\hat{I}_1 \dots \hat{I}_n} = \chi \left[J_{++}^{(n) 1 \dots 1 \hat{J}_1 \dots \hat{J}_n} \right], \quad h = \frac{1}{2}(n+4), \quad J_F = \frac{1}{2}n, \quad n = 0, 1, \dots \quad (4.13)$$

The gravity dual to the vector operators $J^{(n)}$ is furnished by the vector modes called $B_{\mu}^{I_5}$ in table III of [51], with masses

$$m^2 R^2 = n(n+2), \quad n = 0, 1, 2, \dots \quad (4.14)$$

For $n = 0$ the multiplet $\hat{\mathcal{C}}_{0(0,0)}$ contains the $4d$ stress tensor and the Schur operator is a component of the $\mathfrak{su}(2)_R$ symmetry current. The operator $\mathbb{T}^{(0)} \equiv \mathbb{T}$ in the chiral algebra is the $2d$ stress tensor. On the gravity side, we find the massless vectors associated to the Killing vectors of S^5 .

The families discussed above have a natural \mathbb{Z}_2 grading corresponding to even modes $n = 0, 2, \dots$, and odd modes $n = 1, 3, \dots$. The series for even n constitutes a consistent truncation of the chiral algebra. For n even, $J^{(n)}$ and $\mathbb{T}^{(n)}$ have integer holomorphic dimension, while $\mathbf{G}^{(n)}$ and $\tilde{\mathbf{G}}^{(n)}$ have half-integer holomorphic dimension. These assignments obey the standard spin/statistics connection. On the other hand, for n odd the situation is reversed, and the spin/statistics connection is violated. There is of course no contradiction — this is the generic case for chiral algebras associated to $\mathcal{N} = 2$ SCFTs.

4.3 Comments on the full higher-spin algebra

Motivated by the emergence of an AdS_3 Chern-Simons theory in the localization computation of section 3, we believe that the bulk dual of the full chiral algebra is a higher-spin AdS_3 Chern-Simons theory. This expectation is in line with known examples of minimal

model holography (see [5] for a review). From this perspective, we are left with the task of determining the correct higher-spin Lie superalgebra in which the Chern-Simons gauge connection takes values.

Before proceeding, it is useful to review the well-understood case in which the bulk theory is AdS_3 Chern-Simons with gauge algebra $\mathfrak{sl}(n) \oplus \mathfrak{sl}(n)$. This bulk theory describes gravity coupled to massless higher spin fields. In order to identify the states associated to the physical graviton it is necessary to specify an embedding of $\mathfrak{sl}(2)$ in $\mathfrak{sl}(n)$. As explained in [19], the bulk theory must be supplemented by suitable boundary conditions in order to guarantee an asymptotically AdS_3 geometry. The interplay between the $\mathfrak{sl}(2)$ embedding and these boundary conditions determines the asymptotic symmetry algebra of the bulk theory, which is furnished by two copies (left-moving and right-moving) of the same classical infinite-dimensional Poisson algebra. Interestingly, this physical construction based on the asymptotic symmetry algebra is equivalent to the classical Drinfel'd-Sokolov (DS) Hamiltonian reduction of $\mathfrak{sl}(n)$ associated to the prescribed embedding $\mathfrak{sl}(2) \subset \mathfrak{sl}(n)$. In the case of the principal embedding, the outcome of the DS reduction is the classical \mathcal{W}_n algebra, whose quantization yields the quantum \mathcal{W}_n algebra.

If the DS reduction provides the natural way to get the boundary W-algebra from the bulk Lie algebra, the notion of *wedge algebra*, explored in [53] in great generality, proves extremely useful for proceeding in the opposite direction. Let the generators of the W-algebra be denoted as $W^s(\zeta)$, where s labels the integer holomorphic dimension of the generator. Let W_ℓ^s , $\ell \in \mathbb{Z}$ be the modes in the Laurent expansion of $W^s(\zeta)$. The vacuum preserving modes are

$$W_\ell^s, \quad |\ell| < s, \tag{4.15}$$

and preserve both the left and right $\mathfrak{sl}(2)$ invariant vacuum. Our goal is to define a finite-dimensional Lie algebra generated by the vacuum preserving modes (4.15). A naïve truncation of the commutators of the original W-algebra fails in general, due to the non-linear terms that may appear on r.h.s. of the commutators of the vacuum preserving modes. The crucial observation is that, if the W-algebra can be defined for arbitrary values of the central charge c and satisfies additional non-degeneracy assumptions listed in [53],¹⁹ then all non-linear terms on the r.h.s. of commutators of vacuum preserving modes are suppressed in the limit $c \rightarrow \infty$. Furthermore, central terms do not contribute if we restrict to vacuum preserving modes. It follows that the algebra becomes linear and, since associativity of the parent W-algebra holds for any c , we are guaranteed to obtain a *bona fide* finite-dimensional Lie algebra satisfying all Jacobi identities. An essential property of the wedge algebra construction is that, if the starting point is a W-algebra $\mathcal{W}^{\text{DS}}(\mathfrak{g})$ obtained by DS reduction of a finite-dimensional Lie algebra \mathfrak{g} , then the wedge algebra of $\mathcal{W}^{\text{DS}}(\mathfrak{g})$ reproduces \mathfrak{g} itself. In particular, the wedge algebra of \mathcal{W}_n is $\mathfrak{sl}(n)$. Even though we have reviewed a purely bosonic example, the extension of these considerations to graded Lie

¹⁹The additional assumptions of [53] are that both the quantum W-algebra and its classical limit be positive definite, in particular the metric defined by central terms should be positive definite both in the quantum and classical algebras. These conditions can be presumably relaxed for non-unitary W-algebras — non-degeneracy of the metric should suffice.

algebras does not pose any essential difficulty. In particular, the arguments above can be straightforwardly applied to W-superalgebras and Lie superalgebras in the NS sector, which is the sector relevant from the four-dimensional point of view.

In our problem, the role of \mathcal{W}_n is played by the chiral algebra of $\mathcal{N} = 4$ SYM with gauge algebra $\mathfrak{su}(N)$. The existence of this chiral algebra is guaranteed if its central charge is tuned to the value determined by the cohomological construction, $c_{2d} = -3(N^2 - 1)$. It is not clear, however, if for $N \geq 3$ this chiral algebra can be deformed to arbitrary c . As a result, we cannot guarantee the existence of a wedge algebra, which would be the natural candidate for the sought after Lie algebra in the bulk.

If we consider the case of infinite N , however, we can infer the existence of a wedge algebra, which is an ordinary (linear) Lie algebra, albeit infinite dimensional. The argument relies on large N factorization, and goes as follows. We have established in section 4.1 that, in the large N limit, the supergenerators of the chiral algebra are in 1-to-1 correspondence with single trace 1/2 BPS operators of $\mathcal{N} = 4$ SYM theory. Thanks to the protection ensured by supersymmetry, their correlators can be computed in the free field theory limit. We normalize the fundamental adjoint scalars of $\mathcal{N} = 4$ SYM in such a way that their contraction yields schematically

$$\langle X^x{}_y X^z{}_w \rangle \sim g_{\text{YM}}^2 \delta_y^z \delta_w^x, \tag{4.16}$$

where x, y, z, w are fundamental indices of $\mathfrak{su}(N)$, we suppressed all spacetime and R-symmetry dependence, and we restricted to the leading term at large N . With the aid of the standard double-line notation, it is elementary to show the following schematic scalings,

$$\begin{aligned} \langle \text{tr} X^k \text{tr} X^k \rangle &\sim g_{\text{YM}}^{2k} N^k = \lambda^k, \\ \langle \text{tr} X^{k_1} \text{tr} X^{k_2} \text{tr} X^{k_3} \rangle &\sim g_{\text{YM}}^{k_1+k_2+k_3} N^{-1+\frac{1}{2}(k_1+k_2+k_3)} = \lambda^{\frac{1}{2}(k_1+k_2+k_3)} N^{-1}, \\ \langle : \text{tr} X^{k_1} \text{tr} X^{k_2} :: \text{tr} X^{k_1} \text{tr} X^{k_2} : \rangle &\sim g_{\text{YM}}^{2(k_1+k_2)} N^{k_1+k_2} = \lambda^{k_1+k_2}, \\ \langle \text{tr} X^{k_1} \text{tr} X^{k_2} : \text{tr} X^{k_3} \text{tr} X^{k_4} : \rangle &\sim g_{\text{YM}}^{k_1+k_2+k_3+k_4} N^{\frac{1}{2}(k_1+k_2+k_3+k_4)-2} = \lambda^{\frac{1}{2}(k_1+k_2+k_3+k_4)} N^{-2}, \end{aligned} \tag{4.17}$$

where $::$ denotes normal ordering and $\lambda = g_{\text{YM}}^2 N$ is the 't Hooft coupling.²⁰ If we modify the normalization of the single trace operators, setting

$$\mathcal{O}_k = N \lambda^{-\frac{1}{2}k} \text{tr} X^k, \tag{4.18}$$

the previous relations may then be written in the simpler form

$$\begin{aligned} \langle \mathcal{O}_k \mathcal{O}_k \rangle &\sim N^2, \\ \langle \mathcal{O}_{k_1} \mathcal{O}_{k_2} \mathcal{O}_{k_3} \rangle &\sim N^2, \\ \langle : \mathcal{O}_{k_1} \mathcal{O}_{k_2} :: \mathcal{O}_{k_1} \mathcal{O}_{k_2} : \rangle &\sim N^4, \\ \langle \mathcal{O}_{\ell_1} \mathcal{O}_{\ell_2} : \mathcal{O}_{k_1} \mathcal{O}_{k_2} : \rangle &\sim N^2. \end{aligned} \tag{4.19}$$

²⁰The last scaling relation in (4.17) holds for generic ℓ_1, ℓ_2, k_1, k_2 . In the special case $\ell_1 = k_1, \ell_2 = k_2$ the scaling is $\lambda^{k_1+k_2} N^0$.

These relations constrain the N dependence of the OPE coefficients in the OPE of two \mathcal{O}_k operators. Very schematically, we may then write

$$\mathcal{O}_{k_1} \mathcal{O}_{k_2} \sim N^2 \delta_{k_1 k_2} \mathbb{I} + N^0 C_{k_1 k_2}^{k_3} \mathcal{O}_{k_3} + N^{-2} C_{k_1 k_2}^{k_3 k_4} : \mathcal{O}_{k_3} \mathcal{O}_{k_4} : + \dots \quad (4.20)$$

where we have only kept track of the N dependence. Furthermore we have only focused on potential *singular* terms in the OPE, and in particular we supposed $(k_3, k_4) \neq (k_1, k_2)$.²¹ As we can see, if double trace operators enter the singular part of the OPE of two single trace operators, the corresponding OPE coefficient is suppressed by a power of N^{-2} . It is not hard to convince oneself that this pattern persists for all multi-trace operators: if a trace- m operator enters the singular part of the OPE, it appears with a power N^{2-2m} . This argument implies that all non-linear terms in the chiral algebra must be suppressed at large N . As a result, the obstruction to the consistency of the wedge algebra generated by the vacuum preserving modes is removed, and we obtain a well-defined, infinite-dimensional Lie (super)algebra. This is our candidate for the higher-spin Lie (super)algebra in the bulk.

All the necessary information for determining the structure constants of this Lie algebra is contained in the OPE of single-trace operators in the 1/2 BPS chiral ring of $\mathcal{N} = 4$ SYM. It would be desirable, however, to have a more direct construction of this higher-spin algebra, along the lines of [5] in the context of minimal model holography. Such investigation is left for future work. Let us list here the expected vacuum preserving modes of the operators in (4.1) that generate the wedge Lie algebra, suppressing $\mathfrak{su}(2)_F$ indices for simplicity:

	n even:	n odd:
$J_\ell^{(n)}$	$\ell = 0, \pm 1, \dots, \pm \frac{1}{2}n,$	$\ell = \pm \frac{1}{2}, \pm \frac{3}{2}, \dots, \pm \frac{1}{2}n,$
$G_\ell^{(n)}, \tilde{G}_\ell^{(n)}$	$\ell = \pm \frac{1}{2}, \pm \frac{3}{2}, \dots, \pm(\frac{1}{2} + \frac{1}{2}n),$	$\ell = 0, \pm 1, \dots, \pm(\frac{1}{2} + \frac{1}{2}n),$
$T_\ell^{(n)}$	$\ell = 0, \pm 1, \dots, \pm(1 + \frac{1}{2}n),$	$\ell = \pm \frac{1}{2}, \pm \frac{3}{2}, \dots, \pm(1 + \frac{1}{2}n).$

5 Discussion

In this work we have addressed the problem of determining the holographic dual of the protected chiral algebra of $\mathcal{N} = 4$ SYM theory with gauge algebra $\mathfrak{su}(N)$ in the large N limit. The resulting picture is the following. The cohomological construction on the field theory side is mirrored by supersymmetric localization in the bulk. By virtue of this localization, type IIB supergravity on $AdS_5 \times S^5$ reduces to a Chern-Simons theory defined on an AdS_3 slice of the AdS_5 space. The gauge algebra of the Chern Simons theory is an infinite-dimensional supersymmetric higher spin Lie algebra, whose structure can *a priori* be extracted from the coefficients in the OPE of the single-trace 1/2 BPS generators of the chiral ring of $\mathcal{N} = 4$ SYM theory.

²¹In the case $(k_3, k_4) = (k_1, k_2)$ the N scaling is N^0 due to the remark in the previous footnote. This is in accordance with the tautological observation that $: \mathcal{O}_{k_1} \mathcal{O}_{k_2} :$ enters the *regular* part of the $\mathcal{O}_{k_1} \mathcal{O}_{k_2}$ OPE with coefficient one.

Although we were not able to provide a proof for all aspects of the above picture, we have collected several pieces of evidence in favor of it. To begin with, we have implemented the localization program explicitly in a simplified setup, illustrating how an AdS_3 Chern-Simons theory emerges non-trivially from a five-dimensional gauge theory on AdS_5 . Secondly, we have established the super chiral algebra conjecture of [1] in the case of infinite N , providing the correspondence between supergenerators of the chiral algebra, single-trace 1/2 BPS generators of the chiral ring of $\mathcal{N} = 4$ SYM, and Kaluza-Klein modes of type IIB supergravity on $AdS_5 \times S^5$. Finally, we have identified a natural candidate for the higher-spin algebra in which the Chern-Simons connection takes values. It is the wedge algebra of the chiral algebra, i.e. the (infinite-dimensional) Lie algebra generated by the vacuum preserving modes of the generators of the chiral algebra. We furnished an argument based on large N factorization for the existence of this wedge algebra, and we have connected its structure to the OPEs of single-trace 1/2 BPS scalar operators of $\mathcal{N} = 4$ SYM theory.

It is interesting to contrast this four-dimensional setup to the six-dimensional case in which the superconformal field theory is the $(2, 0)$ theory of type A_{N-1} . As established in [54], the protected chiral algebra in this case coincides with \mathcal{W}_N . The latter is defined for arbitrary values of the central charge c and admits $\mathfrak{sl}(N)$ as its wedge algebra. The gravity dual of a \mathcal{W}_N chiral algebra is thus an AdS_3 Chern-Simons theory with gauge algebra $\mathfrak{sl}(N)$. These facts are well-known in the context of minimal model holography [5], and the large N limit is also well understood. This problem is thus simpler from a bottom-up point of view. From a top-down perspective, however, this case is considerably more complicated. In the large N limit we can access the holographic dual of the $(2, 0)$ theory of type A_{N-1} via eleven-dimensional supergravity on $AdS_7 \times S^4$. In contrast to the case studied in this paper, it is not possible to single out a simplified setup without dynamical gravity to perform the localization computation. As a result, a direct check of the emergence of the claimed Chern-Simons theory would require a full-fledged localization computation in supergravity.

The cohomological construction of the protected chiral algebra in $4d$ SCFTs also has a counterpart for $3d$ SCFTs [55, 56]: the protected sector gives rise to a one-dimensional topological algebra. The construction requires at least $\mathcal{N} = 4$ in three dimensions, and is in particular applicable to the maximally supersymmetric case $\mathcal{N} = 8$. In the latter situation the holographic dual can be accessed via eleven-dimensional supergravity on $AdS_4 \times S^7$. In analogy to the case discussed in this work, it is possible to single out a simplified model, involving the dynamics of a vector multiplet in AdS_4 , in order to perform the localization computation. The outcome of the localization procedure is expected to live on an AdS_2 slice of AdS_4 , and it would be interesting to show this explicitly.

A general formalism for defining twisted supergravity theories has been recently introduced in [57], with the motivation to discuss twisted versions of the AdS/CFT correspondence. It would be extremely interesting to apply this formalism to our setup.

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A Conventions and technical material

A.1 Gamma matrices in various dimensions

Gamma matrices in Euclidean $5d$ dimensions are hermitian and satisfy

$$\{\gamma^{\hat{\mu}}, \gamma^{\hat{\nu}}\} = 2\delta^{\hat{\mu}\hat{\nu}} \mathbb{I}_4, \quad C_5 \gamma^{\hat{\mu}} C_5^{-1} = (\gamma^{\hat{\mu}})^\top, \quad C_5^\top = -C_5, \quad \gamma^1 \gamma^2 \gamma^3 \gamma^4 \gamma^5 = \mathbb{I}_4, \quad (\text{A.1})$$

where $\hat{\mu}, \hat{\nu} = 1, \dots, 5$ are flat spacetime indices and C_5 is the unitary charge conjugation matrix. We also need gamma matrices for the R-symmetry directions $\mathcal{I} = 6, \dots, 9, 0$ with signature $(4, 1)$. They satisfy

$$\{\rho^{\mathcal{I}}, \rho^{\mathcal{J}}\} = 2\eta^{\mathcal{I}\mathcal{J}} \mathbb{I}_4, \quad C_{4,1} \rho^{\mathcal{I}} C_{4,1}^{-1} = (\rho^{\mathcal{I}})^\top, \quad C_{4,1}^\top = -C_{4,1}, \quad \rho^6 \rho^7 \rho^8 \rho^9 \rho^0 = i \mathbb{I}_4, \quad (\text{A.2})$$

where $\eta_{IJ} = \text{diag}(+^4, -)$, ρ^0 antihermitian, and the other $\rho^{\mathcal{I}}$ hermitian, $C_{4,1}$ unitary.

Let us combine these objects to obtain a convenient representation of the chiral 16×16 blocks $\Gamma^M, \tilde{\Gamma}^M$ of gamma matrices in ten dimensions, where $M = 1, \dots, 9, 0$ and the flat metric is $\eta_{MN} = \text{diag}(+^9, -)$. Let us set

$$\begin{aligned} \Gamma^{\hat{\mu}} &= C_5 \gamma^{\hat{\mu}} \otimes C_{4,1}, & \Gamma^{\mathcal{I}} &= -i C_5 \otimes C_{4,1} \rho^{\mathcal{I}}, \\ \tilde{\Gamma}^{\hat{\mu}} &= \gamma^{\hat{\mu}} C_5^{-1} \otimes C_{4,1}^{-1}, & \tilde{\Gamma}^{\mathcal{I}} &= i C_5^{-1} \otimes \rho^{\mathcal{I}} C_{4,1}^{-1}. \end{aligned} \quad (\text{A.3})$$

We may then check the Clifford algebra relations and symmetry properties

$$\Gamma^{(M} \tilde{\Gamma}^{N)} = \eta^{MN} \mathbb{I}_{16}, \quad \tilde{\Gamma}^{(M} \Gamma^{N)} = \eta^{MN} \mathbb{I}_{16}, \quad (\Gamma^M)^\top = \Gamma^M, \quad (\tilde{\Gamma}^M)^\top = \tilde{\Gamma}^M, \quad (\text{A.4})$$

as well as the chirality relations

$$\tilde{\Gamma}^1 \Gamma^2 \tilde{\Gamma}^3 \Gamma^4 \tilde{\Gamma}^5 \Gamma^6 \tilde{\Gamma}^7 \Gamma^8 \tilde{\Gamma}^9 \Gamma^0 = \mathbb{I}_{16}, \quad \Gamma^1 \tilde{\Gamma}^2 \Gamma^3 \tilde{\Gamma}^4 \Gamma^5 \tilde{\Gamma}^6 \Gamma^7 \tilde{\Gamma}^8 \Gamma^9 \tilde{\Gamma}^0 = -\mathbb{I}_{16}. \quad (\text{A.5})$$

These relations imply that Γ^M maps positive-chirality spinors to negative-chirality spinors, while $\tilde{\Gamma}^M$ acts in the opposite way. Let us assign a lower Weyl index $\alpha = 1, \dots, 16$ to a positive-chirality spinor Ψ_α , and an upper index to a spinor of negative chirality $\tilde{\Psi}^\alpha$. It follows that the index structure of gamma matrices is $\tilde{\Gamma}_{\alpha\beta}^M, \Gamma^{M\alpha\beta}$. Using Weyl indices we can conveniently formulate the ‘‘trality identities’’ as

$$\Gamma^{M(\alpha\beta} \Gamma_M^{\gamma)\delta} = 0, \quad \tilde{\Gamma}_{(\alpha\beta}^M \tilde{\Gamma}_{M\gamma)\delta} = 0. \quad (\text{A.6})$$

Let us stress that the Lorentz generators are one half of the matrices

$$\Gamma^{MN} = \tilde{\Gamma}^{[M} \Gamma^{N]}, \quad \tilde{\Gamma}^{MN} = \Gamma^{[M} \tilde{\Gamma}^{N]}, \quad (\text{A.7})$$

and since $(\tilde{\Gamma}^{MN})^\top = -\Gamma^{MN}$ positive and negative chirality representations are dual. As a result, Majorana bilinears are simply built contracting Weyl indices on $\Psi_\alpha, \tilde{\Psi}^\alpha, \tilde{\Gamma}_{\alpha\beta}^M, \Gamma^{M\alpha\beta}$.

As a final remark, recall that we adopt an off-shell supersymmetry formalism that realizes manifestly only a subalgebra $\mathfrak{so}(2)_R \times \mathfrak{so}(2,1)_R$ of the full R-symmetry group $\mathfrak{so}(4,1)_R$, associated to the split $\mathcal{I} = (i, A)$, $i = 6, 7$, $A = 8, 9, 0$. It is therefore convenient to specify further the representation of gamma matrices $\rho^{\mathcal{I}}$ by requiring

$$\rho^8 \rho^9 \rho^0 = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}. \tag{A.8}$$

An explicit realization of all the gamma matrices considered in this section in terms of Pauli matrices can be found in subsection A.3 below.

A.2 Embedding formalism for AdS_5 and Killing spinors

Let us review some well-known constructions in order to fix our notations and conventions. We realize Euclidean AdS_5 as a hyperboloid in the embedding space $\mathbb{R}^{5,1}$. Let coordinates on the latter be $X^{\mathcal{A}}$, in which the $6d$ vector index \mathcal{A} is split as $\mathcal{A} \rightarrow (a, \underline{0}, \underline{5})$, with $a = 1, \dots, 4$. The a indices correspond to directions along the conformal boundary of AdS_5 , while $\underline{0}, \underline{5}$ are an auxiliary timelike and spacelike directions respectively (we use underlined indices to avoid possible confusion with other spacetime or R-symmetry indices). The hyperboloid equation is

$$\eta_{AB} X^{\mathcal{A}} X^{\mathcal{B}} = \delta_{ab} X^a X^b - X^{\underline{0}} X^{\underline{0}} + X^{\underline{5}} X^{\underline{5}} = -R^2, \tag{A.9}$$

where R is the radius of AdS_5 . The relation between the embedding coordinates $X^{\mathcal{A}}$ and the Poincaré coordinates x^a, z in (3.10) is

$$X^a = \frac{R x^a}{z}, \quad X^{\underline{0}} + X^{\underline{5}} = \frac{x^a x_a + z^2}{z}, \quad X^{\underline{0}} - X^{\underline{5}} = \frac{R^2}{z}. \tag{A.10}$$

In the discussion of AdS_5 Killing spinors it is convenient to adopt a representation of $5d$ gamma matrices $\gamma^{\hat{\mu}}$ derived from a suitable chiral representation of gamma matrices in the $\mathbb{R}^{5,1}$ embedding space. Let $\tau^{\mathcal{A}}, \tilde{\tau}^{\mathcal{A}}$ be chiral blocks of gamma matrices of $\mathbb{R}^{5,1}$ satisfying

$$\begin{aligned} \tau^{(\mathcal{A} \tilde{\tau}^{\mathcal{B}})} &= \eta^{AB} \mathbb{I}_4, & \tilde{\tau}^{(\mathcal{A} \tau^{\mathcal{B}})} &= \eta^{AB} \mathbb{I}_4, & (\tau^{\mathcal{A}})^{\top} &= -\tau^{\mathcal{A}}, & (\tilde{\tau}^{\mathcal{A}})^{\top} &= -\tilde{\tau}^{\mathcal{A}} \\ \tilde{\tau}^1 \tau^2 \tilde{\tau}^3 \tau^4 \tilde{\tau}^{\underline{0}} \tau^{\underline{5}} &= \mathbb{I}_4, & \tau^1 \tilde{\tau}^2 \tau^3 \tilde{\tau}^4 \tau^{\underline{0}} \tilde{\tau}^{\underline{5}} &= -\mathbb{I}_4, & (\tau^a)^{\dagger} &= \tilde{\tau}^a, & (\tau^{\underline{5}})^{\dagger} &= \tilde{\tau}^{\underline{5}}, & (\tau^{\underline{0}})^{\dagger} &= -\tilde{\tau}^{\underline{0}}. \end{aligned} \tag{A.11}$$

Let us construct the $5d$ gamma matrices $\gamma^{\hat{\mu}} = (\gamma^a, \gamma^{\underline{5}})$ and C_5 from these objects via

$$\gamma^a = \tilde{\tau}^{[a \tau^{\underline{0}]}, \quad \gamma^{\underline{5}} = \gamma^1 \gamma^2 \gamma^3 \gamma^4, \quad C_5 = i \tau^{\underline{0}}. \tag{A.12}$$

The first relation identifies the vector space of $5d$ Dirac 4-component spinors with 4-component Weyl spinors of positive chirality in the $\mathbb{R}^{5,1}$ embedding space. The matrix $\gamma^{\underline{5}}$ is associated to the radial coordinate z , but at the same time plays the role of chirality matrix along the conformal boundary of AdS_5 .

Let $\underline{\lambda}, \tilde{\underline{\lambda}}$ be constant, 4-component Weyl spinors in the embedding space $\mathbb{R}^{5,1}$ of positive and negative chirality, respectively. Their $SO(5,1)$ transformation laws read

$$\delta \underline{\lambda} = \ell \cdot \underline{\lambda} \equiv \frac{1}{4} \ell_{\mathcal{AB}} \tilde{\tau}^{[\mathcal{A} \tau^{\mathcal{B}]} \underline{\lambda}, \quad \delta \tilde{\underline{\lambda}} = \ell \cdot \tilde{\underline{\lambda}} \equiv \frac{1}{4} \ell_{\mathcal{AB}} \tau^{[\mathcal{A} \tilde{\tau}^{\mathcal{B}]} \tilde{\underline{\lambda}}, \tag{A.13}$$

where $\ell_{[AB]}$ is the infinitesimal parameter. Let us consider the maps

$$\underline{\lambda} \mapsto \psi(\underline{\lambda}) = \frac{R}{\sqrt{z}} \left[\frac{\mathbb{I}_4 - \gamma^5}{2} - \frac{x^a \gamma_a + z \mathbb{I}_4}{R} \frac{\mathbb{I}_4 + \gamma^5}{2} \right] \underline{\lambda}, \quad (\text{A.14})$$

$$\tilde{\underline{\lambda}} \mapsto \tilde{\psi}(\tilde{\underline{\lambda}}) = \frac{R}{\sqrt{z}} \left[\frac{\mathbb{I}_4 + \gamma^5}{2} + \frac{x^a \gamma_a - z \mathbb{I}_4}{R} \frac{\mathbb{I}_4 - \gamma^5}{2} \right] \tilde{\tau}^0 \tilde{\underline{\lambda}}. \quad (\text{A.15})$$

First of all, $\psi(\underline{\lambda})$, $\tilde{\psi}(\tilde{\underline{\lambda}})$ satisfy the 5d Killing spinor equations

$$\nabla_\mu \psi = \frac{1}{2R} \gamma_\mu \psi, \quad \nabla_\mu \tilde{\psi} = -\frac{1}{2R} \gamma_\mu \tilde{\psi}, \quad (\text{A.16})$$

provided we adopt the vielbein (3.11) with the radial coordinate z associated to γ^5 . Second of all, the maps (A.14), (A.15) are equivariant under the action of $\mathfrak{so}(5,1)$. At the infinitesimal level we can write

$$\psi(\underline{\lambda} + \ell \cdot \underline{\lambda}) = \psi(\underline{\lambda}) + \ell \cdot \psi(\underline{\lambda}), \quad (\text{A.17})$$

where the action $\ell \cdot \underline{\lambda}$ is defined in (A.13), while the action $\ell \cdot \psi(\underline{\lambda})$ is realized as the negative of the Lie derivative along the vector field ξ^μ associated to ℓ ,

$$\ell \cdot \psi(\underline{\lambda}) \equiv -\mathcal{L}_\xi \psi(\underline{\lambda}) = -\xi^\mu \nabla_\mu \psi(\underline{\lambda}) - \frac{1}{4} \nabla_\mu \xi_\nu \gamma^{\mu\nu} \psi(\underline{\lambda}), \quad \delta X^A = \ell^A_B X^B, \quad \delta x^\mu = \xi^\mu. \quad (\text{A.18})$$

Completely analogous statements hold for the map $\tilde{\underline{\lambda}} \mapsto \tilde{\psi}(\tilde{\underline{\lambda}})$.

In order to implement the off-shell supersymmetry formulation of [25] followed in the main text, we have to promote the 4-component AdS_5 Killing spinors $\psi(\underline{\lambda})$, $\tilde{\psi}(\tilde{\underline{\lambda}})$ to suitable solutions to (3.25), which holds for 16-component spinors in ten dimensions and therefore implicitly involves the R-symmetry directions. Following our realization of 10d gamma matrices (A.3) we seek solutions to (3.25) of the form

$$\epsilon(\underline{\lambda}, \eta) = \psi(\underline{\lambda}) \otimes \eta, \quad \tilde{\epsilon}(\tilde{\underline{\lambda}}, \tilde{\eta}) = \tilde{\psi}(\tilde{\underline{\lambda}}) \otimes \tilde{\eta}, \quad (\text{A.19})$$

where η , $\tilde{\eta}$ are constant 4-component spinors associated to the gamma matrices ρ^I of the R-symmetry space. Using the definition of Λ in (3.18) and the 5d Killing spinor equations (A.16) we see that $\epsilon(\underline{\lambda}, \eta)$, $\tilde{\epsilon}(\tilde{\underline{\lambda}}, \tilde{\eta})$ solve (3.25) as soon as the constant spinors η , $\tilde{\eta}$ satisfy the constraints

$$\rho^8 \rho^9 \rho^0 \eta = \eta, \quad \rho^8 \rho^9 \rho^0 \tilde{\eta} = -\tilde{\eta}. \quad (\text{A.20})$$

It follows that $\epsilon(\underline{\lambda}, \eta)$, $\tilde{\epsilon}(\tilde{\underline{\lambda}}, \tilde{\eta})$ obey the 10d constraints

$$\tilde{\Gamma}^6 \Gamma^7 \epsilon(\underline{\lambda}, \eta) = i \epsilon(\underline{\lambda}, \eta), \quad \tilde{\Gamma}^6 \Gamma^7 \tilde{\epsilon}(\tilde{\underline{\lambda}}, \tilde{\eta}) = -i \tilde{\epsilon}(\tilde{\underline{\lambda}}, \tilde{\eta}), \quad (\text{A.21})$$

readily checked using the last equation in (A.2). The space of solutions to (3.25) is thus explicitly decomposed into eigenspaces of the $\mathfrak{so}(2)_R$ hermitian generator $\frac{i}{2} \tilde{\Gamma}^6 \Gamma^7$. This decomposition mirrors the partition of the boundary supercharges into $\mathcal{Q}^{xI} \equiv (\mathcal{Q}_\alpha^I, \tilde{\mathcal{S}}^{I\dot{\alpha}})$ and $\tilde{\mathcal{Q}}_{xI} \equiv (\mathcal{S}_I^\alpha, \tilde{\mathcal{Q}}_{I\dot{\alpha}})$, where here $\alpha = +, -$, $\dot{\alpha} = \dot{+}, \dot{-}$ are Weyl indices in four dimensions,

which can be collected into an index $x = 1, \dots, 4$ in the (anti)fundamental of the conformal algebra $\mathfrak{su}^*(4) \cong \mathfrak{so}(5, 1)$.

Let us consider the canonical basis $\{\underline{\lambda}_{(x)}\}$, $x = 1, \dots, 4$ of positive-chirality Weyl spinors in the embedding space $\mathbb{R}^{5,1}$, and its counterpart $\{\tilde{\lambda}^{(x)}\}$ for negative chirality, given by

$$\underline{\lambda}_{(1)} = \tilde{\lambda}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \dots, \underline{\lambda}_{(4)} = \tilde{\lambda}^{(4)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \quad (\text{A.22})$$

In light of (A.8) we also consider the objects $\eta_{(I)}, \tilde{\eta}^{(I)}$, $I = 1, 2$

$$\eta_{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \eta_{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \tilde{\eta}^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \tilde{\eta}^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \quad (\text{A.23})$$

We may then set

$$\epsilon_{(xI)} = \epsilon(\underline{\lambda}_{(x)}, \eta_{(I)}), \quad \tilde{\epsilon}^{(xI)} = \tilde{\epsilon}(\tilde{\lambda}^{(x)}, \tilde{\eta}^{(I)}), \quad (\text{A.24})$$

and thus obtain a canonical basis of solutions to (3.25). Thanks to the equivariance of the maps (A.14), (A.15), the label x can be regarded as a bona fide (anti)fundamental index of $\mathfrak{su}^*(4)$. By the same token, I can be regarded as an index in the fundamental of $\mathfrak{sl}(2, \mathbb{R})_R \cong \mathfrak{so}(2, 1)_R$. The most general Killing spinor in ten dimensions may then be written as

$$\epsilon = \kappa^{(xI)} \epsilon_{(xI)} + \tilde{\kappa}_{(xI)} \tilde{\epsilon}^{(xI)}, \quad (\text{A.25})$$

where the constants $\kappa, \tilde{\kappa}$ are in one-to-one correspondence with the boundary supercharges $\mathcal{Q}^{xI}, \tilde{\mathcal{Q}}_{xI}$. In the implementation of the localization technique in section 3 we select the spinor specified by

$$\kappa^{(21)} = \frac{i}{2}, \quad \kappa^{(42)} = \frac{1}{2}, \quad \tilde{\kappa}_{(22)} = -\frac{i}{2}, \quad \tilde{\kappa}_{(41)} = -\frac{1}{2}, \quad (\text{A.26})$$

with all other constants $\kappa, \tilde{\kappa}$ vanishing.

A.3 Some explicit expressions

We adopt the following representation for the chiral blocks $\tau^A, \tilde{\tau}^A$ of gamma matrices in the embedding space $\mathbb{R}^{1,5}$,

$$\begin{aligned} \tau^1 &= -i \sigma_2 \otimes \sigma_1, & \tilde{\tau}^1 &= i \sigma_2 \otimes \sigma_1, \\ \tau^2 &= -\sigma_1 \otimes \sigma_2, & \tilde{\tau}^2 &= -\sigma_1 \otimes \sigma_2, \\ \tau^3 &= i \sigma_2 \otimes \sigma_3, & \tilde{\tau}^3 &= -i \sigma_2 \otimes \sigma_3, \\ \tau^4 &= \sigma_2 \otimes \mathbb{I}_2, & \tilde{\tau}^4 &= \sigma_2 \otimes \mathbb{I}_2, \\ \tau^0 &= \mathbb{I}_2 \otimes \sigma_2, & \tilde{\tau}^0 &= -\mathbb{I}_2 \otimes \sigma_2, \\ \tau^{\tilde{5}} &= \sigma_3 \otimes \sigma_2, & \tilde{\tau}^{\tilde{5}} &= \sigma_3 \otimes \sigma_2. \end{aligned} \quad (\text{A.27})$$

This representation determines the form of spacetime $5d$ gamma matrices according to (A.12),

$$\begin{aligned}
 \gamma^1 &= -\sigma_2 \otimes \sigma_3, \\
 \gamma^2 &= -\sigma_1 \otimes \mathbb{I}_2, \\
 \gamma^3 &= -\sigma_2 \otimes \sigma_1, & C_5 &= \mathbb{I}_2 \otimes i\sigma_2. \\
 \gamma^4 &= \sigma_2 \otimes \sigma_2, \\
 \gamma^5 &= -\sigma_3 \otimes \mathbb{I}_2,
 \end{aligned}
 \tag{A.28}$$

We use a different representation for the gamma matrices $\rho^{\mathcal{I}}$ in the R-symmetry directions, which is better suited for the split $\mathcal{I} = (i, A)$, $i = 6, 7$, $A = 8, 9, 0$, and satisfies (A.8),

$$\begin{aligned}
 \rho^6 &= \sigma_1 \otimes \mathbb{I}_2, \\
 \rho^7 &= \sigma_2 \otimes \mathbb{I}_2, \\
 \rho^8 &= \sigma_3 \otimes \sigma_1, & C_{4,1} &= i\sigma_1 \otimes \sigma_2. \\
 \rho^9 &= \sigma_3 \otimes \sigma_2, \\
 \rho^0 &= -i\sigma_3 \otimes \sigma_3,
 \end{aligned}
 \tag{A.29}$$

The explicit expression for the 16-component Killing spinor ϵ is

$$\epsilon = \frac{1}{2\sqrt{z}}(0, -i\rho e^{i\varphi}, 0, -z, iR, i\bar{\zeta}, 0, 0, 0, 0, -iR, -i\bar{\zeta}, 0, -z, 0, -i\rho^{-i\varphi})^T.
 \tag{A.30}$$

A.4 More on the auxiliary pure spinors

The supersymmetry transformations are presented using an orthonormal set of pure spinors satisfying

$$\nu_m \Gamma^M \epsilon = 0, \quad \nu_m \Gamma^M \nu_n = \delta_{mn} \epsilon \Gamma^M \epsilon.
 \tag{A.31}$$

For our choice of Killing spinor ϵ there is no simple closed form for such a set of orthonormal pure spinors ν_m . We can, however, relax the orthonormality condition and impose the first relation only. The index m can now be regarded as a $\mathfrak{gl}(7)$ index. For any such set of spinors we have

$$\nu_m \Gamma^M \nu_n = \mathcal{M}_{mn} \epsilon \Gamma^M \epsilon,
 \tag{A.32}$$

where the symmetric matrix \mathcal{M}_{mn} is non-degenerate and is used to raise and lower m indices.

We can achieve formal $\mathfrak{gl}(7)$ covariance by introducing a background $\mathfrak{gl}(7)$ connection $Q_\mu{}^m{}_n$ and utilizing covariant derivatives such as

$$\begin{aligned}
 \mathcal{D}_\mu \nu_m &= \nabla_\mu \nu_m - Q_\mu{}^n{}_m \nu_n, & \mathcal{D}_\mu K^m &= D_\mu K^m + Q_\mu{}^m{}_n K^n, \\
 \mathcal{D}_\mu \mathcal{M}_{mn} &= \partial_\mu \mathcal{M}_{mn} - Q_\mu{}^p{}_m \mathcal{M}_{pn} - Q_\mu{}^p{}_n \mathcal{M}_{mp}.
 \end{aligned}
 \tag{A.33}$$

The $\mathfrak{gl}(7)$ transformation laws of ν , K , \mathcal{M} , Q are

$$\begin{aligned}
 K'^m &= M^m{}_n K^n, & \nu'_m &= \nu_n (M^{-1})^n{}_m, & \mathcal{M}'_{mn} &= \mathcal{M}_{pq} (M^{-1})^p{}_m (M^{-1})^q{}_n, \\
 Q'_\mu{}^m{}_n &= M^m{}_r Q_\mu{}^r{}_s (M^{-1})^s{}_n + M^m{}_r \partial_\mu (M^{-1})^s{}_n, & M &\in \mathfrak{gl}(7).
 \end{aligned}
 \tag{A.34}$$

All equations used in main text can be $\mathfrak{gl}(7)$ -covariantized raising and lowering m indices with \mathcal{M}_{mn} rather than δ_{mn} and replacing ∇_μ , D_μ with \mathcal{D}_μ given above whenever acting

on ν_m, K^m . Equivalently, we can regard the equations in the main text as written in a specific class of $\mathfrak{gl}(7)$ frames for which

$$\mathcal{M}_{mn} = \delta_{mn}, \quad Q_\mu{}^m{}_n = 0. \quad (\text{A.35})$$

It is clear that, in such a frame,

$$\mathcal{D}_\mu \mathcal{M}_{mn} = 0, \quad (\text{A.36})$$

but since this equation is covariant it holds in any $\mathfrak{gl}(7)$ frame. From a practical point of view, one first finds a basis of unnormalized pure spinors satisfying the first equation in (A.31) and then computes the associated \mathcal{M}_{mn} via (A.32). One may then compute $Q_\mu{}^m{}_n$ from (A.36) upon imposing the gauge-fixing condition

$$\mathcal{M}_{[m|p} Q_\mu{}^p{}_{|n]} = 0, \quad (\text{A.37})$$

although we do not need the explicit expression for $Q_\mu{}^p{}_n$ in the implementation of localization.

Let us now record a viable set of ν spinors for our Killing spinor ϵ given in (A.30). We have

$$\begin{aligned} \nu_1 &= \frac{1}{\sqrt{R}} (0, \quad \rho e^{i\varphi}, \quad 0, 0, R, 0, 0, 0, \quad 0, \quad 0, \quad -R, 0, 0, 0, \quad 0, \quad \rho e^{-i\varphi})^\top, \\ \nu_2 &= \frac{1}{\sqrt{R}} (0, \quad -iz e^{2i\varphi}, 0, 0, 0, 0, R, 0, \quad -R e^{2i\varphi}, \quad 0, \quad 0, \quad 0, 0, 0, \quad 0, \quad -iz)^\top, \\ \nu_3 &= \frac{1}{\sqrt{R}} (R, \quad \bar{\zeta}, \quad 0, 0, 0, 0, 0, 0, \quad 0, \quad 0, \quad 0, \quad 0, 0, 0, R e^{-2i\varphi}, \bar{\zeta} e^{-2i\varphi})^\top, \\ \nu_4 &= \frac{1}{\sqrt{R}} (0, \quad 0, \quad 0, R, 0, 0, 0, 0, \quad 0, \quad 0, \quad 0, \quad 0, 0, R, \quad 0, \quad 0)^\top, \\ \nu_5 &= \frac{1}{\sqrt{R}} (0, \quad 0, \quad 0, 0, 0, R, 0, 0, \quad 0, \quad 0, \quad 0, \quad -R, 0, 0, \quad 0, \quad 0)^\top, \\ \nu_6 &= \frac{1}{\sqrt{R}} (0, \quad 0, \quad R, 0, 0, 0, 0, 0, \quad 0, \quad 0, \quad 0, \quad 0, R, 0, \quad 0, \quad 0)^\top, \\ \nu_7 &= \frac{1}{\sqrt{R}} (0, \quad 0, \quad 0, 0, 0, 0, 0, R, \quad 0, \quad -R e^{2i\varphi}, 0, \quad 0, 0, 0, \quad 0, \quad 0)^\top. \end{aligned} \quad (\text{A.38})$$

The associated metric \mathcal{M}_{mn} as in (A.32) can be written compactly as

$$\mathcal{M}_{mn} y^m y^n = \frac{4z}{R\rho} \left[e^{i\varphi} (R y^2 y^4 - R y^6 y^7 - iz y^1 y^2) + e^{-i\varphi} y^3 (\bar{\zeta} y^1 - R y^5) + \rho (y^1)^2 \right], \quad (\text{A.39})$$

where y^m are bookkeeping variables.

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