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The β -function for Yukawa theory at large N_f

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ABSTRACT: We compute the β -function for a massless Yukawa theory in a closed form at the order $\mathcal{O}(1/N_f)$ in the spirit of the expansion in a large number of flavours N_f . We find an analytic expression with a finite radius of convergence, and the first singularity occurs at the coupling value $K = 3$.

KEYWORDS: 1/N Expansion, Renormalization Group

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1 Introduction

The success of the Standard Model in describing the electroweak scale phenomena notwithstanding the apparent problems with the high-energy behaviour have lead to revival of interest in better understanding the UV properties of general gauge-Yukawa theories, see e.g. refs. [1–3]. In particular, gauge-Yukawa theories with a large number of fermion flavours, N_f , provide interesting candidates within the asymptotic-safety framework as opposed to the traditional asymptotic-freedom paradigm [4, 5].

The groundwork for these considerations was laid few decades ago with the computation of the leading large- N_f behaviour of the gauge β -functions [6–8] for N_f fermion charged under the gauge group; see also refs. [9, 10]. The leading $1/N_f$ contribution to the β -function is obtained by resumming the gauge self-energy diagrams with ever increasing chain of fermion bubbles constituting a power series in $K = \alpha N_f/\pi$. It was noticed that this series has a finite radius of convergence; in the case of U(1) gauge group $K = 15/2$. Furthermore, the leading $1/N_f$ contribution to the U(1) β -function has a negative pole at $K = 15/2$, thereby suggesting that this behaviour could cure the Landau-pole behaviour of the SM U(1) coupling, see e.g. refs. [9, 11, 12].

Recently, a further step towards a more complete understanding of these models was achieved by working out the leading $1/N_f$ contribution from the gauge sector to a Yukawa coupling [13]; an extension to semi-simple gauge groups was discussed in ref. [14]. However,

only a single fermion flavour was assumed to couple to the scalar, and the scalar self-energy remained unaffected by the N_f fermion bubbles. Our work is the first step to bridge this remaining gap: we provide the leading $1/N_f$ β -function for pure Yukawa theory, where N_f flavours of fermions couple to the scalar field via Yukawa interaction. We leave the more detailed study within a general gauge-Yukawa framework for future work. Interestingly, the pure Yukawa model is closely related to the Gross-Neveu-Yukawa model, whose critical exponents have been recently computed up to $1/N_f^2$ [15, 16]; see also the earlier studies on the Gross-Neveu model e.g. refs. [17, 18].

The paper is organized as follows: in section 2 we introduce the framework and notations and in section 3 give the expressions for the renormalization constants. In section 4 we perform the resummations of the bubble chains and give closed form expressions for the renormalization constants. In section 5 we collect the results, and write down the final expression for the β -function, and in section 6 we conclude. Explicit formulas for the loop integrals are given in appendix A.

2 The framework and definitions

We consider the massless Yukawa theory for a real scalar field, ϕ , and a fermionic multiplet, ψ , consisting of N_f flavours interacting through the usual Yukawa interaction:

$$\mathcal{L}_{\text{Yuk}} = g\bar{\psi}\psi\phi. \quad (2.1)$$

We define the rescaled coupling,

$$K \equiv \frac{g^2}{4\pi^2} N_f, \quad (2.2)$$

which is kept constant in the limit $N_f \rightarrow \infty$. The β -function of the rescaled coupling, K , can then be expanded in powers of $1/N_f$ as

$$\beta(K) \equiv \frac{dK}{d\ln\mu} = K^2 \left[F_0 + \frac{1}{N_f} F_1(K) \right] + \mathcal{O}(1/N_f^2). \quad (2.3)$$

The purpose of this paper is to compute F_0 and $F_1(K)$. The former is entirely fixed at the one-loop level and can be derived just by rescaling the well-known result for the β -function at that order, while the evaluation of $F_1(K)$ requires the resummation of diagrams in figure 1 involving all-order fermion-bubble chains.

The β -function can be obtained from

$$\beta = K^2 \frac{\partial G_1(K)}{\partial K}, \quad (2.4)$$

where G_1 is defined by

$$\ln Z_K \equiv \ln(Z_S^{-1} Z_F^{-2} Z_V^2) = \sum_{n=1}^{\infty} \frac{G_n(K)}{\epsilon^n}, \quad (2.5)$$

and Z_S , Z_F , and Z_V are the renormalization constants for the scalar wave function, the fermion wave function, and the 1PI vertex, respectively. The scalar wave function renormalization constant is determined via

$$Z_S = 1 - \text{div}\{Z_S \Pi_0(p^2, Z_K K, \epsilon)\}, \quad (2.6)$$

Figure 1. Scalar self-energy, fermion self-energy, and vertex corrections due to a chain of fermion bubbles.

where $\Pi_0(p^2, K_0, \epsilon)$ is the scalar self-energy divided by p^2 , where p is the external momentum. Here and in the following, $\text{div} X$ denotes the poles of X in ϵ . The self-energy can be written as

$$\Pi_0(p^2, K_0, \epsilon) = K_0 \Pi^{(1)}(p^2, \epsilon) + \frac{1}{N_f} \sum_{n=2}^{\infty} K_0^n \Pi^{(n)}(p^2, \epsilon), \quad (2.7)$$

where $\Pi^{(1)}$ gives the one-loop result, and $\Pi^{(n)}$ the n -loop part containing $n-2$ fermion bubbles in the chain, and summing over the topologies given in figure 1(a). Other contributions are of higher order in $1/N_f$ and are thus omitted.

For the fermion self-energy and vertex renormalization constants, the lowest non-trivial contributions are already $\mathcal{O}(1/N_f)$, and we, therefore, have

$$Z_F = 1 - \text{div} \{ \Sigma_0(p^2, Z_K K, \epsilon) \}, \quad (2.8)$$

$$\Sigma_0(p^2, K_0, \epsilon) = \frac{1}{N_f} \sum_{n=1}^{\infty} K_0^n \Sigma^{(n)}(p^2, \epsilon), \quad (2.9)$$

where $\Sigma^{(n)}$ is depicted in figure 1(b) with $n-1$ fermion bubbles. Similarly,

$$Z_V = 1 - \text{div} \{ V_0(p^2, Z_K K, \epsilon) \}, \quad (2.10)$$

$$V_0(p^2, K_0, \epsilon) = \frac{1}{N_f} \sum_{n=1}^{\infty} K_0^n V^{(n)}(p^2, \epsilon), \quad (2.11)$$

where $V^{(n)}$ again contains $n-1$ fermion bubbles and is shown diagrammatically in figure 1(c).

Finally, we briefly comment on the scalar three-point and four-point functions, assuming that they are generated via fermion loops: the former exactly vanishes for massless fermions, while the latter is found to be already $\mathcal{O}(1/N_f)$ at the lowest order. Therefore, they can be neglected for the purpose of our analysis.

3 Renormalization constants

In this section our goal is to extract the contributions to the renormalization constants that are $\mathcal{O}(1/N_f)$ and relevant for the computation of the β -function.

Our starting point for Z_S is eq. (2.6). Using the expansion of the scalar self-energy, eq. (2.7), we obtain

$$Z_S = 1 - \text{div} \left\{ Z_S Z_K K \Pi^{(1)}(p^2, \epsilon) + \frac{1}{N_f} \sum_{n=2}^{\infty} Z_S (Z_K K)^n \Pi^{(n)}(p^2, \epsilon) \right\}. \quad (3.1)$$

Recalling that $Z_K \equiv Z_S^{-1} Z_F^{-2} Z_V^2$ and substituting eqs. (2.8) and (2.10), the first term between brackets can be written as

$$\begin{aligned} & \text{div} \left\{ Z_S Z_K \Pi^{(1)}(p^2, \epsilon) K \right\} \\ &= K \text{div} \left\{ \Pi^{(1)} \right\} + \frac{1}{N_f} \text{div} \left\{ 2K \text{div} \left\{ \Sigma_0(p^2, Z_K K, \epsilon) - V_0(p^2, Z_K K, \epsilon) \right\} \Pi^{(1)}(p^2, \epsilon) \right\}. \end{aligned} \quad (3.2)$$

The $\Pi^{(1)}$ part corresponds to the one-loop diagram and is given by

$$\begin{aligned} \Pi^{(1)}(p^2, \epsilon) &\equiv \text{div} \left\{ \Pi^{(1)} \right\} + \Pi_F^{(1)}(p^2, \epsilon) = \frac{1}{(4\pi)^{d/2-2}} \frac{G(1, 1)}{2} (-p^2)^{d/2-2} \\ &= \frac{1}{\epsilon} + \Pi_F^{(1)}(p^2, \epsilon), \end{aligned} \quad (3.3)$$

where $d = 4 - \epsilon$, the loop function, $G(1, 1)$, is defined in eq. (A.2) in appendix A.1, and we have introduced the notation $\Pi_F^{(1)}$ to indicate the finite part of $\Pi^{(1)}$. Then,

$$\begin{aligned} & \text{div} \left\{ Z_S Z_K \Pi^{(1)}(p^2, \epsilon) K \right\} \\ &= \frac{K}{\epsilon} + \frac{1}{N_f} \text{div} \left\{ 2K \text{div} \left\{ \Sigma_0(p^2, Z_K K, \epsilon) - V_0(p^2, Z_K K, \epsilon) \right\} \right. \\ &\quad \left. \times \left(\text{div} \left\{ \Pi^{(1)} \right\} + \Pi_F^{(1)}(p^2, \epsilon) \right) \right\} \\ &= \frac{K}{\epsilon} + \frac{1}{N_f} \text{div} \left\{ 2K \Pi_F^{(1)}(p^2, \epsilon) \left[\Sigma_0(p^2, Z_K K, \epsilon) - V_0(p^2, Z_K K, \epsilon) \right] \right\} \\ &\quad + \frac{1}{N_f} \times \text{higher poles}, \end{aligned} \quad (3.4)$$

where the higher poles, i.e., higher than $1/\epsilon$, arise from the product of two divergent parts and will be omitted because they play no role in what follows. Then, at the lowest order in $1/N_f$,

$$Z_S = 1 - \frac{K}{\epsilon} + \mathcal{O}(1/N_f). \quad (3.5)$$

Therefore, every time $Z_K K$ appears in the argument of Σ_0 and V_0 , it can be replaced by $K \left(1 - \frac{K}{\epsilon}\right)^{-1}$; the additional contributions are higher order in $1/N_f$. For eq. (3.4), we arrive at

$$\begin{aligned} & \text{div} \left\{ Z_S Z_K \Pi^{(1)}(p^2, \epsilon) K \right\} \\ &= \frac{K}{\epsilon} + \sum_{n=1}^{\infty} K^{n+1} \text{div} \left\{ 2\Pi_F^{(1)}(p^2, \epsilon) \left(1 - \frac{K}{\epsilon}\right)^{-n} \left[\Sigma^{(n)}(p^2, \epsilon) - V^{(n)}(p^2, \epsilon) \right] \right\}. \end{aligned} \quad (3.6)$$

Similarly, the second term of eq. (3.1) reads

$$\frac{1}{N_f} \text{div} \left\{ \sum_{n=2}^{\infty} Z_S (Z_S^{-1} K)^n \Pi^{(n)}(p^2, \epsilon) \right\} = \frac{1}{N_f} \sum_{n=2}^{\infty} K^n \text{div} \left\{ \left(1 - \frac{K}{\epsilon}\right)^{1-n} \Pi^{(n)}(p^2, \epsilon) \right\}. \quad (3.7)$$

Altogether, we can write Z_S as

$$Z_S = 1 - \frac{K}{\epsilon} - \frac{1}{N_f} \sum_{n=2}^{\infty} K^n \left\{ \left(1 - \frac{K}{\epsilon}\right)^{1-n} \left(2\Pi_F^{(1)} \left[\Sigma^{(n-1)} - V^{(n-1)} \right] + \Pi^{(n)} \right) \right\}, \quad (3.8)$$

where the explicit functional dependence on (p^2, ϵ) has been omitted to lighten the notation. Using the binomial expansion,

$$\left(1 - \frac{K}{\epsilon}\right)^{1-n} = \sum_{i=0}^{\infty} \binom{n+i-2}{i} \frac{K^i}{\epsilon^i} \quad (3.9)$$

and performing a shift in the summation, $n \rightarrow n - i$, we find our final expression for Z_S :

$$Z_S = 1 - \frac{K}{\epsilon} - \frac{1}{N_f} \sum_{n=2}^{\infty} K^n \text{div} \left\{ \sum_{i=0}^{n-2} \binom{n-2}{i} \frac{1}{\epsilon^i} \left(2\Pi_F^{(1)} \left(\Sigma^{(n-i-1)} - V^{(n-i-1)} \right) + \Pi^{(n-i)} \right) \right\}. \quad (3.10)$$

We notice that eq. (3.10) differs essentially from its counterpart in the QED [7] because of the contribution from the fermion self-energy and the vertex, which exactly cancel in QED because of the Ward identity.

The expression for Z_F can be derived from eq. (2.8) in a similar manner:

$$\begin{aligned} Z_F &= 1 - \frac{1}{N_f} \sum_{n=1}^{\infty} \text{div} \left\{ (Z_K K)^n \Sigma^{(n)}(p^2, \epsilon) \right\} \\ &= 1 - \frac{1}{N_f} \sum_{n=1}^{\infty} K^n \text{div} \left\{ \left(1 - \frac{K}{\epsilon}\right)^{-n} \Sigma^{(n)}(p^2, \epsilon) \right\} \\ &= 1 - \frac{1}{N_f} \sum_{n=1}^{\infty} K^n \text{div} \left\{ \sum_{i=0}^{n-1} \binom{n-1}{i} \frac{1}{\epsilon^i} \Sigma^{(n-i)}(p^2, \epsilon) \right\}, \end{aligned} \quad (3.11)$$

where we have again performed the same shift $n \rightarrow n - i$ in the last line. The derivation of Z_V is completely analogous, and we can readily write the expression for Z_V :

$$Z_V = 1 - \frac{1}{N_f} \sum_{n=1}^{\infty} K^n \text{div} \left\{ \sum_{i=0}^{n-1} \binom{n-1}{i} \frac{1}{\epsilon^i} V^{(n-i)}(p^2, \epsilon) \right\}. \quad (3.12)$$

4 Resummation

In this section we provide closed formulas for eqs. (3.10), (3.11), and (3.12).

4.1 The vertex

By explicit computation, the n -loop contribution to V_0 is

$$V^{(n)}(p^2, \epsilon) = \frac{(-1)^n}{4} \left(\frac{1}{(4\pi)^{d/2-2}} \right)^n \left(\frac{G(1, 1)}{2} \right)^{n-1} (-p^2)^{n(d/2-2)} \times G(1, 1 - (n-1)(d/2-2)), \quad (4.1)$$

where $G(n_1, n_2)$ is defined in eq. (A.2). We notice that, as in ref. [7], eq. (4.1) allows for the following expansion:

$$V^{(n)}(p^2, \epsilon) = (-1)^n \frac{1}{n\epsilon^n} \frac{v(p^2, \epsilon, n)}{2}, \quad (4.2)$$

where

$$v(p^2, \epsilon, n) = \sum_{j=0}^{\infty} v_j(p^2, \epsilon) (n\epsilon)^j, \quad (4.3)$$

and $v_j(p^2, \epsilon)$ are regular in the limit $\epsilon \rightarrow 0$ for all j . In particular, $v_0(\epsilon)$ is independent of p^2 and is explicitly given by

$$v_0(\epsilon) = \frac{2\Gamma(2-\epsilon)}{\Gamma(1-\frac{\epsilon}{2})^2 \Gamma(2-\frac{\epsilon}{2}) \Gamma(\frac{\epsilon}{2}) \epsilon}. \quad (4.4)$$

Substituting eqs. (4.1) and (4.2) in eq. (3.12), we find:

$$Z_V = 1 - \frac{1}{N_f} \sum_{n=1}^{\infty} (-K)^n \text{div} \left\{ \sum_{j=0}^{n-1} \frac{1}{\epsilon^{n-j}} \sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^i (n-i)^{j-1} \frac{v_j(p^2, \epsilon)}{2} \right\}. \quad (4.5)$$

Then, by using the result of ref. [7],

$$\sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^i (n-i)^{j-1} = -\delta_{j,0} \frac{(-1)^n}{n}, \quad j = 0, \dots, n-1, \quad (4.6)$$

eq. (4.5) gets simplified to

$$Z_V = 1 + \frac{1}{2N_f} \sum_{n=1}^{\infty} \frac{K^n}{\epsilon^n} \frac{v_0(\epsilon)}{n}. \quad (4.7)$$

Expanding $v_0(\epsilon)$ as

$$v_0(\epsilon) = \sum_{j=0}^{\infty} v_0^{(j)} \epsilon^j \quad (4.8)$$

and keeping only the $1/\epsilon$ pole of eq. (4.7), we find the closed formula for Z_V :

$$Z_V = 1 + \frac{1}{2\epsilon N_f} \sum_{n=1}^{\infty} \frac{K^n}{n} v_0^{(n-1)} = 1 + \frac{1}{2\epsilon N_f} \int_0^K v_0(t) dt. \quad (4.9)$$

4.2 The fermion self-energy

The n -loop contribution to Σ_0 is found to be

$$\begin{aligned} \Sigma^{(n)}(p^2, \epsilon) = & -\frac{(-1)^n}{8} \left(\frac{1}{(4\pi)^{d/2-2}} \right)^n \left(\frac{G(1, 1)}{2} \right)^{n-1} (-p^2)^{n(d/2-2)} \\ & \times [G(1, 1 - (n-1)(d/2-2)) - G(1, -(n-1)(d/2-2))]. \end{aligned} \quad (4.10)$$

Similarly to eq. (4.1), eq. (4.10) can be expanded as

$$\Sigma^{(n)}(p^2, \epsilon) = -(-1)^n \frac{1}{n\epsilon^n} \frac{\sigma(p^2, \epsilon, n)}{4}, \quad (4.11)$$

where

$$\sigma(n, \epsilon, p^2) = \sum_{j=0}^{\infty} \sigma_j(p^2, \epsilon) (n\epsilon)^j, \quad (4.12)$$

and $\sigma_j(p^2, \epsilon)$ are regular for $\epsilon \rightarrow 0$. Again, $\sigma_0(\epsilon)$ is independent of p^2 , and it is given by

$$\sigma_0(\epsilon) = -\frac{2^{5-\epsilon} \Gamma\left(\frac{3}{2} - \frac{\epsilon}{2}\right)}{\sqrt{\pi}(4-\epsilon)\Gamma\left(-\frac{\epsilon}{2}\right)} \frac{\sin\left(\frac{\pi\epsilon}{2}\right)}{\pi\epsilon}. \quad (4.13)$$

Using the same procedure as in the previous section, we find that only $\sigma_0(\epsilon)$ contributes to Z_F . Keeping only the $1/\epsilon$ pole, the closed formula for Z_F is

$$Z_F = 1 - \frac{1}{4\epsilon N_f} \int_0^K \sigma_0(t) dt. \quad (4.14)$$

4.3 The scalar self-energy

The evaluation of the bubble diagrams in figure 1(a) is quite cumbersome and is discussed in appendix A.2. Here, we notice that the expression for $\Pi^{(n)}(p^2, \epsilon)$, $n \geq 2$, allows for the following expansion:

$$\Pi^{(n)} = -\frac{3}{2} \frac{(-1)^n}{n(n-1)\epsilon^n} \pi(p^2, \epsilon, n), \quad (4.15)$$

where

$$\pi(p^2, \epsilon, n) = \sum_{j=0}^{\infty} \pi_j(p^2, \epsilon) (n\epsilon)^j, \quad (4.16)$$

and $\pi_j(p^2, \epsilon)$ are regular for $\epsilon \rightarrow 0$. Similarly to the previous cases, $\pi_0(\epsilon)$ is independent of p^2 .

In view of eq. (3.10), we define

$$2\Pi_F^{(1)}(p^2, \epsilon) \left(\Sigma^{(n-1)}(p^2, \epsilon) - V^{(n-1)}(p^2, \epsilon) \right) + \Pi^{(n)}(p^2, \epsilon) \equiv \frac{(-1)^n}{n(n-1)\epsilon^n} \xi(p^2, \epsilon, n), \quad (4.17)$$

where

$$\xi(p^2, \epsilon, n) \equiv n\epsilon \Pi_F^{(1)} \left(\frac{\sigma(p^2, \epsilon, n-1)}{2} + v(p^2, \epsilon, n-1) \right) - \frac{3}{2} \pi(p^2, \epsilon, n), \quad (4.18)$$

and

$$\xi(p^2, \epsilon, n) = \sum_{j=0}^{\infty} \xi_j(p^2, \epsilon) (n\epsilon)^j, \quad (4.19)$$

with $\xi_j(\epsilon, p^2)$ regular for $\epsilon \rightarrow 0$ for all j . In particular, $\xi_0(\epsilon)$ is independent of p^2 and is explicitly given by

$$\xi_0(\epsilon) = \Gamma(2 - \epsilon) \left(\frac{2\epsilon}{\Gamma(2 - \frac{\epsilon}{2})} - \frac{6 - 4\epsilon + \epsilon^2}{\Gamma(1 - \frac{\epsilon}{2}) \Gamma(3 - \frac{\epsilon}{2})} \right) \frac{\sin(\frac{\pi\epsilon}{2})}{\pi\epsilon}. \quad (4.20)$$

Then, using the above definitions, eq. (3.10) can be written as

$$\begin{aligned} Z_S &= 1 - \frac{K}{\epsilon} - \frac{1}{N_f} \sum_{n=2}^{\infty} K^n \text{div} \left\{ \sum_{i=0}^{n-2} \binom{n-2}{i} \frac{1}{\epsilon^i} \frac{(-1)^{n-i}}{(n-i)(n-i-1)\epsilon^{n-i}} \xi(p^2, \epsilon, n-i) \right\} \\ &= 1 - \frac{K}{\epsilon} - \frac{1}{N_f} \sum_{n=2}^{\infty} (-K)^n \text{div} \left\{ \sum_{j=0}^{n-1} \frac{1}{\epsilon^{n-j}} \xi_j(p^2, \epsilon) \sum_{i=0}^{n-2} \binom{n-2}{i} (-1)^i \frac{(n-i)^{j-1}}{(n-i-1)} \right\}. \end{aligned} \quad (4.21)$$

Moreover, we find that

$$\sum_{i=0}^{n-2} \binom{n-2}{i} (-1)^i \frac{(n-i)^{j-1}}{(n-i-1)} = \begin{cases} \frac{(-1)^n}{n} & j = 0 \\ \frac{(-1)^n}{n-1} & j = 1, \dots, n-1 \end{cases}, \quad (4.22)$$

and therefore the expression for Z_S can be significantly simplified:

$$\begin{aligned} Z_S &= 1 - \frac{K}{\epsilon} - \frac{1}{N_f} \sum_{n=2}^{\infty} K^n \text{div} \left\{ \frac{1}{\epsilon^n} \left(\frac{\xi_0(\epsilon)}{n} + \frac{1}{n-1} \sum_{j=1}^{n-1} \xi_j(p^2, \epsilon) \epsilon^j \right) \right\} \\ &= 1 - \frac{K}{\epsilon} - \frac{1}{N_f} \sum_{n=2}^{\infty} K^n \text{div} \left\{ \frac{1}{\epsilon^n} \left(\frac{\xi_0(\epsilon)}{n} + \frac{1}{n-1} \sum_{j=1}^{\infty} \xi_j(p^2, \epsilon) \epsilon^j \right) \right\} \\ &= 1 - \frac{K}{\epsilon} - \frac{1}{N_f} \sum_{n=2}^{\infty} K^n \text{div} \left\{ \frac{1}{\epsilon^n} \left(\frac{\xi_0(\epsilon)}{n} + \frac{\xi(p^2, \epsilon, 1) - \xi_0(\epsilon)}{n-1} \right) \right\}, \end{aligned} \quad (4.23)$$

where in the second line we extended the sum over j up to ∞ without affecting the result, since all the terms for $j > n-1$ are finite. The function $\xi(p^2, \epsilon, 1)$, corresponding to

$$\xi(p^2, \epsilon, 1) \equiv \sum_{j=0}^{\infty} \xi_j(p^2, \epsilon) \epsilon^j, \quad (4.24)$$

can be evaluated by taking in eq. (4.18) the limit $n \rightarrow 1$, although the latter is formally defined for $n \geq 2$. We find the following expression:

$$\xi(p^2, \epsilon, 1) = -\frac{\Gamma(4 - \epsilon)}{\Gamma(2 - \frac{\epsilon}{2}) \Gamma(3 - \frac{\epsilon}{2})} \frac{\sin(\frac{\pi\epsilon}{2})}{\pi\epsilon} \equiv \xi(\epsilon, 1). \quad (4.25)$$

Few comments are in order: eq. (4.25) ensures that Z_S is independent of the external momentum p^2 , as it should. This result comes from an exact cancellation among the different contributions of the scalar self-energy, the fermion self-energy, and the vertex in eq. (4.18). In particular, we find that

$$\begin{aligned}\pi(p^2, \epsilon, 1) &= \frac{2}{3} \left(\frac{\sigma(p^2, \epsilon, 0)}{2} + v(p^2, \epsilon, 0) \right) \left[1 + 1 \cdot \epsilon \Pi_F^{(1)}(p^2, \epsilon) \right] \\ &= \frac{2}{3} \left(\frac{\sigma_0(\epsilon)}{2} + v_0(\epsilon) \right) \left[1 + \epsilon \Pi_F^{(1)}(p^2, \epsilon) \right],\end{aligned}\tag{4.26}$$

and therefore

$$\xi(\epsilon, 1) = -\frac{\sigma_0(\epsilon)}{2} - v_0(\epsilon),\tag{4.27}$$

which is equivalent to eq. (4.25). Interestingly, eq. (4.26) only holds for $n = 1$. All in all, the p^2 independence of eq. (4.25) provides a non-trivial check for our computation.

We are now ready to resum the series in eq. (4.23). By expanding $\xi_0(\epsilon)$ as

$$\xi_0(\epsilon) = \sum_{j=0}^{\infty} \xi_0^{(j)} \epsilon^j,\tag{4.28}$$

the $\frac{1}{n}$ term in eq. (4.23) is given by

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{K^n}{\epsilon^n} \frac{\xi_0(\epsilon)}{n} &= \frac{1}{\epsilon} \sum_{n=2}^{\infty} \frac{K^n}{\epsilon^n} \frac{\xi_0^{(n-1)}}{n} + \text{higher poles} \\ &= \frac{1}{\epsilon} \left(\sum_{n=0}^{\infty} K^{n+1} \frac{\xi_0^{(n)}}{n+1} - K \xi_0^{(0)} \right) + \text{higher poles} \\ &= \frac{1}{\epsilon} \int_0^K [\xi_0(t) - \xi_0(0)] dt + \text{higher poles}.\end{aligned}\tag{4.29}$$

As for the $\frac{1}{n-1}$ term, since $\xi(0, 1) = \xi_0(0)$, we can write

$$\xi(\epsilon, 1) - \xi_0(\epsilon) = \sum_{j=1}^{\infty} \tilde{\xi}^{(j)} \epsilon^j,\tag{4.30}$$

and therefore

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{K^n}{\epsilon^n} \frac{\xi(\epsilon, 1) - \xi_0(\epsilon)}{n-1} &= \sum_{n=1}^{\infty} \frac{K^{n+1}}{\epsilon^{n+1}} \frac{\xi(\epsilon, 1) - \xi_0(\epsilon)}{n} \\ &= \frac{1}{\epsilon} \sum_{n=1}^{\infty} K^{n+1} \frac{\tilde{\xi}^{(n)}}{n} + \text{higher poles} \\ &= \frac{K}{\epsilon} \int_0^1 \frac{dx}{x} \sum_{n=1}^{\infty} \tilde{\xi}^{(n)} (xK)^n + \text{higher poles} \\ &= \frac{K}{\epsilon} \int_0^K \frac{\xi(t, 1) - \xi_0(t)}{t} dt + \text{higher poles},\end{aligned}\tag{4.31}$$

where we used $\frac{1}{n} = \int_0^1 x^{n-1} dx$. Finally, the closed formula for Z_S reads

$$Z_S = 1 - \frac{K}{\epsilon} - \frac{1}{\epsilon N_f} \int_0^K \left(\xi_0(t) - \xi_0(0) + \frac{\xi(t, 1) - \xi_0(t)}{t} K \right) dt. \quad (4.32)$$

5 The β -function

Using the results of the previous section together with eq. (2.5), we can finally proceed to evaluating the β -function. First, we find that

$$G_1(K) = K + \frac{1}{N_f} \int_0^K \left(\xi_0(t) - \xi_0(0) + \frac{\xi(t, 1) - \xi_0(t)}{t} K + \frac{\sigma_0(t)}{2} + v_0(t) \right) dt. \quad (5.1)$$

Now, it is straightforward to compute the β -function:

$$\begin{aligned} \beta(K) = K^2 + \frac{K^2}{N_f} \left\{ \xi_0(K) - \xi_0(0) + \xi(K, 1) - \xi_0(K) + \frac{\sigma_0(K)}{2} + v_0(K) \right. \\ \left. + \int_0^K \frac{\xi(t, 1) - \xi_0(t)}{t} dt \right\}. \end{aligned} \quad (5.2)$$

Recalling eq. (4.27) and using $\xi_0(0) = -\frac{3}{2}$, eq. (5.2) can be further simplified to

$$\frac{\beta(K)}{K^2} = 1 + \frac{1}{N_f} \left\{ \frac{3}{2} + \int_0^K \frac{\xi(t, 1) - \xi_0(t)}{t} dt \right\}. \quad (5.3)$$

Finally, by comparison with eq. (2.3), we see that $F_0 = 1$ and

$$F_1(K) = \frac{3}{2} + \int_0^K \frac{\xi(t, 1) - \xi_0(t)}{t} dt \equiv \frac{3}{2} + \int_0^K I_1(t) dt, \quad (5.4)$$

where

$$I_1(t) = \frac{(3 - 2t + \frac{t^2}{2})\Gamma(3 - t) - \Gamma(4 - t) - 2\Gamma(2 - t)\Gamma(3 - \frac{t}{2})t}{\Gamma(2 - \frac{t}{2})\Gamma(3 - \frac{t}{2})t} \times \frac{\sin(\frac{\pi t}{2})}{\pi t}. \quad (5.5)$$

We plot the integrand, $I_1(t)$, in figure 2. We have checked that our β -function agrees at the leading order in N_f up to four-loop level by comparing with the result of ref. [19].

Finally, let us comment on the pole structure: the integrand, $I_1(t)$, has the first pole occurring at $t = 3$, which results in a logarithmic singularity for $F_1(K)$ around $K = 3$. Due to the sign of $I_1(t)$, we see that $F_1(K)$ approaches large negative values for $K \rightarrow 3^-$. This suggests the existence of a UV fixed point at $K_{UV} \lesssim 3$ such that $F_1(K_{UV}) = -N_f$. Moreover, it can be shown that there is an infrared fixed point K_{IR} , symmetric to K_{UV} with respect to $K = 3$; see e.g. ref. [9]. The same qualitative analysis applies to the singularity at $K = 5$.

Figure 2. The function $I_1(t)$.

6 Conclusions

We have computed the leading $1/N_f$ contribution for the β -function in Yukawa theory with N_f fermion flavours coupling to a real scalar. We obtained a closed form expression for the β -function up to order $\mathcal{O}(1/N_f)$. This expression has a finite radius of convergence, and the first singularity occurs at $K = 3$.

The present result adds an interesting ingredient to models with a large number of fermions, and makes a contribution to better understand the UV behaviour of gauge-Yukawa theories. Curiously, the location of the first singularity of the β -function is the same as for the non-abelian gauge theory at the order $1/N_f$ [8, 9], which might suggest an interplay of the gauge and Yukawa contributions. This study is, however, left for future work.

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A Loop integrals

We here provide some explicit formulas. We follow closely the notations of ref. [20].

A.1 The vertex and the fermion self-energy

As shown in eqs. (4.1) and (4.10), the 1PI vertex and the fermion self-energy involve only the function $G(n_1, n_2)$, independently of the number of bubbles. This corresponds to the one-loop integral

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{D_1^{n_1} D_2^{n_2}} = i \frac{1}{(4\pi)^{d/2}} (-p^2)^{d/2-n_1-n_2} (-1)^{n_1+n_2} G(n_1, n_2), \quad (\text{A.1})$$

where $D_1 = (k+p)^2$ and $D_2 = k^2$. Explicitly,

$$G(n_1, n_2) = \frac{\Gamma(-d/2 + n_1 + n_2)\Gamma(d/2 - n_1)\Gamma(d/2 - n_2)}{\Gamma(n_1)\Gamma(n_2)\Gamma(d - n_1 - n_2)}. \quad (\text{A.2})$$

A.2 The scalar self-energy

Unlike the 1PI vertex and the fermion self-energy, the n -loop contribution to the scalar self-energy, Π_0 , indicated by $\Pi^{(n)}$, cannot be written in terms of $G(n_1, n_2)$ functions only. In fact, $\Pi^{(n)}$ is given by ($n \geq 2$):

$$\begin{aligned} p^2 \Pi^{(n)}(p^2, \epsilon) = & - (4\pi^2)^2 (-1)^n \left(\frac{1}{(4\pi)^{d/2-2}} \frac{G(1,1)}{2} \right)^{n-2} (-1)^\alpha \int \frac{d^d k_1}{(2\pi)^d} \int \frac{d^d k_2}{(2\pi)^d} \\ & \left\{ \frac{6}{(p+k_1)^2 k_2^2 ((k_1-k_2)^2)^{1-\alpha}} - \frac{2}{k_1^2 (p+k_1)^2 k_2^2 ((k_1-k_2)^2)^{-\alpha}} \right. \\ & - \frac{2p^2}{k_1^2 (p+k_1)^2 k_2^2 ((k_1-k_2)^2)^{1-\alpha}} + \frac{2p^2}{k_1^4 (p+k_1)^2 k_2^2 ((k_1-k_2)^2)^{-\alpha}} \\ & \left. - \frac{2p^2}{k_1^2 (k_1+p)^2 (k_2+p)^2 k_2^2 ((k_1-k_2)^2)^{-\alpha}} \right\}, \end{aligned} \quad (\text{A.3})$$

where $\alpha = (n-2)(d/2-2) = -(n-2)\epsilon/2$. Eq. (A.3) requires two-loop integrals which can be performed according to the formula in ref. [20]:

$$\int \frac{d^d k_1}{(2\pi)^d} \int \frac{d^d k_2}{(2\pi)^d} \frac{1}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5}} = (-1)^{1+\sum n_i} \frac{\pi^d (-p^2)^{d-\sum n_i}}{(2\pi)^{2d}} G(n_1, n_2, n_3, n_4, n_5), \quad (\text{A.4})$$

where $D_1 = (k_1+p)^2$, $D_2 = (k_2+p)^2$, $D_3 = k_1^2$, $D_4 = k_2^2$, $D_5 = (k_1-k_2)^2$. The functions $G(n_1, n_2, n_3, n_4, n_5)$ are symmetric with respect to the following index exchanges: $(1 \leftrightarrow 2, 3 \leftrightarrow 4)$ and $(1 \leftrightarrow 3, 2 \leftrightarrow 4)$. Moreover, they reduce to a product of $G(n_1, n_2)$ if at least one of the entries is zero:

$$G(n_1, n_2, n_3, n_4, 0) = G(n_1, n_3)G(n_2, n_4), \quad (\text{A.5})$$

$$G(0, n_2, n_3, n_4, n_5) = G(n_3, n_5)G(n_2, n_3 + n_4 + n_5 - d/2). \quad (\text{A.6})$$

It turns out that the first four integrals in eq. (A.3) can always be written in terms of $G(n_1, n_2)$ making use of eqs. (A.5) and (A.6).

However, the last integral in eq. (A.3) involves $G(1, 1, 1, 1, (n-2)\epsilon/2)$ and, for $n > 2$, its expression can be obtained in terms of hypergeometric functions ${}_3F_2$ by means of the Gegenbauer technique [21]:

$$\begin{aligned} G(1, 1, 1, 1, (n-2)\epsilon/2) = & \frac{2\Gamma(1-\epsilon/2)\Gamma(-1+n\epsilon/2)\Gamma(1-(n-1)\epsilon/2)}{(-2+(n-1)\epsilon)\Gamma(2-\epsilon)\Gamma((n-2)\epsilon/2)\Gamma(2-(n+1)\epsilon/2)} \\ & \times \left\{ (2-(n-1)\epsilon)(2\Gamma(1-\epsilon/2)^2\Gamma(-\epsilon) \right. \\ & \left. - (2-(n-2)\epsilon)\Gamma(1-\epsilon)\Gamma(-\epsilon/2) {}_3F_2 \left(\begin{matrix} 1, -\epsilon/2, 1-(n-1)\epsilon/2 \\ 2-(n+1)\epsilon/2, 2-(n-1)\epsilon/2 \end{matrix} \middle| 1 \right) \right\}. \end{aligned} \quad (\text{A.7})$$

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