From Coxeter higher-spin theories to strings and tensor models

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ABSTRACT: A new class of higher-spin gauge theories associated with various Coxeter groups is proposed. The emphasize is on the B_p -models. The cases of B_1 and its infinite graded-symmetric product $\text{sym}(\times B_1)^\infty$ correspond to the usual higher-spin theory and its multi-particle extension, respectively. The multi-particle B_2 -higher-spin theory is conjectured to be associated with String Theory. B_p -higher-spin models with $p > 2$ are anticipated to be dual to the rank- p boundary tensor sigma-models. B_p higher-spin models with $p \geq 2$ possess two coupling constants responsible for higher-spin interactions in AdS background and stringy/tensor effects, respectively. The brane-like idempotent extension of the Coxeter higher-spin theory is proposed allowing to unify in the same model the fields supported by space-times of different dimensions. Consistency of the holographic interpretation of the boundary matrix-like model in the B_2 -higher-spin model is shown to demand $N \geq 4$ SUSY, suggesting duality with the $N = 4$ SYM upon spontaneous breaking of higher-spin symmetries. The proposed models are shown to admit unitary truncations.

KEYWORDS: AdS-CFT Correspondence, Higher Spin Gravity, Higher Spin Symmetry

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To my father

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1 Introduction

Higher-spin (HS) gauge theories, that describe interactions of massless fields of all spins, provide an interesting arena for testing principles of fundamental physics and, in particular, the *AdS/CFT* correspondence conjecture [1–3]. First example of fully nonlinear HS theory was given for the $4d$ case in [4], while its modern formulation was worked out in [5] (see [6] for a review). A specific property of HS gauge theories is that consistent interactions of propagating massless fields demand a curved background providing a length scale in HS interactions that contain higher derivatives. (*A*)*dS* is the most symmetric curved background compatible with HS interactions [7]. The *AdS*₄ HS model is the simplest nontrivial one in the sense that $d = 4$ is the lowest dimension where HS massless fields propagate.

There were several long-standing problems in HS theories. One was the issue of locality raised originally in [8, 9] where it was shown that current interactions in HS theories can be removed by a seemingly local field redefinition containing infinite power series in higher derivatives with the coefficients containing inverse powers of the cosmological constant. This issue has been further analyzed in a number of papers [10]–[16]. The conclusion of [14, 15] was that unfolded HS equations lead to correct local interactions in the lowest order. These results are now extended to higher orders due to the development of the appropriate homotopy technics in [17, 18] where it is explained in particular how to identify the minimally nonlocal frame of [15] and to decrease the level of nonlocality in HS equations of [5] in higher orders.

Once the problem of locality in HS gauge theory has been sorted out, the fundamental remaining question is how to construct more general HS models that could be related to String Theory [19]. This is the problem addressed in this paper where we propose a new class of HS-like gauge theories that contain richer spectra of fields than the standard models of [5, 8]. The proposed models are argued to be rich enough to lead to String Theory-like models with massive HS fields via spontaneous breakdown of HS symmetries. A concrete proposal for the realization of this idea briefly discussed in the end of this paper can shed light on the origin of the most symmetric phase of String Theory discussed long ago in [20, 21]. (For related discussion see also [22]–[25].) Although this conjecture was supported by the analysis of high-energy limit of string amplitudes [21] and passed some nontrivial tests [26]–[29], no satisfactory understanding of this relation beyond the free field sector of the tensionless limit of String Theory [30–32] was available. Note however that an interesting idea of the singleton string whose spectrum is represented by multiple tensor products of singletons was put forward in [33, 34].

The idea of our construction came from the recent development of the *AdS/CFT* holography [1–3]. Indeed, the fact that HS theories are most naturally formulated in the *AdS* background was conjectured to play a role in the context of *AdS/CFT* correspondence [35]–[39]. This expectation conforms to the fundamental result of Flato and Fronsdal [40] on the relation between tensor products of $3d$ conformal fields (singletons) and infinite towers of $4d$ massless fields that appear in HS theories.

In the important work of Klebanov and Polyakov [41] it was argued that the HS gauge theory of [5] should be dual to the $3d$ $O(N)$ sigma-model in the $N \rightarrow \infty$ limit. The

Klebanov-Polyakov conjecture was checked by Giombi and Yin in [42, 43] where it was shown in particular how the bulk computation in HS gauge theory reproduces at least some of conformal correlators in the free 3d theory. (See also important papers [44, 45].)

Original Klebanov-Polyakov conjecture [41] related large- N limit of the 3d boundary sigma-model with the action

$$S = \frac{1}{2} \int d^3x \left(\partial_\nu \varphi_i \partial^\nu \varphi^i + \frac{\lambda}{N} (\varphi_i \varphi^i)^2 \right) \quad (1.1)$$

for scalar fields φ^i in the vector representation of $O(N)$ ($i = 1, 2, \dots, N$) with the particular HS gauge theories of [5], so-called A and B -models, in the bulk. In [46] Klebanov-Polyakov conjecture was extended to the boundary vector models with Chern-Simons interaction where the Chern-Simons level was related to the free phase parameter in the models of [5]. Fermionic analogues of these conjectures were put forward in [47–49] and were partially checked in [50] (for more recent results see [51, 52]), suggesting a nontrivial duality between bosonic and fermionic boundary models, so-called bosonization, [53–55].

Some difficulties of the original analysis of HS holography originated from non-locality of the naive perturbative approach in HS theory leading to divergencies in the holographic tests [42]. In [51, 52, 56] it has been checked that unfolded HS equations in the local frame of [14, 15, 57] properly reproduce the anticipated holographic results including the general case of Chern-Simons boundary theory, thus opening a way towards further analysis of HS holography in the framework of the nonlinear HS system of [5].

AdS_4/CFT_3 HS duality was extended to AdS_3/CFT_2 holography by Gaberdiel and Gopakumar [58] (see [59] for a review) that has been extensively studied in the literature (see e.g. [60]–[69]). Elaborating on AdS_3/CFT_2 duality, Gaberdiel and Gopakumar arrived at the interesting conjecture on the relation between certain compactifications of String Theory and 3d HS theories [70] as well as on the structure of stringy HS symmetry [71], further elaborated in [72]–[75] (see also [76]). Remarkably, symmetries of the string-like HS theories proposed in this paper exhibit interesting similarities with the Gaberdiel-Gopakumar construction.

The vector boundary models were recently extended to so-called tensor models [77] with the scalar and spinor fields $\varphi_{i_1 i_2 \dots i_p}$ carrying several color indices (symmetrized and/or traceless or not). The Lagrangian for the so-called $O(N)^3$ boundary model is

$$S = \frac{1}{2} \int d^d x \left(\partial_n \varphi_{ijk} \partial^n \varphi^{ijk} + g \varphi_{ijk} \varphi^i{}_{nm} \varphi^{jn}{}_l \varphi^{kml} \right). \quad (1.2)$$

For $O(N)^p$ tensor models with any $p > 1$ the interaction vertex is of order $p+1$. One can also consider bosonic and fermionic tensor models of $U(N)^p$ and $USP(2N)^p$ types. Recently, these models have been extensively studied in the literature [78]–[81] in particular due to their relation [82] to the SYK model [83, 84], providing a new class of models possessing an interesting large N regime [85]–[89]. This suggests in turn that they should admit some holographically dual description. The case of $p = 2$ corresponds to models with matrix-valued fields like in $N = 4$ SYM theory. In this case, the holographically dual theory is String Theory.

To the best of our knowledge no models holographically dual to the tensor sigma-models with $p > 2$ beyond the original one-dimensional SYK model [83]–[85] are available in the literature though ideas that there should be some HS holographic description analogous to that for the $O(N)$ model were expressed e.g. in [89, 90]. It is immediately clear that the HS model of [5] as well as its generalizations to three [8] and any [91] dimensions are not appropriate duals since the spectrum of $O(N)$ singlet operators of rank- p tensor models increases with p due to the presence of new multi-particle states [78, 81]. The same time, the sigma-model form of the boundary theory suggests that a bulk dual model has to be of a HS type.

An obvious class of conformal boundary models holographically dual to tensorial HS theories in the bulk first discussed by Becaria and Tseytlin in [78] is provided by the free boundary conformal fields $\varphi_{i_1, i_2, \dots, i_n}$ with the tensorial currents of the type of the vertex in (1.2), i.e., $J = \varphi_{ijk} \varphi_{nm}^i \varphi_l^{jn} \varphi^{kml}$, or similar. Analogously to the Klebanov-Polyakov conjecture [41] for vector models, we conjecture that the proposed Coxeter HS theories are holographically dual to such higher currents in the free boundary models. Moreover, the AdS_4 Coxeter HS models admit an extension to complex coupling constants that should be holographically dual to the $3d$ boundary conformal models with Chern-Simons interactions analogous to the models considered in [46] for the vectorial HS holography. A subtlety of the holographic interpretation of the boundary tensor sigma-models is that their conformal structure analogous to that of critical sigma-model in the Klebanov-Polyakov conjecture [41] is not yet obvious. Note however that, as argued in [92], it is not *a priori* guaranteed that the boundary duals of the class of HS theories proposed in this paper are represented by local boundary sigma-models free from additional interactions via boundary HS conformal fields which may effectively induce nonlocal interactions on the boundary (see also [93]).

An important problem in the context of HS duality consists of the construction of so-called multi-particle HS theories containing multi-particle composite operators of the original HS theory as fundamental operators. In [94] a multi-particle extension of the usual HS symmetry was proposed, conjectured to underly an infinite extension of the conventional HS theory. The aim of this paper is to propose a class of generalized nonlinear HS systems to be associated with the tensor boundary sigma-models and their multi-particle extensions including the vector and tensor ones.

The proposed models are characterized by an integer p analogous to the rank of the tensor $\varphi_{i_1 i_2 \dots i_p}$ and have a much richer spectrum than usual HS theory of [5, 8, 91]. They are based on the deformed oscillator algebras found in [95–97] in the context of p -particle Calogero models, known as Cherednik algebras [98], which in turn are associated with various Coxeter groups. HS-like models of this class could be formulated long ago and potential relevance of the Cherednik algebra to HS theory was mentioned already in [97]. However, naive extension of this class was not formulated explicitly so far because the resulting spectra of states left no room for a massless spin-two state, i.e., graviton, not allowing the resulting models to fit into the standard paradigm of HS gravity. The same time, the naive extension to HS theories of this kind exhibits interesting features indicating their potential relevance as bulk duals of the tensor boundary sigma-models.

In this paper we extend the construction of Cherednik algebras to *framed Cherednik algebras* in such a way that the related HS theories turn out to be free of the aforementioned difficulties, containing massless HS fields in the spectrum. This generalization was inspired by the construction of multi-particle algebras of [94] and, other way around, allows us to formulate nonlinear equations for the multi-particle HS theory. It is based on the q^{th} direct product degree of a chosen rank- p Coxeter group $\mathcal{C}$, thus containing two positive integers as free parameters: p and q are associated, respectively, with the tensor degree of the boundary model and the degree of multi-particle states. For instance $\mathcal{C} = A_1 \sim B_1$ and $q = 1$ or $q = \infty$ correspond to the conventional HS theory or its multi-particle extension, respectively. On the other hand, higher ranks p of the Coxeter group B_p are conjectured to be associated with the rank- p tensor boundary models. Generalization to other Coxeter groups and root systems (for more detail on Coxeter groups see for instance [99]) is also possible and worth to be investigated. The case of $p = 2, q = \infty$ is particularly appealing in connection with the HS interpretation of String Theory. The B_2 -model possesses two coupling constants responsible for HS interactions in AdS background and stringy effects, respectively. It is tempting to speculate that this model can provide a realization of the most symmetric phase of String Theory. A possible HS symmetry breaking mechanism is also discussed in the concluding section of this paper.

The main idea of this work was to look for HS bulk models providing generalizations of the usual HS theory to models with richer spectrum that would qualitatively match the pattern of operators of the tensor boundary models and their multi-particle extensions. It should be stressed that the formal consistency and gauge symmetries along with simple physical conditions like, e.g., Lorentz covariance severely restrict possible higher-rank extensions of HS theories. The HS-like models based on framed Cherednik algebras provide a distinguished and, may be, exhaustive possibility for the construction of models fulfilling these properties.

The paper is organized as follows. We start with recalling the form of standard nonlinear HS equations in section 2. In section 3 we recall the construction of deformed oscillators that underlies the standard formulation of HS equations as well as its Cherednik extension associated with an arbitrary Coxeter root system. In section 4 we explain the difficulty of the naive extension of HS algebras via enlargement of the number of species of oscillators and introduce the notion of the *framed* oscillator algebra free of this difficulty. Then in section 5 we apply this idea to the construction of framed Cherednik algebras underlying the proper extension of HS equations to any Coxeter system presented in section 6. Further extensions of the proposed systems to higher differential forms generating invariant functionals, color systems with Chan-Paton factors, and idempotents associated with brane-like dynamical systems in different dimensions are presented in sections 7, 8 and 9, respectively. Unitarity of the proposed models is discussed in section 10. Graded extension of the construction of multi-particle algebras of [94] along with the discussion of its different frames and new idempotent construction are presented in section 11. General properties of the proposed systems, their interpretation in the context of string-like and tensor-like HS systems and a possible HS symmetry breaking mechanism are discussed in section 12 where it is also argued that the proposed scheme allows one to interpret composite fields

in the original space-time as elementary fields in higher dimensions. Section 13 contains brief conclusions.

2 Standard higher-spin equations

Nonlinear HS equations of [5, 8, 91] are formulated in terms of three types of fields

$$W(Z; Y; k|x) = dx^{\underline{n}} W_{\underline{n}}(Z; Y; k|x), \quad B(Z; Y; k|x), \quad S = dZ^A S_A(Z; Y; k|x). \quad (2.1)$$

Here $x^{\underline{n}}$ are space-time coordinates while the variables Z^A and Y^A ($A, B, \dots = 1, \dots, M$) are auxiliary coordinates acquiring different interpretation in the spinorial HS models of [5, 8] and vectorial HS model of [91]. dZ^A are anticommuting Z -differentials

$$dZ^A dZ^B = -dZ^B dZ^A, \quad dZ^A dx^{\underline{n}} = -dx^{\underline{n}} dZ^A, \quad dx^{\underline{m}} dx^{\underline{n}} = -dx^{\underline{n}} dx^{\underline{m}}. \quad (2.2)$$

HS equations of [5, 8, 91] have the form

$$dW + W * W = 0, \quad (2.3)$$

$$dB + W * B - B * W = 0, \quad (2.4)$$

$$dS + W * S + S * W = 0, \quad (2.5)$$

$$S * B = B * S, \quad (2.6)$$

$$S * S = -i(dZ^A dZ_A + (dz^\alpha dz_\alpha F_*(B) * \kappa + \text{c.c.})). \quad (2.7)$$

Here $d = dx^{\underline{n}} \frac{\partial}{\partial dx^{\underline{n}}}$ is the space-time de Rham differential and $*$ denotes a model-dependent star product. $F_*(B)$ is some star-product function of the field B . Index α takes two values enumerating basis elements of some subspace of the space labelled by A . The complex conjugated term *c.c.* does not contribute in the real models of [8, 91] but has to be added in the $4d$ model of [5]. κ is an element of the algebra defined to commute with everything except for dz^α with which it anticommutes

$$\kappa * dz^\alpha = -dz^\alpha * \kappa. \quad (2.8)$$

The associative star product $*$ acts on functions of two spinor variables

$$(f * g)(Z; Y) = \frac{1}{(2\pi)^4} \int d^4 U d^4 V \exp[iU^A V^B C_{AB}] f(Z + U; Y + U) g(Z - V; Y + V), \quad (2.9)$$

where C_{AB} is a symplectic form and U^A, V^B are real integration variables. It is normalized in such a way that 1 is a unit element of the star-product algebra, i.e., $f * 1 = 1 * f = f$. Star product (2.9) provides a particular realization of the Weyl algebra

$$[Y_A, Y_B]_* = -[Z_A, Z_B]_* = 2iC_{AB}, \quad [Y_A, Z_B]_* = 0 \quad (2.10)$$

($[a, b]_* = a * b - b * a$) resulting from the normal ordering with respect to the elements

$$b_A = \frac{1}{2i}(Y_A - Z_A), \quad a_A = \frac{1}{2}(Y_A + Z_A), \quad (2.11)$$

which satisfy

$$[a_A, a_B]_* = [b_A, b_B]_* = 0, \quad [a_A, b_B]_* = C_{AB}. \quad (2.12)$$

The choice of the star-product realization (2.9) of the Weyl algebra is significant from many perspectives and, in particular, for the analysis of locality in [17, 18]. An important property of the star product (2.9) is that it admits the inner Klein operator

$$\varkappa = \exp iZ_\alpha Y^\alpha, \quad (2.13)$$

which behaves as $(-1)^N$, where N is the oscillator number operator, i.e.,

$$\varkappa * \varkappa = 1, \quad (2.14)$$

$$\varkappa * f(Z; Y) = f(\tilde{Z}; \tilde{Y}) * \varkappa, \quad (2.15)$$

where $\tilde{Z}^A = (-Z^\alpha, Z^a)$ for any $Z^A = (Z^\alpha, Z^a)$ with Z^a being the rest components of Z^A . The Klein operator

$$k = \kappa \varkappa, \quad (2.16)$$

on which all fields (2.1) depend, anticommutes with Y^α , Z^α and dZ^α . Hence, κ in eq. (2.7) is represented as

$$\kappa = k \varkappa. \quad (2.17)$$

(Note that the fields of the 4d model of [5] also depend on the conjugated Klein operator $\bar{k}$.)

3 Deformed oscillator algebras

3.1 Single oscillator

The nontrivial part of the HS equations is represented by eqs. (2.6), (2.7). Since B is covariantly constant by (2.4), and commutes with S by (2.6) and κ since it is a dz -independent zero-form, it behaves as a central element. Replacing $F_*(B)$ by a central element ν and discarding for a moment the complex conjugated term we observe that the nontrivial part of the HS equations has the form

$$S * S = -i(dZ^A dZ_A + \nu * dz^\alpha dz_\alpha * \kappa), \quad \alpha = 1, 2. \quad (3.1)$$

This equation has many important interpretations for particular realizations of S , $*$ and κ (see e.g. [6, 10]). The most important property in the context of this paper is that the r.h.s. of (3.1) commutes with S despite of (2.8) since any trilinear combination of dz^α is zero because $\alpha = 1, 2$ and dz^α anticommute. It is this property that guarantees formal consistency of HS equations (2.3)–(2.7).

Eq. (3.1) induces commutation relations of the deformed oscillator algebra. Indeed, let $S = dz^\alpha q_\alpha + \dots$, where ellipses denotes terms proportional to components of dZ^A different from dz^α , and

$$\tilde{\nu} = \nu k, \quad (3.2)$$

where

$$k dz^\alpha = -dz^\alpha k, \quad k q_\alpha = -q_\alpha k, \quad k^2 = 1. \quad (3.3)$$

So defined k and $\tilde{\nu}$ still commute with S while, setting

$$\varkappa := \kappa k, \quad (3.4)$$

eq. (3.1) implies deformed oscillator commutation relations in the form of [100]

$$[q_\alpha, q_\beta] = -2i\epsilon_{\alpha\beta}(1 + \tilde{\nu}\varkappa), \quad \varkappa q_\alpha = -q_\alpha \varkappa, \quad \varkappa^2 = 1, \quad (3.5)$$

where $\tilde{\nu}$ is central. The enveloping algebra $Aq(\tilde{\nu})$ of these relations, which is the algebra of polynomials $f(q, \varkappa)$ with $\tilde{\nu}$ being a parameter, has the fundamental property [100] that

$$t_{\alpha\beta} := \frac{i}{4}\{q_\alpha, q_\beta\} \quad (3.6)$$

generate $sp(2)$

$$[t_{\alpha\beta}, t_{\gamma\delta}] = \epsilon_{\beta\gamma}t_{\alpha\delta} + \epsilon_{\beta\delta}t_{\alpha\gamma} + \epsilon_{\alpha\gamma}t_{\beta\delta} + \epsilon_{\alpha\delta}t_{\beta\gamma} \quad (3.7)$$

rotating q_α as a $sp(2)$ vector

$$[t_{\alpha\beta}, q_\gamma] = \epsilon_{\beta\gamma}q_\alpha + \epsilon_{\alpha\gamma}q_\beta. \quad (3.8)$$

By these properties, the deformed oscillators (3.5) provide a representation-free realization of the Wigner oscillator [101] considered by many authors (see, e.g., [102–104]).

That the deformed oscillator algebra possesses the $sp(2)$ automorphism algebra implies that nonlinear HS equations (2.3)–(2.7) are also $sp(2)$ covariant. In $4d$ HS theory of [5] and $3d$ HS theory of [8] the respective $sp(2|\mathbb{C})$ and $sp(2|\mathbb{R})$ imply usual Lorentz covariance [6]. In vectorial models in any dimension of [91], $sp(2|\mathbb{R})$ is a Howe dual algebra allowing to truncate away extra states from the spectrum.

3.2 Many oscillators

The deformed oscillator of section (3.1) can be interpreted as describing the relative motion in the two-body Calogero model. Extension to the $\mathcal{N}$ -body Calogero model proposed in [95–97] consist of $\mathcal{N}$ copies of oscillators $q_{\alpha i}$ with $\alpha = 1, 2$, $i = 1, \dots, \mathcal{N}$ and the generators K_{ij} of the symmetric group $S_{\mathcal{N}}$ that obey

$$K_{ij}q_{\alpha i} = q_{\alpha j}K_{ij}, \quad (3.9)$$

$$K_{ij} = K_{ji}, \quad K_{ij}K_{ij} = I, \quad K_{ij}K_{jl} = K_{il}K_{ij} = K_{jl}K_{il} \quad (3.10)$$

(no summation over repeated indices), and K_{ij} commutes with $q_{\alpha l}$ with $i \neq l$, $j \neq l$. Commutation relations of the oscillators $q_{\alpha l}$ have the form [95–97]

$$[q_{\alpha n}, q_{\beta m}] = -i\epsilon_{\alpha\beta} \left(\delta_{nm} \left(2I + \nu \sum_{l=1}^{\mathcal{N}} K_{ln} \right) - \nu K_{nm} \right). \quad (3.11)$$

Remarkably, this deformation still respects Jacobi identities since antisymmetrization over any three two-component indices gives zero as is most obvious from the Coxeter realization presented in section 3.3.

From (3.11) it follows that the *center of mass coordinates*

$$Q_\alpha := \mathcal{N}^{-\frac{1}{2}} \sum_{n=1}^{\mathcal{N}} q_{\alpha n} \quad (3.12)$$

have undeformed commutation relations with themselves and *relative coordinates*

$$[Q_\alpha, Q_\beta] = -2i\epsilon_{\alpha\beta}, \quad [Q_\alpha, q_{\beta n} - q_{\beta m}] = 0. \quad (3.13)$$

The fundamental property of relations (3.11) is that the operators

$$t_{\alpha\beta} := \frac{i}{4} \sum_{n=1}^{\mathcal{N}} \{q_{\alpha n}, q_{\beta n}\} \quad (3.14)$$

obey the $sp(2)$ commutation relations (3.7), properly rotating all indices α by virtue of

$$[t_{\alpha\beta}, q_{\gamma n}] = \epsilon_{\beta\gamma} q_{\alpha n} + \epsilon_{\alpha\gamma} q_{\beta n}. \quad (3.15)$$

Clearly, $t_{\alpha\beta}$ can be represented as a sum of $sp(2)$ generators acting separately on the center of mass and relative coordinates

$$t_{\alpha\beta} = t_{\alpha\beta}^{cm} + t_{\alpha\beta}^{rel}, \quad t_{\alpha\beta}^{cm} := \frac{i}{4} \{Q_\alpha, Q_\beta\}. \quad (3.16)$$

At $\mathcal{N} = 2$, there is only one relative coordinate

$$q_\alpha = \frac{1}{\sqrt{2}}(q_{\alpha 1} - q_{\alpha 2}). \quad (3.17)$$

With $K_{12} = \varkappa$ (3.11) amounts to (3.5) which therefore describes the relative coordinate sector of the two-body Calogero model.

3.3 Coxeter groups and Cherednik algebras

A rank- p Coxeter group $\mathcal{C}$ is generated by reflections with respect to a system of root vectors $\{v_a\}$ in a p -dimensional Euclidean vector space V with the scalar product $(x, y) \in \mathbb{R}$, $x, y \in V$ (for more detail see [99]). An elementary reflection associated with the root vector v_a acts on $x \in V$ as follows

$$\mathcal{R}_{v_a} x^i = x^i - 2v_a^i \frac{(v_a, x)}{(v_a, v_a)}, \quad \mathcal{R}_{v_a}^2 = I \quad (3.18)$$

(no summation over a). p is the rank of the Coxeter root system $\{v_a\}$. Note that $\mathcal{R}_{v_a}$ changes a sign of v_a

$$\mathcal{R}_{v_a} v_a = -v_a. \quad (3.19)$$

Let q_α^n ($\alpha = 1, 2$, $n = 1; \dots, p$) obey Heisenberg commutation relations

$$[q_\alpha^n, q_\beta^m] = -2i\epsilon_{\alpha\beta} \delta^{nm}, \quad q_\alpha^n = \delta^{nm} q_{m\alpha}. \quad (3.20)$$

Cherednik deformation of the semidirect product of the Heisenberg algebra with the group algebra of $\mathcal{C}$ is

$$[q_\alpha^n, q_\beta^m] = -i\epsilon_{\alpha\beta} \left(2\delta^{nm} + \sum_{v \in \mathcal{R}} \nu(v) \frac{v^n v^m}{(v, v)} K_v \right), \quad (3.21)$$

$$K_v q_\alpha^n = \mathcal{R}_v^n q_\alpha^m K_v, \quad (3.22)$$

where K_v generate the Coxeter group, and the coupling constants $\nu(v)$ are invariants of $\mathcal{C}$ being constant on the conjugacy classes of root vectors under the action of $\mathcal{C}$. These relations generalize (3.11) from $A_p = S_{p+1}$ to any Coxeter group. By definition, the operators K_v obey defining relation of the Coxeter group $\mathcal{C}$. In particular, each K_v is involutive

$$K_v K_v = I \quad (3.23)$$

and since the reflections generated by v and $-v$ coincide, the respective reflection generators are also identified

$$K_{-v} = K_v. \quad (3.24)$$

It is not difficult to check that the double commutator of q_α^n in (3.21) respects the Jacobi identities for any (not necessarily $\mathcal{C}$ -invariant) $\nu(v)$. Indeed, the non-zero part of the triple commutator of $q_\alpha^n, q_\beta^m, q_\gamma^k$ resulting from (3.21) is proportional $v^n v^m v^k$ and hence contains the total antisymmetrization over three two-component indices α, β, γ giving zero. Covariance of relations (3.21) under the action of the Coxeter group itself demands $\nu(v)$ be $\mathcal{C}$ -invariant.

In the sector of relative coordinates, the commutation relations (3.11) is a particular case of (3.21) for the A_p root system

$$v^{nm} = e^n - e^m, \quad (3.25)$$

where e^n form an orthonormal frame in $\mathbb{R}^{p+1}$. V is the p -dimensional subspace of relative coordinates in $\mathbb{R}^{p+1}$ spanned by v^{nm} .

Another important case of the Coxeter root system is B_p with the roots

$$R_1 = \{\pm e^n \quad 1 \leq n \leq p\}, \quad R_2 = \{\pm e^n \pm e^m \quad 1 \leq n < m \leq p\}. \quad (3.26)$$

In addition to permutations, B_p contains reflections of any basis axis in $V = \mathbb{R}^p$ generated by $v_\pm^n = \pm e^n$. In this case R_1 and R_2 form two conjugacy classes of B_p . Let K_n, K_{nm} and K_{nm}^+ be associated with $e^n, e^n - e^m$ and $e^n + e^m$, respectively.

The defining relations of B_p are represented by relations (3.10) for the symmetric group generators K_{nm} along with

$$K_n K_{nm} = K_{nm} K_n, \quad K_n K_n = I, \quad K_{nm}^+ = K_n K_{nm} K_n \quad (3.27)$$

(generators K_n, K_{nm} and K_{nm}^+ with pairwise different indices commute).

The Coxeter group underlying $3d$ HS theory of [8] is $A_1 \sim B_1$.¹ The case of B_2 is also of great importance since it is argued in section 12 to be related to the string-like models.

¹More precisely the Coxeter root system $A_1 \sim C_1$ while the root systems B_p and C_p differ by the normalization of root vectors [99] having the same Coxeter group.

For any Coxeter root system the generators

$$t_{\alpha\beta} := \frac{i}{4} \sum_{n=1}^p \{q_{\alpha}^n, q_{\beta}^n\} \quad (3.28)$$

obey the $sp(2)$ commutation relations (3.7), properly rotating all indices α by virtue of

$$[t_{\alpha\beta}, q_{\gamma}^n] = \epsilon_{\beta\gamma} q_{\alpha}^n + \epsilon_{\alpha\gamma} q_{\beta}^n \quad (3.29)$$

as is easy to see using (3.22) and (3.19). This fact is of fundamental importance for HS theories.

4 Higher-spin modules and framed oscillator algebras

A natural way to allow a richer spectrum in the model including, in particular, mixed-symmetry massless and massive states (the latter are known to be present in String Theory [19]) known to allow consistent interactions at least in the lowest orders [105]–[111], is to let more species of oscillators in the model: $y_{\alpha} \rightarrow y_{\alpha}^n$. However, the construction of oscillator algebras with multiple species of oscillators as well as their Cherednik deformation recollected in section 3.3 cannot be directly applied to the construction of physically acceptable HS theories for the following reason.

Consider a HS algebra called for brevity hs_1 with the single set of oscillators (i.e., $p = 1$). It is well known [23, 112] that the Fock hs_1 -module F_1 describes free boundary conformal fields [113, 114], i.e., Dirac singletons [40, 115–119]. Their conformal dimension is identified with the weight h_1 of the vacuum state in F_1 with respect to the dilatation operator D represented by a bilinear of oscillators in hs_1

$$D|0\rangle = h_1|0\rangle. \quad (4.1)$$

(In the AdS unitary frame D is replaced by the energy operator E . For details on the oscillator realization of semisimple Lie algebras and their modules see, e.g., [120].) Correspondingly, lowest weight representations of the naively extended algebras hs_p built from p copies of oscillators will have multiple weights in the respective Fock vacuum F_p ,

$$h_p = ph_1, \quad (4.2)$$

hence carrying higher lowest weights.

For $p = 1$, elements of the tensor product $F_1 \otimes F_1$ are interpreted either as conserved currents on the boundary or as massless fields in the bulk as was originally shown by Flato and Fronsdal [40]. However for $p > 1$ the lowest weights of the submodules in $F_p \otimes F_p$ turn out to be too high to describe both conserved boundary currents and bulk massless fields. In particular, the $p > 1$ modules have no room for a massless spin-two field, i.e., graviton. That the respective theory cannot contain gravity makes it impossible the construction of consistent HS models via allowing more oscillators, i.e., more generating elements of the star-product algebra. (Note that this argument does not apply to $3d$ HS theories since in

this case the graviton is topological [121, 122], carrying no local degrees of freedom, and, hence, having no nontrivial associated module of the space-time symmetry group. As a result, the construction of 3d HS theories turns out to be far less restrictive than in $d > 3$.)

Recently, this difficulty has been resolved in the construction of multi-particle algebra [10] defined as the universal enveloping algebra of the usual HS algebra, i.e., $U(hs_1)$. From this definition it is clear that HS algebra-modules form modules over the multi-particle HS algebra as well. As observed in [10], an important difference between the construction of multi-particle algebra and oscillator algebras with many oscillators is due to appearance of independent unit elements associated with each copy of oscillators.

Namely, in the usual oscillator algebras the fundamental commutation relations are

$$[q_\alpha^n, q_\beta^m] = 2i\delta^{nm}\epsilon_{\alpha\beta}I, \quad (4.3)$$

where I is the unit element of the algebra. In the multi-particle HS algebra these relations are replaced by

$$[q_\alpha^n, q_\beta^m] = 2i\delta^{nm}\epsilon_{\alpha\beta}I_n, \quad (4.4)$$

where “units” I_n assigned to the respective species of the oscillators obey

$$I_n I_n = I_n, \quad I_n I_m = I_m I_n. \quad (4.5)$$

Such algebras will be called *framed oscillator algebras*. Note that the framed oscillator algebras are nothing else as multiple tensor products of the original associative oscillator algebra with no color indices n and m , i.e., of a single pair of oscillators while usual oscillator algebras (4.3) with the single unit element are their quotients.

The same time multi-particle HS algebra is unital being endowed with the unit element I unrelated to I_n . Roughly speaking, I is associated with the physical vacuum with no excited states while I_n is associated with the minimal energy state in the sector of n -th particles.

An important feature of the multi-particle algebras is that their elements are (graded)symmetrized over elementary species. In that case, the only single-particle element built from I_n is

$$e = \sum_{i=1}^p I_i. \quad (4.6)$$

As such e becomes algebraically independent of individual I_n with the consequence that its eigenvalues may be different from multiples of those for I_n which may have no sense at all.

Technically, in the framed oscillator algebras, the problem with vacuum eigenvalues is avoided as follows. The “units” I_n form a set of idempotents allowing to consider Fock-type modules F_p^i generated from vacua $|0^i\rangle$ obeying

$$I_j |0^i\rangle = \delta_i^j |0^i\rangle, \quad a_-^j |0^i\rangle = 0, \quad (4.7)$$

$$a_+^j |0^i\rangle = 0 \quad i \neq j, \quad (4.8)$$

where $a_\pm^i$ are the creation and annihilation operators built from q_α^i . Clearly, the modules of the generators $t_{\alpha\beta}$ formed by F_p^i are equivalent to the $p = 1$ Fock module F_1 . Note that

for usual oscillator algebras this construction is applicable to the oscillators with indices α, β taking any even number of values $2M$ in which case $t_{\alpha\beta}$ (3.28) generate $sp(2M)$.

Since higher-rank HS theories are anticipated to be associated with the Cherednik-like deformation of the oscillator algebras, to let them contain gravity it is necessary to find a framed extension of the Cherednik algebra having a room for units I_n .

5 Framed Cherednik algebras

5.1 A_N

To illustrate the idea let us first consider the A_{p-1} system of section 3.2. In addition to $q_{\alpha n}$ and K_{nm} with $n, m = 1, \dots, p$, we introduce elements I_n that obey

$$I_n I_m = I_m I_n, \quad I_n I_n = I_n, \quad (5.1)$$

$$I_n q_{\alpha n} = q_{\alpha n} I_n = q_{\alpha n}, \quad I_n q_{\alpha m} = q_{\alpha m} I_n. \quad (5.2)$$

In presence of idempotents I_n , the proper modification of the deformed oscillator relations consistent with (5.2) is

$$[q_{\alpha n}, q_{\beta m}] = -i\epsilon_{\alpha\beta} \left(\delta_{nm} \left(2I_n + \nu \sum_{l=1}^p \hat{K}_{ln} \right) - \nu \hat{K}_{nm} \right), \quad (5.3)$$

where

$$\hat{K}_{nm} = I_n I_m K_{nm}. \quad (5.4)$$

Note that operators $\hat{K}_{nm}$ obey all relations of the symmetric group S_p except for the second relation in (3.10) replaced by

$$\hat{K}_{nm} \hat{K}_{nm} = I_n I_m. \quad (5.5)$$

$\hat{K}_{nm}$ are demanded to obey the analogs of (3.9)

$$\hat{K}_{ij} q_{\alpha i} = q_{\alpha j} \hat{K}_{ij} \quad (5.6)$$

and to commute with all I_l

$$I_l \hat{K}_{nm} = \hat{K}_{nm} I_l \quad \forall l, n, m. \quad (5.7)$$

(Note that by (5.1) this is consistent with the naive expectation that $K_{nm} I_m = I_n K_{nm}$.) Also, the following relations held true

$$I_n \hat{K}_{nm} = I_m \hat{K}_{nm} = \hat{K}_{nm}. \quad (5.8)$$

It should be stressed that the unhatted generators of the symmetric group K_{nm} do not appear in the construction of the framed Cherednik algebra. Relation (5.5) is not of the braid group since $I_n I_m$ is not invertible. Hence, $\hat{K}_{nm}$ do not generate a group algebra.

This modification guarantees consistency of defining relations (5.3). Namely, (5.7) implies consistency of (5.3) and (5.2) while the modification (5.5) of the symmetric group relations plays no role in the Jacobi identity check

$$[q_{n\alpha}, [q_{m\beta}, q_{k\gamma}]] + \text{cycle} = 0. \quad (5.9)$$

5.2 General Coxeter group

Extension of commutation relations (3.21) to *framed Cherednik algebra* is

$$[q_\alpha^n, q_\beta^m] = -i\epsilon_{\alpha\beta} \left(2\delta^{nm} I_n + \sum_{v \in \mathcal{R}} \nu(v) \frac{v^n v^m}{(v, v)} \hat{K}_v \right), \quad (5.10)$$

where I_n are idempotents analogous to those of section 4 and

$$\hat{K}_v := K_v \prod I_{i_1(v)} \cdots I_{i_k(v)}, \quad (5.11)$$

where the labels $i_1(v), \dots, i_k(v)$ enumerate those I_n that carry labels affected by the reflection $\mathcal{R}_v$. For instance, in the case of B_p , $\hat{K}_{v_{ij}} = K_{v_{ij}} I_i I_j$ for the root v_{ij} generating the permutation of e_i , and e_j and $\hat{K}_{v_i} = K_{v_i} I_i$ for v_i representing the reflection of e_i of B_p .

It is important that framed Cherednik algebra still possesses inner $sp(2)$ automorphisms generated by

$$t_{\alpha\beta} := \frac{i}{4} \sum_{n=1}^p \{q_\alpha^n, q_\beta^n\} I_n \quad (5.12)$$

obeying (3.29). Evidently,

$$[t_{\alpha\beta}, I_n] = 0. \quad (5.13)$$

Note that usual Cherednik algebra results from the framed one by quotienting out the ideal identifying all I_n with the unit element of the algebra.

6 Coxeter higher-spin equations

6.1 General scheme

The idea is to let all x -dependent fields W , S and B depend on p sets of variables enumerated by the label $n = 1, \dots, p$, that include Y_A^n , Z_A^n ($A = 1, \dots, M$), idempotents I_n , anticommuting differentials dZ_n^A and Klein-like operators $\hat{K}_v$ associated with all root vectors of a chosen Coxeter group $\mathcal{C}$ (at the convention $\hat{K}_{-v} = \hat{K}_v$). As usual in HS theory, the fields W , S and B are allowed to be valued in any associative algebra A (see also section 8). To make contact with the tensorial boundary theory, it is useful to set $A = (Mat_N)^p$ with elements represented by tensorial matrices $a^{u_1 \dots u_p}_{v_1 \dots v_p}$, $u_i, v_i = 1 \dots N$. Here p is the tensor degree of the boundary model while N is the number of values of color indices with respect to which the $N \rightarrow \infty$ limit has to be taken on the boundary. In this limit the leading contribution should be represented by the colorless HS theory in the bulk.

The field equations associated with Cherednik algebra (3.21) are formulated in terms of the star product analogous to (2.9)

$$\begin{aligned} (f * g)(Z; Y; I) &= \frac{1}{(2\pi)^{pM}} \int d^p M S d^p M T \exp [i S_n^A T_m^B \delta^{nm} C_{AB}] \\ &\quad \times f(Z_i + I_i S_i; Y_i + I_i S_i; I) g(Z - T; Y + T; I) \end{aligned} \quad (6.1)$$

with central elements I_n obeying relations

$$I_n * Y_A^n = Y_A^n * I_n = Y_A^n, \quad I_n * Z_A^n = Z_A^n * I_n = Z_A^n, \quad I_n * I_n = I_n. \quad (6.2)$$

With this definition star product (6.1) implies

$$[Y_A^n, Y_B^m]_* = -[Z_A^n, Z_B^m]_* = 2iC_{AB}\delta^{nm}I_n, \quad [Y_A^n, Z_B^m]_* = 0. \quad (6.3)$$

Analogously to eq. (2.13) this star product admits inner Klein operators $\varkappa_v$ associated with the root vectors v

$$\varkappa_v := \exp i \frac{v^n v^m Z_{\alpha n} Y_{\alpha m}}{(v, v)} \quad (6.4)$$

and any subset of indices α among A such that $\det|C_{\alpha\beta}| \neq 0$. It is straightforward to see that the Klein operators $\varkappa_v$ generate the star-product realization of the Coxeter group via

$$\varkappa_v * q_\alpha^n = \mathcal{R}_v{}^n{}_m q_\alpha^m * \varkappa_v, \quad q_\alpha^n = Y_\alpha^n, Z_\alpha^n. \quad (6.5)$$

Nonlinear equations for the generalized HS theory associated with the Coxeter group $\mathcal{C}$ are

$$dW + W * W = 0, \quad (6.6)$$

$$dB + W * B - B * W = 0, \quad (6.7)$$

$$dS + W * S + S * W = 0, \quad (6.8)$$

$$S * B = B * S, \quad (6.9)$$

$$S * S = -i \left(dZ^{An} dZ_{An} + \sum_i \sum_{v \in \mathcal{R}_i} F_{i*}(B) \frac{dZ_n^\alpha v^n dZ_{\alpha m} v^m}{(v, v)} * \hat{\kappa}_v \right), \quad (6.10)$$

where $\hat{\kappa}_v$ act trivially on (i.e., commute with) all elements except for $dZ_{\alpha n}$ on which they act in the standard fashion

$$\hat{\kappa}_v * dZ_\alpha^n = \mathcal{R}_v{}^n{}_m dZ_\alpha^m * \hat{\kappa}_v. \quad (6.11)$$

$F_{i*}(B)$ is any star-product function of the zero-form B on the conjugacy classes R_i of $\mathcal{C}$. For instance in the important case of B_p equation (6.10) reads as

$$\begin{aligned} S * S = -i \left(dZ_{An} dZ^{An} + \sum_{v \in \mathcal{R}_1} F_{1*}(B) \frac{dZ_n^\alpha v^n dZ_{\alpha m} v^m}{(v, v)} * \hat{\kappa}_v \right. \\ \left. + \sum_{v \in \mathcal{R}_2} F_{2*}(B) \frac{dZ_n^\alpha v^n dZ_{\alpha m} v^m}{(v, v)} * \hat{\kappa}_v \right) \end{aligned} \quad (6.12)$$

with arbitrary $F_{1*}(B)$ and $F_{2*}(B)$. In particular, one can set $F_{2*}(B) = 0$ keeping $F_{1*}(B) = \eta B$. In the case of B_1 this gives usual HS equations. $F_{2*}(B)$ can be nonzero for the B_p -HS models at $p \geq 2$. As discussed in more detail in section 12, the roles of the coupling constants contained in $F_{1*}(B)$ and $F_{2*}(B)$ are different: the former are responsible for the HS features of the model in AdS while the latter for the stringy and, more generally, tensorial ones.

Perturbative analysis is performed around the vacuum solution

$$B = 0, \quad S_0 = dZ^{An} Z_{An}, \quad W = W_0(Y|x), \quad (6.13)$$

where $W_0(Y|x)$ is some solution to (6.6) that usually is taken to describe AdS space-time. (In the 3d models of [8] the solution with $B = \text{const}$ is also important.)

The realization of $\hat{\kappa}_v$ can be different in different models and usually needs introduction of the outer Klein operators $\hat{k}_v$ that, by definition, obey (3.22) with $q = Y_{\alpha n}, Z_{\alpha n}$ and $dZ_{\alpha n}$, so that $\hat{\kappa}_v = \varkappa_v \hat{k}_v$ only affects the differentials $dZ_{\alpha n}$.

Equations (6.6)–(6.10) are formally consistent since relations (3.21) respect the Jacobi identities which in terms of the field equations are fulfilled due to the property that the r.h.s. of (6.10) is central. That HS-type equations can be consistently formulated based on Coxeter root systems was clear long ago. (This was mentioned already in [97].) However, such equations were never discussed in the literature so far since their interpretation in the standard HS paradigm was not easy as discussed in section 4 unless the construction is extended to the framed Cherednik algebras proposed in this paper.

Multi-particle extensions of the Coxeter HS systems are associated with the semi-simple Coxeter groups. The simplest option is associated with the Coxeter root system $B_p^{\mathcal{N}}$ that consists of the product of $\mathcal{N}$ B_p systems

$$B_p^{\mathcal{N}} := \underbrace{B_p \times B_p \times \dots}_{\mathcal{N}}. \quad (6.14)$$

The limit $\mathcal{N} \rightarrow \infty$ along with the graded symmetrization of the product factors expressing the spin-statistics relation (for more detail see section 11) is in many respects most natural.

One can consider reductions of the Coxeter HS theories by restricting all fields to invariants of some group S of automorphisms of $\mathcal{C}$. A natural choice is some subgroup $S = S_p \times S_{\mathcal{N}}$ with $S_p \in B_p$ and $S_{\mathcal{N}}$ exchanging the product factors in (6.14). The latter reduction is possible in the case with all functions $F_i(B)$ associated with different product factors in $\mathcal{C}$ equal to each other. Restriction to invariants of S may be important for taking the $\mathcal{N} \rightarrow \infty$ limit.

The choice of S containing the full symmetric group $S_{\mathcal{N}=\infty}$ is distinguished from various perspectives. If $S = S_{\infty}$, the resulting algebra is the (graded symmetric) multi-particle algebra $M(h(\mathcal{C}))$ of [94] of the HS algebra $h(\mathcal{C})$ associated with the $\mathcal{C}$ -framed Cherednik algebra. As explained in [94], $M(h(\mathcal{C}))$ is isomorphic to the universal enveloping algebra of $h(\mathcal{C})$ (see also section 11). This implies in particular that $M(h(\mathcal{C}))$ is a Hopf algebra. Analogously, if $S = S_{\infty} \times G$ with G being some subgroup of the simple Coxeter group $\mathcal{C}$, the resulting multi-particle algebra is $M(h_G(\mathcal{C}))$ where $h_G(\mathcal{C})$ is the subgroup of invariants of G in $h(\mathcal{C})$. Clearly, $M(h_G(\mathcal{C}))$ is also a Hopf algebra.

As discussed in section 12, the particularly important case of B_2^{∞} multi-particle HS model associated with the $\text{sym}(B_2 \times B_2 \times \dots)$ system, where the graded symmetrization is with respect to all elementary B_2 factors, is anticipated to represent a stringy HS model.

Now we are in a position to specify different types of the HS models resulting from the proposed construction.

6.2 Spinor models

Spinor HS models were formulated in three [8] and four [5] dimensions. Naively, these models may look being too far from Superstring living in 10 dimensions [19]. This is

indeed true for the original $3d$ and $4d$ HS models but may not be true for their multi-particle extensions. The point is that, as emphasized in [123, 124], the multi-particle states of a lower-dimensional model can be identified with elementary states in an appropriate higher-dimensional theory. Such interpretation turns out to be most natural within the matrix-space approach to massless field theories elaborated in [114, 125–127] (see also [92] and recent reviews [128, 129]). Moreover, the maximal space-time where such models admit interpretation in terms of local fields is ten-dimensional [125–127] which is just the superstring space-time dimension. (It would be interesting to see whether this phenomenon is related to the twistor-like transform in ten dimensions introduced by Witten in [130].) From this perspective the original $3d$ and $4d$ theories can be interpreted as certain branes in the ten-dimensional theory [126] with the $3d$ HS model serving as an elementary brick from which the others are composed (see also section 12.2).

6.2.1 $3d$ Coxeter higher-spin models

The $3d$ Coxeter extension of the models of [8] is straightforward. It still needs introduction of two Clifford elements $\psi_{1,2}$ as in [8] to induce the doubling of the spectrum of fields as is standard in $3d$ gravity [121, 122] and HS theory [131]. The indices A and α coincide, i.e., A takes just two values representing a $3d$ spinor index. $sp(2)$ (5.12) represents the local Lorentz symmetry. For definiteness we consider the case of the Coxeter group B_p which seems to be most appropriate for HS applications. Extension to other Coxeter groups is straightforward.

The multi-particle extension of the $3d$ Coxeter HS theories results from endowing all variables with the additional flavour index $a = 1, \dots, \mathcal{N}$

$$I_n, z_{\alpha n}, y_{\alpha n}, dz_{\alpha n}, \psi_{1,2}, \hat{\kappa}_n, \hat{\kappa}_{nm} \longrightarrow I_n^a, z_{\alpha n}^a, y_{\alpha n}^a, dz_{\alpha n}^a, \psi_{1,2}^a, \hat{\kappa}_n^a, \hat{\kappa}_{nm}^a. \quad (6.15)$$

One can either consider different functions $F_{1,2*}^a(B)$ or to keep them all equal (i.e., a -independent) to allow the $S_{\mathcal{N}}$ reduction. In the latter case HS equations (6.12) read as

$$\begin{aligned} S * S = & -i \sum_{a=1}^{\mathcal{N}} \sum_{n=1}^p \left(dz_{\alpha n}^a dz^{\alpha n a} + F_{1*}^a(B) * dz_n^{\alpha a} dz_{\alpha n}^a * \hat{\kappa}_n \right. \\ & \left. + 2F_{2*}^a(B) * \sum_{m=1}^p \left(dz_n^{\alpha a} (dz_{\alpha n}^a - dz_{\alpha m}^a) * \hat{\kappa}_{nm}^a + dz_n^{\alpha a} (dz_{\alpha n}^a + dz_{\alpha m}^a) \hat{\kappa}_n^a * \hat{\kappa}_{nm}^a * \hat{\kappa}_n^a \right) \right), \end{aligned} \quad (6.16)$$

where κ_n and κ_{nm} are the generators of the Coxeter group B_p , that act only on $dz_{\alpha n}^a$. The resulting model contains both bosons and fermions for any $\mathcal{N}$ including the $\mathcal{N} = 1$ B_p -HS theory. The master fields are demanded to be graded symmetric under the exchange of odd combinations of spinors $z_{\alpha n}^a, y_{\alpha n}^a$ at different a (see also section 11).

Further reductions can be performed with $S = S_p$ and/or $S = B_p$. Since fermions are represented by odd functions of y the latter choice leads to a bosonic reduced model.

In the graded symmetrically reduced model usual HS gauge fields are represented by the z -independent one-form part $\sum_{i,a}^{p,\mathcal{N}} \omega(y_i^a, \psi_{1i}^a, \hat{k}_i^a | x) * I_i^a$ of W with $y_{\alpha i}^a$ anticommuting at different a . In particular, AdS_3 background fields belong to this field (for more detail see [8].)

Massless matter fields are represented by the z -independent part $\sum_{i,a} C(y_i^a, \psi_{1i}^a, \hat{k}_i^a | x) * \psi_{2i}^a$ of the zero-form B . Indeed, due to the presence of idempotents I_i^a (recall that $\psi_{2i}^a = \psi_{1i}^a * I_i^a$) these obey the same unfolded field equations as the massless fields of [8]. In the absence of I_i^a the additional terms would contribute changing the pattern of the equations in agreement with the discussion of section 4. In the framed system such terms are uplifted to the equations on the two-particle fields containing pairs of units $I_{i_1}^{a_1} I_{i_2}^{a_2}$.

As usual in HS theory [5, 6], one can consider models with fields valued in matrix algebras i.e., carrying Chan-Paton-like matrix indices. Upon imposing appropriate reality conditions the respective HS models possess $U(n)$ gauge symmetry in the spin-one sector. Following the construction of [8, 139] (see also [6]) the resulting models can be truncated to those with $O(n)$ and $Sp(2m)$ spin-one gauge groups. (See also section 8.)

6.2.2 4d Coxeter higher-spin models

In the Minkowski signature 4d Coxeter HS model, $A = \alpha, \dot{\alpha}$ and one introduces two mutually commuting conjugated algebras generated by the left and right elements $y_\alpha^n, z_\alpha^n, \hat{k}_v$ and $\bar{y}_{\dot{\alpha}}^n, \bar{z}_{\dot{\alpha}}^n, \hat{\bar{k}}_{\bar{v}}$, respectively, on which all fields W, S and B depend. In the 4d B_p -HS system, equation (6.12) is

$$S * S = -i \left(dz_{A_n} dz^{A_n} + \sum_{v \in \mathcal{R}_1} F_{1*}(B) \frac{dz_n^\alpha v^n dz_{\alpha m} v^m}{(v, v)} * \hat{k}_v + \sum_{v \in \mathcal{R}_2} F_{2*}(B) \frac{dz_n^\alpha v^n dz_{\alpha m} v^m}{(v, v)} * \hat{k}_v \right. \\ \left. + \sum_{\bar{v} \in \mathcal{R}_1} \bar{F}_{1*}(B) \frac{d\bar{z}_n^{\dot{\alpha}} \bar{v}^n d\bar{z}_{\dot{\alpha} m} \bar{v}^m}{(\bar{v}, \bar{v})} * \hat{\bar{k}}_{\bar{v}} + \sum_{\bar{v} \in \mathcal{R}_2} \bar{F}_{2*}(B) \frac{d\bar{z}_n^{\dot{\alpha}} \bar{v}^n d\bar{z}_{\dot{\alpha} m} \bar{v}^m}{(\bar{v}, \bar{v})} * \hat{\bar{k}}_{\bar{v}} \right). \quad (6.17)$$

In fact, there are several options for the higher-rank extension of the 4d HS theory. The simplest $(B_p \times B_p)'$ -option is with identified idempotents $I_n = \bar{I}_n$ associated with the left and right oscillators y_α^n, z_α^n and $\bar{y}_{\dot{\alpha}}^n, \bar{z}_{\dot{\alpha}}^n$. We will call this HS model *class I*. Recall that by (5.7) both I_n and $\bar{I}_n$ commute with the Klein operators $\hat{k}$ and $\bar{k}$. Hence $I_n - \bar{I}_n$ generates an ideal $\mathcal{I}$ of the $B_p \times B_p$ system with independent I_n and $\bar{I}_n$. The $(B_p \times B_p)'$ system is $(B_p \times B_p)/\mathcal{I}$. In this model the lowest states are associated with 4d massless fields represented by functions of a single copy of oscillators $y_\alpha^n, z_\alpha^n, \bar{y}_{\dot{\alpha}}^n, \bar{z}_{\dot{\alpha}}^n$ and I_n . For instance, in the graded symmetrized case the massless fields are described by the Z -independent parts ω and C of W and B , respectively

$$\omega = \sum_{i,a}^{p,\mathcal{N}} \omega(y_i^a, \hat{k}_i^a; \bar{y}_i^a, \hat{\bar{k}}_i^a | x) * I_i^a \quad \omega(y_i^a, \hat{k}_i^a; \bar{y}_i^a, \hat{\bar{k}}_i^a | x) = \omega(y_i^a, -\hat{k}_i^a; \bar{y}_i^a, -\hat{\bar{k}}_i^a | x), \quad (6.18)$$

$$C = \sum_{i,a}^{p,\mathcal{N}} C(y_i^a, \hat{k}_i^a; \bar{y}_i^a, \hat{\bar{k}}_i^a | x) * I_i^a \quad C(y_i^a, \hat{k}_i^a; \bar{y}_i^a, \hat{\bar{k}}_i^a | x) = -C(y_i^a, -\hat{k}_i^a; \bar{y}_i^a, -\hat{\bar{k}}_i^a | x). \quad (6.19)$$

The 4d model with different $I_n^a \neq \bar{I}_n^a$ which we call *class II* is a particular real form of the complexified 3d $(B_p)^2$ system which construction is fully analogous to the $sl_2(\mathbb{C})$ realization of the 4d Lorentz algebra. In fact this is a simplest manifestation of the fact discussed in the beginning of section 6.2 that tensoring of the lower-dimensional models

can give higher-dimensional ones. Though this phenomenon is less straightforward for the higher tensor products since higher-dimensional Lorentz algebras are not real forms of the direct sums of a number of $sl_2(\mathbb{C})$ the results of [125]–[129] suggest that a proper interpretation can be available in a certain limit in which the contribution of (delocalized) branes associated with the lower rank (dimensional) models can be discarded. In this model, the lower-rank states start with ω and C that depend either only on $y_n^a, \hat{k}_n^a$ or only on $\bar{y}_n^a, \hat{\bar{k}}_n^a$. These states are analogous to the $3d$ massless states to be identified with the boundary states in the AdS_4/CFT_3 holography once the background (i.e., vacuum) fields are chosen appropriately. This opens an interesting possibility for describing both bulk and boundary degrees of freedom in the same model. A natural framework for the realization of this option suggested by the approach of [92] is described in section 9.

Finally, there is an interesting further extension of the $4d$ HS system to the $B_{2p}(\mathbb{C})$ -model, referred to as *class III* which is a specific real form of the complexified B_{2p} $3d$ system where the $\hat{K}_{i\bar{j}}$ reflection that exchanges the spinor variables of opposite chiralities is antilinear to be compatible with the fact that the dotted and undotted spinors exchanged by $\hat{K}_{i\bar{j}}$ are complex conjugated. Note that this leads to a parametric freedom in the model via the phase parameter φ in the complex structure relations

$$\hat{K}_{i\bar{j}} = \exp i\varphi \hat{\bar{K}}_{\bar{i}j}. \quad (6.20)$$

For the $4d$ models the natural choice of the symmetric reduction is with $S = S_p$ being the diagonal subgroup of $S_p \times \bar{S}_p$ in the left and right sector, allowing to keep the frame-like field

$$h^{\alpha\dot{\alpha}} \sum_{n=1}^p y_{n\alpha} \bar{y}_{n\dot{\alpha}} \quad (6.21)$$

S -invariant, as well as its further extension to $S = S_p \times S_N$ in the multi-particle case.

6.3 Vector Coxeter higher-spin models in any dimension

Vector HS model in d dimensions of [91] is formulated in terms of the oscillators Z_n^A, Y_n^A with the double index $A = \nu, \alpha$ where $\nu = 0, \dots, d$ is the vector index of the d -dimensional AdS algebra $o(d-1, 2)$ and $\alpha = 1, 2$ is the vector index of $sp(2)$ with the symplectic form $\epsilon_{\alpha\beta}$ so that

$$C_{AB} = \eta_{\nu\mu} \epsilon_{\alpha\beta}, \quad A = \nu, \alpha, \quad B = \mu, \beta, \quad (6.22)$$

where $\eta_{\nu\mu}$ is the (x -independent) $o(d-1, 2)$ -invariant fiber metric. The index ν is decomposed into $\nu = (\nu^\parallel, \nu^\perp)$ where $\nu^\perp = 0, \dots, d-1$ labels the Lorentz components transforming as a $o(d-1, 1)$ vector while $\nu^\parallel$, taking one value, labels the Lorentz-invariant component. The index α of the nontrivial part of eq. (6.10) is identified with $\nu = (\nu^\parallel, \alpha)$ (i.e., with the $d+1$ th component of the $o(d-1, 2)$ vector index ν in $A = \nu, \alpha$) and still takes two values.

The original HS system of [91] is formulated in terms of the star product

$$(f * g)(Z, Y) = \frac{1}{\pi^{2(d+1)}} \int dS dT e^{-2S_\alpha^A T_A^\alpha} f(Z + S, Y + S) g(Z - T, Y + T), \quad (6.23)$$

which gives rise to the commutation relations

$$[Z_\alpha^\nu, Z_\beta^\mu]_* = -\epsilon_{\alpha\beta}\eta^{\nu\mu}, \quad [Y_\alpha^\nu, Y_\beta^\mu]_* = \epsilon_{\alpha\beta}\eta^{\nu\mu}, \quad [Y_\alpha^\nu, Z_\beta^\mu]_* = 0.$$

The analogue of eq. (2.7) has slightly different normalization

$$S * S = -\frac{1}{2}(dZ_A^\alpha dZ_\alpha^A + 4dZ_{\nu^\parallel\alpha} dZ_\nu^\alpha B * \kappa). \quad (6.24)$$

κ is the product of the inner Klein operator $\varkappa$ and the outer Klein operator k that anti-commutes with all $\nu^\parallel$ components but commutes with the $\nu^\perp$ Lorentz components.

The nontrivially deformed part of (6.24) is of the B_1 type. To extend the $sp(2)$ algebra of automorphisms of the B_1 Cherednik algebra to the full $sp(2)$ rotations of (ν, α) with any ν , for $\nu \neq d$ the action of $sp(2)$ is generated by the $sp(2)$ of undeformed oscillators. The full $sp(2)$ generators are

$$t_{\alpha\beta} := -\frac{1}{2} \sum_{n=1}^{\mathcal{N}} \{S_{\nu\alpha}^n, S_{\mu\beta}^n \eta^{\nu\mu}\}_*. \quad (6.25)$$

$t_{\alpha\beta}^{\text{tot}}$ is the $sp(2)$ generator acting on all indices α, β including those of differentials $dZ_{\nu\alpha}$. In addition, in the model of [91] it is demanded that the generator

$$t_{\alpha\beta}^{\text{int}} := t_{\alpha\beta}^{\text{tot}} - t_{\alpha\beta} \quad (6.26)$$

obeys relations

$$dt_{\alpha\beta}^{\text{int}} - [t_{\alpha\beta}^{\text{int}}, W]_* = 0, \quad [t_{\alpha\beta}^{\text{int}}, S]_* = 0, \quad [t_{\alpha\beta}^{\text{int}}, B]_* = 0 \quad (6.27)$$

(that restricts the spectrum of dynamical fields to two-row rectangular Young diagrams of $o(d-1, 2)$) and that the fields in the ideal formed by the elements of the form $t_{\alpha\beta}^{\text{int}} * f^{\alpha\beta}$ or $f^{\alpha\beta} * t_{\alpha\beta}^{\text{int}}$ have to be factored out, that restricts the spectrum of dynamical fields to traceless two-row rectangular Young diagrams of $o(d-1, 2)$ appropriate for the description of genuine massless HS fields in the unfolded formalism [132, 133].

The Coxeter extension needs several species of oscillators. Here however is a subtlety that, as in the simplest B_1 case of [91], the Lorentz components $\nu^\perp$ of the AdS_d vectors should be treated differently from the $d+1$ th components $\nu^\parallel$. The latter are supposed to be affected by the action of the Coxeter reflections while the former are not to preserve manifest Lorentz covariance. (Note that the latter condition is needed in the Lorentz covariant setup but may be relaxed in a more general situation.) Naively, this suggests that, to use the framed Coxeter algebra construction, one has to introduce different idempotent elements for the $d+1$ th components of oscillators and the Lorentz ones. This is however not necessary since Coxeter HS equations involve hatted Klein operators that obey (5.7) allowing to use $o(d-1, 2)$ -covariant commutation relations

$$[Y_n^{\nu\alpha}, Y_m^{\mu\beta}] = \delta_{nm} \epsilon^{\alpha\beta} \eta^{\nu\mu} I_n. \quad (6.28)$$

In terms of these oscillators, the $o(d-1, 2)$ algebra is realized by the generators

$$T^{\nu\mu} = -\frac{1}{2} \epsilon^{\alpha\beta} \sum_{n=1}^{\mathcal{N}} Y_{n\alpha}^\nu Y_{n\beta}^\mu. \quad (6.29)$$

The $sp(2)$ generators are

$$t_{\alpha\beta} := -\frac{1}{2}\eta_{\nu\mu} \sum_{n=1}^{\mathcal{N}} \{S_{n\alpha}^{\nu}, S_{n\beta}^{\mu}\}_* . \quad (6.30)$$

The generators t^{int} and t^{tot} are defined as in the original model using (6.26).

Analogously to the original HS model, one imposes the $sp(2)$ invariance condition followed by the factorization condition $t_{\alpha\beta}^{\text{int}} * f^{\alpha\beta} \sim f^{\alpha\beta} * t_{\alpha\beta}^{\text{int}} \sim 0$. An important remaining question is whether this is sufficient to remove all unnecessary states from the general Coxeter HS model making it unitary. The natural choice of the symmetric reduction is with the symmetric group S_p that acts on all indices n of Y_{An} , Z_{An} and dZ_{An} irrespectively of the value of A and/or $S_{\mathcal{N}}$ in the multi-particle extension.

Finally let us note that, in agreement with the analysis of [134], it is natural to expect the existence of another HS model in d dimensions which is holographically dual to free conformal fermions on the boundary, as well as the further supersymmetric extension of the bosonic and fermionic models. Though the explicit form of the nonlinear field equations was not presented so far, we anticipate that it can be elaborated along the lines of [91]. (Recently, some low-order results in the construction of the fermionic model were presented in [135].) Such models will also generate a class of Coxeter HS theories in d dimensions.

7 Higher forms and invariants

HS theories admit an important extension to higher differential forms along the lines of [136]. To this end, it is convenient to unify the dZ - and dx -one-forms into the master field

$$\mathcal{W}_1 = d + dx^n W_n(Z; Y; \hat{K}|x) + dZ^A S_A(Z; Y; \hat{K}|x), \quad d = dx^n \frac{\partial}{\partial x^n}, \quad (7.1)$$

extending it further to all higher differential forms of odd total degrees with respect to both dZ and dx ,

$$\mathcal{W} = \sum_{p=1,3,\dots} \mathcal{W}_p. \quad (7.2)$$

Analogously, the zero-form master field B is extended to the field $\mathcal{B}$ containing differential forms of all total even degrees

$$\mathcal{B} = \sum_{p=0,2,\dots} \mathcal{B}_p, \quad \mathcal{B}_0 = B. \quad (7.3)$$

In these terms the appropriately extended system (6.6)–(6.10) takes the form

$$\mathcal{W} * \mathcal{W} = -i \left(dZ^{An} dZ_{An} + F_*(\mathcal{B}, \gamma_i) \right), \quad (7.4)$$

$$[\mathcal{W}, \mathcal{B}]_* = 0, \quad (7.5)$$

where

$$\gamma_i = \sum_{v \in \mathcal{R}_i} \frac{dZ_n^{\alpha} v^n dZ_{\alpha m} v^m}{(v, v)} * \hat{\kappa}_v. \quad (7.6)$$

It is important that elements γ_i (7.6) are central, commuting with any element of the algebra. Together with (7.5) this guarantees formal consistency of the system in the sense that (7.4) is compatible with the associativity of the star product

$$[\mathcal{W} * \mathcal{W}, \mathcal{W}]_* = 0, \quad (7.7)$$

and the consequence of (7.5)

$$[\mathcal{W} * \mathcal{W}, \mathcal{B}]_* = 0. \quad (7.8)$$

In the sector of one-forms $\mathcal{W}_1 = W, S$ and zero-forms $\mathcal{B}_0 = B$ this system with $F_*(\mathcal{B}, \gamma_i)$ linear in γ_i reproduces (6.6)–(6.10). Higher degrees of two-forms γ_i contribute to equations for the higher forms and are important for the construction of invariants as discussed below.

The construction of equations (7.4), (7.5) applies both to irreducible Coxeter systems and to the reducible ones, containing their direct products. In the latter case the label i of γ_i encodes both the conjugacy classes of the irreducible Coxeter systems and different factors in the direct product. This corresponds to the extension $i \rightarrow i, a$. To impose one or another symmetric reduction, $F_*(\mathcal{B}, \gamma_i)$ should be invariant under the chosen symmetry group S . The appearance of higher differential forms in the model is important both for its string-like interpretation since string theory is known to contain higher degree differential forms and for the construction of invariants along the lines of [136].

Invariant functionals are associated with various x -space forms $\mathcal{L}$ that are closed as a consequence of field equations (7.4), (7.5). As shown in [136], these can be associated with various dZ -independent central elements of the star-product algebra. In the case of usual HS theories considered in [136] the only relevant central element was unity of the star-product algebra. In the Coxeter HS theories of this paper the dZ -independent center of the algebra is generated by the elements I_n^a . In the simplest additive case equation (7.4) is modified to

$$\mathcal{W} * \mathcal{W} = -i(dZ^{An} dZ_{An} + F_*(\mathcal{B}, \gamma_i)) + \mathcal{L} \quad (7.9)$$

where

$$\mathcal{L} := \sum_{n_1, \dots, n_k, a_1, \dots, a_k} I_{n_1}^{a_1} * \dots * I_{n_k}^{a_k} \mathcal{L}_{a_1, \dots, a_k}^{n_1, \dots, n_k}(x) \quad (7.10)$$

obeys

$$d\mathcal{L}_{a_1, \dots, a_k}^{n_1, \dots, n_k}(x) = 0. \quad (7.11)$$

The density forms $\mathcal{L}_{a_1, \dots, a_k}^{n_1, \dots, n_k}(x)$ are symmetric under the exchange of pairs of indices n_l, a_l

$$\mathcal{L}_{\dots, a_n, \dots, a_m, \dots}^{\dots, n_n, \dots, n_m, \dots}(x) = \mathcal{L}_{\dots, a_m, \dots, a_n, \dots}^{\dots, n_m, \dots, n_n, \dots}(x) \quad (7.12)$$

and contain differential forms of any positive even degree

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \dots \quad (7.13)$$

generating invariant functionals as integrals over space-time cycles of appropriate dimension

$$S = \int_{\Sigma} \mathcal{L}. \quad (7.14)$$

Let us stress that (7.11) is a consequence of (7.9), i.e., apart from imposing field equations on dynamical fields, eq. (7.9) expresses $\mathcal{L}$ via dynamical fields themselves in such a way that $\mathcal{L}$ turns out to be closed.

System (7.9), (7.11) and (7.5) is invariant under the following gauge transformations with three types of gauge parameters ε , ξ and χ [136]:

$$\delta\mathcal{W} = [\mathcal{W}, \varepsilon]_* + \xi^N \frac{\partial \mathcal{F}_*(\mathcal{B}, \gamma)}{\partial \mathcal{B}^N} + \chi_i \frac{\partial \mathcal{L}}{\partial \mathcal{L}_i}, \quad (7.15)$$

$$\delta\mathcal{B} = \{\mathcal{W}, \xi\}_* + [\mathcal{B}, \varepsilon]_*, \quad (7.16)$$

$$\delta\mathcal{L}_i = d\chi_i, \quad (7.17)$$

where N is the multiindex running over all components of $\mathcal{B}$, the gauge parameters ε and ξ are differential forms of even and odd degrees, respectively, being otherwise arbitrary functions of x and the generating elements of the star-product algebra, while χ_i only depend on x and dx . By (7.17), the functional S (7.14) is gauge invariant.

Generalization of this construction to the systems with non-additive contribution of the Lagrangian forms $\mathcal{L}$ is also possible and can be constructed along the lines of [136].

8 Color extensions

As usual in HS theory [5, 6], one can let all fields be valued in a matrix algebra, i.e., carry Chan-Paton-like [137, 138] matrix indices. Upon imposing appropriate reality conditions the respective HS models possess $U(n)$ gauge symmetry in the spin-one sector. Following the construction of [8, 139] (see also [6]) the resulting models can be truncated to those with $O(n)$ and $Sp(2m)$ spin-one gauge groups. (For instance $O(1)$ -model is the so-called minimal HS model that only contains even spins.)

The local symmetry algebras of different types of 4d HS systems of [5] are $hu(n, m|8)$, $ho(n, m|8)$ and $husp(n, m|8)$.² Analogously, local symmetry algebras of different types of 3d HS systems are $hu_{\alpha\beta}(n, m|4)$, $ho_{\alpha\beta}(n, m|4)$ and $husp_{\alpha\beta}(n, m|4)$ ($\alpha, \beta = 0, 1$, $\alpha + \beta \neq 0$) [8].

In the both cases the meaning of these notations is that the master fields are valued in appropriate $(n + m) \times (n + m)$ matrices in which bosons belong to the diagonal $n \times n$ and $m \times m$ blocks while fermions are in the two off-diagonal $n \times m$ blocks. The respective HS models possess usual supersymmetry at $n = m$.

For multi-particle HS theories one can use algebras $M(h \dots (n, m|2l))$. Also one can add matrix indices to $M(h \dots (n, m|2l))$ in the end which option seems to be less interesting, however.

One proceeds analogously in the rank- p Coxeter HS models. The respective algebras will be called $\mathcal{C}h \dots (n, m|2l)$ with $\mathcal{C}$ denoting a Coxeter group in question. An interesting option available in this case is to let the fields W and B be valued in the matrix algebra $(\times Mat_{N+M})^p$, e.g., $C^{u_1 \dots u_p}_{v_1 \dots v_p}(Y^n)$, $u_i, v_i = 1 \dots N + M$ with the idea to assign any

²These should not be confused with the 4d global symmetry algebras $hu(n, m|4)$, $ho(n, m|4)$ and $husp(n, m|4)$ of [139]. The difference expresses the doubling of spinor variables $Y \rightarrow Y, Z$ in the non-linear system compared to the Z -independent dynamical fields like $\omega(Y; K|x)$.

pair of matrix indices u_n, v_n to the variable Y^n . From the *AdS/CFT* perspective, indices u_i with $i = 1, \dots, N$ and $i = N + 1, \dots, N + M$ can be associated with the boundary scalar and spinor fields, respectively. More generally, one can consider multi-indexed fields with indices u_i and v_i associated with rank- p tensors obeying various symmetry and/or tracelessness conditions. Still, the respective $Ch\ldots(n, m|2l)$ Coxeter HS theories make sense though the relation of n and m with N and M is more complicated.

Usually, indices of this type are not considered in the bulk HS models. The rationale behind this is that the singlet contributions due to the field $C^{u_1 \dots u_p}_{v_1 \dots v_p}(Y^n) \sim \delta^{u_1}_{v_1} \dots \delta^{u_p}_{v_p} C(Y^n)$ decouple from the traceful components, being in a certain sense dominating. This happens because invariants of the color symmetry cannot source components transforming nontrivially under the color group. Careful analysis of this phenomenon may be important for understanding specificities of the $(O(N))^p$ -models associated with boundary fields being symmetric and/or traceless with respect to color indices (as well as their unitary and symplectic analogues) in which case the respective factors $\delta^{u_1}_{v_1} \dots \delta^{u_p}_{v_p}$ have to be appropriately symmetrized and/or projected to the traceless components within the sets of indices $u_1 \dots u_p$ and $v_1 \dots v_p$.

9 Idempotent extension

In this section we propose a further extension of the construction of HS equations allowing to unify brane-like systems of different space-time dimensions in the same HS model. This construction is associated with idempotents (projectors) of the star-product algebra underlying the system in question. Usually, these are Fock idempotents. The idea is illustrated by the fact that the Fock module of the $4d$ HS algebra describes $3d$ conformal fields [113] associated with the boundary fields in the AdS_4/CFT_3 correspondence [92]. Let us stress that in presence of unbroken HS symmetries, the lower-dimensional branes of this type turn out to be delocalized, i.e., reduction of space-time dimension is via a projective identification of astigmatic type not allowing to observe certain directions. However, if the HS symmetries are broken (e.g. by boundary conditions), the HS branes can become space-time localized.

9.1 General scheme

The general construction is as follows. Let A be some associative algebra with the product $*$ and $\pi_i \in A$ be a set of idempotents

$$\pi_i * \pi_i = \pi_i. \quad (9.1)$$

Let $A_i^j \subset A$ be subspaces of A spanned by the elements of the form

$$a_i^j \in A_i^j : \quad a_i^j = \pi_i * a * \pi_j, \quad a \in A. \quad (9.2)$$

The composition law is of the matrix-like form which, in turn, is a particular case of this construction

$$(a * b)_i^j = \sum_k a_i^k * b_k^j. \quad (9.3)$$

Note that in this construction the idempotents π_i need not be orthogonal. The only important condition is (9.1).

The resulting idempotent-extended algebra $A_{\{\pi\}}$ is associative and unital with the unit element

$$Id_i{}^j = \delta_i^j \pi_i. \quad (9.4)$$

If the algebra A was unital it can itself be realized as a particular case of the above construction with $\pi = Id$.

A particular case of this scheme with two idempotents was used in [140] for the formulation of a $d = 2$ HS theory. Here we propose its slightly different application directly to the nonlinear equations of sections 6, 7.

9.2 Realization

In HS theory, A is the algebra of functions of dx, dZ, Z, Y, κ, x with the star product (6.23) underlying the construction of section 6. Idempotents π_i can be identified with the Z -independent Fock projectors of the star-product algebra. We conjecture that exact holographic correspondence can take place for the systems which correspond to consistent nonlinear equations containing both bulk and boundary fields via the idempotent construction of this section.

9.2.1 Vector-like models

For instance, in the $4d$ HS theory

$$\pi_i = 4I_i \exp y_{i\alpha} \bar{y}_i^\alpha \quad (9.5)$$

are such idempotents, i.e.,

$$\pi_i * \pi_i = \pi_i \quad (9.6)$$

(note that here dotted and undotted indices are on equal footing). π_i can indeed be interpreted as Fock idempotents since

$$(y_{i\alpha} - i\bar{y}_{i\alpha}) * \pi_i = 0, \quad \pi_i * (y_{i\alpha} + i\bar{y}_{i\alpha}) = 0. \quad (9.7)$$

The idempotent-extended system has the same form as the systems of sections 6, 7 with the replacement of the original algebra A by $A_{\{\pi\}}$. The unit element of the star-product algebra is considered as one of the idempotents, say, π_0 . Note that with this convention π_0 is not orthogonal to π_i . Luckily, as emphasized in the previous section, they do not need to be orthogonal for the applicability of the general scheme. The important consistency condition is that the full set of idempotents has to be invariant under the Coxeter group in the system and the mutual products $\pi_i * \pi_j$ as well as the elements (9.2) should be well-defined.

In the original HS system, the vacuum solution (6.13) contains S_0 linear in Z that solves (6.10) because Z_A obey Heisenberg commutation relations (2.10). In the idempotent-extended version of the Coxeter HS equations, the respective vacuum solution is

$$S_0 = \sum_i \pi_i * dZ^{A_n} Z_{A_n} * \pi_i, \quad (9.8)$$

where the condition that all π_i are Z -independent and hence $*$ -commute with Z_{An} implies that so defined S_0 solves (7.9) with the unit element (9.4).

If HS fields carry matrix indices, the star-product idempotents π_i^{star} can be accompanied by those in color indices π_i^{color}

$$\pi_i = \pi_i^{\text{star}} \pi_i^{\text{color}}. \quad (9.9)$$

The simplest idempotent π_i^{color} is given by the diagonal matrix with a single unit element on the diagonal, like $\delta_1^u \delta_v^1$. In that case the left and right modules in this sector can be associated with the space of rows and columns with respect to the matrix algebra. This leads to the vector-like fields in A_0^i and A_i^0 .

The physical interpretation of the proposed construction is that the fields valued in new idempotent sectors of $A_{\{\pi\}}$ live in lower space-time dimensions compared to those valued in A . This is so since the HS modules supported by A_i^j -modules are smaller (i.e., have smaller Gelfand-Kirillov dimension) than those of the twisted adjoint A -module supporting dynamical fields in the original system. For instance, the $4d$ twisted adjoint module describes $4d$ massless fields of all spins while the A_0^i -module (9.2), (9.5) describes $3d$ conformal fields (i.e., $4d$ singletons) [113]. This suggests that the idempotent extensions of the Coxeter HS systems are appropriate to describe lower-dimensional objects analogous to branes in String Theory.

From the *AdS/CFT* perspective this construction makes it possible to describe in the same model both fundamental boundary fields as valued in the A_0^i and A_i^0 sectors and composite current-like operators as fundamental fields in the twisted adjoint A -module. Let us stress again that the simplest extension of the idempotent construction to the matrix sector gives vector-like boundary fields.

The condition that the set of idempotents $\{\pi_i\}$ is invariant under the action of the Coxeter group $\mathcal{C}$ restricts significantly the brane (in particular, boundary) sectors. Since the Coxeter group action exchanges different arguments Y and Z the condition is that all idempotents π_i^{star} that belong to the same orbit of $\mathcal{C}$ should be multiplied by the same color idempotent, i.e., $\pi_i^{\text{color}} = \pi_i^{\text{color}}(O_{\mathcal{C}}(\pi_i^{\text{star}}))$, where $\pi_i^{\text{color}}(O_{\mathcal{C}}(\pi_i^{\text{star}}))$ depends on the $\mathcal{C}$ -orbits of π_i^{star} -idempotents. For instance, if there is a $\mathcal{C}$ -invariant π_i^{star} -idempotent, it can be multiplied by any π_i^{color} . On the other hand, if there are many different idempotents related by the action of $\mathcal{C}$, they should all be multiplied by the same color idempotent.

The latter situation is important in the context of usual HS holography. In this case idempotents π_i^{star} associated with the boundary tensor fields have to be of the form

$$\pi_i^{\text{star}} = \Pi_i \pi^{\text{color}}, \quad (9.10)$$

where Π_i is the Fock projector in the sector of Y_i oscillators and π^{color} has to be the same for any π_i^{star} . The most natural option is to choose π^{color} as the rank-one idempotent

$$\delta_1^{u_1} \delta_{v_1}^1 \dots \delta_1^{u_p} \delta_{v_p}^1. \quad (9.11)$$

Here all indices $u_1 \dots u_p$ and $v_1 \dots v_p$ have to take the same number of values N . Correspondingly, the boundary fields are scalars or spinors of the type $\varphi^{u_1 \dots u_p}(x)$ with all indices taking N values.

9.2.2 $N = 4$ SUSY

The original and most important example of holographic correspondence is the duality of Superstring Theory with the most supersymmetric $N = 4$ SYM theory [1]. Here we briefly discuss how this correspondence can be interpreted from the perspective of the proposal of this paper suggesting that string-like HS theories are associated with the Coxeter group B_2 (see also section 12). Remarkably, the case of $N = 4$ supersymmetry turns out to be distinguished from this perspective as well.

Our consideration is based on the unfolded formulation of the linearized $N = 4$ SYM theory proposed in [114, 141] where it was shown that $4d$ conformal massless fields can be described by the fields valued in certain Fock modules. Let the Fock vacuum $\pi_{0\bar{0}}$ be defined by the relations

$$a_\alpha * \pi_{0\bar{0}} = 0, \quad \bar{b}^{\dot{\beta}} * \pi_{0\bar{0}} = 0, \quad \phi_i * \pi_{0\bar{0}} = 0, \quad (9.12)$$

$$\pi_{0\bar{0}} * \bar{a}^{\dot{\alpha}} = 0, \quad \pi_{0\bar{0}} * b_\alpha = 0, \quad \pi_{0\bar{0}} * \bar{\phi}^i = 0 \quad (9.13)$$

with respect to the oscillators obeying the following non-zero relations

$$[a_\alpha, b^\beta]_* = \delta_\alpha^\beta, \quad [\bar{a}_{\dot{\gamma}}, \bar{b}^{\dot{\beta}}]_* = \delta_{\dot{\gamma}}^{\dot{\beta}}, \quad \{\phi_i, \bar{\phi}^j\}_* = \delta_i^j, \quad (9.14)$$

$i, j = 1, \dots, N$. Bilinears of these oscillators form $su(2, 2; N)$ which therefore acts on the Fock module where $4d$ massless fields are valued. In fact, in [141] it was shown that, to have a room for HS potentials for massless fields, one should introduce four Fock modules generated from the Fock vacua $\pi_{p\bar{p}}$ ($p, \bar{p} = 0, 1$) obeying relations analogous to (9.12) with exchanged roles of the creation and annihilation operators a_α, b_β and $\bar{a}^{\dot{\alpha}}, \bar{b}^{\dot{\beta}}$. For instance,

$$b_\alpha * \pi_{1\bar{1}} = 0, \quad \bar{a}^{\dot{\beta}} * \pi_{1\bar{1}} = 0, \quad \bar{\phi}^i * \pi_{1\bar{1}} = 0, \quad (9.15)$$

$$\pi_{1\bar{1}} * \bar{b}^{\dot{\alpha}} = 0, \quad \pi_{1\bar{1}} * a_\alpha = 0, \quad \pi_{1\bar{1}} * \phi_i = 0. \quad (9.16)$$

The Clifford oscillators can be interpreted as generating the color matrix algebra $Mat_{2^{2N}}$ where all fields are valued. Let us now explain how this construction appears in the framework of the B_2 -HS model and why the case of $N = 4$ is distinguished.

The B_2 -HS theory contains a pair of oscillators $y_{i\alpha}, \bar{y}_{i\dot{\alpha}}$. This allows us to build oscillators analogous to (9.7)

$$a_\alpha = y_{1\alpha} + iy_{2\alpha}, \quad b_\alpha = \frac{1}{4i}(y_{1\alpha} - iy_{2\alpha}), \quad \bar{a}_{\dot{\alpha}} = \bar{y}_{1\dot{\alpha}} - i\bar{y}_{2\dot{\alpha}}, \quad \bar{b}_{\dot{\alpha}} = \frac{1}{4i}(\bar{y}_{1\dot{\alpha}} + i\bar{y}_{2\dot{\alpha}}) \quad (9.17)$$

defining vacuum $\pi_{0\bar{0}}$ (9.12). More precisely, since the commutation relations (6.3) contain I_n , to obey (9.12), (9.13) and (9.1) the idempotent $\pi_{0\bar{0}}$ should contain the factor of $I_1 * I_2$.

The fields valued in the left and right modules generated from this vacuum describe $4d$ massless conformal fields. To describe supermultiplets the vacuum $\pi_{0\bar{0}}$ has to be combined with the Clifford vacuum also obeying (9.12), (9.13) [114].

Although, naively, this construction relates the B_2 -HS theory to the massless conformal supermultiplets at any N , this is not quite the case. The problem is that the idempotent $\pi_{0\bar{0}}$ defined this way is not invariant under the B_2 reflection exchanging Y_1^A with Y_2^A or

changing a sign of one of them. Indeed, such reflections map idempotent $\pi_{0\bar{0}}$ to the *opposite idempotent* $\pi_{1\bar{1}}$. In fact, the condition that the set of idempotents has to be invariant under the action of the Coxeter group B_2 demands all $\pi_{p\bar{p}}$ be present in the model.

The problem however is that opposite idempotents may have ill-defined mutual star product so that elements $\pi_{0\bar{0}} * a * \pi_{1\bar{1}}$ (9.2) are ill defined. Namely, for purely bosonic opposite idempotents of this type, i.e., at $N = 0$, $\pi_{0\bar{0}} * \pi_{1\bar{1}}$ is divergent. On the other hand, in the purely Clifford case the product of opposite Fock idempotents is zero. In the supersymmetric case the bosonic and fermionic contributions to the vanishing determinants in the respective Gaussian integrals resulting from the star product of the idempotents have opposite signs. The full compensation of the bosonic divergency occurs just at $N = 4$ when the numbers of bosonic and fermionic oscillators are equal. (Note that for a similar reason, at $N = 4$ the conformal SUSY is $psu(2, 2; 4)$.)

Thus the consistent rank-two conformal boundary system in the B_2 -HS theory must have $N \geq 4$. The modules generated from $\pi_{p\bar{p}}$ describe $4d$ conformal massless fields (supermultiplets) of all spins [114, 141]. It is natural to conjecture that all of them except for those in the lowest-spin $N = 4$ SYM supermultiplet will become massive upon spontaneous breakdown of HS symmetries in the multi-particle B_2 -HS theory. Note that simultaneously the non-zero string tension should result from the spontaneous breaking of HS symmetries

At the condition that the resulting theory is free from massless fields of spins $s \geq 2$, the only remaining option is the $N = 4$ SYM multiplet. This is the case of $N = 4$ SYM which is the only $N = 4$ massless conformal system with spins $s \leq 1$. Thus, a spontaneously broken B_2 -HS theory with non-zero string tension has a chance to be dual to $N = 4$ SYM.

To have a nontrivial gauge group for the $N = 4$ SYM boundary theory of the spontaneously broken HS theory one has to introduce color indices additional to the Clifford algebra. This can be achieved either directly by letting all fields be valued in the matrix algebra (and its further orthogonal/symplectic reductions) or via the construction used for the vector models introducing the fields $C^{u_1 u_2}_{v_1 v_2}(Y)$ and using the additional color idempotent

$$\pi^{col} = \delta^{u_1 u_2} \delta_{v_1 v_2} \delta_{v_1}^{u_1} . \quad (9.18)$$

10 Unitarity

One of the advantages of the unfolded formulation of dynamical equations is that it makes symmetries manifest operating directly with modules of the symmetry h underlying the model in question. In particular, linearized equations on the zero-forms in the system have a form of covariant constancy conditions

$$D_0 C^I(x) = 0 \quad (10.1)$$

with $C^I(x)$ valued in some h -module V , and D_0 being a flat covariant derivative of h . Though V is not unitary in the Lorentz-covariant frame, the condition that the system admits consistent quantization compatible with unitarity demands V be complex equivalent to some unitary h -module U [114]. This makes it straightforward to analyze the pattern of

one or another unfolded system in terms of space-time symmetry algebra modules including the issue of unitarity.

Naively, the proposed Coxeter HS systems contain non-unitary sectors associated with the tensor products of the unitary HS modules with the so-called topological modules of the original HS theory, that describe non-unitary finite-dimensional modules of the AdS algebra.

For instance, in the $4d$ B_2 -HS theory, consider a rank-two field $C(Y_1, k_1; Y_2, k_2) := C_{0,1}(Y_1, Y_2)k_2$ proportional to the Klein operator k_2 of the second type and independent of k_1 . The covariant derivative with respect to AdS background connection contains commutator with the frame field $h(Y)$ with respect to the first argument and anticommutator with respect to the second. This is manifested in the form of covariant derivative

$$D_0(C_{1,2}(Y_1, Y_2)) = D^L C_{1,2}(Y_1, Y_2) - h^{\alpha\dot{\beta}} \left(y_{1\alpha} \frac{\partial}{\partial \bar{y}_1^{\dot{\beta}}} + \frac{\partial}{\partial y_1^\alpha} \bar{y}_{1\dot{\beta}} - i y_{2\alpha} \bar{y}_{2\dot{\beta}} + i \frac{\partial^2}{\partial y_2^\alpha \partial \bar{y}_2^{\dot{\beta}}} \right) C_{1,2}(Y_1, Y_2) \quad (10.2)$$

with the Lorentz covariant derivative

$$D^L A = d - \sum_{i=1,2} \left(\omega^{\alpha\beta} y_{i\alpha} \frac{\partial}{\partial y_i^\beta} + \bar{\omega}^{\dot{\alpha}\dot{\beta}} \bar{y}_{i\dot{\alpha}} \frac{\partial}{\partial \bar{y}_i^{\dot{\beta}}} \right). \quad (10.3)$$

The first term in (10.2) acts on homogeneous polynomials of Y_1 of any definite degree while the second mixes polynomials of different degrees of Y_2 . This structure is tantamount to the fact that $C_{1,2}(Y_1, Y_2)$ is valued in the tensor product of the adjoint module of the HS algebra with respect to Y_1 and twisted adjoint module with respect to Y_2 . The twisted adjoint module and its tensor products correspond to unitary particle-like states [142] and their multi-particle tensor products, respectively.

Having a form of the infinite sum of finite-dimensional modules, the zero-form fields valued in the adjoint module or even containing it as a factor as in (10.2) cannot correspond to a unitary particle-like representation. The only exception is when the field $C_{1,2}(Y_1, Y_2)$ is independent of Y_1 . Then it contains the factor of I_1 and, in accordance with the discussion of section 6, corresponds to unitary massless states of the usual HS theory in the sector of variables Y_2 .

Naively, non-singlet states in the adjoint module factors break down unitarity of the system. However, these potentially non-unitary states can be consistently truncated away. Indeed, consider for instance product of the factors $C_{1,2}(Y_1, Y_2)k_2$. There are two options. Either the arguments Y_1, k_1 and Y_2, k_2 are permuted, in which case one ends up with the unitary rank-two field, or they are not permuted. In the latter case, the product will still contain the singlet component I_1 in the adjoint factor describing nonlinear corrections to usual massless HS equations, i.e., non-singlet elements of the adjoint factor are never generated. Note that, to account properly the contribution of the Klein operators, their explicit appearance in eq. (7.4) via the factors of γ_i should be taken into account.

In the original HS theory topological fields were interpreted as moduli (coupling constants) rather than propagating degrees of freedom. Similarly, in the Coxeter HS theories,

the fields associated with non-unitary sectors resulting from the tensor products of unitary and topological modules should be interpreted as non-propagating by imposing appropriate boundary conditions in the topological sector somewhat analogously to the analysis of conformal gravity in [143]. On the other hand assigning non-zero VEVs to the fields in potentially non-unitary sectors may significantly affect the interpretation of the bulk HS theory similarly to the $3d$ HS model of [8] in which the value of the singlet topological field is interpreted as the mass parameter in the theory. Analogous suggestion for the mass-generating spontaneous breaking mechanism in multi-particle HS theories is discussed in section 12. The question whether the non-unitary fluctuations can be consistently truncated away in presence of non-zero VEVs of topological fields remains to be investigated.

A related comment is that the issue of whether a HS model is unitary or not may depend on the choice of reality conditions and vacuum solution. For instance, in the sector of zero-forms C of the $4d$ HS theory, condition (6.19) just implies that, from the perspective of the $3d$ model, the field C is valued in the tensor product of (complexified) $3d$ particle module with the topological one. Nonetheless, the resulting module of the $4d$ HS model turns out to be unitary due to imposing appropriate reality conditions compatible with the $4d$ vacuum solution.

To summarize, to make the proposed models unitary, potentially dangerous fields containing products with non-singlet factors of the adjoint HS module have to be truncated away. Remarkably, this is possible in all orders so that the reduced model turns out to be unitary. This factorization is somewhat analogous to that of non-singlet color sectors in the HS theory with colorful fields.

Let us stress that the HS models dual to the free tensor boundary theories with $O(N^p)$ broken to $O(N)^3$ at the level of observables have to be unitary [78]. This condition has little to do with potential non-unitarity of interacting tensor models with ill-defined potentials. On the other hand, it would be interesting to see whether the non-unitary boundary tensor models can be dual to B_p -HS theories with non-unitary bulk fields switched on.

11 Field frames of multi-particle theories

In this section we recall the construction of [94] for the multi-particle algebra extending it to the graded-symmetric case and analysis of idempotents.

11.1 Definition and field equations

Let A be an associative algebra with the product law $*$ and basis elements t_i obeying

$$t_I * t_J = f_{IJ}^K t_K, \quad f_{IJ}^K f_{KL}^N = f_{IK}^N f_{JL}^K. \quad (11.1)$$

In the Coxeter HS case, A is (matrix-valued extension of) the algebra of functions of $I_i, Z_i^A, Y_i^A, K_v, dZ_i^A$ as well as of π_i for the idempotent extension of section 9 (space-time differential dx^n is not included).

As a linear space, multi-particle algebra $M(A)$ is the direct sum of all graded-symmetric tensor degrees of A

$$M(A) = \sum_{n=0}^{\infty} \oplus \text{Sym} \underbrace{A \otimes \dots \otimes A}_n, \quad (11.2)$$

where the $\mathbb{Z}_2$ -grading is identified with the power in Z_i^A and Y_i^A ,

$$f(-Z_i^A, -Y_i^A) = (-1)^{\pi_f} f(Z_i^A, Y_i^A), \quad (11.3)$$

i.e., odd polynomials in Z_i^A and Y_i^A are antisymmetrized under the symbol Sym in (11.2). Let us stress that this does not contradict to the fact that Z_i^A and Y_i^A obey star-product commutation relations in A . So defined graded symmetrization accounting the total number of spinorial indices carried by Z_i^A and Y_i^A expresses the Pauli principle. (Note that in the vectorial model of [91] considered in section 6.3 only integer-spin states contribute.)

Let α_I be odd (even) elements of the Grassmann algebra associated with the odd (even) elements t_I , respectively,

$$\alpha_I \cdot \alpha_J = (-1)^{\pi_I \pi_J} \alpha_J \cdot \alpha_I. \quad (11.4)$$

The product $\cdot$ denotes usual (juxtaposition) product in the Grassmann algebra of α introduced to distinguish between the graded symmetric tensor product $\cdot$ and the commutative product in the definition of the star-product algebra underlying A of $M(A)$. The latter is the algebra of functions $F(\alpha)$ with the product law

$$F(\alpha) \circ G(\alpha) = F(\alpha) \cdot \exp. \left(\frac{\overleftarrow{\partial}}{\partial \alpha_I} f_{IJ}^N \alpha_N \frac{\overrightarrow{\partial}}{\partial \alpha_J} \right) \cdot G(\alpha). \quad (11.5)$$

An elementary computation shows that associativity of A implies associativity of the product $\circ$ of $M(A)$. (For the even case see [94].)

Algebra $M(A)$ is unital, with the unit element Id identified with $F(\alpha) = 1$. The $\mathbb{Z}_2$ -grading of $M(A)$ is induced by that of A

$$F((-1)^{\pi(\alpha)} \alpha) = (-1)^{\pi(F)} F(\alpha). \quad (11.6)$$

The graded commutativity (11.4) implies the graded symmetrization in (11.2).

The power-series expansion of the functions $F(\alpha)$

$$F(\alpha) = \sum_{n=0}^{\infty} F^{I_1 \dots I_n} \cdot \alpha_{I_1} \cdot \dots \cdot \alpha_{I_n} \quad (11.7)$$

will be assumed to have anticommuting coefficients $F^{I_1 \dots I_n}$

$$F^{I_1 \dots I_n} \cdot G^{J_1 \dots J_m} = (-1)^{(\sum_{k=1}^n \pi_{I_k})(\sum_{l=1}^m \pi_{J_l})} G^{J_1 \dots J_m} \cdot F^{I_1 \dots I_n}. \quad (11.8)$$

The same time it is convenient to demand them to commute with α_I

$$F^{I_1 \dots I_n} \cdot \alpha_J = \alpha_J \cdot F^{I_1 \dots I_n}. \quad (11.9)$$

Since components of $F(\alpha)$ are identified with the physical fields in the multi-particle HS theory, convention (11.8) expresses the Pauli principle.

Since $A \subset M(A)$ is represented by linear functions of α_I , the $*$ product acts on linear functions of α_I according to

$$\alpha_J * \alpha_K = f_{JK}^M \alpha_M. \quad (11.10)$$

Note that $A \circ A$ does not belong to A .

$M(A)$ is isomorphic to the universal enveloping algebra $U(l(A))$,

$$M(A) \sim U(l(A)), \quad (11.11)$$

where $l(A)$ is the Lie superalgebra associated with A ,

$$[t_I, t_J]_* = g_{IJ}^K t_K, \quad g_{IJ}^K = f_{IJ}^K - (-1)^{\pi_I \pi_J} f_{JI}^K. \quad (11.12)$$

This is because $M(A)$ is isomorphic to $U(l(A))$ as a linear space and

$$\alpha_I \circ \alpha_J - (-1)^{\pi_I \pi_J} \alpha_J \circ \alpha_I = g_{IJ}^K \alpha_K. \quad (11.13)$$

Concise form of the product law (11.5) is specific for the case of Lie superalgebras $l = l(A)$ associated with associative algebras A .

11.2 Frame generating functions

The following useful property of $M(A)$ is a simple consequence of eq. (11.5)

$$\forall f, g \in A: \quad \exp. f(\alpha) \circ \exp. g(\alpha) = \exp. (f \bullet g)(\alpha), \quad (11.14)$$

where

$$f \bullet g := f + g + f * g = (f + e_*) * (g + e_*) - e_* \in A \quad (11.15)$$

and e_* is the unit element of A if the latter is unital (recall that $f, g \in A$ implies that $f(\alpha)$ and $g(\alpha)$ are linear in α). Associativity of $*$ implies associativity of $\bullet$ even if A is not unital.

Let

$$G_{\cdot\nu} = \exp.(\nu) \in M(A), \quad \nu = \nu^I \cdot \alpha_I, \quad (11.16)$$

where ν^I are free Grassmann-odd (even) parameters for odd (even) α_I , which however commute with α_i

$$\nu^I \cdot \nu^J = (-1)^{\pi_I \pi_J} \nu^J \cdot \nu^I, \quad \nu^I \cdot \alpha_J = \alpha_J \cdot \nu^I. \quad (11.17)$$

Eq. (11.14) gives

$$G_{\cdot\nu} \circ G_{\cdot\mu} = G_{\cdot\nu \bullet \mu}. \quad (11.18)$$

This formula is convenient for practical computations via differentiation over ν^I with $G_{\cdot\nu}$ serving as the generating function for elements of $M(A)$.

As any universal enveloping algebra $M(A)$ is double filtered. Indeed, let V_n be the linear space of order- n polynomials of α_I . From eqs. (11.5), (11.13) it follows that

$$F_n \circ F_m \in V_{n+m}, \quad F_n \circ F_m - (-1)^{\pi(F_n)\pi(F_m)} F_m \circ F_n \in V_{n+m-1}, \quad F_m \in V_m, \quad F_n \in V_n. \quad (11.19)$$

A linear map of $M(A)$ to itself can be represented by

$$\varkappa(\alpha, a)[F.] = \varkappa\left(\alpha, \frac{\vec{\partial}}{\partial\alpha}\right)F(\alpha), \quad (11.20)$$

where derivatives $\frac{\vec{\partial}}{\partial\alpha}$ act on $F(\alpha)$. This formula can be interpreted as representing the action of the normal-ordered oscillator algebra with the generating elements α_I and $\bar{\alpha}^J$ acting on the Fock module spanned by $F(\alpha)|0\rangle$ with $\bar{\alpha}^I|0\rangle = 0$. To respect the double filtration property mapping order- n polynomials to order- n polynomials, $\varkappa(\alpha, \bar{\alpha})$ should be *filtered* obeying

$$\varkappa(\alpha, \bar{\alpha}) = \sum_{m,n=0}^{\infty} \varkappa^{I_1 \dots I_m}_{J_1 \dots J_n} \alpha_{I_1} \dots \alpha_{I_m} \cdot \bar{\alpha}^{J_1} \dots \bar{\alpha}^{J_n}, \quad \varkappa^{I_1 \dots I_m}_{J_1 \dots J_n} = 0 \quad \text{at} \quad m > n. \quad (11.21)$$

In these terms, the unit map is $\mathbf{Id} = 1$. Consider maps of the form

$$U(f) \equiv \varkappa(\alpha, \bar{\alpha}|f) = \phi \exp.(\alpha_I \cdot f^I(\bar{\alpha})) \quad (11.22)$$

with some α -independent coefficients $f^I(\bar{\alpha})$ and constant ϕ . The map $U(f)$ is filtered if $f^I(\bar{\alpha})$ is at least linear in $\bar{\alpha}$, i.e.,

$$f^I(0) = 0. \quad (11.23)$$

Interpreting $\bar{\alpha}$ as parameters, we can identify any $f(\alpha) := \sum_I f^I \alpha_I$ with $f(t) = \sum_I f^I t_I \in A$. For $U(f)$ (11.22) acting on G_ν (11.16) we obtain

$$U(f)(G_\nu) = \exp.(\tilde{f}^I(\nu) \alpha_I), \quad \tilde{f}^I(\nu) = \nu^I + f^I(\nu). \quad (11.24)$$

Hence, eq. (11.14) gives

$$U(f)(G_\nu) \circ U(g)(G_\mu) = \exp.((\tilde{f}(\nu) \bullet \tilde{g}(\mu))(\alpha)), \quad (11.25)$$

where $\tilde{f}(\nu)$ and $\tilde{g}(\mu)$ are interpreted as elements of A , i.e., $\tilde{f}(\nu) = \tilde{f}^I(\nu) \alpha_I$.

Consider map (11.22) with

$$f^I(\bar{\alpha}) \alpha_I = f(a_n), \quad a_n := \alpha_{I_1} * \dots * \alpha_{I_n} \cdot \bar{\alpha}^{I_1} \dots \bar{\alpha}^{I_n}, \quad a_1 = a = \alpha_I \cdot \bar{\alpha}^I, \quad a_0 = e_*, \quad (11.26)$$

where $f(a_n)$ is a linear function of a_n ($n \geq 1$). Such a map has the form (11.22) since $a_n \in A$. A particularly important class of maps (11.22) is represented by $\mathbf{U}_u$ of the form

$$\mathbf{U}_u(a) =: \exp[u(a) - a] :, \quad (11.27)$$

where normal ordering is with respect to α_I and $\bar{\alpha}^I$ and for $a \in A$

$$u(a) = (u_1^1 a + u_1^2 e_*) * (u_2^1 a + u_2^2 e_*)_*^{-1}, \quad (e_* + \beta a)_*^{-1} := \sum_{n=0}^{\infty} (-\beta)^n a_n \quad (11.28)$$

with $u_i^j \in \mathbb{R}$. Composition of such maps gives a map of the same class

$$\mathbf{U}_u \mathbf{U}_v = \mathbf{U}_{uv}, \quad (11.29)$$

where $(uv)_i^j = u_i^k v_k^j$ is the matrix product. The maps $\mathbf{U}_u$ with $\det|u| \neq 0$ are invertible and form Mobius group.

From eq. (11.24) it follows that

$$\mathbf{U}_u(G_{\cdot\nu}) = G_{\cdot u(\nu)}. \quad (11.30)$$

To find the composition law of $M(A)$ in the frame $T_{I_1 \dots I_n}^u$ associated with $G_{\cdot u(\nu)}$ via

$$T_{I_1 \dots I_n}^u = \frac{\partial^n}{\partial \nu^{I_1} \dots \partial \nu^{I_n}} G_{\cdot u(\nu)} \Big|_{\nu=0} \quad (11.31)$$

we compute

$$G_{\cdot\nu} \diamond G_{\cdot\mu} := \mathbf{U}_u^{-1}(G_{\cdot u(\nu)} \circ G_{\cdot u(\mu)}). \quad (11.32)$$

eq. (11.15) gives

$$G_{\cdot\nu} \diamond G_{\cdot\mu} = G_{\cdot u^{-1}(u(\nu) \bullet u(\mu))}. \quad (11.33)$$

General map (11.27) is not filtered, not respecting condition (11.23). The subgroup P of filtered maps (11.27) is represented by the lower triangular matrices

$$u_{b,\beta}(f) = bf * (e_* + \beta f)_*^{-1} \quad (11.34)$$

with the composition law

$$b_{1,2} = b_1 b_2, \quad \beta_{1,2} = \beta_2 + \beta_1 b_2. \quad (11.35)$$

Using for these transformations notation $U_{b,\beta}$ instead of U_u we observe that the unit element is

$$\mathbf{Id} = \mathbf{U}_{1,0} \quad (11.36)$$

and

$$\mathbf{U}_{b,\beta}^{-1} = \mathbf{U}_{b^{-1}, -\beta b^{-1}}. \quad (11.37)$$

The map

$$\mathbf{R} = \mathbf{U}_{-1,1} \quad (11.38)$$

is involutive

$$\mathbf{R}^2 = \mathbf{Id} \quad (11.39)$$

describing the principal antiautomorphism of $M(A)$ [94].

For the maps (11.34), the composition law (11.33) takes the form [94]

$$G_{\cdot\nu} \diamond G_{\cdot\mu} = G_{\cdot \sigma_{b,\beta}(\nu, \mu)}, \quad (11.40)$$

where

$$\sigma_{b,\beta}(\nu, \mu) = -\beta^{-1}(e_* - (e_* + \beta\mu) * (e_* - \beta(b + \beta)\nu * \mu)^{-1} * (e_* + \beta\nu)). \quad (11.41)$$

Three most important cases include

$$\sigma_{1,0}(\nu, \mu) = \nu + \mu + \nu * \mu = \nu \bullet \mu, \quad (11.42)$$

$$\sigma_{-1,1}(\nu, \mu) = \nu + \mu + \mu * \nu = \mu \bullet \nu, \quad (11.43)$$

$$\sigma_{1,-\frac{1}{2}}(\nu, \mu) = 2(e_* - (2e_* - \mu) * (4e_* + \nu * \mu)_*^{-1} * (2e_* - \nu)). \quad (11.44)$$

Here $\sigma_{1,0}(\nu, \mu)$ corresponds to the identity map reproducing the basis in $M(A)$ resulting just from the symmetrized tensor product of the framed oscillator algebras. $\sigma_{-1,1}(\nu, \mu)$ corresponds to the basis resulting from the action of principal antiautomorphism $\mathbf{R}$. The case of $\sigma_{1,-\frac{1}{2}}(\nu, \mu)$ is most interesting, reproducing the current operator algebra of [144].

Practically, the difference between the frames (11.42) and (11.44) is as follows. The product law associated with (11.42) is the original product law (11.4), from which it is easy to see that the product $\circ$ of two polynomials of α of degrees n and m gives degree- k polynomials with $\max(n, m) \leq k \leq n + m$. This implies that, with the product law (11.4), the higher-rank fields (say, rank-two) will not contribute to the equations for the lower-rank ones (say, rank-one). This property is inconsistent both with structure of the boundary operator algebra analyzed in [144] and with the idea that VEVs of the higher-rank fields can modify field equation for the lower-rank ones to yield a Higgs phenomenon. The product law associated with (11.43) has analogous properties.

However, in [94] it was shown that the multi-particle algebra in the frame (11.44) properly reproduces the boundary operator algebra inducing such a product law that the product of two polynomials of α of degrees n and m gives degree- k polynomials with $|m - n| \leq k \leq n + m$ which is consistent both with the structure of the boundary operator algebra of [144] and with the idea of higgsing by virtue of VEVs of higher-rank fields. Hence, we anticipate that frame (11.44) is most appropriate for the analysis of the multi-particle Coxeter HS theories. Let us stress that since the multi-particle algebra is infinite dimensional it is not *a priori* guaranteed that the multi-particle HS theories associated with different frames described in this section are physically equivalent.

In application to nonlinear equations of the multi-particle HS theory, system (7.4), (7.5) preserves its form with the $*$ replaced by $\diamond$

$$\mathcal{W} \diamond \mathcal{W} = -i \left(dZ^{A_n} dZ_{A_n} + F_\diamond(\mathcal{B}, \gamma_i) \right), \quad [\mathcal{W}, \mathcal{B}]_\diamond = 0, \quad (11.45)$$

where $\diamond$ is built via (11.44) from the star product of the Coxeter HS theory underlying its multi-particle extension.

11.3 Unity idempotent map

Multi-particle algebra has an important property that every idempotent $\pi \in A$ induces an idempotent $\Pi \in M(A)$. This immediately follows from formulae (11.14), (11.15) with

$$\Pi := \exp. -\pi, \quad (11.46)$$

i.e., $\Pi \circ \Pi = \Pi$ if $\pi * \pi = \pi$. In particular, application of this construction to $\pi = e$ gives the *unity idempotent* of $M(A)$

$$\Pi_e := \exp. -e. \quad (11.47)$$

Note that being built from the central element e , Π_e is central in $M(A)$.

Unity idempotent Π_e has an interesting property that it is $\circ$ -orthogonal to any $a \in A$

$$\Pi_e \circ a = a \circ \Pi_e = 0, \quad a \in A, \quad (11.48)$$

which is obvious from (11.5) and (11.47). Analogously one can check that

$$(a \cdot \Pi_e) \circ b = (a * b) \cdot \Pi_e, \quad \forall a \in A \quad (11.49)$$

and

$$a \cdot \Pi_e \circ b \cdot \Pi_e = a * b \cdot \Pi_e, \quad \forall a, b \in A. \quad (11.50)$$

This relation provides a homomorphism of A to $M(A)$. It should be stressed however that this map is not polynomial. (No homomorphism of A to $M(A)$ realized as polynomial functions of A exists.)

Relation (11.50) has the consequence that

$$\Pi_e^1 := e \cdot \Pi_e \quad (11.51)$$

is also an idempotent

$$\Pi_e^1 \circ \Pi_e^1 = \Pi_e^1. \quad (11.52)$$

From (11.49) it follows

$$\Pi_e^1 \circ a = a \cdot \Pi_e, \quad \forall a \in A. \quad (11.53)$$

The consequence of this construction is that if A had a Lie (super)algebra l associated with some $t_i \in A$, then $M(A)$ admits a symmetry algebra $l \oplus l$ generated by t_i and $t_i \cdot \Pi_e$ with t_i generating the diagonal subalgebra $l \subset l \oplus l$. This fact may have important implications for the space-time interpretation of the string-like HS theories discussed in the next section.

12 Interpretation

12.1 Strings and tensor models

Coxeter HS equations and their multi-particle extensions have a number of features indicating their relation to string-like models and their further tensor-like extensions anticipated to be holographic duals of the boundary tensor sigma-models considered in [77, 78].

First of all, the spectra of fields described by the rank- p Coxeter HS models with $p > 1$ are far larger than of usual rank-one HS gauge theories. This is most obvious from the fact that the zero-form fields $C(Y_\alpha^n; k_v)$ depend on p copies of the oscillators Y_α^n as well as on the Klein operators k_v associated with all roots of the underlying Coxeter system (modulo identification $k_{-v} = k_v$). This enlargement of the spectrum is in qualitative agreement with the observation of [78, 81] that the spectrum of the boundary operators in tensor boundary models is far richer than in the vector sigma-model. Most importantly, however, the Klein operators generating Coxeter reflections effectively permute the arguments of the elementary master fields like $C(Y_1, \dots, Y_p; k|x)$.

In the absence of the Klein operators, products of the fields would correspond to the tensor product of p copies of star-product algebras valued in Mat_{N+M} . However, in the process of solving Coxeter HS equations, relations of the type (3.22) will permute the variables Y_n and Y_m with different n and m , not affecting the matrix indices a_n, b_n

and a_m, b_m . For instance, in the case of $p = 2$, the star product of two master fields $C(Y_1, Y_2|x)k_{12}$ gives

$$(C(Y_1, Y_2|x)k_{12}) * (C(Y_1, Y_2|x)k_{12}) = C(Y_1, Y_2|x) * C(Y_2, Y_1|x). \quad (12.1)$$

As a result, nonlinear corrections to the $p = 2$ system will contain products of elementary strings of master fields with repeatedly permuted arguments Y_1 and Y_2

$$C_{string}^n := \underbrace{C(Y_1, Y_2|x) * C(Y_2, Y_1|x) * C(Y_1, Y_2|x) \dots}_n. \quad (12.2)$$

Such strings are analogous to the product of elementary matrix factors and can be identified with the letters of an infinite Alphabet with $n = 0, 1, \dots, \infty$. These are analogues of the single-trace operators in the ordinary *AdS/CFT* dictionary. General product of operators decomposes into products of elementary letters analogous to the multi-trace operators. For instance, operator (12.1) as well as $C(Y_1, Y_2|x)$ are single-trace while $C(Y_1, Y_2|x) * C(Y_1, Y_2|x)$ is double-trace. Thus the spectrum of operators of the $p = 2$ HS model is analogous to that of String Theory with the infinite set of Regge trajectories. More precisely, to interpret these nonlinear combinations of operators as corresponding to elementary states of some string-like model one has to consider a multi-particle HS model associated with the infinite set of elementary B_2 systems

$$sym(B_2 \times B_2 \times \dots), \quad (12.3)$$

where the graded symmetrization is with respect to all elementary B_2 factors in the sense that all respective master fields $C(Y_1^1, Y_2^1, k_v^1; Y_1^2, Y_2^2, k_v^2; Y_1^3, Y_2^3, k_v^3; \dots |x)$ are demanded to be graded symmetric under the exchange of the variables Y_1^a, Y_2^a, k_v^a associated with different factors of B_2 (i.e., index a).

The $p = 2$ Coxeter HS theory has deep parallels with the analysis of stringy HS models by Gaberdiel and Gopakumar [70]–[74]. In particular, the master fields of this theory depend on the two sets of oscillators $Y_{1,2}^A$ which is close to saying that the stringy HS theory is based on two different HS symmetry algebras being one of the conclusions of Gaberdiel and Gopakumar. It should be stressed that these algebras do not commute with each other if the stringy coupling constant identified with the vacuum expectation value of $F_{2*}(B)$ in (6.16) is non-zero inducing nontrivial Cherednik-like deformation (5.10) of the oscillator commutation relations. (Note that for conformal models considered by Gaberdiel and Gopakumar the VEV of $F_{1*}(B)$ must be non-zero.)

Usual HS theory and its multi-particle extension result from the analogous construction applied to B_1 . Rank- p tensor HS theories and their multi-particle extensions result analogously from B_p with $p > 2$ as well as from other higher-rank Coxeter root systems. Clearly the pattern of elementary operators of the $p > 2$ higher-rank Coxeter HS models will increase enormously compared to the $p = 2$ string-like models which is in agreement with [78, 81].

To make the holographic correspondence more explicit it is necessary to compare the spectrum of singlet boundary operators with that of the singlet sector of the tensor bulk

model. As explained in section 6, due to using framed algebras, these spectra do match at least in the massless sector containing usual massless states dual to the boundary currents in the holographic interpretation. The pattern of the full spectrum of the higher-rank Coxeter HS theories remains to be elaborated.

Let us stress that the Coxeter HS theories proposed in this paper are formulated in the anti-de Sitter space of appropriate dimension. In particular, this is true for the stringy B_2 -HS models. Hence the construction of this paper is different from that of genuine String Theory formulated in the flat rather than, say, AdS_{10} space. In fact, the reason why it is difficult to formulate String Theory in AdS_d is analogous to that discussed in section 4 for HS theory: a naive attempt to deform the string spectrum to AdS would immediately lead to infinite vacuum energy since in this case all string modes have to contribute to the momentum generators to ensure that their commutator is proportional to the Lorentz ones

$$[P_n, P_m] \sim -\Lambda M_{nm} \quad (12.4)$$

that act on all modes. (Recall that the usual string theory momentum operator is built from the zero modes [19] which is not possible in AdS.) Hence, the extension to the framed oscillator algebra can also be crucial to reach a string-like theory in AdS.

An important related feature of the B_p -HS models with $p \geq 2$, is that they have two independent coupling constants instead of one in the usual B_1 -HS theories. These are the coefficients $\eta_{1,2}$ in the linearized parts of the functions $F_{1,2*}(B)$ in (6.12). The function F_{1*} is analogous to that of the rank-one (i.e., B_1) HS theory. The respective term is important for the proper interpretation of the equations in AdS_4 leading to the so-called central on-shell theorem of [142] representing the unfolded equations for massless fields in AdS_4 . The function F_{2*} first appears in the rank-two stringy model and, containing the Klein operators that permute different species of Y -variables, is responsible for the appearance of single-trace-like strings of operators. The presence of two different coupling constants is anticipated to have important implications for establishing relation with usual string theory in flat space. One option is that to reach the latter theory one has to take the limit with $\eta_2/\eta_1 \rightarrow \infty$ which in turn may select String Theory in critical dimension.

It should be stressed that it is a distinguishing property of the $B_p \sim C_p$ Coxeter group that the related Cherednik system has two types of coupling constants responsible for HS and stringy effects. It is not clear, in particular, whether the A_p and D_p systems admit a meaningful HS interpretation since, due to the absence of usual Klein operators reflecting signs of different species of oscillators Y_A^i , the free field equations unlikely have a room for massless fields.

The full-fledged string theory is conjectured to be related to a rank-two multi-particle B_2 Coxeter HS model. As explained in section 11 following [94], the form of the multi-particle algebra significantly depends on the chosen basis. It remains to be analysed to which extent the formulations in different frames are equivalent to each other. More precisely, different frames of the Coxeter HS models of finite rank are equivalent. However, for the multi-particle extensions the respective frame changes are in the infinite-dimensional space and different frames may not be equivalent. The details of description of multi-

particle algebras in different frames are presented in section 11 where the construction of [94] is extended to the graded-symmetric case expressing the Pauli principle.

This question may be related to the fundamental issue of the breaking of HS symmetries in the Coxeter HS theory. Indeed, it is plausible to expect [145, 146] that spontaneous breaking of HS symmetries resulting in the appearance of massive HS fields is only possible in string-like models with the infinite number of Regge trajectories. As conjectured in this paper the appropriate Coxeter HS theory is the B_2 multi-particle theory. The goal is to break down HS symmetries to usual space-time symmetry of AdS or Minkowski type. The simplest way to do so is to let a rank-two zero-form topological field $B(Y_1; Y_2)$ acquire a non-zero VEV

$$B_0 = Y_{iA} \cdot Y_j^A (\gamma \sigma_1^{ij} + \rho \delta^{ij}), \quad (12.5)$$

where the Pauli matrix σ_1^{ij} and δ^{ij} are the two symmetric matrices invariant under the exchange $1 \leftrightarrow 2$ in i, j . (Note that this proposal is somewhat reminiscent of the Girardello-Porrati-Zaffaroni mechanism [147].) Such B_0 preserves the AdS symmetry but breaks down the HS one. Note that so defined B_0 is nonzero because elementary oscillators Y^{iA} are anticommuting with respect to the product $\cdot$ (11.4). It remains to be seen how this VEV would affect (deform) the structure of the AdS modules (field equations) of the originally massless fields. Spontaneous symmetry breaking would correspond to the mixture between the originally massless rank-one particle module and the rank-two current module. In the infinite-dimensional case of the multi-particle algebra the mechanism of such mixing may depend on the choice of the frame in the multi-particle algebra. As argued in section 11.2, it is natural to anticipate that the proper frame is defined by (11.44), being associated with the boundary current algebra of [144]. Detailed analysis of this issue is one of the most urgent problems on the agenda.

12.2 Space-time metamorphoses

The formulation of HS equations in the unfolded form expressing the space-time exterior derivative d via the values of other fields like in equations (2.3)–(2.5) on the fields W , B and S subjected to the constraints (2.6) and (2.7) allows us to unify in the same framework the systems that live in space-times of different dimensions [92, 124, 128]. This is achieved by letting the de Rham derivative d be defined in the infinite-dimensional space-time. Then the physical space-times like Minkowski or AdS appear when the background (vacuum) connection $W_0(x)$ of the respective symmetry group G is nontrivial with non-degenerate frame-like components along $x_{\parallel}^n$ associated with the translation (or transvection) generators. In that case the coordinates $x_{\parallel}^n$ are observable while the rest ones $x_{\perp}^n$ are not: in the absence of components of forms $W_0(x)$ along $x_{\perp}^n$ the respective unfolded equations treated perturbatively would imply that, locally, all other differential form fields in the system are either $x_{\perp}^n$ -independent zero-forms or pure gauge (i.e., exact) $p > 0$ -forms. By this mechanism, the system formulated in the infinite-dimensional space-time is visualized by the coordinates $x_{\parallel}^n$ representing one or another G -invariant space described by $W_0(x_{\parallel}^n)$.

In [124] it was observed that the higher-rank fields in lower dimension can be interpreted as elementary fields in higher dimension. In the framework of Coxeter HS theory

this phenomenon acquires a direct realization as we explain now. Consider a spinorial Coxeter HS theory. In this case, the rank-one sector consists of the fields depending on a single spinor variable Y^A with $A = 1, \dots, M$ ($M = 2$ in the $3d$ model and $M = 4$ in the $4d$ model). Bilinears of Y^A form generators of $sp(M)$ with respect to the star product. This is the AdS_4 algebra $sp(4)$ in the $4d$ model and a $sp(2)$ half of the AdS_3 algebra in the $3d$ model (to be doubled via introducing the Clifford element ψ_1 [8]). The diagonal embedding of the star-product generators of $sp^{\text{diag}}(M)$ into a rank- p system is

$$t_{AB} = \sum_{i=1}^p \{Y_{Ai}, Y_{Bi}\}_* . \quad (12.6)$$

Vacuum connection $W_0(x)$ that describes usual space-time geometry like AdS_4 or AdS_3 is a flat $sp(M)$ connection ($sp(2) \oplus sp(2)$ in the $3d$ case).

Let $\Omega = A, i$. Operators

$$T_{\Omega\Lambda} = \{Y_\Omega, Y_\Lambda\}_* * I_1 * \dots * I_p \quad (12.7)$$

are generators of $sp(pM)$. The diagonal embedding $sp^{\text{diag}}(M)$ into $sp(pM)$ is realized by the generators

$$t'_{AB} = t_{AB} * I_1 * \dots * I_p . \quad (12.8)$$

The same time the $sp(M)$ generated by t_{AB} (12.6) acts diagonally on the full framed Cherednik algebra. Evidently,

$$(t_{AB} - t'_{AB}) * T_{\Omega\Lambda} = T_{\Omega\Lambda} * (t_{AB} - t'_{AB}) = 0 . \quad (12.9)$$

Let $\tau_{\Omega'\Lambda'}$ form a basis of $sp(pM)/sp^{\text{diag}}(M)$. Due to (12.9), it is possible to identify the space-time components of the flat connections of t_{AB} and t'_{AB} demanding them to have non-zero components along coordinates $x_{\parallel}^n$ of the space-time $\mathcal{M}$ originally associated with the $sp(M)$ symmetry. Now it is possible to choose a flat connection $W'_0(X)$ on some $sp'(pM)$ -invariant space-time $\mathcal{M}'$ with local coordinates X in such a way that the pushforward of $W'_0(X)$ to $\mathcal{M}$ gives $W_0(x)$. This makes it possible to treat the rank- p fields in $\mathcal{M}$ as elementary fields in $\mathcal{M}'$.

One can proceed analogously with the additional species of oscillators in the multi-particle algebra construction. From the analysis of section 11.3 it follows that the multi-particle B_2 -HS model has $sp(8) \oplus sp(8)$ as a finite-dimensional symmetry. The natural homogeneous space that admits this symmetry is the group manifold $Sp(8)$ which is 36-dimensional. Moreover, the conformal-like symmetry of $Sp(8)$ is $Sp(16)$ [114, 148, 149].

This has an interesting consequence that the usual space-time interpretation of the Coxeter HS theories and their multi-particle extensions is likely to have maximal Minkowski dimension ten. This follows from the analysis of the $sp(2M)$ invariant theories in [126, 127] where it was shown how usual Minkowski space-time dimensions emerges from the $sp(2M)$ -invariant equations with the conclusion that the maximal Minkowski space known to emerge from the $sp(16)$ -invariant equations is ten dimensional. Though it is tempting to conjecture that this is how the Superstring dimension ten emerges from the Coxeter HS theory, details

of this phenomenon need further investigation. In particular, it would be interesting to see whether this phenomenon is related to the twistor-like transform in ten dimensions introduced by Witten in [130].

Note that the mechanism explained in this section is based on the extension of the space-time symmetries due to appearance of the additional species of oscillators in the higher-rank Coxeter HS theories and/or multi-particle extension of HS theories. As such it is not quite the same as the holographic correspondence within the idempotent construction of section 9 where the symmetry algebra g remains the same but the g -module pattern changes depending on the idempotent sector in question.

13 Conclusion

We propose a class of Coxeter HS models conjectured to underly a symmetric phase of String Theory (Coxeter group B_2) and its further tensor-like extensions (Coxeter group B_p). These models contain two coupling constants one of which is responsible for stringy effects (absent in the conventional HS theory) and exhibit a number of interesting parallelisms with String Theory. In particular, consistency of the holographic interpretation of the boundary matrix-like model demands the latter to have $N = 4$ SUSY.

The main idea of our construction was to find a formally consistent extension of the known HS gauge systems [5, 8, 91] possessing a richer spectrum and having a room for massless fields of all spins including spin-two gravitational field. The former goal was reached via extension of the Coxeter groups Z_2 or $Z_2 \times Z_2$ generated by the Klein elements of the models of [5, 8, 91] to any Coxeter group including the most important case of B_p . To let massless fields be present in the model this construction was further extended to the framed algebras containing additional idempotent elements.

Though the main emphasize in this paper was on the spinorial HS models somewhat analogous to Green-Schwarz superstring, bosonic Coxeter HS theories of vectorial type considered in section 6.3 and their fermionic counterparts to be elaborated are also of interest as analogues of the bosonic and fermionic strings. (For more detail see [134].) We believe that formal consistency in presence massless fields in the spectrum is so restrictive that it hardly leaves a room for consistent HS models beyond the list presented in this paper, supplemented by the Coxeter extensions of to be constructed fermionic generalizations of the model of [91] in any dimension. In particular it determines the structure of extended HS symmetries as well as the field pattern.

Many of the important aspects of the proposed models, such as detailed analysis of field spectra, spontaneous breaking of HS symmetries, holographic interpretation, proper space-time interpretation, the choice of appropriate frame in the multi-particle HS theories and others were only briefly sketched in this paper, demanding a more detailed study delegated to the future as well as some other issues including, for instance, the analysis of locality along the lines of [17, 18].

An interesting open question is to give an interpretation to the Coxeter HS theories based on the Coxeter groups different from B_p which were of most interest in this paper. In particular, it would be interesting to study more carefully the case of Dihedral group

$I_2(n)$ which is the symmetry group of the n -gon on the plane. In this relation it should be noted that the case of B_2 is special due to the isomorphism $B_2 \sim I_2(4)$. An interesting feature of the $B_2 \sim I_2(4)$ HS theory is that, as shown recently in [150, 151], for certain linear relations between the coupling constants ν_1 and ν_2 the respective Cherednik algebra acquires ideals. As a result, on this locus of the plane of coupling constants some states in the system should decouple. It would be interesting to investigate this phenomenon in detail, especially in the context of unitarity.

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