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Hidden isometry of "T-duality without isometry"

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Abstract: We study the T-dualisability criteria of Chatzistavrakidis, Deser and Jonke [3] who recently used Lie algebroid gauge theories to obtain sigma models exhibiting a "T-duality without isometry". We point out that those T-dualisability criteria are not written invariantly in [3] and depend on the choice of the algebroid framing. We then show that there always exists an isometric framing for which the Lie algebroid gauging boils down to standard Yang-Mills gauging. The "T-duality without isometry" of [3] is therefore nothing but traditional isometric non-Abelian T-duality in disguise.

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Contents

1 Introduction

T-dualisability is a rare property of non-linear sigma-models and it is not known what necessary conditions must be imposed on a target space metric G, and closed 3-form field H, such that the corresponding sigma-model has a T-dual with $(\widehat{G}, \widehat{H})$. On the other hand, several sufficient conditions are known, giving various T-dualities like the Abelian one [9, 16] or non-Abelian one [1, 5–7], both in turn included as special cases of Poisson-Lie T-duality [10, 11]. Chatzistavrakidis, Deser and Jonke (CDJ in what follows) recently proposed a new set of sufficient conditions which, they claimed, would give rise to new examples of T-dual pairs [3]. Their dualisability conditions appear much less restrictive than those previously described in T-duality research. It is the purpose of the present work to show that, in reality, they are not less restrictive as they give rise to the same duality pattern as that of traditional non-Abelian T-duality.

The proposal of CDJ for dualising a given sigma model on a target M, is an extension of the Roček-Verlinde approach [5, 15], which amounts to the introduction of an intermediate gauge theory yielding the T-dual pair of sigma models upon eliminating different sets of fields. It was traditionally thought that the Roček-Verlinde intermediate gauge theory can be constructed only if the background of the sigma-model is isometric with respect to the action of the Lie algebra $\mathfrak{g}$ of the gauge group. However, CDJ have argued that more general gaugings are possible if one uses the recently introduced Lie algebroid gauge theory [12, 14, 17]. The construction of the Lie algebroid generalisation of the Roček-Verlinde intermediate gauge theory requires the existence of a Lie algebroid bundle Q, over the target M, as well as a fixed connection ∇^ω on Q compatible with the sigma model

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background. As CDJ show, the compatibility of ∇^ω, G and H can be expressed in a particularly simple way for exact 3-form backgrounds $H = dB$ where it reads:

$$\mathcal{L}_{\rho(e_a)}G = \omega^b{}_a \vee \iota_{\rho(e_b)}G, \qquad \mathcal{L}_{\rho(e_a)}B = \omega^b{}_a \wedge \iota_{\rho(e_b)}B\,. \tag{1.1}$$

Here e_a form local frames of the Lie algebroid, the Lie derivatives are taken with respect to the anchored frames $\rho(e_a)$, the symbols $\vee$ and $\wedge$ stand respectively for the symmetrised and anti-symmetrised direct products of 1-forms on M, and the 1-forms $\omega^b{}_a$ are defined by the relations

$$\nabla^\omega e_a := \omega^b{}_a \otimes e_b\,. \tag{1.2}$$

Since the choice of the connection ∇^ω seems largely arbitrary, it may appear from (1.1) that a vast set of non-isometric backgrounds could be gauged, thus producing a new and rich T-duality pattern. However, as we shall argue in this paper, this is not the case. The simplest way to understand what is happening is to realise that the compatibility conditions (1.1), as given by CDJ in ref. [3] are not written invariantly; upon a local changes of frames $e'_a = P^b{}_a e_b$, $P^b{}_a \in C^\infty(M)$, they change to

$$\mathcal{L}_{\rho(e'_a)}G = \omega'^b{}_a \vee \iota_{\rho(e'_b)}G, \qquad \mathcal{L}_{\rho(e'_a)}B = \omega'^b{}_a \wedge \iota_{\rho(e'_b)}B\,, \tag{1.3}$$

where

$$\nabla^{\omega'} e'_a := \omega'^b{}_a \otimes e'_b\,. \tag{1.4}$$

The components of the connection form $\omega^b{}_a$ transform non-homogeneously upon a change of the framing, and we may naturally question whether there exists a distinguished frame $\hat{e}_a$ for which they all vanish. This question can be answered in the affirmative, and this fact follows from the Lie algebroid gauge invariance of the Roček-Verlinde intermediate gauge theory. It is therefore always possible to write down an equivalent version of the CDJ compatibility conditions (1.1) in the standard isometric form

$$\mathcal{L}_{\rho(\hat{e}_a)}G = 0\,, \qquad \mathcal{L}_{\rho(\hat{e}_a)}B = 0\,. \tag{1.5}$$

Moreover, the gauge invariance of the intermediate gauge theory also requires that the structure functions $\hat{C}^c{}_{ab}$ defined by the Lie algebroid brackets

$$[\hat{e}_a, \hat{e}_b] \equiv \widehat{C}^c{}_{ab}\; \hat{e}_c, \tag{1.6}$$

be constants, and we thus recover the standard intermediate Yang-Mills gauge theory leading to traditional non-Abelian T-duality [5, 7].

The plan of our paper is as follows: in section 2 we expose some useful preliminary background on traditional non-Abelian T-duality. In section 3 we review the "T-duality without isometry" proposal of CDJ and detail the field redefinitions which reproduce standard non-Abelian T-duality. In section 4 we work out the case of non-exact 3-form background H. In section 5 we provide a geometric interpretation of the field redefinitions from the invariant perspective of Lie algebroid gauge theory. In section 6, we illustrate a few examples where, by simple field redefinitions, the traditional isometric Roček-Verlinde gauge theory may look like a non-trivial Lie algebroid gauge theory. In particular, we unmask the "non-isometric T-duality" example of CDJ presented in [3]. Finally, we end with a short discussion.

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2 Preliminaries on the non-Abelian T-duality

To set up some technical and notational background, as well as remind the reader of the gauging approach to T-duality, we review traditional non-Abelian T-duality obtained by the Roček-Verlinde procedure [5–7]. We first restrict our attention to backgrounds for which $H = dB$ is an exact 3-form, postponing the study of cohomologically non-trivial backgrounds to section 4.

Let a Lie group G act from the right on the target manifold M, let T_a be a basis of the Lie algebra $\mathfrak{g} \equiv \mathrm{Lie}(\mathsf{G})$, and v_a the set of vector fields on M corresponding to the infinitesimal right actions of the elements T_a. The Lie derivatives $\mathcal{L}_{v_a} v_b$ then satisfy

$$\mathcal{L}_{v_a} v_b = [v_a, v_b] = C^c{}_{ab}\, v_c \,, \tag{2.1}$$

where $C^c{}_{ab}$ are the structure constants of $\mathfrak{g}$ in the basis T_a.

Denoting the (Lorentzian) cylindrical world-sheet by Σ and introducing coordinates X^i on M, we write the sigma model action with the background metric G and the 3-form field $H = dB$ as

$$S(X^i) = \frac{1}{2} \int_\Sigma dX^i \wedge \left(G_{ij} * dX^j + B_{ij} dX^j\right) . \tag{2.2}$$

Here d denotes the de Rham differential, $*$ ($*^2 = 1$) the Hodge star on the world-sheet Σ, and the X^i are viewed as functions on Σ describing a string moving in M.

If the Lie derivatives of the metric and the B field vanish

$$\mathcal{L}_{v_a} G = 0, \qquad \mathcal{L}_{v_a} B = 0 \,, \tag{2.3}$$

then the sigma model (2.2) can be gauged in the standard Yang-Mills way. This means that one introduces a world-sheet one-form A valued in the Lie algebra $\mathfrak{g} \equiv \mathrm{Lie}(\mathsf{G})$, a world-sheet scalar η valued in the dual $\mathfrak{g}^*$, and the gauged action

$$S(X^i, A, \eta) = \frac{1}{2} \int_\Sigma DX^i \wedge \left(G_{ij} * DX^j + B_{ij} DX^j\right) + \int_\Sigma \langle \eta, F(A)\rangle \,. \tag{2.4}$$

Here $\langle \cdot\,, \cdot\rangle$ is the canonical pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$, $F(A)$ is the standard Yang-Mills field strength

$$F(A) := dA + A \wedge A \equiv \left(dA^a + \frac{1}{2} C^a{}_{bc}\, A^b \wedge A^c\right) T_a \,, \tag{2.5}$$

and DX^i are the covariant derivatives

$$DX^i := dX^i - v^i_a A^a \,. \tag{2.6}$$

If the isometry conditions (2.3) hold, the action (2.4) is gauge invariant with respect to the following local infinitesimal gauge transformations:

$$\delta_\epsilon X^i = v^i_a \epsilon^a, \quad \delta_\epsilon A = d\epsilon + [A, \epsilon] \equiv \left(d\epsilon^a + C^a{}_{bc} A^b \epsilon^c\right) T_a, \quad \delta_\epsilon \eta = -\,\mathrm{ad}^*_\epsilon\, \eta \equiv -C^c{}_{ab} \eta_c \epsilon^b T^{*a} \,. \tag{2.7}$$

Here ϵ is a function on the world-sheet valued in $\mathfrak{g}$, and ad^* denotes the co-adjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$.

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Varying the Lagrange multiplier η forces the field strength to vanish, thereby imposing that the gauge field A be pure gauge $A = -dg\, g^{-1}$, and the action (2.4) becomes that of the original model (2.2)

$$S(X^i, -dg\; g^{-1}, \eta) = \frac{1}{2}\int_\Sigma dY^i \wedge \left(G_{ij} * dY^j + B_{ij} dY^j\right) . \tag{2.8}$$

Here $Y^i = {}^g X^i$ which means that Y^i is obtained from X^i by applying the gauge transformation g. If instead, we eliminate the non-dynamical fields A from (2.4), as well as fixing the gauge, we obtain the dual sigma model. The exact form of the dual action depends on several factors; like whether or not the G action on M is free, or the presence of so-called spectator fields. We do not give the complete account of all possible cases (the interested reader can find it in [1, 2, 5, 8]), because our concern is different. We will show that the Lie algebroid generalisation of the intermediate gauge theory proposed by CDJ in [3] can be rewritten, using appropriate field redefinitions, in the standard non-Abelian T-duality form (2.4). It follows that the CDJ proposal cannot describe more general T-duality patterns than that of traditional non-Abelian T-duality.

3 CDJ gauge theory

CDJ generalised the structural data M, G, B, v_a considered in section 2 by including an additional matrix valued 1-form $\omega^a{}_b \equiv \omega^a{}_{bi} dX^i$ on M, and by promoting the structure constants $C^c{}_{ab}$ to functions on M. The action of the intermediate gauge theory is then proposed to be the following expression:[1]

$$S(X^i, A, \eta) = \frac{1}{2}\int_\Sigma DX^i \wedge \left(G_{ij} * DX^j + B_{ij} DX^j\right) + \int_\Sigma \langle \eta, F_\omega(A, X)\rangle , \tag{3.1}$$

where the covariant derivatives DX^i are as before (cf. (2.6)) and the generalized field strength $F_\omega(A, X)$ (borrowed from ref. [12]) is given by the formula

$$F^a_\omega(A, X) := dA^a + \frac{1}{2} C^a{}_{bc}(X) A^b \wedge A^c - \omega^a{}_{bi} A^b \wedge (dX^i - v^i_c A^c) . \tag{3.2}$$

CDJ then argued that a necessary condition for the infinitesimal gauge invariance of the theory (3.1), required for T-duality applications, is given by

$$\mathcal{L}_{v_a} G = \omega^b{}_a \vee \iota_{v_b} G , \qquad \mathcal{L}_{v_a} B = \omega^b{}_a \wedge \iota_{v_b} B . \tag{3.3}$$

The infinitesimal gauge transformations themselves depend on $\omega^a{}_b$ and they read (cf. [3])

$$\begin{aligned} \delta_\epsilon X^i &= v^i_a \epsilon^a , \\ \delta_\epsilon A^a &= d\epsilon^a + C^a{}_{bc} A^b \epsilon^c + \omega^a{}_{bi}(dX^i - v^i_a A^a)\epsilon^b , \\ \delta_\epsilon \eta_a &= \left(-C^c{}_{ab}\eta_c + v^i_a \omega^c{}_{bi}\eta_c\right)\epsilon^b . \end{aligned} \tag{3.4}$$

[1] The formulas appearing here are for exact 3-form $H = dB$, and are equivalent to the equations appearing in [3] under this assumption. Non-trivial H, as considered in [3], is treated in section 4.

We point out that the conditions (3.3) are not sufficient to guarantee the gauge invariance. This can be seen by evaluating the variation $\delta_\epsilon F_\omega(A,X)$ of the field strength:

$$\delta_\epsilon F^a_\omega(A,X) = (d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b)\epsilon^b + O(A) + O(A^2)\,, \tag{3.5}$$

where $O(A)$ and $O(A^2)$ stand for the terms linear and quadratic in A, respectively. The variation $\delta_\epsilon\langle\eta, F_\omega(A,X)\rangle$ is required to vanish,

$$\begin{aligned} 0 = \delta_\epsilon\langle\eta, F_\omega(A,X)\rangle &= \langle\delta_\epsilon\eta, F_\omega(A,X)\rangle + \langle\eta, \delta_\epsilon F_\omega(A,X)\rangle \\ &= \eta_a(d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b)\epsilon^b + O(A) + O(A^2)\,. \end{aligned} \tag{3.6}$$

All three terms must vanish separately which means that the conditions (3.3) of the gauge invariance have to be supplemented by, at least, one other one

$$d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = 0\,. \tag{3.7}$$

The condition (3.7) is easy to solve since it has the Maurer-Cartan form. Therefore, there must exist a matrix $K^a{}_b(X)$ such that

$$\omega^a{}_b = (K^{-1})^a{}_c\, dK^c{}_b\,. \tag{3.8}$$

It turns out that the conditions (3.3), together with (3.7), are necessary but still not sufficient to guarantee the gauge invariance. In order to find the full set of conditions to be imposed, we perform the following field redefinitions:

$$\widehat{A}^a = K^a{}_b A^b\,, \qquad \hat{\eta}_a = \eta_b (K^{-1})^b{}_a\,. \tag{3.9}$$

In terms of the new fields X^i, $\widehat{A}^a$ and $\hat{\eta}_a$, the action (3.1) of CDJ acquires the following form:

$$S(X^i, \widehat{A}, \hat{\eta}) = \frac{1}{2}\int_\Sigma DX^i \wedge \left(G_{ij} * DX^j + B_{ij} DX^j\right) + \int_\Sigma \hat{\eta}_a \left(d\widehat{A}^a + \frac{1}{2}\widehat{C}^a{}_{bc}(X)\widehat{A}^b \wedge \widehat{A}^c\right)\,, \tag{3.10}$$

where

$$\widehat{C}^a{}_{bc} := K^a{}_d \left((K^{-1})^e{}_b (K^{-1})^f{}_c C^d{}_{ef} + (K^{-1})^e{}_b v^i_e \partial_i (K^{-1})^d{}_c - (K^{-1})^e{}_c v^i_e \partial_i (K^{-1})^d{}_b\right) \tag{3.11}$$

and

$$DX^i = dX^i - v^i_a A^a = dX^i - \hat{v}^i_a \widehat{A}^a, \qquad \hat{v}^i_a := v^i_b (K^{-1})^b{}_a\,. \tag{3.12}$$

Furthermore, upon the field redefinitions $(A,\eta) \to (\widehat{A},\hat{\eta})$, the gauge transformation formulas (3.4) simplify[2]

$$\begin{aligned} \delta_{\hat{\epsilon}} X^i &= \hat{v}^i_a \hat{\epsilon}^a\,, \\ \delta_{\hat{\epsilon}} \widehat{A}^a &= d\hat{\epsilon}^a + \widehat{C}^a{}_{bc}(X)\widehat{A}^b \hat{\epsilon}^c\,, \\ \delta_{\hat{\epsilon}} \hat{\eta}_a &= -\widehat{C}^c{}_{ab}(X)\hat{\eta}_c \hat{\epsilon}^b\,, \end{aligned} \tag{3.13}$$

[2] Our field redefinitions are in some sense inverse to those considered in [13] in the different context of the Yang-Mills-Higgs gauge theory.

where

$$\hat{\epsilon}^a = K^a{}_b\,\epsilon^b\,. \tag{3.14}$$

Remarkably, upon the field redefinitions, the gauge invariance conditions (3.3) guaranteeing the gauge invariance of the first term in the action (3.10) become the isometry conditions

$$\mathcal{L}_{\hat{v}_a} G = 0, \qquad \mathcal{L}_{\hat{v}_a} B = 0, \tag{3.15}$$

which can be established directly from (3.3):

$$K^a{}_b\,\mathcal{L}_{\hat{v}_a} G = K^a{}_b \mathcal{L}_{v_c (K^{-1})^c{}_a} G = \mathcal{L}_{v_b} G - (K^{-1})^a{}_c\, dK^c{}_b \vee \iota_{v_a} G = \mathcal{L}_{v_b} G - \omega^a{}_b \vee \iota_{v_a} G = 0\,,$$

and similarly,

$$K^a{}_b\,\mathcal{L}_{\hat{v}_a} B = K^a{}_b \mathcal{L}_{v_c (K^{-1})^c{}_a} B = \mathcal{L}_{v_b} B - (K^{-1})^a{}_c dK^c{}_b \wedge \iota_{v_a} B = \mathcal{L}_{v_b} B - \omega^a{}_b \wedge \iota_{v_a} B = 0\,.$$

Now we turn to the second term in the action (3.10), i.e. the one containing the Lagrange multiplier η. It is easy to calculate the variation of the action (3.10) with respect to the gauge transformations (3.13) provided that the isometry conditions (3.15) are fulfilled:

$$\delta_{\hat{\epsilon}} S(X^i, \widehat{A}, \hat{\eta}) = \int_\Sigma \hat{\eta}_a (\partial_i \widehat{C}^a{}_{bc}) \hat{\epsilon}^c DX^i \wedge \widehat{A}^b\,. \tag{3.16}$$

The gauge invariance thus require that the structure functions $\widehat{C}^a{}_{bc}$ be constants.[3] We observe that our field redefinitions (3.9) permit us to rewrite the action (3.1) of CDJ gauge theory precisely in the Roček-Verlinde form (2.4) corresponding to isometric non-Abelian T-duality. Consequently, the approach of CDJ cannot give any T-duality pattern not already contained in the traditional non-Abelian T-duality story.

4 Inclusion of non-exact 3-form background H

Let H be a closed 3-form on the target manifold M which is not exact. In this case H cannot be written globally as dB for any 2-form field B but we can introduce an auxiliary 2-form field C on M and write down the following action:[4]

$$\begin{aligned} S(X^i, A, \eta) = &\frac{1}{2}\int_\Sigma DX^i \wedge \big(G_{ij} * DX^j + C_{ij} DX^j\big) + \int_\Sigma \langle \eta, F_\omega(A, X)\rangle \\ &+ \frac{1}{6}\int_{\Sigma_3} H_{ijk} dX^i \wedge dX^j \wedge dX^k - \frac{1}{2}\int_\Sigma C_{ij} dX^i \wedge dX^j\,. \end{aligned} \tag{4.1}$$

Here Σ_3 is a volume for which the world-sheet Σ is the boundary.

[3] It is not difficult to conclude that there exists no modification of the gauge transformation formula $\delta_\epsilon \eta_a = \big(-C^c{}_{ab}\eta_c + v^i_a \omega^c{}_{bi}\eta_c\big)\,\epsilon^b$ such that the modified gauge invariance criteria would permit any possibility other than $\omega^a{}_b = (K^{-1})^a{}_c dK^c{}_b$ and $\widehat{C}^a{}_{bc}$ constants.

[4] Note that if $H = dB$ then C can be identified with B and the action (4.1) reduces to (3.1) as it should.

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The action (4.1) was introduced by CDJ in order to study "T-duality without isometry" in the presence of the non-exact 3-form background H. The gauge invariance conditions[5] of the action (4.1) with respect to the gauge transformations (3.4) read

$$\mathcal{L}_{v_a} G = \omega^b{}_a \vee \iota_{v_b} G, \qquad \mathcal{L}_{v_a} C = \omega^b{}_a \wedge \iota_{v_b} C\,, \qquad \iota_{v_a}(H - dC) = 0, \qquad d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = 0\,. \tag{4.2}$$

The last condition can be solved as in section 3, yielding

$$\omega^a{}_b = (K^{-1})^a{}_c\, dK^c{}_b\,. \tag{4.3}$$

With the field redefinitions

$$\widehat{A}^a = K^a{}_b\, A^b, \qquad \hat{\eta}_a = \eta_b (K^{-1})^b{}_a\,, \tag{4.4}$$

the action (4.1) and the gauge transformations (3.4) acquire the following form

$$\begin{aligned} S(X^i, \widehat{A}, \hat{\eta}) = \frac{1}{2}\int_\Sigma DX^i \wedge \left(G_{ij} * DX^j + C_{ij} DX^j\right) + \int_\Sigma \hat{\eta}_a \left(d\widehat{A}^a + \frac{1}{2}\widehat{C}^a{}_{bc}(X)\widehat{A}^b \wedge \widehat{A}^c\right) \\ + \frac{1}{6}\int_{\Sigma_3} H_{ijk} dX^i \wedge dX^j \wedge dX^k - \frac{1}{2}\int_\Sigma C_{ij} dX^i \wedge dX^j\,, \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} \delta_{\hat{\epsilon}} X^i &= \hat{v}^i_a \hat{\epsilon}^a\,, \\ \delta_{\hat{\epsilon}} \widehat{A}^a &= d\hat{\epsilon}^a + \widehat{C}^a{}_{bc}(X)\widehat{A}^b \hat{\epsilon}^c\,, \\ \delta_{\hat{\epsilon}} \hat{\eta}_a &= -\widehat{C}^c{}_{ab}(X)\hat{\eta}_c \hat{\epsilon}^b\,. \end{aligned} \tag{4.6}$$

Here

$$\hat{\epsilon}^a = K^a{}_b\, \epsilon^b\,, \tag{4.7}$$

$$\widehat{C}^a{}_{bc} \equiv K^a{}_d \left((K^{-1})^e{}_b (K^{-1})^f{}_c C^d{}_{ef} + (K^{-1})^e{}_b v^i_e \partial_i (K^{-1})^d{}_c - (K^{-1})^e{}_c v^i_e \partial_i (K^{-1})^d{}_b\right)\,, \tag{4.8}$$

and

$$DX^i = dX^i - v^i_a A^a = dX^i - \hat{v}^i_a \widehat{A}^a\,, \qquad \hat{v}^i_a = v^i_b (K^{-1})^b{}_a\,. \tag{4.9}$$

As in section 3, the gauge invariance requires that the structure functions $\widehat{C}^a{}_{bc}$ be constants and the conditions (4.2) become the standard isometry conditions

$$\mathcal{L}_{\hat{v}_a} G = 0\,, \qquad \mathcal{L}_{\hat{v}_a} C = 0\,, \qquad \iota_{\hat{v}_a}(H - dC) = 0\,. \tag{4.10}$$

The gauge theory (4.5) thus boils down to the Roček-Verlinde Yang-Mills theory underlying standard non-Abelian T-duality in the presence of the WZ term.

[5] The gauge invariance conditions originally written in [3], or in [4], use the notation $\theta_a := -\iota_{v_a} C$ and therefore look slightly different.

5 Lie algebroid gauged sigma models

So far we have been using the coordinates X^i on M, in order to compare more directly our calculations with those of CDJ. However, we have obtained our principal insights invariantly, using the language of the Lie algebroid gauged sigma models [12]. We consider it appropriate to include this invariant perspective in the present paper, as it highlights the geometric origin of the field redefinitions (3.9).

Recall that a Lie algebroid is a vector bundle Q over the manifold M, equipped with a Lie bracket $[\cdot\,,\cdot]_Q$ on the space of sections $\Gamma(Q)$ and with an anchor homomorphism $\rho : Q \to TM$ intertwining the bracket $[\cdot\,,\cdot]_Q$ with the standard Lie bracket of vector fields $[\cdot\,,\cdot]_{TM}$. Given a background metric G, a 2-form B on M, and a connection $\nabla^\omega : \Gamma(Q) \to \Omega^1(M) \otimes \Gamma(Q)$ on the vector bundle Q, we can define the CDJ gauge theory invariantly. It is a classical field theory on the world-sheet Σ with the action

$$S(X,A,\eta) = \frac{1}{2}\int_\Sigma \|TX - \rho(A)\|^2_G + \int_\Sigma X^*_A B + \int_\Sigma \langle \eta, (X^* d^{\nabla^\omega})A - \frac{1}{2}\, {}^Q\mathcal{T}^\omega(A \stackrel{\wedge}{,} A)\rangle\,. \quad (5.1)$$

The dynamical fields of the theory are: a map $X : \Sigma \to M$, a 1-form A on Σ with values in the pull-back bundle X^*Q and a section η of the dual pull-back bundle X^*Q^*. The expression $F_\omega(A,X) := (X^* d^{\nabla^\omega})A - \frac{1}{2}\, {}^Q\mathcal{T}^\omega(A \stackrel{\wedge}{,} A)$ takes values in $\Lambda^2 T^*\Sigma \otimes X^*Q$ and it is conveniently referred to as the Lie algebroid field strength.

Let us explain in more detail the notation. First we note that the connection $\nabla^\omega : \Gamma(Q) \to \Omega^1(M) \otimes \Gamma(Q)$ on the vector bundle Q induces the so-called linear connection ${}^Q\nabla^\omega : \Gamma(Q) \to \Gamma(Q^*) \otimes \Gamma(Q)$ on the Lie algebroid Q, defined by the relation

$${}^Q\nabla^\omega_{s_1} s_2 := \nabla^\omega_{\rho(s_1)} s_2, \qquad s_1, s_2 \in \Gamma(Q).$$

To the linear connection ${}^Q\nabla^\omega$ on Q is then associated the Q-torsion ${}^Q\mathcal{T}^\omega$ which is $C^\infty(M)$-bilinear form on $\Gamma(Q) \times \Gamma(Q)$ with values in $\Gamma(Q)$:

$${}^Q\mathcal{T}^\omega(s_1, s_2) := {}^Q\nabla^\omega_{s_1} s_2 - {}^Q\nabla^\omega_{s_2} s_1 - [s_1, s_2]_Q. \quad (5.2)$$

It must be mentioned that by writing ${}^Q\mathcal{T}^\omega(A \stackrel{\wedge}{,} A)$ as in eq. (5.1) we have used somewhat short-hand notation, with the purpose not to make the expression too heavy from the notational point of view. In reality, we should have written rather $X^*\, {}^Q\mathcal{T}^\omega(\breve{A} \stackrel{\wedge}{,} \breve{A})\big|_{X(u)}$, where $u \in \Sigma$ and $\breve{A}$ is any section of $\Omega^1(M,Q)$ with the property $X^*\breve{A}\big|_{X(u)} = A(u)$ (here X^* stands for the pull-back of differential forms and $\cdot\big|_{X(u)}$ means the restriction to the algebroid fiber over the point $X(u)$). It is the crucial property of $C^\infty(M)$-bilinearity of the torsion which guarantees that the ambiguity of the choice of the lifted section $\breve{A}$ does not influence the value of the field strength $F_\omega(A,X)$.

Recall also that $X^* d^{\nabla^\omega}$ stands for the standard pull-back of the extension of the connection ∇^ω to the differential forms valued in Q and $\|TX - \rho(A)\|_G$ means taking simultaneously the G-norm of the covariant tangent map $TX - \rho(A)$ in X^*TM and the Minkowski (indefinite) norm of 1-forms on the world-sheet Σ. Finally, $X^*_A B$ stands for the covariant pull-back of the differential form, which is the 2-form on Σ given at every point $u \in \Sigma$ by contracting B in $X(u)$ with $(TX - \rho(A)) \wedge (TX - \rho(A))$.

To achieve our invariant description of CDJ theory, we have to define infinitesimal gauge transformations of the fields X, A and η. The infinitesimal parameters ϵ of this transformations must be sections of the pull-back bundle X^*Q and the first of the transformations (3.4) evidently reads

$$\delta_\epsilon X = \rho(\epsilon). \tag{5.3}$$

To write invariantly the second and the third of the transformations (3.4) is, however, more subtle, since putative invariant variations $\delta_\epsilon A$ or $\delta_\epsilon \eta$ do not make sense, because both A and η live in the pull-back bundles which themselves change with the variation of X. This means that, at a given $u \in \Sigma$, we cannot subtract the field $A(u)$ from the transformed field $A_\epsilon(u)$ in order to define a variation $\delta_\epsilon A(u)$, being unable to subtract two vectors living in different spaces: $A(u)$ lives in the algebroid fiber over $X(u)$ whereas $A_\epsilon(u)$ lives in the algebroid fiber over $X(u) + \delta_\epsilon X(u)$. Fortunately, we have the connection ∇^ω which we can use to parallel transport the vector $A_\epsilon(u)$ from $X(u) + \delta_\epsilon X(u)$ to $X(u)$; the result of this parallel transport we denote as $A_\epsilon^{||}(u)$ and it now makes perfect sense to define a "parallel" invariant variation $\delta_\epsilon^{||} A$ by the formula

$$\delta_\epsilon^{||} A := A_\epsilon^{||}(u) - A(u). \tag{5.4}$$

It is clear that the knowledge of the parallel variation $\delta_\epsilon^{||} A$ fully determines the infinitesimally transformed gauge field $A_\epsilon(u)$ and vice versa, since those two quantities are tied by the parallel transport. The concrete formula for $\delta_\epsilon^{||} A$ is then given by the following nice invariant expression

$$\delta_\epsilon^{||} A = (X^* \nabla^\omega)\epsilon - {}^Q\mathcal{T}^\omega(A, \epsilon). \tag{5.5}$$

Here $X^*\nabla^\omega$ stands for the standard pull-back of the connection ∇^ω and the torsion term at some $u \in \Sigma$ should be understood as $X^* \, {}^Q\mathcal{T}^\omega(\check{A}, \check{\epsilon})\big|_{X(u)}$, where $\check{A}$ and $\check{\epsilon}$ are any sections of the respective bundles $\Omega^1(M, Q)$ and Q fulfilling the properties $X^*\check{A}\big|_{X(u)} = A(u)$ and $\check{\epsilon}\big|_{X(u)} = \epsilon(u)$. As before, it is the $C^\infty(M)$-bilinearity of the torsion which guarantees that the ambiguities in the choices of the lifted sections $\check{A}$ and $\check{\epsilon}$ do not influence the value of the parallel variation $\delta_\epsilon^{||} A$.

The same philosophy we use for the description of the infinitesimal gauge transformation of the Lagrange multiplier η with the result

$$\delta_\epsilon^{||} \eta = - \, {}^Q\mathcal{T}_\epsilon^{*\omega} \eta. \tag{5.6}$$

Here the operator ${}^Q\mathcal{T}_\epsilon^{*\omega} : Q^* \to Q^*$ is obtained by the transposition of the $C^\infty(M)$-linear operator ${}^Q\mathcal{T}_\epsilon^{\omega} : Q \to Q$ defined itself in terms of the torsion as

$${}^Q\mathcal{T}_\epsilon^{\omega} s := {}^Q\mathcal{T}^\omega(\epsilon, s), \qquad s \in \Gamma(Q). \tag{5.7}$$

Of course, the invariant formulas can be worked out in components, upon a choice of some local coordinates X^i on M and local frames e_a on the algebroid Q.[6] Explicitly, we

[6]By the local frames we mean $C^\infty(M)$ bases in the spaces of the local sections of the algebroid bundle Q.

introduce 1-forms $A^a(u)$ on Σ, scalars $\eta_a(u)$ on Σ, vector fields v_a on M, 1-forms $\omega^b{}_a$ on M and structure functions $C^a{}_{bc}(X)$ on M by the relations:

$$A = A^a(u)e_a\big|_{X(u)},\ \eta = \eta_a(u)e^*_a\big|_{X(u)},\ v_a = \rho(e_a),\ \nabla^\omega e_a = \omega^b{}_a \otimes e_b,\ [e_b,e_c]_Q = C^a{}_{bc}(X)e_a. \tag{5.8}$$

Inserting all those data in our invariant action (5.1), we recover straightforwardly the component action (3.1) of CDJ.

We now give a more detailed calculation, illustrating how to recover the component gauge transformation (3.4) from the invariant formulas (5.3), (5.5) and (5.6). First we concentrate on the most involved case of the transformation of the gauge field A. Using the parametrisation $A = A^a(u)e_a\big|_{X(u)}$, we set, respectively,

$$A_\epsilon(u) \equiv A^a_\epsilon(u)e_a\big|_{X(u)+\rho(\epsilon)(u)},\quad \delta^{||}_\epsilon A(u) \equiv \delta^{||}_\epsilon A^a(u)e_a\big|_{X(u)},\quad \delta_\epsilon A^a(u) := A^a_\epsilon(u) - A^a(u). \tag{5.9}$$

If the frames e_a were covariantly constant, then $A^{||}_\epsilon(u)$ would be equal to $A^a_\epsilon(u)e_a\big|_{X(u)}$, hence, following (5.4), the components of the parallel variation $\delta^{||}_\epsilon A^a(u)$ would be equal to the ordinary variations $\delta_\epsilon A^a(u)$. However, if e_a are not covariantly constant, there is a correction proportional to the covariant derivative $\nabla^\omega_\epsilon e_a = \iota_\epsilon \omega^b{}_a e_b$ which we find to be equal to

$$\delta^{||}_\epsilon A^a = \delta_\epsilon A^a + \iota_\epsilon \omega^a{}_b A^b. \tag{5.10}$$

Now the formula (5.5) worked out in components yields

$$\delta^{||}_\epsilon A^a = d\epsilon^a + \omega^a{}_{bi} dX^i \epsilon^b - \left(\iota_{v_b}\omega^a{}_c - \iota_{v_c}\omega^a{}_b - C^a{}_{bc}(X)\right) A^b \epsilon^c. \tag{5.11}$$

Combining the equations (5.10) and (5.11), we find

$$\delta_\epsilon A^a = d\epsilon^a + \omega^a{}_{ci}(dX^i - v^i_b A^b)\epsilon^c + C^a{}_{bc}(X)A^b\epsilon^c, \tag{5.12}$$

which is nothing but the CDJ gauge transformation (3.4).

The case of the field η is even simpler. First we find

$$\delta^{||}_\epsilon \eta_a = \delta_\epsilon \eta_a - \iota_\epsilon \omega^c{}_a \eta_c, \tag{5.13}$$

and then eq. (5.6) written in components gives

$$\delta^{||}_\epsilon \eta_a = \left(-C^c{}_{ab} + \iota_{v_a}\omega^c{}_b - \iota_{v_b}\omega^c{}_a\right)\epsilon^b \eta_c\,. \tag{5.14}$$

Combining the equations (5.13) and (5.14), we finally find

$$\delta_\epsilon \eta_a = \left(-C^c{}_{ab}\eta_c + v^i_a \omega^c{}_{bi}\eta_c\right)\epsilon^b\,, \tag{5.15}$$

which is nothing but the third of the CDJ gauge transformation (3.4).

Next we have to clarify the question of the gauge symmetry of the invariant CDJ action (5.1) with respect to the gauge transformations (5.3), (5.5) and (5.6). We have already learned from the component calculations, that this gauge symmetry is not automatic but requires some compatibility of the background data G, B, Q and ∇^ω. We have found three compatibility conditions which, written invariantly, become:

1. For every section χ of the bundle Q, it must hold
$$\mathcal{L}_{\rho(\chi)}G = \left(\rho(\nabla^\omega\chi)\otimes \mathrm{Id} + \mathrm{Id}\otimes\rho(\nabla^\omega\chi)\right) G\,, \tag{5.16}$$
$$\mathcal{L}_{\rho(\chi)}B = \left(\rho(\nabla^\omega\chi)\otimes \mathrm{Id} + \mathrm{Id}\otimes\rho(\nabla^\omega\chi)\right) B\,, \tag{5.17}$$
where $\rho(\nabla^\omega\chi)\in T^*M\otimes TM$ is viewed as the linear operator on T^*M;
2. The connection ∇^ω on the algebroid bundle Q must be flat;
3. For any two sections $s,t\in\Gamma(Q)$, the following implication must hold
$$\nabla^\omega s = \nabla^\omega t = 0 \implies \nabla^\omega[s,t]_Q = 0\,. \tag{5.18}$$

If the conditions (2) and (3) are fulfilled we say that the Lie algebroid (Q,∇^ω) is admissible. By the component calculations, we have established that to every admissible Lie algebroid (Q,∇^ω) there is a naturally associated Lie algebra $\mathfrak{g}(Q,\omega)$, consisting of covariantly constant local sections of Q. The structure of the Lie algebra $\mathfrak{g}(Q,\omega)$ is induced from the Lie algebroid bracket $[\cdot\,,\cdot]_Q$ and the condition (1) implies that the action of $\mathfrak{g}(Q,\omega)$ on the sigma model background (G,B) — via the anchor map — is isometric. Therefore every CDJ theory is locally equivalent to the standard intermediate gauge theory used to derive the traditional non-Abelian T-duality. From the global point of view, it may happen that the Lie algebra $\mathfrak{g}(Q,\omega)$ action on non-simply connected targets M cannot be made global by parallel transport.[7] In such a case, the CDJ proposal would give a new insight on subtle topological issues related to the standard non-Abelian T-duality, rather than a recipe to produce new genuinely non-isometric T-dual pairs of sigma models.

Let us finish this section by stressing that, from the Lie algebroid vantage point, it does not have any invariant meaning to say that "a non-isometric action of a Lie algebra on M is gauged". This is because the local imput data $M,G,B,v_a,\omega^a{}_b,C^c{}_{ab}(X)$ used by CDJ to construct their intermediate gauge theory can be equally well replaced by equivalent data $M,G,B,\hat{v}_a,\hat{\omega}^a{}_b,\widehat{C}^c{}_{ab}(X)$ without changing the T-duality pattern. Given two local anchored frames v_a and $\hat{v}_a$, the structure *constants* $C^c{}_{ab}$ and $\widehat{C}^c{}_{ab}$ may be related by a *non-constant* matrix $K^a{}_b(X)$, and as we shall illustrate with examples in the next section, those two sets of structure constants may even define two non-isometrically acting non-isomorphic Lie algebras, the gauging of which yields the same T-dual pair of sigma models!

We conclude this section by emphasising that it is solely the Lie algebroid structure (Q,∇^ω) that has invariant meaning in the CDJ theory. Recall, however, that the gauge invariance of the CDJ theory based on the admissible Lie algebroid requires the existence of the preferred covariantly constant framing on Q for which the anchored action on the sigma model background (G,B) is isometric. Moreover, the structure functions of the covariantly constant frames must be constant and they thus define the preferred Lie algebra $\mathfrak{g}(Q,\omega)$. In this case saying "the isometric action of the Lie algebra $\mathfrak{g}(Q,\omega)$ on M is gauged" does have an invariant meaning, and it is in this way that the standard non-Abelian T-duality input (i.e. the isometric action of some Lie algebra on the target) is recovered from the structure of the admissible Lie algebroid (Q,∇^ω).

[7]This could happen, for example, by considering targets of the form of coset of Lie group by its discrete subgroup.

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6 Examples

Let M be the manifold $\mathbb{R}^3$ parametrised by global Cartesian coordinates (X^1, X^2, X^3), equipped with the metric

$$ds^2 = (dX^1)^2 + \left(dX^2 - X^1 dX^3\right)^2 + (dX^3)^2 \tag{6.1}$$

and vanishing B-field.

Let $Q = TM$ be the tangent Lie algebroid of M (with the identity anchor map) and let $\widehat{\nabla}$ be a flat connection on TM defined by declaring that the following global frame $\hat{e}_a$ is covariantly constant:

$$\hat{e}_a = \{\partial_1 + X^3\partial_2, \partial_2, \partial_3\}\,. \tag{6.2}$$

This structure $(TM, \widehat{\nabla})$ defines the admissible Lie algebroid in the sense of section 5, since the structure functions of the covariantly constant framing $\hat{e}_a$ are all constant. In fact the only non-vanishing ones are $\widehat{C}^2{}_{31} = -\widehat{C}^2{}_{13} = 1$. Let us now establish that the Lie derivatives $\mathcal{L}_{\hat{e}_a}$ of all three differentials $dX^1, dX^2 - X^1 dX^3, dX^3$ vanish for every $a = 1, 2, 3$. This can be seen either by a direct calculation or by remarking that the forms $dX^1, dX^2 - X^1 dX^3, dX^3$ form a basis in the space of the left-invariant 1-forms on the Heisenberg group H consisting of the matrices of the following form:

$$\mathsf{H} = \left\{ \begin{pmatrix} 1 & X^1 & X^2 \\ 0 & 1 & X^3 \\ 0 & 0 & 1 \end{pmatrix},\ X^1, X^2, X^3 \in \mathbb{R} \right\},$$

while $\hat{e}_a$ form the basis in the space of right-invariant vector fields in H. Either way, we conclude that the Lie derivatives $\mathcal{L}_{\hat{e}_a}$ of the metric (6.1) vanish since the metric is constructed from the left-invariant forms $dX^1, dX^2 - X^1 dX^3, dX^3$.

We have now all ingredients to define the Lie algebroid gauged sigma model and its action reads

$$\begin{aligned}\hat{S}(X, \widehat{A}, \hat{\eta}) = &\frac{1}{2}\int_\Sigma \Big((dX^1 - \widehat{A}^1)\wedge *(dX^1 - \widehat{A}^1) + (dX^3 - \widehat{A}^3)\wedge *(dX^3 - \widehat{A}^3)\Big) \\ &+ \frac{1}{2}\int_\Sigma (dX^2 - X^3\widehat{A}^1 - \widehat{A}^2 - X^1(dX^3 - \widehat{A}^3))\wedge *(dX^2 - X^3\widehat{A}^1 - \widehat{A}^2 - X^1(dX^3 - \widehat{A}^3)) \\ &+ \int_\Sigma (\hat{\eta}_1 d\widehat{A}^1 + \hat{\eta}_2(d\widehat{A}^2 + \widehat{A}^3\wedge\widehat{A}^1) + \hat{\eta}_3 d\widehat{A}^3)\,.\end{aligned} \tag{6.3}$$

The gauge transformations are

$$\begin{aligned} &\delta_\epsilon X^1 = \epsilon^1, && \delta_\epsilon X^2 = \epsilon^2 + X^3\epsilon^1, && \delta_\epsilon X^3 = \epsilon^3, \\ &\delta_\epsilon \widehat{A}^1 = d\epsilon^1, && \delta_\epsilon \widehat{A}^2 = d\epsilon^2 + \widehat{A}^3\epsilon^1 - \widehat{A}^1\epsilon^3, && \delta_\epsilon \widehat{A}^3 = d\epsilon^3, \\ &\delta_\epsilon \hat{\eta}_1 = \epsilon^3\hat{\eta}_2, && \delta_\epsilon \hat{\eta}_2 = 0, && \delta_\epsilon \hat{\eta}_3 = -\epsilon^1\hat{\eta}_2\,. \end{aligned} \tag{6.4}$$

Since we are gauging the isometry, we are doing the standard non-Abelian T-duality, nevertheless, for the sake of illustration, we go on further with the well-known procedure how to recover from (6.3) the T-dual pair of sigma models.

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Varying the Lagrange multipliers $\hat{\eta}_a$ we obtain the Maurer Cartan relations

$$d\widehat{A}^1 = 0\,, \qquad d\widehat{A}^2 + \widehat{A}^3 \wedge \widehat{A}^1 = 0\,, \qquad d\widehat{A}^3 = 0\,, \tag{6.5}$$

which can be solved in full generality as

$$\widehat{A}^1 = -dx^1, \qquad \widehat{A}^2 = -dx^2 + x^1 dx^3, \qquad \widehat{A}^3 = -dx^3. \tag{6.6}$$

Plugging the solution (6.6) into the action (6.3), we recover the original sigma model corresponding to the metric (6.1) and the vanishing B-field:

$$\widehat{S}(Y^k) = \frac{1}{2}\int_\Sigma \left(dY^1 \wedge *dY^1 + (dY^2 - Y^1 dY^3) \wedge *(dY^2 - Y^1 dY^3) + dY^3 \wedge *dY^3\right), \tag{6.7}$$

where[8]

$$Y^1 := X^1 + x^1\,, \qquad Y^2 := X^2 + x^2 + x^1 X^3\,, \qquad Y^3 := X^3 + x^3\,. \tag{6.8}$$

In order to recover the dual sigma model, we first make in (6.3) the following field redefinitions:

$$\widehat{B}^1 := \widehat{A}^1 - dX^1, \qquad \widehat{B}^2 := \widehat{A}^2 - dX^2 + X^3\widehat{A}^1 + X^1(dX^3 - \widehat{A}^3), \qquad \widehat{B}^3 := \widehat{A}^3 - dX^3, \tag{6.9}$$

$$\hat{\mu}_1 := \hat{\eta}_1 - X^3\hat{\eta}_2, \qquad \hat{\mu}_2 := \hat{\eta}_2, \qquad \hat{\mu}_3 := \hat{\eta}_3 + X^1\hat{\eta}_2, \tag{6.10}$$

in terms of which the action (6.3) becomes

$$\widehat{S}(\widehat{B},\hat{\mu}) = \frac{1}{2}\int_\Sigma (\widehat{B}^1 \wedge *\widehat{B}^1 + \widehat{B}^2 \wedge *\widehat{B}^2 + \widehat{B}^3 \wedge *\widehat{B}^3) - \int_\Sigma (d\hat{\mu}_1 \wedge \widehat{B}^1 + d\hat{\mu}_2 \wedge \widehat{B}^2 + d\hat{\mu}_3 \wedge \widehat{B}^3 - \hat{\mu}_2 \widehat{B}^3 \wedge \widehat{B}^1)\,. \tag{6.11}$$

Varying $\widehat{B}$ in $\widehat{S}(\widehat{B},\hat{\mu})$ gives

$$\widehat{B}^1 = -\frac{1}{1+\hat{\mu}_2^2}(\hat{\mu}_2 d\hat{\mu}_3 + *d\hat{\mu}_1), \qquad \widehat{B}^2 = -*d\hat{\mu}_2, \qquad \widehat{B}^3 = \frac{1}{1+\hat{\mu}_2^2}(\hat{\mu}_2 d\hat{\mu}_1 - *d\hat{\mu}_3). \tag{6.12}$$

Inserting (6.12) back into $\widehat{S}(\widehat{B},\hat{\mu})$ gives the dual sigma model:

$$\widehat{S}(\hat{\mu}) = \frac{1}{2}\int_\Sigma \frac{1}{1+\hat{\mu}_2^2}\left(d\hat{\mu}_1 \wedge *d\hat{\mu}_1 + (1+\hat{\mu}_2^2) d\hat{\mu}_2 \wedge *d\hat{\mu}_2 + d\hat{\mu}_3 \wedge *d\hat{\mu}_3 + 2\hat{\mu}_2 d\hat{\mu}_1 \wedge d\hat{\mu}_3\right)\,. \tag{6.13}$$

There is a simple reason why we have chosen to work out this particular example of the non-Abelian T-duality. It is because CDJ have applied in [3] their "T-duality without isometry" recipe exactly on the sigma model background (6.1).[9] Our point is to emphasize that CDJ did not obtain anything new but just the same dual model (6.13) as we did by

[8]The field redefinitions (6.8) reflect the group multiplication law in the Heisenberg group H and the new fields Y can be interpreted as X acted upon by the non-infinitesimal gauge transformation "x" obtained by exponentiating the gauge transformations (6.4).

[9]CDJ wrote in [3] that they dualised the Heisenberg nilmanifold but the computation that they performed therein concerns, in reality, the Heisenberg group target.

gauging the isometry (cf. eq. (4.19) of ref. [3]). We also understand the reason why this fact is not coincidental. Indeed, it occurs because CDJ just used a different framing in order to write the same invariant Lie algebroid $(TM, \hat{\nabla})$ gauge theory in components. Actually, the framing of CDJ was formed by the following basis of the left-invariant vector fields on the group H

$$e'_a = \{\partial_1, \partial_2, X^1\partial_2 + \partial_3\}, \tag{6.14}$$

and because the Lie derivatives of the left-invariant forms with respect to the left-invariant vector fields do not vanish (for non-Abelian Lie groups), the background sigma model metric (6.1) is not invariant. Of course, the non-invariance of the metric can be seen also from the formula (5.16) since the framing e'_a is not covariantly constant.

CDJ used the frame e'_a to carry out the non-isometric gauging of the Heisenberg group acting on itself from the right, but we have already explained in section 5 that only gauging with respect to the covariantly constant frame has invariant meaning. In fact, the gauging of CDJ is not even the only possible non-isometric gauging based on an action of a Lie group. Indeed, we can even consider the non-isometric gauging of the sigma model (6.7) by the action of the Abelian translation group $\mathbb{R}^3$ on itself! We achieve that by choosing yet a third frame e_a for which the structure functions $C^a{}_{bc}$ all vanish. It reads simply

$$e_a = \{\partial_1, \partial_2, \partial_3\}. \tag{6.15}$$

The framing e_a is neither covariantly constant nor does it leave the metric (6.1) invariant. From the relation

$$\hat{e}_a = e_b (K^{-1})^b{}_a, \qquad K = \begin{pmatrix} 1 & 0 & 0 \\ -X^3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{6.16}$$

it easily follows that the input data of the CDJ gauge theory is the metric (6.1), $B = 0$, the frame e_a and the matrix valued 1-form ω given by

$$\omega = \begin{pmatrix} 0 & 0 & 0 \\ -dX^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6.17}$$

The action (3.1) of the CDJ gauge theory can be specified for those data and the direct calculation yields, without surprise, again the dual pair (6.7), (6.13) of sigma models.

Remark. We note that whenever an n-dimensional group manifold G admits a global coordinate system $X^1, \ldots, X^n$, then every G-isometric background can be either T-dualised by the standard isometric gauging based on the G action on itself or by the CDJ non-isometric gauging of the Abelian group $\mathbb{R}^n$ generated by the coordinate vector fields $\partial_1, \ldots, \partial_n$.

Our second example directly generalizes the first one in the sense we replace the Heisenberg group H by an arbitrary Lie group G. We shall work invariantly since we shall no longer need to make comparison with the coordinate calculations of CDJ.

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Our Lie algebroid Q is now the tangent bundle $T\mathsf{G}$ and we pick the flat connection ∇^ω by declaring that the right-invariant vector fields on G are covariantly constant. We write the sigma model action, which we want to gauge, as

$$S(g) = \frac{1}{2}\int_\Sigma (g^{-1}dg \wedge *g^{-1}dg)_G + \frac{1}{2}\int_\Sigma (g^{-1}dg \wedge g^{-1}dg)_B. \tag{6.18}$$

Here $g^{-1}dg$ is the left-invariant Maurer-Cartan form on G and the background geometry is encoded in the choice of two non-invariant bilinear forms $(\cdot,\cdot)_G$ and $(\cdot,\cdot)_B$ defined on the Lie algebra $\mathfrak{g}$ of a Lie group G, the former symmetric and the latter antisymmetric. In other words, the metric G and the 2-form B underlying this particular sigma model action are obtained by the left-transport of the bilinear forms $(\cdot,\cdot)_G$ and $(\cdot,\cdot)_B$ to every point of the group manifold.

The CDJ gauge theory corresponding to the right-invariant framing is now the standard Roček-Verlinde Yang-Mills theory, traditionally used to work out the standard non-Abelian T-dual of the model (6.18). It can be also obtained without knowing anything about Lie algebroids, just by gauging the rigid left action of the group G on itself which is the symmetry of the action (6.18):

$$S(g,\widehat{A},\eta) = \frac{1}{2}\int_\Sigma (g^{-1}Dg \wedge *g^{-1}Dg)_G + \frac{1}{2}\int_\Sigma (g^{-1}Dg \wedge g^{-1}Dg)_B + \int_\Sigma \langle \hat{\eta}, d\widehat{A} - \widehat{A}\wedge\widehat{A}\rangle. \tag{6.19}$$

Here $\widehat{A}$ and $\hat{\eta}$ are respectively $\mathfrak{g}$-valued 1-form and $\mathfrak{g}^*$-valued 0-form on the world-sheet Σ and the covariant derivative is defined as

$$g^{-1}Dg := g^{-1}dg - g^{-1}\widehat{A}g. \tag{6.20}$$

Note also the form of the field strength $d\widehat{A} - \widehat{A}\wedge\widehat{A}$. From the (invariant) CDJ point of view, it comes from the fact that the torsion of the flat connection leaving the right-invariant vector fields covariantly constant coincides precisely with the commutator on the Lie algebra $\mathfrak{g}$. For that matter, this is coherent with the appearance of the minus sign in the field strength which reflects the fact that the structure constants corresponding to the infinitesimal left action of G on itself pick a minus sign with respect to the structure constants of the Lie algebra $\mathfrak{g}$.

The gauged action (6.19) has the following left gauge symmetry

$$(g,\widehat{A},\hat{\eta}) \to (\hat{h}g, \hat{h}\widehat{A}\hat{h}^{-1} + d\hat{h}\hat{h}^{-1}, \mathrm{Ad}^*_{\hat{h}}\,\hat{\eta}), \tag{6.21}$$

or, infinitesimally,

$$\delta_{\hat{\epsilon}}(g,\widehat{A},\hat{\eta}) = (\hat{\epsilon}g, d\hat{\epsilon} - [\widehat{A},\hat{\epsilon}], \mathrm{ad}^*_{\hat{\epsilon}}\,\hat{\eta}). \tag{6.22}$$

Here $\hat{h}$ and $\hat{\epsilon}$ are smooth maps from the world-sheet Σ to the Lie group G and the Lie algebra $\mathfrak{g}$, respectively. Note in particular, that the expression $g^{-1}Dg$ turns out to be gauge invariant, which immediately explains the gauge invariance of the part of the action (6.19) not containing the Lagrange multiplier.

We now switch from the natural isometric right-invariant framing to non-natural non-isometric left-invariant one, and wish to rewrite the CDJ gauge theory (6.19) accordingly.

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To work this out, we can depart directly from the invariant action (5.1), but it is simpler to do it by making the appropriate field redefinitions in the action (6.19), induced by the change of the frame. For that, it is enough to note that the "new" gauge transformation now hits the sigma model configuration g from the right, which implies that the g-dependent frame-changing operator K in eq. (3.14) is simply $K = \mathrm{Ad_g}$. Indeed, writing

$$h = g^{-1}\hat{h}g\,, \tag{6.23}$$

gives infinitesimally

$$\epsilon = \mathrm{Ad}_{\mathrm{g}^{-1}}\,\hat{\epsilon}, \tag{6.24}$$

which, following eq. (3.14), fixes $K = \mathrm{Ad_g}$.

With this choice of K the remaining field redefinitions are dictated by eq. (3.9):

$$A = g^{-1}\widehat{A}g, \qquad \eta = \mathrm{Ad}^*_{g^{-1}}\,\hat{\eta}\,. \tag{6.25}$$

The covariant derivative $g^{-1}Dg$, the action (6.19) and the gauge tranformations (6.21) then become, respectively,

$$g^{-1}Dg = g^{-1}\,dg - A\,, \tag{6.26}$$

$$S(g,A,\eta) = \frac{1}{2}\int_\Sigma (g^{-1}Dg \wedge *g^{-1}Dg)_G + \frac{1}{2}\int_\Sigma (g^{-1}Dg\wedge g^{-1}Dg)_B$$
$$+\int_\Sigma \langle \eta, dA + A\wedge A + g^{-1}Dg\wedge A + A\wedge g^{-1}Dg\rangle\,. \tag{6.27}$$

$$(g,A,\eta) \to (gh, A - g^{-1}dg + (gh)^{-1}\,d(gh), \eta)\,. \tag{6.28}$$

Note that the infinitesimal version of the gauge transformations (6.28) read

$$\delta_\epsilon(g,A,\eta) = (g\epsilon, d\epsilon + [A,\epsilon] + \mathrm{Ad}_{g^{-1}Dg}\,\epsilon, 0)\,. \tag{6.29}$$

We remark that the gauged sigma model action (6.27), as well as the infinitesimal gauge transformations (6.29), have now indeed the CDJ form (3.1), (3.4), where the 1-form ω with values in $\mathrm{End}(\mathfrak{g})$ is invariantly written as

$$\omega = \mathrm{ad}_{g^{-1}dg}\ . \tag{6.30}$$

If a reader would look just at the gauge theory action (6.27) as well as at the infinitesimal gauge transformations (6.29), without knowing how we have obtained them, they would probably believe they have some exotic gauging of the right action of the group G on itself. Our point is that this CDJ non-isometric exotic right gauging is just the standard isometric left gauging in disguise, therefore it cannot give rise to any new T-duality pattern.

7 Conclusion and outlook

We have ruled out the proposal of Chatzistavrakidis, Deser and Jonke in the sense that it does not give rise to a new pattern of genuinely non-isometric T-duality. Nevertheless,

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we see some room to apply the invariantly formulated CDJ gauge theory of section 5 to study some subtle topological effects within the framework of the traditional non-Abelian T-duality. This possibility might take place if there exist admissible Lie algebroids (Q, ∇^{ω}) for which the invariant Lie algebra $\mathfrak{g}(Q, \omega)$ would act just locally on the target manifold M and could not be extended to a global action. It is plausible to expect that this situation may occur for which the targets M are of the form of the quotient of a Lie group by one of its discrete subgroups. The CDJ theory could then take into account the phenomena of winding strings on non-contractible cycles of M.

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