

On C_J and C_T in conformal QED

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ABSTRACT: QED with a large number N of massless fermionic degrees of freedom has a conformal phase in a range of space-time dimensions. We use a large N diagrammatic approach to calculate the leading corrections to C_T , the coefficient of the two-point function of the stress-energy tensor, and C_J , the coefficient of the two-point function of the global symmetry current. We present explicit formulae as a function of d and check them versus the expectations in 2 and $4 - \epsilon$ dimensions. Using our results in higher even dimensions we find a concise formula for C_T of the conformal Maxwell theory with higher derivative action $F_{\mu\nu}(-\nabla^2)^{\frac{d}{2}-2}F^{\mu\nu}$. In $d = 3$, QED has a topological symmetry current, and we calculate the correction to its two-point function coefficient, C_J^{top} . We also show that some RG flows involving QED in $d = 3$ obey $C_T^{\text{UV}} > C_T^{\text{IR}}$ and discuss possible implications of this inequality for the symmetry breaking at small values of N .

KEYWORDS: 1/N Expansion, Renormalization Group

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1 Introduction and summary

One of the important observables in Conformal Field Theory (CFT) is C_T , the coefficient of the two-point function of the stress-energy tensor $T_{\mu\nu}$, defined via [1]

$$\langle T_{\mu\nu}(x_1) T_{\lambda\rho}(x_2) \rangle = C_T \frac{I_{\mu\nu,\lambda\rho}(x_{12})}{(x_{12}^2)^d}, \quad (1.1)$$

where

$$\begin{aligned} I_{\mu\nu,\lambda\rho}(x) &\equiv \frac{1}{2} (I_{\mu\lambda}(x) I_{\nu\rho}(x) + I_{\mu\rho}(x) I_{\nu\lambda}(x)) - \frac{1}{d} \delta_{\mu\nu} \delta_{\lambda\rho}, \\ I_{\mu\nu}(x) &\equiv \delta_{\mu\nu} - 2 \frac{x_\mu x_\nu}{x^2}. \end{aligned} \quad (1.2)$$

If the CFT has a global symmetry generated by conserved currents J_μ^a , then another interesting observable is C_J , the coefficient of their two-point functions:

$$\langle J_\mu^a(x_1) J_\nu^b(x_2) \rangle = C_J \frac{I_{\mu\nu}(x_{12})}{(x_{12}^2)^{d-1}} \delta^{ab}. \quad (1.3)$$

In CFTs with a large number of degrees of freedom, N , these observables typically admit $1/N$ expansions of the form

$$\begin{aligned} C_J &= C_{J_0} \left(1 + \frac{C_{J_1}}{N} + \frac{C_{J_2}}{N^2} + \mathcal{O}(1/N^3) \right), \\ C_T &= C_{T_0} \left(1 + \frac{C_{T_1}}{N} + \frac{C_{T_2}}{N^2} + \mathcal{O}(1/N^3) \right). \end{aligned} \quad (1.4)$$

The values of C_{J_1} and C_{T_1} have been calculated in a variety of models. Petkou [2] has used large N methods and operator products expansions to calculate them as a function of d in the scalar $O(N)$ model. Very recently, these results were reproduced using the large N diagrammatic approach in [3], where the same technique was also used to calculate C_{J_1} and C_{T_1} as a function of d in the conformal Gross-Neveu model. An important feature of the diagrammatic approach, which was uncovered in [3], is the necessity, in the commonly used regularization scheme [4–8], of a divergent multiplicative “renormalization” Z_T for the stress-energy tensor. This factor is required by the conformal Ward identities in the regularized theory.

In this paper we extend the methods of [3] to calculate $C_{J_1}(d)$ and $C_{T_1}(d)$ in the conformal QED in d dimensions. This theory, which is reviewed in section 2, may be thought of as the Maxwell field coupled to N_f massless 4-component Dirac fermions continued from 4 dimensions to a more general dimension d . The large N expansion in this model runs in powers of the total number of fermionic components, which is $N = 4N_f$. In the physically interesting dimension $d = 3$, this corresponds to an even number $2N_f$ of two-component Dirac fermions.

Our main results are

$$C_{J_1}(d) = \eta_{m1} \left(\frac{3d(d-2)}{8(d-1)} \Theta(d) + \frac{d-2}{d} \right), \quad (1.5)$$

$$C_{T_1}(d) = \eta_{m1} \left(\frac{3d(d-2)}{8(d-1)} \Theta(d) + \frac{d(d-2)}{(d-1)(d+2)} \Psi(d) - \frac{(d-2)(3d^2+3d-8)}{2(d-1)^2 d(d+2)} \right), \quad (1.6)$$

$$\Theta(d) \equiv \psi'(d/2) - \psi'(1), \quad \Psi(d) \equiv \psi(d-1) + \psi(2-d/2) - \psi(1) - \psi(d/2-1),$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$. Here $\eta_{m1}(d)$ encodes the electron mass anomalous dimension; it is [9]¹

$$\eta_{m1}(d) = -\frac{2(d-1)\Gamma(d)}{\Gamma(\frac{d}{2})^2 \Gamma(\frac{d}{2}+1) \Gamma(2-\frac{d}{2})}. \quad (1.7)$$

In the physically interesting case of $d = 3$ we find

$$\begin{aligned} C_{J_1}(3) &= \frac{736}{9\pi^2} - 8 \approx 0.285821, \\ C_{T_1}(3) &= \frac{4192}{45\pi^2} - 8 \approx 1.43863. \end{aligned} \quad (1.8)$$

Let us compare our results with the earlier diagrammatic calculations [10, 11], which were carried out in $d = 3$ using a regulator different from ours. Our result for $C_{J_1}(3)$ agrees with that given by Huh and Strack in [11].² However, our value of $C_{T_1}(3)$ does not agree with that given in [11], which after translating to our convention for N is

¹We define the anomalous dimension of the electron mass operator $O_m = \bar{\psi}\psi$ as $\Delta_{O_m} = d - 1 + \eta_m$, where $\eta_m = \eta_{m1}/N + \mathcal{O}(1/N^2)$.

²In [11] only a numerical value $\tilde{C}_J^{(1)} \approx 0.59322699$ was given. We have found the exact expression behind this number: $\tilde{C}_J^{(1)} = \frac{136}{3\pi^2} - 4$, which leads to $C_{J_1}^{\text{H\&S}} = \frac{368}{9\pi^2} - 4$. The relative factor of 2 between this and our (1.8) is due to the different conventions: in [11] $N_f^{\text{H\&S}}$ is the number of $d = 3$ Dirac doublets. Therefore, our $N = 4N_f = 2N_f^{\text{H\&S}}$.

$C_{T1}^{\text{H\&S}} = \frac{3808}{45\pi^2} - 8 \approx 0.574024$.³ The source of the disagreement is the effect of Z_T , which was not included in [11].

A nontrivial check of our results (1.5) and (1.6) comes from comparing them with the known exact values in $d = 2$ and the $4 - \epsilon$ expansions, see sections 3 and 5. Had we not included Z_T , there would be no agreement with the $4 - \epsilon$ expansion. In higher even d , the conformal QED reduces to a free theory of N fermions and a conformal higher-derivative Maxwell theory with the action (see e.g. [12])

$$F_{\mu\nu}(-\nabla^2)^{\frac{d}{2}-2} F^{\mu\nu}. \tag{1.9}$$

Using the value of C_{T1} in general even dimensions, we extract the C_T of this conformal Maxwell theory

$$C_T^{\text{conf. Maxwell}}|_{\text{even } d} = (-1)^{\frac{d}{2}} \frac{d}{S_d^2} \binom{d}{\frac{d}{2} - 1}, \tag{1.10}$$

where $S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$.

In $d = 3$ the QED has a special ‘‘topological’’ U(1) symmetry current $j^{\text{top}} = \frac{1}{2\pi} * F$. In section 4 we calculate its two-point function to order $1/N^2$, and obtain the associated C_J^{top} coefficient, in the normalization (1.3), to be

$$C_J^{\text{top}} = \frac{16}{\pi^4 N} \left(1 + \frac{1}{N} \left(8 - \frac{736}{9\pi^2} \right) + \mathcal{O}(1/N^2) \right), \tag{1.11}$$

where $N = 4N_f$ is twice the number of two-components Dirac fermions. The leading order term is in agreement with [10, 13].

The QED₃ Lagrangian also has an enhanced SU(2N_f) global symmetry, and for small N_f this symmetry may be broken spontaneously to SU(N_f) × SU(N_f) × U(1) [14, 15]. In section 6 we present a new estimate for the critical value of N_f above which the symmetry breaking cannot occur by using the RG inequality $C_T^{\text{UV}} > C_T^{\text{IR}}$. It implies that the chiral symmetry cannot be broken for $N_f > 1 + \sqrt{2}$. The status of this conclusion is uncertain, since there are known violations of the inequality in some supersymmetric RG flows [16]. Nevertheless, it is interesting that the critical value of N_f it yields is close to other available estimates [12, 17–20] and is consistent with the results available from lattice gauge theory [21, 22].

2 Large N expansion for conformal QED_d

The action for Maxwell theory coupled to N_f massless charged fermions in flat Euclidean space

$$S = \int d^d x \left(\frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu} - \sum_{i=1}^{N_f} \bar{\psi}_i \gamma^\mu (\partial_\mu + iA_\mu) \psi^i \right). \tag{2.1}$$

³In [11] only a numerical value $\tilde{C}_T^{(1)} \approx -0.41548168$ was given. We have found the exact expression behind this number: $\tilde{C}_T^{(1)} = \frac{1592}{45\pi^2} - 4$.

Here the fermions ψ^i are taken to be four-component complex spinors. We define the dimensional continuation of the theory by keeping the number of fermion components fixed. In other words, we take γ^μ to be 4×4 matrices satisfying $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu} \mathbf{1}$, with $\text{Tr}\mathbf{1} = 4$. All vector indices are formally continued to d dimensions, i.e. $\delta^{\mu\nu}\delta_{\mu\nu} = d$, $\gamma^\mu\gamma_\mu = d \cdot \mathbf{1}$, etc.

One may develop the $1/N$ expansion of the theory by integrating out the fermions [15, 23]. This produces an effective action for the gauge field of the form

$$S_{\text{eff}} = \int d^d x \frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu} + \int d^d x d^d y \left(\frac{1}{2} A^\mu(x) A^\nu(y) \langle J_\mu(x) J_\nu(y) \rangle_0 + \mathcal{O}(A^3) \right), \quad (2.2)$$

where

$$J_\mu = \bar{\psi}_i \gamma_\mu \psi^i \quad (2.3)$$

is the conserved U(1) current. Using the bare fermion propagator

$$\delta_j^i G(p) = \langle \psi^i(p) \bar{\psi}_j(-p) \rangle = \delta_j^i \frac{i\not{p}}{p^2}, \quad (2.4)$$

where $\not{p} \equiv \gamma^\mu p_\mu$, the current two-point function in the free fermion theory is found to be

$$\langle J_\mu(p) J_\nu(-p) \rangle_0 = N \frac{2\Gamma(2 - \frac{d}{2})\Gamma(\frac{d}{2})^2}{(4\pi)^{\frac{d}{2}}\Gamma(d)} \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) (p^2)^{\frac{d}{2}-1}. \quad (2.5)$$

Thus, when $d < 4$, one sees that the non-local kinetic term in (2.2) is dominant in the low momentum (IR) limit compared to the two-derivative Maxwell term. Hence, the latter can be dropped at low energies, and one may develop the $1/N$ expansion of the critical theory by using the induced quadratic term

$$S_{\text{crit QED}} = \int \frac{d^d p}{(2\pi)^d} \left(\frac{1}{2} A^\mu(p) \langle J_\mu(p) J_\nu(-p) \rangle_0 A^\nu(-p) - \bar{\psi}_i i\not{p} \psi^i - i\bar{\psi}_i \gamma^\mu A_\mu \psi^i \right). \quad (2.6)$$

Note that this effective action is gauge invariant as it should, due to conservation of the current.

The induced photon propagator is obtained by inverting the non-local kinetic term in (2.6). As usual, this requires gauge-fixing. Working in a generalized Feynman gauge, the propagator is

$$D_{\mu\nu}(p) = \frac{C_A}{N(p^2)^{\frac{d}{2}-1+\Delta}} \left(\delta_{\mu\nu} - (1-\xi) \frac{p_\mu p_\nu}{p^2} \right), \quad (2.7)$$

where ξ is an arbitrary gauge parameter ($\xi = 0$ corresponds to Landau gauge $\partial_\mu A^\mu = 0$). The normalization constant C_A is given by

$$C_A = \frac{(4\pi)^{\frac{d}{2}}\Gamma(d)}{2\Gamma(\frac{d}{2})^2\Gamma(2 - \frac{d}{2})} \quad (2.8)$$

and in (2.7) we have introduced, as in [3], a regulator Δ to handle divergences [4–8], which should be sent to zero at the end of the calculation. This makes the interaction vertex in (2.6) dimensionful, and one should introduce a renormalization scale μ so that $S_{\text{vertex}} = -i\mu^\Delta \int \bar{\psi}_i \gamma^\mu A_\mu \psi^i$.

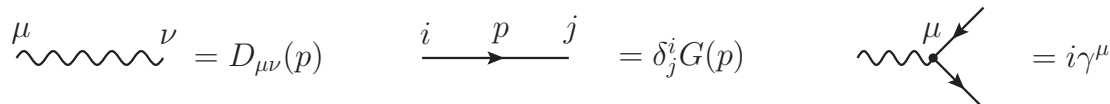


Figure 1. Feynman rules for the Large N QED.

The Feynman rules of the model are summarized in figure 1. In what follows we calculate the two-point function of the $SU(N_f)$ current and stress-energy tensor, which are given by⁴

$$\begin{aligned}
 J_\nu^a &= -\bar{\psi}_i(t^a)_j^i \gamma_\nu \psi^j, \\
 T_{\mu\nu} &= -\frac{1}{4}(\bar{\psi}_i \gamma_{(\mu} D_{\nu)} \psi^i - D_{(\mu}^* \bar{\psi}_i \gamma_{\nu)} \psi^i),
 \end{aligned}
 \tag{2.9}$$

where $\gamma_{(\mu} D_{\nu)} \equiv \gamma_\mu D_\nu + \gamma_\nu D_\mu$ and $D_\mu = \partial_\mu + iA_\mu$. Note that there is no Maxwell term contribution in $T_{\mu\nu}$, as this term was dropped in (2.6) in the critical limit.

We will work in flat Euclidean d -dimensional metric and introduce a null vector z^μ , which satisfies

$$z^2 = z^\mu z^\nu \delta_{\mu\nu} = 0. \tag{2.10}$$

From (1.1), (1.3), we see that the two-point functions of the projected operators $T \equiv z^\mu z^\nu T_{\mu\nu}$ and $J \equiv z^\mu J_\mu$ have the form

$$\begin{aligned}
 \langle T(x)T(0) \rangle &= \frac{4C_T}{(x^2)^d} \frac{x_z^4}{x^4}, \\
 \langle J^a(x)J^b(0) \rangle &= \delta^{ab} \frac{-2C_J}{(x^2)^{d-1}} \frac{x_z^2}{x^2},
 \end{aligned}
 \tag{2.11}$$

where we have introduced the notation $x_z \equiv z^\mu x_\mu$. It will be also useful to report the form of these two-point functions in momentum space, which may be obtained by Fourier transform and reads

$$\begin{aligned}
 \langle T(p)T(-p) \rangle &= C_T \frac{\pi^{\frac{d}{2}} \Gamma(2 - \frac{d}{2})}{2^{d-2} \Gamma(d+2)} \frac{p_z^4}{(p^2)^{2-\frac{d}{2}}}, \\
 \langle J^a(p)J^b(-p) \rangle &= C_J \frac{\pi^{\frac{d}{2}} \Gamma(2 - \frac{d}{2})}{2^{d-3} \Gamma(d)} \frac{p_z^2}{(p^2)^{2-\frac{d}{2}}} \delta^{ab},
 \end{aligned}
 \tag{2.12}$$

where $p_z \equiv z^\mu p_\mu$.

⁴As it was pointed out in [24], for correlation functions with only gauge invariant operators we can omit the gauge fixing part and ghost part of the stress-energy tensor. This was explicitly checked in QCD in $d = 4$ up to three-loops in [25].

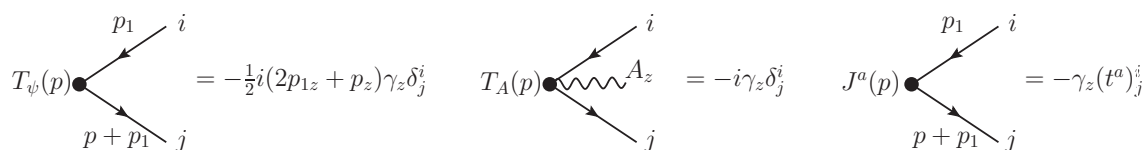


Figure 2. Diagrammatic representation for $T = T_\psi + T_A$ and J^a .

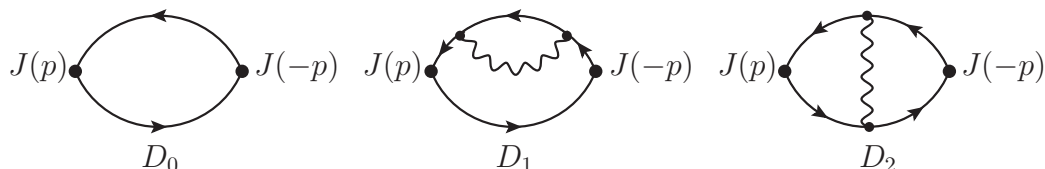


Figure 3. Diagrams contributing to C_J up to order $1/N$.

For the stress-tensor of conformal QED, we may write $T = T_\psi + T_A$, where the two terms are given in momentum space by

$$\begin{aligned}
 T_\psi(p) &= -\frac{1}{2} \int \frac{d^d p_1}{(2\pi)^d} \bar{\psi}_i(-p_1) i\gamma_z (2p_{1z} + p_z) \psi^i(p + p_1), \\
 T_A(p) &= - \int \frac{d^d p_1}{(2\pi)^d} \bar{\psi}_i(-p_1) i\gamma_z A_z \psi^i(p + p_1), \\
 J^a(p) &= - \int \frac{d^d p_1}{(2\pi)^d} \bar{\psi}_i(-p_1) (t^a)_j^i \gamma_z \psi^j(p + p_1).
 \end{aligned} \tag{2.13}$$

The diagrammatic representation is shown in figure 2.

3 Calculation of C_{J_1} and C_{T_1}

The diagrams contributing to $\langle JJ \rangle$ up to order $1/N$

$$\langle J^a(p) J^b(-p) \rangle = D_0 + D_1 + D_2 + \mathcal{O}(1/N^2) \tag{3.1}$$

are shown in figure 3. Their expressions in momentum space and explicit results are listed in appendix B.

Putting together the results, we find

$$\langle J^a(p) J^b(-p) \rangle = -\text{tr}(t^a t^b) C_{J_0} \left(1 + \frac{C_{J_1}(d)}{N} + \mathcal{O}(1/N^2) \right) \frac{\pi^{\frac{d}{2}} \Gamma(2 - \frac{d}{2})}{2^{d-3} \Gamma(d)} \frac{p_z^2}{(p^2)^{2 - \frac{d}{2}}}, \tag{3.2}$$

where $C_{J_1}(d)$ is given in (1.5), and

$$C_{J_0} = \text{Tr} \mathbf{1} \frac{1}{S_d^2} \tag{3.3}$$

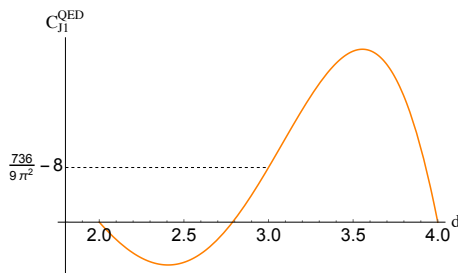


Figure 4. Plot of C_{J1} .

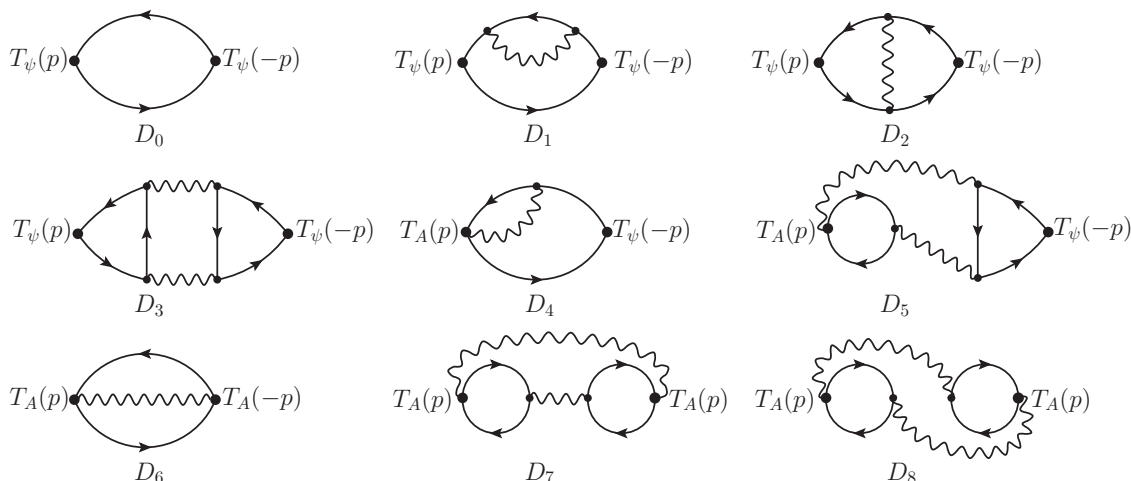


Figure 5. Diagrams contributing to C_T up to N^0 order.

is the free fermion contribution. A plot of C_{J1} as a function of d is given in figure 4. The value in $d = 3$ was given in (1.8) above. One may also extract the following ϵ -expansions

$$C_{J1}|_{d=2+\epsilon} = -\epsilon + \mathcal{O}(\epsilon^2), \quad C_{J1}|_{d=4-\epsilon} = \frac{9\epsilon}{2} + \left(\frac{9}{2} - 9\zeta(3)\right)\epsilon^2 + \mathcal{O}(\epsilon^3). \quad (3.4)$$

In $d = 3$ the leading correction is quite small even for small N ; for $N = 4$, corresponding to $N_f = 1$, it makes C_J around 7% bigger than the free fermion result.

Let us now turn to the calculation of C_T . Up to order N^0 , the stress-tensor two-point function receives contribution from the diagrams shown in figure 5. Note that for some topologies we did not draw explicitly diagrams with the opposite fermion loop direction, but they have to be included. We list the integrands and results for these diagrams in appendix B. We have

$$\langle T^{\text{ren}}(p)T^{\text{ren}}(-p) \rangle = Z_T^2 \langle T(p)T(-p) \rangle = Z_T^2 \left(\sum_{n=0}^8 D_n + \mathcal{O}(1/N) \right), \quad (3.5)$$

where we have introduced a “ Z_T -factor” [3], which is computed in appendix A from the Ward identity. It reads $Z_T = 1 + (Z_{T1}/\Delta + Z'_{T1})/N + \mathcal{O}(1/N^2)$, with

$$Z_{T1} = -\frac{d(d-2)\eta_{m1}}{2(d+2)(d-1)}, \quad Z'_{T1} = -\frac{(d-2)\eta_{m1}}{(d+2)(d-1)}, \quad (3.6)$$

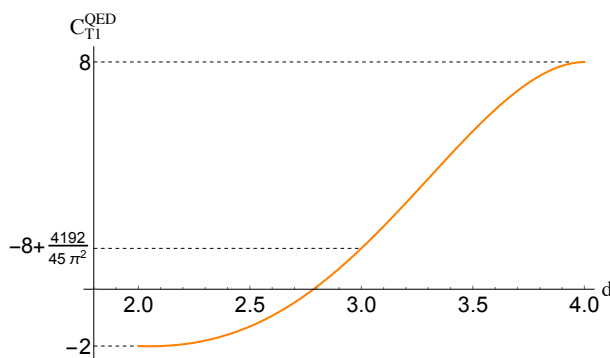


Figure 6. Plot of C_{T1} .

where η_{m1} is given in (1.7). Putting together the results for the diagrams given in appendix B, we obtain

$$\langle T^{\text{ren}}(p)T^{\text{ren}}(-p) \rangle = C_{T0} \left(1 + \frac{C_{T1}(d)}{N} + \mathcal{O}(1/N^2) \right) \frac{\pi^{\frac{d}{2}} \Gamma(2 - \frac{d}{2})}{2^{d-2} \Gamma(d+2)} \frac{p_z^4}{(p^2)^{2-\frac{d}{2}}}, \quad (3.7)$$

where $C_{T1}(d)$ is given in (1.6), and the free fermion contribution is

$$C_{T0} = N \frac{d}{2S_d^2}. \quad (3.8)$$

As a check of our calculation, we note that the final result does not depend on the gauge parameter ξ .

A plot of $C_{T1}(d)$ in $2 < d < 4$ is given in figure 6. We see that C_{T1} is negative for $2 < d < 2.79$. This means that the inequality $C_T^{\text{UV}} > C_T^{\text{IR}}$ is violated for the flow from conformal QED_d (which may be thought of as the UV fixed point of the Thirring model) to the free fermion theory for $2 < d < 2.79$. However, it holds for $2.79 < d < 4$, including in particular $d = 3$.

Near some even dimensions we find

$$C_{T1}|_{d=2+\epsilon} = -2 - \frac{\epsilon}{4}, \quad C_{T1}|_{d=4-\epsilon} = 8 - \frac{\epsilon}{6}, \quad C_{T1}|_{d=6-\epsilon} = -30 + \frac{61\epsilon}{6}. \quad (3.9)$$

Note that in $d = 2$ we get

$$C_T|_{d=2} = \frac{N}{S_2^2} \left(1 - \frac{2}{N} \right). \quad (3.10)$$

This result is precisely as expected, since the conformal QED_2 corresponds to the multiflavor Schwinger model with $2N_f$ Dirac fermions, which is described by a CFT with central charge $c = 2N_f - 1$ [26, 27]. Normalizing (3.10) by the free scalar contribution $C_T^{\text{sc}} = d/((d-1)S_d^2)$, and recalling $N = 4N_f$, we obtain precisely this central charge. In section 5 we will see that $C_{T1}|_{d=4-\epsilon}$ also agrees with the $4 - \epsilon$ expansion.

Near even dimensions the QED_d theory is expected to be described by the free fermions weakly coupled to a $U(1)$ gauge theory with the local kinetic term (1.9). For example, in

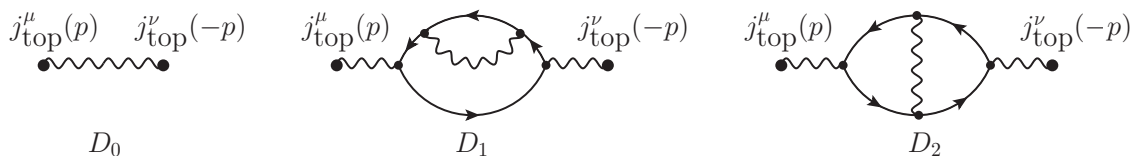


Figure 7. Diagrams contributing to C_J^{top} up to $1/N^2$ order.

$d = 6$ this higher-derivative theory was explored in [12, 28–33]. We may use (1.6) to extract the C_T coefficient for the conformal Maxwell theory (1.9). From (1.6) it follows that

$$C_{T1}^{\text{QED}}|_{\text{even } d} = \frac{2(-1)^{\frac{d}{2}} d!}{(\frac{d}{2} - 1)! (\frac{d}{2} + 1)!} = 2(-1)^{\frac{d}{2}} \binom{d}{\frac{d}{2} - 1}. \quad (3.11)$$

Recalling that the contribution of the free massless fermions is given by (3.8), we find that the C_T of the conformal Maxwell theory is

$$C_T^{\text{conf. Maxwell}}|_{\text{even } d} = \frac{d}{2S_d^2} C_{T1}^{\text{QED}}|_{\text{even } d} = (-1)^{\frac{d}{2}} \frac{d}{S_d^2} \binom{d}{\frac{d}{2} - 1}. \quad (3.12)$$

In $d = 4, 6, 8, 10, \dots$ this formula gives 16, $-90, 448, -2100, \dots$ times $1/S_d^2$. In $d = 4$ this agrees with the standard answer for the Maxwell theory. In $d = 6, 8, \dots$, eq. (3.12) gives new results for the values of C_T in the free conformal theory with the higher-derivative action (1.9).

4 C_J^{top} for the topological current in $d = 3$

In $d = 3$, it is interesting to compute C_J^{top} for the “topological” U(1) current

$$j_{\text{top}}^\mu = \frac{i}{4\pi} \epsilon^{\mu\nu\lambda} F_{\nu\lambda}, \quad (4.1)$$

where the factor of i arises because we are working in Euclidean signature, and the normalization is such that the associated charges are integers. The diagrams contributing to the current two-point function up to order $1/N^2$ are shown in figure 7. The diagrams D_1 and D_2 have the same structure as the corresponding ones in figure 4 for the $SU(N_f)$ current,⁵ with the difference that at the external points we now have the gauge U(1) current, to which we attach the two induced photon propagators. Thus, using the results from appendix B, we find

$$\begin{aligned} \langle j_{\text{top}}^\mu(p) j_{\text{top}}^\nu(-p) \rangle &= -\frac{1}{(4\pi)^2} \epsilon^{\mu\rho\sigma} \epsilon^{\nu\tau\lambda} \langle (p_\rho A_\sigma(p) - p_\sigma A_\rho(p)) (p_\tau A_\lambda(-p) - p_\lambda A_\tau(-p)) \rangle \\ &= -\frac{|p|}{4\pi^2} \frac{C_A}{N} \left(1 - \frac{C_{J1}(3)}{N} + \mathcal{O}(1/N^2) \right) \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right), \end{aligned} \quad (4.2)$$

⁵In fact these diagrams can be extracted from the evaluation of the polarization operator, which was computed in case of QCD in [34].

N_f	1	2	3	4	5	10	20
C_T/C_{T0}	1.3597	1.1798	1.1199	1.0899	1.0719	1.0360	1.0180
C_J/C_{J0}	1.0715	1.0357	1.0238	1.0179	1.0143	1.0072	1.0036
$8\pi^2 C_J^{\text{top}}$	3.0106	1.5632	1.0550	0.7961	0.63919	0.3219	0.1615

Table 1. Results for C_T , C_J and C_J^{top} in $d = 3$ for different values of N_f , the number of 4-component fermions (half the number of 2-component Dirac spinors). C_T and C_J are normalized by the free field values in (3.8) and (3.3). To facilitate the comparison with [13], C_J^{top} is normalized by the free fermion contribution (3.3) for 2-component spinors ($\text{Tr}\mathbf{1} = 2$), which is $\text{Tr}\mathbf{1}/S_3^2 = 1/(8\pi^2)$.

where C_A and $C_{J1}(d)$ are given in (2.8) and (1.5), which yield $C_A|_{d=3} = 32$ and the value of $C_{J1}(3)$ given in (3.4). Therefore, we finally get

$$\langle j_{\text{top}}^\mu(p) j_{\text{top}}^\nu(-p) \rangle = -\frac{8|p|}{\pi^2 N} \left(1 + \frac{1}{N} \left(8 - \frac{736}{9\pi^2} \right) + \mathcal{O}(1/N^2) \right) \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right). \quad (4.3)$$

Comparing with the momentum space normalization in (2.12), we find the result given in eq. (1.11). We note that this is related to C_J in (1.4)–(1.8) by an inversion, $C_A \sim 1/C_J$. This essentially follows from the fact that in the large N critical QED, A_μ and J_μ are related by a Legendre transformation [35, 36], see eq. (2.6).

The conformal bootstrap constraints on the values of C_J , C_T and C_J^{top} in QED₃ for $N_f = 1, 2, 3$ were recently discussed in [13]. In table 1 we summarize our results for these coefficients in $d = 3$ and for different values of N_f (the number of 4-component fermions). These results appear to fall within the regions allowed by the bootstrap for $N_f = 1, 2, 3$.

5 4 – ϵ expansion of C_J and C_T

To find C_J in the $4 - \epsilon$ expansion to the leading non-trivial order, we have to compute diagrams with the same topology as those in the large N approach, figure 3, but now the photon propagator is the standard one obtained from the Maxwell term. It reads

$$D_{\mu\nu}(p) = \frac{1}{p^2} \left(\delta_{\mu\nu} - (1 - \xi) \frac{p_\mu p_\nu}{p^2} \right), \quad (5.1)$$

where we have introduced an arbitrary gauge parameter ($\xi = 1$ is the usual Feynman gauge, and $\xi = 0$ Landau gauge).

The renormalization of the electric charge is well-known, and in minimal subtraction scheme it reads [37]:

$$e_0 = \mu^{\frac{\epsilon}{2}} \left(e + \frac{4N_f}{3\epsilon} \frac{e^3}{(4\pi)^2} + \left(\frac{8N_f^2}{3\epsilon^2} + \frac{2N_f}{\epsilon} \right) \frac{e^5}{(4\pi)^4} + \dots \right), \quad (5.2)$$

where e is the renormalized coupling, and the corresponding beta function is

$$\beta = -\frac{\epsilon}{2} e + \frac{4N_f}{3} \frac{e^3}{(4\pi)^2} + \frac{4N_f e^5}{(4\pi)^4} - \frac{2N_f(22N_f + 9)}{9} \frac{e^7}{(4\pi)^6} + \dots \quad (5.3)$$

Then, one finds an IR stable perturbative fixed point at

$$e_* = \pi \sqrt{\frac{6\epsilon}{N_f}} \left(1 - \frac{9}{16N_f} \epsilon + \frac{3(44N_f + 207)}{512N_f^2} \epsilon^2 + \mathcal{O}(\epsilon^3) \right). \quad (5.4)$$

Computing the diagrams in figure 3 with the photon propagator (5.1), taking a Fourier transform to coordinate space, and setting $e = e_*$ at the end, we obtain in $d = 4 - \epsilon$

$$C_J/C_J^{\text{free}} = 1 + \frac{9\epsilon}{8N_f} + \mathcal{O}(1/N_f^2), \quad (5.5)$$

which precisely agrees with (3.4) (recall that in this case we have $N = N_f \text{Tr} \mathbf{1} = 4N_f$).

To calculate the $4 - \epsilon$ expansion of C_T to order ϵ , we will use as a shortcut the fact that in $d = 4$ the C_T coefficient may be obtained as (see e.g. [38, 39])

$$C_T = \frac{640}{\pi^2} \beta_a, \quad (5.6)$$

where β_a is the beta function for the Weyl-squared term, which is known to be [40, 41]

$$\beta_a = \frac{N_f + 2}{20(4\pi)^2} + \frac{7N_f}{36} \frac{e^2}{(4\pi)^4} + \dots \quad (5.7)$$

The first term corresponds to the contributions of the free fermions and of the Maxwell field, while the second one encodes the leading interaction corrections. The second term, when evaluated at the IR fixed point (5.4) in $d = 4 - \epsilon$, gives $\frac{7\epsilon}{6(16\pi)^2}$. However, this is not the only contribution of order ϵ because the free field contributions need to be evaluated in $4 - \epsilon$ dimensions. The contribution of free massless fermions is given in (3.8). The contribution of the Maxwell field is more subtle, since this theory is scale invariant but not conformal away from four dimensions [42]. However, defining the projected stress-tensor $T_{\text{Maxwell}} = z^\mu z^\nu F_{\mu\alpha} F_{\nu}^\alpha$ (this selects the traceless part of $T_{\mu\nu}$), and using the field strength two-point function [42]

$$\langle F_{\mu\nu}(x) F_{\rho\sigma}(0) \rangle = \frac{(2d-4)\Gamma(\frac{d}{2}-1)}{4\pi^{\frac{d}{2}}(x^2)^{d/2}} \left[\left(\delta_{\mu\rho} - \frac{d}{2} x_\mu x_\rho / x^2 \right) \left(\delta_{\nu\sigma} - \frac{d}{2} x_\nu x_\sigma / x^2 \right) - \mu \leftrightarrow \nu \right] \quad (5.8)$$

we find that $\langle T_{\text{Maxwell}}(x) T_{\text{Maxwell}}(0) \rangle$ takes the form (2.11), just as in a conformal field theory, with the normalization given by

$$C_T^{\text{Maxwell}} = \frac{d^2(d-2)}{2S_d^2}. \quad (5.9)$$

This serves as the natural definition of C_T for the Maxwell theory (in $d = 4$, it agrees with the well-known result [1]). Putting these results together we find

$$C_T^{\text{QED}} = C_T^{\text{free ferm}} \left(1 + \frac{d(d-2) + 35\epsilon/6}{N} + \dots \right) = C_T^{\text{free ferm}} \left(1 + \frac{8 - \epsilon/6}{N} + \dots \right), \quad (5.10)$$

which exactly agrees with (3.9). This gives a highly non-trivial test of the dimension dependence of C_{T1} .

6 A new estimate for symmetry breaking in QED₃

In $d = 3$, the QED Lagrangian has $SU(2N_f)$ global symmetry. For $N_f < N_{f,\text{crit}}$ it may be broken via the generation of vacuum expectation value of the operator $\sum_{j=1}^{N_f} \bar{\psi}_j \psi^j$ (this is written using the 4-component spinors ψ^i and gamma-matrices) [14, 15]. This operator preserves the 3-d time reversal symmetry, but it breaks the global symmetry to $SU(N_f) \times SU(N_f) \times U(1)$.

In an earlier paper [12], using the F -theorem inequality $F^{\text{UV}} > F^{\text{IR}}$ [43–47] we showed that theories with $N_f = 5$ and higher must be in the conformal phase. The F -theorem method is inconclusive, however, for theories with $N_f \leq 4$. There is lattice evidence that theories with $N_f = 1, 2$ are not conformal [21, 22],⁶ but little is known about theories with $N_f = 3, 4$.

Let us now consider a different RG inequality:

$$C_T^{\text{UV}} > C_T^{\text{IR}}, \quad (6.1)$$

which is sometimes called “the C_T theorem”. While there is a known $d = 3$ counterexample to this inequality [16], which involves theories with $\mathcal{N} = 2$ supersymmetry, many known RG flows appear to obey (6.1). For example, it is obeyed for flows involving the scalar $O(N)$ [2, 3] and the Gross-Neveu model [3]. If we think of the conformal QED₃ theory as the UV fixed point of the Thirring model, then the inequality (6.1) is obeyed by the flow to the free fermion theory because $C_{T1}(3) > 0$. We may also test this inequality for the flow from the QED theory in the extreme UV, which consists of the free decoupled Maxwell field and N_f 4-component fermions, to the conformal QED₃. For the former we find using (5.9) and (3.8)

$$C_T^{\text{UV}} = \frac{12N_f + 9}{32\pi^2}. \quad (6.2)$$

For the interacting conformal phase, using our result (1.8), we have

$$C_T^{\text{IR}} = \frac{6N_f}{16\pi^2} \left(1 + \frac{\frac{4192}{45\pi^2} - 8}{4N_f} + \mathcal{O}(1/N_f^2) \right). \quad (6.3)$$

We see that at large N_f (6.1) is obeyed to order N_f^0 because $9 > 3 \left(\frac{4192}{45\pi^2} - 8 \right) \approx 4.32$.

Let us now try applying (6.1) to the $d = 3$ flow from QED in the extreme UV to the broken symmetry phase. For the former we have (6.2). The latter is a free conformal field theory of $2N_f^2 + 1$ scalar fields; therefore, it has

$$C_T^{\text{IR}} = \frac{3(2N_f^2 + 1)}{32\pi^2}. \quad (6.4)$$

We find that the two expressions are equal for $N_f = N_{f,\text{crit}} = 1 + \sqrt{2} \approx 2.414$. This suggests that theories with $N_f = 3$ and higher are in the conformal phase. The inequality (6.1), however, does not require the $N_f = 1, 2$ theories to be conformal, and indeed there is lattice evidence that they are not [21, 22].⁷

⁶See, however, the recent lattice work [48] suggesting that they are conformal.

⁷A more stringent value $N_{f,\text{crit}} = 3/2$ follows from the RG inequality based on the coefficient of the thermal free energy [49]. This appears to be in contradiction with the lattice gauge theory work [22] claiming that the $N_f = 2$ theory is not conformal. However, both $N_{f,\text{crit}} = 3/2$ and $N_{f,\text{crit}} = 1 + \sqrt{2} \approx 2.414$ are consistent with the recent paper [48] claiming that the symmetry breaking does not take place even for $N_f = 1$.

7 C_T for large N_f QCD $_d$

To the leading nontrivial order, the large N_f computations for QCD look similar to those in the QED case. The results for large N_f QCD at the critical point can be deduced from the lagrangian [8, 33, 34, 50–55]

$$\mathcal{L}_{\text{crit QCD}} = -\bar{\psi}_i \gamma^\mu (\partial_\mu + iA_\mu^a t^a) \psi^i + \frac{N_f}{2\xi} (\square^{(d-4)/2} \partial A)^2 + \partial_\mu \bar{c}^a \partial^\mu c^a + f^{abc} \partial^\mu \bar{c}^a A_\mu^b c^c, \quad (7.1)$$

where ψ^i with $i = 1, \dots, N_f$ are the quark fields belonging to the fundamental representation of the colour group G , A_μ^a is the gluon field and c^a and \bar{c}^a are the ghost fields in the adjoint representation of the colour group. We will use the following notation for the Casimirs of the Lie group generators t^a ($[t^a, t^b] = i f^{abc} t^c$):

$$\text{tr}(t^a t^b) = C(r) \delta^{ab}, \quad t^a t^a = C_2(r) \cdot I, \quad f^{acd} f^{bcd} = C_2(G) \cdot I \quad (7.2)$$

and also $\text{tr}(I) = d(r)$ and $\delta^{ab} \delta^{ab} = d(G)$. The stress-energy tensor is (2.9) with $D_\mu = \partial_\mu + iA_\mu^a t^a$, and as we mentioned above we can omit the gauge fixing and ghost parts of $T_{\mu\nu}$ when computing correlation functions of gauge invariant operators. The diagrams contributing to C_T to order $1/N_f$ are the same as in the QED case (see figure 5). It is not hard to show that the relations between QED and QCD diagrams are

$$D_0^{\text{QCD}} = d(r) D_0^{\text{QED}}, \quad D_n^{\text{QCD}} = d(G) D_n^{\text{QED}}, \quad n = 1, \dots, 8, \quad (7.3)$$

where for some diagrams we used the identity $d(r)C_2(r) = d(G)C(r)$. Therefore, we find

$$C_T^{\text{QCD}} = d(r) C_{T0} \left(1 + \frac{1}{N} \frac{d(G)}{d(r)} C_{T1} + \mathcal{O}(1/N^2) \right), \quad (7.4)$$

where C_{T0} and C_{T1} are the results for QED given in (3.8) and (1.6). For $\text{SU}(N_c)$ gauge group we have $d(r) = N_c$ and $d(G) = N_c^2 - 1$, thus

$$C_T^{\text{QCD}} = N_c C_{T0} \left(1 + \frac{1}{N} \frac{N_c^2 - 1}{N_c} C_{T1} + \mathcal{O}(1/N^2) \right). \quad (7.5)$$

Let us check that this agrees with the known exact result for central charge in $d = 2$ gauge theory with massless flavors. The conformal limit of $\text{SU}(N_c)$ gauge theory has central charge [26, 27, 56, 57]

$$c = c_{\text{free}} - \frac{(N_c^2 - 1)k}{k + N_c}. \quad (7.6)$$

The subtraction of the second term is due to the gauging of the $\text{SU}(N_c)$ Kac-Moody algebra with level k . Since there are $2N_f$ 2-d Dirac flavors in the fundamental representation of $\text{SU}(N_c)$, we have $k = 2N_f$. This theory may be described by a $\text{SU}(2N_f)_{N_c} \times \text{U}(1)$ WZW model [26, 27]. Its central charge is

$$c = 2N_f \frac{2N_f N_c + 1}{2N_f + N_c} = N_f N_c - \frac{2(N_c^2 - 1)N_f}{2N_f + N_c} = 2N_f N_c \left(1 - \frac{1}{2N_f} \frac{N_c^2 - 1}{N_c} + \dots \right), \quad (7.7)$$

which is in agreement with (7.5) evaluated in $d = 2$. For a general gauge group G we have

$$c = 2N_f d(r) - \frac{2d(G)N_f}{2N_f + d(r)} = 2N_f d(r) \left(1 - \frac{1}{2N_f} \frac{d(G)}{d(r)} + \dots \right), \quad (7.8)$$

which agrees with (7.4). Analogously, one can easily see that we have the same relation between C_J in QCD and QED:

$$C_J^{\text{QCD}} = d(r)C_{J_0} \left(1 + \frac{1}{N} \frac{d(G)}{d(r)} C_{J_1} + \mathcal{O}(1/N^2) \right), \quad (7.9)$$

where C_{J_0} and C_{J_1} are the results for QED given in (3.3) and (1.5).

Note added. After the first version of this paper appeared, the value of C_T for the $d = 6$ conformal Maxwell theory was calculated directly in [58]. The result is in agreement with our (3.12), providing a check of our methods.

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A Calculation of Z_T

In this appendix we present the computation of the Z_T factor for the stress-energy tensor in the theory of Critical QED. As we show below, a non-trivial Z_T is required for the Ward identity to hold. We define the “renormalized” stress-energy tensor $T_{\mu\nu}^{\text{ren}}$ by

$$T_{\mu\nu}^{\text{ren}}(x) = Z_T T_{\mu\nu}(x), \quad (A.1)$$

where $Z_T = 1 + (Z_{T_1}/\Delta + Z'_{T_1})/N + \mathcal{O}(1/N^2)$, and $T_{\mu\nu}$ is the “bare” stress-tensor. To find Z_T we will use the three-point function $\langle T_{\mu\nu}^{\text{ren}}(x_1) O_m^{\text{ren}}(x_2) O_m^{\text{ren}}(x_3) \rangle$, where $O_m^{\text{ren}} = Z_{O_m} O_m$ is the electron mass operator, Z_{O_m} is its renormalization constant and the bare operator is

$$O_m = \bar{\psi}\psi. \quad (A.2)$$

This three point function is gauge invariant. So using conformal invariance and conservation of the stress-tensor, one has the general expression for the three-point function

$$\langle T_{\mu\nu}^{\text{ren}}(x_1) O_m^{\text{ren}}(x_2) O_m^{\text{ren}}(x_3) \rangle = \frac{-C_{TO_m O_m}}{(x_{12}^2 x_{13}^2)^{\frac{d}{2}-1} (x_{23}^2)^{\Delta_{O_m} - \frac{d}{2} + 1}} \left((X_{23})_\mu (X_{23})_\nu - \frac{1}{d} \delta_{\mu\nu} (X_{23})^2 \right), \quad (A.3)$$

where

$$(X_{23})_\nu = \frac{(x_{12})_\nu}{x_{12}^2} - \frac{(x_{13})_\nu}{x_{13}^2}. \quad (A.4)$$

The conformal Ward identity gives

$$C_{TO_m O_m} = \frac{1}{S_d} \frac{d\Delta_{O_m}}{d-1} C_{O_m}, \quad (A.5)$$

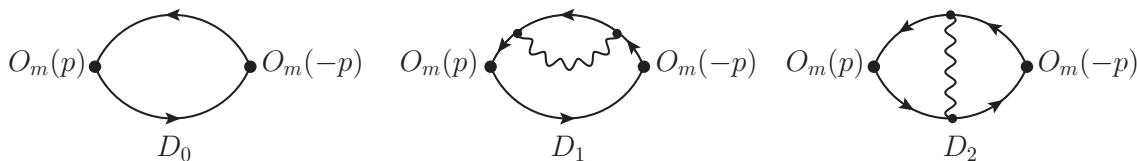


Figure 8. Diagrams contributing to $\langle O_m(p)O_m(-p) \rangle$ up to order $1/N$.

where C_{O_m} and Δ_{O_m} are two-point constant and anomalous dimension of the operator O_m in coordinate space:

$$\langle O_m^{\text{ren}}(x)O_m^{\text{ren}}(0) \rangle = \frac{C_{O_m}}{(x^2)^{\Delta_{O_m}}}. \quad (\text{A.6})$$

Taking the Fourier transform of (A.3) and setting the momentum of the stress-energy tensor to zero for simplicity, one finds in terms of the projected stress tensor $T = z^\mu z^\nu T_{\mu\nu}$

$$\langle T^{\text{ren}}(0)O_m^{\text{ren}}(p)O_m^{\text{ren}}(-p) \rangle = (d - 2\Delta_{O_m})\tilde{C}_{O_m} \frac{p_z^2}{(p^2)^{\frac{d}{2} - \Delta_{O_m} + 1}}, \quad (\text{A.7})$$

where \tilde{C}_{O_m} is the two-point constant of $\langle O_m^{\text{ren}}O_m^{\text{ren}} \rangle$ correlator in the momentum space:

$$\langle O_m^{\text{ren}}(p)O_m^{\text{ren}}(-p) \rangle = \frac{\tilde{C}_{O_m}}{(p^2)^{1 - \frac{d}{2} - \eta_m}}, \quad (\text{A.8})$$

and $\Delta_{O_m} = d - 1 + \eta_m$, where $\eta_m = \eta_{m1}/N + \mathcal{O}(1/N^2)$. In order to find \tilde{C}_{O_m} , Z_{O_m} and η_{m1} up to $1/N$ order, we have to calculate the diagrams depicted in figure 8. The expressions for the diagrams are

$$D_0 = \int \frac{d^d p_1}{(2\pi)^d} (-1) \text{Tr}(G(p + p_1)G(p_1)), \quad (\text{A.9})$$

$$D_1 = 2(i)^2 \mu^{2\Delta} \int \frac{d^d p_1 d^d p_2}{(2\pi)^{2d}} (-1) \text{Tr}(G(p + p_1)G(p_1)\gamma^{\nu_1}G(p_2)\gamma^{\nu_2}G(p_1))D_{\nu_1\nu_2}(p_1 - p_2),$$

$$D_2 = (i)^2 \mu^{2\Delta} \int \frac{d^d p_1 d^d p_2}{(2\pi)^{2d}} (-1) \text{Tr}(G(p + p_1)\gamma^{\nu_1}G(p + p_2)G(p_2)\gamma^{\nu_2}G(p_1))D_{\nu_1\nu_2}(p_1 - p_2)$$

and

$$\langle O_m^{\text{ren}}(p)O_m^{\text{ren}}(-p) \rangle = Z_{O_m}^2 \langle O_m(p)O_m(-p) \rangle = Z_{O_m}^2 (D_0 + D_1 + D_2 + \mathcal{O}(1/N^2)). \quad (\text{A.10})$$

Computing these diagrams one finds

$$2Z_{O_m1} = \eta_{m1} = -\frac{2(d-1)\Gamma(d)}{\Gamma(\frac{d}{2})^2\Gamma(\frac{d}{2}+1)\Gamma(2-\frac{d}{2})} \quad (\text{A.11})$$

and

$$\tilde{C}_{O_m} = \frac{4^{1-d}\pi^{\frac{3-d}{2}}\text{Tr}\mathbf{1}}{\Gamma(\frac{d-1}{2})\sin(\pi\frac{d}{2})} \left(1 + \frac{1}{N}\eta_{m1} \left(\frac{3d(d-2)}{8(d-1)}\Theta(d) - \Psi(d) + \frac{d-2}{d} \right) \right), \quad (\text{A.12})$$

where $\Theta(d) \equiv \psi'(d/2) - \psi'(1)$ and $\Psi(d) \equiv \psi(d-1) + \psi(2-d/2) - \psi(1) - \psi(d/2-1)$.

Now we can calculate the three-point function $\langle T^{\text{ren}}(0)O_m^{\text{ren}}(p)O_m^{\text{ren}}(-p) \rangle$ using Feynman diagrams, namely we have

$$\langle T^{\text{ren}}(0)O_m^{\text{ren}}(p)O_m^{\text{ren}}(-p) \rangle = Z_T Z_{O_m}^2 \langle T(0)O_m(p)O_m(-p) \rangle \quad (\text{A.13})$$

and the diagrams contributing to $\langle T(0)O_m(p)O_m(-p) \rangle$ up to order $1/N$ are shown in figure 9, and the explicit results are listed in eq. (A.14) below. Putting these diagrams together and equating the expression (A.7) required by conformal symmetry with the diagrammatic result for (A.13), we find that the required Z_T factor is the one given in (3.6). As a check of our calculation, we note that dependence on the gauge parameter ξ drops out from the final result.

Let us end this section by listing the results for the diagrams in figure 9. They are given by

$$\begin{aligned} D_0 &= -\text{Tr} \mathbf{1} \frac{\pi \csc(\pi \frac{d}{2}) \Gamma(\frac{d}{2})}{(4\pi)^{\frac{d}{2}} \Gamma(d-2)} \frac{p_z^2}{(p^2)^{2-\frac{d}{2}}}, \\ D_1 &= \frac{1}{N} D_0 \eta_{m1} \left(\left(\frac{1}{\Delta} - \log \left(\frac{p^2}{\mu^2} \right) \right) \left(\frac{d-4}{4} + \frac{d\xi}{4(d-1)} \right) + \left(\left(\frac{d-4}{4} + \frac{d\xi}{4(d-1)} \right) \Psi(d) \right. \right. \\ &\quad \left. \left. - \frac{d^3 - 8d^2 + 16d - 16}{4(d-2)d} - \frac{d^2 \xi}{4(d-2)(d-1)} \right) \right), \\ D_2 &= \frac{1}{N} D_0 \eta_{m1} \left(- \left(\frac{1}{\Delta} - \log \left(\frac{p^2}{\mu^2} \right) \right) \left(\frac{d^3 - 7d^2 + 10d - 8}{8(d-1)(d+2)} + \frac{d\xi}{8(d-1)} \right) - \left(\left(\frac{d^3 - 7d^2 + 10d - 8}{8(d-1)(d+2)} + \right. \right. \right. \\ &\quad \left. \left. \frac{d\xi}{8(d-1)} \right) \Psi - \frac{2d^7 - 21d^6 + 63d^5 - 68d^4 - 60d^3 + 192d^2 - 160d + 64}{8(d-2)(d-1)^2 d(d+2)^2} - \frac{(2d^3 - 7d^2 + 12d - 8)\xi}{8(d-2)(d-1)^2} \right), \\ D_3 &= \frac{1}{N} D_0 \eta_{m1} \left(- \left(\frac{1}{\Delta} - \log \left(\frac{p^2}{\mu^2} \right) \right) \left(\frac{d}{4} + \frac{d\xi}{4(d-1)} \right) + \left(\frac{3d(d-2)^2}{8(d-1)^2} \Theta(d) - \left(\frac{d}{4} + \frac{d\xi}{4(d-1)} \right) \Psi(d) \right. \right. \\ &\quad \left. \left. + \frac{d^3 - d^2 + 2d - 4}{4(d-2)(d-1)^2} + \frac{(3d^2 - 6d + 4)\xi}{4(d-2)(d-1)^2} \right) \right), \\ D_4 &= \frac{1}{N} D_0 \eta_{m1} \left(\left(\frac{1}{\Delta} - \log \left(\frac{p^2}{\mu^2} \right) \right) \left(\frac{d-4}{8} + \frac{d\xi}{8(d-1)} \right) + \left(\left(\frac{d-4}{8} + \frac{d\xi}{8(d-1)} \right) \Psi(d) \right. \right. \\ &\quad \left. \left. + \frac{3d^3 - 16d^2 + 32d - 16}{8(d-2)(d-1)d} - \frac{d^2 \xi}{8(d-2)(d-1)^2} \right) \right), \\ D_5 &= \frac{1}{N} D_0 \eta_{m1} \left(- \left(\frac{1}{\Delta} - 2 \log \left(\frac{p^2}{\mu^2} \right) \right) \left(\frac{(d-2)^2}{4(d-1)(d+2)} \right) - \left(\frac{(d-2)^2}{2(d-1)(d+2)} \Psi(d) \right. \right. \\ &\quad \left. \left. - \frac{(d-2)(5d^4 - 9d^3 + 4d^2 + 28d - 16)}{4(d-1)^2 d(d+2)^2} - \frac{(d-2)\xi}{2(d-1)^2} \right) \right), \\ D_6 &= \frac{1}{N} D_0 \eta_{m1} \left(\frac{3d(d-2)}{8(d-1)^2} \Theta(d) + \frac{d-2}{4(d-1)} - \frac{(d-2)\xi}{2(d-1)^2} \right), \\ D_7 &= \frac{1}{N} D_0 \eta_{m1} \left(\frac{3d(d-2)}{8(d-1)^2} \Theta(d) + \frac{1}{2(d-1)} - \frac{\xi}{2(d-1)} \right), \\ D_8 &= \frac{1}{N} D_0 \eta_{m1} \left(\left(\frac{1}{\Delta} - \log \left(\frac{p^2}{\mu^2} \right) \right) \left(\frac{d-2}{2(d-1)} \right) + \left(\frac{(d-2)}{2(d-1)} \Psi(d) - \frac{d^2 - 3d + 4}{2(d-1)d} + \frac{\xi}{2(d-1)} \right) \right), \\ D_9 &= \frac{1}{N} D_0 \eta_{m1} \left(- \left(\frac{1}{\Delta} - 2 \log \left(\frac{p^2}{\mu^2} \right) \right) \left(\frac{d-2}{4(d-1)} \right) - \left(\frac{(d-2)}{2(d-1)} \Psi(d) - \frac{d^3 - 3d^2 + 5d - 4}{2(d-1)^2 d} + \frac{(d-2)\xi}{2(d-1)^2} \right) \right), \\ D_{10} &= \frac{1}{N} D_0 \eta_{m1} \left(- \frac{3d(d-2)}{8(d-1)^2} \Theta(d) - \frac{2d-3}{2(d-1)^2} + \frac{(d-2)\xi}{2(d-1)^2} \right), \end{aligned} \quad (\text{A.14})$$

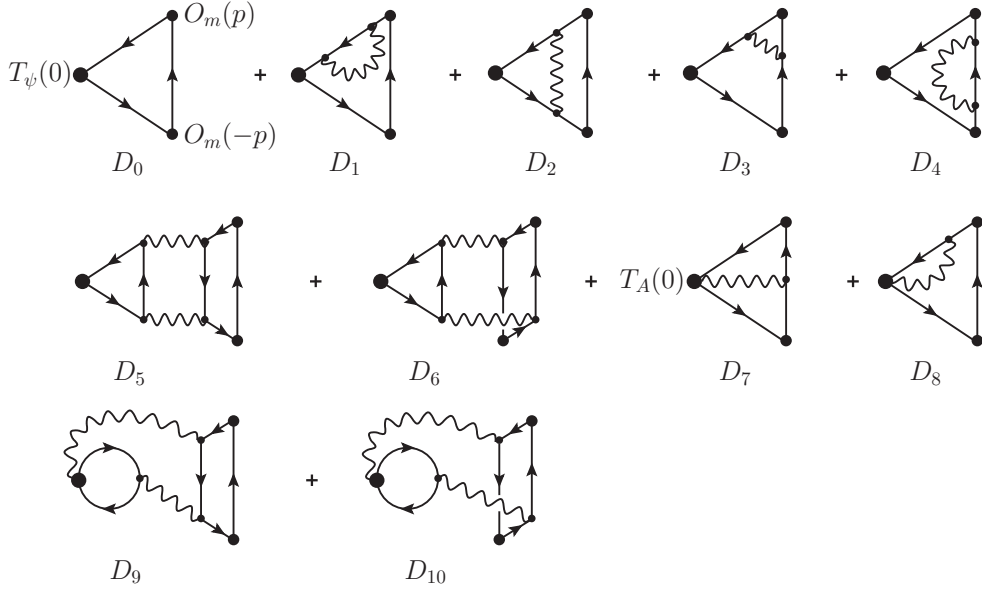


Figure 9. Diagrams contributing to $\langle T(0)O_m(p)O_m(-p) \rangle$ up to order $1/N$.

where $\Theta(d) \equiv \psi'(d/2) - \psi'(1)$ and $\Psi(d) \equiv \psi(d-1) + \psi(2-d/2) - \psi(1) - \psi(d/2-1)$ and η_{m1} is given in (A.11). We notice that

$$D_7 + D_8 + D_9 + D_{10} = \frac{D_0 \eta_{m1}}{N \Delta} \frac{(d-2)}{4(d-1)}. \quad (\text{A.15})$$

B Results for $\langle JJ \rangle$ and $\langle TT \rangle$ diagrams

The diagrams for $\langle JJ \rangle$ shown in figure 3 are given explicitly by

$$D_0 = \text{tr}(t^a t^b) \int \frac{d^d p_1}{(2\pi)^d} (-1) \text{Tr}(\gamma_z G(p_1) \gamma_z G(p+p_1)), \quad (\text{B.1})$$

$$D_1 = 2 \text{tr}(t^a t^b) (i)^2 \mu^{2\Delta} \int \frac{d^d p_1 d^d p_2}{(2\pi)^{2d}} (-1) \text{Tr}(\gamma_z G(p+p_1) \gamma_z G(p_1) \gamma^{\nu_1} G(p_2) \gamma^{\nu_2} G(p_1)) D_{\nu_1 \nu_2}(p_1-p_2),$$

$$D_2 = \text{tr}(t^a t^b) (i)^2 \mu^{2\Delta} \int \frac{d^d p_1 d^d p_2}{(2\pi)^{2d}} (-1) \text{Tr}(\gamma_z G(p+p_1) \gamma^{\nu_1} G(p+p_2) \gamma_z G(p_2) \gamma^{\nu_2} G(p_1)) D_{\nu_1 \nu_2}(p_1-p_2)$$

and the results are

$$D_0 = \text{tr}(t^a t^b) \text{Tr} \mathbf{1} \frac{\pi \csc(\frac{\pi d}{2}) \Gamma(\frac{d}{2})}{(4\pi)^{\frac{d}{2}} (d-1) \Gamma(d-2)} \frac{p_z^2}{(p^2)^{2-\frac{d}{2}}},$$

$$D_1 = \frac{1}{N} D_0 \eta_{m1} \left(\left(\frac{1}{\Delta} - \log\left(\frac{p^2}{\mu^2}\right) \right) \left(\frac{d-4}{4} + \frac{d\xi}{4(d-1)} \right) + \left(\left(\frac{d-4}{4} + \frac{d\xi}{4(d-1)} \right) \Psi(d) + \frac{3d^3 - 16d^2 + 32d - 16}{4(d-2)(d-1)d} - \frac{d^2 \xi}{4(d-2)(d-1)^2} \right) \right),$$

$$D_2 = \frac{1}{N} D_0 \eta_{m1} \left(- \left(\frac{1}{\Delta} - \log\left(\frac{p^2}{\mu^2}\right) \right) \left(\frac{d-4}{4} + \frac{d\xi}{4(d-1)} \right) + \left(- \left(\frac{d-4}{4} + \frac{d\xi}{4(d-1)} \right) \Psi(d) + \frac{3d(d-2)}{8(d-1)} \Theta(d) + \frac{(d-4)d}{4(d-2)(d-1)} + \frac{d^2 \xi}{4(d-2)(d-1)^2} \right) \right). \quad (\text{B.2})$$

The diagrams for $\langle TT \rangle$ depicted in figure 5 are given explicitly by

$$\begin{aligned}
 D_0 &= N_f \left(\frac{-i}{2} \right)^2 \int \frac{d^d p_1}{(2\pi)^d} (2p_{1z} + p_z)^2 (-1) \text{Tr}(\gamma_z G(p + p_1) \gamma_z G(p_1)), \\
 D_1 &= 2N_f \mu^{2\Delta} \left(\frac{-i}{2} \right)^2 (i)^2 \int \frac{d^d p_1 d^d p_2}{(2\pi)^{2d}} (2p_{1z} + p_z)^2 (-1) \\
 &\quad \times \text{Tr}(\gamma_z G(p + p_1) \gamma_z G(p_1) \gamma^{\nu_1} G(p_2) \gamma^{\nu_2} G_{p_1}) D_{\nu_1 \nu_2}(p_1 - p_2), \\
 D_2 &= N_f \mu^{2\Delta} \left(\frac{-i}{2} \right)^2 (i)^2 \int \frac{d^d p_1 d^d p_2}{(2\pi)^{2d}} (2p_{1z} + p_z)(2p_{2z} + p_z) (-1) \\
 &\quad \times \text{Tr}(\gamma_z G(p + p_1) \gamma^{\nu_1} G(p + p_2) \gamma_z G(p_2) \gamma^{\nu_2} G_{p_1}) D_{\nu_1 \nu_2}(p_1 - p_2), \\
 D_3 &= N_f^2 \mu^{4\Delta} \left(\frac{-i}{2} \right)^2 (i)^4 \int \frac{d^d p_1 d^d p_2 d^d p_3}{(2\pi)^{3d}} (2p_{1z} + p_z) (-1) \\
 &\quad \times \text{Tr}(\gamma_z G(p + p_1) \gamma^{\nu_1} G(p_1 - p_3) \gamma^{\nu_2} G(p_1)) \\
 &\quad \times D_{\nu_1 \nu_3}(p + p_3) D_{\nu_2 \nu_4}(p_3) (2p_{2z} + p_z) (-1) \text{Tr}(\gamma_z G(p_2) \gamma^{\nu_4} G(p_2 - p_3) \gamma^{\nu_3} G(p + p_2)) + \dots, \\
 D_4 &= 2N_f \mu^{2\Delta} \left(\frac{-i}{2} \right) (-i)(i) \int \frac{d^d p_1 d^d p_2}{(2\pi)^{2d}} (2p_{1z} + p_z) (-1) \\
 &\quad \times \text{Tr}(\gamma_z G(p + p_1) \gamma_z G(p_2) \gamma^{\nu_1} G(p_1)) D_{\nu_1 z}(p_1 - p_2), \\
 D_5 &= 2N_f^2 \mu^{4\Delta} \left(\frac{-i}{2} \right) (-i)(i)^3 \int \frac{d^d p_1 d^d p_2 d^d p_3}{(2\pi)^{3d}} (-1) \\
 &\quad \times \text{Tr}(\gamma_z G(p_1 - p_3) \gamma^{\nu_1} G(p_1)) D_{\nu_1 \nu_2}(p_3) D_{z\nu_3}(p + p_3) \\
 &\quad \times (2p_{2z} + p_z) (-1) \text{Tr}(\gamma_z G(p_2) \gamma^{\nu_2} G(p_2 - p_3) \gamma^{\nu_3} G(p + p_2)) + \dots, \\
 D_6 &= N_f \mu^{2\Delta} (-i)^2 \int \frac{d^d p_1 d^d p_2}{(2\pi)^{2d}} (-1) \text{Tr}(\gamma_z G(p + p_2) \gamma_z G(p_1)) D_{zz}(p_1 - p_2), \\
 D_7 &= N_f^2 \mu^{4\Delta} (-i)^2 (i)^2 \int \frac{d^d p_1 d^d p_2 d^d p_3}{(2\pi)^{3d}} (-1) \text{Tr}(\gamma_z G(p_1 - p_3) \gamma^{\nu_1} G(p_1)) D_{zz}(p + p_3) D_{\nu_1 \nu_2}(p_3) \\
 &\quad \times (-1) \text{Tr}(\gamma_z G(p_2) \gamma^{\nu_2} G(p_2 - p_3)), \\
 D_8 &= N_f^2 \mu^{4\Delta} (-i)^2 (i)^2 \int \frac{d^d p_1 d^d p_2 d^d p_3}{(2\pi)^{3d}} (-1) \text{Tr}(\gamma_z G(p_1 - p_3) \gamma^{\nu_1} G(p_1)) D_{\nu_1 z}(p_3) D_{z\nu_2}(p + p_3) \\
 &\quad \times (-1) \text{Tr}(\gamma_z G(p_2 - p_3) \gamma^{\nu_2} G(p + p_2)) \tag{B.3}
 \end{aligned}$$

where dots mean that there is also an expression which corresponds to the opposite direction of the fermion loop. After carrying out the momentum integrals using techniques similar to the ones described in the appendices of [3], we find

$$\begin{aligned}
 D_0 &= -N \frac{\pi^{1-\frac{d}{2}} \csc(\pi \frac{d}{2}) \Gamma(\frac{d}{2})}{4^{\frac{d}{2}+1} (d-1)(d+1) \Gamma(d-2)} \frac{p_z^4}{(p^2)^{2-\frac{d}{2}}}, \\
 D_1 &= \frac{1}{N} D_0 \eta_{m1} \left(\left(\frac{1}{\Delta} - \log \left(\frac{p^2}{\mu^2} \right) \right) \left(\frac{d-4}{4} + \frac{d\xi}{4(d-1)} \right) + \left(\left(\frac{d-4}{4} + \frac{d\xi}{4(d-1)} \right) \Psi(d) \right. \right. \\
 &\quad \left. \left. + \frac{2d^4 - 10d^3 + 15d^2 + 4d - 8}{2(d-2)(d-1)d(d+1)} - \frac{d(2d-1)\xi}{2(d-2)(d-1)^2(d+1)} \right) \right),
 \end{aligned}$$

$$\begin{aligned}
D_2 &= \frac{1}{N} D_0 \eta_{m1} \left(- \left(\frac{1}{\Delta} - \log \left(\frac{p^2}{\mu^2} \right) \right) \left(\frac{d^3 - 7d^2 + 10d - 8}{4(d-1)(d+2)} + \frac{d\xi}{4(d-1)} \right) + \left(\frac{3d(d-2)}{8(d-1)} \Theta(d) \right. \right. \\
&\quad \left. \left. - \left(\frac{d^3 - 7d^2 + 10d - 8}{4(d-1)(d+2)} + \frac{d\xi}{4(d-1)} \right) \Psi(d) + \frac{5d^5 - 27d^4 + 44d^3 - 30d^2 - 12d + 16}{2(d-2)(d-1)^2 d(d+1)(d+2)} \right. \right. \\
&\quad \left. \left. - \frac{(d^3 - 4d^2 + 2)\xi}{2(d-2)(d-1)^2(d+1)} \right) \right), \\
D_3 &= \frac{1}{N} D_0 \eta_{m1} \left(- \left(\frac{1}{\Delta} - 2 \log \left(\frac{p^2}{\mu^2} \right) \right) \left(\frac{(d-2)^2}{2(d-1)(d+2)} \right) + \left(\frac{2(d-2)}{(d-1)(d+2)} \Psi(d) \right. \right. \\
&\quad \left. \left. - \frac{d(d^3 - 8d + 11)}{(d-1)^2(d+1)(d+2)} + \frac{\xi}{2(d-1)} \right) \right) + dD_0/(2N), \tag{B.4}
\end{aligned}$$

and

$$\begin{aligned}
D_4 &= \frac{1}{N} D_0 \eta_{m1} \left(\left(\frac{1}{\Delta} - \log \left(\frac{p^2}{\mu^2} \right) \right) \left(\frac{d-2}{d-1} \right) + \left(\frac{(d-2)\Psi(d)}{d-1} - \frac{d^4 - 4d^3 + 5d^2 + 2d - 2}{(d-1)^2 d(d+1)} + \frac{\xi}{d-1} \right) \right), \\
D_5 &= \frac{1}{N} D_0 \eta_{m1} \left(- \left(\frac{1}{\Delta} - 2 \log \left(\frac{p^2}{\mu^2} \right) \right) \left(\frac{d-2}{2(d-1)} \right) - \left(\frac{(d-2)\Psi(d)}{d-1} - \frac{d^4 - 4d^3 + 5d^2 + 2d - 2}{(d-1)^2 d(d+1)} + \frac{\xi}{d-1} \right) \right) \\
&\quad - dD_0/N, \\
D_6 &= \frac{1}{N} D_0 \eta_{m1} \left(- \frac{\xi - 1}{2(d-1)} \right), \\
D_7 &= \frac{1}{N} D_0 \eta_{m1} \left(\frac{\xi - 1}{2(d-1)} \right), \\
D_8 &= dD_0/(2N), \tag{B.5}
\end{aligned}$$

where $\Theta(d) \equiv \psi'(d/2) - \psi'(1)$ and $\Psi(d) \equiv \psi(d-1) + \psi(2-d/2) - \psi(1) - \psi(d/2-1)$ and η_{m1} is given in (A.11). We notice that

$$D_4 + D_5 + D_6 + D_7 + D_8 = \frac{D_0 \eta_{m1}}{N \Delta} \frac{(d-2)}{2(d-1)} - dD_0/(2N). \tag{B.6}$$

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References

- [1] H. Osborn and A.C. Petkou, *Implications of conformal invariance in field theories for general dimensions*, *Annals Phys.* **231** (1994) 311 [[hep-th/9307010](#)] [[INSPIRE](#)].
- [2] A.C. Petkou, *C(T) and C(J) up to next-to-leading order in 1/N in the conformally invariant O(N) vector model for 2 < d < 4*, *Phys. Lett.* **B 359** (1995) 101 [[hep-th/9506116](#)] [[INSPIRE](#)].
- [3] K. Diab, L. Fei, S. Giombi, I.R. Klebanov and G. Tarnopolsky, *On C_J and C_T in the Gross-Neveu and O(N) Models*, [arXiv:1601.07198](#) [[INSPIRE](#)].
- [4] A.N. Vasiliev and M.Yu. Nalimov, *Analog of Dimensional Regularization for Calculation of the Renormalization Group Functions in the 1/n Expansion for Arbitrary Dimension of Space*, *Theor. Math. Phys.* **55** (1983) 423 [[INSPIRE](#)].

- [5] A.N. Vasiliev, Yu.M. Pismak and Yu.R. Khonkonen, *Simple Method of Calculating the Critical Indices in the $1/N$ Expansion*, *Theor. Math. Phys.* **46** (1981) 104 [INSPIRE].
- [6] A.N. Vasiliev, Yu.M. Pismak and Yu.R. Khonkonen, *$1/N$ Expansion: Calculation of the Exponents η and ν in the Order $1/N^2$ for Arbitrary Number of Dimensions*, *Theor. Math. Phys.* **47** (1981) 465 [INSPIRE].
- [7] S.E. Derkachov and A.N. Manashov, *The simple scheme for the calculation of the anomalous dimensions of composite operators in the $1/N$ expansion*, *Nucl. Phys.* **B 522** (1998) 301 [[hep-th/9710015](#)] [INSPIRE].
- [8] M. Ciuchini, S.E. Derkachov, J.A. Gracey and A.N. Manashov, *Quark mass anomalous dimension at $O(1/N_f^2)$ in QCD*, *Phys. Lett.* **B 458** (1999) 117 [[hep-ph/9903410](#)] [INSPIRE].
- [9] J.A. Gracey, *Electron mass anomalous dimension at $O(1/N_f^2)$ in quantum electrodynamics*, *Phys. Lett.* **B 317** (1993) 415 [[hep-th/9309092](#)] [INSPIRE].
- [10] Y. Huh, P. Strack and S. Sachdev, *Conserved current correlators of conformal field theories in $2+1$ dimensions*, *Phys. Rev.* **B 88** (2013) 155109 [Erratum *ibid.* **B 90** (2014) 199902] [[arXiv:1307.6863](#)] [INSPIRE].
- [11] Y. Huh and P. Strack, *Stress tensor and current correlators of interacting conformal field theories in $2+1$ dimensions: Fermionic Dirac matter coupled to $U(1)$ gauge field*, *JHEP* **01** (2015) 147 [Erratum *ibid.* **03** (2016) 054] [[arXiv:1410.1902](#)] [INSPIRE].
- [12] S. Giombi, I.R. Klebanov and G. Tarnopolsky, *Conformal QED_d , F -Theorem and the ϵ Expansion*, *J. Phys.* **A 49** (2016) 135403 [[arXiv:1508.06354](#)] [INSPIRE].
- [13] S.M. Chester and S.S. Pufu, *Towards bootstrapping QED_3* , *JHEP* **08** (2016) 019 [[arXiv:1601.03476](#)] [INSPIRE].
- [14] R.D. Pisarski, *Chiral Symmetry Breaking in Three-Dimensional Electrodynamics*, *Phys. Rev.* **D 29** (1984) 2423 [INSPIRE].
- [15] T. Appelquist, D. Nash and L.C.R. Wijewardhana, *Critical Behavior in $(2+1)$ -Dimensional QED*, *Phys. Rev. Lett.* **60** (1988) 2575 [INSPIRE].
- [16] T. Nishioka and K. Yonekura, *On RG Flow of τ_{RR} for Supersymmetric Field Theories in Three-Dimensions*, *JHEP* **05** (2013) 165 [[arXiv:1303.1522](#)] [INSPIRE].
- [17] K. Kaveh and I.F. Herbut, *Chiral symmetry breaking in $QED(3)$ in presence of irrelevant interactions: A renormalization group study*, *Phys. Rev.* **B 71** (2005) 184519 [[cond-mat/0411594](#)] [INSPIRE].
- [18] C.S. Fischer, R. Alkofer, T. Dahm and P. Maris, *Dynamical chiral symmetry breaking in unquenched $QED(3)$* , *Phys. Rev.* **D 70** (2004) 073007 [[hep-ph/0407104](#)] [INSPIRE].
- [19] J. Braun, H. Gies, L. Janssen and D. Roscher, *Phase structure of many-flavor QED_3* , *Phys. Rev.* **D 90** (2014) 036002 [[arXiv:1404.1362](#)] [INSPIRE].
- [20] L. Di Pietro, Z. Komargodski, I. Shamir and E. Stamou, *Quantum Electrodynamics in $D = 3$ from the ϵ Expansion*, *Phys. Rev. Lett.* **116** (2016) 131601 [[arXiv:1508.06278](#)] [INSPIRE].
- [21] C. Strouthos and J.B. Kogut, *The Phases of Non-Compact QED_3* , *PoS(LATTICE 2007)278* [[arXiv:0804.0300](#)] [INSPIRE].
- [22] O. Raviv, Y. Shamir and B. Svetitsky, *Nonperturbative β -function in three-dimensional electrodynamics*, *Phys. Rev.* **D 90** (2014) 014512 [[arXiv:1405.6916](#)] [INSPIRE].

- [23] T. Appelquist and R.D. Pisarski, *High-Temperature Yang-Mills Theories and Three-Dimensional Quantum Chromodynamics*, *Phys. Rev. D* **23** (1981) 2305 [INSPIRE].
- [24] N.K. Nielsen, *The Energy Momentum Tensor in a Nonabelian Quark Gluon Theory*, *Nucl. Phys. B* **120** (1977) 212 [INSPIRE].
- [25] M.F. Zoller and K.G. Chetyrkin, *OPE of the energy-momentum tensor correlator in massless QCD*, *JHEP* **12** (2012) 119 [arXiv:1209.1516] [INSPIRE].
- [26] D. Gepner, *Nonabelian Bosonization and Multiflavor QED and QCD in Two-dimensions*, *Nucl. Phys. B* **252** (1985) 481 [INSPIRE].
- [27] I. Affleck, *On the Realization of Chiral Symmetry in (1+1)-dimensions*, *Nucl. Phys. B* **265** (1986) 448 [INSPIRE].
- [28] E.A. Ivanov, A.V. Smilga and B.M. Zupnik, *Renormalizable supersymmetric gauge theory in six dimensions*, *Nucl. Phys. B* **726** (2005) 131 [hep-th/0505082] [INSPIRE].
- [29] E.A. Ivanov and A.V. Smilga, *Conformal properties of hypermultiplet actions in six dimensions*, *Phys. Lett. B* **637** (2006) 374 [hep-th/0510273] [INSPIRE].
- [30] A.V. Smilga, *6D superconformal theory as the theory of everything*, [hep-th/0509022](#) [INSPIRE].
- [31] A.V. Smilga, *Chiral anomalies in higher-derivative supersymmetric 6D theories*, *Phys. Lett. B* **647** (2007) 298 [hep-th/0606139] [INSPIRE].
- [32] M. Beccaria and A.A. Tseytlin, *Conformal a-anomaly of some non-unitary 6d superconformal theories*, *JHEP* **09** (2015) 017 [arXiv:1506.08727] [INSPIRE].
- [33] J.A. Gracey, *Six dimensional QCD at two loops*, *Phys. Rev. D* **93** (2016) 025025 [arXiv:1512.04443] [INSPIRE].
- [34] M. Ciuchini, S.E. Derkachov, J.A. Gracey and A.N. Manashov, *Computation of quark mass anomalous dimension at $O(1/N_f^2)$ in quantum chromodynamics*, *Nucl. Phys. B* **579** (2000) 56 [hep-ph/9912221] [INSPIRE].
- [35] E. Witten, *SL(2, Z) action on three-dimensional conformal field theories with Abelian symmetry*, [hep-th/0307041](#) [INSPIRE].
- [36] R.G. Leigh and A.C. Petkou, *SL(2, Z) action on three-dimensional CFTs and holography*, *JHEP* **12** (2003) 020 [hep-th/0309177] [INSPIRE].
- [37] M. Moshe and J. Zinn-Justin, *Quantum field theory in the large-N limit: A review*, *Phys. Rept.* **385** (2003) 69 [hep-th/0306133] [INSPIRE].
- [38] A. Cappelli, D. Friedan and J.I. Latorre, *C theorem and spectral representation*, *Nucl. Phys. B* **352** (1991) 616 [INSPIRE].
- [39] A. Petkou, *Conserved currents, consistency relations and operator product expansions in the conformally invariant $O(N)$ vector model*, *Annals Phys.* **249** (1996) 180 [hep-th/9410093] [INSPIRE].
- [40] S.J. Hathrell, *Trace Anomalies and QED in Curved Space*, *Annals Phys.* **142** (1982) 34 [INSPIRE].
- [41] I. Jack and H. Osborn, *Analogs for the c Theorem for Four-dimensional Renormalizable Field Theories*, *Nucl. Phys. B* **343** (1990) 647 [INSPIRE].

- [42] S. El-Showk, Y. Nakayama and S. Rychkov, *What Maxwell Theory in $D \neq 4$ teaches us about scale and conformal invariance*, *Nucl. Phys. B* **848** (2011) 578 [[arXiv:1101.5385](#)] [[INSPIRE](#)].
- [43] D.L. Jafferis, I.R. Klebanov, S.S. Pufu and B.R. Safdi, *Towards the F-Theorem: $N = 2$ Field Theories on the Three-Sphere*, *JHEP* **06** (2011) 102 [[arXiv:1103.1181](#)] [[INSPIRE](#)].
- [44] R.C. Myers and A. Sinha, *Seeing a c-theorem with holography*, *Phys. Rev. D* **82** (2010) 046006 [[arXiv:1006.1263](#)] [[INSPIRE](#)].
- [45] H. Casini, M. Huerta and R.C. Myers, *Towards a derivation of holographic entanglement entropy*, *JHEP* **05** (2011) 036 [[arXiv:1102.0440](#)] [[INSPIRE](#)].
- [46] I.R. Klebanov, S.S. Pufu and B.R. Safdi, *F-Theorem without Supersymmetry*, *JHEP* **10** (2011) 038 [[arXiv:1105.4598](#)] [[INSPIRE](#)].
- [47] H. Casini and M. Huerta, *On the RG running of the entanglement entropy of a circle*, *Phys. Rev. D* **85** (2012) 125016 [[arXiv:1202.5650](#)] [[INSPIRE](#)].
- [48] N. Karthik and R. Narayanan, *No evidence for bilinear condensate in parity-invariant three-dimensional QED with massless fermions*, *Phys. Rev. D* **93** (2016) 045020 [[arXiv:1512.02993](#)] [[INSPIRE](#)].
- [49] T. Appelquist, A.G. Cohen and M. Schmaltz, *A new constraint on strongly coupled gauge theories*, *Phys. Rev. D* **60** (1999) 045003 [[hep-th/9901109](#)] [[INSPIRE](#)].
- [50] A. Hasenfratz and P. Hasenfratz, *The equivalence of the $SU(N)$ Yang-Mills theory with a purely fermionic model*, *Phys. Lett. B* **297** (1992) 166 [[hep-lat/9207017](#)] [[INSPIRE](#)].
- [51] J.A. Gracey, *Quark, gluon and ghost anomalous dimensions at $O(1/N_f)$ in quantum chromodynamics*, *Phys. Lett. B* **318** (1993) 177 [[hep-th/9310063](#)] [[INSPIRE](#)].
- [52] D.I. Kazakov and G.S. Vartanov, *Renormalizable $1/N_f$ Expansion for Field Theories in Extra Dimensions*, *JHEP* **06** (2007) 081 [[arXiv:0707.2564](#)] [[INSPIRE](#)].
- [53] D.B. Ali and J.A. Gracey, *Anomalous dimension of nonsinglet quark currents at $O(1/N_f^2)$ in QCD*, *Phys. Lett. B* **518** (2001) 188 [[hep-ph/0105038](#)] [[INSPIRE](#)].
- [54] D. Dudal, J.A. Gracey, V.E.R. Lemes, R.F. Sobreiro, S.P. Sorella and H. Verschelde, *Renormalization properties of the mass operator $A_\mu^a A_\mu^a$ in three dimensional Yang-Mills theories in the Landau gauge*, *Annals Phys.* **317** (2005) 203 [[hep-th/0409254](#)] [[INSPIRE](#)].
- [55] J.F. Bennett and J.A. Gracey, *Three loop renormalization of the $SU(N_c)$ nonAbelian Thirring model*, *Nucl. Phys. B* **563** (1999) 390 [[hep-th/9909046](#)] [[INSPIRE](#)].
- [56] E. Witten, *Nonabelian Bosonization in Two-Dimensions*, *Commun. Math. Phys.* **92** (1984) 455 [[INSPIRE](#)].
- [57] G. Bhanot, K. Demeterfi and I.R. Klebanov, *$(1+1)$ -dimensional large- N QCD coupled to adjoint fermions*, *Phys. Rev. D* **48** (1993) 4980 [[hep-th/9307111](#)] [[INSPIRE](#)].
- [58] H. Osborn and A. Stergiou, *C_T for non-unitary CFTs in higher dimensions*, *JHEP* **06** (2016) 079 [[arXiv:1603.07307](#)] [[INSPIRE](#)].