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# Constraints on perturbative RG flows in six dimensions

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Abstract: When conformal field theories (CFTs) are perturbed by marginally relevant deformations, renormalization group (RG) flows ensue that can be studied with perturbative methods, at least as long as they remain close to the original CFT. In this work we study such RG flows in the vicinity of six-dimensional unitary CFTs. Neglecting effects of scalar operators of dimension two and four, we use Weyl consistency conditions to prove the $a$-theorem in perturbation theory, and establish that scale implies conformal invariance. We identify a quantity that monotonically decreases in the flow to the infrared due to unitarity, showing that it does not agree with the one studied recently in the literature on the six-dimensional $\phi^3$ theory.

Keywords: Anomalies in Field and String Theories, Field Theories in Higher Dimensions, Renormalization Group

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## 1 Introduction

The evolution of physical quantities with energy in quantum and statistical field theories is described by the renormalization group (RG). According to the Wilsonian picture, the RG flow from the ultraviolet (UV) to the infrared (IR) corresponds to a coarse-graining of degrees of freedom, and should therefore be irreversible. It is interesting to ask whether there is any physical observable in quantum field theory (QFT) that can be understood as the "number of degrees of freedom", and which decreases along the RG flow.

This intuition has been beautifully borne out in $d = 2$ spacetime dimensions by Zamolodchikov [1], who established that a certain combination of two-point functions of the stress-energy tensor, called $C$, monotonically decreases in the flow to the IR in unitary QFTs. This so called $C$-function is stationary at fixed points of the RG, where conformal field theories (CFTs) live, and is equal to the central charge of the corresponding CFT there.

Soon after Zamolodchikov's work, Cardy attempted a generalization to $d = 4$ [2], where he suggested that $a$, the coefficient of the Euler term in the four-dimensional trace anomaly, plays the role of the monotonically decreasing quantity. Although a general proof of the monotonicity of $a$, commonly referred to as the $a$-theorem, was not obtained in [2], significant differences with the $d = 2$ case were elucidated, and further support was given to the intuition that results similar to Zamolodchikov's should hold in any even spacetime dimension.

Osborn [3] later analyzed the case of a unitary CFTs in $d = 4$ deformed by a set of marginally relevant operators. By studying the Wess-Zumino consistency conditions for the anomalous Ward indentity of Weyl rescalings, within the formalism of the local Callan-Symankzik (CS) equation, Osborn derived a perturbative proof of the $a$-theorem, using

also results of [4].[1,2] More specifically, an equation of the form

$$\frac{\partial}{\partial\lambda^I}\hat{a} = (\chi_{IJ} + \xi_{IJ})\beta^J \tag{1.1}$$

was derived in [3, 4], where $\hat{a}$ is a local function of the coupling constants $\lambda^I$ of the theory, which reduces to $a$ at fixed points, $\chi_{IJ}$ and $\xi_{IJ}$ are symmetric and antisymmetric tensors respectively, defined also in terms of $\lambda^I$, and $\beta^I$ is the beta function associated with $\lambda^I$. By unitarity, $\chi_{IJ}$ is positive-definite at leading order in perturbation theory, as it can be related to the two-point function $\langle\mathcal{O}_I(x)\mathcal{O}_J(0)\rangle$ of marginal operators. Upon contracting (1.1) with $\beta^I$, one gets

$$\mu\frac{d}{d\mu}\hat{a} = \chi_{IJ}\beta^I\beta^J \geq 0\,, \tag{1.2}$$

thereby establishing the monotonicity of $\hat{a}$ along perturbative RG flows. The inequality is saturated only if $\beta^I = 0$.

Recently, Komargodski and Schwimmer [9] demonstrated that $a$-theorem holds true beyond perturbation theory in $d = 4$, more specifically that $a_{\rm UV} - a_{\rm IR}$ must be positive in unitary theories. Their argument relies on dispersion relations for four-point scattering amplitudes for the dilaton, i.e. the background source for the trace of the stress-energy tensor. The connection between the non-perturbative and perturbative arguments was made in [5], where it was shown how equation (1.1) can be extended beyond leading order by employing the dilaton effective action.

A question closely related to the $a$-theorem is that of the relation between scale and conformal invariance, in particular whether scale invariant field theories (SFTs) enjoy the full conformal symmetry under the assumptions of locality and unitarity. Polchinski proved the equivalence SFT = CFT in $d = 2$ [10]. In $d = 4$, perturbative checks were performed in [10] as well as [11], and general perturbative arguments were later given in [12, 13]. Beyond perturbation theory in $d = 4$ conditions for the equivalence of scale and conformal invariance have been analysed in [14].

Due to the importance of the $a$-theorem and its consequences for the structure of QFTs, it is of great interest to continue the exploration of these ideas to higher spacetime dimensions, in particular $d = 6$. Some important results have been obtained in [15, 16], but in this work we will focus on the approach pioneered by Osborn in [3], which relies on the local CS equation. This formalism was recently generalized to $d = 6$ [17], where complications arise due to the large number of terms that have to be considered in the Weyl anomaly.

In the present work we study the RG flow in the proximity of a six-dimensional CFT by deforming it by a set of marginally relevant operators $\mathcal{O}_I$,

$$S[\Phi,\lambda] = S_{\rm CFT}[\Phi] + \int d^6x\,\lambda^I\mathcal{O}_I(x)\,. \tag{1.3}$$

[1] Although the arguments in [3, 4] rely on perturbative computations around the free theory, they can be generalized to the case where the RG flow lies perturbatively close to any interacting CFT, weakly or strongly coupled [5–7]. In fact, the CFT need not even have a Lagrangian description.

[2] A recent review of this approach to the $a$-theorem can be found in [8].



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For simplicity, we assume that relevant operators of dimension two and four are absent from the theory. We plan to include their contributions in future work.

By analyzing the Wess-Zumino consistency conditions in the context of the local CS equation, we will be able to identify a function of the coupling constants, $\hat{a}$, satisfying an equation analogous to (1.1), thereby proving the $a$-theorem in perturbation theory. In fact, we find a one-parameter family of functions, $\hat{a} + \lambda\hat{b}$, satisfying an equation of the form

$$\mu\frac{d}{d\mu}(\hat{a} + \lambda\hat{b}) = \chi_{IJ}\beta^I\beta^J + \mathrm{O}(\beta^3, \beta^2\partial\beta)\,. \tag{1.4}$$

This result dispels the concerns on the validity of the perturbative $a$-theorem in $d = 6$ raised by [18], where a different function of the coupling constants was proposed as the monotonically decreasing quantity. As a direct consequence of the $a$-theorem we prove the equivalence SFT = CFT in our setup.

## 2 The local Callan-Symanzik equation

In this section we review briefly the local CS equation formalism, which we will use to derive constraints on the RG flow. The local CS equation was first derived in the seminal work [3]. We refer the reader to [5] and [6] for a detailed and thorough analysis of this technology in four dimensions.

The RG flow is equivalent to a global rescaling of distances, which is controlled by the properties of the trace of the stress-energy tensor, $T$. In perturbation theory, the stress-energy tensor can be expanded in a basis of operators of the CFT. Schematically,

$$T \sim \beta^I\mathcal{O}_I + S^A\nabla_\mu J^\mu_A - \eta^a\nabla^2\mathcal{O}_a + C^\alpha\nabla^2\nabla^2\varphi_\alpha\,, \tag{2.1}$$

where $\mathcal{O}_I$ are marginal scalar operators of dimension six, $J^\mu_A$ are currents of dimension five generating an exact flavor symmetry $G_F$ at the fixed point $\lambda^I = 0$, while $\mathcal{O}_a$ and $\varphi_\alpha$ are scalar operators of dimensions four and two.[3]

For simplicity, in this work we will assume that the lower-dimensional scalar operators $\mathcal{O}_a$ and $\varphi_\alpha$ are absent. It would be interesting to include them in the future, also to further test results in the perturbative $\phi^3$ theory [19].

To express the response of the theory (1.3) under local changes of the renormalization scale, it is necessary to turn on sources for the renormalized operators in (2.1). We lift the theory to curved spacetime, such that the metric $g_{\mu\nu}(x)$ sources the stress-energy tensor $T^{\mu\nu}$. In addition, we promote the couplings $\lambda^I(x)$ to spacetime dependent sources of the marginal operators $\mathcal{O}_I$, and we introduce the background gauge fields $A^A_\mu(x)$ sourcing the currents $J^\mu_A$. The $G_F$ symmetry is thus gauged and $\lambda^I$ transform under the symmetry. The quantum effective action then reads

$$\mathcal{W}[\mathcal{J}] = -i\log\int\mathcal{D}\Phi\, e^{iS[\Phi,\mathcal{J}]}\,, \tag{2.2}$$

where we collectively denote the sources as $\mathcal{J} \equiv (g^{\mu\nu}(x), \lambda^I(x), A^\mu_A(x))$.

[3] By the unitarity bound $\varphi_\alpha$ can only be free fields satisfying $\nabla^2\varphi_\alpha = 0$ at the fixed point.



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The connected correlation functions can be expressed as functional derivatives with respect to the $\mathcal{J}$'s,

$$\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}(x)} \to [T_{\mu\nu}(x)]\,, \qquad \frac{1}{\sqrt{-g}}\frac{\delta}{\delta \lambda^I(x)} \to [\mathcal{O}_I(x)]\,, \qquad \frac{1}{\sqrt{-g}}\frac{\delta}{\delta A^A_\mu(x)} \to \left[J^\mu_A(x)\right]\,, \tag{2.3}$$

where the square brackets denote the operator insertion inside a renormalized correlation function. For instance, the time-ordered renormalized correlators of the scalar marginal operators are obtained as

$$\langle \mathbf{T}\left\{\mathcal{O}_{I_1}(x_1)\cdots\mathcal{O}_{I_n}(x_n)\right\}\rangle = \frac{(-i)^{n-1}}{\sqrt{-g(x_1)}\cdots\sqrt{-g(x_n)}}\frac{\delta}{\delta\lambda^{I_1}(x_1)}\cdots\frac{\delta}{\delta\lambda^{I_n}(x_n)}\mathcal{W}\,. \tag{2.4}$$

To evaluate these correlation functions in the perturbed theory (1.3) in flat space, one has to take $g^{\mu\nu}(x)\to\eta^{\mu\nu}$, $\lambda^I(x)\to\lambda^I=\text{const}$, $A^A_\mu(x)\to 0$ after the variation.

To derive constraints on the RG flow we will consider the response of the quantum effective action to a local change of the renormalization scale. The local CS equation [3] reads

$$\begin{aligned}&\Delta_\sigma\mathcal{W} \equiv \int d^6x\sqrt{-g}\left(2\sigma g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}} + \delta_\sigma\lambda^I\frac{\delta}{\delta\lambda^I} + \delta_\sigma A^A_\mu\cdot\frac{\delta}{\delta A^A_\mu}\right)\mathcal{W} = \int d^6x\sqrt{-g}\,\mathcal{A}_\sigma\,,\\ &\delta_\sigma\lambda^I = -\sigma\,\beta^I\,, \qquad \delta_\sigma A^A_\mu = -\sigma\rho^A_I\,\nabla_\mu\lambda^I + \partial_\mu\sigma\,S^A\,,\end{aligned} \tag{2.5}$$

where $\Delta_\sigma$ contains the most general terms allowed by covariance and power counting, $\nabla$ is a gauge covariant derivative, and the anomaly $\mathcal{A}_\sigma$ is a local functional of the sources, whose form is constrained by diff-invariance and power counting. The Wess-Zumino consistency conditions,

$$\Delta_\sigma\mathcal{A}_{\sigma'} - \Delta_{\sigma'}\mathcal{A}_\sigma = 0\,, \tag{2.6}$$

expressing the commutativity of Weyl rescalings, impose further constraints among the coefficients of the various terms that appear in $\mathcal{A}_\sigma$. At the fixed point, i.e. for $\lambda^I=\text{const}$, $\beta^I = S^A = 0$ and $A^\mu_A = 0$, $\mathcal{A}_\sigma$ reduces to the usual conformal anomaly [20],

$$\mathcal{A}_\sigma = \sigma\left(-a\,E_6 + c_1 I_1 + c_2 I_2 + c_3 I_3\right), \tag{2.7}$$

up to six contributions (trivial anomalies) that can be eliminated by adding local counterterms to the effective action. In (2.7) $E_6$ is the Euler term while $I_1$, $I_2$, $I_3$ are Weyl invariant tensors. Their explicit form can be found in appendix B. The condition (2.6) at the fixed point imposes the vanishing of seven other possible anomalies (analogous to the $R^2$ anomaly in $d=4$).

In the next section we are going to derive constraints on the RG flow implied by the consistency conditions for the anomaly outside the fixed point.

## 3 Constraints on RG flows

Consistency conditions that follow from the commutativity of Weyl rescalings impose constraints among the various terms that appear in the anomaly $\mathcal{A}_\sigma$. In $d=2,4$ these



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conditions were originally considered in [3], and were recently also studied in detail in [5, 6, 12, 13], and holographically in [21, 22]. The consistency conditions were also studied in supersymmetric theories in [23, 24]. In $d = 6$ they were first considered in [17]. Here we derive the consistency conditions from the results of (2.6), as obtained in [17], and perform a detailed analysis of those. We find that some consistency conditions obtained in [17] were incomplete.

For the moment, we will neglect the contributions to equation (2.5) related to the gauge fields $A^A_\mu$ sourcing the currents $J^\mu_A$. However, as will be shown in section 4, this will not change our conclusions. The complete form of $\mathcal{A}_\sigma$ can be found in appendix B. After decomposing (2.6) in a linearly-independent basis, it is possible to read off constraint equations for the anomaly coefficients. This is technically challenging, particularly due to difficulties related to integration by parts and Bianchi identities.[4] The consistency conditions obtained here were checked at two loops in the $\phi^3$ theory against the results of [19].[5] We have also checked that they are satisfied by the general form of the trace anomaly on the conformal manifold as constucted in [7].

In this work we exploit all constraints imposed on anomaly coefficients with up to two indices. This requires us to decompose the consistency conditions and isolate the ones that stem from terms involving up to two couplings $\lambda$. For example, we are interested in the consistency condition arising from contributions to the left-hand side of (2.6) proportional to $(\sigma\,\partial_\mu\sigma' - \sigma'\,\partial_\mu\sigma)\nabla^2\lambda^I\,\partial^\mu\nabla^2\lambda^J$, but not in the one arising from contributions proportional to $(\sigma\,\partial_\mu\sigma' - \sigma'\,\partial_\mu\sigma)\,\partial^\mu\lambda^I\,\nabla^2\lambda^J\,\nabla^2\lambda^K$.

A particularly important equation contained in (2.6) is obtained from terms proportional to $(\sigma\,\partial_\mu\sigma' - \sigma'\,\partial_\mu\sigma)H_1{}^{\mu\nu}\partial_\nu\lambda^I$, where $H_1{}^{\mu\nu}$ is a generalization of the Einstein tensor in $d = 6$ [25] (see (B.4) for its explicit form), namely

$$\partial_I \check{a} = \frac{1}{6}\mathcal{H}_{IJ}\beta^J + \frac{1}{6}\mathcal{H}_I\,, \tag{3.1}$$

where

$$\begin{aligned} \check{a} &= a + \frac{1}{6}b_1 - \frac{1}{90}b_3 + \frac{1}{6}b_{11} + \frac{1}{12}\mathcal{A}_J\beta^J + \frac{1}{6}\mathcal{H}^1_J\beta^J\,, \\ \mathcal{H}_I &= -\mathcal{H}^5_I - \frac{1}{2}\mathcal{H}^6_I - \frac{1}{2}\mathcal{I}^7_I\,, \qquad \mathcal{H}_{IJ} = \frac{1}{4}\mathcal{A}_{JI} + \mathcal{H}^1_{IJ} + \partial_I\mathcal{A}_J + \partial_{[I}\mathcal{H}^1_{J]}\,, \end{aligned} \tag{3.2}$$

with the definition

$$\partial_{[I}X_{J]} = \partial_I X_J - \partial_J X_I\,. \tag{3.3}$$

All tensors appearing above are local functions of the couplings, and their definition can be found in appendix B. Use of the consistency condition arising from $(\sigma\nabla^\mu\nabla^\nu\partial^\rho\sigma' -$

[4] All our computations were performed in *Mathematica* using the package `xAct`, and details on the derivation of the consistency conditions can be found in appendix A. Due to the large number of terms appearing in (2.6) and related consistency conditions, we do not report most of them in the text. The interested reader can find them in a separate *Mathematica* file attached to the submission.

[5] To extend the check to higher loops it will be necessary to include the effects of the operators of dimension two and four.



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$\sigma' \nabla^\mu \nabla^\nu \partial^\rho \sigma) \nabla_\mu \nabla_\nu \partial_\rho \lambda^I$ allows us to put (3.1) in the form

$$\partial_I \tilde{a} = \frac{1}{6}\left(\mathcal{H}^1_{IJ} - \frac{1}{4}\hat{\mathcal{A}}''_{IJ}\right)\beta^J + \frac{1}{6}\partial_{[I}\mathcal{H}^1_{J]}\beta^J - \frac{1}{12}\mathcal{I}^7_I\,, \qquad \tilde{a} = a + \frac{1}{6}b_1 - \frac{1}{90}b_3 + \frac{1}{6}\mathcal{H}^1_I\beta^I\,, \tag{3.4}$$

which contains fewer anomaly coefficients than (3.1) with (3.2). Unlike in the two and four-dimensional cases, (3.4) does not present itself in the form of (1.1), due to the presence of the vector anomaly $\mathcal{I}^7_I$. Notice that this contribution was missed in [17], which led to consider $\tilde{a}$ as the candidate for a monotonically-decreasing function in [18]. However, $\tilde{a}$ cannot be such a candidate, even more so because it is scheme-dependent[6] at order $\beta$.[7]

In this work we consider linear combinations of the consistency conditions in order to find all independent equations having the form of (1.1). Most importantly, we find the equation[8]

$$\partial_I \hat{a} = (\chi_{IJ} + \xi_{IJ})\beta^J\,, \tag{3.5}$$

where

$$\begin{aligned}
\hat{a} &= a - \frac{5}{6}b_1 + \frac{1}{10}b_2 + \frac{1}{45}b_3 + \frac{1}{10}b_4 \\
&\quad + \left(\frac{1}{10}\mathcal{B}_I + \frac{1}{24}\mathcal{C}_I + \frac{1}{20}\mathcal{E}_I + \frac{1}{24}\mathcal{F}_I + \frac{1}{6}\mathcal{H}^1_I + \frac{1}{20}\mathcal{H}^2_I + \frac{1}{12}\mathcal{H}^3_I + \frac{1}{8}\mathcal{H}^4_I - \frac{1}{40}\mathcal{H}^6_I\right)\beta^I\,, \\
\chi_{IJ} &= \frac{1}{20}\partial_{(I}\mathcal{B}_{J)} - \frac{1}{40}\hat{\mathcal{B}}'_{IJ} + \frac{1}{48}\hat{\mathcal{C}}'_{IJ} + \frac{1}{20}\hat{\mathcal{E}}_{(IJ)} + \frac{1}{24}\mathcal{F}_{(IJ)} + \frac{1}{6}\mathcal{H}^1_{IJ} \\
&\quad + \frac{1}{20}\mathcal{H}^2_{IJ} + \frac{1}{12}\mathcal{H}^3_{IJ} + \frac{1}{8}\mathcal{H}^4_{IJ} - \frac{1}{40}\mathcal{H}^6_{IJ}\,, \\
\xi_{IJ} &= \frac{1}{20}\partial_{[I}\mathcal{B}_{J]} + \frac{1}{48}\mathcal{C}_{[IJ]} + \frac{1}{40}\hat{\mathcal{E}}_{[IJ]} + \frac{1}{48}\mathcal{F}_{[IJ]} + \frac{1}{48}\mathcal{F}'_{[IJ]} \\
&\quad + \frac{1}{6}\partial_{[I}\mathcal{H}^1_{J]} + \frac{1}{20}\partial_{[I}\mathcal{H}^2_{J]} + \frac{1}{12}\partial_{[I}\mathcal{H}^3_{J]} + \frac{1}{8}\partial_{[I}\mathcal{H}^4_{J]} - \frac{1}{40}\partial_{[I}\mathcal{H}^6_{J]}\,,
\end{aligned} \tag{3.6}$$

and we use (3.3) and

$$\partial_{(I}X_{J)} = \partial_I X_J + \partial_J X_I\,, \qquad X_{(IJ)} = X_{IJ} + X_{JI}\,, \qquad X_{[IJ]} = X_{IJ} - X_{JI}\,. \tag{3.7}$$

$\hat{a}$ equals $a$ at the fixed point, for the anomalies $b_{1,\dots,7}$ are all proportional to $\beta$. $\chi_{IJ}$ and $\xi_{IJ}$ are symmetric and antisymmetric tensors, respectively.[9] Note that, by virtue of equation (3.5), $\hat{a}$ is scheme independent at order $\beta$, while $\chi_{IJ}$ and $\xi_{IJ}$ are scheme-independent at order $\beta^0$, i.e. they are not affected to that order by adding local counterterms to the effective action.

[6] In this paper, by "scheme-dependent" quantities we mean those which change under the addition of purely background-dependent counterterms to the effective action.

[7] For example, the addition of a term $\int d^6x\sqrt{\gamma}\,X_I\,\partial_\mu\lambda^I\,\nabla_\nu H_4^{\mu\nu}$ in $\mathcal{W}[\mathcal{J}]$, with $X_I$ arbitrary, induces, among others, the shifts $\mathcal{I}^7_I \to \mathcal{I}^7_I + \mathscr{L}_\beta X_I$, where $\mathscr{L}_\beta$ is the Lie derivative along the beta function, and $\mathcal{H}^1_I \to \mathcal{H}^1_I - \frac{1}{2}X_I$. The shift of $\mathcal{H}^1_I$ affects $\tilde{a}$ at order $\beta$.

[8] The linear combination of the consistency conditions leading to (3.5) is explicitly reported in the *Mathematica* file attached to the submission.

[9] Using the consistency conditions we have checked that $\xi_{IJ}$ cannot be written as $\partial_{[I}X_{J]}$ for some vector $X_J$.



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Now we can show that the metric $\chi_{IJ}$ in (3.5) is positive-definite. Indeed, consider the RG derivative of the two-point correlator of the marginal operators $\langle \mathcal{O}_I(x)\mathcal{O}_J(0)\rangle$ (in Euclidean signature). First notice that, since lengths are entirely controlled by $g_{\mu\nu}$, this operation can be expressed as a Weyl rescaling,

$$\mu\frac{\partial}{\partial\mu}\mathcal{W} = -2\int d^6x\, g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}}\mathcal{W}\,. \tag{3.8}$$

Then, neglecting terms involving the $\beta$ function,

$$\mu\frac{\partial}{\partial\mu}\langle\mathcal{O}_I(x)\mathcal{O}_J(0)\rangle = -\int d^6y\frac{1}{\sqrt{-g(x)}\sqrt{-g(0)}}\left(2g^{\mu\nu}(y)\frac{\delta}{\delta g^{\mu\nu}(y)}\right)\frac{\delta}{\delta\lambda^I(x)}\frac{\delta}{\delta\lambda^J(0)}\mathcal{W} \tag{3.9}$$

$$= -\int d^6y\sqrt{-g(y)}\,\frac{1}{\sqrt{-g(x)}}\frac{\delta}{\delta\lambda^I(x)}\frac{1}{\sqrt{-g(0)}}\frac{\delta}{\delta\lambda^J(0)}\mathcal{A}_{\sigma=1} \tag{3.10}$$

$$= g_{IJ}(\partial^2)^3\delta^{(6)}(x)\,, \tag{3.11}$$

where in the last line we go to flat space, $\delta^{(6)}(x)$ is the six-dimensional delta function, and $g_{IJ}$ is evaluated via the anomaly in appendix B,

$$g_{IJ} = -\partial_{(I}\mathcal{A}_{J)} - \hat{\mathcal{A}}_{(IJ)} + \hat{\mathcal{A}}'_{IJ} + \hat{\mathcal{A}}''_{IJ}\,. \tag{3.12}$$

It can be shown that $g_{IJ}$ is proportional to the Zamolodchikov metric and is thus positive-definite by unitarity [7]. Furthermore, the consistency conditions relate the tensors $\chi_{IJ}$ and $g_{IJ}$ via

$$\chi_{IJ} = \frac{1}{6}g_{IJ} + \mathrm{O}(\beta,\partial\beta)\,. \tag{3.13}$$

With this result, and upon contracting equation (3.5) with $\beta^I$, we get the desired monotonicity constraint in perturbation theory for $\hat{a}$,

$$\mu\frac{d}{d\mu}\hat{a} = \chi_{IJ}\beta^I\beta^J \geq 0\,, \tag{3.14}$$

where the inequality is saturated only if $\beta^I = 0$. This proves the $a$-theorem in perturbation theory (in theories with no relevant scalar operators of dimension two and four).

Additionally, we find another, independent equation of the form[10]

$$\partial_I\hat{b} = (\chi'_{IJ} + \xi'_{IJ})\beta^J\,, \tag{3.15}$$

where

$$\begin{aligned}
\hat{b} &= 4b_1 - \frac{4}{5}b_2 - \frac{4}{15}b_3 - \frac{4}{5}b_4 - \left(\frac{4}{5}\mathcal{B}_I + \frac{1}{2}\mathcal{C}_I + \frac{2}{5}\mathcal{E}_I + \frac{2}{5}\mathcal{H}^2_I + \frac{2}{3}\mathcal{H}^3_I + \frac{2}{3}\mathcal{H}^4_I - \frac{1}{5}\mathcal{H}^6_I\right)\beta^I\,,\\
\chi'_{IJ} &= -\frac{2}{5}\partial_{(I}\mathcal{B}_{J)} + \frac{1}{3}\hat{\mathcal{A}}''_{IJ} + \frac{1}{5}\hat{\mathcal{B}}'_{IJ} - \frac{1}{6}\hat{\mathcal{C}}'_{IJ} - \frac{1}{5}\hat{\mathcal{E}}_{(IJ)} - \frac{2}{5}\mathcal{H}^2_{IJ} - \frac{2}{3}\mathcal{H}^3_{IJ} - \frac{2}{3}\mathcal{H}^4_{IJ} + \frac{1}{5}\mathcal{H}^6_{IJ}\,,\\
\xi'_{IJ} &= -\frac{2}{5}\partial_{[I}\mathcal{B}_{J]} - \frac{1}{5}\hat{\mathcal{E}}_{[IJ]} - \frac{2}{5}\partial_{[I}\mathcal{H}^2_{J]} - \frac{2}{3}\partial_{[I}\mathcal{H}^3_{J]} - \frac{2}{3}\partial_{[I}\mathcal{H}^4_{J]} + \frac{1}{5}\partial_{[I}\mathcal{H}^6_{J]}\,.
\end{aligned} \tag{3.16}$$

[10]The linear combination of the consistency conditions leading to (3.15) is explicitly reported in the *Mathematica* file attached to the submission.



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$\hat{b}$ is of order $\beta$ and so vanishes at fixed points, and $\chi'_{IJ}$, $\xi'_{IJ}$ are symmetric and antisymmetric respectively. The existence of the metric $\chi'_{IJ}$ is related to the fact that in $d = 6$ there are three rank-two conformally covariant operators one can define on the conformal manifold [7], corresponding to just as many scheme-independent rank-two tensors at the fixed point. This is in contrast with the two- and four-dimensional cases where there is only a unique rank-two tensor related to the Zamolodchikov metric. Nevertheless, we found that the consistency conditions impose an orthogonality constraint on $\chi'_{IJ}$,

$$\chi'_{IJ}\beta^J = \mathrm{O}(\beta^2, \beta\partial\beta)\,, \tag{3.17}$$

even though, in general, $\chi'_{IJ}$ does not vanish at fixed points. Equations (3.5), (3.15), (3.17) imply that there exists a one-parameter family of monotonically decreasing functions at leading order in perturbation theory,

$$\mu\frac{d}{d\mu}(\hat{a} + \lambda\hat{b}) = \frac{1}{6}g_{IJ}\beta^I\beta^J + \mathrm{O}(\beta^3, \beta^2\partial\beta)\,. \tag{3.18}$$

## 4 Scale versus conformal invariance

We could ask whether the theory (1.3) can flow to a nearby scale invariant field theory without conformal invariance. This question becomes nontrivial in presence of dimension five currents, as we see from equation (2.1). Indeed, at the fixed point it is $\beta^I = 0$, and $T$ has the operatorial form

$$T \sim S^A\nabla_\mu J^\mu_A \equiv \nabla_\mu V^\mu\,, \tag{4.1}$$

where $V^\mu$ is the so-called virial current. If $S^A \neq 0$ the theory is scale but not conformally invariant, for $T$ is a total divergence.

Before proceeding, it is useful to rewrite the Weyl operator in a more convenient form. By making use of the Ward identities for the $G_F$ symmetries represented by the broken generators $T^A$, it possible to redefine the Weyl operator in (2.5) to encapsulate both a Weyl rescaling and a $G_F$ transformation [3, 5, 6, 13],

$$\Delta'_\sigma\mathcal{W} \equiv \int d^6x\sqrt{-g}\,\sigma\left(2g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}} - B^I\frac{\delta}{\delta\lambda^I} - P^A_I\nabla_\mu\lambda^I\cdot\frac{\delta}{\delta A^A_\mu}\right)\mathcal{W} = \int d^6x\sqrt{-g}\,\mathcal{A}_\sigma\,, \tag{4.2}$$

$$B^I = \beta^I - (S^AT_A\lambda)^I\,, \qquad P^A_I = \rho^A_I + \partial_I S^A\,, \tag{4.3}$$

with the constraint $B^IP^A_I = 0$ due to the commutativity of Weyl rescaling, $[\Delta'_\sigma, \Delta'_{\sigma'}] = 0$. In this parametrization a scale invariant field theory corresponds to $B^I = -(S^AT_A\lambda)^I$, while a conformal invariant field theory to $B^I = 0$.

Now, let us generalize the equation (3.5) in the presence of dimension five currents. By covariance, at leading order in $B^I$ it takes the form

$$\partial_I\hat{a} = (\chi_{IJ} + \xi_{IJ})B^J + P^A_I f_A\,, \tag{4.4}$$

where $f_A$ is an generic combination of anomaly coefficients of terms involving the gauge fields $A^A_\mu$. Upon contracting (4.4) with $B^I$ and using the condition $B^IP^A_I = 0$ we get

$$B^I\partial_I\hat{a} = \mu\frac{d}{d\mu}\hat{a} = \frac{1}{6}g_{IJ}B^IB^J \geq 0\,. \tag{4.5}$$



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Therefore, we reach the same conclusion we found in section 3. Furthermore, suppose we are in a scale invariant field theory. Then, $G_F$-invariance of $\hat{a}$ implies that $B^I\partial_I\hat{a} = -(S^A T_A\lambda)^I\partial_I\hat{a} = 0$, so that (4.5) gives

$$g_{IJ}B^I B^J = 0\,. \tag{4.6}$$

Due to the positive-definiteness of $g_{IJ}$ this can only be true for $B^I = 0$. This proves that scale invariance implies conformal invariance in our setup, in analogy with the four-dimensional case [12, 13]. Our proof here follows the logic of [13].

## 5 Conclusions

In this work we studied the properties of RG flows originating from marginal deformations to unitary conformal field theories in six dimensions. For simplicity, we restricted the analysis to a class of CFTs where relevant scalar operators of dimension two and four are absent. Even though we work in perturbation theory, the UV CFT can in general be strongly coupled and may not admit a Lagrangian description.

The results obtained here can be summarized as follows:

- We derived all the consistency conditions with up to two powers of the coupling outside the fixed point. We solved those to find all the constraints among the anomaly coefficients which can be put in the form of a flow equation.

- We identified a one-parameter family of scheme-independent functions of the coupling constants of the theory, $\hat{a} + \lambda\hat{b}$ with $\lambda \in \mathbb{R}$, equal to the $a$-anomaly coefficient plus $\mathrm{O}(\beta)$ corrections, which flow monotonically in the proximity of a fixed point thanks to unitarity. There is no parameter $\lambda$ for which the combination $\hat{a} + \lambda\hat{b}$, agrees with the quantity analyzed in [18] in the context of $\phi^3$ theory, therefore we dispel the doubts cast on the perturbative $a$-theorem in six dimensions.

- As a direct consequence of the $a$-theorem we proved, using standard arguments, that scale implies conformal invariance in our setup.

The dynamics of perturbative QFTs in six dimensions appears structurally different with respect to the four-dimensional case, due to the presence of multiple scheme-independent rank two tensors at the fixed point. Nevertheless, we were able to find a class of physical quantity whose RG flow is governed uniquely by the positive definite Zamolodchikov metric. We presume that extending our argument beyond perturbation theory would single out the monotonically-decreasing function in the one-parameter family that we found.

In the future, it will be interesting to extend our results in the presence of scalar operators of dimension two and four. First, that could highlight possible differences with the lower spacetime dimensional cases, where relevant operators do not affect the monotonicity constraints [3, 5, 6].[11] Second, that will be necessary to test our results in the

[11] In four dimensions that is made clear by the argument employing the on-shell dilaton amplitude, which is manifestly insensitive to those effects [5].



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$\phi^3$ theory, which is the only perturbatively calculable theory in six dimensions. It should be straightforward to generalize our computations to include those contributions, with the only difficulties arising due to the proliferation of terms in the anomaly functional and in the Weyl operator.

It would also be of interest to analyze $\hat{a}$ and $\hat{b}$ to higher-loop orders in $\phi^3$ theory with the use of the consistency conditions, along the lines of [26]. The effects of dimension two and four operators as described in the previous paragraph may be necessary for such an analysis.

The question stands whether the $a$-theorem and the equivalence of scale and conformal invariance is valid beyond perturbation theory in six dimensions. So far no counterexamples are known. In four dimensions, certain dilaton scattering amplitudes provide a powerful tool to address these questions [9, 14]. Attempts were made to use dilaton scattering amplitudes [15] in six dimensions, but it is not clear what the right approach would be.


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## A Derivation of the consistency conditions

In this work, in order to derive the consistency conditions it was necessary to write the variation 2.6 in a linearly independent basis. This was technically nontrivial due to the large number of terms ($\sim$ O(100)) and redundancies related to integration by parts. Our approach is outlined in this appendix. First, by integrating by parts, we took all the derivatives off either $\sigma$ or $\sigma'$. As a result, we ended up with terms such as

$$(\sigma\,\partial_\mu\sigma' - \sigma'\,\partial_\mu\sigma)\,f_I(\lambda)\,\partial^\mu\lambda^I\,R^2\,, \qquad (\sigma\,\nabla_\mu\partial_\nu\sigma' - \sigma'\,\nabla_\mu\partial_\nu\sigma)\,f(\lambda)\,H_1^{\mu\nu}\,. \tag{A.1}$$

However, there are still redundancies related to antisymmetrization with respect to $\sigma, \sigma'$. For example, consider the trivial equation

$$(\partial_\mu\sigma\,\partial_\nu\sigma' - \partial_\mu\sigma'\,\partial_\nu\sigma)\,f(\lambda)\,H_1^{\mu\nu} = 0\,, \tag{A.2}$$

where $H_1^{\mu\nu}$ is symmetric. Upon integrating by parts and writing this equation in the same basis as (A.1), we get

$$(\sigma\,\nabla_\mu\partial_\nu\sigma' - \sigma'\,\nabla_\mu\partial_\nu\sigma)\,f(\lambda)\,H_1^{\mu\nu} + (\sigma\,\partial_\nu\sigma' - \sigma'\,\partial_\nu\sigma)\,\partial_I f(\lambda)\,\partial_\mu\lambda^I\,H_1^{\mu\nu} = 0\,, \tag{A.3}$$

since $\nabla_\mu H_1^{\mu\nu} = 0$ in this example. This allows to eliminate the second term in (A.1). Similarly one can get rid of all the terms with an even number of derivatives on $\sigma$, $\sigma'$. This prescription fixes unambiguously a complete basis for (2.6).

## B Conventions and basis for the anomaly

We define the Riemann tensor via

$$[\nabla_\mu, \nabla_\nu] A^\rho = R^\rho{}_{\sigma\mu\nu} A^\sigma \,, \tag{B.1}$$

and the Ricci tensor and Ricci scalar as $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$ and $R = g^{\mu\nu} R_{\mu\nu}$. The Einstein tensor is defined in $d \geq 2$ by

$$G_{\mu\nu} = \frac{2}{d-2}(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) \,, \tag{B.2}$$

while the Weyl tensor is defined in $d \geq 3$ by

$$W_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{2}{d-2}(g_{\mu[\sigma} R_{\rho]\nu} + g_{\nu[\rho} R_{\sigma]\mu}) + \frac{2}{(d-1)(d-2)} g_{\mu[\rho} g_{\sigma]\nu} R \,. \tag{B.3}$$

At dimension four we consider the tensors

$$\begin{aligned}
E_4 &= \frac{2}{(d-2)(d-3)}(R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - 4R^{\mu\nu} R_{\mu\nu} + R^2) \,, \qquad I = W^{\mu\nu\rho\sigma} W_{\mu\nu\rho\sigma} \,, \\
H_{1\mu\nu} &= \frac{(d-2)(d-3)}{2} E_4 \, g_{\mu\nu} - 4(d-1) H_{2\mu\nu} + 8 H_{3\mu\nu} + 8 H_{4\mu\nu} - 4 R^{\rho\sigma\tau}{}_\mu R_{\rho\sigma\tau\nu} \,, \\
H_{2\mu\nu} &= \frac{1}{d-1} R R_{\mu\nu} \,, \qquad H_{3\mu\nu} = R_\mu{}^\rho R_{\rho\nu} \,, \qquad H_{4\mu\nu} = R^{\rho\sigma} R_{\rho\mu\sigma\nu} \,, \\
H_{5\mu\nu} &= \nabla^2 R_{\mu\nu} \,, \qquad H_{6\mu\nu} = \frac{1}{d-1} \nabla_\mu \partial_\nu R \,.
\end{aligned} \tag{B.4}$$

A complete basis of scalar dimension-six curvature terms consists of [20]

$$\begin{aligned}
&K_1 = R^3 \,, && K_2 = R R^{\mu\nu} R_{\mu\nu} \,, && K_3 = R R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} \,, && K_4 = R^{\mu\nu} R_{\nu\rho} R^\rho{}_\mu \,, \\
&K_5 = R^{\mu\nu} R^{\rho\sigma} R_{\mu\rho\sigma\nu} \,, && K_6 = R^{\mu\nu} R_{\mu\rho\sigma\tau} R_\nu{}^{\rho\sigma\tau} \,, && K_7 = R^{\mu\nu\rho\sigma} R_{\rho\sigma\tau\omega} R^{\tau\omega}{}_{\mu\nu} \,, && \\
&K_8 = R^{\mu\nu\rho\sigma} R_{\tau\nu\rho\omega} R_\mu{}^{\tau\omega}{}_\sigma \,, && K_9 = R \nabla^2 R \,, && K_{10} = R^{\mu\nu} \nabla^2 R_{\mu\nu} \,, && K_{11} = R^{\mu\nu\rho\sigma} \nabla^2 R_{\mu\nu\rho\sigma} \,, \\
&K_{12} = R^{\mu\nu} \nabla_\mu \partial_\nu R \,, && K_{13} = \nabla^\mu R^{\nu\rho} \nabla_\mu R_{\nu\rho} \,, && K_{14} = \nabla^\mu R^{\nu\rho} \nabla_\nu R_{\mu\rho} \,, && \\
&K_{15} = \nabla^\mu R^{\nu\rho\sigma\tau} \nabla_\mu R_{\nu\rho\sigma\tau} \,, && K_{16} = \nabla^2 R^2 \,, && K_{17} = (\nabla^2)^2 R \,. &&
\end{aligned} \tag{B.5}$$

In $d = 6$ a convenient basis is given by

$$I_1 = \frac{19}{800} K_1 - \frac{57}{160} K_2 + \frac{3}{40} K_3 + \frac{7}{16} K_4 - \frac{9}{8} K_5 - \frac{3}{4} K_6 + K_8 \,, \tag{B.6}$$

$$I_2 = \frac{9}{200} K_1 - \frac{27}{40} K_2 + \frac{3}{10} K_3 + \frac{5}{4} K_4 - \frac{3}{2} K_5 - 3K_6 + K_7 \,, \tag{B.7}$$

$$I_3 = -\frac{11}{50} K_1 + \frac{27}{10} K_2 - \frac{6}{5} K_3 - K_4 + 6K_5 + 2K_7 - 8K_8 \tag{B.8}$$

$$+ \frac{3}{5} K_9 - 6K_{10} + 6K_{11} + 3K_{13} - 6K_{14} + 3K_{15} \,, \tag{B.9}$$

$$E_6 = K_1 - 12K_2 + 3K_3 + 16K_4 - 24K_5 - 24K_6 + 4K_7 + 8K_8 \,, \tag{B.10}$$

$$J_1 = 6K_6 - 3K_7 + 12K_8 + K_{10} - 7K_{11} - 11K_{13} + 12K_{14} - 4K_{15}\,, \tag{B.11}$$

$$J_2 = -\frac{1}{5}K_9 + K_{10} + \frac{2}{5}K_{12} + K_{13}\,, \qquad J_3 = K_4 + K_5 - \frac{3}{20}K_9 + \frac{4}{5}K_{12} + K_{14}\,, \tag{B.12}$$

$$J_4 = -\frac{1}{5}K_9 + K_{11} + \frac{2}{5}K_{12} + K_{15}\,, \qquad J_5 = K_{16}\,, \qquad J_6 = K_{17}\,, \tag{B.13}$$

$$L_1 = -\frac{1}{30}K_1 + \frac{1}{4}K_2 - K_6\,, \qquad L_2 = -\frac{1}{100}K_1 + \frac{1}{20}K_2\,, \tag{B.14}$$

$$L_3 = -\frac{37}{6000}K_1 + \frac{7}{150}K_2 - \frac{1}{75}K_3 + \frac{1}{10}K_5 + \frac{1}{15}K_6\,, \qquad L_4 = -\frac{1}{150}K_1 + \frac{1}{20}K_3\,, \tag{B.15}$$

$$L_5 = \frac{1}{30}K_1\,, \qquad L_6 = -\frac{1}{300}K_1 + \frac{1}{20}K_9\,, \qquad L_7 = K_{15}\,, \tag{B.16}$$

where the first three transform covariantly under Weyl variations, and $E_6$ is the Euler term in $d = 6$. The $J$'s are trivial anomalies in a six-dimensional CFT defined in curved space, and the first six $L$'s are constructed based on the relation $\delta_\sigma \int d^6x\sqrt{-g}\, L_{1,\ldots,6} = \int d^6x\sqrt{-g}\,\sigma J_{1,\ldots,6}$.

In six spacetime dimensions there are ninety four independent terms that can contribute to the anomaly [17]. In general, we can write

$$\int d^6x\sqrt{-g}\,\mathcal{A}_\sigma = \sum_{p=1}^{65}\int d^6x\sqrt{-g}\,\sigma\mathscr{T}_p + \sum_{q=1}^{30}\int d^6x\sqrt{-g}\,\partial_\mu\sigma\,\mathscr{Z}^\mu_q, \tag{B.17}$$

where $\mathscr{T}_p$ and $\mathscr{Z}^\mu_q$ are dimension-six and dimension-five terms respectively, that can involve curvatures as well as derivatives on the couplings $\lambda^I$. In writing down the various terms below, we neglect total derivatives.

If only curvatures are included, then we have the terms

$$\mathscr{T}_1 = -c_1 I_1\,, \quad \mathscr{T}_2 = -c_2 I_2\,, \quad \mathscr{T}_3 = -c_3 I_3\,, \quad \mathscr{T}_4 = -aE_6\,, \quad \mathscr{T}_{5,\ldots,11} = -b_{1,\ldots,7}L_{1,\ldots,7}\,. \tag{B.18}$$

We also have the terms

$$\begin{aligned} &\mathscr{Z}^\mu_1 = -b_8\,\partial^\mu E_4\,, \qquad \mathscr{Z}^\mu_2 = -b_9\,\partial^\mu I\,, \qquad \mathscr{Z}^\mu_3 = -\frac{1}{25}b_{10}\,R\,\partial^\mu R\,,\\ &\mathscr{Z}^\mu_4 = -\frac{1}{5}b_{11}\,\partial^\mu\nabla^2 R\,, \quad \mathscr{Z}^\mu_{5,6,7} = -b_{12,13,14}\,\nabla_\nu H^{\mu\nu}_{2,3,4}\,. \end{aligned} \tag{B.19}$$

Actually, the terms in (B.19) overcomplete the basis of trivial anomalies. This is because there are six trivial anomalies, but seven terms in (B.19). If we integrate the (B.19) terms by parts, then we may require that $\nabla_\mu\mathscr{Z}^\mu_{1,\ldots,7}$ do not affect the coefficients of $L_{1,\ldots,7}$. This forces us to impose

$$b_{13} = -\frac{24}{d^2-5d+6}b_8 + \frac{4(d-6)}{d-2}b_9 - \frac{5}{d-1}b_{12}. \tag{B.20}$$

With (B.20) it is guaranteed that $L_{1,\ldots,7}$ are vanishing anomalies, and we also see that the coefficients of $E_6, I_{1,2,3}$ are unaffected by $\nabla_\mu\mathscr{Z}^\mu_{1,\ldots,7}$. Thus, with the condition (B.20) the terms $\mathscr{Z}^\mu_{1,\ldots,7}$ substitute exactly the trivial anomalies $J_{1,\ldots,6}$.



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Next, we have

$$\begin{aligned}
&\mathscr{T}_{12} = \mathcal{I}_I^1\,\partial_\mu\lambda^I\,\partial^\mu E_4\,, \qquad \mathscr{T}_{13} = \mathcal{I}_I^2\,\partial_\mu\lambda^I\,\partial^\mu I\,, \qquad \mathscr{T}_{14} = \frac{1}{25}\mathcal{I}_I^3\,\partial_\mu\lambda^I\,R\,\partial^\mu R\,,\\
&\mathscr{T}_{15} = \frac{1}{5}\mathcal{I}_I^4\,\partial_\mu\lambda^I\,\partial^\mu\nabla^2 R\,, \quad \mathscr{T}_{16,17,18} = \mathcal{I}_I^{5,6,7}\,\partial_\mu\lambda^I\,\nabla_\nu H^{\mu\nu}_{2,3,4}\,,
\end{aligned} \tag{B.21}$$

and

$$\begin{aligned}
&\mathscr{Z}_8^\mu = \mathcal{G}_I^1\,\partial^\mu\lambda^I\,E_4, \qquad \mathscr{Z}_9^\mu = \mathcal{G}_I^2\,\partial^\mu\lambda^I\,I, \qquad \mathscr{Z}_{10}^\mu = \frac{1}{25}\mathcal{G}_I^3\,\partial^\mu\lambda^I\,R^2,\\
&\mathscr{Z}_{11}^\mu = \frac{1}{5}\mathcal{G}_I^4\,\partial^\mu\lambda^I\,\nabla^2 R, \qquad \mathscr{Z}^\mu_{12,\ldots,17} = \mathcal{H}_I^{1,\ldots,6}\,\partial_\nu\lambda^I H^{\mu\nu}_{1,\ldots,6}\,,\\
&\mathscr{Z}_{18}^\mu = \mathcal{F}_I\,\nabla_\kappa\partial_\lambda\lambda^I\,\nabla^\mu G^{\kappa\lambda}\,, \qquad \mathscr{Z}_{19}^\mu = \frac{1}{5}\mathcal{E}_I\,\nabla^2\lambda^I\,\partial^\mu R\,.
\end{aligned} \tag{B.22}$$

With more $\partial\lambda$'s we have

$$\begin{aligned}
&\mathscr{T}_{19} = \frac{1}{2}\mathcal{G}^1_{IJ}\,\partial_\mu\lambda^I\partial^\mu\lambda^J\,E_4\,, \qquad \mathscr{T}_{20} = \frac{1}{2}\mathcal{G}^2_{IJ}\,\partial_\mu\lambda^I\partial^\mu\lambda^J\,I\,, \qquad \mathscr{T}_{21} = \frac{1}{50}\mathcal{G}^3_{IJ}\,\partial_\mu\lambda^I\partial^\mu\lambda^J\,R^2\,,\\
&\mathscr{T}_{22} = \frac{1}{10}\mathcal{G}^4_{IJ}\,\partial_\mu\lambda^I\partial^\mu\lambda^J\,\nabla^2 R\,, \qquad \mathscr{T}_{23,\ldots,28} = \frac{1}{2}\mathcal{H}^{1,\ldots,6}_{IJ}\,\partial_\mu\lambda^I\partial_\nu\lambda^J\,H^{\mu\nu}_{1,\ldots,6}\,,\\
&\mathscr{T}_{29} = \mathcal{F}_{IJ}\,\partial_\kappa\lambda^I\nabla_\lambda\partial_\mu\lambda^J\,\nabla^\kappa G^{\lambda\mu}\,, \qquad \mathscr{T}_{30} = \mathcal{F}'_{IJ}\,\partial_\kappa\lambda^I\nabla_\lambda\partial_\mu\lambda^J\,\nabla^\lambda G^{\kappa\mu}\,,
\end{aligned} \tag{B.23}$$

and

$$\begin{aligned}
&\mathscr{Z}_{20}^\mu = \frac{1}{5}\mathcal{E}_{IJ}\,\partial^\mu\lambda^I\partial_\nu\lambda^J\,\partial^\nu R\,, \quad \mathscr{Z}_{21}^\mu = \mathcal{D}_{IJ}\,\partial_\kappa\lambda^I\nabla_\lambda\partial_\nu\lambda^J\,R^{\mu\lambda\kappa\nu}\,,\\
&\mathscr{Z}_{22}^\mu = \mathcal{C}_I\,\partial_\nu\nabla^2\lambda^I\,G^{\mu\nu}, \qquad \mathscr{Z}_{23}^\mu = \mathcal{C}_{IJ}\,\partial_\kappa\lambda^I\nabla_\nu\partial^\kappa\lambda^J\,G^{\mu\nu}, \qquad \mathscr{Z}_{24}^\mu = \mathcal{C}'_{IJ}\,\partial_\nu\lambda^I\nabla^2\lambda^J\,G^{\mu\nu},\\
&\mathscr{Z}_{25}^\mu = \frac{1}{5}\mathcal{B}_{IJ}\,\partial^\mu\lambda^I\nabla^2\lambda^J\,R \qquad \mathscr{Z}_{26}^\mu = \mathcal{A}_{IJ}\,\partial_\nu\nabla^2\lambda^I\nabla^\mu\partial^\nu\lambda^J, \qquad \mathscr{Z}_{27}^\mu = \mathcal{A}'_{IJ}\,\partial^\mu\lambda^I(\nabla^2)^2\lambda^J\,.
\end{aligned} \tag{B.24}$$

Furthermore, we have

$$\begin{aligned}
&\mathscr{T}_{31} = \frac{1}{2}\mathcal{F}_{IJK}\,\partial_\kappa\lambda^I\partial_\lambda\lambda^J\partial_\mu\lambda^K\,\nabla^\kappa G^{\lambda\mu}, \qquad \mathscr{T}_{32} = \frac{1}{5}\hat{\mathcal{E}}_{IJ}\,\partial_\mu\lambda^I\nabla^2\lambda^J\,\partial^\mu R\,,\\
&\mathscr{T}_{33} = \frac{1}{10}\mathcal{E}_{IJK}\,\partial_\mu\lambda^I\partial_\nu\lambda^J\partial^\nu\lambda^K\,\partial^\mu R\,, \qquad \mathscr{T}_{34} = \mathcal{D}_{IJK}\,\partial_\kappa\lambda^I\partial_\mu\lambda^J\nabla_\lambda\partial_\nu\lambda^K\,R^{\kappa\lambda\mu\nu},\\
&\mathscr{T}_{35} = \frac{1}{4}\mathcal{D}_{IJKL}\,\partial_\kappa\lambda^I\partial_\lambda\lambda^J\partial_\mu\lambda^K\partial_\nu\lambda^L\,R^{\kappa\lambda\mu\nu}, \quad \mathscr{T}_{36} = \hat{\mathcal{C}}_{IJ}\,\nabla_\mu\partial_\nu\lambda^I\nabla^2\lambda^J\,G^{\mu\nu},\\
&\mathscr{T}_{37} = \frac{1}{2}\hat{\mathcal{C}}'_{IJ}\,\nabla_\kappa\partial_\mu\lambda^I\nabla^\kappa\partial_\nu\lambda^J\,G^{\mu\nu}, \qquad \mathscr{T}_{38} = \frac{1}{2}\mathcal{C}_{IJK}\,\partial_\mu\lambda^I\partial_\nu\lambda^J\nabla^2\lambda^K\,G^{\mu\nu},\\
&\mathscr{T}_{39} = \mathcal{C}'_{IJK}\,\partial_\mu\lambda^I\partial_\kappa\lambda^J\nabla^\kappa\partial_\nu\lambda^K\,G^{\mu\nu}, \qquad \mathscr{T}_{40} = \frac{1}{2}\mathcal{C}''_{IJK}\,\partial_\kappa\lambda^I\partial^\kappa\lambda^J\nabla_\mu\partial_\nu\lambda^K\,G^{\mu\nu},\\
&\mathscr{T}_{41} = \frac{1}{4}\mathcal{C}_{IJKL}\,\partial_\mu\lambda^I\partial_\nu\lambda^J\partial_\kappa\lambda^K\partial^\kappa\lambda^L\,G^{\mu\nu}, \qquad \mathscr{T}_{42} = \frac{1}{5}\mathcal{B}_I\,(\nabla^2)^2\lambda^I\,R\,,\\
&\mathscr{T}_{43} = \frac{1}{10}\hat{\mathcal{B}}_{IJ}\,\nabla^2\lambda^I\nabla^2\lambda^J\,R\,, \qquad \mathscr{T}_{44} = \frac{1}{10}\hat{\mathcal{B}}'_{IJ}\,\nabla_\mu\partial_\nu\lambda^I\nabla^\mu\partial^\nu\lambda^J\,R\,,\\
&\mathscr{T}_{45} = \frac{1}{10}\mathcal{B}_{IJK}\,\partial_\mu\lambda^I\partial^\mu\lambda^J\nabla^2\lambda^K\,R\,, \qquad \mathscr{T}_{46} = \frac{1}{10}\mathcal{B}'_{IJK}\,\partial_\mu\lambda^I\partial_\nu\lambda^J\nabla^\mu\partial^\nu\lambda^K\,R\,,\\
&\mathscr{T}_{47} = \frac{1}{20}\mathcal{B}_{IJKL}\,\partial_\mu\lambda^I\partial^\mu\lambda^J\partial_\nu\lambda^K\partial^\nu\lambda^L\,R\,,
\end{aligned} \tag{B.25}$$



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and

$$\mathscr{Z}^{\mu}_{28} = \mathcal{A}_{IJK}\,\partial_\nu\lambda^I\nabla^\mu\partial^\nu\lambda^J\nabla^2\lambda^K, \qquad \mathscr{Z}^{\mu}_{29} = \mathcal{A}'_{IJK}\,\partial_\kappa\lambda^I\nabla^\mu\partial_\lambda\lambda^J\nabla^\kappa\partial^\lambda\lambda^K,$$
$$\mathscr{Z}^{\mu}_{30} = \frac{1}{2}\mathcal{A}_{IJKL}\,\partial_\nu\lambda^I\partial^\nu\lambda^J\partial^\mu\lambda^K\nabla^2\lambda^L. \tag{B.26}$$

Finally, we also have the terms

$$\mathscr{T}_{48} = \mathcal{A}_I\,(\nabla^2)^3\lambda^I, \qquad \mathscr{T}_{49} = \hat{\mathcal{A}}_{IJ}\,(\nabla^2)^2\lambda^I\nabla^2\lambda^J, \qquad \mathscr{T}_{50} = \frac{1}{2}\hat{\mathcal{A}}'_{IJ}\,\partial_\mu\nabla^2\lambda^I\partial^\mu\nabla^2\lambda^J,$$
$$\mathscr{T}_{51} = \frac{1}{2}\hat{\mathcal{A}}''_{IJ}\,\nabla_\kappa\nabla_\lambda\partial_\mu\lambda^I\nabla^\kappa\nabla^\lambda\partial^\mu\lambda^J, \qquad \mathscr{T}_{52} = \frac{1}{8}\hat{\mathcal{A}}_{IJK}\,\nabla^2\lambda^I\nabla^2\lambda^J\nabla^2\lambda^K,$$
$$\mathscr{T}_{53} = \frac{1}{2}\hat{\mathcal{A}}'_{IJK}\,\nabla_\kappa\partial_\mu\lambda^I\nabla^\kappa\partial_\nu\lambda^J\nabla^\mu\partial^\nu\lambda^K, \qquad \mathscr{T}_{54} = \hat{\mathcal{A}}''_{IJK}\,\partial_\mu\lambda^I\nabla^2\lambda^J\partial^\mu\nabla^2\lambda^K,$$
$$\mathscr{T}_{55} = \check{\mathcal{A}}_{IJK}\,\partial_\mu\lambda^I\nabla^\mu\partial_\nu\lambda^J\partial^\nu\nabla^2\lambda^K, \qquad \mathscr{T}_{56} = \frac{1}{2}\check{\mathcal{A}}'_{IJK}\,\partial_\mu\lambda^I\partial^\mu\lambda^J(\nabla^2)^2\lambda^K,$$
$$\mathscr{T}_{57} = \frac{1}{2}\check{\mathcal{A}}''_{IJK}\,\partial_\mu\lambda^I\partial_\nu\lambda^J\nabla^\mu\partial^\nu\nabla^2\lambda^K, \qquad \mathscr{T}_{58} = \frac{1}{4}\hat{\mathcal{A}}_{IJKL}\,\partial_\mu\lambda^I\partial^\mu\lambda^J\nabla^2\lambda^K\nabla^2\lambda^L,$$
$$\mathscr{T}_{59} = \frac{1}{4}\hat{\mathcal{A}}'_{IJKL}\,\partial_\kappa\lambda^I\partial^\kappa\lambda^J\nabla_\mu\partial_\nu\lambda^K\nabla^\mu\partial^\nu\lambda^L, \qquad \mathscr{T}_{60} = \frac{1}{2}\hat{\mathcal{A}}''_{IJKL}\,\partial_\kappa\lambda^I\partial_\lambda\lambda^J\nabla^\kappa\partial_\mu\lambda^K\nabla^\lambda\partial^\mu\lambda^L,$$
$$\mathscr{T}_{61} = \frac{1}{2}\check{\mathcal{A}}_{IJKL}\,\partial_\mu\lambda^I\partial_\nu\lambda^J\nabla^\mu\partial^\nu\lambda^K\nabla^2\lambda^L, \qquad \mathscr{T}_{62} = \frac{1}{2}\check{\mathcal{A}}'_{IJKL}\,\partial_\kappa\lambda^I\partial_\lambda\lambda^J\partial_\mu\lambda^K\nabla^\kappa\nabla^\lambda\partial^\mu\lambda^L,$$
$$\mathscr{T}_{63} = \frac{1}{4}\mathcal{A}_{IJKLM}\,\partial_\mu\lambda^I\partial^\mu\lambda^J\partial_\nu\lambda^K\partial^\nu\lambda^L\nabla^2\lambda^M, \qquad \mathscr{T}_{64} = \frac{1}{4}\mathcal{A}'_{IJKLM}\,\partial_\kappa\lambda^I\partial^\kappa\lambda^J\partial_\lambda\lambda^K\partial_\mu\lambda^L\nabla^\lambda\partial^\mu\lambda^M,$$
$$\mathscr{T}_{65} = \frac{1}{8}\mathcal{A}_{IJKLMN}\,\partial_\kappa\lambda^I\partial^\kappa\lambda^J\partial_\lambda\lambda^K\partial^\lambda\lambda^L\partial_\mu\lambda^M\partial^\mu\lambda^N. \tag{B.27}$$



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