

A rotating hairy AdS₃ black hole with the metric having only one Killing vector field

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ABSTRACT: We perturbatively construct a three-dimensional rotating AdS black hole with a real scalar hair. We choose the mass of a scalar field slightly above the Breitenlohner-Freedman bound and impose a general boundary condition for the bulk scalar field at AdS infinity. We first show that rotating BTZ black holes are unstable against scalar field perturbations under our more general boundary condition. Next we construct a rotating hairy black hole perturbatively with respect to a small amplitude ϵ of the scalar field, up to $O(\epsilon^4)$. Our hairy black hole is stationary and exhibits no dissipation, but the lumps of the non-linearly perturbed geometry break axial symmetry, thus providing the first example of a rotating black hole whose metric admits only one Killing vector field. Furthermore, we numerically show that the entropy of our hairy black hole is larger than that of the BTZ black hole with the same energy and the angular momentum. We briefly discuss if our rotating hairy black hole in lumpy geometry could be the endpoint of the instability.

KEYWORDS: Black Holes, Classical Theories of Gravity

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1 Introduction

After the discovery of the gauge-gravity duality [1], Anti-de Sitter (AdS) spacetime has attracted significant interest in both the gravity and high energy physics community. Asymptotically AdS space-times exhibit a number of characteristic features which are absent in asymptotically flat or de Sitter spacetimes. One of such key features is the occurrence of particular types of instability, such as weakly turbulent instability [2] and superradiant instability [3] (see also ref. [4] and references therein), in which the AdS boundary plays a crucial role as a confining box. It is clearly important to consider consequences — in particular, possible final states — of such instabilities.

Consider, for example, superradiant instability of a rotating AdS black hole with angular velocity Ω . Suppose a monochromatic wave of the form $e^{-i(\omega t - k\varphi)}$ falls into the black hole. If $\text{Re}[\omega] < k\Omega$, then such an incident wave gets amplified by the black hole rotation and reflected at AdS infinity towards the black hole where it again gets amplified, and the process repeats. It was suggested that the end point of this superradiant instability may be a rotating black hole which admits only one Killing vector field [5]. In fact, the possibility of such a less symmetric black hole was first suggested in [6] in the context of asymptotically flat black holes in higher dimensions. Motivated by this, rotating black hole solutions with complex scalar hair that break axial symmetry were numerically constructed [7, 8, 10]. The proposed solutions are, however, not completely satisfactory in the sense that although the scalar field configuration is indeed invariant only under one Killing vector field, the metric itself admits more than one Killing vector field.

In this paper, we address this issue in a three-dimensional setup and construct, for the first time, a rotating AdS black hole whose metric possesses only a single Killing vector field. For this purpose, we first consider scalar field perturbation of a rotating BTZ black

hole [13], which is the simplest model of rotating AdS black holes.¹ When the mass-squared of the scalar field takes a certain negative value, we can impose, rather than the Dirichlet condition, more general boundary conditions at AdS infinity under which the rotating BTZ black hole exhibits instabilities. Next, by inspecting quasi-normal modes of the scalar field, we identify the marginally stable BTZ solution. Then, based on that, we perturbatively construct a rotating AdS black hole whose metric possesses only one Killing vector field. Interestingly, we find that the total energy of our *lumpy* black hole is lower than that of the BTZ black hole with the same entropy and the angular momentum. This in turn implies that our hairy black hole is entropically more favorable than the background BTZ black hole with the same mass and angular momentum.

2 “Superradiant” instability of rotating BTZ black hole

Consider the three-dimensional model with Lagrangian

$$L = R + \frac{2}{l^2} - 2(\nabla_\mu \phi \nabla^\mu \phi + m^2 \phi^2 + \eta \phi^4), \quad (2.1)$$

where l represents the AdS scale, R the Einstein-Hilbert term, and ϕ a real scalar field with some constants m and η . In the absence of the real scalar field, our model (2.1) admits a rotating BTZ black hole [13] with the metric

$$ds^2 = -\frac{(r_+^2 - r_-^2)^2 z}{l^2(1-z)(r_+^2 - r_-^2 z)} dt^2 + \frac{l^2}{4z(1-z)^2} dz^2 + \frac{r_+^2 - r_-^2 z}{1-z} \left(d\varphi - \frac{r_+ r_- (1-z)}{l(r_+^2 - r_-^2 z)} dt \right)^2, \quad (2.2)$$

where the outer horizon and the infinity (*i.e.*, boundary) are located at $z = 0$ and $z = 1$ as in [15], and φ is the angular coordinate with period 2π , and r_+ (r_-) denotes the outer (inner) horizon radius.

Making the ansatz $\phi = \text{Re}[\Pi_1(z)] \cos(\omega t - k\varphi)$ and for the meantime setting $\eta = 0$, we find the general solution for $\Pi_1(z)$ on the background (2.2) expressed in terms of the hypergeometric function [15],

$$\begin{aligned} \Pi_1 &= C_1 z^{\zeta i} (1-z)^{\frac{1-\sigma}{2}} {}_2F_1(a, b, c; z) + C_2 z^{-\zeta i} (1-z)^{\frac{1-\sigma}{2}} {}_2F_1(1+a-c, 1+b-c, 2-c; z), \\ a &= \frac{il(l\omega - k)}{2(r_+ - r_-)} + \frac{1-\sigma}{2}, & b &= \frac{il(l\omega + k)}{2(r_+ + r_-)} + \frac{1-\sigma}{2}, \\ c &= 1 + 2i\zeta, & \sigma &= \sqrt{1 + m^2 l^2}, & \zeta &= \frac{l(lr_+ \omega - kr_-)}{2(r_+^2 - r_-^2)}. \end{aligned} \quad (2.3)$$

Imposing an ingoing boundary condition on the horizon, we set $C_1 = 0$. Then, with the help of [16], the asymptotic behavior is given by

$$\begin{aligned} \Pi_1 &\simeq \alpha(1-z)^{\frac{1-\sigma}{2}} + \beta(1-z)^{\frac{1+\sigma}{2}}, \\ \alpha &= \frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)}, & \beta &= \frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(1+a-c)\Gamma(1+b-c)}, \end{aligned} \quad (2.4)$$

¹Compare [11, 12] for some attempts in three-dimensions along the same line of [7].

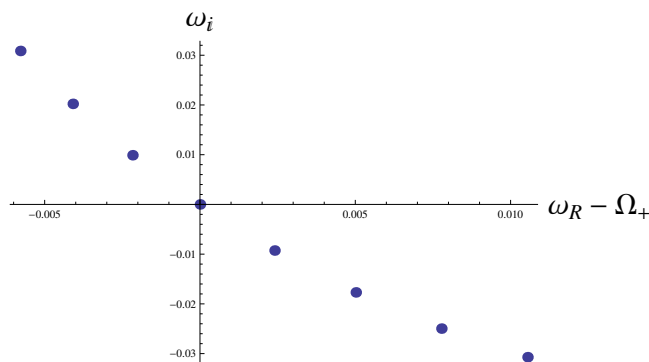


Figure 1. The horizontal line represents $\omega_R - \Omega_+$, while the vertical line represents ω_I . The parameters are chosen as $r_- = 3$, $k = l = 1$, and $\kappa = -0.414$.

where we have set $C_2 = 1$. The Breitenlohner-Freedman (BF) bound [17] corresponds to the case $m^2 l^2 = -1$. In the range $-1 < m^2 l^2 < 0$, both the modes are normalizable near the infinity and therefore we can impose a more general boundary condition at infinity given by $\alpha = \kappa^{-1} \beta$ with κ being some constant. Such a more general boundary condition corresponds to adding a double-trace interaction $\sim (1/2\kappa^2) \int dx^2 \mathcal{O}^2$ to the dual field theory [14], where \mathcal{O} is an operator dual to ϕ . We will see later that the total energy defined in an appropriate manner [18] is indeed conserved. Therefore, in the present context, there is nothing wrong with imposing such a more general boundary condition.

Now we are concerned with the behavior of quasi normal modes, which can be computed by fixing the geometric parameters, say r_- , k , l , and by imposing the ingoing condition at the horizon and our more general boundary condition at infinity. For the purpose of numerical computation below, let us set $m^2 = -8/9l^2$, $l = 1 = k$, $r_- = 3$, and choose κ as the following particular value:

$$\kappa = -\frac{4\sqrt{3}\pi^3 \left(\cosh \frac{\pi}{5} + \frac{1}{2}\right)^{-1}}{\left(\Gamma\left(\frac{1}{3}\right)\right)^2 \left|\Gamma\left(\frac{1}{3} + \frac{i}{10}\right)\right|^4} = -0.414. \tag{2.5}$$

In this setup, we compute quasi normal modes $\omega = \omega_R + i\omega_I$ and plot some of our results in figure 1.

Note that the outer horizon radius r_+ (and thus the angular velocity Ω_+) also varies depending on the value of ω . In particular, we can find that when $r_+ = 5$, ω becomes real and equal to $\Omega_+ = 3/5$ as shown in figure 1, and also that there is no flux across the horizon. In fact, to achieve this, we have purposely chosen the particular value of κ in eq. (2.5). Thus the rotating BTZ black holes become unstable under our more general boundary condition against modes with $\omega_I > 0$ shown in figure 1. In particular, the solution with $r_+ = 5$ and $r_- = 3$ can be viewed as the marginally stable solution. This is not the case under the standard Dirichlet boundary conditions [19](see also [20]). We also note that the BTZ background admits a Killing vector field which is causal everywhere outside the event horizon, and therefore there would not be superradiant instability in the standard sense, according to the argument of [9]. However note also that our model violates the

dominant energy condition and thus the argument of [9] does not straightforwardly apply to the present case. Since our unstable modes $\omega_I > 0$ appear only when $\omega_R < \Omega_+$, i.e., the standard superradiant condition is apparently satisfied as can be seen in figure 1, we loosely call our instability “superradiant instability” in the rest of the paper. This instability might also be called a “scalar condensation instability” [23] as it can occur due to our general boundary condition.

3 Perturbative construction of a hairy black hole in lumpy geometry

Having obtained the marginally stable solution, we can expect that there should exist a hairy black hole solution dressed with a condensed real scalar field. We perturbatively construct such a hairy black hole solution by expanding the metric functions and scalar field in a series of small amplitude ϵ of the scalar field, up to $O(\epsilon^4)$, starting from our marginal solution.

In the probe limit of the scalar field, where the backreaction onto the spacetime is ignored, ϕ depends on the coordinate $(r_-/r_+l)t - \varphi$ and z only, we expect that the resultant hairy black hole also depends only on $y = \omega_* t - \varphi$ and z . Thus, we make the metric ansatz as

$$ds^2 = -f(y, z)e^{-2\delta(y, z)}dt^2 + \frac{g'(z)^2 dz^2}{4g(z)f(y, z)} + g(z)(d\varphi - \Omega(y, z)dt)^2, \quad (3.1)$$

where $f = 0$ on the horizon $z = 0$. The scalar field ϕ and the metric functions, collectively denoted by F , can be expanded as

$$\begin{aligned} \phi(y, z) &= \epsilon\phi_1(y, z) + \epsilon^3\phi_2(y, z) + \dots, \\ F &= F_0(y, z) + \epsilon^2F_1(y, z) + \epsilon^4F_2(y, z) + \dots, \\ \omega_* &= \frac{3}{5l} + \epsilon^2\omega_1 + \dots, \quad F = f, g, \Omega, \delta, \end{aligned} \quad (3.2)$$

where F_0 and the leading term of ω_* respectively denote the corresponding values of the marginal BTZ black hole. (Remember that we have set $r_+ = 5$, $r_- = 3$, $l = 1$.) We require the following asymptotic conditions for the metric functions,

$$\lim_{z \rightarrow 1} \Omega = \lim_{z \rightarrow 1} \delta = 0 \quad (3.3)$$

and the condition (2.5) for ϕ . Near the horizon, the scalar curvature R becomes $R \sim (\omega_* - \Omega)^2(\partial_y\phi)^2/f$. To remove the singularity on the horizon, we require

$$\omega_* = \Omega|_{z=0}, \quad (3.4)$$

where $\Omega|_{z=0}$ is in fact independent of y as well, thus a true constant as can be seen in eq. (3.6) below. The solution for ϕ_1 is given by $\phi_1 = \Pi_1(z)\cos(ky)$ and using (3.4), the equation of motion for $\Pi_1(z)$ reduces to $\mathcal{L}_k\Pi_1 = 0$ with \mathcal{L}_k being the k -dependent differential operator

$$\mathcal{L}_k := z\partial_z^2 + \partial_z - \frac{l^2(m^2r_+^2 + k^2(1-z))}{4r_+^2(1-z)^2}. \quad (3.5)$$

Note that the solution to this equation corresponds to eq. (2.3) with $\eta = 0$. From the structure of the Einstein equations, $F_1(y, z)$ can be expanded as

$$\begin{aligned} f_1(y, z) &= z(P(z) \cos(2ky) + Q(z)) + a_1 f_{\text{btz}}^{(1)}(z), \\ \delta_1(y, z) &= R(z) \cos(2ky) + S(z), \quad g_1(z) = a_1 g_{\text{btz}}^{(1)}(z), \\ \Omega_1(y, z) &= zT(z) \cos(2ky) + U(z), \end{aligned} \tag{3.6}$$

where $f_{\text{btz}}^{(1)}(z)$ and $g_{\text{btz}}^{(1)}(z)$ are trivial perturbed solutions² of the BTZ black hole (2.2) such that $r_+ \rightarrow r_+ + a_1 \epsilon^2 + a_2 \epsilon^4 + \dots$ and $r_- \rightarrow r_- + a_1 r_- / r_+ \epsilon^2 + \dots$. Note that this deviation does not change the velocity of the black hole. Then, the parameter a_1 is not included in the perturbed functions (P, Q, R, S, T, U) .

The functions (P, R, T) representing an oscillating mode are decoupled from the zero mode functions (Q, S, U) at $O(\epsilon^2)$. From the tz , φz , and $\varphi\varphi$ components of the Einstein equations, we obtain a master equation for the oscillating mode

$$zT'' + \left(4 + \frac{2k^2 l^2}{r_+^2 - r_-^2} - \frac{2r_+^2}{r_+^2 - r_-^2 z}\right) T' - \frac{2r_-^2 (r_+^2 - r_-^2 + k^2 l^2)}{(r_+^2 - r_-^2)(r_+^2 - r_-^2 z)} T = S_T(\Pi_1, \Pi_1'). \tag{3.7}$$

P and R are expressed by T, T', Π_1 , and Π_1' only [See appendix].

Similarly, we obtain the equations of motion for the zero mode and find the regular solutions satisfying (3.3) as

$$\begin{aligned} h(z) &= -\frac{k^2 r_-^2}{r_+^2} \int_0^z \frac{\Pi_1'^2}{1-z} dz - \frac{4r_+^2}{l^2} \int_0^z (1-z) \Pi_1'^2 dz \\ &\quad + \frac{2r_+ r_-}{l} [U(z) - U(0)] - \frac{2r_+^2 (r_+^2 - r_-^2)}{l^2} \left[\frac{S(z)}{r_+^2 - r_-^2 z} - \frac{S(0)}{r_+^2} \right], \\ S &= \frac{1}{r_+^2 - r_-^2} \int_z^1 \left[\frac{k^2 r_-^2 l^2}{2r_+^2} \Pi_1'^2 + 2(1-z)(r_+^2 - r_-^2 z) \Pi_1'^2 \right] dz, \\ U &= \frac{r_+^4 (z-1) B}{(r_+^2 - r_-^2 z)(r_+^2 - r_-^2)} + \frac{z-1}{(r_+^2 - r_-^2 z)(r_+^2 - r_-^2)} \int_0^z (r_+^2 - r_-^2 z')^2 \mathcal{S}_U(z') dz', \\ &\quad + \frac{1}{r_+^2 - r_-^2} \int_z^1 (r_+^2 - r_-^2 z')(z'-1) \mathcal{S}_U(z') dz', \\ \mathcal{S}_U &= -\frac{r_- l k^2}{2r_+(1-z)(r_+^2 - r_-^2 z)} \Pi_1'^2 - \frac{2r_+ r_- (1-z)}{l(r_+^2 - r_-^2 z)} \Pi_1'^2, \end{aligned} \tag{3.8}$$

where $h = zQ + \frac{2(r_+^2 - r_-^2)z}{l^2} \Pi_1 \Pi_1'$ and B is an integration constant. For simplicity, setting $B = 0$, we easily find that

$$h(1) = 2\Omega_+(r_+^2 - r_-^2)U'(1), \quad \Omega_+ = \frac{r_-}{r_+ l}. \tag{3.9}$$

As shown later, this relation agrees with the first law of our hairy black hole.

²Here we mean that $f_{\text{btz}}^{(1)}(z)$ and $g_{\text{btz}}^{(1)}(z)$ are solutions of source-free differential equations for metric perturbation.

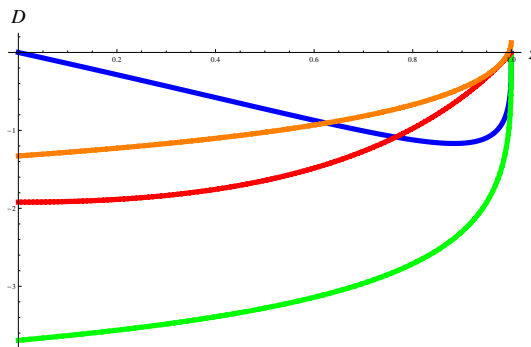


Figure 2. $D = \phi_{21}$ (solid, blue), $D = \phi_{23}/2$ (dashed, green), and $D = \Omega_{20}$ (dotted, red), $D = 10^{-3} \cdot h_{20}$ (dotdashed, orange) for the parameter choice $r_+ = 5$, $r_- = 3$, $k = 1$, $l = 1$, and $\eta = 1$.

4 Higher order solutions for ϵ

The equations of motion for $\phi_2(y, z)$ are given by

$$\begin{aligned} \phi_2(y, z) &= \phi_{21}(z) \cos(ky) + \phi_{23}(z) \cos(3ky), \\ \mathcal{L}_k \phi_{21} &= \mathcal{S}_{\phi_1}, \quad \mathcal{L}_{3k} \phi_{23} = \mathcal{S}_{\phi_3}, \end{aligned} \tag{4.1}$$

where the source terms \mathcal{S}_{ϕ_1} and \mathcal{S}_{ϕ_3} include the metric functions for $O(\epsilon^2)$, Π_1 , and the parameters a_1 , B . Here we simply omit the precise expression of those source terms, as their explicit form is rather complicated and not essential for the rest of our arguments. Here, ω_1 is replaced by $U(0)$ by eq. (3.4). Again just for simplicity, we hereafter set $B = 0$. The second equation can be formally solved by constructing two independent homogeneous solution λ_i ($i = 1, 2$) satisfying $\mathcal{L}_{3k} \lambda_i = 0$, $i = 1, 2$, where λ_1 and λ_2 are solutions satisfying our more general boundary condition (2.5) at infinity and the regularity at the horizon. The explicit form of ϕ_{23} is given in appendix.

The solution for ϕ_{21} in (4.1) cannot be obtained by using a similar manner of the case ϕ_{23} . This is because there is no regular homogeneous solution, independent of Π_1 .³ By shooting the parameter a_1 , however, we can numerically find the regular solution satisfying our more general boundary condition. Figure 2 shows the numerical data for ϕ_{23} and ϕ_{21} for $r_+ = 5$, $r_- = 3$, $k = 1$, $l = 1$, and $\eta = 1$. a_1 is approximately $\simeq 473$. The asymptotic functions of ϕ_{21} and ϕ_{23} are well approximated as $\phi_{21} \simeq -4.31(1-z)^{\frac{1}{3}} + 1.79(1-z)^{\frac{2}{3}}$ and $\phi_{23} \simeq -12.5(1-z)^{\frac{1}{3}} + 5.18(1-z)^{\frac{2}{3}}$ near infinity, agreeing with our more general boundary condition (2.5).

Similarly, we also numerically construct the metric functions for $O(\epsilon^4)$. Since we are interested in the energy difference between the hairy BH and BTZ solution, we only construct the zero mode solution (F_2 solutions which are independent of y) at $O(\epsilon^4)$. Denoting Ω_{20} , f_{20} as the zero mode solutions of Ω_2 , f_2 , figure 1. shows the functions of

³Note that Π_1 is the homogeneous regular solution satisfying our more general boundary condition (2.5).

Ω_{20} and h_{20} for the same parameter choice, where h_{20} is defined by

$$h_{20}(z) := f_{20} + \frac{2(r_+^2 - r_-^2)}{l^2} \left(\Pi_1 \phi_{21} + \frac{a_1}{r_+} \Pi_1^2 \right)'. \quad (4.2)$$

Note that h_{20} becomes finite at $z \rightarrow 1$ since the divergent term in f_{20} ;

$$f_{20} \simeq \frac{4(r_+^2 - r_-^2)}{3r_+ l^2} \alpha (a_1 \alpha + r_+ \alpha_2) (1-z)^{-\frac{1}{3}}, \quad (4.3)$$

canceling with the second divergent term in (4.2), where α_2 is the asymptotic leading coefficient of $\phi_{21} \simeq \alpha_2 (1-z)^{1/3}$.

5 Calculation of effective holographic energy

Following the re-normalization method [21] and [18], we calculate the effective holographic energy momentum tensor

$$T_{ij} = K_{ij} - K h_{ij} - \frac{1}{l} h_{ij} - \sqrt{z}(1-z) \{ (1-\sigma)(1-z)^{-\sigma} + 2\kappa \} \tilde{\alpha}^2 \frac{h_{ij}}{l}, \quad (5.1)$$

where h_{ij} and $K_{ij} := -\nabla_{(i} n_{j)}$ ($i, j = t, \varphi$) are the induced metric and the extrinsic curvature defined on a $z = \text{const.}$ surface with n^μ being the outward pointing unit normal vector to it, and then the limit $z \rightarrow 1$ is taken in eq. (5.1) as a whole. Here $\tilde{\alpha}$ is the leading coefficient of the asymptotic scalar field $\phi \simeq \tilde{\alpha}(t, \varphi)(1-z)^{\frac{1}{3}} + \kappa \tilde{\alpha}(t, \varphi)(1-z)^{\frac{2}{3}}$. Note that the terms in the second line in eq. (5.1) come from our deformation of the dual field theory by the double-trace interaction. Since the divergent term in the first line in T_{ij} cancels with the divergent term in the second line in eq. (5.1), T_{ij} is well defined on the dual field theory side (note that h and h_{20} in (3.8), (4.2) do not diverge asymptotically). Furthermore, for an arbitrary function $\tilde{\alpha}$, it can be easily shown that T_{ij} is conserved by Codacci's equation [22], i. e., $\nabla_i T^{ij} = 0$. Thus, in this paper, we adopt T_{ij} as the definition of the effective holographic energy momentum tensor. Let us define the spatially averaged quantity \overline{H} as $\overline{H} := \int_0^{2\pi} H(t, \varphi) d\varphi / (2\pi)$. Then, at $O(\epsilon^2)$, we find that

$$\begin{aligned} E_{\text{hair}}^{(2)} &:= \overline{T^{(2)}_{tt}} \simeq -\frac{h(1)}{2l} \epsilon^2 + \frac{r_+^2 + r_-^2}{r_+ l^3} \epsilon^2 a_1, \\ J_{\text{hair}}^{(2)} &:= -\overline{T^{(2)}_{t\varphi}} = -\epsilon^2 \left(\frac{r_+^2 - r_-^2}{l} U'(1) - \frac{2a_1 r_-}{l^2} \right). \end{aligned} \quad (5.2)$$

Combining with (3.9) yields the first law of the hairy black hole;

$$E_{\text{hair}}^{(2)} = T S_{\text{hair}}^{(2)} + \Omega_+ J_{\text{hair}}^{(2)}, \quad T := \frac{r_+^2 - r_-^2}{2\pi r_+ l^2}, \quad (5.3)$$

where T is the temperature and $S_{\text{hair}}^{(2)}$ is the deviation of the entropy of the black hole given by $S_{\text{hair}}^{(2)} = 2\pi a_1 \epsilon^2 / l$.

Now, let us derive the fourth order corrections of the energy $E_{\text{hair}}^{(4)}$ and compare with the energy of BTZ black hole with the same angular momentum and entropy. For the

outer (inner) horizon radius R_+ and R_- , the energy, the angular momentum, and the entropy of the BTZ black hole are

$$E_{\text{btz}} = \frac{R_+^2 + R_-^2}{2l^3}, \quad J_{\text{btz}} = \frac{R_+ R_-}{l^2}, \quad S_{\text{btz}} = \frac{2\pi}{l} R_+. \quad (5.4)$$

Let us expand R_{\pm} as $R_+ = r_+ + a_1 \epsilon^2 + a_2 \epsilon^4 + \dots$ and $R_- = r_- + r_{1-} \epsilon^2 + r_{2-} \epsilon^4 + \dots$ so that the entropy of the BTZ is equal to the one of the hairy BH, $S_{\text{btz}} = S_{\text{hair}}$. Equating $E_{\text{btz}}^{(2)} = E_{\text{hair}}^{(2)}$, and $J_{\text{btz}}^{(4)} = J_{\text{hair}}^{(4)}$, r_{1-} and r_{2-} are expressed by a_1 , a_2 , $E_{\text{hair}}^{(2)}$, and $J_{\text{hair}}^{(4)}$, and we obtain

$$E_{\text{btz}}^{(4)} = \frac{l^3 (E_{\text{hair}}^{(2)})^2}{2r_-^2} - \frac{\epsilon^2 a_1 (r_+^2 + r_-^2)}{r_+ r_-^2} E_{\text{hair}}^{(2)} + \frac{r_- J_{\text{hair}}^{(4)}}{l r_+} + \frac{\epsilon^4 a_1^2}{2l^3} \left(3 + \frac{r_+^2}{r_-^2} \right) + \frac{\epsilon^4 a_2 (r_+^2 - r_-^2)}{r_+ l^3}. \quad (5.5)$$

Note that $\Delta^{(4)} E := E_{\text{hair}}^{(4)} - E_{\text{btz}}^{(4)}$ is independent of the second order deviation parameter a_2 in (3.2). The numerical calculation shows that $\Delta^{(4)} E \simeq -5.8 \times 10^2 \cdot \epsilon^4 < 0$. This implies that, if we increase $E_{\text{hair}}^{(4)}$ so that it becomes equal to $E_{\text{btz}}^{(4)}$, then we have the relationship $S_{\text{hair}}^{(4)} > S_{\text{btz}}^{(4)}$, due to the thermodynamic laws.

6 Summary and discussions

In this paper, aiming at constructing a rotating black hole whose metric admits only a single Killing vector field, we have investigated three-dimensional AdS black holes with a real scalar field. First we showed that under our more general boundary condition, the scalar field perturbation shows instabilities. Then we have constructed rotating hairy black holes perturbatively with respect to the scalar perturbation amplitude, ϵ , from the onset of the instability, up to $O(\epsilon^4)$. We showed that the entropy of our hairy black hole is larger than the one of BTZ, with the same energy and the angular momentum. Judging merely from this entropically favored nature, one might expect our hairy black hole to be a possible final fate of superradiant instability in AdS₃. However, since our solution is constructed perturbatively, there is a possibility that once non-linear effects are fully taken into account, some new unstable modes might show up. Also we should mention that under our general boundary condition, the background AdS₃ itself turns out to be unstable: one can check that our boundary condition $\kappa = -0.414$ violates the linear stability criterion established in [24] (see eq. (169) in that reference), which in the present case ($m^2 l^2 = -8/9$, $k = 0$) reads off

$$\kappa \geq -\frac{|\Gamma(-1/3)| \Gamma(2/3)^2}{\Gamma(1/3)^3} = -0.387. \quad (6.1)$$

In particular, the $k = 0$ mode appears to be a dominant unstable mode over other non-axisymmetric $k \neq 0$ unstable modes.⁴ As can be seen in eq. (4.1), the equations of motion for higher order perturbations include a source term that consists only of higher (odd-number $k \neq 0$) modes. This implies that if there is no zero mode in the initial data for our

⁴The authors thank Oscar Dias, Jorge Santos, and Benson Way for pointing out this instability [23].

perturbations, the zero mode would not be excited by higher order effects. Having such an unstable property, our hairy black hole may have some interesting applications in the context of holographic superconductor [25], in which the most dangerous $k = 0$ mode of our AdS background would be irrelevant (or stabilized) due to, e.g., lattice structure. It would be interesting to pursue this possibility.

The most striking feature of our solution is that lumpy geometry oscillates periodically and co-rotate with the black hole accompanied by the non-linear scalar perturbations. Although it is not strictly thermodynamically equilibrium, the system never dissipates [26], i.e., the entropy is always constant. This is in contrast to the solutions of refs. [27, 28], which actually show dissipation. In this sense, our solution may be viewed as an extension of the time-periodic solutions of [29] to the case with a black hole added. It would also be interesting to generalize the present analysis to higher dimensions.

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A Solutions for P , R , and ϕ_{23}

Here we give the expressions of P and R in terms of T , T' , Π_1 , and Π'_1 , and the solution ϕ_{23} of eq. (4.1).

$$\begin{aligned}
 P = & \frac{r_+(r_+^2 - r_-^2)(r_+^2 + 3r_-^2 + 4k^2l^2)z}{r_-l(r_+^2 + r_-^2 + 2k^2l^2)}T' - \frac{r_+r_-(r_+^2 - r_-^2)\{(5r_-^2 + 4k^2l^2)z - r_+^2(2-z)\}}{l(r_+^2 + r_-^2 + 2k^2l^2)(r_+^2 - r_-^2z)}T \\
 & - \frac{(r_+^2 - r_-^2)^2(r_+^2m^2 - k^2(1-z))}{2(r_+^2 + r_-^2 + 2k^2l^2)(1-z)(r_+^2 - r_-^2z)}\Pi_1^2 - \frac{2(r_+^2 - r_-^2)^2(r_-^2 + 2k^2l^2)z}{l^2(r_+^2 + r_-^2 + 2k^2l^2)(r_+^2 - r_-^2z)}\Pi_1\Pi'_1 \\
 & + \frac{2r_+^2(r_+^2 - r_-^2)^2(1-z)z}{l^2(r_+^2 + r_-^2 + 2k^2l^2)(r_+^2 - r_-^2z)}\Pi_1'^2, \tag{A.1}
 \end{aligned}$$

$$\begin{aligned}
 R = & -\frac{l(r_+^2 - r_-^2z)z\{(r_+^4 + r_-^4)z - 2r_+^2(r_-^2(2-z) + 2k^2l^2(1-z))\}}{2r_-r_+(r_+^2 - r_-^2)(r_+^2 + r_-^2 + 2k^2l^2)}T' \\
 & + \frac{r_-l\{4r_+^4 - 8r_+^2r_-^2z + ((r_+^2 + r_-^2)^2 + 4(r_+^2 - r_-^2)k^2l^2)z^2\}}{2r_+(r_+^2 - r_-^2)(r_+^2 + r_-^2 + 2k^2l^2)}T \\
 & + \frac{l^2(r_+^2m^2 - k^2(1-z))(r_-^2z - r_+^2(2-z))}{4r_+^2(r_+^2 + r_-^2 + 2k^2l^2)(1-z)}\Pi_1^2 + \frac{(r_+^2 - r_-^2 - 2k^2l^2(1-z))z}{r_+^2 + r_-^2 + 2k^2l^2}\Pi_1\Pi'_1 \\
 & + \frac{(1-z)z(r_+^2(2-z) - r_-^2z)}{r_+^2 + r_-^2 + 2k^2l^2}\Pi_1'^2, \tag{A.2}
 \end{aligned}$$

$$\phi_{23} = \frac{1}{\sigma(\hat{\beta}_3 - \kappa\hat{\alpha}_3)} \times \left(\int_0^z \mathcal{S}_{\phi_3}(z')\lambda_2(z')dz'\lambda_1(z) + \int_z^1 \mathcal{S}_{\phi_3}(z')\lambda_1(z')dz'\lambda_2(z) \right),$$

$$\hat{\alpha}_3 = \frac{\Gamma(\sigma)}{\Gamma\left(\frac{1+\sigma}{2} + \frac{3il}{2r_+}\right)\Gamma\left(\frac{1+\sigma}{2} - \frac{3il}{2r_+}\right)},$$

$$\hat{\beta}_3 = \frac{\Gamma(-\sigma)}{\Gamma\left(\frac{1-\sigma}{2} - \frac{3il}{2r_+}\right)\Gamma\left(\frac{1-\sigma}{2} + \frac{3il}{2r_+}\right)}, \tag{A.3}$$

where $\lambda_1(z)$ and $\lambda_2(z)$ are homogeneous solutions of $\mathcal{L}_{3k}\lambda_i = 0$, $i = 1, 2$;

$$\lambda_1 = (1-z)^{\frac{1-\sigma}{2}} F(\alpha_3, \beta_3, \alpha_3 + \beta_3; 1-z) + \kappa(1-z)^{\frac{1+\sigma}{2}} F(1-\alpha_3, 1-\beta_3, 2-\alpha_3-\beta_3; 1-z),$$

$$\lambda_2 = (1-z)^{\frac{1-\sigma}{2}} F(\alpha_3, \beta_3, 2; z),$$

$$\alpha_3 := \frac{3ikl(l\Omega_+ - 1)}{2(r_+ - r_-)} + \frac{1-\sigma}{2}, \quad \beta_3 := \frac{3ikl(l\Omega_+ - 1)}{2(r_+ - r_-)} - \frac{1-\sigma}{2}. \tag{A.4}$$

Here, λ_1 is the solution satisfying our more general boundary condition (2.5), while λ_2 is a regular solution at the horizon $z = 0$.

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References

- [1] J.M. Maldacena, *The large- N limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113 [[hep-th/9711200](#)] [[INSPIRE](#)].
- [2] P. Bizon and A. Rostworowski, *On weakly turbulent instability of anti-de Sitter space*, *Phys. Rev. Lett.* **107** (2011) 031102 [[arXiv:1104.3702](#)] [[INSPIRE](#)].
- [3] V. Cardoso, O.J.C. Dias, J.P.S. Lemos and S. Yoshida, *The black hole bomb and superradiant instabilities*, *Phys. Rev. D* **70** (2004) 044039 [*Erratum ibid.* **70** (2004) 049903] [[hep-th/0404096](#)] [[INSPIRE](#)].
- [4] R. Brito, V. Cardoso and P. Pani, *Superradiance*, [arXiv:1501.06570](#) [[INSPIRE](#)].
- [5] H.K. Kunduri, J. Lucietti and H.S. Reall, *Gravitational perturbations of higher dimensional rotating black holes: Tensor perturbations*, *Phys. Rev. D* **74** (2006) 084021 [[hep-th/0606076](#)] [[INSPIRE](#)].
- [6] H.S. Reall, *Higher dimensional black holes and supersymmetry*, *Phys. Rev. D* **68** (2003) 024024 [*Erratum ibid.* **70** (2004) 089902] [[hep-th/0211290](#)] [[INSPIRE](#)].
- [7] O.J.C. Dias, G.T. Horowitz and J.E. Santos, *Black holes with only one Killing field*, *JHEP* **07** (2011) 115 [[arXiv:1105.4167](#)] [[INSPIRE](#)].
- [8] S. Stotyn, M. Park, P. McGrath and R.B. Mann, *Black Holes and Boson Stars with One Killing Field in Arbitrary Odd Dimensions*, *Phys. Rev. D* **85** (2012) 044036 [[arXiv:1110.2223](#)] [[INSPIRE](#)].
- [9] S.W. Hawking and H.S. Reall, *Charged and rotating AdS black holes and their CFT duals*, *Phys. Rev. D* **61** (2000) 024014 [[hep-th/9908109](#)] [[INSPIRE](#)].
- [10] C.A.R. Herdeiro and E. Radu, *Kerr black holes with scalar hair*, *Phys. Rev. Lett.* **112** (2014) 221101 [[arXiv:1403.2757](#)] [[INSPIRE](#)].

- [11] S. Stotyn and R.B. Mann, *Another Mass Gap in the BTZ Geometry?*, *J. Phys. A* **45** (2012) 374025 [[arXiv:1203.0214](#)] [[INSPIRE](#)].
- [12] S. Stotyn, M. Chanona and R.B. Mann, *Numerical Boson Stars with a Single Killing Vector II: the $D = 3$ Case*, *Phys. Rev. D* **89** (2014) 044018 [[arXiv:1309.2911](#)] [[INSPIRE](#)].
- [13] M. Bañados, C. Teitelboim and J. Zanelli, *The black hole in three-dimensional space-time*, *Phys. Rev. Lett.* **69** (1992) 1849 [[hep-th/9204099](#)] [[INSPIRE](#)].
- [14] E. Witten, *Multitrace operators, boundary conditions and AdS/CFT correspondence*, [hep-th/0112258](#) [[INSPIRE](#)].
- [15] I. Ichinose and Y. Satoh, *Entropies of scalar fields on three-dimensional black holes*, *Nucl. Phys. B* **447** (1995) 340 [[hep-th/9412144](#)] [[INSPIRE](#)].
- [16] M. Abramowitz and I.A. Stegun eds., *Handbook of Mathematical Functions*, Dover, New York (1965).
- [17] P. Breitenlohner and D.Z. Freedman, *Stability in Gauged Extended Supergravity*, *Annals Phys.* **144** (1982) 249 [[INSPIRE](#)].
- [18] O.J.C. Dias, G.T. Horowitz, N. Iqbal and J.E. Santos, *Vortices in holographic superfluids and superconductors as conformal defects*, *JHEP* **04** (2014) 096 [[arXiv:1311.3673](#)] [[INSPIRE](#)].
- [19] D. Birmingham, *Choptuik scaling and quasinormal modes in the AdS/CFT correspondence*, *Phys. Rev. D* **64** (2001) 064024 [[hep-th/0101194](#)] [[INSPIRE](#)].
- [20] S. Stotyn and R.B. Mann, *Another Mass Gap in the BTZ Geometry?*, *J. Phys. A* **45** (2012) 374025 [[arXiv:1203.0214](#)] [[INSPIRE](#)].
- [21] V. Balasubramanian and P. Kraus, *A stress tensor for anti-de Sitter gravity*, *Commun. Math. Phys.* **208** (1999) 413 [[hep-th/9902121](#)] [[INSPIRE](#)].
- [22] S.W. Hawking and G.F.R. Ellis, *The large scale structure of space-time*, Cambridge University Press, (1973).
- [23] O.J.C. Dias, J.E. Santos and B. Way, private communication with their unpublished notes.
- [24] A. Ishibashi and R.M. Wald, *Dynamics in nonglobally hyperbolic static space-times. 3. Anti-de Sitter space-time*, *Class. Quant. Grav.* **21** (2004) 2981 [[hep-th/0402184](#)] [[INSPIRE](#)].
- [25] S.A. Hartnoll, C.P. Herzog and G.T. Horowitz, *Building a Holographic Superconductor*, *Phys. Rev. Lett.* **101** (2008) 031601 [[arXiv:0803.3295](#)] [[INSPIRE](#)].
- [26] N. Iizuka, A. Ishibashi and K. Maeda, *Persistent Superconductor Currents in Holographic Lattices*, *Phys. Rev. Lett.* **113** (2014) 011601 [[arXiv:1312.6124](#)] [[INSPIRE](#)].
- [27] P. Figueras and T. Wiseman, *Stationary holographic plasma quenches and numerical methods for non-Killing horizons*, *Phys. Rev. Lett.* **110** (2013) 171602 [[arXiv:1212.4498](#)] [[INSPIRE](#)].
- [28] S. Fischetti, D. Marolf and J.E. Santos, *AdS flowing black funnels: Stationary AdS black holes with non-Killing horizons and heat transport in the dual CFT*, *Class. Quant. Grav.* **30** (2013) 075001 [[arXiv:1212.4820](#)] [[INSPIRE](#)].
- [29] M. Maliborski and A. Rostworowski, *Time-Periodic Solutions in an Einstein AdS-Massless-Scalar-Field System*, *Phys. Rev. Lett.* **111** (2013) 051102 [[arXiv:1303.3186](#)] [[INSPIRE](#)].