

Three-family particle physics models from global F-theory compactifications

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ABSTRACT: We construct four-dimensional, globally consistent F-theory models with three chiral generations, whose gauge group and matter representations coincide with those of the Minimal Supersymmetric Standard Model, the Pati-Salam Model and the Trinification Model. These models result from compactification on toric hypersurface fibrations X with the choice of base \mathbb{P}^3 . We observe that the F-theory conditions on the G_4 -flux restrict the number of families to be at least three. We comment on the phenomenology of the models, and for Pati-Salam and Trinification models discuss the Higgsing to the Standard Model. A central point of this work is the construction of globally consistent G_4 -flux. For this purpose we compute the vertical cohomology $H_V^{(2,2)}(X)$ in each case and solve the conditions imposed by matching the M- and F-theoretical 3D Chern-Simons terms. We explicitly check that the expressions found for the G_4 -flux allow for a cancellation of D3-brane tadpoles. We also use the integrality of 3D Chern-Simons terms to ensure that our G_4 -flux solutions are adequately quantized.

KEYWORDS: Beyond Standard Model, F-Theory, Supersymmetric Standard Model, GUT

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Contents

1	Introduction	1
2	Tools & strategies for four-dimensional model building	4
3	Minimal supersymmetric standard model: $G_{F_{11}} = \text{SU}(3) \times \text{SU}(2) \times \text{U}(1)$	9
3.1	The geometry of gauge symmetry and particle representations	9
3.2	G_4 -flux and matter chiralities	11
3.3	4D anomaly cancelation and family structure	14
3.4	Phenomenological discussion	16
4	Pati-Salam model: $G_{F_{13}} = (\text{SU}(4) \times \text{SU}(2)^2)/\mathbb{Z}_2$	18
4.1	The geometry of gauge group and matter representations	18
4.2	G_4 -flux and chiral generations	19
4.3	Phenomenological discussion	22
5	Trinification model: $G_{F_{16}} = (\text{SU}(3)^3)/\mathbb{Z}_3$	26
5.1	The geometry of gauge group and matter representations	26
5.2	G_4 -flux and chiralities	28
5.3	Phenomenological discussion	30
6	Conclusions	33
A	Hodge numbers of CY-fourfolds	35
B	Concrete toric lattice polyhedra of 4D chiral models	37

1 Introduction

The construction of fully fledged particle physics models which reproduce the phenomenology of the Standard Model, while providing generic predictions for its behavior at higher scales remains a very active part of the research in string theory. One active front in this direction is F-theory [1], where the geometrization of certain properties of non-perturbative Type IIB string theory allows for a very systematic understanding and engineering of the gauge symmetry, particle content and the interactions in a given model.

Most model building endeavors in F-theory have appealed to an underlying $\text{SU}(5)$ gauge group (for a non exhaustive list of works see [2–16]). This is partly due to the simple group theoretical embedding of the standard model gauge group and its representations into $\text{SU}(5)$, with its well earned merit for gauge coupling unification.¹ In addition this gives

¹For a detailed discussion on gauge coupling unification in $\text{SU}(5)$ GUTs see e.g. [17, 18].

the advantage of having the full gauge symmetry concentrated on a single divisor in the base that allows for a local treatment of certain features of the model [2–5]. Nevertheless, the increasing understanding of many global issues has prompted interest in alternative models which aim either at the direct construction of the bare MSSM² [21–23], as well as alternative grand unification schemes such as the Pati-Salam model or Trinification, among others. These models have the advantage that they do not suffer from the pathological group theoretical puzzles inherent to SU(5), such as the doublet triplet splitting problem. These schemes constitute also a very promising alternative in other corners of the string landscape such as perturbative Type IIA/B and the heterotic string (see e.g. [24–26]).

In the model building program one aims at reproducing certain features of the particle physics models such as appropriate gauge symmetries, particle representations, the right number of generations and, at least, the possibility to generate a hierarchy in the Yukawa textures (for reviews on this topic in an F-theory context see e.g. [27, 28]). As for the gauge symmetries, the appearance of non-Abelian factors has been understood since the beginning of F-theory and can be tracked by the degeneration of the elliptic fiber over codimension one surfaces on the base [29–31], see [32] for recent refinements. Abelian gauge symmetries are due to the presence of sections of the elliptic fibration, in addition to the so-called zero section of the Weierstrass model [30, 33, 34]. Being related to global objects, as well as being essential tools for controlling the phenomenology of particle physics models, U(1) symmetries have pushed the F-theory model building program towards a more global picture. Similarly, the charged matter representations can be tracked by degenerations of the elliptic curve at codimension two in the base [35], see also [36] for the discussion of higher symmetric representation. In order to achieve a chiral spectrum, the addition of G_4 -flux is necessary, see [37, 38] for first global examples. The intersection of matter curves at codimension three leads to the geometrically allowed couplings of the model. In F-theory the hierarchy for the Yukawas is possible since generically, the Yukawas for one family are generated geometrically while for the other two of these couplings arise from instanton or flux contributions that are significantly small.

The compactification spaces which are commonly used for constructing 4D $\mathcal{N} = 1$ effective theories in F-theory are genus one fibered Calabi-Yau (CY) fourfolds. The global model building process is divided into two steps, the first step being the construction of appropriate compactification manifolds exhibiting the desired fiber degenerations which lead to the appropriate gauge symmetry, matter and interactions. The second step is concerned with the construction of appropriate G_4 -flux to account for the desired chirality in the spectrum.

Regarding the construction of suitable CY-fourfolds, the efforts have been divided in two fronts: the first pushes towards the classification and construction of all admissible bases [39–41], see also [42–45] for the related study of non-compact bases. The second is more concerned with the construction of genus-one or elliptic fibers \mathcal{C} which naturally allow for certain generic features of the compactification. In this direction, there have been two

²See [19, 20] for earlier attempts to get the standard model gauge group from a deformation of the SU(5) singularity.

major conceptual extensions: one is related to the construction of elliptic curves exhibiting an ever growing number of rational points [13–15, 46–62]. These permit the construction of elliptically fibered CY-manifolds with a certain number of rational sections and hence, a non trivial Mordell-Weil (MW) group of rational sections. While the free part of the MW-group yields the U(1)-gauge fields in F-theory [29], the torsion part is responsible for the presence of non-simply connected gauge groups [33] and its effects are seen at codimension two as it forbids certain representations to be part of the theory. The other conceptual extensions has to do with fibers which entirely lack rational points. These lead to genus-one fibrations which do not have any section [63–65]. Nevertheless, this type of fibrations is suitable for F-theory as their associated Jacobian fibration does have a section and describes the same physics. In the genus-one fibrations, the presence of an m -sections has been unveiled as the geometric object responsible for the presence of discrete gauge symmetries in their associated effective SUGRA theory [22, 66–69].

There is a natural framework which provides the simplest examples for the two kinds of fibers described above: the 16 inequivalent 2D toric varieties [70, 71]. By describing the genus-one fiber as an algebraic curve in any of these toric ambient spaces one ends up with any of the possible cases of elliptic fibers with up to four independent rational points as well as genus-one curves with two- and three-sections. In [22], a systematic analysis of the effective six dimensional theories stemming from compactification of F-theory on any of these 16 toric hypersurface fibrations over an arbitrary complex-two dimensional base was done. Abandoning the paradigm of the holomorphic zero section and allowing it to be simply rational, all resolution divisors inherited from the corresponding ambient space descend to Cartan divisors on the corresponding CY-manifold. In this fashion, it was possible to deduce the intrinsic gauge symmetry of each toric hypersurface fibration, without the necessity to introduce further specializations of the geometry such as tops [72–74], which appear in algorithmic approaches to F-theory model building [58]. Among these intrinsic gauge symmetries, groups which are familiar for particle physicists such as $SU(3) \times SU(2) \times U(1)$, $SU(4) \times SU(2)^2/\mathbb{Z}_2$ and $SU(3)^3/\mathbb{Z}_3$ appear naturally. Even more interestingly, the matter representations that are possible in each of these models coincide with those needed for the MSSM, Pati-Salam model and Trinification, respectively. While the analysis of [22] was made in six dimensions, these findings carry over to 4D as well. Another important feature of the six-dimensional models is the existence of a so called “toric Higgsing” (or a chain of those) which allows for a pictorial, toric interpretation of Higgsings from the Pati-Salam and Trinification models down to the MSSM.

In this work we study the four dimensional-version of the MSSM, the Pati-Salam model and Trinification engineered by F-theory on toric hypersurface fibrations X . For the first time we construct explicit, globally consistent, three-family models with the chiral matter content of the MSSM, Pati-Salam model and Trinification. These models result from three different toric hypersurface fibrations over a fixed base $B = \mathbb{P}^3$. We construct the vertical cohomology $H_V^{(2,2)}(X)$ and provide the most general expression for the G_4 -flux in each case. These expressions are shown to comply with all conditions on M-theory Chern-Simons terms imposed by duality with F-theory and allow for a D3-brane tadpole canceling solution which involves an integral and positive number of D3-branes. Since

the determination of an integral basis for the vertical cohomology $H_V^{(2,2)}(X)$, G_4 -flux quantization is checked indirectly by ensuring quantization of the induced 3D Chern-Simons terms. Regarding the phenomenology of the considered models we have two main drawbacks: since we have no control over the vector-like sector of the theory, it is not guaranteed that we have the vector-like pairs needed for the breaking of electroweak symmetry neither for the breaking of the Pati-Salam or Trinification groups down to the Standard Model gauge group. In addition, we have only stated the trilinear couplings which are generated geometrically and have argued that under the assumption of the presence of a light pair of Higgses, these would allow for the generation of a hierarchy. However, the currently applicable tools do not allow us to provide to perform a quantitative analysis.

This paper is organized as follows: in section 2 we summarize the general procedure to construct 4D chiral models from F-theory and shortly review the tools needed for our analysis: the general features of elliptic fibrations, G_4 -flux and G_4 -flux consistency conditions in F-theory. In section 3 we discuss the elliptic fibration leading to the gauge group and matter content of the MSSM. There, we also compute G_4 -flux and the resulting 4D matter chiralities. Similar as in the standard model, we see that the exact cancelation of anomalies (without Green Schwarz counterterms for anomalies involving $U(1)_Y$) enforces a family structure. In this regime we scan over all allowed strata in the moduli space of the CY-manifold $X_{F_{11}}$ with base \mathbb{P}^3 and compute the smallest number of families for which the D3 brane tadpole is canceled with a positive integral number of D3-branes. We observe that three is in fact the smallest permitted number of families in this model. The same observation holds also for Pati-Salam and Trinification models. In addition we give closed formulas for the Hodge structure of all fibrations for the choice of base $B = \mathbb{P}^3$ in appendix A. We generically find $h^{(2,1)} = 0$ which restricts cosmological applications. We conclude this section with a brief discussion of the phenomenology of the model under the assumption that a light pair of Higgs fields is present. In sections 4 and 5 we present a similar discussion of the Pati-Salam and Trinification models as in section 3. In addition, we comment on the Higgsing down to the Standard Model gauge group. We indeed find that there exist three-family models both for Pati-Salam and Trinification, that Higgs down to three-family Standard models. Finally in section 6 we present our conclusions and discuss future possible directions of research. Appendix A contains a brief account on the computation of Hodge numbers of CY-fourfolds given as toric hypersurfaces and applications to the considered cases $X_{F_{11}}$, $X_{F_{13}}$ and $X_{F_{16}}$ with base $B = \mathbb{P}^3$. Appendix B contains the explicit lattice polytopes for the five-dimensional toric ambient spaces of all considered toric hypersurface fibrations.

2 Tools & strategies for four-dimensional model building

In this section, which serves as a preparation for the analysis in sections 3, 4 and 5, we outline the basic techniques for building F-theoretic models of particle physics. Although many of the presented methods are applicable to general Calabi-Yau (CY) fourfolds X , we focus here on the case of the toric hypersurface fibrations as studied in [22]. Except for the discussion of G_4 -flux quantization, this section is mainly a concise review on the

construction of chiral 4D F-theory models, following closely [15] to which we refer for further details. In an accompanying appendix A we discuss the computation of Hodge numbers of CY-fourfolds X given as toric hypersurfaces. The reader interested in the phenomenological results can safely skip this section and continue with section 3.

F-theory geometry: on the geometry side, the starting point for the construction of an F-theory model is the choice of a three-dimensional base manifold B as well as the genus-one or elliptic fiber of the Calabi-Yau fourfold X . As a next step all codimension one, two and three singularities of the fibration have to be analyzed in order to determine the gauge group G , matter content, i.e. the representations and their matter curves in X , and the Yukawa couplings of the 4D effective theory of F-theory.

In order to being able to construct G_4 -flux we have to compute the cohomology ring of X . For a CY-fourfold given as a toric hypersurface (or complete intersection) fibration, which is the case of interest of this work, the full cohomology ring of X as a quotient polynomial ring generated by $H^{1,1}(X)$ [15, 75–77]. Concretely, for a CY-fourfold X with a given toric base B , we choose a basis D_A , $A = 0, \dots, h^{(1,1)}(X) - 1$, for the divisor group $H^{1,1}(X)$. Then the cohomology ring of X is given by the polynomial ring in the D_A divided by the Stanley-Reisner (SR) ideal of the ambient toric variety³ of X . The reduction by the Stanley-Reisner ideal works in our case as we have no non-toric divisors. There can be non-toric divisors at corners of the allowed region for \mathcal{S}_7 and \mathcal{S}_9 for the base \mathbb{P}^3 , cf. the discussion in [15]. As our focus is on chirality inducing G_4 -flux, we are primarily interested in the subgroup $H_V^{2,2}(X)$ of the fourth cohomology of X , the primary vertical cohomology [78]. It is given by the quotient ring at grade two, that is constructed by forming all possible products $D_A \cdot D_B$. These are linearly dependent. Thus, we compute the rank of the inner product matrix on these elements, which yields the dimension $h_V^{(2,2)}(X)$, and choose an appropriate basis. The topological metric on this basis is denoted by $\eta^{(2)}$.

We emphasize that the full Chern class of the CY-manifold X can be computed independently of the base B by using the adjunction formula and the total Chern class of the ambient space, see [15] for more details. Of particular interest for F-theory are the second Chern class $c_2(X)$ and the Euler number $\chi(X)$ of X .

Constraints on G_4 -flux in F-theory: F-theory on $X \times S^1$ and M-theory on X are dual to each other [79]. Thus, consistent G_4 -flux in a four-dimensional F-theory compactification on X is understood as G_4 -flux in the dual M-theory compactification on the same X to three dimensions, so that the G_4 -flux obeys additional consistency conditions. These consistency conditions follow from requiring that the three-dimensional effective actions of F- and M-theory agree, which can be used to derive the full effective action of F-theory [80].

In M-theory, G_4 -flux has to fulfill two basic conditions. First, it must obey the following quantization condition [81]:

$$G_4 + \frac{c_2(X)}{2} \in H^4(X, \mathbb{Z}) . \tag{2.1}$$

³The reduction by the Stanley-Reisner ideal works in our case as we have no non-toric divisors. There can be non-toric divisors at corners of the allowed region for \mathcal{S}_7 and \mathcal{S}_9 for the base \mathbb{P}^3 , cf. the discussion in [15].

Second, the cancelation of M2-brane tadpoles, which lift to D3-brane tadpoles in Type IIB strings and F-theory, requires the equality [82, 83]

$$\frac{\chi(X)}{24} = n_{\text{D3}} + \frac{1}{2} \int_X G_4 \wedge G_4, \tag{2.2}$$

where n_{D3} denotes the number of D3-branes. As mentioned before, we will focus here on special G_4 -flux that is entirely in the subgroup $H_V^{(2,2)}(X)$.⁴

For compatibility with the duality between M- and F-theory, we need to impose additional conditions on the G_4 -flux. These are most easily formulated in terms of conditions on the Chern-Simons (CS) terms for the three-dimensional vectors on the Coulomb branch of the effective action of the M-theory compactification on the CY-fourfold X . On the M-theory side, these CS-terms are given by [87]

$$\Theta_{AB}^M = \int_X G_4 \wedge D_A \wedge D_B, \tag{2.3}$$

where here and in the following, Poincaré duality is always understood. We note that the 3D CS-terms have to obey the quantization condition $\Theta_{AB}^M \in \mathbb{Z}$ or $\mathbb{Z}/2$, see e.g. [88, 89] for recent discussions. We note that these quantization conditions are expected to be equivalent to the G_4 -flux quantization conditions (2.1) [85].

In the dual F-theory side the same CS-terms, denoted now by Θ_{AB}^F , have two contributions. First, we can have classical CS-terms $\Theta_{\text{cl}, AB}^F$, which either descend from 4D to 3D from gaugings of axions or which correspond to circle fluxes [90]. Second, CS-terms on the 3D Coulomb branch receive one-loop corrections from integrating out massive fermions [91–93]. In the duality between M- and F-theory, it is crucial to include all Kaluza-Klein (KK) states in the loop [15, 54],⁵ yielding the full loop corrected CS-terms expression

$$\Theta_{AB}^F = \Theta_{\text{cl}, AB}^F + \frac{1}{2} \sum_{\underline{q}} n(\underline{q}) q_A q_B \text{sign}(q_A \zeta^A). \tag{2.4}$$

Here $n(\underline{q})$ is the number of 3D fermions with charge vector $\underline{q} = (q_0, q_\alpha, q_i, q_m)$. It includes the charge q_0 w.r.t. the 3D graviphoton, i.e. the KK-level of states, the charges q_α , $\alpha = 1, \dots, h^{(1,1)}(B)$, under 3D vectors dual to the Kähler moduli of B , the charges q_i , $i = 1, \dots, \text{rk}(G)$, and q_m , $m = 1, \dots, r$, w.r.t. to 4D Cartan gauge fields of the non-Abelian gauge group G of F-theory and the r U(1) gauge fields, respectively. The real parameters ζ^A are the Coulomb branch parameters.

Duality requires an identification of the CS-terms on the F-theory side with those in (2.3) on the M-theory side [14, 15, 51, 54, 94, 95],

$$\Theta_{AB} \equiv \Theta_{AB}^M \stackrel{!}{=} \Theta_{AB}^F. \tag{2.5}$$

This immediately leads to additional restrictions on the CS-terms in F-theory [15, 37, 51, 54], because certain CS-terms Θ_{AB}^F in F-theory computed according to (2.4) are identically

⁴For recent analysis of horizontal G_4 -flux in F-theory, see [77, 84–86].

⁵See also [57] for the case of CS-terms in 5D M-/F-theory duality.

zero. Physically, the implied constraints on the G_4 -flux ensure the absence of circle flux in the circle compactification from F- to M-theory, an unbroken non-Abelian gauge group in 4D due to the absence of axion gaugings and the absence of non-geometric effects,

$$\Theta_{0\alpha} = \Theta_{i\alpha} = \Theta_{\alpha\beta} = 0 . \tag{2.6}$$

Here we have to choose the basis D_A of $H^{(1,1)}(X)$ so that index 0 corresponds to the zero section \hat{s}_0 of the fibration of X , $\alpha = 1, \dots, h^{(1,1)}(B)$, labels the vertical divisors induced from the base B , $i = 1, \dots, \text{rk}(G)$ labels the Cartan divisors of X , where G as before is non-Abelian part of the F-theory gauge group, and $m = 1, \dots, r$ labels the r $U(1)$ -factors corresponding to Shioda maps $\sigma(\hat{s}_m)$ of the rank r Mordell-Weil (MW) group of rational sections \hat{s}_m of X .

Chiralities in F-theory and G_4 -flux quantization: in order to calculate the matter chiralities $\chi(\mathbf{R})$ for a given matter representation \mathbf{R} in a four-dimensional F-theory compactification, we need to integrate the G_4 -flux over a corresponding matter surface in X . The relevant matter surface $\mathcal{C}_{\mathbf{R}}^w$ is given as the rational surface constructed by fibering a \mathbb{P}^1 carrying the weight \mathbf{w} of the representation \mathbf{R} over the corresponding matter curve in the base B . The 4D chirality of \mathbf{R} is computed as

$$\chi(\mathbf{R}) = n(\mathbf{R}) - n(\bar{\mathbf{R}}) = \int_{\mathcal{C}_{\mathbf{R}}^w} G_4 , \tag{2.7}$$

where $n(\mathbf{R})$ denotes the number of left-chiral Weyl fermions in the representation \mathbf{R} .

Technically, the determination of the $\mathcal{C}_{\mathbf{R}}^w$ can be involved and requires the computation of the homology class of prime ideals describing the given matter surface. This can be done using the resultant technique that was applied first in [15, 56] for F-theory and will be exemplify for the three examples studied in this work. As a consistency check of our geometric computations, following [15, 51, 54], we use the matching condition (2.5) of the CS-terms to double-check the 4D chiralities calculated using (2.7).

Finally, let us comment on G_4 -flux quantization. In principal, in order to address G_4 -flux quantization we have to expand G_4 and $c_2(X)$ in an integral basis for $H_V^{(2,2)}(X)$ and check the condition (2.1). This integral basis can be determined employing mirror symmetry techniques [77, 84, 86]. Since this is beyond the scope of this work, we will apply an indirect approach to ensure integral G_4 -flux.

Here we exploit that G_4 -flux quantization (2.1), the integrality of the number n_{D3} of D3-branes, that is a necessary condition for quantized G_4 -flux [81], the integrality of the CS-terms (2.3) and of the chiralities (2.7) are obviously linked to each other. Thus, our strategy will be the following. First, we compute all chiralities $\chi(\mathbf{R})$ using (2.7). Then, we parametrize the coefficients in the expansion of the G_4 -flux w.r.t. a basis of $H_V^{(2,2)}(X)$ in terms of these integral chiralities. We then impose the necessary condition of integrality and positivity of n_{D3} . This will yield in turn constraints in form of lower bounds on the 4D chiralities. Next, we impose, if possible, a family structure on our model. Finally, we check that for this phenomenologically preferred choice of G_4 -flux all CS-terms are integral, which ensures that the quantization condition (2.1) is obeyed. We remark that there might exist

a finer basis non-diagonal in vertical and horizontal cohomology for which more integral n_{D3} chiral family solutions exist. In contrast we only focus on the vertical part, that might be a multiple of this minimal component, such that it is integral on its own.

Toric hypersurface fibrations for 4D chiral F-theory models: in order to introduce some notation used throughout this work, we conclude this introductory section with a very brief review of CY-fourfolds X constructed as toric hypersurface fibrations. A detailed account on this subject can be found in [22].

We consider here elliptically fibered Calabi-Yau manifolds X_{F_i} whose elliptic fiber is realized as the general CY-hypersurface in a 2D toric variety \mathbb{P}_{F_i} associated to one of the 2D reflexive polyhedra F_i . Here we focus on the polyhedra F_{11} , F_{13} and F_{16} in [22], that naturally yield phenomenologically interesting models. In these cases, the corresponding toric ambient varieties \mathbb{P}_{F_i} of the elliptic fiber are blow-ups of \mathbb{P}^2 . The elliptic curves in all considered cases are consequently given as an appropriate specialization of the general cubic

$$p = s_1 u^3 + s_2 u^2 v + s_3 u v^2 + s_4 v^3 + s_5 u^2 w + s_6 u v w + s_7 v^2 w + s_8 u w^2 + s_9 v w^2 + s_{10} w^3. \quad (2.8)$$

Here the coefficients s_i take values in a field K and $[u : v : w]$ are projective coordinates on \mathbb{P}^2 .

An elliptic fibration X_{F_i} with fiber given by (2.8) or specializations thereof is constructed by first fibering the toric ambient space \mathbb{P}_{F_i} over a chosen base B , then imposing (2.8) and finally demanding the CY-condition. In this procedure, the coordinates $[u : v : w]$ and the coefficients s_i in (2.8) are lifted to sections of appropriate line bundles on B . The CY-condition fixes these line bundles to the following:

section	Line Bundle
s_1	$\mathcal{O}_B(3[K_B^{-1}] - \mathcal{S}_7 - \mathcal{S}_9)$
s_2	$\mathcal{O}_B(2[K_B^{-1}] - \mathcal{S}_9)$
s_3	$\mathcal{O}_B([K_B^{-1}] + \mathcal{S}_7 - \mathcal{S}_9)$
s_4	$\mathcal{O}_B(2\mathcal{S}_7 - \mathcal{S}_9)$
s_5	$\mathcal{O}_B(2[K_B^{-1}] - \mathcal{S}_7)$
s_6	K_B^{-1}
s_7	$\mathcal{O}_B(\mathcal{S}_7)$
s_8	$\mathcal{O}_B([K_B^{-1}] + \mathcal{S}_9 - \mathcal{S}_7)$
s_9	$\mathcal{O}_B(\mathcal{S}_9)$
s_{10}	$\mathcal{O}_B(2\mathcal{S}_9 - \mathcal{S}_7)$

section	Line Bundle
u	$\mathcal{O}(H + \mathcal{S}_9 + [K_B])$
v	$\mathcal{O}(H + \mathcal{S}_9 - \mathcal{S}_7)$
w	$\mathcal{O}(H)$

}
(2.9)

Here, $\mathcal{O}(D)$ denotes the line bundle associated to a divisor D ,⁶ H is the hyperplane on \mathbb{P}^2 , $[K_B^{-1}]$ is the anti-canonical divisor of B and \mathcal{S}_7 , \mathcal{S}_9 are the divisor classes of s_7 , s_9 , respectively. We note that the table on the right hand side in (2.9) applies for all examples studied below.

⁶A subscript indicates the space over which this line bundle is defined, e.g. $\mathcal{O}_B(D)$ denotes a line bundle over B . If a subscript is omitted, the line bundle lives on the ambient space of X .

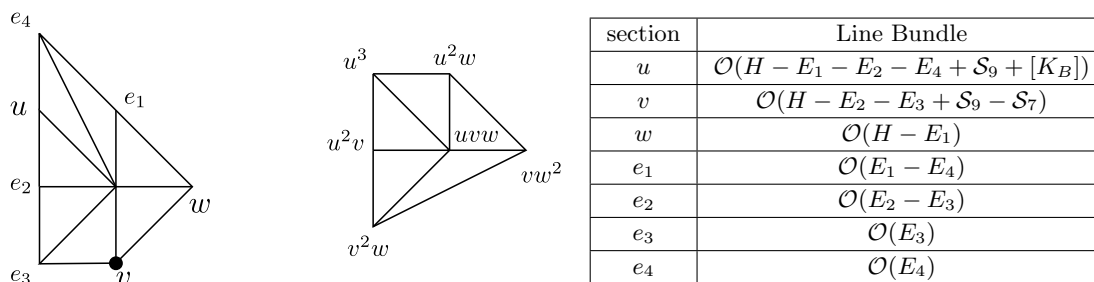


Figure 1. The toric diagram of polyhedron F_{11} and its dual. The zero section is indicated by the dot. In the accompanying table we indicate the divisor classes of the fiber coordinates.

3 Minimal supersymmetric standard model: $G_{F_{11}} = \text{SU}(3) \times \text{SU}(2) \times \text{U}(1)$

In this section we discuss an F-theory compactification on the elliptically fibered CY-manifold $X_{F_{11}}$ which yields precisely the gauge group and representation content of the Minimal Supersymmetric Standard Model (MSSM) [22].

In section 3.1 we elaborate on the basic geometrical properties of $X_{F_{11}}$ that encode the gauge symmetry, including the U(1) generator, as well as the matter representations. While these observations are model independent, we further specialize to the simple base $B = \mathbb{P}^3$. For this specific case we compute the vertical cohomology $H_V^{(2,2)}(X_{F_{11}})$ in section 3.2. Using these results, we explicitly construct G_4 -flux consistent with all F-theory consistency constraints. We compute the induced 4D chiralities of the matter representations, that we double-check employing 3D CS-terms and M-/F-theory duality. Next in section 3.3 we discuss 4D anomaly cancelation and the properties of models which exhibit a complete family structure, in particular the existence of three family models with positive and integral D3-brane charge and quantized G_4 -flux. In section 3.4 we conclude with some comments on the phenomenology of the three family models we found.

The elliptic fibration $X_{F_{11}}$ has been completely analyzed in [22], to which we refer for more details on its codimension one, two and three singularities and the corresponding 6D F-theory compactification. The relevant results are summarized in section 3.1. The reader less interested in the technical details can directly jump to the 4D chiralities in (3.17) and the following discussions.

3.1 The geometry of gauge symmetry and particle representations

The elliptic fiber which is used to engineer F-theory models that naturally exhibit the gauge symmetry of the standard model is given as the CY-hypersurface

$$\begin{aligned}
 p_{F_{11}} = & s_1 e_1^2 e_2^2 e_3 e_4^4 u^3 + s_2 e_1 e_2^2 e_3^2 e_4^2 u^2 v + s_3 e_2^2 e_3^2 u v^2 + s_5 e_1^2 e_2 e_4^3 u^2 w \\
 & + s_6 e_1 e_2 e_3 e_4 u v w + s_9 e_1 v w^2
 \end{aligned} \tag{3.1}$$

in the toric ambient space $\mathbb{P}_{F_{11}}$. Its toric data is summarized in figure 1. The divisor classes in $\mathbb{P}_{F_{11}}$ are H , the hyperplane class of \mathbb{P}^2 , as well as the four exceptional divisors E_1 , E_2 , E_3 and E_4 .

Representation	Locus
$(\mathbf{3}, \mathbf{2})_{1/6}$	$V(I_{(1)}) := \{s_3 = s_9 = 0\}$
$(\mathbf{1}, \mathbf{2})_{-1/2}$	$V(I_{(2)}) := \{s_3 = s_2 s_5^2 + s_1(s_1 s_9 - s_5 s_6) = 0\}$
$(\bar{\mathbf{3}}, \mathbf{1})_{-2/3}$	$V(I_{(3)}) := \{s_5 = s_9 = 0\}$
$(\bar{\mathbf{3}}, \mathbf{1})_{1/3}$	$V(I_{(4)}) := \{s_9 = s_3 s_5^2 + s_6(s_1 s_6 - s_2 s_5) = 0\}$
$(\mathbf{1}, \mathbf{1})_1$	$V(I_{(5)}) := \{s_1 = s_5 = 0\}$

Table 1. Charged matter representations under $SU(3) \times SU(2) \times U(1)$ and corresponding codimension two loci in $X_{F_{11}}$. The charge under the $U(1)_Y$ generator is indicated by a subscript.

Next, an elliptically fibered CY-fourfold $X_{F_{11}}$ with the elliptic fiber (3.1) is constructed by promoting the coefficients s_i in the CY-equation to sections of the line bundles of B given in (2.9). The elliptic fibration of $X_{F_{11}}$ is equipped with two independent rational sections

$$\begin{aligned} \hat{s}_0 &= X_{F_{11}} \cap \{v = 0\} : [1 : 0 : s_1 : 1 : 1 : -s_5 : 1], \\ \hat{s}_1 &= X_{F_{11}} \cap \{e_4 = 0\} : [s_9 : 1 : 1 : -s_3 : 1 : 1 : 0], \end{aligned} \quad (3.2)$$

where we have chosen \hat{s}_0 as the zero section. By computation of the discriminant of the fibration (3.1), one can check that over the loci $\mathcal{S}_{SU(2)} = \{s_3 = 0\}$ and $\mathcal{S}_{SU(3)} = \{s_9 = 0\}$ the fiber degenerates to I_2 - and I_3 -fibers giving rise to $SU(2)$ and $SU(3)$ gauge symmetries, respectively. The Cartan divisors of these gauge groups are

$$D_1^{SU(2)} = [e_1], \quad D_1^{SU(3)} = [e_2] \quad D_2^{SU(3)} = [u]. \quad (3.3)$$

Having these divisors at hand, one can show that the generator of the $U(1)$ symmetry, that is the Shioda map of \hat{s}_1 , is given by

$$\sigma(\hat{s}_1) = S_1 - \tilde{S}_0 + [K_B] + \frac{1}{2}D_1^{SU(2)} + \frac{1}{3}\left(D_1^{SU(3)} + 2D_2^{SU(3)}\right). \quad (3.4)$$

Here, S_1 denotes the class of \hat{s}_1 and we used $\tilde{S}_0 = S_0 + \frac{1}{2}[K_B^{-1}]$ [96], where S_0 is the class of \hat{s}_0 and K_B^{-1} denotes the anti-canonical bundle of the base B . The corresponding Néron-Tate height pairing reads

$$b_{11} = \frac{3}{2}[K_B^{-1}] - \frac{1}{2}\mathcal{S}_7 - \frac{1}{6}\mathcal{S}_9. \quad (3.5)$$

Furthermore, there are codimension two singularities in the elliptic fibration $X_{F_{11}}$ which support all matter representations of the Standard Model⁷ as one can see in table 1.

We note that the second Chern class c_2 and the Euler number of $X_{F_{11}}$ can be computed base independently [61]. They are needed to check the G_4 -flux quantization condition (2.1)

⁷At this stage, over the different codimension two loci, matter comes in vector-like pairs. It is the G_4 -flux that induces chiralities for the fields.

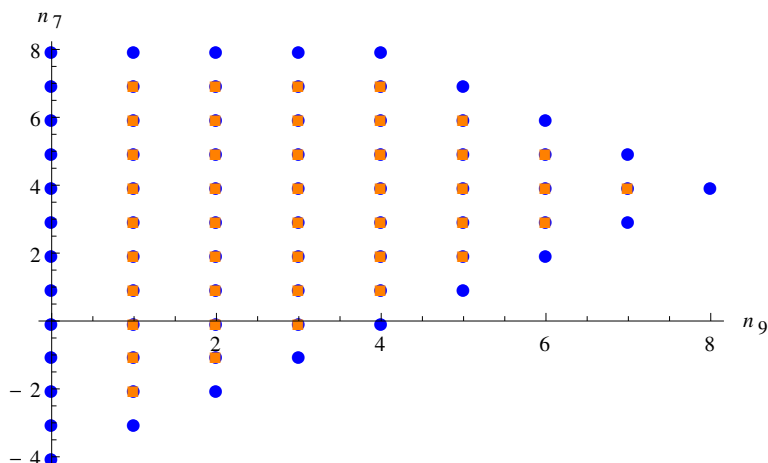


Figure 2. Allowed region for (n_7, n_9) for the CY-fourfold $X_{F_{11}}$ with base \mathbb{P}^3 . Orange dots indicate that all SM representations are present and a G_4 -flux admitting b families exists.

as well as the cancelation of D3 tadpoles (2.2). We obtain

$$\begin{aligned}
 c_2(X_{F_{11}}) = & -c_1^2 + c_2 + c_1 E_2 - c_1 E_3 - 7c_1 E_4 - 7E_4^2 + 4c_1 H + 2c_1 \mathcal{S}_7 + 4E_1 \mathcal{S}_7 \\
 & + E_2 \mathcal{S}_7 + E_3 \mathcal{S}_7 + 6E_4 \mathcal{S}_7 - 4H \mathcal{S}_7 - c_1 \mathcal{S}_9 - 5E_1 \mathcal{S}_9 - 3E_2 \mathcal{S}_9 \\
 & - E_3 \mathcal{S}_9 + 3H \mathcal{S}_9 - 3\mathcal{S}_7 \mathcal{S}_9 + 3\mathcal{S}_9^2, \tag{3.6}
 \end{aligned}$$

$$\begin{aligned}
 \chi(X_{F_{11}}) = & 3(24c_1^3 + 4c_1 c_2 - 16c_1^2 \mathcal{S}_7 + 8c_1 \mathcal{S}_7^2 - 18c_1^2 \mathcal{S}_9 \\
 & + 3c_1 \mathcal{S}_7 \mathcal{S}_9 - 3\mathcal{S}_7^2 \mathcal{S}_9 + 6c_1 \mathcal{S}_9^2 + \mathcal{S}_7 \mathcal{S}_9^2), \tag{3.7}
 \end{aligned}$$

where c_1 and c_2 denote the first and second Chern class of the base B , respectively. The divisors \mathcal{S}_7 and \mathcal{S}_9 are introduced in (2.9).

For the remainder of this section, we chose the base of the fibration to be $B = \mathbb{P}^3$. For this simple choice of base the only vertical divisor is the pullback of the hyperplane class \mathbb{P}^3 , which we denote by H_B . In this case, we have $c_1 = 4[H_B]$ and $c_2 = 6[H_B^2]$ in (3.7). In addition, we readily expand the divisors $\mathcal{S}_7, \mathcal{S}_9$ in (2.9) needed to specify the fibration $X_{F_{11}}$ in terms of H_B ,

$$\mathcal{S}_7 = n_7 H_B, \quad \mathcal{S}_9 = n_9 H_B, \quad [K_B^{-1}] = 4H_B, \tag{3.8}$$

where n_7 and n_9 denote integers. These integers are constrained by requiring effectiveness of all divisor classes in (2.9), that enter the CY-constraint (3.1). This determines a region of allowed values for the pair (n_7, n_9) , to which we refer to as the allowed region, as depicted in figure 2.

3.2 G_4 -flux and matter chiralities

For the specific base $B = \mathbb{P}^3$, the full SR-ideal of the toric ambient space of $X_{F_{11}}$ is given by⁸

$$SR_{F_{11}} = \{ue_1, uw, uv, ue_3, e_4w, e_4v, e_4e_3, e_4e_2, e_1v, e_1e_3, e_1e_2, we_3, we_2, ve_2, x_0x_1x_2x_3\} \tag{3.9}$$

⁸The SR-ideal of the fiber alone can be found in [22].

where $[u, v, w]$ and the e_i , $i = 1, \dots, 4$, are the projective coordinates on the fiber and x_j ($j = 0, 1, 2, 3$) are the homogeneous coordinates on the \mathbb{P}^3 base. A basis for $H^{(1,1)}(X_{F_{11}})$ is given by

$$H^{(1,1)}(X_{F_{11}}) = \langle H_B, \tilde{S}_0, D_1^{\text{SU}(2)}, D_1^{\text{SU}(3)}, D_2^{\text{SU}(3)}, \sigma(\hat{s}_1) \rangle, \quad (3.10)$$

where we denote, by abuse of notation, divisors and their Poincare dual $(1, 1)$ -forms by the same symbol.

Next we proceed with the computation of the full vertical cohomology ring of $X_{F_{11}}$ following [15]. To set up its computation as a quotient ring, we need the SR ideal (3.9), the basis of divisors (3.8) as well as the intersection numbers,

$$H_B^3 \cdot S_0^2 = -1, \quad H_B^3 \cdot S_1^2 = -1, \quad (3.11)$$

which follow from the toric intersections in $\mathbb{P}_{F_{11}}$. The quartic intersections of $X_{F_{11}}$ can be readily computed and a canonical basis for $H^{(3,3)}(X_{F_{11}})$ is obtained by duality to $H^{(1,1)}(X_{F_{11}})$. We obtain generators for $H_V^{(2,2)}(X_{F_{11}})$ by constructing all possible products of two divisors in $H^{(1,1)}(X_{F_{11}})$. We evaluate the rank of the inner product on these generators as

$$\dim H^{2,2}(X_{F_{11}}) = 7. \quad (3.12)$$

As a basis for $H_V^{(2,2)}(X_{F_{11}})$ we choose the seven elements

$$H_V^{(2,2)}(X_{F_{11}}) = \langle (H_B)^2, H_B \tilde{S}_0, D_1^{\text{SU}(2)} H_B, D_1^{\text{SU}(3)} H_B, D_2^{\text{SU}(3)} H_B, H_B \sigma(\hat{s}_1), \tilde{S}_0^2 \rangle. \quad (3.13)$$

We can then expand the G_4 -flux in terms of this basis. Imposing the conditions (2.6) required by a match of M- and F-theory CS-terms leads to five conditions on the G_4 -flux, yielding the following two parameter G_4 -flux on $X_{F_{11}}$:

$$G_4 = a_6 H_B \cdot \sigma(\hat{s}_1) - a_7 \left[\tilde{S}_0^2 + (20n_7 - n_7^2 + 8n_9 - n_7 n_9 - 92) H_B^2 \right]. \quad (3.14)$$

Here a_6 and a_7 are free discrete parameters entering the G_4 -flux. Their quantization is fixed by the G_4 -flux quantization condition (2.1) using the expression (3.7) for $c_2(X_{F_{11}})$. Solving flux quantization in general requires the knowledge of the integral basis for $H_V^{(2,2)}(X_{F_{11}})$. Since the determination of the general integral basis is beyond the scope of this work, we will check G_4 -flux quantization in dependence on the number of chiral families indirectly by ensuring an integral and positive number n_{D3} of D3-branes and quantization of the 3D CS-terms (2.3). For the detailed discussion, we refer to section 3.3.

In order to compute the 4D matter chiralities we have to compute the homology classes of the matter surfaces for all representations in table 1. For those codimension two matter surfaces given as complete intersections in the toric ambient space of $X_{F_{11}}$, the homology classes of the corresponding matter surfaces follow directly from the second column in

table 1 and the splitting of the fiber (3.1) at the respective locus, cf. [22]. We obtain

$$\begin{aligned}
 \mathcal{C}_{(\mathbf{3},\mathbf{2})_{1/6}}^w &= \mathcal{S}_9 \cdot ([K_B^{-1}] + \mathcal{S}_7 - \mathcal{S}_9) \cdot E_4, \\
 \mathcal{C}_{(\bar{\mathbf{3}},\mathbf{1})_{-2/3}}^w &= -\mathcal{S}_9 \cdot (2[K_B^{-1}] - \mathcal{S}_7) \cdot (2H - E_2 - E_3 - \mathcal{S}_7 + \mathcal{S}_9 + [K_B^{-1}]), \\
 \mathcal{C}_{(\mathbf{1},\mathbf{1})_1}^w &= -(2[K_B^{-1}] - \mathcal{S}_7) \cdot (3[K_B^{-1}] - \mathcal{S}_7 - \mathcal{S}_9) \cdot (2H - E_1 - E_4 + \mathcal{S}_9),
 \end{aligned} \tag{3.15}$$

where we have used (2.9). Here, we have chosen a node in the respective fiber at codimension two that is not intersected by the zero section. The matter surfaces for the two matter loci supporting the representations $(\mathbf{1}, \mathbf{2})_{-1/2}$ and $(\bar{\mathbf{3}}, \mathbf{1})_{1/3}$ are not complete intersections. Their associated prime ideals are computed by a primary decomposition. Their respective homology classes are obtained by choosing a suitable complete intersection containing a given matter surface and by subtracting all its other irreducible components with their corresponding multiplicities as determined by the resultant. We obtain

$$\begin{aligned}
 [\mathcal{C}_{(\mathbf{1},\mathbf{2})_{-1/2}}^w] &= [(D_1^{\text{SU}(2)} + D_1^{\text{SU}(3)} + 2D_2^{\text{SU}(3)} + 4[K_B^{-1}] + 3\mathcal{S}_1 - 2\mathcal{S}_7) \cdot (6[K_B^{-1}] - 2\mathcal{S}_7 - \mathcal{S}_9) \\
 &\quad - 2(2[K_B^{-1}] - \mathcal{S}_7) \cdot (3[K_B^{-1}] - \mathcal{S}_7 - \mathcal{S}_9) \\
 &\quad + 2(-2[K_B^{-1}] + \mathcal{S}_7) \cdot \mathcal{S}_9] \cdot (\mathcal{S}_9 - \mathcal{S}_7 - [K_B^{-1}]), \\
 [\mathcal{C}_{(\bar{\mathbf{3}},\mathbf{1})_{1/3}}^w] &= [2[K_B^{-1}] \cdot (\mathcal{S}_7 - 2[K_B^{-1}]) + (D_1^{\text{SU}(2)} + D_2^{\text{SU}(3)} + 2[K_B^{-1}] + 2\mathcal{S}_1 - \mathcal{S}_7) \\
 &\quad \times (5[K_B^{-1}] - \mathcal{S}_7 - \mathcal{S}_9)] \cdot \mathcal{S}_9.
 \end{aligned} \tag{3.16}$$

Finally, we compute the integrals (2.7) of the G_4 -flux in (3.14) over the various matter surfaces in (3.15) and (3.16), yielding the following 4D matter chiralities:

$$\begin{aligned}
 \chi_{(\mathbf{3},\mathbf{2})_{1/6}} &= \frac{1}{6}(4 + n_7 - n_9)n_9a_6, \\
 \chi_{(\mathbf{1},\mathbf{2})_{-1/2}} &= \frac{1}{2}(4 + n_7 - n_9)((2n_7 + n_9 - 24)a_6 + 4(n_7 - 8)(n_7 + n_9)a_7 - 12) \\
 \chi_{(\bar{\mathbf{3}},\mathbf{1})_{-2/3}} &= \frac{1}{3}(n_7 - 8)n_9(2a_6 + 3(n_7 + n_9 - 12)a_7), \\
 \chi_{(\bar{\mathbf{3}},\mathbf{1})_{1/3}} &= -\frac{1}{3}n_9((n_7 + n_9 - 20)a_6 + 3(n_7 - 8)(n_7 + n_9 - 12)a_7), \\
 \chi_{(\mathbf{1},\mathbf{1})_1} &= (n_7 - 8)(n_7 + n_9 - 12)(a_6 + (2n_7 + n_9 - 16)a_7).
 \end{aligned} \tag{3.17}$$

As a cross-check of these geometric results, we use the duality between M- and F-theory in three dimensions and the implied matching of 3D CS-terms [15, 51, 54] to compute the 4D chiralities. For the spectrum in table (1), we compute all nonzero CS-terms on the F-theory side as

$$\begin{aligned}
 \Theta_{m=1,n=1}^F &= \frac{1}{2} \left(\frac{1}{6}\chi_{(\mathbf{3},\mathbf{2})_{1/6}} - \frac{1}{2}\chi_{(\mathbf{1},\mathbf{2})_{-1/2}} - \frac{4}{3}\chi_{(\bar{\mathbf{3}},\mathbf{1})_{-2/3}} + \frac{1}{3}\chi_{(\bar{\mathbf{3}},\mathbf{1})_{1/3}} + 3\chi_{(\mathbf{1},\mathbf{1})_1} \right), \\
 \Theta_{i=2,j=2}^F &= 3\chi_{(\mathbf{3},\mathbf{2})_{1/6}} - \chi_{(\mathbf{1},\mathbf{2})_{-1/2}}, \\
 \Theta_{i=3,j=3}^F &= 2\chi_{(\mathbf{3},\mathbf{2})_{1/6}} - \chi_{(\bar{\mathbf{3}},\mathbf{1})_{-2/3}} + \chi_{(\bar{\mathbf{3}},\mathbf{1})_{1/3}}, \\
 \Theta_{i=3,j=4}^F &= -\frac{1}{2}(2\chi_{(\mathbf{3},\mathbf{2})_{1/6}} - \chi_{(\bar{\mathbf{3}},\mathbf{1})_{-2/3}} + \chi_{(\bar{\mathbf{3}},\mathbf{1})_{1/3}}), \\
 \Theta_{i=4,j=4}^F &= 2\chi_{(\mathbf{3},\mathbf{2})_{1/6}} - \chi_{(\bar{\mathbf{3}},\mathbf{1})_{-2/3}} + \chi_{(\mathbf{3},\mathbf{1})_{1/3}},
 \end{aligned} \tag{3.18}$$

where the label $m = 1$ corresponds to the divisor of the 4D $U(1)$ gauge fields, $i = 2$ to the Cartan divisor $D_1^{\text{SU}(2)}$ and $i = 3, 4$ to the Cartan divisors $D_1^{\text{SU}(3)}$ and $D_2^{\text{SU}(3)}$, respectively. We note that only the SM-singlet has a non-trivial KK-charge, cf. [22]. We readily compute the CS-terms (2.3) on the M-theory side for the G_4 -flux (3.14). The matching $\Theta_{AB}^F \stackrel{!}{=} \Theta_{AB}^M$ precisely reproduces the chiralities in (3.17).

3.3 4D anomaly cancelation and family structure

As an additional cross-check of our computations, we can verify that all 4D anomalies are canceled by a generalized Green-Schwarz mechanism [97, 98].

For $X_{F_{11}}$, we have the following conditions implied by cancelation of the purely non-Abelian, mixed Abelian-non-Abelian, purely Abelian and mixed Abelian-gravitational anomalies:

$$\begin{aligned}
 \text{SU}(3)^3 &: -2\chi_{(\mathbf{3},\mathbf{2})_{1/6}} + \chi_{(\bar{\mathbf{3}},\mathbf{1})_{-2/3}} + \chi_{(\bar{\mathbf{3}},\mathbf{1})_{1/3}} = 0, \\
 \text{SU}(2)^2 - \text{U}(1) &: \frac{1}{2} \left[3\chi_{(\mathbf{3},\mathbf{2})_{1/6}} \left(\frac{1}{6} \right) + \chi_{(\mathbf{1},\mathbf{2})_{-1/2}} \left(-\frac{1}{2} \right) \right] = -\frac{1}{8} b_{\text{SU}(2)}^\alpha \Theta_{\alpha,m=1}, \\
 \text{SU}(3)^2 - \text{U}(1) &: \frac{1}{2} \left[2\chi_{(\mathbf{3},\mathbf{2})_{1/6}} \left(\frac{1}{6} \right) + \chi_{(\bar{\mathbf{3}},\mathbf{1})_{-2/3}} \left(-\frac{2}{3} \right) \right. \\
 &\quad \left. + \chi_{(\bar{\mathbf{3}},\mathbf{1})_{1/3}} \left(\frac{1}{3} \right) \right] = -\frac{1}{8} b_{\text{SU}(3)}^\alpha \Theta_{\alpha,m=1}, \\
 \text{U}(1)^3 &: \frac{1}{6} \left[6\chi_{(\mathbf{3},\mathbf{2})_{1/6}} \left(\frac{1}{6} \right)^3 + 2\chi_{(\mathbf{1},\mathbf{2})_{-1/2}} \left(-\frac{1}{2} \right)^3 + 3\chi_{(\bar{\mathbf{3}},\mathbf{1})_{-2/3}} \left(-\frac{2}{3} \right)^3 \right. \\
 &\quad \left. + 3\chi_{(\bar{\mathbf{3}},\mathbf{1})_{1/3}} \left(\frac{1}{3} \right)^3 + \chi_{(\mathbf{1},\mathbf{1})_1} (1)^3 \right] = -\frac{1}{8} b_{11}^\alpha \Theta_{\alpha,m=1}, \\
 \text{Grav.}^2 - \text{U}(1) &: \frac{1}{48} \left[6\chi_{(\mathbf{3},\mathbf{2})_{1/6}} \left(\frac{1}{6} \right) + 2\chi_{(\mathbf{1},\mathbf{2})_{-1/2}} \left(-\frac{1}{2} \right) + 3\chi_{(\bar{\mathbf{3}},\mathbf{1})_{-2/3}} \left(-\frac{2}{3} \right) \right. \\
 &\quad \left. + 3\chi_{(\bar{\mathbf{3}},\mathbf{1})_{1/3}} \left(\frac{1}{3} \right) + \chi_{(\mathbf{1},\mathbf{1})_1} (1) \right] = \frac{1}{32} a^\alpha \Theta_{\alpha,m=1}.
 \end{aligned} \tag{3.19}$$

We recall that the index α runs over base divisors. For \mathbb{P}^3 we only have $\alpha = 1$ for the single vertical divisor H_B . The coefficients $b_{11}^{\alpha=1}$, $b_{\text{SU}(2)}^{\alpha=1}$ and $b_{\text{SU}(3)}^{\alpha=1}$ can be computed as intersections of H_B^2 with the Néron-Tate height pairing b_{11} (3.5), and the GUT divisors $S_{\text{SU}(2)}$, $S_{\text{SU}(3)}$, respectively. They read, written in terms of the integers n_7 , n_9 introduced in (3.8), as

$$b_{11}^{\alpha=1} = 6 - \frac{1}{2}n_7 - \frac{1}{6}n_9, \quad b_{\text{SU}(2)}^{\alpha=1} = 4 + n_7 - n_9, \quad b_{\text{SU}(3)}^{\alpha=1} = n_9. \tag{3.20}$$

The coefficient a^α in (3.19) appearing in the mixed Abelian-gravitational anomaly stems from expanding K_B in terms of the vertical divisors. For this particular case we have $a^{\alpha=1} = -4$, according to (3.8). Finally, the CS-term $\Theta_{\alpha=1,m=1}$ for the axion gauging is given by

$$\Theta_{\alpha=1,m=1} = \frac{1}{6} [(-36 + 3n_7 + n_9)a_6 + 6(-8 + n_7)(-12 + n_7 + n_9)a_7], \tag{3.21}$$

as we compute using the G_4 -flux in (3.14) and the general formula (2.3). Using these results and the chiralities (3.17), we find that all anomalies in relations (3.19) are indeed satisfied.

Regarding the anomalies there are two remarks in order. First, we recall that even though the pure $SU(2)$ anomaly is trivial, one has to guarantee that the model does not have a Witten anomaly, i.e. the spectrum of the theory must always exhibit an even number of doublets [99]. In our case, the Witten anomaly takes the form

$$3\chi_{(\mathbf{3},\mathbf{2})_{1/6}} + \chi_{(\mathbf{1},\mathbf{2})_{-1/2}} \in 2\mathbb{Z}. \tag{3.22}$$

Using again the expressions given in (3.17), we see that the Witten anomaly is canceled only if $a_6(n_7 + n_9)$ is an even number. This is in contrast to the anomalies (3.19), which are canceled independently of the specific values for n_7, n_9, a_6 and a_7 . We expect that (3.22) is automatically obeyed for appropriately quantized G_4 -flux. For a model exhibiting a family structure, which is the case of interest in the following, we have $\chi_{(\mathbf{3},\mathbf{2})_{1/6}} = \chi_{(\mathbf{1},\mathbf{2})_{-1/2}}$ for which (3.22) is trivially satisfied.

Second, for general G_4 -flux, the axion gauging (3.21) is non-zero, as required by anomalies. This induces a mass-term for the $U(1)$ gauge field of $X_{F_{11}}$. As the $U(1)$ in this model corresponds to the hypercharge $U(1)_Y$, we have to impose that it is massless for the sake of the phenomenology of our model. For this reason we have to require $\Theta_{\alpha=1, m=1} = 0$, which reduces (3.14) to a one-parameter G_4 -flux. In the absence of the axion gauging all anomalies in (3.19) must vanish identically. Precisely as in the Standard Model, the cancelation of all anomalies can only be achieved if the chiralities for all fields coincide, i.e. if the matter fields come in a certain number b of complete families. Written in terms of this parameter $b \equiv \chi_{(\mathbf{3},\mathbf{2})_{1/6}}$, the coefficients a_6 and a_7 take the following form:

$$a_6 = \frac{6b}{(4 + n_7 - n_9)n_9}, \quad a_7 = -\frac{b(3n_7 + n_9 - 36)}{n_9(n_7 - 8)(4 + n_7 - n_9)(n_7 + n_9 - 12)}. \tag{3.23}$$

We note that the vanishing of the factors appearing in the denominator of the above equations define the boundaries of the allowed region for (n_7, n_9) , see figure 2. Recall also that at some of the boundaries of the allowed region, one of the base divisors supporting a non-Abelian gauge group disappears. For this reasons, the boundary region has to be excluded to begin with since there one or more matter curves disappear, leading to a model unsuited for phenomenological applications.

Using (3.23), we can write the G_4 -flux (3.14) as a function of b, n_7 and n_9 . In this parametrization, we check, for every allowed value for (n_7, n_9) , whether there are integral values of b for which the number n_{D3} of D3-branes needed to cancel the tadpole (2.2) is a positive integer, as expected for a smooth CY-fourfold $X_{F_{11}}$ and appropriately quantized G_4 -flux [81]. Additionally, we impose that all CS-terms (2.3) are integral, which is equivalent to G_4 -flux quantization as discussed in section 2. Without adding additional horizontal G_4 -flux, these two conditions impose the lower bounds on the number b of families shown in table 2 for all values of (n_7, n_9) in the allowed region and together with the corresponding numbers n_{D3} of D3-branes. Remarkably, this simple analysis shows that the minimal value of generations b obeying these constraints is three. We find $b = 3$ generations with

$n_7 \setminus n_9$	1	2	3	4	5	6	7
7	-	(27; 16)	-	-			
6	-	(12; 81)	(21; 42)	-	-		
5	-	-	(12; 57)	(30; 8)	-	(3; 46)	
4	(42; 4)	-	(30; 32)	-	-	-	-
3	-	(21; 72)	-	-	-	(15; 30)	
2	(45; 16)	(24; 79)	(21; 66)	(24; 44)	(3; 64)		
1	-	-	-	-			
0	-	-	(12; 112)				
-1	(36; 91)	(33; 74)					
-2	-						

Table 2. The entries (b, n_{D3}) show the minimal number of families b for which the number n_{D3} of D3-branes is integral and positive for integral 3D CS-terms. At the allowed points for (n_7, n_9) marked as “-” the number of D3-branes is negative for all positive integral values of b .

$n_{D3} = 64$ and $n_{D3} = 46$ for the two strata with $(n_7, n_9) = (2, 5)$ and $(n_7, n_9) = (5, 6)$, respectively.⁹

3.4 Phenomenological discussion

The discussion in the previous section shows that only the two models with $(n_7, n_9) = (2, 5), (5, 6)$ admit three chiral families, cf. table 2. Note also that $b = 3$ is the smallest permitted number of generations. Having found these three family solutions, we proceed in this section with the discussion of the phenomenology of the model.

We begin by identifying the representations from table 1 with the Standard Model particles they correspond to:

Q_i	\bar{u}_i	\bar{d}_i	L_i	\bar{e}_i	H_u, H_d	(3.24)
$(\mathbf{3}, \mathbf{2})_{1/6}$	$(\bar{\mathbf{3}}, \mathbf{1})_{-2/3}$	$(\bar{\mathbf{3}}, \mathbf{1})_{1/3}$	$(\mathbf{1}, \mathbf{2})_{-1/2}$	$(\mathbf{1}, \mathbf{1})_1$	$(\mathbf{1}, \mathbf{2})_{\pm 1/2}$	

Here the index $i = 1, 2, 3$ labels the families and we use the common notation to denote quarks by Q_i, \bar{u}_i and \bar{d}_i , leptons by L_i and \bar{e}_i and the two Higgses by H_u and H_d , respectively.

In $X_{F_{11}}$ the Higgs fields emerge as a vector-like pair from the same matter curve as the leptons L_i . In order to check geometrically that there is indeed a massless vector-like pair supported on the corresponding matter curve we need to be able to go beyond the chiral index and compute the individual numbers of left- and right-chiral fermions for the G_4 -flux (3.14). Unfortunately, these techniques are not available as of now, see however [100] for promising recent advancements in this direction. Thus, we work in the following under the assumption that the desired vector-like pair is indeed part of the massless spectrum.

⁹Adding horizontal G_4 -flux can lower the number of D3-branes further.

Yukawa	Locus
$(\mathbf{3}, \mathbf{2})_{1/6} \cdot (\overline{\mathbf{3}}, \mathbf{1})_{-2/3} \cdot (\mathbf{1}, \mathbf{2})_{-1/2}$	$s_3 = s_5 = s_9 = 0$
$(\mathbf{3}, \mathbf{2})_{1/6} \cdot (\overline{\mathbf{3}}, \mathbf{1})_{1/3} \cdot (\mathbf{1}, \mathbf{2})_{-1/2}$	$s_3 = s_9 = 0 = s_1 s_6 - s_2 s_5$
$(\overline{\mathbf{3}}, \mathbf{1})_{-2/3} \cdot (\overline{\mathbf{3}}, \mathbf{1})_{1/3} \cdot (\mathbf{1}, \mathbf{1})_1$	$s_1 = s_5 = s_9 = 0$
$(\mathbf{3}, \mathbf{2})_{1/6} \cdot (\mathbf{3}, \mathbf{2})_{1/6} \cdot (\overline{\mathbf{3}}, \mathbf{1})_{1/3}$	$s_3 = s_9 = s_6 = 0$
$(\mathbf{1}, \mathbf{2})_{-1/2} \cdot (\mathbf{1}, \mathbf{2})_{-1/2} \cdot (\mathbf{1}, \mathbf{1})_1$	$s_1 = s_5 = s_3 = 0$
$(\overline{\mathbf{3}}, \mathbf{1})_{1/3} \cdot (\overline{\mathbf{3}}, \mathbf{1})_{1/3} \cdot (\overline{\mathbf{3}}, \mathbf{1})_{-2/3}$	$s_5 = s_6 = s_9$

Table 3. Codimension three loci and corresponding Yukawa couplings for $X_{F_{11}}$.

Then it would be possible to induce the following bilinear coupling

$$\mathcal{W} \subset \mu H_u H_d + \beta^i H_u L_i. \tag{3.25}$$

These two terms could be generated by tuning the complex structure of our model to a model with enhanced (non-Abelian or Abelian) gauge symmetry and a SM-singlet $\mathbf{1}$, that admits Yukawa couplings with H_u , H_d and L_i , respectively. Then if $\mathbf{1}$ acquires a VEV, which breaks the enhanced gauge symmetry, the superpotential (3.25) could be generated. While the μ -term has to be very small in order to be consistent with electroweak symmetry breaking, the β^i terms are lepton violating and hence they must be adequately suppressed. We note that both these coefficients are moduli dependent functions, that cannot be computed by known techniques. However, we expect that in a sufficiently generic geometry the moduli of $X_{F_{11}}$ allow for appropriate tunings providing a phenomenologically viable scenario. At this point, we must remark that the geometry of $X_{F_{11}}$ offers no obvious way by which we could assign a quantum number to forbid the μ -term or the β^i terms.

Regarding the trilinear couplings we note that it was shown in [22] that all gauge invariant trilinear couplings are realized geometrically, see table 3. Thus, all MSSM Yukawas are geometrically allowed, giving rise to the superpotential terms

$$\mathcal{W} \subset Y_{i,j}^u Q_i \bar{u}_j H_u + Y_{i,j}^d Q_i \bar{d}_j H_d + Y_{i,j}^L \bar{e}_i \bar{L}_j H_d. \tag{3.26}$$

Since all three copies of each SM field live on the same matter curve, it is expected that the hierarchies in the Yukawas are generated in a similar fashion as in most SU(5) F-theory GUTs, where the Yukawa matrix has rank one, so that the geometrical coupling gives the mass for the heavy generation while the lighter ones pick their masses from instanton contributions [101–103]

Note also that since we cannot distinguish between H_d and L_i , the following dimension four proton decay operators are also geometrically allowed

$$\mathcal{W} \subset \lambda_{i,j,k}^{(0)} Q_i \bar{d}_j L_k + \lambda_{i,j,k}^{(1)} \bar{e}_i L_j L_k + \lambda_{i,j,k}^{(2)} \bar{u}_i \bar{d}_j \bar{d}_k. \tag{3.27}$$

Here the coupling $\lambda^{(2)}$ can only be suppressed by appropriate tunings of moduli. However, this is not possible for $\lambda^{(0)}$, $\lambda^{(1)}$ as their geometric origin is the same as that of the

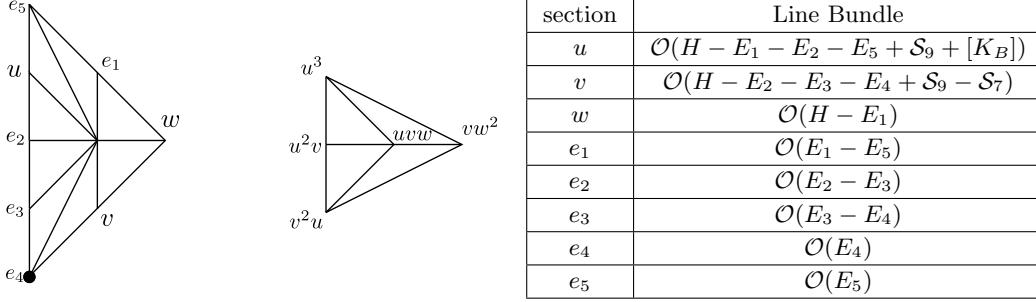


Figure 3. The toric diagram of polyhedron F_{13} and its dual. The zero section is indicated by the dot. In the accompanying table we indicate the divisor classes of the fiber coordinates.

Yukawa couplings (3.26). It then seems very challenging to suppress them to orders of $\lambda \leq 10^{-10}$ [104], while keeping the Yukawa couplings at values of about 0.1.

4 Pati-Salam model: $G_{F_{13}} = (\text{SU}(4) \times \text{SU}(2)^2)/\mathbb{Z}_2$

The fibration $X_{F_{13}}$ has been shown to exhibit the gauge symmetry and matter representation of the Pati-Salam (PS) model [22]. In section 4.1 we review the geometrical properties of $X_{F_{13}}$. In section 4.2 we explicitly construct G_4 -flux for the base $B = \mathbb{P}^3$. There we also compute the homology class of all matter surfaces and the 4D chiralities for all matter representations. We also determine the minimal number of generations which allow for D3-brane tadpole cancelation and integral G_4 -flux. Finally, the phenomenology of F-theoretic PS-models with three generations and their Higgsing down to the MSSM are described in section 4.3.

Readers directly interested in the 4D chiralities and phenomenological aspects of the models can directly jump to (4.14) and the following discussions.

4.1 The geometry of gauge group and matter representations

The elliptic fiber which yields F-theory models that naturally give rise to the gauge group and charge pattern needed for models of Pati-Salam (PS) unification is given by the following CY-hypersurface

$$p_{F_{13}} = s_1 e_1^2 e_2^2 e_3 e_5^4 u^3 + s_2 e_1 e_2^2 e_3^2 e_4^2 e_5^2 u^2 v + s_3 e_2^2 e_3^3 e_4^4 u v^2 + s_6 e_1 e_2 e_3 e_4 e_5 u v w + s_9 e_1 v w^2, \quad (4.1)$$

defined in the toric ambient space $\mathbb{P}_{F_{13}}$. The toric diagram of the ambient space as well as the divisor classes of the fiber coordinates are summarized in figure 3, where as before H is the hyperplane on \mathbb{P}^2 and E_i , $i = 1, \dots, 5$, denote the exceptional divisors.

The elliptically fibered Calabi-Yau fourfold $X_{F_{13}}$ is constructed by promoting the coefficients s_i to sections of the line bundles over the base B given in (2.9). The elliptic fibration of $X_{F_{13}}$ is equipped with a zero section given by

$$\hat{s}_0 = X_{F_{13}} \cap \{e_4 = 0\} : [1 : s_1 : 1 : 1 : 1 : -s_9 : 0 : 1]. \quad (4.2)$$

Representation	Locus
$(\mathbf{1}, \mathbf{2}, \mathbf{2})$	$V(I_{(1)}) := \{s_1 = s_3 = 0\}$
$(\mathbf{4}, \mathbf{2}, \mathbf{1})$	$V(I_{(2)}) := \{s_1 = s_9 = 0\}$
$(\mathbf{4}, \mathbf{1}, \mathbf{2})$	$V(I_{(3)}) := \{s_3 = s_9 = 0\}$
$(\mathbf{6}, \mathbf{1}, \mathbf{1})$	$V(I_{(4)}) := \{s_6 = s_9 = 0\}$

Table 4. Charged matter representations under $(\mathrm{SU}(4) \times \mathrm{SU}(2)^2)/\mathbb{Z}_2$ and their associated codimension two loci on $X_{F_{13}}$.

Furthermore, the elliptic fibration admits a section of order two, giving rise to \mathbb{Z}_2 Mordell-Weil group [22, 62]. In addition, one can see by computing the discriminant that over the locus $\mathcal{S}_{\mathrm{SU}(4)} = \{s_9 = 0\}$ the fiber degenerates to an I_4 -fiber. The corresponding Cartan divisors of the resulting $\mathrm{SU}(4)$ gauge symmetry are given by

$$D_1^{\mathrm{SU}(4)} = [s_9] - [u] - [e_2] - [e_3], \quad D_2^{\mathrm{SU}(4)} = [u], \quad D_3^{\mathrm{SU}(4)} = [e_2]. \quad (4.3)$$

Similarly, at the loci $\mathcal{S}_{\mathrm{SU}(2)_1} = \{s_1 = 0\}$ and $\mathcal{S}_{\mathrm{SU}(2)_2} = \{s_3 = 0\}$ we obtain I_2 -fibers. The resulting two $\mathrm{SU}(2)$ factors have the following associated Cartan divisors:

$$D_1^{\mathrm{SU}(2)_1} = [s_1] - [v], \quad D_1^{\mathrm{SU}(2)_2} = [e_1]. \quad (4.4)$$

The codimension two loci where the singularities of the fibration enhance, corresponding to the presence of matter fields, are given in table 4. We readily observe that the codimension two loci of $X_{F_{13}}$ support the matter representations characteristic for the Pati-Salam model. Similar as in section 3.1 we provide base independent expressions for the second Chern class as well as the Euler number of $X_{F_{13}}$, which are needed for G_4 -flux quantization and D3-brane tadpoles. We obtain

$$\begin{aligned} c_2(X_{F_{13}}) = & -c_1^2 + c_2 - 6c_1E_1 - 6E_1^2 - 2c_1E_3 - 2E_4^2 + 4c_1H + c_1\mathcal{S}_7 + 9E_1\mathcal{S}_7 + E_2\mathcal{S}_7 \\ & + E_3\mathcal{S}_7 - E_4\mathcal{S}_7 - E_5\mathcal{S}_7 - 3H\mathcal{S}_7 - 4E_1\mathcal{S}_9 - 2E_2\mathcal{S}_9 + 2H\mathcal{S}_9 \\ & - 2\mathcal{S}_7\mathcal{S}_9 + 2\mathcal{S}_9^2, \end{aligned} \quad (4.5)$$

$$\chi(X_{F_{13}}) = 12(6c_1^3 + c_1c_2 - 4c_1^2\mathcal{S}_7 + 2c_1\mathcal{S}_7^2 - 6c_1^2\mathcal{S}_9 + 2c_1\mathcal{S}_7\mathcal{S}_9 - \mathcal{S}_7^2\mathcal{S}_9 + 2c_1\mathcal{S}_9^2), \quad (4.6)$$

where, as before, c_1 and c_2 denote the first and second Chern class of the base B , respectively, and the divisors \mathcal{S}_7 and \mathcal{S}_9 are introduced in (2.9).

Again, we fix the base of the fibration to be $B = \mathbb{P}^3$ for the remainder of this section. We expand the divisor \mathcal{S}_7 , \mathcal{S}_9 and the anti-canonical class $[K_B^{-1}]$ w.r.t. H_B as in 3.8. Demanding effectiveness of all sections in (2.9) entering the CY-constraint (4.1), we find the allowed values for the pair (n_7, n_9) depicted in figure 4.

4.2 G_4 -flux and chiral generations

For the base $B = \mathbb{P}^3$, the SR-ideal of the toric ambient space of the fourfold $X_{F_{13}}$ is given by

$$\begin{aligned} SR_{F_{13}} = & \{ue_1, uw, uv, ue_4, ue_3, e_5w, e_5v, e_5e_4, e_5e_3, e_5e_2, e_1v, e_1e_4, e_1e_3, e_1e_2, \\ & we_4, we_3, we_2, ve_3, ve_2, e_4e_2, x_0x_1x_2x_3\}, \end{aligned} \quad (4.7)$$

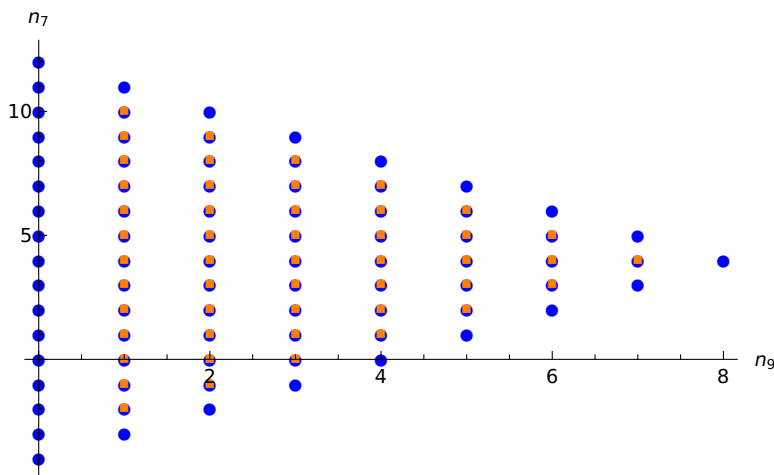


Figure 4. Allowed region of (n_7, n_9) for the CY-fourfold $X_{F_{13}}$ with $B = \mathbb{P}^3$. For points in orange, all representations of the PS-model are present and a G_4 -flux admitting b families exists.

where, again, $[u, v, w]$ and the e_i , $i = 1, \dots, 5$, are the projective coordinates on the fiber and x_j ($j = 0, 1, 2, 3$) are the homogeneous coordinates on the \mathbb{P}^3 base. As a basis of $H^{1,1}(X_{F_{13}})$ we choose the hyperplane H_B , the zero section and the five Cartan divisors,

$$H^{(1,1)}(X_{F_{13}}) = \langle H_B, S_0, D_1^{\text{SU}(2)_1}, D_1^{\text{SU}(2)_2}, D_1^{\text{SU}(4)}, D_2^{\text{SU}(4)}, D_3^{\text{SU}(4)} \rangle, \quad (4.8)$$

where, as before, the divisors and their dual $(1, 1)$ -forms are denoted by the same symbol. Here, S_0 is the class of \hat{s}_0 .

For the computation of the vertical cohomology ring of $X_{F_{13}}$ we use the SR-ideal (4.7), the basis (4.8) as well as the following intersections,

$$H_B^3 \cdot S_0^2 = -1, \quad H_B^3 \cdot (D_1^{\text{SU}(2)_2})^2 = -2, \quad H_B^3 \cdot (D_3^{\text{SU}(4)})^2 = -2, \quad (4.9)$$

which follow from intersections in $\mathbb{P}_{F_{13}}$. We readily obtain the quartic intersections of $X_{F_{13}}$ and the basis of $H^{(3,3)}(X_{F_{13}})$ is canonically determined by (4.8). Considering all products of two divisors in $H^{(1,1)}(X_{F_{13}})$, we obtain generators for $H_V^{(2,2)}(X_{F_{13}})$. The rank of their inner product matrix reveals that

$$\dim H_V^{(2,2)}(X_{F_{13}}) = 8. \quad (4.10)$$

We choose the following eight-dimensional basis for $H_V^{(2,2)}(X_{F_{13}})$:

$$H_V^{(2,2)}(X_{F_{13}}) = \langle H_B^2, H_B S_0, H_B D_1^{\text{SU}(2)_1}, H_B D_1^{\text{SU}(2)_2}, H_B D_1^{\text{SU}(4)}, H_B D_2^{\text{SU}(4)}, H_B D_3^{\text{SU}(4)}, S_0^2 \rangle. \quad (4.11)$$

Next, we make an ansatz for the G_4 -flux in terms of this basis. Following the description in section 2 we impose the condition (2.6). We find seven conditions on the G_4 -flux which leaves us with the following one parameter flux:

$$G_4 = -a_8 \left[S_0^2 + H_B \cdot \left(4S_0 + (12n_9 - n_7 n_9 - n_9^2) H_B - \frac{1}{2} n_9 D_1^{\text{SU}(2)_1} + \frac{1}{4} (n_7 + n_9 - 12) (3D_1^{\text{SU}(4)} + 2D_2^{\text{SU}(4)} + D_3^{\text{SU}(4)}) \right) \right]. \quad (4.12)$$

Here a_8 is a free discrete parameter, whose quantization is determined by G_4 -flux quantization. As in section 3 we will address G_4 -flux quantization indirectly in dependence on the number of chiral families by ensuring an integral and positive number n_{D3} of D3-branes and quantization of 3D CS-terms (2.3). We present the findings of this analysis below.

As a next step, we need the homology classes of all matter surfaces for the representations in table 4. Two of the matter surfaces are given as complete intersections in the toric ambient space $X_{F_{13}}$. Their homology classes are given by

$$C_{(4,2,1)}^w = \mathcal{S}_9 \cdot (3[K_B^{-1}] - \mathcal{S}_7 - \mathcal{S}_9) \cdot (H + [K_B^{-1}]), \quad C_{(4,1,2)}^w = \mathcal{S}_9([K_B^{-1}] + \mathcal{S}_7 - \mathcal{S}_9)E_5, \quad (4.13)$$

where we chose those nodes which are not intersected by the zero section. These matter surfaces lead to the following chiralities

$$\chi_{(4,2,1)} = -\chi_{(4,1,2)} = \frac{1}{4}n_9(4 + n_7 - n_9)(n_7 + n_9 - 12)a_8. \quad (4.14)$$

We note that the representations $(\mathbf{1}, \mathbf{2}, \mathbf{2})$ and $(\mathbf{6}, \mathbf{1}, \mathbf{1})$ (see table 4) are real representations. Therefore, their chiralities (2.7) are zero by definition. This is manifested in the geometry of $X_{F_{13}}$ by the fact that the fiber at the corresponding codimension two loci is non-split [22], so that precisely the nodes carrying the weights of the respective representation are interchanged by codimension three monodromies. This implies that the integral of the G_4 -flux over the matter surfaces associated to one node is equal to the integral over the matter surface associated to the other node. However, as the sum of the weights of the two nodes has to equal a root of the gauge group, the two integrals add up to zero, see (2.6), and we get zero chirality. It is important to stress that this does not necessarily mean that there are no massless fields transforming under real representations, in F-theory. Indeed, there can be vector-like fields in the theory, which are counted by the number $n(\mathbf{R})$ of individual Weyl fermions. For the above reasons, however, the multiplicity of fermions in real representations is not accessible by the methods described in this work. It would be interesting to develop methods based on those introduced in [100] to compute the number $n(\mathbf{R})$ for real representations explicitly.

We note that according to (4.14), the chiralities of the two complex representations $(\mathbf{4}, \mathbf{2}, \mathbf{1})$ and $(\mathbf{4}, \mathbf{1}, \mathbf{2})$ are equal up to a sign, which guarantees the cancelation of the cubic SU(4) anomaly.¹⁰ Thus, the number b of chiral generations coincides with the chirality of one of these two representations, i.e. we set $b = \chi_{(4,2,1)}$. This allows us to express the parameter a_8 in the G_4 -flux in (4.12) as

$$a_8 = \frac{4b}{n_9(n_9 - n_7 - 4)(12 - n_7 - n_9)}. \quad (4.15)$$

We note again that those values for (n_7, n_9) , for which the denominator vanishes, are excluded by the requirement to keep all non-Abelian gauge factors and to have non-vanishing

¹⁰Note that the cubic SU(4) anomaly, together with the Witten anomaly of the SU(2) factors are the only anomalies to check in this model. One can also see that the cancelation of the SU(4) anomaly guarantees that the Witten anomaly cancels as well.

$n_7 \setminus n_9$	1	2	3	4	5	6	7
10	(13; 204)						
9	-	(11; 140)					
8	(33; 94)	(10; 119)	(9; 90)				
7	-	(9; 100)	(6; 77)	(14; 48)			
6	(15; 108)	(8; 86)	(21; 52)	(12; 46)	(5; 44)		
5	(6; 106)	(35; 44)	-	(30; 16)	-	(3; 44)	
4	(7; 102)	(6; 75)	(15; 50)	(8; 42)	(15; 30)	(6; 41)	(7; 42)
3	(6; 106)	(35; 44)	-	(30; 16)	-	(3; 44)	
2	(15; 108)	(8; 86)	(21; 52)	(12; 46)	(5; 44)		
1	-	(9; 100)	(6; 77)	(14; 48)			
0	(33; 94)	(10; 119)	(9; 90)				
-1	-	(11; 140)					
-2	(13; 204)						

Table 5. The entries $(b; n_{D3})$ show the minimal number of families b for which the number of D3 branes n_{D3} is integral and positive for integral 3D CS terms. At the points marked with “-” the number of D3 branes is negative for all positive integral values of b .

chiralities (4.14). Thus, we are restricted to the interior (orange dots) of the allowed region for the fibration over \mathbb{P}^3 in figure 4.

As in section 3.3, we parametrize the G_4 -flux in terms of the integral number b in (4.15) of families. In this parametrization we evaluate the D3-brane tadpole (2.2) and require integral, positive n_{D3} and integral CS-terms (2.3). Without further horizontal G_4 -flux, this yields the lower bounds on the number b in dependence of (n_7, n_9) shown in table 5, where also the corresponding numbers n_{D3} of D3-branes are displayed. We find that three families are possible at two values for n_7 and n_9 . Again, we find that in the context of this simple analysis three is the minimal number b of families compatible with the D3-brane tadpole (2.2). We also note that table 5 has inherited the symmetries of the polyhedron F_{14} .

We conclude by noting that we double-check the chiralities in (4.14) using the matching (2.5) of CS-terms. The real representations in table 4 do not contribute to the loop-corrections in (2.4). We obtain the non-vanishing CS-terms

$$\Theta_{i,i}^F = -2\chi_{4,1,2} \quad (i = 1, \dots, 5), \quad \Theta_{i=3,j=4}^F = \Theta_{i=4,j=5}^F = -\chi_{4,1,2}, \quad (4.16)$$

where the indices label the five Cartan divisor $\{D_1^{\text{SU}(2)_1}, D_1^{\text{SU}(2)_2}, D_1^{\text{SU}(4)}, D_2^{\text{SU}(4)}, D_3^{\text{SU}(4)}\}_i$. Equating this with the CS-terms (2.3) on the M-theory side we readily reproduces (4.14).

4.3 Phenomenological discussion

In table 5 we find only two models with $(n_7, n_9) = (5, 6), (3, 6)$ that allow for three chiral Pati-Salam families. As in the standard model we see that three is the minimum allowed number of generations. These two models are equivalent under the reflection along the

$(4, n_9)$ line, which reflects the invariance of the theory under exchange of the two $SU(2)$ gauge groups.

The Higgs transition from $X_{F_{13}}$ to $X_{F_{11}}$ has been considered for the six dimensional case [22]. However, some of the observations made there immediately carry over to four dimensions. Similar to the 6D case, the transition happens due to a toric blow-down in the ambient space of the elliptic fiber of $X_{F_{13}}$. In this case we see that blowing down either e_4 or e_5 in figure 3 leads to the toric diagram of F_{11} in figure 1. These two transitions are equivalent up to redefinitions of the coordinates on the fiber, and for this reason we focus only on the blow down of e_4 which leads to F_{11} in its canonical form.

There are some subtleties that have to be discussed before proceeding with the detailed discussion of the Higgsings. First, we note that the blow-down process requires a restriction of the allowed region in figure 4 for $X_{F_{13}}$ because the section s_5 has to be effective, which is present in the CY-equation (3.1) for $X_{F_{11}}$, but not in the one (4.1) for $X_{F_{13}}$. This reduction amounts to excluding only two models below the $(8, n_9)$ line in table 9. To understand from an effective field theory point view, why the Higgsing is not possible for these two models is elusive. We recall that in the six dimensional case, the exclusion of certain points in the allowed region for (n_7, n_9) is explained from the field theory point of view by the fact that these models lack a sufficient number of Higgses for carrying out a D-flat Higgsing. In four dimensions one expects the same to occur. However, here the Higgses are pairs of vector-like fields and due to our lack of control over this sector of the theory, we can not verify this statement at this point.

Second, we also note that while in $X_{F_{11}}$ certain points in table 2 do not permit a family structure, in $X_{F_{13}}$ some of these points do allow for a certain number of families. Since the net chirality is preserved by Higgsings, it remains the question of why it is not possible to have the Higgsed models from $X_{F_{13}}$ as consistent tadpole canceling solutions¹¹ in $X_{F_{11}}$. It is expected that this seeming contradiction can be resolved by the inclusion of horizontal G_4 -flux. For example, it has been argued in [62, 85], at least in simple situations, that horizontal G_4 -flux exists that compensates for the change of the Euler number of a CY-fourfold in an extremal transition. Adding this G_4 -flux to F-theory on $X_{F_{11}}$, its D3-brane tadpole (2.2) becomes effectively identical to that on $X_{F_{13}}$ and we expect that our search strategy for three family models on both $X_{F_{11}}$ and $X_{F_{13}}$ will be consistent with the Higgs effect relating their effective actions.

In the following, we consider the special point $(n_7, n_9) = (5, 6)$ which has three as the smallest number of families both in $X_{F_{11}}$ and $X_{F_{13}}$. Therefore, the toric Higgsing is possible without the necessity of adding further G_4 -flux.¹² We discuss the field theoretical Higgsing in more detail and make some remarks about the phenomenology of this three-family model.

¹¹A logically possible explanation is again the absence of enough Higgs fields for performing the transition to begin with, which can not be tested with the tools at hand. The analysis of the vector-like spectrum using e.g. techniques available at weak coupling is beyond the scope of this work.

¹²Note that the point $(n_7, n_9) = (2, 5)$ allows for three families in $X_{F_{11}}$, while in $X_{F_{13}}$ the smallest number is five. Hence, based on these simple arguments the Higgsing is not possible without discussing horizontal G_4 -flux.

Name	Representation	SM decomposition
PS Higgs	$1 \times (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2})_{H^1}$	$\rightarrow \bar{d}_H : (\bar{\mathbf{3}}, \mathbf{1})_{\frac{1}{3}}, \bar{u}_H : (\bar{\mathbf{3}}, \mathbf{1})_{-\frac{2}{3}}, \bar{e}_H : (\mathbf{1}, \mathbf{1})_1, \bar{\nu}_H : (\mathbf{1}, \mathbf{1})_0$
PS Higgs	$1 \times (\mathbf{4}, \mathbf{1}, \mathbf{2})_{H^2}$	$\rightarrow d_H : (\mathbf{3}, \mathbf{1})_{-\frac{1}{3}}, u_H : (\mathbf{3}, \mathbf{1})_{\frac{2}{3}}, e_H : (\mathbf{1}, \mathbf{1})_{-1}, \nu_H : (\mathbf{1}, \mathbf{1})_0$
Exotic	$1 \times (\mathbf{6}, \mathbf{1}, \mathbf{1})$	$\rightarrow \bar{D} : (\bar{\mathbf{3}}, \mathbf{1})_{\frac{1}{3}}, D : (\mathbf{3}, \mathbf{1})_{-\frac{1}{3}}$
SM Higgs	$1 \times (\mathbf{1}, \mathbf{2}, \mathbf{2})_H$	$\rightarrow H_u : (\mathbf{1}, \mathbf{2})_{1/2}, H_d : (\mathbf{1}, \mathbf{2})_{-\frac{1}{2}}$
SM Matter	$3 \times (\mathbf{4}, \mathbf{2}, \mathbf{1})_M$	$\rightarrow Q_i : (\mathbf{3}, \mathbf{2})_{\frac{1}{6}}, L_i : (\mathbf{1}, \mathbf{2})_{-\frac{1}{2}}$
SM Matter	$3 \times (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2})_M$	$\rightarrow \bar{d} : (\bar{\mathbf{3}}, \mathbf{1})_{\frac{1}{3}}, \bar{u} : (\bar{\mathbf{3}}, \mathbf{1})_{-\frac{2}{3}}, \bar{e} : (\mathbf{1}, \mathbf{1})_1, \bar{\nu} : (\mathbf{1}, \mathbf{1})_0$

Table 6. The Pati-Salam matter content and its decomposition into standard model fields. The MSSM spectrum originates purely from the PS representations of the last three rows, which also provide candidates for right handed neutrinos ν . The representation in the first three rows are the Pati-Salam Higgses and the sextet needed to decouple exotic triplets.

On the field theory side the Higgsing is triggered by a VEV in the $(\mathbf{4}, \mathbf{1}, \mathbf{2})$ representation at the locus $s_3 = s_9 = 0$. A supersymmetric Higgsing requires at least one vector-like pair of fields in the representation $(\mathbf{4}, \mathbf{1}, \mathbf{2})$ in addition to the three chiral families. However, as mentioned already, we can not determine the massless vector-like spectrum with the techniques presented in section 2. Therefore, we work under the assumption that this vector-like pair of Higgs fields is indeed part of the spectrum. In table 6 we summarize the desired spectrum of the PS-model and its decomposition in terms of representations under the SM gauge group. In addition to the three chiral pairs $(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2})_M, (\mathbf{4}, \mathbf{2}, \mathbf{1})_M$, we require the presence of a light vector-like pair $(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2})_{H^1}$ and $(\mathbf{4}, \mathbf{1}, \mathbf{2})_{H^2}$ to serve as the PS-Higgses, whose neutral components develop VEVs $\langle H^1 \rangle = \langle H^2 \rangle$. In this type of breaking the hypercharge generator of the SM corresponds to the combination

$$Q_Y = \frac{2}{\sqrt{6}} T_4^{15} - T_2^3 \tag{4.17}$$

of the broken Cartans $T_4^{15} = \frac{1}{2\sqrt{6}} \text{diag}(1, 1, 1, -3)$ in $SU(4)$ and $T_2^3 = \frac{\sigma^3}{2}$ in $SU(2)$. Note also that in addition to these fields, we also expect a sextet and a bidoublet to be part of the massless spectrum.¹³ From the bidoublet $(\mathbf{1}, \mathbf{2}, \mathbf{2})_H$ we get the SM-Higgses and from the $(\mathbf{6}, \mathbf{1}, \mathbf{1})$ we get a pair of color triplets which serve to decouple some otherwise massless fields arising from the PS-Higgs multiplet [105].

Regarding the decomposition of the Higgs fields H^1 and H^2 in table 6, we note that the fields $\bar{u}_H, u_H, \bar{e}_H, e_H$ are removed from the massless spectrum as half of them become the longitudinal modes of the massive bosons of the broken $SU(4) \times SU(2)$ and the other half become massive Higgs bosons [106]. Therefore one only has to care about the lifting of the states \bar{d}_H and d_H . For that purpose we use the exotic $(\mathbf{6}, \mathbf{1}, \mathbf{1})$. In table 7 we present the geometrically allowed Yukawa couplings in $X_{F_{13}}$. Writing the couplings involving the

¹³We recall again that the presence of real representations and of a single vector-like pair of PS-Higgses is just introduced as an optimistic possibility in our discussion. It is to be seen if the hypersurface fibration over \mathbb{P}^3 actually allows for these desired fields.

Yukawa	Locus
$(\mathbf{1}, \mathbf{2}, \mathbf{2}) \cdot \overline{(\mathbf{4}, \mathbf{2}, \mathbf{1})} \cdot (\mathbf{4}, \mathbf{1}, \mathbf{2})$	$s_1 = s_3 = s_9 = 0$
$\overline{(\mathbf{6}, \mathbf{1}, \mathbf{1})} \cdot (\mathbf{4}, \mathbf{2}, \mathbf{1}) \cdot (\mathbf{4}, \mathbf{2}, \mathbf{1})$	$s_1 = s_6 = s_9 = 0$
$\overline{(\mathbf{6}, \mathbf{1}, \mathbf{1})} \cdot (\mathbf{4}, \mathbf{1}, \mathbf{2}) \cdot (\mathbf{4}, \mathbf{1}, \mathbf{2})$	$s_3 = s_6 = s_9 = 0$

Table 7. Codimension three loci and corresponding Yukawa couplings for $X_{F_{13}}$.

sextet and the Higgses and decomposing them into SM representations we find

$$\langle H^1 \rangle \bar{d}_H D + \langle H^2 \rangle \bar{D} d_H \subset (\mathbf{6}, \mathbf{1}, \mathbf{1}) \cdot \overline{(\mathbf{4}, \mathbf{1}, \mathbf{2})}_{H^1}^2 + (\mathbf{6}, \mathbf{1}, \mathbf{1}) \cdot (\mathbf{4}, \mathbf{1}, \mathbf{2})_{H^2}^2. \quad (4.18)$$

With these couplings, the masses of all exotics can be pushed towards the grand unification scale Λ_{PS} .

For the real representations we generically expect the following field theoretical bilinears in the superpotential

$$\mathcal{W} \supset M_2 (\mathbf{1}, \mathbf{2}, \mathbf{2})_H \cdot (\mathbf{1}, \mathbf{2}, \mathbf{2})_H + M_6 (\mathbf{6}, \mathbf{1}, \mathbf{1}) \cdot (\mathbf{6}, \mathbf{1}, \mathbf{1}). \quad (4.19)$$

Similar to the bilinear couplings (3.25) in the SM, these terms are expected to be generated by a VEV of a PS-singlet, that can be made visible by tuning the complex structure of $X_{F_{13}}$.¹⁴ Thus, the masses M_2 and M_6 are expected to be generically above the PS unification scale. However, after decomposing the bidoublets in terms of the SM gauge group, we see that the mass M_2 is related to the mass coefficient in front of the bilinear $H_u H_d$ in (3.25). Therefore, some fine tuning is needed in order to guarantee that the μ term is small enough. Similarly, the coupling M_6 enforces a kind of see-saw mechanism for the triplets d_H and D . The lowest mass eigenstate would have a mass of the order $\Lambda_{\text{PS}}^2/M_6$. Hence, we have to guarantee that M_6 is not too far beyond the PS scale as otherwise the exotic masses will be pushed towards observable mass scales. Therefore, we have to rely on the possibility that certain points in the complex structure moduli space of $X_{F_{13}}$ allow for a configuration in which both M_2 and M_6 are sufficiently small.

The Yukawa couplings for the SM fields are also generated geometrically as one can see from table 7. Indeed, we have the couplings

$$\mathcal{W} \supset (\mathbf{1}, \mathbf{2}, \mathbf{2})_H \cdot (\mathbf{4}, \mathbf{2}, \mathbf{1})_{M,i} \cdot \overline{(\mathbf{4}, \mathbf{1}, \mathbf{2})}_{M,j}. \quad (4.20)$$

Since both up-and down-type Yukawas arise from the same Yukawa point, the masses for up and down type quarks coincide at the PS-scale. The observed mass splitting is then due to the RG-running of the masses to the infrared. Again, since all matter fields of the same type arise from the same matter curve, the rank of the Yukawa matrix is expected to be one, with the lighter families picking their masses from instanton effects.

We conclude with a final important remark about the Pati-Salam model. We note that Higgses and leptons arise from different representations of the PS-group, which is

¹⁴In fact, we can confirm, that there are codimension three components in the matter curves of the real representations of the PS-model, that could geometrically support the couplings (4.19).

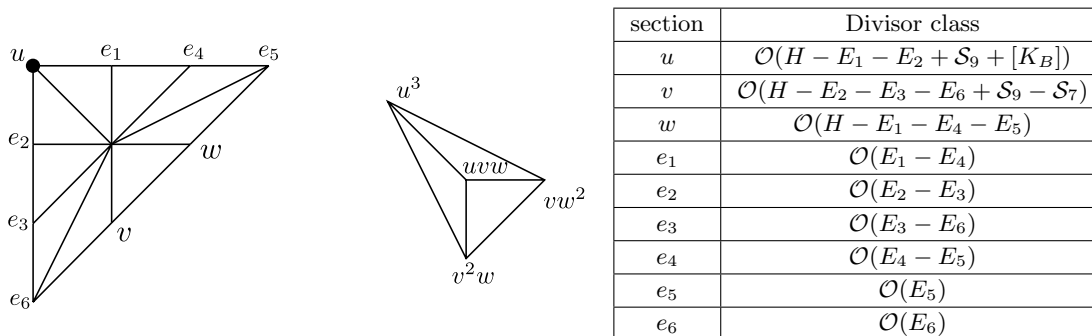


Figure 5. The toric diagram of polyhedron F_{16} and its dual. The zero section is indicated by the dot. In the accompanying table we indicate the divisor classes of the fiber coordinates.

in contrast to GUT schemes such based on $SO(10)$ or $SU(5)$. This has the advantage that in PS-models, there is no Yukawa coupling which induces the dimension four proton decay operators (3.27), which are only generated below Λ_{PS} after integrating out the heavy triplets. Therefore, these couplings are suppressed by a factor Λ_{PS}/M_6 . The dimension five proton decay operators are generated in a similar fashion. To see this in more detail let us consider the couplings of the SM fields to the exotic sextet

$$\begin{aligned}
 (\mathbf{6}, \mathbf{1}, \mathbf{1})(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2})_{M,i}(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2})_{M,j} &\supset \bar{D}\bar{d}_i\bar{u}_j + D\bar{d}_i\nu_j + D\bar{u}_i\bar{e}_j, \\
 (\mathbf{6}, \mathbf{1}, \mathbf{1})(\mathbf{4}, \mathbf{2}, \mathbf{1})_{M,i}(\mathbf{4}, \mathbf{2}, \mathbf{1})_{M,j} &\supset \bar{D}Q_iL_j + DQ_iQ_j.
 \end{aligned}
 \tag{4.21}$$

Upon integration of the exotic states D, \bar{D} we obtain the effective five operators,

$$Q_iQ_jQ_kL_m + Q_iL_j\bar{u}_k\bar{e}_i + \bar{d}_i\bar{u}_j\bar{u}_k\bar{e}_m
 \tag{4.22}$$

all of which are suppressed by a factor $(1/M_6^2)$.

5 Trinification model: $G_{F_{16}} = (SU(3)^3)/\mathbb{Z}_3$

The last type of models we consider are F-theory compactifications on $X_{F_{16}}$, which exhibit the gauge group of Trinification as well as its characteristic bitriplet spectrum. The geometrical information relevant for the discussion of the matter and gauge content of the theory is provided in section 5.1. The construction of the G_4 -flux for this models and the induced 4D matter chiralities is discussed in section 5.2. There we also determine the smallest allowed numbers of families for all of the allowed strata in moduli space. Finally, in section 5.3 we focus on a particular model which allows for three generations. We describe the Higgsings to the F-theoretic SM obtained from $X_{F_{11}}$, both on the geometry and field theory sides, and comment on the phenomenology of the model.

Readers directly interested in the 4D matter chiralities and the subsequent phenomenological discussion can start reading at (5.13).

5.1 The geometry of gauge group and matter representations

The elliptic fiber used for an F-theoretic realization of the Trinification model is a toric CY-hypersurface in the toric ambient space $\mathbb{P}_{F_{16}}$ given by

$$p_{F_{16}} = s_1e_1^2e_2^2e_3e_4u^3 + s_6e_1e_2e_3e_4e_5e_6uvw + s_7e_2^2e_3^3e_6^3v^2w + s_9e_1e_4^2e_5^3vw^2.
 \tag{5.1}$$

Representation	Locus
$(\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1})$	$V(I_{(1)}) := \{s_1 = s_7 = 0\}$
$(\mathbf{3}, \mathbf{1}, \bar{\mathbf{3}})$	$V(I_{(2)}) := \{s_1 = s_9 = 0\}$
$(\mathbf{1}, \mathbf{3}, \bar{\mathbf{3}})$	$V(I_{(3)}) := \{s_7 = s_9 = 0\}$

Table 8. Charged matter representations under $SU(3)^3/\mathbb{Z}_3$ and corresponding codimension two loci in $X_{F_{16}}$.

The relevant toric data is provided in figure 5. The divisor classes on the fiber are the hyperplane H and the exceptional divisors E_1 to E_5 .

In constructing elliptic fibrations $X_{F_{16}}$ with the curve (5.1) we promote the coefficients s_i to sections of the line bundles over the base B given in (2.9). The elliptic fibration of $X_{F_{16}}$ exhibits three toric sections with two torsional relations among them, so that the Mordell-Weil group of the fibration is \mathbb{Z}_3 [22, 62]. The fibration admits a zero section that we choose as

$$\hat{s}_0 = X_{F_{16}} \cap \{u = 0\} : [0 : 1 : 1 : s_7 : -s_9 : 1 : 1 : 1 : 1]. \quad (5.2)$$

As can be seen by inspecting the Weierstrass model of $X_{F_{16}}$, there are three codimension one loci over which the fiber becomes singular, namely $\mathcal{S}_{SU(3)^1} = \{s_1 = 0\}$, $\mathcal{S}_{SU(3)^2} = \{s_7 = 0\}$ and $\mathcal{S}_{SU(3)^3} = \{s_9 = 0\}$. The fiber degenerates to I_3 over these three divisors, giving rise to gauge group $SU(3)^3/\mathbb{Z}_3$ which is characteristic of the Trinitification model. The MW-torsion \mathbb{Z}_3 acts simultaneous on the centers of the $SU(3)$ factors. The Cartan divisors for the three $SU(3)$ factors read

$$\begin{aligned} D_1^{SU(3)_1} &= [v], & D_2^{SU(3)_1} &= [w], \\ D_1^{SU(3)_2} &= [e_4], & D_2^{SU(3)_2} &= [s_7] - [e_1] - [e_4], \\ D_1^{SU(3)_3} &= [e_3], & D_2^{SU(3)_3} &= [s_9] - [e_2] - [e_3]. \end{aligned} \quad (5.3)$$

At codimension two we find three loci in B , over which the singularity type of the elliptic fibration enhances and bifundamental matter is supported,¹⁵ see table 8.

We complete the base independent analysis of $X_{F_{16}}$ with the computation of its second Chern Class as well as its Euler number. Using the methods of [61], we obtain

$$\begin{aligned} c_2(X_{F_{16}}) &= -c_1^2 + c_2 - 7c_1E_1 + 2c_1E_2 - 9E_4^2 - 2c_1E_5 - 2c_1E_6 + 4c_1H + 3c_1\mathcal{S}_7 \\ &\quad + 5E_1\mathcal{S}_7 + E_2\mathcal{S}_7 + E_3\mathcal{S}_7 + 7E_4\mathcal{S}_7 + E_6\mathcal{S}_7 - 5H\mathcal{S}_7 - 2c_1\mathcal{S}_9 + E_1\mathcal{S}_9 \\ &\quad - 4E_2\mathcal{S}_9 - 2E_3\mathcal{S}_9 - 8E_4\mathcal{S}_9 + E_5\mathcal{S}_9 + 4H\mathcal{S}_9 - 4\mathcal{S}_7\mathcal{S}_9 + 4\mathcal{S}_9^2, \end{aligned} \quad (5.4)$$

$$\begin{aligned} \chi(X_{F_{16}}) &= 3(24c_1^3 + 4c_1c_2 - 24c_1^2\mathcal{S}_7 + 8c_1\mathcal{S}_7^2 - 24c_1^2\mathcal{S}_9 + 17c_1\mathcal{S}_7\mathcal{S}_9 \\ &\quad - 3\mathcal{S}_7^2\mathcal{S}_9 + 8c_1\mathcal{S}_9^2 - 3\mathcal{S}_7\mathcal{S}_9^2), \end{aligned} \quad (5.5)$$

¹⁵The effect of the \mathbb{Z}_3 torsion is manifest at codimension two as the only representations which are manifest are singlets under torsion.

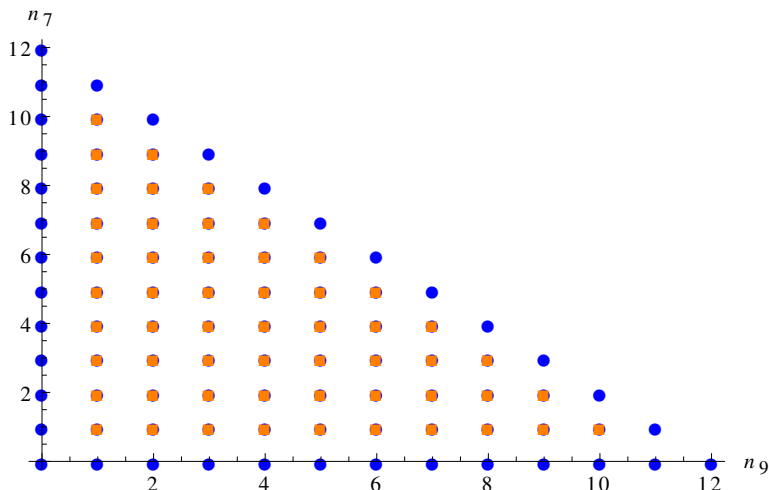


Figure 6. The allowed region (n_7, n_9) for the CY-fourfold $X_{F_{16}}$ for all points that are not on the boundary (orange) allow for b families in the Trinification model.

where, as before, c_1 and c_2 denoted the first and second Chern class of B , respectively.

Next we choose the base $B = \mathbb{P}^3$ and expand the divisors \mathcal{S}_7 and \mathcal{S}_9 in terms of the pullback of the hyperplane class H_B , cf. (3.8). Then we use conditions implied by effectiveness of the section s_i in (2.9), that enter the CY-equation (5.1), to obtain the allowed region for the pair (n_7, n_9) . It is depicted in figure 6.

5.2 G_4 -flux and chiralities

For the specific base \mathbb{P}^3 , the full SR-ideal of the ambient space of the CY-fourfold $X_{F_{16}}$ reads

$$SR = \{ue_4, ue_5, uw, uv, ue_6, ue_3, e_1e_5, e_1w, e_1v, e_1e_6, e_1e_3, e_1e_2, e_4w, e_4v, e_4e_6, e_4e_3, e_4e_2, e_5v, e_5e_6, e_5e_3, e_5e_2, we_6, we_3, we_2, ve_3, ve_2, e_6e_2, x_0x_1x_2x_3\}. \tag{5.6}$$

Here $[u : v : w]$ and the e_i , $i = 1 \dots, 5$ are the homogeneous coordinates on $\mathbb{P}_{F_{16}}$ and x_j , $j = 0, \dots, 3$, denote homogeneous coordinates on \mathbb{P}^3 . As a basis for $H^{(1,1)}(X_{F_{16}})$ we choose

$$H^{(1,1)}(X_{F_{16}}) = \langle H_B, S_0, D_1^{\text{SU}(3)_1}, D_2^{\text{SU}(3)_1}, D_1^{\text{SU}(3)_2}, D_2^{\text{SU}(3)_2}, D_1^{\text{SU}(3)_3}, D_2^{\text{SU}(3)_3} \rangle. \tag{5.7}$$

Next, we compute the vertical cohomology ring of $X_{F_{16}}$ as described in section 2 as a quotient ring using the SR-ideal (5.6) together with the following intersection numbers, that descend from the toric intersections in $\mathbb{P}_{F_{16}}$:

$$\begin{aligned} H_B^3 \cdot S_0^2 &= -1, & H_B^3 \cdot (D_2^{\text{SU}(3)_1})^2 &= -2, \\ H_B^3 \cdot (D_2^{\text{SU}(3)_2})^2 &= -2, & H_B^3 \cdot (D_2^{\text{SU}(3)_3})^2 &= -2. \end{aligned} \tag{5.8}$$

With this information at hand we can obtain all quartic intersections in $X_{F_{16}}$. The dimension of $H_V^{(2,2)}(X_{F_{16}})$ is found after taking all possible products of two elements in $H^{(1,1)}$ and evaluating the rank of their inner product matrix, yielding

$$\dim(H_V^{(2,2)}(X_{F_{16}})) = 9. \tag{5.9}$$

As a basis for $H_V^{(2,2)}(X_{F_{16}})$ we choose

$$H_V^{(2,2)}(X_{F_{16}}) = \langle H_B^2, H_B S_0, D_1^{\text{SU}(3)_1} H_B, D_2^{\text{SU}(3)_1} H_B, D_1^{\text{SU}(3)_2} H_B, D_2^{\text{SU}(3)_2} H_B, D_1^{\text{SU}(3)_3} H_B, D_2^{\text{SU}(3)_3} H_B, S_0^2 \rangle, \quad (5.10)$$

which we use to make an ansatz for the most general G_4 -flux. Since the G_4 -flux must be consistent with the matching of M- and F-theoretical CS terms, we have to impose the constraints (2.6), which amount to eight independent constraints. We are left with the following one parameter G_4 -flux:

$$G_4 = a_9 \left[-\frac{1}{3} H_B \cdot (n_7 D_1^{\text{SU}(3)_3} + 2n_7 D_2^{\text{SU}(3)_3} + n_9 D_1^{\text{SU}(3)_2} + 2n_9 D_2^{\text{SU}(3)_2}) + n_7 n_9 H_B^2 + 4H_B \cdot S_0 + S_0^2 \right]. \quad (5.11)$$

The parameter a_9 must be consistently quantized so that the G_4 -flux quantization condition (2.1) with $c_2(X_{F_{16}})$ as given in (5.4) is obeyed. Again, we ensure the quantization indirectly by choosing a_9 so that the number n_{D3} is a positive integer and that all 3D CS-terms (2.3) are integral. This issue is discussed below.

As a first step towards the computation of the 4D matter chiralities we provide the homology classes for the matter surfaces. Using the results from table 8, they read

$$\begin{aligned} C_{(\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1})}^w &= \mathcal{S}_7(3[K_B^{-1}] - \mathcal{S}_7 - \mathcal{S}_9)(H - E_1 + \mathcal{S}_9), \\ C_{(\mathbf{3}, \mathbf{1}, \bar{\mathbf{3}})}^w &= \mathcal{S}_9(3[K_B^{-1}] - \mathcal{S}_7 - \mathcal{S}_9)(H - E_2 + \mathcal{S}_9), \\ C_{(\mathbf{1}, \mathbf{3}, \bar{\mathbf{3}})}^w &= \mathcal{S}_7 \mathcal{S}_9(2H - E_1 - E_2 - E_3 - E_4 + K_B^{-1}) - \mathcal{S}_7 + \mathcal{S}_9, \end{aligned} \quad (5.12)$$

where for each matter surface we have chosen a node in the fiber at codimension two with weight w of the respective representation which is not intersected by the zero section. Integrating the G_4 -flux over the surfaces (5.12) leads according to (2.7) the following chiral indices:

$$\chi_{(\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1})} = \chi_{(\mathbf{3}, \mathbf{1}, \bar{\mathbf{3}})} = \chi_{(\mathbf{1}, \mathbf{3}, \bar{\mathbf{3}})} = \frac{1}{3} n_7 n_9 (-12 + n_7 + n_9) a_9. \quad (5.13)$$

Note that all chiralities are equal, as expected in order for the cubic $\text{SU}(3)$ anomalies to cancel. We also emphasize that at the boundary of the allowed region (see figure 6) we can not have a chiral theory, as all chiralities (5.13) vanish there.

As before, we express the parameter a_9 in terms of the number of families $b \equiv \chi_{(\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1})}$ as

$$a_9 = \frac{3b}{n_7 n_9 (-12 + n_7 + n_9)}. \quad (5.14)$$

For all allowed values of (n_7, n_9) , we explore which positive integral values for b lead to a canceled D3-brane tadpole with a positive integral number n_{D3} of D3-branes without adding horizontal G_4 -flux. As shown in table 6, we find that for three families ($b = 3$), which is also the minimal value of families, the D3-brane tadpole is canceled at nine different values for

$n_7 \backslash n_9$	1	2	3	4	5	6	7	8	9	10
10	(5; 120)									
9	(3; 94)	(3; 94)								
8	(4; 72)	(8; 69)	(4; 72)							
7	(14; 48)	(7; 54)	(7; 54)	(14; 48)						
6	(5; 50)	(8; 44)	(3; 44)	(8; 44)	(5; 50)					
5	(5; 50)	(5; 42)	(10; 36)	(10; 36)	(5; 42)	(5; 50)				
4	(14; 48)	(8; 44)	(10; 36)	(16; 30)	(10; 36)	(8; 44)	(14; 48)			
3	(4; 72)	(7; 54)	(3; 44)	(10; 36)	(10; 36)	(3; 44)	(7; 54)	(4; 72)		
2	(3; 94)	(8; 69)	(7; 54)	(8; 44)	(5; 42)	(8; 44)	(7; 54)	(8; 69)	(3; 94)	
1	(5; 120)	(3; 94)	(4; 72)	(14; 48)	(5; 50)	(5; 50)	(14; 48)	(4; 72)	(3; 94)	(5; 120)

Table 9. The entries $(b; n_{D3})$ show the minimal number of families b for which the number of D3 branes n_{D3} is integral and the 3D CS-terms are quantized.

(n_7, n_9) . We have also checked for the three-family models, that all 3D CS-terms (2.3) are integral. We observe that table 6 is symmetric as expected by the symmetries of polyhedron F_{16} and that there is a family structure for every allowed value of (n_7, n_9) .

As a cross-check of our results (5.13) we verify the matching (2.5) of CS-terms in F- and M-theory. The non-vanishing CS-terms on the F-theory side computed using (2.4) read

$$\begin{aligned}
 \Theta_{11}^F &= -(\chi_{(3, \bar{3}, 1)} + \chi_{(3, 1, \bar{3})}), & \Theta_{12}^F &= \frac{1}{2}(\chi_{(3, \bar{3}, 1)} + \chi_{(3, 1, \bar{3})}), \\
 \Theta_{22}^F &= -(\chi_{(3, \bar{3}, 1)} + \chi_{(3, 1, \bar{3})}), & & \\
 \Theta_{33}^F &= -2\chi_{(1, 3, \bar{3})}, & \Theta_{34}^F &= \frac{1}{2}(3\chi_{(3, \bar{3}, 1)} - \chi_{(1, 3, \bar{3})}), \\
 \Theta_{44}^F &= -(3\chi_{(3, \bar{3}, 1)} - \chi_{(1, 3, \bar{3})}), & & \\
 \Theta_{55}^F &= -(3\chi_{(3, \bar{3}, 1)} - \chi_{(1, 3, \bar{3})}), & \Theta_{56}^F &= \frac{1}{2}(3\chi_{(3, \bar{3}, 1)} - \chi_{(1, 3, \bar{3})}), \\
 \Theta_{66}^F &= -2\chi_{(1, 3, \bar{3})}. & &
 \end{aligned} \tag{5.15}$$

We readily compute the CS-terms (2.3) in M-theory, which allows us to reproduce precisely the chiralities in (5.13).

5.3 Phenomenological discussion

The breaking from the Trinification model to the SM proceeds via two successive Higgsings. Geometrically, the Higgsings correspond to blow-downs in $X_{F_{16}}$ induced by toric blow-downs in the ambient space $\mathbb{P}_{F_{16}}$ of the elliptic fiber. Thus, we can geometrically visualize the Higgsing directly in the toric diagram of F_{16} , see [22] for details.

More concretely, in order to obtain the CY-hypersurface $X_{F_{11}}$ starting from $X_{F_{16}}$, we have to perform two blow-downs in the fiber of $X_{F_{16}}$ that are identified by requiring that the fiber polyhedron F_{16} is mapped to F_{11} . There are three possible ways to achieve this. However all of these are equivalent due to the symmetries of the polyhedron. For concreteness we choose here the transition $X_{F_{16}} \rightarrow X_{F_{11}}$ induced by blowing down the

divisor $e_5 = 0$ and subsequently the divisor $e_4 = 0$ in the toric diagram of F_{16} , see figure 5. Note that after the blow downs we indeed get the toric diagram of F_{11} in figure 1 reflected along the horizontal axis passing through the origin.

The blow-down process restricts the allowed region in figure 6 of $X_{F_{16}}$ due to the requirement that the divisors associated to the sections s_3 and s_5 , which are part of (3.1) but not of (5.1), are effective. This removes all models which lie above the line $4 + n_7 - n_9 = 0$ in figure 6. Once we exclude these un-Higgsable models, we can compare the number of generations in $X_{F_{16}}$ and $X_{F_{11}}$ in figures 2 and 9. In order to perform such a comparison we have to bear in mind that the polyhedron obtained by Higgsing $X_{F_{16}}$ and the polyhedron specifying $X_{F_{11}}$ are related by a reflection, as mentioned above. Thus, one has to perform a redefinition of the integers (n_7, n_9) before and after Higgsing which amounts to the shift $(n_7, n_9) \rightarrow (8 - n_7, n_9)$. Under this map we see that the point $(n_7, n_9) = (3, 6)$ in $X_{F_{16}}$ maps to $(5, 6)$ in $X_{F_{11}}$, both of supporting models with $b = 3$. This suggests that there is a toric Higgsing of a Trinification model to a SM with three families. We take this as the example for the following phenomenological discussion.

On the field theory side the described transition $X_{F_{16}} \rightarrow X_{F_{11}}$ proceeds by a VEV of the field in the representation $(\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1})$ at the matter curve $s_1 = s_7 = 0$, cf. table 8. For reasons that will become clear in the following, the Higgsing down to the MSSM is only possible if in addition to the three chiral families one also has two vector-like pairs $(\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1})$, $(\bar{\mathbf{3}}, \mathbf{3}, \mathbf{1})$ [107]. While one pair, which we denote as H^1, \bar{H}^1 is needed for the intermediate breaking $SU(3)^3/\mathbb{Z}_3 \rightarrow SU(3) \times SU(2)^2 \times U(1)$, the second pair H^2, \bar{H}^2 is needed because the representation, which breaks the $SU(2) \times U(1)$ down to the hypercharge generator, is contained in the $(\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1})$ representation, too. Therefore, in addition to the three generations of chiral fields $(\lambda_i, Q_i, \bar{Q}_i)$, we assume that our model allows for two massless vector-like pairs of Higgs fields H^1, \bar{H}^1 and H^2, \bar{H}^2 .

In the first Higgsing inducing the symmetry breaking $SU(3)^3/\mathbb{Z}_3 \rightarrow SU(3) \times SU(2)^2 \times U(1)$ we arrive at the following decomposition of fields composing the chiral families of our model:

$$\begin{aligned} \lambda_i &= (\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1}) \rightarrow (\mathbf{2}, \mathbf{2}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{2}, \mathbf{1})_{-3} + (\mathbf{2}, \mathbf{1}, \mathbf{1})_3 + (\mathbf{1}, \mathbf{1}, \mathbf{1})_0, \\ Q_i &= (\bar{\mathbf{3}}, \mathbf{1}, \mathbf{3}) \rightarrow (\mathbf{2}, \mathbf{1}, \mathbf{3})_{-1} + (\mathbf{1}, \mathbf{1}, \mathbf{3})_{+2}, \\ \bar{Q}_i &= (\mathbf{1}, \mathbf{3}, \bar{\mathbf{3}}) \rightarrow (\mathbf{1}, \mathbf{2}, \bar{\mathbf{3}})_1 + (\mathbf{1}, \mathbf{1}, \bar{\mathbf{3}})_{-2}. \end{aligned} \tag{5.16}$$

Similarly the Higgses of the Trinification model decompose as

$$\begin{aligned} H^1, H^2 &= (\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1}) \rightarrow (\mathbf{2}, \mathbf{2}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{2}, \mathbf{1})_{-3} + (\mathbf{2}, \mathbf{1}, \mathbf{1})_{+3} + (\mathbf{1}, \mathbf{1}, \mathbf{1})_0, \\ \bar{H}^1, \bar{H}^2 &= (\bar{\mathbf{3}}, \mathbf{3}, \mathbf{1}) \rightarrow (\mathbf{2}, \mathbf{2}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{2}, \mathbf{1})_{+3} + (\mathbf{2}, \mathbf{1}, \mathbf{1})_{-3} + (\mathbf{1}, \mathbf{1}, \mathbf{1})_0. \end{aligned} \tag{5.17}$$

Here, the generator of the unbroken $U(1)$ is given by

$$Q = T_1^8 + T_2^8, \quad \text{with} \quad T_{1,2}^8 = \text{diag}(1, 1, -2), \tag{5.18}$$

where the subscripts 1, 2 label the corresponding $SU(3)$ factors.

Next we need to break one of the $SU(2)$ factors together with the $U(1)$ -factor to the hypercharge $U(1)_Y$. To this end, one of the fields in either the representation $(\mathbf{1}, \mathbf{2}, \mathbf{1})_{-3}$

Tri-Rep	SM decomposition
λ_i	$\rightarrow H_{d,i} : (\mathbf{1}, \mathbf{2})_{-1/2}, H_{u,i} : (\mathbf{1}, \mathbf{2})_{1/2}, \bar{e}_i : (\mathbf{1}, \mathbf{1})_1,$ $L_i : (\mathbf{1}, \mathbf{2})_{-1/2}, \bar{\nu}_i^1 : (\mathbf{1}, \mathbf{1})_0, \bar{\nu}_i^2 : (\mathbf{1}, \mathbf{1})_0$
\mathcal{Q}_i	$\rightarrow Q_i : (\mathbf{3}, \mathbf{2})_{\frac{1}{6}}, D_i : (\bar{\mathbf{3}}, \mathbf{1})_{-\frac{1}{3}}$
$\bar{\mathcal{Q}}_i$	$\rightarrow \bar{u} : (\bar{\mathbf{3}}, \mathbf{1})_{-\frac{2}{3}}, \bar{d}_i : (\bar{\mathbf{3}}, \mathbf{1})_{\frac{1}{3}}, \bar{D}_i : (\bar{\mathbf{3}}, \mathbf{1})_{\frac{1}{3}}$
H^1, \bar{H}^1	$\rightarrow H_d^1, \bar{H}_u^1 : (\mathbf{1}, \mathbf{2})_{-1/2}, H_u^1, \bar{H}_d^1 : (\mathbf{1}, \mathbf{2})_{1/2}$
H^2, \bar{H}^2	$\rightarrow H_d^2, \bar{H}_u^2, H_d^{2l} : (\mathbf{1}, \mathbf{2})_{-1/2}, H_u^1, \bar{H}_d^1, H_d^{2l'} : (\mathbf{1}, \mathbf{2})_{1/2}$

Table 10. The Trinification representations decomposed in terms of their MSSM constituents. We have also included the charged fields inside of the fields H^1, H^2 which do not participate in the Higgs process.

Yukawa	Locus
$(\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1}) \cdot (\bar{\mathbf{3}}, \mathbf{1}, \bar{\mathbf{3}}) \cdot (\mathbf{1}, \mathbf{3}, \bar{\mathbf{3}})$	$s_1 = s_7 = s_9 = 0$
$(\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1}) \cdot (\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1}) \cdot (\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1})$	$s_1 = s_6 = s_7 = 0$
$(\mathbf{3}, \mathbf{1}, \bar{\mathbf{3}}) \cdot (\mathbf{3}, \mathbf{1}, \bar{\mathbf{3}}) \cdot (\mathbf{3}, \mathbf{1}, \bar{\mathbf{3}})$	$s_1 = s_6 = s_8 = 0$
$(\mathbf{1}, \mathbf{3}, \bar{\mathbf{3}}) \cdot (\mathbf{1}, \mathbf{3}, \bar{\mathbf{3}}) \cdot (\mathbf{1}, \mathbf{3}, \bar{\mathbf{3}})$	$s_6 = s_7 = s_9 = 0$

Table 11. Codimension three loci and corresponding Yukawa points for $X_{F_{16}}$.

or $(\mathbf{2}, \mathbf{1}, \mathbf{1})_{+3}$ in the decomposition of the representation $(\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1})$ has to pick a VEV. Note also that for the pair H^1, \bar{H}^1 , both of these fields (and their complex conjugates) will be absorbed as the longitudinal modes of the massive vector fields. Hence, we have to pick the fields for the second Higgsing as irreducible representations stemming from the decomposition of H^2 and \bar{H}^2 in (5.17). For concreteness we take the states $(\mathbf{1}, \mathbf{2}, \mathbf{1})_{-3} \subset H^2$, $(\mathbf{1}, \mathbf{2}, \mathbf{1})_{+3} \subset \bar{H}^2$ to be responsible for the second breaking. In that case the hypercharge generator written in terms of the U(1) generator Q and the Cartan generator T^3 of the first SU(2) reads

$$Q_Y = -(T^3 + Q/6). \quad (5.19)$$

The decomposition of the chiral matter representations in terms of the SM gauge group is given in table 10. Here we immediately see that each family at the Trinification level provides an entire SM family, extended by two right handed neutrinos $\bar{\nu}_{1,i}, \bar{\nu}_{2,i}$, a vector-like pair of color triplets D_i, \bar{D}_i and a pair of SM-like Higgs fields $H_{u,i}, H_{d,j}$. For the decoupling of the exotics and the discussion of the couplings of the fields, we have to refer to the geometrically allowed Yukawa couplings in $X_{F_{16}}$ which are given in table 11 [22]. First notice that in general one has two scales Λ_1 and Λ_2 associated to each of the two symmetry breakings from Trinification to SM. However, radiative corrections will push these scales towards each other, and due to that we can simply assume that both Higgsings occur simultaneously at some scale Λ . Given the allowed couplings $H^1 \mathcal{Q}_i \bar{\mathcal{Q}}_i$ and $H^2 \mathcal{Q}_i \bar{\mathcal{Q}}_j$

we see that the exotic triplets D_i, \bar{D}_i pick up a mass of order Λ . We also note from table 10 that we get eight pairs of SM-like Higgses. Nevertheless, working out all bilinears among them, which result from the three point couplings of the Trinification model with VEV insertions, one sees that essentially all of the Higgses get lifted at the Trinification scale Λ . One might hope that this can be fixed by tuning the complex structure of $X_{F_{16}}$ to suppress the corresponding Yukawa couplings, however, we have to recall that there are only two structurally different types of Yukawa couplings. Indeed, on the one hand we have a three point coupling involving a single matter curve. On the other hand we have a coupling involving the three different matter curves, which if suppressed, will imply that Yukawa couplings for quarks and leptons are suppressed.

We conclude with another remark regarding the presence of vector-like pairs in the model. As we discussed above, we need vector-like matter H_1, \bar{H}_1, H_2 and \bar{H}_2 in the representations $(\bar{\mathbf{3}}, \mathbf{3}, \mathbf{1})$ and $(\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1})$, respectively, for the Higgsing down to the MSSM. It is then natural to expect also additional vector-like matter in the representations $(\bar{\mathbf{3}}, \mathbf{1}, \mathbf{3})$ and $(\mathbf{1}, \mathbf{3}, \bar{\mathbf{3}})$, which would give rise to exotics beyond those presented in table 10.¹⁶ Indeed, this can be motivated geometrically by the \mathbb{Z}_3 -symmetry of the polyhedron F_{16} . We have seen that this symmetry is realized on the chiral spectrum in table 9, relating different theories that only differ by a permutation of their SU(3) factors. We expect this to hold more generally for the vector-like sectors of different theories related by the \mathbb{Z}_3 symmetry. In particular for the stratum $n_7 = n_9 = 4$, which is a fixed point of that a symmetry, one expects the same number of vector-like pairs for all representations. For these type of models one expects unification of the SU(3) gauge couplings above the Trinification scale [108]. Furthermore, in [107, 109] the possibility of having a light SM Higgs pair has been related to the presence of additional discrete R and non- R symmetries in Trinification models with vector-like pairs. These symmetries could reduce to the standard matter parity after the Trinification group is broken down to the SM gauge group, and hence they could prevent the model from an exceedingly fast proton decay. It would be interesting to find ways to realize these additional symmetries in F-theory.

6 Conclusions

In this work we have presented explicit, globally consistent four-dimensional F-theory compactifications that have the standard model gauge group and three chiral families. Moreover we considered embeddings of the standard model into the Pati-Salam or trinification models using a toric realization of the relevant Higgs effects.

The models considered in this work result from F-theory compactifications on Calabi-Yau fourfolds, with their elliptic fibers defined as toric hypersurfaces in the three 2D toric ambient spaces $\mathbb{P}_{F_{11}}, \mathbb{P}_{F_{13}}$ and $\mathbb{P}_{F_{16}}$, see e.g. [22]. While the gauge symmetry, the type of matter representations and the possible Yukawa couplings are properties independent of the choice for the base manifold, we fix the base manifold to be $B = \mathbb{P}^3$ in order to construct the vertical cohomology in each case. This allows us to find explicit solutions for the chirality-inducing G_4 -flux and ultimately, to determine the possible 4D chiralities of

¹⁶Again, it is expected that these exotics acquire masses of order Λ , too.

matter fields. The expressions for the G_4 -flux in each type of compactification are shown to satisfy the consistency conditions imposed by the matching of F- and M-theoretical Chern-Simons terms, as well as the D3-brane tadpole cancellation condition with a positive, integral number of D3-branes. The G_4 -flux quantization condition is ensured indirectly by the analogous quantization condition of the 3D CS-terms. We also show that the obtained chiralities are consistent with field theoretic calculations of CS-terms and anomalies.

In this work we have only considered the vertical part of the middle cohomology, which is responsible for the generation of chirality. With no G_4 -flux along the horizontal components we observe that the D3-brane tadpole cancellation imposes very stringent constraints on the minimal number of generations permitted in each of the models, which happens to be precisely three. We explicitly identify the consistent three-family models in our list of concrete models. Although addition of horizontal G_4 -flux G_4^H does not directly change the chiralities, the divisibility properties of $\chi(X_4)/24 - 1/2(G_4^H)^2$ might be improved compared to that of $\chi(X_4)/24$ alone, yielding weaker constraints on the number of families imposed by the integrality of n_{D_3} .¹⁷ Thus, we expect a greater number of consistent three-family models.

For the Pati-Salam and Trinification models we also described the Higgs transition to the Standard Model. On the geometric side, effectiveness conditions of the divisor classes needed to specify the models after the transition determine points in the allowed region for (n_7, n_9) of $X_{F_{13}}$ and $X_{F_{16}}$, for which the Higgsing is possible. While in the six-dimensional case, these conditions are known to ensure a D-flat Higgsing, in four dimensions, a similar field theoretical understanding remains elusive and requires a better understanding of vector-like matter in F-theory, beyond chiral indices. Indeed, we presume that for values of (n_7, n_9) , where some divisors fail to be effective, one does not have a light pair of vector-like fields to carry out the Higgsing. Due to the current lack of control of the vector-like sector of F-theory, we simply assume that over the points where all divisors defining the model remain effective after the transition, one has the necessary vector-like pairs of Higgses and that the transition is indeed possible. The verification of this assumption is left for future works. Since the supersymmetric Higgs mechanism does not change the net chiralities, we expect that a model based on $X_{F_{13}}$ or $X_{F_{16}}$ with three families at a given point (n_7, n_9) maps to a model with three families in $X_{F_{11}}$, at the same point (n_7, n_9) . Remarkably for both $X_{F_{13}}$ and $X_{F_{16}}$ we find a point with three families before and after Higgsing. If we in addition require that the number of D3-branes remains constant in the transition, we must add horizontal G_4 -flux after the Higgsing in order to compensate for the change in the Euler number of the CY-fourfold. The systematic inclusion of horizontal G_4 -flux and their effect on the Higgsing, as well as the connection to G_4 -flux quantization, would be interesting to study in future works.

In this work we have also made some remarks on the phenomenology of the studied models. A rough look at the superpotential at quadratic and cubic order of the resulting effective field theories lead to the well known observation that the bare models have some

¹⁷There might be a finer integral basis of $H^4(X_4)$ that is off-diagonal in the splitting into horizontal and vertical cohomology.

phenomenologically unappealing features, such as the prediction of a fast proton decay or the difficulty to retain a light pair of electroweak Higgs fields in the spectrum. However, it is likely that the proton decay operators can be kept under control if one goes to special points in complex structure moduli space of the CY-fourfolds. On the field theory side this usually corresponds to the existence of an accidental discrete symmetry. It would be very interesting whether there exists horizontal G_4 -flux that stabilizes the complex structure of the CY-fourfold at these points in moduli, following the general arguments discussed in [86].

Finally, we have also computed the Hodge numbers of the considered CY-fourfolds with three-families. For our simple choice of base \mathbb{P}^3 we always have $h^{(2,1)} = 0$ which constraints cosmological F-theory applications of our models. Clearly, it is an interesting future direction to extend the phenomenological analysis carried out in this work to other bases B allowing for richer possible applications to cosmology.

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A Hodge numbers of CY-fourfolds

In this appendix we discuss the computation of the Hodge numbers of CY-fourfolds given as toric hypersurfaces. We readily use these methods for the calculation of the Hodge numbers of the three families of CY-fourfolds $X_{F_{11}}$, $X_{F_{13}}$ and $X_{F_{16}}$.

A CY-fourfold has the independent Hodge numbers $h^{(1,1)}$, $h^{(2,1)}$ and $h^{(3,1)}$. They are related to the Euler number as

$$\chi(X) = 6 \left(8 + h^{(1,1)} + h^{(3,1)} - h^{(2,1)} \right). \tag{A.1}$$

Since one can compute $\chi(X)$ independently by integrating the top Chern class $c_4(X)$ over X , it is sufficient to calculate $h^{(2,1)}$ and $h^{(3,1)}$ to obtain all independent Hodge numbers.

For a CY-fourfold given as a toric hypersurface specified by a pair of dual five dimensional reflexive lattice polyhedra (Δ, Δ^*) , one can use the combinatorial Batyrev formulas

(see [76] and references therein) to calculate the Hodge numbers as

$$\begin{aligned}
 h^{(1,1)} &= l(\Delta) - (4 + 2) - \sum_{\dim\Theta=4} l'(\Theta) + \sum_{\text{codim}\Theta_i=2} l'(\Theta_i)l'(\Theta_i^*), \\
 h^{(3,1)} &= l(\Delta^*) - (4 + 2) - \sum_{\dim\Theta^*=4} l'(\Theta^*) + \sum_{\text{codim}\Theta_i=2} l'(\Theta_i)l'(\Theta_i^*), \\
 h^{(2,1)} &= \sum_{\text{codim}\Theta_i=3} l'(\Theta_i)l'(\Theta_i^*).
 \end{aligned}
 \tag{A.2}$$

Here, Θ (Θ^*) denote faces of Δ (Δ^*), while the sum is over pairs (Θ_i, Θ_i^*) of dual faces. The $l(\Theta)$ and $l'(\Theta)$ count the total number of integral points of a face Θ and the number inside the face Θ , respectively. Finally, $l(\Delta)$ is the total number of integral points of the polyhedron Δ .

In the following we are considering fourfolds that have the elliptic curves in $\mathbb{P}_{F_{11}}$, $\mathbb{P}_{F_{13}}$ and $\mathbb{P}_{F_{16}}$ as a fiber over a \mathbb{P}^3 base space. An explicit expression for the total polyhedra of the 5D toric ambient space of these fibrations can be found in appendix B. We only need some general observations about these polyhedra here. The polytope of the \mathbb{P}^3 base always contributes four points. Hence we can write

$$l(\Delta) = 4 + \#\text{Points}(F_i). \tag{A.3}$$

For all fibrations of this type we find that there are never points within codimension two and three facets of the polyhedra i.e.

$$l'(\Theta_{\text{codim}=2,i}) = 0 \quad (\forall i), \quad l'(\Theta_{\text{codim}=3,i}) = 0 \quad (\forall i). \tag{A.4}$$

Furthermore we use the following observation that we made in [22], namely

$$\#\text{Points}(F_i) - 4 = \text{Rank}(G_{F_i}), \tag{A.5}$$

at a *generic* stratum of the fibration, i.e. on a bulk stratum of the allowed region. Hence the above formula (A.2) for the Hodge number $h^{(1,1)}$ simplifies to

$$h^{(1,1)} = \#\text{Points}(F_i) - 2 - \sum_{\dim\Theta=4} l'(\Theta) = \text{Rank}(G_{F_i}) + 2. \tag{A.6}$$

Note that, in contrast to this formula, the formula (A.2) is valid for all strata of the fibration even at the boundary of the allowed regions where divisors are switched off and the rank of the gauge group is reduced. This rank reduction is precisely taken care of by vertices that move into the interior of dimension four facets and therefore correct the above formula for the $h^{(1,1)}$ numbers.

Similarly we find, that for all of our models

$$h^{(2,1)} = 0. \tag{A.7}$$

Using (A.1) we can thus give a closed formula for all $h^{(3,1)}$

$$h^{(3,1)} = \frac{\chi}{6} - 10 - \text{Rank}(G_{F_i}). \tag{A.8}$$

variable	Base Vertex			Fiber Vertex	
z_1	1	1	1	$n_9 - 4$	$4 - n_7$
z_2	-1	0	0	0	0
z_4	0	-1	0	0	0
z_3	0	0	-1	0	0
\hat{v}_0	0	0	0	0	0
\hat{v}_i	0	0	0	v_i	

Table 12. The five dimensional polyhedron describing a \mathbb{P}^3 fibered fourfold with the two dimensional fiber coordinates v_i . The first four coordinates describe the \mathbb{P}^3 and the choice of n_7 and n_9 fix the fibration.

Furthermore, by having fixed the above Hodge numbers we can obtain the Hodge number $h^{(2,2)}$ as well using

$$h^{(2,2)} = 2 \left(22 + 2h^{(1,1)} + 2h^{(3,1)} - h^{(2,1)} \right) = 2 \left(6 + \frac{\chi}{3} \right). \tag{A.9}$$

By using the explicit presentation of the vertical cohomology ring $H_V^{(2,2)}(X_{F_i})$ as a quotient ring, we can compute $h_V^{(2,2)}$. With the knowledge of the full $h^{(2,2)}$ we can compute the dimension of the horizontal cohomology.

We note that the absence of any $(2, 1)$ -forms is of particular interest for cosmological applications as the resulting three-forms could be used to obtain axions to drive inflation as discussed in an F-theory context in [110, 111]. However these specific type of axions are absent in all of our models. We conclude that the \mathbb{P}^3 base is too simple to allow for these features.¹⁸

B Concrete toric lattice polyhedra of 4D chiral models

Using the algorithm of [58] we can construct the CY-manifolds X_{F_i} as concrete toric hypersurfaces associated to a five dimensional lattice polyhedron that specifies the underlying 5D ambient space. In our case the ambient space is a \mathbb{P}_{F_i} -fibration with a \mathbb{P}^3 base specified by two numbers n_7 and n_9 . The polyhedron is given in table 12. From the above polyhedron one can deduce dimension one, two, three and four facets and the points within them. From that information we can calculate the amount of Hodge numbers and in particular we find the vanishing of $h^{(2,1)}$ numbers for all n_7 and n_9 . In the table 13 we specify the Euler and Hodge numbers for the specific fibrations that allow for three-family G_4 -flux. We see that only a transition between loci is possible when it is possible for the Euler number to increase.

¹⁸We still find generically a lot of complex structure moduli $h^{(3,1)}$ that can be used as well. However the specific scenario in [110] is excluded.

Fiber F_i	Stratum (n_7, n_9)	χ	$h^{(3,1)}$	$h^{(1,1)}$	$h^{(2,1)}$	$h_v^{(2,2)}$	$h^{2,2}$
F_{11}	(5, 6)	1134	175	6	0	7	768
F_{11}	(2, 5)	1554	245	6	0	7	1048
F_{13}	(5, 6)	1080	165	7	0	8	732
F_{16}	(3, 6)	1062	161	8	0	9	720
F_{16}	(2, 9)	2274	363	8	0	9	820

Table 13. The Euler number and Hodge numbers for all inequivalent strata that support three family fluxes.

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