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6D gauge-Higgs unification on T^2/Z_N with custodial symmetry

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Abstract: We investigate the gauge-Higgs unification models compactified on T^2/Z_N that have the custodial symmetry. We select possible gauge groups, orbifolds and representations of the matter fermions that are consistent with the custodial symmetry, by means of the group theoretical analysis. The best candidate we found is 6D $SU(3)_C \times U(4)$ gauge theory on T^2/Z_3 and the third generation quarks are embedded into bulk fermions that are the symmetric traceless rank-2 tensor of SO(6).

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1 Introduction

The gauge-Higgs unification [1–4] is an interesting candidate for the new physics beyond the standard model. This solves the gauge hierarchy problem because a higher dimensional gauge symmetry protects the Higgs mass against quantum corrections. Chiral fermions in four-dimensional (4D) effective theory can be obtained by compactifying the extra dimensions on an orbifold.

The simplest models of this category are based on five-dimensional (5D) gauge theories whose gauge groups are U(3) in the flat spacetime [5–7], and SO(5) × U(1) in the warped spacetime [8–10]. In these models, the electroweak symmetry is broken by the vacuum expectation value (VEV) of the Wilson line phase $\theta_{\rm H} \equiv \int_C dy\, A_y$, where C is a non-contractible cycle along the extra dimension and A_y is the extra-dimensional component of the gauge field. The mass scale of the Kaluza-Klein (KK) excitation modes $m_{\rm KK}$ is determined by $\langle\theta_{\rm H}\rangle$ as $m_{\rm KK} \simeq m_W/|\langle\theta_{\rm H}\rangle|$ (m_W: the W boson mass) in the flat spacetime, and $m_{\rm KK} \simeq m_W \pi\sqrt{k\pi R}/|\sin\langle\theta_{\rm H}\rangle|$ ($e^{k\pi R}$: the warp factor) in the warped spacetime [9]. The current experimental constraints require $\langle\theta_{\rm H}\rangle$ to be small, i.e., $\langle\theta_{\rm H}\rangle \lesssim \mathcal{O}(0.1)$. In order to realize such small values of $\langle\theta_{\rm H}\rangle$, we need some amount of fine-tuning among the model parameters, such as 5D mass parameters for fermions. This stems from the fact that the effective potential for $\theta_{\rm H}$ does not exist at tree level and is induced at one-loop level in 5D gauge-Higgs unification models. The one-loop potential is typically expressed as a sum of periodic functions of $\theta_{\rm H}$ with the period of π and 2π (or $\pi/2$), which are roughly approximated as the cosine functions [11, 12]. Therefore, $\langle\theta_{\rm H}\rangle = \mathcal{O}(1)$ is realized unless the model parameters are fine-tuned. This problem can be evaded in the six-dimensional (6D) gauge-Higgs unification models. In this case, quartic terms in the Wilson line phases exist at tree level, while quadratic terms are induced at one-loop level. In the flat spacetime, for example, the effective potential has a form of

$$V(\theta_{\rm H}) = -\frac{c_2 g^2}{l_6 R^2}\left(\frac{\theta_{\rm H}}{g\pi R}\right)^2 + c_4 g^2\left(\frac{\theta_{\rm H}}{g\pi R}\right)^4 + \mathcal{O}(\theta_{\rm H}^6), \tag{1.1}$$

where $c_2, c_4 = \mathcal{O}(1)$ are numerical constants, g is the $\mathrm{SU(2)_L}$ gauge coupling constant, $l_6 \equiv 128\pi^3$ is the 6D loop factor, and R is a typical radius of the extra-dimensional space. By minimizing this, we find that

$$\langle\theta_{\rm H}\rangle \simeq \frac{g\pi\sqrt{c_2}}{\sqrt{2 l_6 c_4}} \simeq \frac{0.02\sqrt{c_2}}{\sqrt{c_4}} \ll 1, \tag{1.2}$$

and the KK modes are estimated to be around a few TeV.

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In extra-dimensional models, coupling constants in 4D effective theories generally deviate from the standard model values even at tree level due to mixing with the KK modes [13–15]. Unless $m_{\rm KK}$ is very high, models need some mechanisms to suppress such deviations. Especially a requirement that the ρ parameter and the Z boson coupling to the left-handed bottom quark (the $Zb_L\bar{b}_L$ coupling) do not deviate too much often imposes severe constraints on the model building. It is known that the custodial symmetry can protect them against the corrections induced by the mixing with the KK modes [8, 16]. Hence we focus on 6D gauge-Higgs unification models that has the custodial symmetry in this paper.

The purpose of this paper is to select candidates for realistic 6D gauge-Higgs unification models by means of the group theoretical analysis. The analysis is useful to investigate the gauge-Higgs unification models because the Higgs sector is determined by the gauge group structure. There are some works along this direction. 5D models are analyzed in ref. [17], the tree-level Higgs potentials in 6D models are calculated in ref. [18], and models in arbitrary dimensions are discussed in ref. [19]. In these works, the custodial symmetry is not considered and the electroweak gauge symmetry $\mathrm{SU}(2)_{\rm L} \times \mathrm{U}(1)_Y$ is embedded into a simple group. Thus the Weinberg angle θ_W is determined only by the group structure, and they found that no simple group realizes the observed value of θ_W. However, the assumption that $\mathrm{SU}(2)_{\rm L} \times \mathrm{U}(1)_Y$ is embedded into a simple group is not indispensable because the color symmetry $\mathrm{SU}(3)_C$ is not unified anyway. Besides, any brane localized terms allowed by the symmetries are not introduced in refs. [17, 19]. In fact, the realistic models constructed so far allow both an extra U(1) gauge symmetry, which is relevant to the realization of the experimental value of θ_W, and various terms and fields localized at the fixed points of the orbifolds [6, 7, 10, 11]. Therefore, we include both ingredients in our analysis. Since larger gauge groups contain more unwanted exotic particles, we consider a case that the 6D gauge group is $\mathrm{SU}(3)_C \times G \times \mathrm{U}(1)$, where G is a simple group whose rank is less than four.

The paper is organized as follows. In the next section, we explain our setup and derive conditions for zero-modes. In section 3, we list the zero-modes in the bosonic sector for all the rank-two and the rank-three groups that include the custodial symmetry. In section 4, we find a condition to preserve the custodial symmetry, and provide explicit expressions of the W and Z boson masses. In section 5, we discuss embeddings of quarks into 6D fermions, and search for appropriate representations of G that the 6D fermions should belong to. In section 6, we calculate the Higgs potential at tree level. Section 7 is devoted to the summary. In appendix A, we collect formulae in the Cartan-Weyl basis of the gauge group generators. In appendix B, general forms of the orbifold boundary conditions are shown. In appendix C, we list irreducible decompositions of various G representations into the $\mathrm{SU}(2)_{\rm L} \times \mathrm{SU}(2)_{\rm R}$ multiplets.

2 Setup

2.1 Compactified space

The 6D spacetime is assumed to be flat, and the metric is given by

$$ds^2 = \eta_{MN} dx^M dx^N = \eta_{\mu\nu} dx^\mu dx^\nu + (dx^4)^2 + (dx^5)^2, \tag{2.1}$$

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where $M, N = 0, 1, \cdots, 5$, $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ is the 4D Minkowski metric, and a point in the extra space (x^4, x^5) is identified as

$$\begin{pmatrix} x^4 \\ x^5 \end{pmatrix} \sim \begin{pmatrix} x^4 \\ x^5 \end{pmatrix} + 2\pi n_1 R_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\pi n_2 R_2 \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \tag{2.2}$$

where n_1 and n_2 are integers, and $R_1, R_2 > 0$ and $0 < \theta < \pi$ are constants. In order to obtain a 4D chiral theory at low energies, we compactify the extra space on a two-dimensional orbifold. All possible orbifolds are T^2/Z_N $(N = 2, 3, 4, 6)$ [20]. It is convenient to use a complex (dimensionless) coordinate $z \equiv \frac{1}{2\pi R_1}(x^4 + ix^5)$. Then, the orbifold obeys the identification,

$$z \sim \omega z + n_1 + n_2 \tau, \tag{2.3}$$

where $\omega = e^{2\pi i/N}$ and $\tau \equiv \frac{R_2}{R_1} e^{i\theta}$. Note that an arbitrary value of τ is allowed when $N = 2$ while it must be equal to ω when $N \neq 2$.

The orbifold T^2/Z_N has the following fixed points in the fundamental domain [21, 22].

$$z = z_{\mathrm{f}} \equiv \begin{cases} 0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2} & (\text{on } T^2/Z_2) \\ 0, \frac{2+\tau}{3}, \frac{1+2\tau}{3} & (\text{on } T^2/Z_3) \\ 0, \frac{1+\tau}{2} & (\text{on } T^2/Z_4) \\ 0 & (\text{on } T^2/Z_6) \end{cases} \tag{2.4}$$

4D fields or interactions are allowed to be introduced on these fixed points.

2.2 Field content

We consider a 6D gauge theory whose gauge group is $\mathrm{SU}(3)_C \times G \times \mathrm{U}(1)_Z$, where G is a simple group. Since G must include $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}}$, its rank r is greater than one. In this paper, we investigate cases of $r = 2, 3$. In the following, we omit $\mathrm{SU}(3)_C$ since it is irrelevant to the discussion. The 6D gauge fields for G and $\mathrm{U}(1)_Z$ are denoted as A_M and B_M^Z, and the field strengths and the covariant derivative are defined as $F_{MN}^{(A)} \equiv \partial_M A_N - \partial_N A_M - i[A_M, A_N]$, $F_{MN}^{(Z)} \equiv \partial_M B_N^Z - \partial_N B_M^Z$, and $\mathcal{D}_M \equiv \partial_M - iA_M - iq_Z B_M^Z$, where q_Z is a $\mathrm{U}(1)_Z$ charge. The 6D Lagrangian is expressed as

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4g_A^2}\mathrm{tr}\left(F^{(A)MN}F_{MN}^{(A)}\right) - \frac{1}{4g_Z^2}F^{(Z)MN}F_{MN}^{(Z)} + i\sum_f \bar{\Psi}^f \Gamma^M \mathcal{D}_M \Psi^f \\ & + \sum_{z_{\mathrm{f}}} \mathcal{L}^{(z_{\mathrm{f}})}\delta^{(2)}(z - z_{\mathrm{f}}), \end{aligned} \tag{2.5}$$

where g_A and g_Z are the 6D gauge coupling constants for G and $\mathrm{U}(1)_Z$, Γ^M are the 6D gamma matrices, and $\mathcal{L}^{(z_{\mathrm{f}})}$ are 4D Lagrangians localized at the fixed points $z = z_{\mathrm{f}}$.

The G gauge field A_M is decomposed as

$$A_M = \sum_i C_M^i H_i + \sum_\alpha W_M^\alpha E_\alpha, \tag{2.6}$$

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where $\{H_i, E_\alpha\}$ are the generators in the Cartan-Weyl basis, i.e., H_i $(i = 1, \cdots, r)$ are the Cartan generators and α runs over all the roots of G. Since A_M is Hermitian, C_M^i are real and $W_M^{-\alpha} = (W_M^\alpha)^*$. In the complex coordinate (x^μ, z), the extra-dimensional components of the gauge fields are expressed as

$$\begin{aligned} A_z &= \pi R_1 \left(A_4 - iA_5\right), \qquad & A_{\bar{z}} &= A_z^\dagger, \\ B_z^Z &= \pi R_1 \left(B_4^Z - iB_5^Z\right), \qquad & B_{\bar{z}}^Z &= B_z^{Z\dagger}. \end{aligned} \tag{2.7}$$

2.3 Orbifold conditions for gauge fields

As shown in appendix B, the general orbifold boundary conditions for the gauge fields can be expressed as

$$\begin{aligned} A_M(x, z+1) &= A_M(x,z), & B_M^Z(x, z+1) &= B_M^Z(x,z), \\ A_M(x, z+\tau) &= A_M(x,z), & B_M^Z(x, z+\tau) &= B_M^Z(x,z), \\ A_\mu(x, \omega z) &= P A_\mu(x,z) P^{-1}, & A_z(x, \omega z) &= \omega^{-1} P A_z(x,z) P^{-1}, \\ B_\mu^Z(x, \omega z) &= B_\mu^Z(x,z), & B_z^Z(x, \omega z) &= \omega^{-1} B_z^Z(x,z), \end{aligned} \tag{2.8}$$

where P is an element of G. The orbifold conditions for 6D fermions are provided in (5.2).

Since zero-modes of the gauge fields have flat profiles over the extra dimensional space, we can see from (2.8) that B_μ^Z has a zero-mode while B_z^Z does not. Namely $\mathrm{U}(1)_Z$ is unbroken by the orbifold conditions. The condition for A_M to have zero-modes is determined by the choice of the matrix P in (2.8). It is always possible to choose the generators so that P is expressed as

$$P = \exp\left(ip \cdot H\right), \tag{2.9}$$

where $p \cdot H \equiv \sum_i p_i H_i$ and p_i are real constants. Thus $P H_i P^{-1} = H_i$ and $P E_\alpha P^{-1} = e^{ip\cdot\alpha} E_\alpha$, and the relevant conditions in (2.8) to the zero-mode conditions are rewritten as

$$\begin{aligned} C_\mu^i(x, \omega z) &= C_\mu^i(x,z), & C_z^i(x, \omega z) &= \omega^{-1} C_z^i(x,z), \\ W_\mu^\alpha(x, \omega z) &= e^{ip\cdot\alpha} W_\mu^\alpha(x,z), & W_z^\alpha(x, \omega z) &= e^{i(p\cdot\alpha - \frac{2\pi}{N})} W_z^\alpha(x,z). \end{aligned} \tag{2.10}$$

This indicates that C_μ^i always have zero-modes while C_z^i do not irrespective of the choice of the matrix P. Therefore the orbifold boundary conditions cannot reduce the rank of G as pointed out in ref. [23]. In contrast, whether W_μ^α and W_z^α have zero-modes depend on the choice of P. Since (2.10) is the Z_N transformation, p_i must satisfy $e^{iNp\cdot\alpha} = \mathbf{1}$. Thus possible values of $p \cdot \alpha$ are

$$p \cdot \alpha = \frac{2n_\alpha \pi}{N}, \tag{2.11}$$

where n_α is an integer.

In this paper, we focus on P such that the orbifold bondary conditions break G to $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R} \times \mathrm{U}(1)^{r-2}$. We denote the positive roots that specify $\mathrm{SU}(2)_\mathrm{L}$ and $\mathrm{SU}(2)_\mathrm{R}$ as α_L and α_R, respectively. The $\mathrm{SU}(2)_\mathrm{L}$ and $\mathrm{SU}(2)_\mathrm{R}$ generators are given by (4.1). Then (2.11) is further restricted as

$$\begin{aligned} p \cdot \alpha_L = p \cdot \alpha_R &= 0, \quad (\mathrm{mod}\ 2\pi) \\ p \cdot \beta &= \frac{2n_\beta \pi}{N}. \qquad (\beta \neq \alpha_L, \alpha_R, \quad n_\beta \in \mathbb{Z}, \quad n_\beta \notin N\mathbb{Z}) \end{aligned} \tag{2.12}$$

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From the last condition in (2.10), the zero-mode condition for W_z^β is

$$p \cdot \beta = \frac{2\pi}{N}. \tag{2.13}$$

3 Zero-modes of gauge and Higgs fields

In this section, we investigate the field content of the zero-modes from the 6D gauge fields.

3.1 Rank-two groups

First we consider a case of $r = 2$, i.e., $G = \mathrm{SO}(5), \mathrm{G}_2$. In this case, the unbroken gauge group by the orbifold conditions is $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R} \times \mathrm{U}(1)_Z$. We do not consider $G =$SU(3) because it does not contain $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}$ as a subgroup. The roots of G can be expressed as linear combinations of two-dimensional basis vectors $\boldsymbol{e}^i$ $(i = 1, 2)$.

3.1.1 SO(5)

The roots are $\{\pm\boldsymbol{e}^i \pm \boldsymbol{e}^j, \pm\boldsymbol{e}^i\}$ $(1 \leq i \neq j \leq 2)$. We can choose the unbroken subgroup $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}$ as

$$(\alpha_L, \alpha_R) = (\boldsymbol{e}^1 + \boldsymbol{e}^2, \boldsymbol{e}^1 - \boldsymbol{e}^2). \tag{3.1}$$

The other possible choices are essentially equivalent to this case.[1] Then the adjoint representation of G is decomposed into the irreducible representations of $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}$ as

$$\mathbf{10} = \mathbf{(3,1)} + \mathbf{(1,3)} + \mathbf{(2,2)}. \tag{3.2}$$

A candidate for the Higgs fields is a bidoublet $\mathbf{(2,2)}$, which consists of $\pm\boldsymbol{e}^1$ and $\pm\boldsymbol{e}^2$. The conditions in (2.12) are now expressed as

$$\begin{aligned} p_1 + p_2 = p_1 - p_2 = 0, \quad & (\mathrm{mod}\ 2\pi) \\ p_1 = \frac{2n_P\pi}{N}. \quad & (n_P \in \mathbb{Z}, \quad n_P \notin N\mathbb{Z}) \end{aligned} \tag{3.3}$$

It is enough to find a solution in a range: $0 \leq p_1, p_2 < 2\pi$. A solution exists when $N \neq 3$, and it is

$$(p_1, p_2) = (\pi, \pi), \tag{3.4}$$

or

$$P = \exp\{i\pi(H_1 + H_2)\}. \tag{3.5}$$

Therefore the zero-mode condition (2.13) for $\mathbf{(2,2)}$ is expressed as

$$\pi = \frac{2\pi}{N}. \tag{3.6}$$

Namely, we have one Higgs bidoublet when $N = 2$, while no Higgs exists in the other cases.

[1] We cannot choose them as $(\alpha_L, \alpha_R) = (\boldsymbol{e}^1, \boldsymbol{e}^2)$ because $\alpha_L + \alpha_R$ is a root in such a case.

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3.1.2 G_2

The roots are $\{\pm(\boldsymbol{e}^1 \pm \sqrt{3}\boldsymbol{e}^2)/2, \pm(\boldsymbol{e}^1 \pm \frac{1}{\sqrt{3}}\boldsymbol{e}^2)/2, \pm\boldsymbol{e}^1, \pm\boldsymbol{e}^2/\sqrt{3}\}$. We can choose the $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}$ subgroup as

$$(\alpha_L, \alpha_R) = \left(\boldsymbol{e}^1, \frac{\boldsymbol{e}^2}{\sqrt{3}}\right), \quad \left(\frac{\boldsymbol{e}^2}{\sqrt{3}}, \boldsymbol{e}^1\right). \tag{3.7}$$

The other possible choices are essentially equivalent to these cases.

Let us first consider the case of $(\alpha_L, \alpha_R) = (\boldsymbol{e}^1, \boldsymbol{e}^2/\sqrt{3})$. The irreducible decomposition of the adjoint representation of G is

$$\mathbf{14} = \mathbf{(3,1)} + \mathbf{(1,3)} + \mathbf{(2,4)}. \tag{3.8}$$

A candidate for the Higgs fields is $\mathbf{(2,4)}$. The conditions in (2.12) become

$$\begin{aligned} p_1 = \frac{p_2}{\sqrt{3}} &= 0, \qquad\qquad (\mathrm{mod}\ 2\pi) \\ \frac{p_1}{2} + \frac{p_2}{2\sqrt{3}} &= \frac{2n_P\pi}{N}. \quad (n_P \in \mathbb{Z}, \quad n_P \notin N\mathbb{Z}) \end{aligned} \tag{3.9}$$

It is enough to find a solution in a range $0 \le p_1, \frac{p_2}{\sqrt{3}} < 2\pi$. A solution exists when $N \neq 3$, and it is

$$P = \exp\left(2\sqrt{3}\pi i H_2\right). \tag{3.10}$$

Therefore the zero-mode condition (2.13) for $\mathbf{(2,4)}$ is expressed as

$$\pi = \frac{2\pi}{N}. \tag{3.11}$$

Namely, we have a $\mathbf{(2,4)}$ multiplet as the Higgs fields when $N = 2$, while no Higgs exists in the other cases.

In the case of $(\alpha_L, \alpha_R) = (\boldsymbol{e}^2/\sqrt{3}, \boldsymbol{e}^1)$, the results are obtained by exchanging $\mathrm{SU}(2)_\mathrm{L}$ and $\mathrm{SU}(2)_\mathrm{R}$ in the above resuts. Hence we do not have $\mathrm{SU}(2)_\mathrm{L}$-doublet Higgses.

3.2 Rank-three groups

Next we consider a case of $r = 3$, i.e., G =SU(4),SO(7),Sp(6). In this case, the unbroken gauge group by the orbifold conditions is $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R} \times \mathrm{U}(1)_X \times \mathrm{U}(1)_Z$. The roots of G can be expressed as linear combinations of three-dimensional basis vectors $\boldsymbol{e}^i$ $(i = 1, 2, 3)$.

3.2.1 SU(4)

The roots are $\{\sqrt{2}\boldsymbol{e}^1,\ \sqrt{2}\boldsymbol{e}^2,\ \pm\frac{\boldsymbol{e}^1}{\sqrt{2}} \pm \frac{\boldsymbol{e}^2}{\sqrt{2}} + \boldsymbol{e}^3\}$.[2] We can choose the $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}$ subgroup as

$$(\alpha_L, \alpha_R) = (\sqrt{2}\boldsymbol{e}^1, \sqrt{2}\boldsymbol{e}^2). \tag{3.12}$$

[2] It is sometimes convenient to embed these roots into a four-dimensional vector space. Then they are expressed as $\hat{\boldsymbol{e}}^I - \hat{\boldsymbol{e}}^J$ $(1 \le I \neq J \le 4)$, where $\hat{\boldsymbol{e}}^I$ are the basis vectors of the embeded space. The original basis vectors are expressed as $\boldsymbol{e}^1 = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{e}}^1 - \hat{\boldsymbol{e}}^2)$, $\boldsymbol{e}^2 = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{e}}^3 - \hat{\boldsymbol{e}}^4)$ and $\boldsymbol{e}^3 = \frac{1}{2}(\hat{\boldsymbol{e}}^1 + \hat{\boldsymbol{e}}^2 - \hat{\boldsymbol{e}}^3 - \hat{\boldsymbol{e}}^4)$.

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The other choices are essentially equivalent to this case. The $\mathrm{U}(1)_X$ generator Q_X is identified as

$$Q_X = 2\boldsymbol{e}_3 \cdot H = 2H_3. \tag{3.13}$$

The irreducible decomposition of the adjoint representation of G is

$$\mathbf{15} = \mathbf{(3,1)_0} + \mathbf{(1,3)_0} + \mathbf{(2,2)_{+2}} + \mathbf{(2,2)_{-2}} + \mathbf{(1,1)_0}, \tag{3.14}$$

where $\mathbf{(3,1)_0}$, $\mathbf{(1,3)_0}$ and $\mathbf{(1,1)_0}$ correspond to $\mathrm{SU}(2)_{\mathrm{L}}$, $\mathrm{SU}(2)_{\mathrm{R}}$ and $\mathrm{U}(1)_X$ generators, respectively. Thus the candidates for the Higgs fields are two bidoublets. The conditions in (2.12) become

$$\begin{aligned} &\sqrt{2}p_1 = \sqrt{2}p_2 = 0, && (\mathrm{mod}\ 2\pi) \\ &\frac{p_1}{\sqrt{2}} + \frac{p_2}{\sqrt{2}} + p_3 = \frac{2n_P\pi}{N}, && (n_P \in \mathbb{Z},\ \ n_P \notin N\mathbb{Z})\,. \end{aligned} \tag{3.15}$$

Solutions are

$$P = \exp\left(\frac{2n_P\pi i}{N} H_3\right), \tag{3.16}$$

where $n_P = 1, \cdots, N-1$. Therefore the zero-mode conditions (2.13) for $\mathbf{(2,2)_{\pm 2}}$ are

$$\pm\frac{2n_P\pi}{N} = \frac{2\pi}{N}. \quad (\mathrm{mod}\ 2\pi) \tag{3.17}$$

Namely, the scalar zero-modes we have are

$$\begin{aligned} \mathbf{(2,2)_{+2}},\ \mathbf{(2,2)_{-2}} &: (\text{when } N = 2) \\ \mathbf{(2,2)_{+2}} &: (\text{when } N = 3,4,6 \text{ and } n_P = 1) \\ \mathbf{(2,2)_{-2}} &: (\text{when } N = 3,4,6 \text{ and } n_P = N-1) \\ \text{Nothing} &: (\text{in the other cases})\,. \end{aligned} \tag{3.18}$$

3.2.2 SO(7)

The roots are $\{\pm\boldsymbol{e}^i \pm \boldsymbol{e}^j, \pm\boldsymbol{e}^i\}$ $(1 \le i \ne j \le 3)$. Essentially inequivalent choices of the $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}}$ subgroup are

$$(\alpha_L, \alpha_R) = (\boldsymbol{e}^1 + \boldsymbol{e}^2, \boldsymbol{e}^1 - \boldsymbol{e}^2),\ \ (\boldsymbol{e}^1 + \boldsymbol{e}^2, \boldsymbol{e}^3),\ \ (\boldsymbol{e}^3, \boldsymbol{e}^1 + \boldsymbol{e}^2). \tag{3.19}$$

(I) $\boldsymbol{(\alpha_L, \alpha_R) = (e^1 + e^2, e^1 - e^2)}$

The $\mathrm{U}(1)_X$ generator is

$$Q_X = \boldsymbol{e}^3 \cdot H = H_3. \tag{3.20}$$

The irreducible decomposition of the adjoint representation of G is

$$\begin{aligned} \mathbf{21} = &\mathbf{(3,1)_0} + \mathbf{(1,3)_0} + \mathbf{(2,2)_{+1}} + \mathbf{(2,2)_{-1}} + \mathbf{(2,2)_0} \\ &+\mathbf{(1,1)_{+1}} + \mathbf{(1,1)_{-1}} + \mathbf{(1,1)_0}, \end{aligned} \tag{3.21}$$

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where $\mathbf{(3,1)_0}$, $\mathbf{(1,3)_0}$ and $\mathbf{(1,1)_0}$ correspond to $\mathrm{SU(2)_L}$, $\mathrm{SU(2)_R}$ and $\mathrm{U(1)_X}$ generators, respectively. Thus candidates for the scalar zero-modes are three bidoublets and two singlets. Independent conditions in (2.12) are expressed as

$$\begin{aligned} p_1+p_2=p_1-p_2=0, \qquad & (\mathrm{mod}\ 2\pi) \\ p_1+p_3, p_1, p_3=\frac{2n_P\pi}{N}. \qquad & (n_P\in\mathbb{Z}, \quad n_P\notin N\mathbb{Z}) \end{aligned} \tag{3.22}$$

Solutions exist only when $N=4,6$, and they are

$$P=\exp\left\{i\pi\left(H_1+H_2+\frac{2n_P}{N}H_3\right)\right\}, \tag{3.23}$$

where $n_P\neq 0, N/2$. Therefore the zero-mode conditions (2.13) for $\mathbf{(2,2)_{\pm 1}}$, $\mathbf{(2,2)_0}$ and $\mathbf{(1,1)_{\pm 1}}$ are

$$\pi\pm\frac{2n_P\pi}{N}=\frac{2\pi}{N}, \qquad \pi=\frac{2\pi}{N}, \qquad \pm\frac{2n_P\pi}{N}=\frac{2\pi}{N}, \tag{3.24}$$

respectively. Here the double signs correspond.

When $N=4$, the scalar zero-modes we have are

$$\begin{aligned} &\mathbf{(2,2)_{-1}},\ \mathbf{(1,1)_{+1}} : (\text{when } n_P=1) \\ &\mathbf{(2,2)_{+1}},\ \mathbf{(1,1)_{-1}} : (\text{when } n_P=3) \end{aligned} \tag{3.25}$$

When $N=6$, they are

$$\begin{aligned} &\mathbf{(1,1)_{+1}} : (\text{when } n_P=1) \\ &\mathbf{(2,2)_{-1}} : (\text{when } n_P=2) \\ &\mathbf{(2,2)_{+1}} : (\text{when } n_P=4) \\ &\mathbf{(1,1)_{-1}} : (\text{when } n_P=5) \end{aligned} \tag{3.26}$$

(II) $\boldsymbol{(\alpha_L,\alpha_R)=(e^1+e^2,e^3)}$

The $\mathrm{U(1)}_X$ generator is

$$Q_X=(e^1-e^2)\cdot H=H_1-H_2. \tag{3.27}$$

The irreducible decomposition of the adjoint representation of G is

$$\begin{aligned} \mathbf{21}=&\mathbf{(3,1)_0}+\mathbf{(1,3)_0}+\mathbf{(2,3)_{+1}}+\mathbf{(2,3)_{-1}} \\ &+\mathbf{(1,1)_{+2}}+\mathbf{(1,1)_{-2}}+\mathbf{(1,1)_0}. \end{aligned} \tag{3.28}$$

Candidates for the scalar zero-modes are $\mathbf{(2,3)_{\pm 1}}$ and $\mathbf{(1,1)_{\pm 1}}$. Independent conditions in (2.12) are expressed as

$$\begin{aligned} p_1+p_2=p_3=0, \qquad & (\mathrm{mod}\ 2\pi) \\ p_1+p_3, p_2+p_3, p_1-p_2=\frac{2n_P\pi}{N}. \qquad & (n_P\in\mathbb{Z}, \quad n_P\notin N\mathbb{Z}) \end{aligned} \tag{3.29}$$

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Solutions exist when $N = 3, 4, 6$, and they are

$$P = \exp\left\{\frac{2n_P\pi i}{N}(H_1 - H_2)\right\}, \tag{3.30}$$

where $n_P \neq 0, N/2$. Therefore the zero-mode conditions (2.13) for $\mathbf{(2,3)_{\pm 1}}$ and $\mathbf{(1,1)_{\pm 1}}$ are

$$\pm\frac{2n_P\pi}{N} = \frac{2\pi}{N}, \qquad \pm\frac{4n_P\pi}{N} = \frac{2\pi}{N}, \tag{3.31}$$

respectively.

When $N = 3$, the scalar zero-modes we have are

$$\begin{aligned} &\mathbf{(2,3)_{+1}},\ \ \mathbf{(1,1)_{-2}} : (\text{when } n_P = 1) \\ &\mathbf{(2,3)_{-1}},\ \ \mathbf{(1,1)_{+2}} : (\text{when } n_P = 2) \end{aligned} \tag{3.32}$$

When $N = 4$, they are

$$\begin{aligned} &\mathbf{(2,3)_{+1}} : (\text{when } n_P = 1) \\ &\mathbf{(2,3)_{-1}} : (\text{when } n_P = 3) \end{aligned} \tag{3.33}$$

When $N = 6$, they are

$$\begin{aligned} &\mathbf{(2,3)_{+1}} : (\text{when } n_P = 1) \\ &\text{Nothing} : (\text{when } n_P = 2, 4) \\ &\mathbf{(2,3)_{-1}} : (\text{when } n_P = 5) \end{aligned} \tag{3.34}$$

(III) $\boldsymbol{(\alpha_L, \alpha_R) = (e^3, e^1 + e^2)}$

The results are obtained by exchanging $\mathrm{SU}(2)_{\mathrm{L}}$ and $\mathrm{SU}(2)_{\mathrm{R}}$ in the case (II). Hence we do not have $\mathrm{SU}(2)_{\mathrm{L}}$-doublet Higgses.

3.2.3 Sp(6)

The roots are $\{\pm\boldsymbol{e}^i \pm \boldsymbol{e}^j, \pm 2\boldsymbol{e}^i\}$ $(1 \leq i \neq j \leq 3)$. Essentially inequivalent choices of the $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}}$ are

$$(\alpha_L, \alpha_R) = (2\boldsymbol{e}^1, 2\boldsymbol{e}^2),\ \ (\boldsymbol{e}^1 + \boldsymbol{e}^2, 2\boldsymbol{e}^3),\ \ (2\boldsymbol{e}^3, \boldsymbol{e}^1 + \boldsymbol{e}^2). \tag{3.35}$$

(I) $\boldsymbol{(\alpha_L, \alpha_R) = (2e^1, 2e^2)}$

The $\mathrm{U}(1)_X$ generator is

$$Q_X = \boldsymbol{e}^3 \cdot H = H_3. \tag{3.36}$$

The irreducible decomposition of the adjoint representation of G is

$$\begin{aligned} \mathbf{21} = &\mathbf{(3,1)_0} + \mathbf{(1,3)_0} + \mathbf{(2,2)_0} + \mathbf{(2,1)_{+1}} + \mathbf{(2,1)_{-1}} \\ &+\mathbf{(1,2)_{+1}} + \mathbf{(1,2)_{-1}} + \mathbf{(1,1)_{+2}} + \mathbf{(1,1)_{-2}} + \mathbf{(1,1)_0}. \end{aligned} \tag{3.37}$$

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Independent conditions in (2.12) are expressed as

$$\begin{aligned} 2p_1 = 2p_2 = 0, \qquad & (\text{mod } 2\pi) \\ p_1+p_2, p_1 \pm p_3, p_2 \pm p_3, 2p_3 = \frac{2n_P\pi}{N}, \qquad & (n_P \in \mathbb{Z}, \ \ n_P \notin N\mathbb{Z}) \end{aligned} \tag{3.38}$$

Solutions exist only when $N = 4, 6$. They are

$$P = \begin{cases} P^{(1)}_{n_P} \equiv \exp\left\{ i\pi \left(H_2 + \dfrac{2n_P\pi}{N} H_3 \right) \right\}, \\ P^{(2)}_{n_P} \equiv \exp\left\{ i\pi \left(H_1 + \dfrac{2n_P\pi}{N} H_3 \right) \right\}, \end{cases} \tag{3.39}$$

where $n_P \neq 0, N/2$.

When $N = 4$, the scalar zero-modes we have are

$$\begin{aligned} &\mathbf{(2,1)_{+1}}, \ \ \mathbf{(1,2)_{-1}} : (\text{for } P^{(1)}_1 \text{ or } P^{(2)}_3) \\ &\mathbf{(2,1)_{-1}}, \ \ \mathbf{(1,2)_{+1}} : (\text{for } P^{(1)}_3 \text{ or } P^{(2)}_1) \end{aligned} \tag{3.40}$$

When $N = 6$, they are

$$\begin{aligned} &\mathbf{(2,1)_{+1}} : (\text{for } P^{(1)}_1 \text{ or } P^{(2)}_4) \\ &\mathbf{(1,2)_{-1}} : (\text{for } P^{(1)}_2 \text{ or } P^{(2)}_5) \\ &\mathbf{(1,2)_{+1}} : (\text{for } P^{(1)}_4 \text{ or } P^{(2)}_1) \\ &\mathbf{(2,1)_{-1}} : (\text{for } P^{(1)}_5 \text{ or } P^{(2)}_2) \end{aligned} \tag{3.41}$$

(II) $\boldsymbol{(\alpha_L, \alpha_R) = (e^1 + e^2, 2e^3)}$

The $\mathrm{U}(1)_X$ generator is

$$Q_X = \left(e^1 - e^2\right) \cdot H = H_1 - H_2. \tag{3.42}$$

The irreducible decomposition of the adjoint representation of G is

$$\mathbf{21} = \mathbf{(3,1)_0} + \mathbf{(1,3)_0} + \mathbf{(3,1)_{+2}} + \mathbf{(3,1)_{-2}} + \mathbf{(2,2)_{+1}} + \mathbf{(2,2)_{-1}} + \mathbf{(1,1)_0}. \tag{3.43}$$

Independent conditions in (2.12) are expressed as

$$\begin{aligned} p_1 + p_2 = 2p_3 = 0, \qquad & (\text{mod } 2\pi) \\ p_1+p_3, p_2 + p_3, 2p_1, 2p_2 = \frac{2n_P\pi}{N}, \qquad & (n_P \in \mathbb{Z}, \ \ n_P \notin N\mathbb{Z}) \end{aligned} \tag{3.44}$$

where $n_P \neq 0, N/2$. Solutions exist only when $N = 3, 4, 6$. They are

$$P = \begin{cases} P^{(1)}_{n_P} \equiv \exp\left\{ \dfrac{2n_P\pi i}{N} (H_1 - H_2) \right\}, \\ P^{(2)}_{n_P} \equiv \exp\left\{ i\pi \left(\dfrac{2n_P - N}{N} (H_1 - H_2) + H_3 \right) \right\}. \end{cases} \tag{3.45}$$

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When $N = 3$, the scalar zero-modes we have are

$$\begin{aligned} &\mathbf{(3,1)_{-2}},\ \mathbf{(2,2)_{+1}} : (\text{for } P_1^{(1)} \text{ or } P_1^{(2)}) \\ &\mathbf{(3,1)_{+2}},\ \mathbf{(2,2)_{-1}} : (\text{for } P_2^{(1)} \text{ or } P_2^{(2)}) \end{aligned} \tag{3.46}$$

When $N = 4$, they are

$$\begin{aligned} &\mathbf{(2,2)_{+1}} : (\text{for } P_1^{(1)} \text{ or } P_1^{(2)}) \\ &\mathbf{(2,2)_{-1}} : (\text{for } P_3^{(1)} \text{ or } P_3^{(2)}) \end{aligned} \tag{3.47}$$

When $N = 6$, they are

$$\begin{aligned} &\mathbf{(2,2)_{+1}} : (\text{for } P_1^{(1)} \text{ or } P_1^{(2)}) \\ &\mathbf{(2,2)_{-1}} : (\text{for } P_5^{(1)} \text{ or } P_5^{(2)}) \\ &\text{Nothing} : (\text{in the other cases}) \end{aligned} \tag{3.48}$$

(III) $(\boldsymbol{\alpha_L}, \boldsymbol{\alpha_R}) = (\mathbf{2}\boldsymbol{e^3}, \boldsymbol{e^1} + \boldsymbol{e^2})$

The results are obtained by exchanging $\mathrm{SU}(2)_\mathrm{L}$ and $\mathrm{SU}(2)_\mathrm{R}$ in the case (II).

4 Custodial symmetry and Weinberg angle

4.1 Custodial symmetry

Here we consider a condition that the custodial symmetry is preserved after the electroweak symmetry is broken. The $\mathrm{SU}(2)_\mathrm{L}$ and $\mathrm{SU}(2)_\mathrm{R}$ generators are

$$(T_L^\pm, T_L^3) = \left(\frac{E_{\pm\alpha_L}}{|\alpha_L|}, \frac{\alpha_L \cdot H}{|\alpha_L|^2}\right), \qquad (T_R^\pm, T_R^3) = \left(\frac{E_{\pm\alpha_R}}{|\alpha_R|}, \frac{\alpha_R \cdot H}{|\alpha_R|^2}\right), \tag{4.1}$$

respectively. Thus (2.6) is rewritten as

$$\begin{aligned} A_\mu = {} & W_{L\mu}^+ T_L^+ + W_{L\mu}^- T_L^- + W_{L\mu}^3 T_L^3 + W_{R\mu}^+ T_R^+ + W_{R\mu}^- T_R^- + W_{R\mu}^3 T_R^3 \\ & + B_\mu^X \mathrm{x} \cdot H + \cdots, \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} W_{L\mu}^\pm &\equiv |\alpha_L|\, W_\mu^{\pm\alpha_L}, \qquad W_{L\mu}^3 \equiv \alpha_L \cdot C_\mu, \\ W_{R\mu}^\pm &\equiv |\alpha_R|\, W^{\pm\alpha_R}, \qquad W_{R\mu}^3 \equiv \alpha_R \cdot C_\mu, \end{aligned} \tag{4.3}$$

and $B_\mu^X \equiv \frac{\mathrm{x}\cdot C_\mu}{|\mathrm{x}|^2}$ is the $\mathrm{U}(1)_X$ gauge field that does not exist when $r = 2$. The ellipsis denotes components that do not have zero-modes. Since the generators in (4.1) are normalized as

$$\mathrm{tr}\left(T_L^+ T_L^-\right) = \mathrm{tr}\left((T_L^3)^2\right) = \frac{1}{|\alpha_L|^2}, \qquad \mathrm{tr}\left(T_R^+ T_R^-\right) = \mathrm{tr}\left((T_R^3)^2\right) = \frac{1}{|\alpha_R|^2}, \tag{4.4}$$

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the canonically normalized zero-mode gauge fields are

$$\hat{W}^{\pm,3}_{L\mu} \equiv \frac{\sqrt{\mathcal{A}}}{g_A\,|\alpha_L|} W^{\pm,3}_{L\mu}, \qquad \hat{W}^{\pm,3}_{R\mu} \equiv \frac{\sqrt{\mathcal{A}}}{g_A\,|\alpha_R|} W^{\pm,3}_{R\mu}, \qquad \hat{B}^{Z}_{\mu} \equiv \frac{\sqrt{\mathcal{A}}}{g_Z} B^{Z}_{\mu}, \tag{4.5}$$

where $\mathcal{A}$ is the area of the fundamental domain of T^2/Z_N.

Since we have assumed that $\mathrm{SU}(2)_\mathrm{R} \times \mathrm{U}(1)_Z$ is unbroken by the orbifold boundary conditions, we introduce some 4D scalar fields at one of the fixed points of T^2/Z_N in order to break it to $\mathrm{U}(1)_Y$. We demand that the custodial symmetry $\mathrm{SU}(2)_V \subset \mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}$ remains unbroken after the Higgs fields have VEVs. The generators of $\mathrm{SU}(2)_V$ are

$$\begin{aligned} T^{\pm}_V &\equiv T^{\pm}_L + T^{\pm}_R = \frac{E_{\pm\alpha_L}}{|\alpha_L|} + \frac{E_{\pm\alpha_R}}{|\alpha_R|}, \\ T^{3}_V &\equiv T^{3}_L + T^{3}_R = \frac{\alpha_L \cdot H}{|\alpha_L|^2} + \frac{\alpha_R \cdot H}{|\alpha_R|^2}. \end{aligned} \tag{4.6}$$

Thus the conditions for $\mathrm{SU}(2)_V$ to be unbroken are

$$\begin{aligned} \left[T^{\pm}_V, \langle A_z \rangle\right] &= \sum_\beta \langle W^\beta_z \rangle \left(\frac{N_{\pm\alpha_L,\beta} E_{\beta\pm\alpha_L}}{|\alpha_L|} + \frac{N_{\pm\alpha_R,\beta} E_{\beta\pm\alpha_R}}{|\alpha_R|} \right) = 0, \\ \left[T^{3}_V, \langle A_z \rangle\right] &= \sum_\beta \langle W^\beta_z \rangle \left(\frac{\alpha_L \cdot \beta}{|\alpha_L|^2} + \frac{\alpha_R \cdot \beta}{|\alpha_R|^2} \right) E_\beta = 0, \end{aligned} \tag{4.7}$$

since C^i_z do not have zero-modes and thus $\langle C^i_z \rangle = 0$.

4.1.1 Rank-two groups

Let us first consider the rank-two groups. We introduce the following Lagrangian at $z = 0$.[3]

$$\mathcal{L}_\mathrm{loc} = \left\{ -\mathcal{D}_\mu \phi^\dagger \mathcal{D}^\mu \phi - V(\phi) \right\} \delta(z), \tag{4.8}$$

where ϕ is a complex scalar field belonging to $\mathbf{(1,2)_{+1/2}}$ under $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R} \times \mathrm{U}(1)_Z$, and $V(\phi)$ is a potential that force ϕ to have a nonvanishing VEV. After ϕ gets a VEV, $\mathrm{SU}(2)_\mathrm{R} \times \mathrm{U}(1)_Z$ is broken to $\mathrm{U}(1)_Y$, and the corresponding massless gauge field is expressed as

$$\hat{B}^Y_\mu \equiv \sin\theta_Z \hat{W}^3_{R\mu} + \cos\theta_Z \hat{B}^Z_\mu, \tag{4.9}$$

where a mixing angle θ_Z is determined by $\tan\theta_Z = g_Z/(g_A\,|\alpha_R|)$. The hypercharge operator Y is identified as

$$Y = T^3_R + Q_Z = \frac{\alpha_R \cdot H}{|\alpha_R|^2} + Q_Z. \tag{4.10}$$

After W^β_z have nonvanishing VEVs, $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_Y$ is broken to the electromagnetic symmetry $\mathrm{U}(1)_\mathrm{em}$. Since W^β_z is $\mathrm{U}(1)_Z$ neutral and only $U(1)_\mathrm{em}$ neutral W^β_z can have nonvanishing VEVs, the root β must satisfy

$$\frac{\alpha_L \cdot \beta}{|\alpha_L|^2} + \frac{\alpha_R \cdot \beta}{|\alpha_R|^2} = 0, \tag{4.11}$$

[3]Of course, $\mathcal{L}_\mathrm{loc}$ can be localized at other fixed point.

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if $\langle W_z^\beta \rangle \neq 0$. Thus the second condition in (4.7) is automatically satisfied. The roots that satisfy (4.11) are $\pm \boldsymbol{e}^2 \in \mathbf{(2,2)}$ in SO(5), and $\pm\left(\frac{\boldsymbol{e}^1}{2} - \frac{\boldsymbol{e}^2}{2\sqrt{3}}\right) \in \mathbf{(2,4)}$ in G_2. Then, from the first condition in (4.7), we obtain a condition,

$$\left|\langle W_z^{e^2}\rangle\right| = \left|\langle W_z^{-e^2}\rangle\right|, \qquad \langle W_z^\beta \rangle = 0, \quad (\beta \neq \pm \boldsymbol{e}^2) \tag{4.12}$$

for SO(5), while no nonvanishing VEV is allowed for G_2.

4.1.2 Rank-three groups

Next consider the rank-three groups. Since the unbroken gauge symmetry by the orbifold conditions is $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R} \times \mathrm{U}(1)_X \times \mathrm{U}(1)_Z$, let us first assume that ϕ in (4.8) also has a nonzero $\mathrm{U}(1)_X$ charge in order to obtain $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_Y$ at low energies. Then the $\mathrm{U}(1)_Y$ gauge field B_μ^Y becomes a linear combination of $W_{R\mu}^3$, B_μ^X and B_μ^Z, and the hypercharge is identified as

$$Y = T_R^3 + Q_X + Q_Z = \frac{\alpha_R \cdot H}{|\alpha_R|^2} + \mathrm{x} \cdot H + Q_Z. \tag{4.13}$$

Thus the condition (4.11) now becomes

$$\frac{\alpha_L \cdot \beta}{|\alpha_L|^2} + \frac{\alpha_R \cdot \beta}{|\alpha_R|^2} + \mathrm{x} \cdot \beta = 0. \tag{4.14}$$

From this and the second condition in (4.7), both (4.11) and $\mathrm{x} \cdot \beta = 0$ must be satisfied if $\langle W_z^\beta \rangle \neq 0$. Such roots do not exist among the zero-modes listed in section 3.2. Therefore we introduce two complex scalar fields ϕ_1 and ϕ_2 instead of ϕ on the fixed point,

$$\mathcal{L}_\mathrm{loc} = \left\{ -\mathcal{D}_\mu \phi_1^\dagger \mathcal{D}^\mu \phi_1 - \mathcal{D}_\mu \phi_2^\dagger \mathcal{D}^\mu \phi_2 - V(\phi_1, \phi_2) \right\} \delta(z), \tag{4.15}$$

where ϕ_1 and ϕ_2 are complex scalars belonging to $\mathbf{(1,2)_{0,+1/2}}$ and $\mathbf{(1,1)_{+1,0}}$ respectively under $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R} \times \mathrm{U}(1)_X \times \mathrm{U}(1)_Z$, and $V(\phi_1, \phi_2)$ is a potential for them. Since ϕ_1 is neutral for $\mathrm{U}(1)_X$, the $\mathrm{U}(1)_Y$ gauge field B_μ^Y is now independent of B_μ^X. Hence the hypercharge is identified as (4.10). The $\mathrm{U}(1)_X$ charges are no longer relevant to the $\mathrm{U}(1)_Y$ and $\mathrm{U}(1)_\mathrm{em}$ charges because $\mathrm{U}(1)_X$ is completely broken by a VEV of another scalar ϕ_2. Thus the $\mathrm{U}(1)_Y$ gauge field is given by (4.9). In this case, the $\mathrm{U}(1)_\mathrm{em}$ neutral condition becomes (4.11), which is consistent with the second condition in (4.7). As a result, possible nonvanishing VEVs are as follows.

$$\begin{aligned}
&\left|\langle W_z^{\pm(e^1-e^3)}\rangle\right| = \left|\langle W_z^{\pm(e^2-e^4)}\rangle\right| \in \mathbf{(2,2)_{\pm 2}} && \text{in SU(4)},\\
&\left|\langle W_z^{\pm(e^2+e^3)}\rangle\right| = \left|\langle W_z^{\pm(-e^2+e^3)}\rangle\right| \in \mathbf{(2,2)_{\pm 1}}, \quad \left|\langle W_z^{\pm e^3}\rangle\right| \in \mathbf{(1,1)_{\pm 1}} && \text{in SO(7) (I)},\\
&\left|\langle W_z^{\pm(e^1-e^3)}\rangle\right| = \left|\langle W_z^{\pm(-e^2+e^3)}\rangle\right| \in \mathbf{(2,2)_{\pm 1}} && \text{in Sp(6) (II), Sp(6) (III)},
\end{aligned} \tag{4.16}$$

where the double signs correspond.

In summary, fields that can have nonzero VEVs are the neutral components of a bidoublet $\mathbf{(2,2)}$ or a singlet $\mathbf{(1,1)}$. The above conditions indicate that a bidoublet $\mathcal{H}_a$ must have a VEV:

$$\langle \mathcal{H}_a \rangle = \frac{1}{2} \begin{pmatrix} v_a & \\ & v_a \end{pmatrix}, \tag{4.17}$$

where $v_a > 0$, if we redefine a phase of each field component appropriately.

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4.2 Weinberg angle and weak gauge boson masses

In the approximation that the W and Z bosons have constant profiles over the extra dimensions, the 4D $\mathrm{SU}(2)_{\mathrm{L}}$ and $\mathrm{U}(1)_Y$ gauge coupling constants are read off from couplings to the matter zero-modes, and are identified as

$$g = \frac{g_A\,|\alpha_L|}{\sqrt{\mathcal{A}}}, \qquad g' = \frac{g_A g_Z\,|\alpha_R|}{\sqrt{\mathcal{A}(g_A^2\,|\alpha_R|^2 + g_Z^2)}}. \tag{4.18}$$

Thus the Weinberg angle is calculated as

$$\tan^2\theta_W \equiv \frac{g'}{g} = \frac{g_Z^2\,|\alpha_R|^2}{|\alpha_L|^2\,(g_A^2\,|\alpha_R|^2 + g_Z^2)}. \tag{4.19}$$

We can obtain the experimental value $\tan^2\theta_W \simeq 0.30$ by tuning the ratio g_Z/g_A.

Next we derive the expressions of the W and Z boson masses. From (4.5) and (4.9), the expression (4.2) becomes

$$A_\mu = W^+_{L\mu}T^+_L + W^-_{L\mu}T^-_L + W^3_{L\mu}T^3_L + \sin\theta_Z B^Y_\mu T^3_R + \cdots, \tag{4.20}$$

where $B^Y_\mu \equiv \frac{g_A|\alpha_R|}{\sqrt{\mathcal{A}}}\hat{B}^Y_\mu$, after the breaking $\mathrm{SU}(2)_{\mathrm{R}} \times \mathrm{U}(1)_Z \to \mathrm{U}(1)_Y$. Then it follows that

$$\begin{aligned}[A_\mu, \langle A_z\rangle] = \sum_\beta W_z^\beta \Bigg\{ & W^+_{L,\mu}\frac{N_{\alpha_L,\beta}}{|\alpha_L|}E_{\beta+\alpha_L} + W^-_{L\mu}\frac{N_{-\alpha_L,\beta}}{|\alpha_L|}E_{\beta-\alpha_L} \\ & + \left(W^3_{L\mu}\frac{\alpha_L\cdot\beta}{|\alpha_L|^2} + B^Y_\mu \sin\theta_Z \frac{\alpha_R\cdot\beta}{|\alpha_R|^2}\right)E_\beta\Bigg\}.\end{aligned} \tag{4.21}$$

From the results in the previous subsections, the only components that contribute to the W and Z boson masses are the neutral components of the bidoublets. Since the roots that form a bidoublet are expressed as

$$\begin{pmatrix} \gamma_a + \alpha_L & \xrightarrow{\alpha_R} & \gamma_a + \alpha_L + \alpha_R \\ \uparrow_{\alpha_L} & & \uparrow_{\alpha_L} \\ \gamma_a & \xrightarrow{\alpha_R} & \gamma_a + \alpha_R \end{pmatrix}, \tag{4.22}$$

where a labels the bidoublets, (4.21) are rewritten as

$$\begin{aligned}[A_\mu, \langle A_z\rangle] = \sum_a \Bigg[& \langle W_z^{\gamma_a+\alpha_L}\rangle \left\{\frac{e^{i\zeta}}{\sqrt{2}}W^-_{L\mu}E_{\gamma_a} + \left(\frac{1}{2}W^3_{L\mu} - \frac{\sin\theta_Z}{2}B^Y_\mu\right)E_{\gamma_a+\alpha_L}\right\} \\ & + \langle W_z^{\gamma_a+\alpha_R}\rangle \left\{\frac{e^{i\eta}}{\sqrt{2}}W^+_{L\mu}E_{\gamma_a+\alpha_L+\alpha_R} - \left(\frac{1}{2}W^3_{L\mu} - \frac{\sin\theta_Z}{2}B^Y_\mu\right)E_{\gamma_a+\alpha_R}\right\}\Bigg],\end{aligned} \tag{4.23}$$

where γ_a is the $T^3_L = T^3_R = -1/2$ component of the zero-mode bidoublets $\mathcal{H}_a$. We have used that $|N_{-\alpha_L,\gamma_a+\alpha_L}|^2 = |N_{\alpha_L,\gamma_a+\alpha_R}|^2 = |\alpha_L|^2/2$, and $\zeta \equiv \arg(N_{-\alpha_L,\gamma_a+\alpha_L})$ and $\eta \equiv$

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$\arg(N_{\alpha_L,\gamma_a+\alpha_R})$. Thus the relevant terms in 6D Lagrangian are calculated as

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4g_A^2}\mathrm{tr}(F^{(A)MN}F^{(A)}_{MN}) + \cdots = -\frac{1}{2g_A^2\pi^2R_1^2}\mathrm{tr}\left([A^\mu,\langle A_z\rangle]\,[A_\mu,\langle A_z\rangle]^\dagger\right) + \cdots \\ &= -\sum_a \frac{\left|\langle W_z^{\gamma_a+\alpha_L}\rangle\right|^2 + \left|\langle W_z^{\gamma_a+\alpha_R}\rangle\right|^2}{2g_A^2\pi^2R_1^2}\left\{\frac{1}{2}W_L^{+\mu}W^-_{L\mu} + \left(\frac{1}{2}W^3_{L\mu} - \frac{\sin\theta_Z}{2}B^Y_\mu\right)^2\right\} + \cdots \\ &= -\frac{g^2\sum_a v_a^2}{4\mathcal{A}}\left\{\hat{W}_L^{+\mu}\hat{W}^-_{L\mu} + \frac{1}{2}\left(\hat{W}^3_{L\mu} - \frac{|\alpha_R|\sin\theta_Z}{|\alpha_L|}\hat{B}^Y_\mu\right)^2\right\} + \cdots . \end{aligned} \tag{4.24}$$

At the last step, we have used that (4.5), and $\left|\langle W_z^{\gamma_a+\alpha_L}\rangle\right| = \left|\langle W_z^{\gamma_a+\alpha_R}\rangle\right| \equiv g\pi R_1 v_a/\sqrt{2}\,|\alpha_L|$ (g: 4D $\mathrm{SU(2)_L}$ gauge coupling), which follows from (4.12) or (4.16). We obtain the W and Z boson mass terms by integrating (4.24) over the extra dimensions, and their masses are read off as

$$\begin{aligned} m_W &= \frac{g}{2}\sqrt{\sum_a v_a^2}, \\ m_Z &= \left(1 + \frac{|\alpha_R|^2\sin^2\theta_Z}{|\alpha_L|^2}\right)^{1/2} m_W = \left(1 + \frac{g_Z^2\,|\alpha_R|^2}{|\alpha_L|^2\,(g_A^2\,|\alpha_R|^2 + g_Z^2)}\right)^{1/2} m_W. \end{aligned} \tag{4.25}$$

From these and (4.19), we find that $\rho \equiv m_W^2/(m_Z^2\cos^2\theta_W) = 1$. This is expected because we have assumed that only $\mathrm{SU(2)_L}$ doublets and singlets have nonzero VEVs and neglected the z-dependence of the mode functions for the W and Z bosons. The custodial symmetry will play a crucial role when such z-dependence is taken into account.

5 Matter field

We consider a case that quarks and leptons live in the bulk. This case is interesting because the hierarchical structure of the Yukawa coupling constants can be realized by the wave function localization [24, 25], and the generation structure can also be obtained by a background magnetic flux [22]. In the following, we focus on the quark sector, but a similar argument is also applicable to the lepton sector.

5.1 Zero-mode condition

A 6D Weyl fermion Ψ_{χ_6} with the 6D chirality $\chi_6 = \pm$ is decomposed as

$$\Psi_{\chi_6} = \sum_{\chi_4=\pm}\Psi_{\chi_6,\chi_4}, \tag{5.1}$$

where $\chi_4 = +(R), -(L)$ is the 4D chirality. The orbifold boundary conditions for Ψ_{χ_6,χ_4} are given by [5]

$$\begin{aligned} \Psi_{\chi_6,\chi_4}(x,z+1) &= \Psi_{\chi_6,\chi_4}(x,z), \\ \Psi_{\chi_6,\chi_4}(x,z+\tau) &= \Psi_{\chi_6,\chi_4}(x,z), \\ \Psi_{\chi_6,\chi_4}(x,\omega z) &= \omega^{-\frac{\chi_4\chi_6}{2}}e^{i\varphi_\omega}P\Psi_{\chi_6,\chi_4}(x,z). \end{aligned} \tag{5.2}$$

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A factor $\omega^{-\frac{\chi_4\chi_6}{2}}$ appears because a 6D spinor is charged under a rotation in the extra-dimensional space. The phase φ_ω satisfies (B.4).

As pointed out in ref. [26], the generations and the hierarchy among the Yukawa couplings can be obtained by introducing an extra gauge symmetry G_F and assuming a magnetic flux on T^2/Z_N and the Wilson line phases for G_F. The zero-modes are contained in Ψ_{χ_6,χ_4} as

$$\Psi_{\chi_6,\chi_4}(x,z) = \sum_{j=1}^{j_{\max}} \sum_{\mu} f_{\chi_6}^{(j)\mu}(z)|\mu\rangle \psi_{\chi_4}^{(j)\mu}(x) + \cdots , \tag{5.3}$$

where μ runs over the weights of the zero-mode states,[4] and the ellipsis denotes the nonzero KK modes. The number of the zero-modes $j_{\max}$ is determined by the magnetic flux [22]. The zero-mode functions $f_{\chi_6}^{(j)\mu}(z)$ are determined so that (5.3) satisfies the first two conditions in (5.2). From the last condition in (5.2), we obtain

$$\psi_{\chi_4}^{(j)\mu}(x) = \omega^{-\frac{\chi_4\chi_6}{2}} e^{i\varphi_\omega} P \psi_{\chi_4}^{(j)\mu}(x). \tag{5.4}$$

Namely, the zero-mode is an eigenvector of $\omega^{-\frac{\chi_4\chi_6}{2}} e^{i\varphi_\omega} P$ with an eigenvalue 1. Denote the highest weight of a representation $\mathcal{R}$ that Ψ_{χ_4,χ_6} belongs to as $\mu_{\max}$. Then μ is expressed as

$$\mu = \mu_{\max} - \sum_i k_i \alpha_i, \tag{5.5}$$

where k_i are non-negative integers, and α_i are the simple roots. Since $P^N|\mu\rangle = e^{iNp\cdot\mu}|\mu\rangle = e^{iNp\cdot\mu_{\max}}|\mu\rangle$,[5] the phase φ_ω is determined by (B.4) as $\varphi_\omega = \frac{\pi}{N}(2m_\omega + 1) - p\cdot\mu_{\max}$, where $m_\omega = 0, 1, \cdots, N-1$. Thus we find that

$$\begin{aligned}
\omega^{-\frac{\chi_4\chi_6}{2}} e^{i\varphi_\omega} P|\mu\rangle &= e^{-\frac{2\pi i}{N}\cdot\frac{\chi_4\chi_6}{2}} \exp\left(\frac{(2m_\omega+1)\pi i}{N} - ip\cdot\mu_{\max}\right) e^{ip\cdot\mu}|\mu\rangle \\
&= \exp\left(\frac{\pi i(2m_\omega + 1 - \chi_4\chi_6)}{N} - i\sum_i k_i(p\cdot\alpha_i)\right)|\mu\rangle.
\end{aligned} \tag{5.6}$$

Namely, the zero-mode condition for the state $|\mu\rangle$ is

$$\frac{\pi(2m_\omega + 1 - \chi_4\chi_6)}{N} - \sum_i k_i(p\cdot\alpha_i) = 0. \pmod{2\pi} \tag{5.7}$$

5.2 $Zb_L\bar{b}_L$ coupling

When the quarks live in the bulk, the $Zb_L\bar{b}_L$ coupling often receives a large correction induced by mixing with the KK modes. The authors of ref. [16] pointed out that the custodial symmetry plays an important role to suppress the deviation of this coupling from the standard model value. The $Zb_L\bar{b}_L$ coupling is protected if the theory has a parity symmetry $\mathcal{P}_{\rm LR}$ that exchanges $\mathrm{SU}(2)_{\rm L}$ and $\mathrm{SU}(2)_{\rm R}$, and b_L is the component of $T_L^3 = T_R^3 = -\frac{1}{2}$ in a bidoublet $\mathbf{(2,2)}$. Since the Higgs fields also belong to $\mathbf{(2,2)}$, the right-handed quarks should belong to $\mathbf{(1,1)}$ or $\mathbf{(1,3)} + \mathbf{(3,1)}$.

[4] Do not confuse it with the 4D Lorentz index.

[5] We have used (2.11) at the second equality.

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Cases in which the bosonic sector has the parity symmetry $\mathcal{P}_{\rm LR}$ and a scalar bidoublet are SO(5), SU(4) and SO(7) (I) in section 3. In appendix C, we list the irreducible representations of these groups whose dimensions are less than 30, and their decomposition into the $\mathrm{SU}(2)_{\rm L} \times \mathrm{SU}(2)_{\rm R}(\times \mathrm{U}(1)_X)$ multiplets. There is no $\mathbf{(1,3)} + \mathbf{(3,1)}$ multiplets included in the list. Hence the left-handed and the right-handed quarks should be embedded into $\mathbf{(2,2)}$ and $\mathbf{(1,1)}$, respectively.

5.3 Yukawa couplings

5.3.1 General expression

The Yukawa couplings originate from the 6D minimal couplings in the kinetic term, $i\bar{\Psi}_{\chi_6}\Gamma^M \mathcal{D}_M \Psi_{\chi_6} = -\frac{i\chi_6}{\pi R_1}\bar{\Psi}_{\chi_6,\chi_4=\chi_6} A_z \Psi_{\chi_6,\chi_4=-\chi_6} + \text{h.c.} + \cdots$. The canonically normalized Higgs zero-mode H^β is contained in A_z as $A_z = \sum_\beta \frac{\sqrt{2}}{|\alpha_L|} g\pi R_1 H^\beta E_\beta + \cdots$, where g is the $\mathrm{SU}(2)_{\rm L}$ gauge coupling constant. (See (6.5).) Then the Yukawa couplings in 4D effective Lagrangian are expressed as

$$\mathcal{L}_{\rm yukawa} = \begin{cases} \displaystyle\sum_{i,j}\left(\sum_{\beta,\mu_L} y^{\mathcal{R}_H\mathcal{R}_L\mathcal{R}_R}_{(+)ij}\left(H^\beta\right)^* \bar{\psi}_L^{(i)\mu_L}\psi_R^{(j)\mu_L+\beta} + \text{h.c.}\right) & (\chi_6 = +) \\ \displaystyle\sum_{i,j}\left(\sum_{\beta,\mu_L} y^{\mathcal{R}_H\mathcal{R}_L\mathcal{R}_R}_{(-)ij} H^\beta \bar{\psi}_L^{(i)\mu_L}\psi_R^{(j)\mu_L-\beta} + \text{h.c.}\right) & (\chi_6 = -) \end{cases}, \tag{5.8}$$

where $\mathcal{R}_H$, $\mathcal{R}_L$ and $\mathcal{R}_R$ are irreducible representations of $\mathrm{SU}(2)_{\rm L}\times\mathrm{SU}(2)_{\rm R}(\times\mathrm{U}(1)_X)\times\mathrm{U}(1)_Z$ that $|\beta\rangle$, $|\mu_L\rangle$ and $|\mu_R\rangle = |\mu_L + \chi_6\beta\rangle$ belong to, and

$$\begin{aligned} y^{\mathcal{R}_H\mathcal{R}_L\mathcal{R}_R}_{(+)ij} &\equiv i\sqrt{2}g\langle\mu_L|E_{-\beta}|\mu_L+\beta\rangle \int d^2z\, f_{+,0}^{(i)\mu_L *}(z) f_{+,0}^{(j)\mu_L+\beta}(z), \\ y^{\mathcal{R}_H\mathcal{R}_L\mathcal{R}_R}_{(-)ij} &\equiv i\sqrt{2}g\langle\mu_L|E_{\beta}|\mu_L-\beta\rangle \int d^2z\, f_{-,0}^{(i)\mu_L *}(z) f_{-,0}^{(j)\mu_L-\beta}(z). \end{aligned} \tag{5.9}$$

Note that these coupling constants only depend on the representations $\{\mathcal{R}_H, \mathcal{R}_L, \mathcal{R}_R\}$, and take common values for all $\beta \in \mathcal{R}_H$ and $\mu_L \in \mathcal{R}_L$.

Exponentially small Yukawa couplings can be obtained by using the wave function localization in the extra dimensions [24, 25]. For the third generation, we assume that the overlap integrals in (5.9) do not provide any suppression factors, i.e., equal one. Then the Yukawa couplings are determined only by the group-theoretical factors. In the following, we focus on the third generation quarks.

Consider a 6D Dirac fermion $\Psi = \Psi_+ + \Psi_-$ that belongs to the representation $\mathcal{R}$. The theory is assumed to be symmetric under an exchange: $\Psi_+ \leftrightarrow -\Psi_-$ so that a 6D mass term $M_\Psi(\bar{\Psi}_+\Psi_- + \bar{\Psi}_-\Psi_+)$ is prohibited. Let us also assume that $\Psi_{\chi_6,-}$ and $\Psi_{\chi_6,+}$ have zero-modes $\mathcal{Q}_L^{(\chi_6)} \in \mathbf{(2,2)}$ and $\lambda_R^{(\chi_6)} \in \mathbf{(1,1)}$. The Higgs fields H^β that couple to them form bidoublets $\mathcal{H}_a$. Then, from (5.8), the Yukawa couplings from $i\sum_{\chi_6=\pm}\bar{\Psi}_{\chi_6}\Gamma^M\mathcal{D}_M\Psi_{\chi_6}$ before the breaking of $\mathrm{SU}(2)_{\rm R}\times\mathrm{U}(1)_Z$ at the fixed point are expressed as

$$\mathcal{L}_{\rm yukawa} = \sum_a \left\{ y_a^{(+)}\mathrm{tr}\left(\bar{\mathcal{Q}}_L^{(+)}\tilde{\mathcal{H}}_a\right)\lambda_R^{(+)} + y_a^{(-)}\mathrm{tr}\left(\bar{\mathcal{Q}}_L^{(-)}\mathcal{H}_a\right)\lambda_R^{(-)} + \text{h.c.}\right\}, \tag{5.10}$$

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where $\tilde{\mathcal{H}}_a \equiv \sigma_2 \mathcal{H}_a^* \sigma_2$ and

$$y_a^{(+)} = i\sqrt{2}g\langle \mu_L | E_{-\beta} | \mu_L + \beta \rangle = i\sqrt{2}g N^*_{\beta,\mu_L},$$
$$y_a^{(-)} = i\sqrt{2}g\langle \nu_L | E_{\beta} | \nu_L - \beta \rangle = i\sqrt{2}g N^*_{-\beta,\nu_L}. \tag{5.11}$$

Here $|\mu_L\rangle, |\nu_L\rangle \in \mathbf{(2,2)}$, $|\mu_L+\beta\rangle, |\nu_L-\beta\rangle \in \mathbf{(1,1)}$, and a complex constant $N_{\beta,\mu}$ is defined below (A.3). Note that $\mathcal{Q}_L^{(+)}$ and $\mathcal{Q}_L^{(-)}$ ($\lambda_R^{(+)}$ and $\lambda_R^{(-)}$) belong to different $\mathbf{(2,2)}$ ($\mathbf{(1,1)}$) multiplets in $\mathcal{R}$ because the same $\mathbf{(2,2)}$ ($\mathbf{(1,1)}$) cannot satisfy (5.7) for $\chi_6 = \pm$ simultaneously. We discriminate the two different $\mathbf{(2,2)}$ and $\mathbf{(1,1)}$ multiplets by denoting them as $\mathcal{Q}_L^{(\chi_6)} \in \mathbf{(2,2)_{\chi_6}}$ and $\lambda_R^{(\chi_6)} \in \mathbf{(1,1)_{\chi_6}}$. The Yukawa couplings depend on how the quark fields are embedded into $\mathcal{Q}_L^{(\pm)}$ and $\lambda_R^{(\pm)}$.

5.3.2 Embedding of quarks

As we will see in section 6, the Higgs potential at tree level only contains quartic terms. The electroweak symmetry breaking occurs at one-loop level, and the top Yukawa coupling provides a dominant contribution to the one-loop Higgs potential. In general, such one-loop potential breaks $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}$, and thus the Higgs VEVs are not aligned as (4.17). Namely the custodial symmetry is broken. A simple way to avoid this difficulty is to assume that the quark fields couple to the Higgs fields only through a combination $\mathcal{H}_a + \tilde{\mathcal{H}}_a$. This is achieved when $y_a^{(+)} = y_a^{(-)*}$ and the quark fields are equally contained in both Ψ_+ and Ψ_-. Specifically, consider a case that $\nu_L = -\mu_L$ and 4D fermions $\zeta_R \in \mathbf{(2,2)}$ and $\eta_L \in \mathbf{(1,1)}$ are localized at a fixed point, which transform as $\zeta_R \to -\zeta_R$ and $\eta_L \to -\eta_L$ under $\Psi_\pm \to -\Psi_\mp$. Then combinations $\mathcal{Q}'_L \equiv (-\mathcal{Q}_L^{(+)} + \mathcal{Q}_L^{(-)})/\sqrt{2}$ and $\lambda'_R \equiv (-\lambda_R^{(+)} + \lambda_R^{(-)})/\sqrt{2}$ have masses with them at the fixed point and are decoupled at low energies. Since $y_a^{(+)} = y_a^{(-)*}$ due to the property (A.2), we can redefine the overall phases of $\mathcal{H}_a$ so that $y_a^{(+)} = y_a^{(-)} > 0$. Then we obtain the desired form of the Yukawa coupling,[6]

$$\mathcal{L}_\mathrm{yukawa} = \frac{y_\lambda}{2} \sum_a \mathrm{tr}\left\{ \bar{\mathcal{Q}}_L \left(\mathcal{H}_a + \tilde{\mathcal{H}}_a \right) \right\} \lambda_R + \mathrm{h.c.} + \cdots, \tag{5.12}$$

where $y_\lambda \equiv y_a^{(+)} = y_a^{(-)}$, $\mathcal{Q}_L \equiv (\mathcal{Q}_L^{(+)} + \mathcal{Q}_L^{(-)})/\sqrt{2}$ and $\lambda_R \equiv (\lambda_R^{(+)} + \lambda_R^{(-)})/\sqrt{2}$.

Now we will see how the quark fields should be embedded into 6D fields. For simplicity, we consider a case that there is one Higgs bidoublet $\mathcal{H}$ as a zero-mode for a while. We introduce two 6D Dirac fermions $\Psi^{(2/3)} = \Psi_+^{(2/3)} + \Psi_-^{(2/3)}$ and $\Psi^{(-1/3)} = \Psi_+^{(-1/3)} + \Psi_-^{(-1/3)}$, whose $\mathrm{U}(1)_Z$ charges are $2/3$ and $-1/3$, respectively. Let us assume that $\Psi^{(q_Z)}$ ($q_Z = 2/3, -1/3$) contain $\mathcal{Q}_L^{(q_Z)} \in \mathbf{(2,2)}$ and $\lambda_R^{(q_Z)} \in \mathbf{(1,1)}$ as zero-modes. The bidoublets are decomposed as

$$\mathcal{Q}_L^{(2/3)} = (Q_L^{(1)}, Q_L^{(2)}), \qquad \mathcal{Q}_L^{(-1/3)} = (Q_L^{(3)}, Q_L^{(4)}), \qquad \mathcal{H} = (\tilde{H}_2, H_1), \tag{5.13}$$

where $\tilde{H}_2^i \equiv \epsilon_{ij} H_2^{j*}$, and $\{Q_L^{(1)}, Q_L^{(3)}, \tilde{H}_2\}$ and $\{Q_L^{(2)}, Q_L^{(4)}, H_1\}$ are $\mathrm{SU}(2)_\mathrm{L}$ doublets whose T_R^3 eigenvalues are $-1/2$ and $1/2$, respectively. Then the Yukawa couplings in the form

[6] Notice that $\mathcal{Q}_R^{(\mp)}$ and $\lambda_L^{(\mp)}$ also satisfy the zero-mode condition (5.7) when $\mathcal{Q}_L^{(\pm)}$ and $\lambda_R^{(\pm)}$ are zero-modes. So we also need additional 4D localized fermions to decouple them.

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of (5.12) are expressed as

$$\begin{aligned}\mathcal{L}_{\text{yukawa}} &= \frac{y_t}{2}\text{tr}\left\{\bar{\mathcal{Q}}_L^{(2/3)}\left(\mathcal{H}+\tilde{\mathcal{H}}\right)\right\}t_R + \frac{y_b}{2}\text{tr}\left\{\bar{\mathcal{Q}}_L^{(-1/3)}\left(\mathcal{H}+\tilde{\mathcal{H}}\right)\right\}b_R + \text{h.c.} \\ &= \frac{y_t}{2}\left\{\bar{Q}_L^{(1)}\left(\tilde{H}_2+\tilde{H}_1\right)+\bar{Q}_L^{(2)}\left(H_1+H_2\right)\right\}t_R \\ &\quad + \frac{y_b}{2}\left\{\bar{Q}_L^{(3)}\left(\tilde{H}_2+\tilde{H}_1\right)+\bar{Q}_L^{(4)}\left(H_1+H_2\right)\right\}b_R + \text{h.c.},\end{aligned} \tag{5.14}$$

where y_t and y_b are calculated from (5.11). Since only the combination H_1+H_2 couples to the quarks, this combination obtains a tachyonic mass while the other combination H_1-H_2 does not at one-loop level. Therefore the latter does not have a nonzero VEV, and $\langle H_1\rangle = \langle H_2\rangle$ is realized. Namely, the alignment (4.17) is achieved. (See section 6.2.) Since $Q_L^{(1)}$ and $Q_L^{(4)}$ have the same quantum numbers for $\text{SU(2)}_\text{L}\times\text{U(1)}_Y$, they are mixed with each other after the breaking $\text{SU(2)}_\text{R}\times\text{U(1)}_Z \to \text{U(1)}_Y$ occurs at the fixed point. The left-handed quark is identified as a linear combination,

$$q_L = \cos\theta_q Q_L^{(1)} + \sin\theta_q Q_L^{(4)}, \tag{5.15}$$

where θ_q is a mixing angle. The orthogonal combination and $Q_L^{(2)}$ and $Q_L^{(3)}$ are exotic fields that must be decoupled at low energies. Hence we need to introduce 4D localized fermions that couple with those exotic components. As a result, the following Yukawa couplings are obtained at low energies.

$$\mathcal{L}_{\text{yukawa}}^{\text{SU(2)}_L\times\text{U(1)}_Y} = \frac{y_t}{2}\cos\theta_q q_L^\dagger\left(\tilde{H}_2+\tilde{H}_1\right)t_R + \frac{y_b}{2}\sin\theta_q q_L^\dagger\left(H_1+H_2\right)b_R + \text{h.c.}. \tag{5.16}$$

When $y_t = y_b$, the large ratio of the top quark mass m_t to the bottom quark mass m_b is obtained if $\theta_q = \mathcal{O}(m_b/m_t)$.[7] In such a case, m_t is calculated as

$$m_t = \left|\frac{y_t}{2}v\cos\theta_q\right| \simeq \left|\frac{y_t v}{2}\right| = \frac{g\left|N_{\beta,\mu_L}\right|v}{\sqrt{2}} = \sqrt{2}\left|N_{\beta,\mu_L}\right|m_W, \tag{5.17}$$

where v is defined as $\langle H_1\rangle = \langle H_2\rangle = (0, v/2)^t$ (see (4.17)). We have used that $\cos\theta_q \simeq 1$, (4.25) and (5.11). Therefore, the observed top quark mass is obtained if $|N_{\beta,\mu_L}| = \sqrt{2}$.[8] We can extend this result to the two-Higgs-bidoublet case straightforwardly.

5.3.3 Available representations for matter fermions

In summary, the quark multiplets should be embedded into two 6D Dirac fermions $\Psi^{(2/3)}$ and $\Psi^{(-1/3)}$ whose U(1)_Z charges are $2/3$ and $-1/3$, respectively. Irreducible representations $\mathcal{R}$ which they belong to must satisfy the following conditions.

1. $\mathcal{R}$ includes two bidoublets and two singlets, which are denoted as $\mathbf{(2,2)}_\pm$ and $\mathbf{(1,1)}_\pm$, respectively.

[7] In contrast to the mixing between $\mathcal{Q}_L^{(+)}$ and $\mathcal{Q}_L^{(-)}$, the mixing angle θ_q can take arbitrary values because there is no symmetry to fix it.

[8] A small deviation from the observed value of m_t is expected to be explained by quantum correction. (See ref. [27]).

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2. There are weights μ_L and $\nu_L = -\mu_L$ that satisfy $|\mu_L\rangle \in \mathbf{(2,2)}_+$, $|\mu_L + \beta\rangle \in \mathbf{(1,1)}_+$, $|\nu_L\rangle \in \mathbf{(2,2)}_-$, $|\nu_L - \beta\rangle \in \mathbf{(1,1)}_-$, and $|N_{\beta,\mu_L}| = |N_{-\beta,\nu_L}| = \sqrt{2}$, where β is a root in the Higgs bidoublet.

3. The states in $\mathbf{(2,2)}_\pm$ and $\mathbf{(1,1)}_\pm$ satisfy the zero-mode condition (5.7).

We will search for $\mathcal{R}$ that satisfies these conditions from the list in appendix C. We focus on the cases of $G = \mathrm{SO}(5), \mathrm{SU}(4), \mathrm{SO}(7)(\mathrm{I})$, which have the $\mathcal{P}_{\mathrm{LR}}$ symmetry.

SO(5)

There is no irreducible representation that satisfies the condition 1 among the list in appendix C.1.

SU(4)

Only $\mathbf{20'}$ satisfies the condition 1 among the list in appendix C.2. The weights of $\mathbf{20'}$ that form $\mathbf{(2,2)}$ and $\mathbf{(1,1)}$ are

$$\begin{aligned}
\mathbf{(2,2)}_{\pm\mathbf{2}} &: \begin{pmatrix} \frac{\boldsymbol{e}^1-\boldsymbol{e}^2}{\sqrt{2}} \pm \boldsymbol{e}^3 & \xrightarrow{\alpha_R} & \frac{\boldsymbol{e}^1+\boldsymbol{e}^2}{\sqrt{2}} \pm \boldsymbol{e}^3 \\ \uparrow{\scriptstyle \alpha_L} & & \uparrow{\scriptstyle \alpha_L} \\ \frac{-\boldsymbol{e}^1-\boldsymbol{e}^2}{\sqrt{2}} \pm \boldsymbol{e}^3 & \xrightarrow{\alpha_R} & \frac{-\boldsymbol{e}^1+\boldsymbol{e}^2}{\sqrt{2}} \pm \boldsymbol{e}^3 \end{pmatrix}, \\
\mathbf{(1,1)}_{\pm\mathbf{4}} &: \pm 2\boldsymbol{e}^3, \qquad \mathbf{(1,1)}_{\mathbf{0}} \ : \ \mathbf{0}.
\end{aligned} \tag{5.18}$$

where the double signs correspond. Notice that the weights that form bidoublets are the same as the roots that form the Higgs bidoublets.

When the Higgs bidoublet $\mathbf{(2,2)}_{\pm\mathbf{2}}$ appears as a zero-mode, one example of (β, μ_L, ν_L) is chosen as

$$(\beta, \mu_L, \nu_L) = \left(\frac{\boldsymbol{e}^1 - \boldsymbol{e}^2}{\sqrt{2}} \pm \boldsymbol{e}^3, \frac{-\boldsymbol{e}^1 + \boldsymbol{e}^2}{\sqrt{2}} \pm \boldsymbol{e}^3, \frac{\boldsymbol{e}^1 - \boldsymbol{e}^2}{\sqrt{2}} \mp \boldsymbol{e}^3 \right), \tag{5.19}$$

where the double signs correspond. Then $\{\mu_L - \beta, \mu_L, \mu_L + \beta\}$ and $\{\nu_L - \beta, \nu_L, \nu_L + \beta\}$ are the weights, but $\mu_L \pm 2\beta$ and $\nu_L \pm 2\beta$ are not. Therefore the condition 2 is satisfied (see (A.3)).

Since $(p_1, p_2, p_3) = (0, 0, 2n_P\pi/3)$ and the simple roots are $(\alpha_1, \alpha_2, \alpha_3) = (\sqrt{2}\boldsymbol{e}^1, -\frac{\boldsymbol{e}^1}{\sqrt{2}} - \frac{\boldsymbol{e}^2}{\sqrt{2}} + \boldsymbol{e}^3, \sqrt{2}\boldsymbol{e}^2)$, the zero-mode condition (5.7) becomes

$$\frac{\pi(2m_\omega + 1 - \chi_4\chi_6)}{N} - \frac{2n_P k_2 \pi}{N} = 0, \qquad (\mathrm{mod}\ 2\pi) \tag{5.20}$$

where $m_\omega = 0, 1, \cdots, N-1$. The decomposition of $\mathbf{20'}$ is given by (C.13), and

$$\begin{aligned}
&k_2 = 0 : \mathbf{(1,1)}_{+\mathbf{4}}, \\
&k_2 = 1 : \mathbf{(2,2)}_{+\mathbf{2}}, \\
&k_2 = 2 : \mathbf{(3,3)}_{\mathbf{0}}, \ \mathbf{(1,1)}_{\mathbf{0}}, \\
&k_2 = 3 : \mathbf{(2,2)}_{-\mathbf{2}}, \\
&k_2 = 4 : \mathbf{(1,1)}_{-\mathbf{4}}.
\end{aligned} \tag{5.21}$$

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Thus the condition 3 is satisfied only when the model is compactified on T^2/Z_3. In fact, when $(N, n_P, m_\omega) = (3, 1, 0)$, the fermionic zero-modes from each 6D Dirac fermion contain

$$\begin{aligned} \mathcal{Q}_L^{(+)} &\in \mathbf{(2,2)_{+2}}, \quad & \lambda_R^{(+)} &\in \mathbf{(1,1)_{+4}}, \\ \mathcal{Q}_L^{(-)} &\in \mathbf{(2,2)_{-2}}, \quad & \lambda_R^{(-)} &\in \mathbf{(1,1)_{-4}}, \end{aligned} \tag{5.22}$$

and when $(N, n_P, m_\omega) = (3, 2, 2)$, they contain

$$\begin{aligned} \mathcal{Q}_L^{(+)} &\in \mathbf{(2,2)_{-2}}, \quad & \lambda_R^{(+)} &\in \mathbf{(1,1)_{-4}}, \\ \mathcal{Q}_L^{(-)} &\in \mathbf{(2,2)_{+2}}, \quad & \lambda_R^{(-)} &\in \mathbf{(1,1)_{+4}}. \end{aligned} \tag{5.23}$$

By introducing 4D localized fermions with appropriate quantum numbers to decouple unwanted zero-modes, the desired Yukawa couplings (5.16) are obtained. For the other choices of (N, n_P, m_ω), we cannot obtain the necessary multiplets.

SO(**7**) **(I)**

The irreducible representations that satisfy the condition 1 among the list in appendix C.3 are **21** and **27**. These also satisfy the condition 2, but they cannot satisfy the condition 3 for any choice of (N, n_P, m_ω).

6 Higgs potential

In contrast to the 5D gauge-Higgs unification model, we have quartic couplings of the Higgs fields at tree level. The relevant terms in the 6D Lagrangian are

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4g_A^2}\mathrm{tr}\left(F^{(A)MN}F^{(A)}_{MN}\right) + \cdots \\ &= -\frac{1}{2g_A^2(\pi R_1)^2}\mathrm{tr}\left((\partial^\mu A_z)^\dagger \partial_\mu A_z\right) - \frac{1}{8g_A^2(\pi R_1)^4}\mathrm{tr}\left([A_z, A_{\bar{z}}]^2\right) + \cdots . \end{aligned} \tag{6.1}$$

In this section, we calculate the classical Higgs potential V_{tree} focusing on the Higgs bidoublets, which are relevant to the electroweak symmetry breaking. In the previous section, we have shown that only a model of G =SU(4) compactified on T^2/Z_3 has required zero-mode spectrum for the quarks. For the sake of completeness, however, we will also calculate V_{tree} in the other cases that have Higgs bidoublets. We have one Higgs bidoublet in the cases of SO(5) on T^2/Z_2, SU(4) on T^2/Z_N $(N = 3, 4, 6)$, SO(7) (I) on T^2/Z_N $(N = 4, 6)$, and Sp(6) (II) or (III) on T^2/Z_N $(N = 3, 4, 6)$, and we have two Higgs bidoublets in the case of SU(4) on T^2/Z_2.

6.1 SO(5) case

First we consider the SO(5) case. In this case, the roots that form the bidoublet are

$$\begin{pmatrix} \boldsymbol{e}^2 & \xrightarrow{\alpha_R} & \boldsymbol{e}^1 \\ \uparrow_{\alpha_L} & & \uparrow_{\alpha_L} \\ -\boldsymbol{e}^1 & \xrightarrow{\alpha_R} & -\boldsymbol{e}^2 \end{pmatrix}. \tag{6.2}$$

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From (6.1), the kinetic terms of the zero-modes W_z^β in the 4D effective Lagrangian are

$$\mathcal{L}_{\rm eff} = -\frac{\mathcal{A}}{2(g_A \pi R_1)^2} \sum_\beta (\partial^\mu W_z^\beta)^* \partial_\mu W_z^\beta + \cdots . \tag{6.3}$$

We have used (A.4), and $\mathcal{A}$ is the area of T^2/Z_N. Thus the canonically normalized Higgs bidoublet is defined as

$$\mathcal{H} = \begin{pmatrix} H_2^{2*} & H_1^1 \\ -H_2^{1*} & H_1^2 \end{pmatrix} \equiv \frac{\sqrt{\mathcal{A}}}{\sqrt{2} g_A \pi R_1} \begin{pmatrix} W_z^{e^2} & W_z^{e^1} \\ -W_z^{-e^1} & W_z^{-e^2} \end{pmatrix} . \tag{6.4}$$

Then it follows that

$$\begin{aligned} A_z &= \frac{\sqrt{2} g_A \pi R_1}{\sqrt{\mathcal{A}}} \left(H_1^1 E_{e^1} + H_2^{2*} E_{e^2} + H_1^2 E_{-e^2} + H_2^{1*} E_{-e^1} \right) , \\ [A_z, A_{\bar{z}}] &= \frac{2(g_A \pi R_1)^2}{\mathcal{A}} \Big[\left(\left| H_1^1 \right|^2 - \left| H_2^1 \right|^2 \right) H_1 + \left(\left| H_2^2 \right|^2 - \left| H_1^2 \right|^2 \right) H_2 \\ &\qquad + \left\{ N_{e^1, e^2} \left(H_1^1 H_1^{2*} - H_2^1 H_2^{2*} \right) E_{\alpha_L} \right. \\ &\qquad \left. + N_{e^1, -e^2} \left(H_1^1 H_2^2 - H_1^2 H_2^1 \right) E_{\alpha_R} + \text{h.c.} \right\} \Big] , \end{aligned} \tag{6.5}$$

where we have used (A.2). Hence, from (6.1), $V_{\rm tree}$ is calculated as

$$\begin{aligned} V_{\rm tree} &= \frac{\mathcal{A}}{8 g_A^2 (\pi R_1)^4} \text{tr} \left([A_z, A_{\bar{z}}]^2 \right) \\ &= \frac{g_A^2}{2\mathcal{A}} \Big[\left(\left| H_1^1 \right|^2 - \left| H_2^1 \right|^2 \right)^2 + \left(\left| H_2^2 \right|^2 - \left| H_1^2 \right|^2 \right)^2 \\ &\qquad + 2 \left| N_{e^1, e^2} \right|^2 \left| H_1^1 H_1^{2*} - H_2^1 H_2^{2*} \right|^2 + 2 \left| N_{e^1, -e^2} \right|^2 \left| H_1^1 H_2^2 - H_1^2 H_2^1 \right|^2 \Big] \\ &= \frac{g^2}{4} \left\{ \left(H_2^\dagger H_2 - H_1^\dagger H_1 \right)^2 + 4 \left| \tilde{H}_2^\dagger H_1 \right|^2 \right\} \\ &= \frac{g^2}{4} \left[\left\{ \text{tr} \left(\mathcal{H}^\dagger \mathcal{H} \right) \right\}^2 - 4 \det \left(\mathcal{H}^\dagger \mathcal{H} \right) \right] , \end{aligned} \tag{6.6}$$

where $H_2 \equiv (H_2^1, H_2^2)^t$ and $H_1 \equiv (H_1^1, H_1^2)^t$ are the $\text{SU}(2)_{\rm L}$ doublets with the hypercharge $Y = 1/2$. We have used that (4.18) with $|\alpha_L|^2 = 2$, and $\left| N_{e^1, e^2} \right|^2 = \left| N_{e^1, -e^2} \right|^2 = 1$. The above result agrees with eq. (7) in ref. [18]. The final expression in (6.6) is manifestly invariant under the transformation: $\mathcal{H} \to U_L \mathcal{H} U_R^\dagger$ ($U_L \in \text{SU}(2)_{\rm L}$ and $U_R \in \text{SU}(2)_{\rm R}$).

6.2 Cases of rank-three groups

Next we consider the cases of the rank-three groups. In these cases, the candidates for the zero-mode Higgs bidoublets consist of the following roots.

$$\begin{pmatrix} \gamma + \alpha_L & \xrightarrow{\alpha_R} & \gamma + \alpha_L + \alpha_R \\ \uparrow \alpha_L & & \uparrow \alpha_L \\ \gamma & \xrightarrow{\alpha_R} & \gamma + \alpha_R \end{pmatrix} , \quad \begin{pmatrix} -\gamma - \alpha_R & \xrightarrow{\alpha_R} & -\gamma \\ \uparrow \alpha_L & & \uparrow \alpha_L \\ -\gamma - \alpha_L - \alpha_R & \xrightarrow{\alpha_R} & -\gamma - \alpha_L \end{pmatrix} , \tag{6.7}$$

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where $\gamma = -\frac{\boldsymbol{e}^1}{\sqrt{2}} - \frac{\boldsymbol{e}^2}{\sqrt{2}} + \boldsymbol{e}^3$ for SU(4), $\gamma = -\boldsymbol{e}^1 + \boldsymbol{e}^3$ for SO(7) (I), and $\gamma = -\boldsymbol{e}^2 - \boldsymbol{e}^3$ for Sp(6) (II) or (III). The canonically normalized Higgs bidoublets are defined as

$$\begin{aligned}
\mathcal{H}_+ &= \begin{pmatrix} H^{2*}_{2+} & H^1_{1+} \\ -H^{1*}_{2+} & H^2_{1+} \end{pmatrix} \equiv \frac{\sqrt{\mathcal{A}}}{\sqrt{2} g_A \pi R_1} \begin{pmatrix} W_z^{\gamma+\alpha_L} & W_z^{\gamma+\alpha_L+\alpha_R} \\ -W_z^{\gamma} & W_z^{\gamma+\alpha_R} \end{pmatrix}, \\
\mathcal{H}_- &= \begin{pmatrix} H^{2*}_{2-} & H^1_{1-} \\ -H^{1*}_{2-} & H^2_{1-} \end{pmatrix} \equiv \frac{\sqrt{\mathcal{A}}}{\sqrt{2} g_A \pi R_1} \begin{pmatrix} W_z^{-\gamma-\alpha_R} & W_z^{-\gamma} \\ -W_z^{-\gamma-\alpha_L-\alpha_R} & W_z^{-\gamma-\alpha_L} \end{pmatrix},
\end{aligned} \tag{6.8}$$

where the signs in the suffixes denote the signs of the $\mathrm{U}(1)_X$ charges. Then it follows that

$$\begin{aligned}
A_z = \frac{\sqrt{2} g_A \pi R_1}{\sqrt{\mathcal{A}}} & \left(H^1_{1+} E_{\gamma_{LR}} + H^{2*}_{2+} E_{\gamma_L} + H^2_{1+} E_{\gamma_R} + H^{1*}_{2+} E_{\gamma} \right. \\
& \left. + H^1_{1-} E_{-\gamma} + H^{2*}_{2-} E_{-\gamma_R} + H^2_{1-} E_{-\gamma_L} + H^{1*}_{2-} E_{-\gamma_{LR}} \right) + \cdots, \\
[A_z, A_{\bar{z}}] = \frac{2 (g_A \pi R_1)^2}{\mathcal{A}} & \Big[\left(\left| H^1_{2+} \right|^2 - \left| H^1_{1-} \right| \right) \gamma \cdot H + \left(\left| H^2_{2+} \right|^2 - \left| H^2_{1-} \right|^2 \right) \gamma_L \cdot H \\
& + \left(\left| H^2_{1+} \right|^2 - \left| H^2_{2-} \right|^2 \right) \gamma_R \cdot H + \left(\left| H^1_{1+} \right|^2 - \left| H^1_{2-} \right|^2 \right) \gamma_{LR} \cdot H \\
& + \left\{ N_{\gamma_{LR}, -\gamma_R} \left(-H^1_{1+} H^{2*}_{1+} + H^1_{2-} H^{2*}_{2-} \right) E_{\alpha_L} \right. \\
& \quad + N_{\gamma_{LR}, -\gamma_L} \left(-H^1_{1+} H^2_{2+} + H^2_{1-} H^1_{2-} \right) E_{\alpha_R} \\
& \quad + N_{\gamma_L, -\gamma} \left(-H^1_{2+} H^{2*}_{2+} + H^1_{1-} H^{2*}_{1-} \right) E_{\alpha_L} \\
& \quad \left. + N_{\gamma_R, -\gamma} \left(-H^2_{1+} H^1_{2+} + H^1_{1-} H^2_{2-} \right) E_{\alpha_R} + \text{h.c.} \right\} \Big] + \cdots,
\end{aligned} \tag{6.9}$$

where $\gamma_L \equiv \gamma + \alpha_L$, $\gamma_R \equiv \gamma + \alpha_R$ and $\gamma_{LR} \equiv \gamma + \alpha_L + \alpha_R$, and the ellipses denote fields belonging to other multiplets, if any. After some calculations, we obtain

$$\begin{aligned}
V_{\text{tree}} &= \frac{g^2}{2} \left[\left(|H_{1+}|^2 - |H_{2-}|^2 \right)^2 + \left(|H_{2+}|^2 - |H_{1-}|^2 \right)^2 \right. \\
&\quad + \left| H^\dagger_{1+} \tilde{H}_{2+} \right|^2 + \left| H^\dagger_{1+} \tilde{H}_{2-} \right|^2 + \left| H^\dagger_{1-} \tilde{H}_{2+} \right|^2 + \left| H^\dagger_{1-} \tilde{H}_{2-} \right|^2 \\
&\quad \left. - \left| \tilde{H}^t_{2+} H_{2-} \right|^2 - \left| \tilde{H}^t_{1+} H_{1-} \right|^2 + \left| \tilde{H}^\dagger_{2+} H_{1+} + \tilde{H}^\dagger_{2-} H_{1-} \right|^2 \right] \\
&= \frac{g^2}{2} \left[\left\{ \mathrm{tr} \left(\mathcal{H}^\dagger_+ \mathcal{H}_+ \right) \right\}^2 + \left\{ \mathrm{tr} \left(\mathcal{H}^\dagger_- \mathcal{H}_- \right) \right\}^2 - \mathrm{tr} \left(\tilde{\mathcal{H}}^\dagger_+ \tilde{\mathcal{H}}_+ \mathcal{H}^\dagger_- \mathcal{H}_- \right) \right. \\
&\quad \left. - \mathrm{tr} \left(\mathcal{H}^\dagger_- \tilde{\mathcal{H}}_+ \tilde{\mathcal{H}}^\dagger_+ \mathcal{H}_- \right) - 2 \det \left(\mathcal{H}^\dagger_+ \mathcal{H}_+ \right) - 2 \det \left(\mathcal{H}^\dagger_- \mathcal{H}_- \right) \right] + \cdots,
\end{aligned} \tag{6.10}$$

where $\tilde{H}^i_{1,2+} \equiv \epsilon_{ij} H^{j*}_{1,2+}$, and $\tilde{\mathcal{H}}_\pm \equiv \sigma_2 \mathcal{H}^*_\pm \sigma_2$. We have used that

$$\begin{aligned}
\gamma \cdot \gamma_{LR} &= \gamma_L \cdot \gamma_R = 0, \\
|\gamma_{LR}|^2 &= |\gamma_L|^2 = |\gamma_R|^2 = |\gamma|^2 = 2, \\
\gamma \cdot \gamma_L &= \gamma \cdot \gamma_R = \gamma_L \cdot \gamma_{LR} = \gamma_R \cdot \gamma_{LR} = \frac{|\gamma|^2}{2} = 1, \\
|N_{\gamma_{LR}, -\gamma_L}|^2 &= |N_{\gamma_{LR}, -\gamma_R}|^2 = |N_{\gamma_L, -\gamma}|^2 = |N_{\gamma_R, -\gamma}|^2 = \frac{|\gamma|^2}{2} = 1, \\
\frac{N_{\gamma_L, -\gamma}}{N_{\gamma_{LR}, -\gamma_R}} &= \frac{N^*_{\gamma, \alpha_L}}{N^*_{\gamma_R, \alpha_L}} = \frac{N^*_{\gamma, \alpha_R}}{N^*_{\gamma_L, \alpha_R}} = \frac{N_{\gamma_R, -\gamma}}{N_{\gamma_{LR}, -\gamma_L}},
\end{aligned} \tag{6.11}$$

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which are followed by (A.2), (A.3) and the fact that $\alpha_L \cdot \alpha_R = 0$ and $[E_{\alpha_L}, E_{\alpha_R}] = 0$. We have also chosen the phases of the Higgs fields so that $N_{\gamma_L,-\gamma}/N_{\gamma_{LR},-\gamma_R} = -1$.

The final expression in (6.10) is manifestly invariant under the transformation: $\mathcal{H}_\pm \to U_L \mathcal{H}_\pm U_R$ ($U_L \in \mathrm{SU}(2)_\mathrm{L}$ and $U_R \in \mathrm{SU}(2)_\mathrm{R}$). Except for the case of SU(4) on T^2/Z_2, one of the bidoublets $\mathcal{H}_\pm$ is absent due to the orbifold boundary conditions. In such cases, the model becomes a two-Higgs-doublet model. In contrast to the SO(5) case, the potential (6.10) with $\mathcal{H}_+ = 0$ or $\mathcal{H}_- = 0$ does not agree with (7) of ref. [18]. This is because they have assumed $\gamma + \alpha_L + \alpha_R = -\gamma$, which only holds in the SO(5) case.

Finally we comment on the Higgs mass. We consider a case of SU(4) on T^2/Z_3. The tree-level Higgs potential (6.10) becomes

$$\begin{aligned} V_\mathrm{tree} &= \frac{g^2}{2}\left[\left\{\mathrm{tr}\left(\mathcal{H}^\dagger\mathcal{H}\right)\right\}^2 - 2\det\left(\mathcal{H}^\dagger\mathcal{H}\right)\right] \\ &= \frac{g^2}{2}\left\{\left(H_1^\dagger H_1\right)^2 + \left(H_2^\dagger H_2\right)^2 + 2\left|\tilde{H}_2^\dagger H_1\right|^2\right\}, \end{aligned} \tag{6.12}$$

where $\mathcal{H} = (\tilde{H}_2, H_1)$ is one of $\mathcal{H}_\pm$. Since only the $\mathrm{U}(1)_\mathrm{em}$ neutral components H_1^2 and H_2^2 can have nonzero VEVs, we focus on them. As discussed in section 5.3.2, we expect that $h_+ \equiv (H_1^2 + H_2^2)/\sqrt{2}$ has a tachyonic mass while $h_- \equiv (H_1^2 - H_2^2)/\sqrt{2}$ does not at one-loop level. Including such mass terms, the potential becomes

$$V = -m_+^2 |h_+|^2 + m_-^2 |h_-|^2 + \frac{g^2}{4}\left\{\left(|h_+|^2 + |h_-|^2\right)^2 + \left(h_+^* h_- + h_+ h_-^*\right)^2\right\} + \cdots, \tag{6.13}$$

where $m_\pm > 0$, and the ellipsis denotes terms involving the charged components. We can always redefine the phase of fields so that $\langle h_+ \rangle > 0$. Then, from the minimization condition for the potential, we obtain

$$\langle h_+ \rangle = \frac{\sqrt{2} m_+}{g}, \qquad \langle h_- \rangle = 0. \tag{6.14}$$

Therefore, the alignment (4.17) is actually achieved. Note that the one-loop induced quadratic terms do not have the $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}$ symmetry. Thus the custodial symmetry is broken in the Higgs sector. This does not cause a problem for the protection of the ρ parameter at tree level as long as the relevant terms to the W and Z boson masses in (4.24) have the custodial symmetry.

The mass of the lightest neutral Higgs boson is

$$m_H = g\,|\langle h_+ \rangle| = \frac{gv}{\sqrt{2}} = \sqrt{2} m_W, \tag{6.15}$$

where v is defined as $\langle H_1^2 \rangle = \langle H_2^2 \rangle = v/2 > 0$, and we have used (4.25) at the last equality. We expect that the deviation from the observed value $m_H \simeq 125\,\mathrm{GeV}$ is explained by quantum corrections.[9]

[9] Although the quantum correction to the Higgs quartic couplings is divergent, the ratio m_H/m_W is proven to be finite and calculable in ref. [28].

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7 Summary

We have investigated 6D gauge-Higgs unification models compactified on T^2/Z_N ($N = 2, 3, 4, 6$) that have the custodial symmetry. The gauge group is assumed to be $\mathrm{SU}(3)_C \times G \times \mathrm{U}(1)_Z$, where G is a simple group. Since G includes $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}$, its rank must be more than one. The Higgs fields originate from the extra-dimensional components of the G gauge field. In contrast to 5D models [8–10], we have at least two Higgs doublets. Thus their VEVs need to be aligned as (4.17) to preserve the custodial symmetry. This severely constrains the structure of models.

In order to select candidates for viable models, we demanded the following requirements.

- The model has a scalar bidoublet zero-mode as the Higgs fields.
- The bosonic sector has a symmetry under a parity $\mathcal{P}_\mathrm{LR}$ that exchanges $\mathrm{SU}(2)_\mathrm{L}$ and $\mathrm{SU}(2)_\mathrm{R}$ in order to protect the $Zb_L\bar{b}_L$ coupling against a large deviation induced by mixing with the KK modes.
- The quark fields are embedded into 6D fermions in such a way that they couple to the Higgs bidoublet $\mathcal{H}$ only through a combination $\mathcal{H} + \sigma_2\mathcal{H}^*\sigma_2$.
- The representation $\mathcal{R}$ that the 6D fermions belong to provides a right size group factor to realize the top Yukawa coupling constant.

The third requirement is demanded in order for the Higgs VEVs to be aligned as (4.17). The third and the fourth requirements can be achieved if $\mathcal{R}$ satisfies the three conditions in section 5.3.3. Our results are summarized in table 1. In the cases with blank, there is no choice of the orbifold boundary conditions so that G is broken to $\mathrm{SU}(2)_\mathrm{L}\times\mathrm{SU}(2)_\mathrm{R}(\times\mathrm{U}(1)_X)$. There is only one candidate that satisfies the above requirements if we restrict ourselves to the cases that $\mathrm{rank}\, G \leq 3$ and $\dim \mathcal{R} < 30$. It is the case of $G =$SU(4), $N = 3$ and $\mathcal{R} = \mathbf{20'}$. Namely, the model is 6D $\mathrm{SU}(3)_C \times \mathrm{U}(4)$ gauge theory compactified on T^2/Z_3, and the top and bottom quarks are embedded into the symmetric traceless rank-2 tensor of SO(6).[10] We have focused on the third generation quarks to restrict G, N and $\mathcal{R}$. Embeddings of other fermions are much less constrained.

There are many issues that we have not discussed in this paper. We have approximated the mode functions of the W and Z bosons as constants. However, after the electroweak symmetry is broken, they are no longer constants and depend on z. This z-dependence causes the deviation of the ρ parameter and the $Zb_L\bar{b}_L$ coupling from the standard model values. We have to check that the custodial symmetry actually suppresses these deviations by using the exact mode functions. We should also calculate the one-loop effective potential to check that the vacuum alignment (4.17) is actually achieved, and to evaluate the Higgs mass spectrum. The moduli stabilization in the gauge-Higgs unification is also an important subject [12, 29]. It is interesting to investigate this subject in the presence of a background magnetic flux on T^2/Z_N. All these issues are left for our future works.

[10] Since the zero-mode spectrum is non-chiral in this case, the chiral structure of the effective theory must be attributed to the chiral field content of the localized fermions at the fixed point.

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	SO(5)	G_2	SU(4)	SO(7)		Sp(6)	
				(I)	(II), (III)	(I)	(II), (III)
T^2/Z_2	1 (S)	0	2 (S)				
T^2/Z_3			1 (S) ✓		0		1
T^2/Z_4	0 (S)	0	1 (S)	1 (S)	0	0 (S)	1
T^2/Z_6	0 (S)	0	1 (S)	1 (S)	0	0 (S)	1

Table 1. Summary of the results. The numbers denote those of the Higgs bidoublets. (I), (II) and (III) represent three different ways of choosing the $SU(2)_L \times SU(2)_R$ subgroup in section 3.2. "(S)" indicates that the spectrum is symmetric under $SU(2)_L \leftrightarrow SU(2)_R$. The check mark is added to a case that there is an appropriate embedding of quarks into 6D fermions.

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A Cartan-Weyl basis

The generators of a simple group G whose rank is r in the Cartan-Weyl basis are H_i $(i = 1, \cdots, r)$ and E_α, which satisfy

$$\begin{aligned} H_i^\dagger &= H_i, & E_\alpha^\dagger &= E_{-\alpha}, \\ [H_i, H_j] &= 0, & [H_i, E_\alpha] &= \alpha_i E_\alpha, \\ [E_\alpha, E_\beta] &= N_{\alpha,\beta} E_{\alpha+\beta}, & [E_\alpha, E_{-\alpha}] &= \alpha \cdot H, \end{aligned} \tag{A.1}$$

where α, β are the root vectors, and $\alpha \neq -\beta$. A complex constant $N_{\alpha,\beta}$ is nonzero only when $\alpha + \beta$ is a root, and satisfies the following equations.

$$N_{\alpha,\beta} = -N_{\beta,\alpha} = -N^*_{-\alpha,-\beta} = N_{\beta,-\alpha-\beta} = N_{-\alpha-\beta,\alpha}. \tag{A.2}$$

For a series of the weights $\{\mu - q\alpha, \cdots, \mu - \alpha, \mu, \mu + \alpha, \cdots, \mu + p\alpha\}$, where p and q are integers and neither $\mu - (q+1)\alpha$ nor $\mu + (p+1)\alpha$ is a weight, it follows that

$$\frac{2\alpha \cdot \mu}{|\alpha|^2} = q - p, \qquad |N_{\alpha,\mu}|^2 = \frac{p(q+1)\,|\alpha|^2}{2}, \tag{A.3}$$

where a complex constant $N_{\alpha,\mu}$ is defined as $E_\alpha|\mu\rangle = N_{\alpha,\mu}|\mu + \alpha\rangle$. The generators are normalized as

$$\mathrm{tr}(H_i H_j) = \delta_{ij}, \qquad \mathrm{tr}(H_i E_\alpha) = 0, \qquad \mathrm{tr}(E_\alpha E_\beta) = \delta_{\alpha,-\beta}. \tag{A.4}$$

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B Orbifold boundary conditions

The orbifold T^2/Z_N is defined by identifying points of $\mathbb{R}^2$ by a discrete group Γ which is generated by three descrete transformations $\mathcal{O}_1$: $z \to z+1$, $\mathcal{O}_\tau$: $z \to z+\tau$ and $\mathcal{O}_\omega$: $z \to \omega z$. Field values of a 6D field at Γ-equivalent points must be related to each other through gauge transformations[11] in order for the Lagrangian to be single-valued on T^2/Z_N. Thus the most general orbifold boundary conditions are given by

$$\begin{aligned} A_M(x, z+1) &= T_1 A_M(x,z) T_1^{-1}, \qquad B_M^Z(x, z+1) = B_M^Z(x,z), \\ \Psi_{\chi_6}(x, z+1) &= e^{i\varphi_1} T_1 \Psi_{\chi_6}(x,z), \end{aligned} \tag{B.1}$$

for the translation $\mathcal{O}_1$,

$$\begin{aligned} A_M(x, z+\tau) &= T_\tau A_M(x,z) T_\tau^{-1}, \qquad B_M^Z(x, z+\tau) = B_M^Z(x,z), \\ \Psi_{\chi_6}(x, z+\tau) &= e^{i\varphi_\tau} T_\tau \Psi_{\chi_6}(x,z), \end{aligned} \tag{B.2}$$

for the translation $\mathcal{O}_\tau$, and

$$\begin{aligned} A_\mu(x,\omega z) &= P A_\mu(x,z) P^{-1}, & A_z(x,\omega z) &= \omega^{-1} P A_z(x,z) P^{-1}, \\ B_\mu^Z(x,\omega z) &= B_\mu^Z(x,z), & B_z^Z(x,\omega z) &= \omega^{-1} B_z^Z(x,z), \\ \Psi_{\chi_4,\chi_6}(x,\omega z) &= \omega^{-\frac{\chi_4\chi_6}{2}} e^{i\varphi_\omega} P \Psi_{\chi_4,\chi_6}, \end{aligned} \tag{B.3}$$

for the Z_N twist $\mathcal{O}_\omega$. Matrices T_1, T_τ and P are elements of G, and φ_1 and φ_τ are the Scherk-Schwarz phases. A factor ω^{-1} and $\omega^{-\frac{\chi_4\chi_6}{2}}$ in (B.3) appears because A_z, B_z^Z and Ψ_{χ_4,χ_6} are charged under the rotation in the extra-dimensional space. Since $(\omega^{-\frac{\chi_4\chi_6}{2}})^N = -1$, the phase φ_ω is determined so that

$$e^{iN\varphi_\omega} P^N = -\mathbf{1}. \tag{B.4}$$

The matrices T_1, T_τ and P satisfy the relations,

$$\begin{aligned} &[T_1, T_\tau] = 0, \qquad\qquad P^N = \mathbf{1}, \\ &P^{-1} T_1 P = \begin{cases} T_1^{-1} & (N=2) \\ T_\tau^{-1} T_1^{-1} & (N=3) \\ T_\tau^{-1} & (N=4) \\ T_\tau^{-1} T_1 & (N=6) \end{cases}, \quad P^{-1} T_\tau P = \begin{cases} T_\tau^{-1} & (N=2) \\ T_1 & (N=3,4,6) \end{cases}, \end{aligned} \tag{B.5}$$

which reflect the properties of $\mathcal{O}_1$, $\mathcal{O}_\tau$ and $\mathcal{O}_\omega$.

Here we perform a gauge transformation,

$$A_M \to U A_M U^{-1} + iU\partial_M U^{-1}, \qquad \Psi \to U\Psi, \tag{B.6}$$

[11] More properly, they are related through automorphisms of the Lie algebra of G. For simplicity, we do not consider a case of outer automorphisms [23].

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where

$$\mathrm{U}(z) \equiv \exp\left\{-\frac{\mathrm{Im}\,(\tau\bar{z})}{\mathrm{Im}\,\tau}\ln T_1 - \frac{\mathrm{Im}\,z}{\mathrm{Im}\,\tau}\ln T_\tau\right\}, \tag{B.7}$$

Using (B.5), we can show that

$$\begin{aligned} \mathrm{U}(z+1) &= \mathrm{U}(z)T_1^{-1}, & \mathrm{U}(z+\tau) &= \mathrm{U}(z)T_\tau^{-1}, \\ P^{-1}\mathrm{U}(\omega z)P &= \mathrm{U}(z), & P^{-1}\left(iU\partial_z U^{-1}\right)P &= \omega^{-1}\left(iU\partial_z U^{-1}\right). \end{aligned} \tag{B.8}$$

Thus, the matrices T_1 and T_τ in (B.1) and (B.2) can be absorbed by this gauge transformation, while the conditions in (B.3) are unchanged. Since we need the fermionic zero-modes, we assume that $\varphi_1 = \varphi_\tau = 0$ for the fermion that the quarks are embedded. Then the orbifold boundary conditions are reexpressed as (2.8) and (5.2).

C Decomposition of G representations

Here we list various representations of G =SO(5),SU(4),SO(7), and their irreducible decompositions to multiplets of the $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}(\times\mathrm{U}(1)_X)$ subgroup.

Each representation is specified by the Dynkin coefficients m_i $(i = 1, \cdots, r)$, and the highest weight is expressed as $\mu_{\max} = \sum_i m_i\mu_i$, where μ_i are the fundamental weights. The dimension of the representation is calculated by the Weyl dimension formula:

$$\dim\mathcal{R} = \prod_l \frac{\sum_i (m_i+1)l_i\,|\alpha_i|^2}{\sum_i l_i\,|\alpha_i|^2}, \tag{C.1}$$

where α_i are the simple roots, and l_i are numbers such that $\sum_i l_i\alpha_i$ are positive roots. We focus on irreducible representations whose dimensions are less than 30 in the following.[12]

C.1 SO(5)

The simple roots are $(\alpha_1, \alpha_2) = (\boldsymbol{e}^1 - \boldsymbol{e}^2, \boldsymbol{e}^2)$, and the fundamental weights are $(\mu_1, \mu_2) = (\boldsymbol{e}^1, \frac{\boldsymbol{e}^1+\boldsymbol{e}^2}{2})$. The dimension formula (C.1) becomes

$$\dim\mathcal{R} = \frac{1}{6}(m_1+1)(m_2+1)(m_1+m_2+2)(2m_1+m_2+3). \tag{C.2}$$

The decompositions to the irreducible representations of $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}$ are as follows.

$[\boldsymbol{m_1}, \boldsymbol{m_2}] = [\mathbf{1}, \mathbf{0}]$

$$\mathbf{5} = (\mathbf{2}, \mathbf{2}) + (\mathbf{1}, \mathbf{1}). \tag{C.3}$$

$[\boldsymbol{m_1}, \boldsymbol{m_2}] = [\mathbf{0}, \mathbf{1}]$

$$\mathbf{4} = (\mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{2}). \tag{C.4}$$

$[\boldsymbol{m_1}, \boldsymbol{m_2}] = [\mathbf{2}, \mathbf{0}]$

$$\mathbf{14} = (\mathbf{3}, \mathbf{3}) + (\mathbf{2}, \mathbf{2}) + (\mathbf{1}, \mathbf{1}). \tag{C.5}$$

[12] The irreducible decompositions of other representations and the weights of each representation are easily obtained by using LieART [30].

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$[m_1, m_2] = [1, 1]$

$$\mathbf{16} = (\mathbf{3}, \mathbf{2}) + (\mathbf{2}, \mathbf{3}) + (\mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{2}). \tag{C.6}$$

$[m_1, m_2] = [0, 2]$

This is the adjoint representation and decomposed as (3.2).

$[m_1, m_2] = [0, 3]$

$$\mathbf{20} = (\mathbf{4}, \mathbf{1}) + (\mathbf{3}, \mathbf{2}) + (\mathbf{2}, \mathbf{3}) + (\mathbf{1}, \mathbf{4}). \tag{C.7}$$

C.2 SU(4)

The simple roots are $(\alpha_1, \alpha_2, \alpha_3) = (\sqrt{2}\boldsymbol{e}^1, -\frac{\boldsymbol{e}^1}{\sqrt{2}} - \frac{\boldsymbol{e}^2}{\sqrt{2}} + \boldsymbol{e}^3, \sqrt{2}\boldsymbol{e}^2)$, and the fundamental weights are $(\mu_1, \mu_2, \mu_3) = (\frac{\boldsymbol{e}^1}{\sqrt{2}} + \frac{\boldsymbol{e}^3}{2}, \boldsymbol{e}^3, \frac{\boldsymbol{e}^2}{\sqrt{2}} + \frac{\boldsymbol{e}^3}{2})$. The dimension formula (C.1) becomes

$$\begin{aligned}\dim \mathcal{R} = \frac{1}{12}&(m_1+1)(m_2+1)(m_3+1)(m_1+m_2+2)\\ &\times(m_2+m_3+2)(m_1+m_2+m_3+3).\end{aligned} \tag{C.8}$$

The decompositions to the irreducible representations of $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R} \times \mathrm{U}(1)_X$ are as follows.

$[m_1, m_2, m_3] = [1, 0, 0]$

$$\mathbf{4} = (\mathbf{2}, \mathbf{1})_{+\mathbf{1}} + (\mathbf{1}, \mathbf{2})_{-\mathbf{1}}. \tag{C.9}$$

$[m_1, m_2, m_3] = [0, 1, 0]$

$$\mathbf{6} = (\mathbf{2}, \mathbf{2})_{\mathbf{0}} + (\mathbf{1}, \mathbf{1})_{+\mathbf{2}} + (\mathbf{1}, \mathbf{1})_{-\mathbf{2}}. \tag{C.10}$$

$[m_1, m_2, m_3] = [0, 0, 1]$

$$\bar{\mathbf{4}} = (\mathbf{2}, \mathbf{1})_{-\mathbf{1}} + (\mathbf{1}, \mathbf{2})_{+\mathbf{1}}. \tag{C.11}$$

$[m_1, m_2, m_3] = [1, 0, 1]$

This is the adjoint representation and decomposed as (3.14).

$[m_1, m_2, m_3] = [0, 1, 1]$

$$\mathbf{20} = (\mathbf{3}, \mathbf{2})_{-\mathbf{1}} + (\mathbf{2}, \mathbf{3})_{+\mathbf{1}} + (\mathbf{2}, \mathbf{1})_{+\mathbf{1}} + (\mathbf{2}, \mathbf{1})_{-\mathbf{3}} + (\mathbf{1}, \mathbf{2})_{+\mathbf{3}} + (\mathbf{1}, \mathbf{2})_{-\mathbf{1}}. \tag{C.12}$$

$[m_1, m_2, m_3] = [0, 2, 0]$

$$\mathbf{20'} = (\mathbf{3}, \mathbf{3})_{\mathbf{0}} + (\mathbf{2}, \mathbf{2})_{+\mathbf{2}} + (\mathbf{2}, \mathbf{2})_{-\mathbf{2}} + (\mathbf{1}, \mathbf{1})_{+\mathbf{4}} + (\mathbf{1}, \mathbf{1})_{-\mathbf{4}} + (\mathbf{1}, \mathbf{1})_{\mathbf{0}}. \tag{C.13}$$

$[m_1, m_2, m_3] = [1, 1, 0]$

$$\overline{\mathbf{20}} = (\mathbf{3}, \mathbf{2})_{+\mathbf{1}} + (\mathbf{2}, \mathbf{3})_{-\mathbf{1}} + (\mathbf{2}, \mathbf{1})_{+\mathbf{3}} + (\mathbf{2}, \mathbf{1})_{-\mathbf{1}} + (\mathbf{1}, \mathbf{2})_{+\mathbf{1}} + (\mathbf{1}, \mathbf{2})_{-\mathbf{3}}. \tag{C.14}$$

$[m_1, m_2, m_3] = [0, 0, 3]$

$$\mathbf{20''} = (\mathbf{4}, \mathbf{1})_{-\mathbf{3}} + (\mathbf{3}, \mathbf{2})_{-\mathbf{1}} + (\mathbf{2}, \mathbf{3})_{+\mathbf{1}} + (\mathbf{1}, \mathbf{4})_{+\mathbf{3}}. \tag{C.15}$$

$[m_1, m_2, m_3] = [3, 0, 0]$

$$\overline{\mathbf{20}}'' = (\mathbf{4}, \mathbf{1})_{+\mathbf{3}} + (\mathbf{3}, \mathbf{2})_{+\mathbf{1}} + (\mathbf{2}, \mathbf{3})_{-\mathbf{1}} + (\mathbf{1}, \mathbf{4})_{-\mathbf{3}}. \tag{C.16}$$

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C.3 SO(7)

The simple roots are $(\alpha_1, \alpha_2, \alpha_3) = (\boldsymbol{e}^1 - \boldsymbol{e}^2, \boldsymbol{e}^2 - \boldsymbol{e}^3, \boldsymbol{e}^3)$, and the fundamental weights are $(\mu_1, \mu_2, \mu_3) = (\boldsymbol{e}^1, \boldsymbol{e}^1 + \boldsymbol{e}^2, \frac{\boldsymbol{e}^1+\boldsymbol{e}^2+\boldsymbol{e}^3}{2})$. The dimension formula (C.1) becomes

$$\begin{aligned}\dim \mathcal{R} = \frac{1}{720}&(m_1+1)(m_2+1)(m_3+1)(m_1+m_2+2)(m_2+m_3+2)(2m_2+m_3+3)\\ &\times(m_1+m_2+m_3+3)(m_1+2m_2+m_3+4)(2m_1+2m_2+m_3+5).\end{aligned} \tag{C.17}$$

The $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}}$ subgroup is chosen as $(\alpha_L, \alpha_R) = (\boldsymbol{e}^1 + \boldsymbol{e}^2, \boldsymbol{e}^1 - \boldsymbol{e}^2)$. The decompositions to the irreducible representations of $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}} \times \mathrm{U}(1)_X$ are as follows.

$[\boldsymbol{m_1}, \boldsymbol{m_2}, \boldsymbol{m_3}] = [\mathbf{1}, \mathbf{0}, \mathbf{0}]$

$$\mathbf{7} = \mathbf{(2,2)_0} + \mathbf{(1,1)_{+1}} + \mathbf{(1,1)_{-1}} + \mathbf{(1,1)_0}. \tag{C.18}$$

$[\boldsymbol{m_1}, \boldsymbol{m_2}, \boldsymbol{m_3}] = [\mathbf{0}, \mathbf{1}, \mathbf{0}]$

This is the adjoint representation and decomposed as (3.21).

$[\boldsymbol{m_1}, \boldsymbol{m_2}, \boldsymbol{m_3}] = [\mathbf{0}, \mathbf{0}, \mathbf{1}]$

$$\mathbf{8} = \mathbf{(2,1)_{+1/2}} + \mathbf{(2,1)_{-1/2}} + \mathbf{(1,2)_{+1/2}} + \mathbf{(1,2)_{-1/2}}. \tag{C.19}$$

$[\boldsymbol{m_1}, \boldsymbol{m_2}, \boldsymbol{m_3}] = [\mathbf{2}, \mathbf{0}, \mathbf{0}]$

$$\begin{aligned}\mathbf{27} = &\mathbf{(3,3)_0} + \mathbf{(2,2)_{+1}} + \mathbf{(2,2)_{-1}} + \mathbf{(2,2)_0} + \mathbf{(1,1)_{+2}}\\ &+\mathbf{(1,1)_{+1}} + \mathbf{(1,1)_0} + \mathbf{(1,1)_0} + \mathbf{(1,1)_{-1}} + \mathbf{(1,1)_{-2}}.\end{aligned} \tag{C.20}$$

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