The one-loop worldsheet S-matrix for the $AdS_n \times S^n \times T^{10-2n}$ superstring

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ABSTRACT: We compute the massive-sector worldsheet S-matrix for superstring theories in $AdS_n \times S^n \times T^{10-2n}$ (with $n = 2, 3, 5$) in the near BMN expansion up to one-loop order in inverse string tension. We show that, after taking into account the wave function renormalization, the one-loop S-matrix is UV finite. In an appropriate regularization scheme the S-matrix is consistent with the underlying symmetries of the superstring theory, i.e. for the $n = 3, 5$ cases it coincides with the one implied by the light-cone gauge symmetries with the dressing phases determined from the crossing equations. For the $n = 2, 3$ cases we observe that the massless modes decouple from the one-loop calculation of massive mode scattering, i.e. the $2n$-dimensional supercoset sigma model and the full 10-dimensional superstring happen to have the same massive one-loop S-matrix.

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Contents

1 Introduction

The Green-Schwarz (GS) superstring sigma model corresponding to a consistent 10d super-gravity background should be one-loop UV finite when considered in conformal gauge and restricted to on-shell values of background worldsheet fields. This applies, in particular, to the one-loop partition function of the $AdS_5 \times S^5$ superstring evaluated on a classical string solution. Divergences may appear if one first solves for the 2d metric (i.e. starts with Nambu-Goto-type action) or considers off-shell correlators. Then the computation of the worldsheet S-matrix near some vacuum such as the one provided by the BMN point-like string (usually done in a "light-cone" or mixed coordinate-momentum gauge) may not a priori produce a UV finite result.

Indeed, past attempts of one-loop BMN S-matrix computations led to UV divergent results. This is puzzling as one would like to provide a perturbative one-loop check of

formal constructions of BMN vacuum S-matrices in $AdS_n \times S^n \times T^{10-2n}$ theories[1] which are based on symmetry considerations and general properties (integrability, unitarity, crossing) and assume that the S-matrix should be UV finite. The aim of the present paper is to resolve this problem by showing that the one-loop worldsheet S-matrix computed directly from the superstring action and properly defined to account for non-trivial wave-function renormalization is indeed UV finite.[2]

Previous work on one-loop BMN S-matrices in $AdS_n \times S^n \times T^{10-2n}$ include:

- The near flat space (NFS) limit computations ($n = 2, 3, 5$) [5–12]

- Constructions based on generalized unitarity ($n = 3, 5$) [13–15]

- Computations of some finite BMN amplitudes ($n = 2, 3$) [12, 16].

Below we will present the direct computation of the full near-BMN 2-particle S-matrix not relying on truncations or assumptions. We will find that the divergences which appear at intermediate stages may be interpreted as wave-function renormalization of the bosons and they cancel in the S-matrix defined according to standard rules. Furthermore, there exists a symmetry-preserving regularization scheme in which the resulting finite S-matrix matches the (massive sector) S-matrix found previously from symmetries and crossing considerations for $n = 3, 5$ theories. In the $n = 2$ case the perturbative S-matrix agrees with earlier calculations performed in [12] and the recent suggestion to fix the S-matrix using symmetries and the Yang-Baxter equation [17].

Let us summarize our main results. We are interested in the S-matrix for scattering of massive excitations at one loop in the near-BMN expansion. A naive direct calculation shows that some of the one-loop scattering amplitudes appear to diverge

$$\mathcal{A}^{(\text{naive})}(zz \to zz) = \text{infinite}, \quad \mathcal{A}^{(\text{naive})}(yz \to yz) = \text{finite}, \quad \mathcal{A}^{(\text{naive})}(yy \to yy) = \text{infinite}.$$

Here z and y denote the AdS_n and S^n bosonic excitations respectively. However, to properly define the amplitudes and S-matrix one needs to take into account the field (or "wave-function") renormalization.[3] The latter is computed from the (unrenormalized) one-loop *off-shell* two-point functions[4]

$$\langle zz \rangle = \frac{iZ_z}{p^2 - m^2} + \mathcal{O}(g^{-2}), \qquad \langle yy \rangle = \frac{iZ_y}{p^2 - m^2} + \mathcal{O}(g^{-2}). \tag{1.1}$$

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[1]We focus on these particular examples as they are the simplest to analyze perturbatively, having for example no cubic interaction vertices. The integrability of the string in these backgrounds was pointed out in [1–3].

[2]A finite theory may still require infinite wave-function renormalization, a well known example being $\mathcal{N} = 4$ super Yang-Mills theory in dimensional regularization, see, e.g., [4] and references there.

[3]If it were possible to argue that the y^2z^2 vertices should not be renormalized, then the finiteness of $\mathcal{A}^{(\text{naive})}(yz \to yz)$ would imply that the renormalization factors of the z and y fields should obey $Z_zZ_y = finite$, i.e. that the corresponding one-loop divergences should have opposite sign.

[4]The masses will be set to 1 in our conventions but we keep them here for clarity.

Explicit calculations show that the wave-function renormalization factors are given by

$$Z_z = 1 + \frac{1}{4\pi\hat{g}}\left(-\frac{2}{\epsilon} + \dots\right), \quad Z_y = 1 - \frac{1}{4\pi\hat{g}}\left(-\frac{2}{\epsilon} + \dots\right), \quad \hat{g} = \begin{cases} g & \text{for} \quad n = 3,5 \\ 2g & \text{for} \quad n = 2 \end{cases} \quad (1.2)$$

where g is the string tension (the effective worldsheet coupling is g^{-1}).[5] The UV divergence comes from the tadpole integral

$$\int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2 - m^2} \to \frac{i}{4\pi}\left(-\frac{2}{\epsilon} + \gamma_E + \log\frac{m^2}{4\pi}\right), \quad (1.3)$$

where we evaluated the integral in dimensional regularization in $2 - \epsilon$ dimensions. There is no independent mass renormalization, which is consistent with the BMN vacuum being 1/2 BPS.

Note that the bosonic field renormalization is of opposite signs for the AdS_n and the S^n excitations. While this may appear at odds with the non-manifest BMN vacuum symmetry, e.g. $[\text{PSU}(2|2)]^2$ for $n = 5$, all that we can ask is that the S-matrix have this symmetry, which it does.[6] It is also interesting to observe that the results are universal in n assuming that in the AdS_2 case the string tension g is effectively replaced by $2g$. A similar effect has been noticed earlier at one [12] and two [11] loops.[7] The two-point function of the fermions turns out not to get renormalized at the one-loop order.

Taking this wave-function renormalization into account, the scattering amplitudes are given by[8]

$$\begin{aligned} \mathcal{A}(zz \to zz) &= (\sqrt{Z_z})^4 \mathcal{A}^{(\text{naive})}(zz \to zz), \\ \mathcal{A}(yz \to yz) &= (\sqrt{Z_y})^2(\sqrt{Z_z})^2 \mathcal{A}^{(\text{naive})}(yz \to yz), \\ \mathcal{A}(yy \to yy) &= (\sqrt{Z_y})^4 \mathcal{A}^{(\text{naive})}(yy \to yy), \end{aligned} \quad (1.4)$$

and these are found to be finite, implying that no other (coupling or vertex) renormalizations are indeed required.

Equivalently, given a field theory with quartic (and higher-point) interaction vertices, one may start with a Lagrangian with Z-factors introduced for all terms. Requiring the two-point functions to be finite determines the wave-function Z-factors as in (1.2). Next, requiring that the on-shell four-point function is finite determines the Z-factors in front of the quartic coupling. In our case their divergent part is given by $Z_{\phi_1\phi_2\phi_3\phi_4} = \sqrt{Z_{\phi_1}Z_{\phi_2}Z_{\phi_3}Z_{\phi_4}}$.

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[5]We will carry out the calculations in light-cone gauge, which is the special case $a = 1/2$ of the interpolating a-gauge of [18]. In general, the renormalization factors Z_y and Z_z may depend on the gauge-fixing parameter a and, because of absence of a Z_2 symmetry between y and z fields, are not expected to be related by simply changing the sign of the one-loop term.

[6]It is worth mentioning that, in theories in which symmetries are not manifest or realized only on shell, fields belonging to the same representation/multiplet may still be renormalized differently without spoiling the symmetry. An example is provided by $\mathcal{N} = 4$ super Yang-Mills theory where, in a component formulation, vector and scalar fields have different renormalization factors [4].

[7]It was slightly hidden there due to the fact that the string tension was called $g/2$ instead of g.

[8]For a standard definition of renormalized S-matrix elements see, e.g., [19].

This structure implies that there is in fact no genuine renormalization of the quartic couplings as this is controlled by the ratio $Z_{\phi_1\phi_2\phi_3\phi_4}/\sqrt{Z_{\phi_1}Z_{\phi_2}Z_{\phi_3}Z_{\phi_4}}$ which is finite.[9]

While this wave-function renormalization renders the S-matrix finite one still has to be careful with how one regularizes the divergent integrals that appear in intermediate steps: the regularization should be consistent with underlying symmetries.[10] A naive approach based on computing all integrals in dimensional regularization gives an S-matrix which differs from the one determined by the symmetries, i.e. this regularization breaks (or at least gives a different realization of) the symmetries preserved by the BMN vacuum. As we shall explain below, there is an improved regularization prescription based on first reducing the one-loop integrals to a few divergent (tadpole) integrals by using algebraic identities in $d = 2$ and then computing the latter integrals in dimensional regularization. This regularization scheme leads to the same one-loop S-matrix as determined by the symmetries and crossing equations for $AdS_5 \times S^5$ ($n = 5$) (see, e.g., [23]) and $AdS_3 \times S^3 \times T^4$ ($n = 3$) [24] theories. In $AdS_2 \times S^2 \times T^6$ ($n = 2$) the result is compatible with previous calculations performed in [12] and the recent derivation of the S-matrix from symmetries and the Yang-Baxter equation in [17].

This regularization prescription is therefore compatible with the symmetries of the BMN vacuum and with integrability, at least up to one loop order. It also has the interesting feature that the massless modes present in the $n = 2$ and $n = 3$ cases decouple completely from the computation of the massive S-matrix, i.e. completely cancel out from internal lines of one-loop graphs. In that sense the supercoset model appears to be equivalent to the full superstring as far as the massive one-loop S-matrix is concerned. This feature should no longer be true at two-loop order (see for example [11]).

The outline of this paper is as follows. In section 2 we shall describe the general structure of the 10d superstring action to quartic order in fermions. In section 3 we shall specify to the case of $AdS_n \times S^n \times T^{10-2n}$ theories and fix the light-cone gauge adapted to the BMN vacuum. Section 4 describes our regularization procedure. The results for the one-loop massive sector S-matrix are presented in section 5 with details in appendix B. Appendix A contains some relations between one-loop integrals. In appendix C we comment on the computation of the near BMN S-matrix and dispersion relation in conformal gauge.

2 Superstring action

The Green-Schwarz superstring action can be expanded in powers of fermions (here g denotes the string tension)

$$S = g \int d^2\xi \left(\mathcal{L}^{(0)} + \mathcal{L}^{(2)} + \dots \right). \tag{2.1}$$

In the $AdS_5 \times S^5$ case it is known to all orders in fermions [25] due to the background being maximally supersymmetric. However, in a general 10d supergravity background it

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[9]This is consistent with the corresponding beta-function being zero since it is determined in terms of the same ratio $Z_{\phi_1\phi_2\phi_3\phi_4}/\sqrt{Z_{\phi_1}Z_{\phi_2}Z_{\phi_3}Z_{\phi_4}}$.

[10]Equivalently, preservation of symmetries (including hidden ones related to integrability) may require a particular choice of finite counterterms, see, e.g., [20–22] for the complex Sine-Gordon theory example.

is only known to quartic order [26]. The purely bosonic terms in the Lagrangian are

$$\mathcal{L}^{(0)} = \frac{1}{2}\gamma^{ij}e_i{}^a e_j{}^b \eta_{ab} + \frac{1}{2}\varepsilon^{ij}B_{ij}^{(0)}, \qquad \gamma^{ij} = \sqrt{-h}\,h^{ij}, \tag{2.2}$$

where we denote the bosonic vielbein pulled back to the worldsheet by $e_i{}^a$ ($a = 0, \ldots, 9$; $i = 0, 1$) and $B_{ij}^{(0)} = e_i{}^a e_j{}^b B_{ab}^{(0)}$ is the lowest component in the Grassmann parameter Θ-expansion of the NSNS two-form superfield B. For the terms involving fermions we will follow [26] and write the expressions appropriate to type IIA supergravity, i.e. $\Theta^{\underline{\alpha}}$ will be a 32-component Majorana spinor. At the end we will describe how to get the type IIB expressions by performing some simple substitutions.

The terms quadratic in fermions take the form

$$\mathcal{L}^{(2)} = \frac{i}{2}e_i{}^a\,\Theta\Gamma_a K^{ij}\mathcal{D}_j\Theta, \qquad K^{ij} = \gamma^{ij} - \varepsilon^{ij}\Gamma_{11}, \tag{2.3}$$

where

$$\mathcal{D}\Theta = \left(d - \frac{1}{4}\omega^{ab}\Gamma_{ab} + \frac{1}{8}e^a M_a\right)\Theta, \qquad M_a = H_{abc}\,\Gamma^{bc}\Gamma_{11} + \mathcal{S}\Gamma_a. \tag{2.4}$$

Here ω^{ab} is the spin connection, $H = dB$ is the NSNS three-form field strength and the RR fields enter the action through the bispinor[11]

$$\mathcal{S} = e^\phi\left(\frac{1}{2}F_{ab}^{(2)}\Gamma^{ab}\Gamma_{11} + \frac{1}{4!}F_{abcd}^{(4)}\Gamma^{abcd}\right). \tag{2.5}$$

The quartic fermionic terms are somewhat more complicated [26]

$$\mathcal{L}^{(4)} = -\frac{1}{8}\Theta\Gamma^a\mathcal{D}_i\Theta\,\Theta\Gamma_a K^{ij}\mathcal{D}_j\Theta + \frac{i}{24}e_i{}^a\,\Theta\Gamma_a K^{ij}\mathcal{M}\mathcal{D}_j\Theta + \frac{i}{192}e_i{}^a e_j{}^b\,\Theta\Gamma_a K^{ij}(M + \tilde{M})\mathcal{S}\Gamma_b\Theta$$

$$+ \frac{1}{192}e_i{}^c e_j{}^d\,\Theta\Gamma_c{}^{ab}K^{ij}\Theta\,(3\Theta\Gamma_d U_{ab}\Theta - 2\Theta\Gamma_a U_{bd}\Theta)$$

$$- \frac{1}{192}e_i{}^c e_j{}^d\,\Theta\Gamma_c{}^{ab}\Gamma_{11}K^{ij}\Theta\,(3\Theta\Gamma_d\Gamma_{11}U_{ab}\Theta + 2\Theta\Gamma_a\Gamma_{11}U_{bd}\Theta). \tag{2.6}$$

Here $\tilde{M} = \Gamma_{11}M\Gamma_{11}$ and we defined

$$\mathcal{M}^\alpha{}_\beta = M^\alpha{}_\beta + \tilde{M}^\alpha{}_\beta + \frac{i}{8}(M^a\Theta)^\alpha\,(\Theta\Gamma_a)_\beta - \frac{i}{32}(\Gamma^{ab}\Theta)^\alpha\,(\Theta\Gamma_a M_b)_\beta - \frac{i}{32}(\Gamma^{ab}\Theta)^\alpha\,(C\Gamma_a M_b\Theta)_\beta,$$

$$M^\alpha{}_\beta = \frac{1}{2}\Theta T\Theta\,\delta^\alpha_\beta - \frac{1}{2}\Theta\Gamma_{11}T\Theta\,(\Gamma_{11})^\alpha{}_\beta + \Theta^\alpha\,(CT\Theta)_\beta + (\Gamma^a T\Theta)^\alpha\,(\Theta\Gamma_a)_\beta, \tag{2.7}$$

$$T = \frac{i}{2}\nabla_a\phi\,\Gamma^a + \frac{i}{24}H_{abc}\,\Gamma^{abc}\Gamma_{11} + \frac{i}{16}\Gamma_a\mathcal{S}\Gamma^a, \tag{2.8}$$

$$U_{ab} = \frac{1}{4}\nabla_{[a}M_{b]} + \frac{1}{32}M_{[a}M_{b]} - \frac{1}{4}R_{ab}{}^{cd}\,\Gamma_{cd}. \tag{2.9}$$

The dilatino equation is $T\xi = 0$ while the integrability condition for the gravitino equation is $U_{ab}\xi = 0$, where ξ is a Killing spinor [26, 27].

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[11]Here ϕ is the dilaton and we use the convention $F^{(n)} = \frac{1}{n!}dx^{m_n}\wedge\cdots\wedge dx^{m_1}F_{m_1\ldots m_n}$ for the form fields.

To find the corresponding type IIB string expressions the 32-component Majorana spinor $\Theta^{\underline{\alpha}}$ should be replaced by a doublet of 16-component Majorana-Weyl spinors $\Theta^{\alpha 1}, \Theta^{\alpha 2}$. Similarly, the 32×32 Dirac matrices are replaced by the 16×16 ones as follows

$$\Gamma_a \to \gamma_a\,, \qquad\qquad \Gamma_{11} \to \sigma^3\,, \tag{2.10}$$

with one exception: $\Gamma_{11} T \to -\sigma^3 T$. Finally, instead of the bispinor $\mathcal{S}$ defined in (2.5) one should use the expression appropriate to the type IIB theory[12]

$$\mathcal{S} = -e^{\phi}\left(i\sigma^2 \gamma^a F_a^{(1)} + \frac{1}{3!}\sigma^1 \gamma^{abc} F_{abc}^{(3)} + \frac{1}{2 \cdot 5!} i\sigma^2 \gamma^{abcde} F_{abcde}^{(5)} \right)\,. \tag{2.11}$$

With these replacements all the previous expressions apply also for the superstring in a type IIB supergravity background.

The superstring action simplifies in the cases we are considering in this paper as all RR background fields are constant (and there is no NSNS flux, $H_{abc} = 0$)[13]

$$AdS_5 \times S^5: \quad F^{(5)} = 4e^{-\phi}(\Omega_{AdS_5} + \Omega_{S^5})\,,$$

$$AdS_3 \times S^3 \times T^4: \quad F^{(4)} = 2e^{-\phi} dx^9 \wedge (\Omega_{AdS_3} + \Omega_{S^3})\,, \tag{2.12}$$

$$AdS_2 \times S^2 \times T^6: \quad F^{(2)} = e^{-\phi}\Omega_{AdS_2}\,, \quad F^{(4)} = -e^{-\phi}\Omega_{S^2} \wedge \left(dx^5 \wedge dx^4 + dx^7 \wedge dx^6 + dx^9 \wedge dx^8 \right)$$

Here the AdS_n and S^n radii are set to be 1. We also find from (2.5), (2.11) and (2.8)

$$AdS_5 \times S^5: \quad \mathcal{S} = -4i\sigma^2 \gamma^{01234}\,, \ T = 0\,,$$

$$AdS_3 \times S^3 \times T^4: \quad \mathcal{S} = -4\mathcal{P}_{16}\Gamma^{0129}\,, \ T = -\frac{i}{2}\Gamma^{0129}(1 - \mathcal{P}_{16})\,, \tag{2.13}$$

$$AdS_2 \times S^2 \times T^6: \quad \mathcal{S} = -4\mathcal{P}_8\Gamma^{01}\Gamma_{11}\,, \ T = \frac{i}{2}\Gamma^{01}\Gamma_{11}(1 - \mathcal{P}_8)\,,$$

where we have defined the following three projection operators, with the dimension of the subspace they project on, i.e. the number of supersymmetries preserved by the background, indicated

$$\mathcal{P}_{16} = \frac{1}{2}(1 + \Gamma^{012345})\,, \qquad \mathcal{P}_8 = \frac{1}{4}(1 - \Gamma^{4567} - \Gamma^{4589} - \Gamma^{6789})\,. \tag{2.14}$$

We will take the metric of AdS_n in the form

$$ds^2_{AdS_n} = -\left(\frac{1 + \frac{1}{2}|z_I|^2}{1 - \frac{1}{2}|z_I|^2} \right)^2 dt^2 + \frac{2|dz_I|^2}{(1 - \frac{1}{2}|z_I|^2)^2} \qquad I = 1, \ldots, (n-1)/2\,, \tag{2.15}$$

where the spatial coordinates are grouped together into two complex coordinates in AdS_5, one in AdS_3 and one real coordinate $x_1 = \sqrt{2}z$ in AdS_2. Similarly, the S^n metric is

$$ds^2_{S^n} = \left(\frac{1 - \frac{1}{2}|y_I|^2}{1 + \frac{1}{2}|y_I|^2} \right)^2 d\varphi^2 + \frac{2|dy_I|^2}{(1 + \frac{1}{2}|y_I|^2)^2} \qquad I = 1, \ldots, (n-1)/2\,. \tag{2.16}$$

Again, we use $x_2 = \sqrt{2}y$ for the real coordinate in S^2.

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[12]Here σ^n are Pauli matrices. For more details and definitions of the gamma-matrices see [26].

[13]We follow the conventions of [27]. Note that for the AdS_2 and AdS_3 case we give the fluxes of the type IIA solution. The corresponding type IIB solution is obtained by T-duality in a torus direction. For the AdS_5 case the full superstring action is known in the form of a supercoset model [25]. This supercoset model coincides with the GS action described above up to quartic order in Θ provided the coset representative is chosen as $g = e^{x^m P_m} e^{\Theta^{\alpha} Q_{\alpha}}$ [28]. Since we will need the Θ^6-terms for the one-loop S-matrix in the fermionic sector we will use the supercoset model for our calculations in this case.

3 Near BMN expansion of the $AdS_n \times S^n \times T^{10-2n}$ string action

Given the form of the background fields we can now expand the string action around the BMN vacuum $t = \varphi = \tau$ [29]. We shall fix the light-cone gauge and the corresponding kappa symmetry gauge as

$$x^+ \equiv \frac{1}{2}(t + \varphi) = \tau\,, \qquad \Gamma^+ \Theta = 0\,, \tag{3.1}$$

where the complete gauge fixing also includes the conditions

$$p^+ \equiv -\frac{1}{2}\frac{\partial \mathcal{L}}{\partial \dot{x}^-} = 1\,, \qquad \frac{\partial \mathcal{L}}{\partial x^{-\prime}} = 0\,. \tag{3.2}$$

In this gauge the worldsheet metric in (2.2) takes the form $\gamma_{ij} = \eta_{ij} + \hat{\gamma}_{ij}$, where $\hat{\gamma}_{ij}$ denotes higher order corrections to be determined from the above conditions.[14]

Next, let us consider the near BMN expansion of the action, i.e. in powers of the transverse coordinates and fermions. Scaling all transverse fields with a factor $g^{-1/2}$ yields

$$\mathcal{L} = \mathcal{L}_2 + \frac{1}{g}\mathcal{L}_4 + \frac{1}{g^2}\mathcal{L}_6 + \ldots\,,$$

where the subscript denotes the number of transverse coordinates in each term. Note that only terms with an even number of fields appear in the expansion. This fact simplifies the perturbative expansion in the $AdS_n \times S^n$ case compared to more complicated backgrounds. The quadratic Lagrangian $\mathcal{L}_2$ takes the form[15]

$$\mathcal{L}_2 = |\partial_i z_I|^2 - |z_I|^2 + |\partial_i y_I|^2 - |y_I|^2 + |\partial_i u_{I'}|^2 + i\bar{\chi}_L^r \partial_- \chi_L^r + i\bar{\chi}_R^r \partial_+ \chi_R^r - \bar{\chi}_L^r \chi_R^r - \bar{\chi}_R^r \chi_L^r$$
$$+ i\bar{\chi}_L^{r'} \partial_- \chi_L^{r'} + i\bar{\chi}_R^{r'} \partial_+ \chi_R^{r'}\,. \tag{3.3}$$

The field content of the $n = 5, 3, 2$ theories is summarized in table 1 and the U(1) charges are summarized in tables 2–4. For the interaction terms we will only give the bosonic terms quartic in fields due to the length of the expressions:

$$\mathcal{L}_4^B = \frac{1}{2}(|y_I|^2 - |z_I|^2)\left(|\partial_0 z_I|^2 + |\partial_1 z_I|^2 + |\partial_0 y_I|^2 + |\partial_1 y_I|^2 + |\partial_0 u_{I'}|^2 + |\partial_1 u_{I'}|^2\right)$$
$$+ |z_I|^2 |\partial_i z_I|^2 - |y_I|^2 |\partial_i y_I|^2\,. \tag{3.4}$$

4 Regularization procedure

For the $AdS_n \times S^n \times T^{10-2n}$ backgrounds under consideration the string Lagrangian expanded near the BMN vacuum contains fourth and sixth order interaction vertices. The one-loop contribution to the two-point function comes from tadpole diagrams with topology

$$(4.1)$$

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[14]The Virasoro constraints can be used to solve for x^- whose explicit form we will not need here.

[15]Here $\partial_\pm = \partial_0 \pm \partial_1$ and massless modes have a primed index.

	$m = 1$			$m = 0$	
	AdS	Sphere	Fermions	Torus	Fermions
$AdS_5 \times S^5$	z_1, z_2	y_1, y_2	$\chi^{1,2,3,4}$	—	—
$AdS_3 \times S^3 \times T^4$	z_1	y_1	$\chi^{1,2}$	u_1, u_2	$\chi^{3,4}$
$AdS_2 \times S^2 \times T^6$	x_1	x_2	χ^1	u_1, u_2, u_3	$\chi^{2,3,4}$

Table 1. Summary of the field content. All fields are complex except $(x_1, x_2) = \sqrt{2}(z, y)$ in the $AdS_2 \times S^2 \times T^6$ case. The massive fields ($m = 1$) come from the supercoset model while the massless ones ($m = 0$) are only present in the full 10d superstring theory.

	y_1	y_2	z_1	z_2	χ^1	χ^2	χ^3	χ^4
$U(1)_1$	-1	0	0	0	$-1/2$	$1/2$	$1/2$	$1/2$
$U(1)_2$	0	-1	0	0	$1/2$	$-1/2$	$1/2$	$1/2$
$U(1)_3$	0	0	-1	0	$1/2$	$1/2$	$-1/2$	$1/2$
$U(1)_4$	0	0	0	-1	$1/2$	$1/2$	$1/2$	$-1/2$

Table 2. Summary of U(1) charges for $AdS_5 \times S^5$.
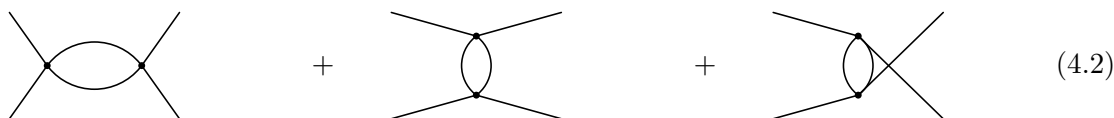
	y_1	z_1	u_1	u_2	χ^1	χ^2	χ^3	χ^4
$U(1)_1$	-1	0	0	0	$-1/2$	$1/2$	$1/2$	$1/2$
$U(1)_2$	0	-1	0	0	$1/2$	$-1/2$	$1/2$	$1/2$
$U(1)_3$	0	0	-1	0	$1/2$	$1/2$	$-1/2$	$1/2$

Table 3. Summary of U(1) charges for $AdS_3 \times S^3 \times T^4$. The U(1)'s associated to T^4 are compatible with the fluxes in (2.12) assuming $u_1 = \frac{1}{\sqrt{2}}(x^6 + ix^7)$ and $u_2 = \frac{1}{\sqrt{2}}(x^8 + ix^9)$.

	x_1, x_2	u_1	u_2	u_3	χ^1	χ^2	χ^3	χ^4
$U(1)_1$	0	-1	0	0	$-1/2$	$-1/2$	$1/2$	$1/2$
$U(1)_2$	0	0	-1	0	$-1/2$	$1/2$	$-1/2$	$1/2$
$U(1)_3$	0	0	0	-1	$-1/2$	$1/2$	$1/2$	$-1/2$

Table 4. Summary of U(1) charges for $AdS_2 \times S^2 \times T^6$. The U(1)'s associated to T^6 are compatible with the fluxes in (2.12) assuming $u_1 = \frac{1}{\sqrt{2}}(x^4 + ix^5)$, $u_2 = \frac{1}{\sqrt{2}}(x^6 + ix^7)$ and $u_3 = \frac{1}{\sqrt{2}}(x^8 + ix^9)$.

while the contribution to the four-point function comes from the three (s, t and u-channel) bubble diagrams

(4.2)

and one tadpole diagram arising from the sixth order interaction term

$$(4.3)$$

We will now describe how we evaluate these.

In the calculation of the one-loop Feynman diagrams involving only massive fields one encounters the following bubble integrals, corresponding to the diagrams in eq. (4.2),

$$B^{r,s}(P) \equiv \int \frac{d^2k}{(2\pi)^2} \frac{k_+^r k_-^s}{(k^2 - m^2)((k-P)^2 - m^2)} \,, \qquad (4.4)$$

where P is a combination of the external momenta, and also the tadpole integrals (corresponding to eq. (4.3))

$$T^{r,s}(P) \equiv \int \frac{d^2k}{(2\pi)^2} \frac{k_+^r k_-^s}{(k-P)^2 - m^2} \,. \qquad (4.5)$$

Many of these integrals are UV-divergent and need to be regularized. For the two-point function determining the wave function renormalization we only have a tadpole contribution in eq. (4.1) which we simply evaluate in dimensional regularization.

Given a sum of loop integrals one has several options to evaluate it. One may simply introduce Feynman parameters and evaluate the integrals one by one in, e.g., dimensional regularization. In the presence of power-like divergences this is typically dangerous as dimensional regularization amounts to an uncontrolled subtraction of such divergences which may include finite terms as well. A safer alternative is to employ the reduction to master integrals. In this approach one uses algebraic identities as well as identities valid only after integration to express the original regularized integrals as linear combinations of a smaller set of integrals which are in some sense linearly independent (e.g. they do not have overlapping branch cuts). For the same reason as before, use of integral identities for dimensionally-regulated power-divergent integrals may lead to an uncontrolled elimination of finite terms with rational momentum dependence. Here we will use a variant of this approach which makes use of only algebraic identities and is similar in spirit to what is sometimes called "implicit regularization". It proceeds in the following steps:[16]

1. Use algebraic identities on the integrands to reduce the result to a minimal set of divergent integrals, in our case tadpole integrals.[17]

2. Evaluate these in a suitable regularization scheme consistent with the algebraic identities used in the first step and the symmetries we want to preserve; in our case this regularization is dimensional regularization.

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[16] A similar procedure was used in [16] but tadpoles were written in terms of bubbles instead of the other way around. We have checked that our present procedure does not change any of the results obtained there.

[17] There is typically no standard choice for this set of integrals. One simply has to find (if possible) a set which leads to a result compatible with the symmetries one wants to preserve. Note also that we do not allow shifts of loop variables as this can be problematic in divergent integrals.

Let us now apply this procedure to the integrals appearing in the problem of two-particle scattering. The first step will be to use the identity $k_+k_- = k^2 - m^2 + m^2$, which implies

$$B^{r,s}(P) = T^{r-1,s-1}(P) + m^2 B^{r-1,s-1}(P) \,. \tag{4.6}$$

This allows us to reduce all relevant bubble integrals to the following set

$$B^{r,0}(P) \,, \qquad B^{0,s}(P) \,, \qquad r,s = 0,1,2,3 \,. \tag{4.7}$$

These are still (potentially) divergent[18] for $r,s \geq 2$ so we want to reduce them further. This can be done by using the identity

$$\frac{1}{(k-P)^2 - m^2} = \frac{1}{k^2 - m^2} + \frac{2k \cdot P - P^2}{((k-P)^2 - m^2)(k^2 - m^2)} \,, \tag{4.8}$$

which implies

$$P_- B^{r+1,s}(P) + P_+ B^{r,s+1}(P) = T^{r,s}(P) - T^{r,s}(0) + P^2 B^{r,s}(P) \,. \tag{4.9}$$

Combining this with the previous identity (4.6) we can reduce all bubble integrals to B^{00} and B^{01}, which are finite without any regularization, and tadpole integrals.

So far we have used only algebraic identities and made no shifts in loop variables. The next step is to note that since B^{01} is finite we are allowed to shift the integration variable. Making the shift $k \to -k + P$ gives

$$B^{01}(P) = \frac{1}{2} P_- B^{00}(P) \,. \tag{4.10}$$

We now note the important fact that for this identity to be consistent with eq. (4.9) for $(r,s) = (0,0)$ we must have

$$T^{00}(P) - T^{00}(0) = 0 \,, \tag{4.11}$$

i.e. we should be allowed to shift the loop variable in the T^{00} tadpole integral. It is then consistent to also allow shifts of loop variables in other tadpole integrals[19] which reduces them further to an even smaller set.

In the end we are left only with $B^{00}(P)$, $T^{11}(0)$ and $T^{00}(0)$ (see appendix A). The two tadpole integrals $T^{11}(0)$ and $T^{00}(0)$ can be computed in dimensional regularization which respects (4.11) and the remaining bubble integral $B^{00}(P)$ is manifestly finite.[20]

Let us note that the fact that computing all integrals in dimensional regularization (without using any algebraic identities) gives a different answer can be seen by looking, for

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[18]In a Lorentz-invariant regularization scheme such as dimensional regularization they are finite. Here, however, we are not interested in preserving Lorentz invariance but rather a non-relativistic symmetry of the BMN vacuum.

[19]We could of course instead just compute them directly in dimensional regularization.

[20]One reason for using dimensional regularization in this last step is that it removes the quadratic divergence in $T^{11}(0)$. This quadratic divergence appears not to be consistent with the symmetries of the BMN vacuum. In general, there may be additional quadratically divergent terms coming from the measure and local field redefinition factors, and use of dimensional regularization allows us to ignore them too.

example, at $T^{01}(P)$ which is linearly divergent. In dimensional regularization we can shift the loop variable to get

$$T^{01}(P) = T^{01}(0) + P_- T^{00}(0) = P_- T^{00}(0) \,. \tag{4.12}$$

On the other hand, we could use the algebraic identities (4.9) and (4.6) to write

$$T^{01}(P) = P_- T^{00}(P) + P_+ B^{02}(P) + \left(m^2 - \frac{1}{2}P^2\right)P_- B^{00}(P) \,. \tag{4.13}$$

The right hand sides in these two expressions are not equal in dimensional regularization — they differ by rational terms coming from $B^{02}(P)$.[21] We find it natural to require that algebraic identities should always hold and only allow shifts in loop momenta when it is consistent with this requirement.

In the $AdS_3 \times S^3 \times T^4$ and $AdS_2 \times S^2 \times T^6$ cases we also have massless modes in the near-BMN action which means that we will have integrals of the form (4.4) and (4.5) with $m = 0$. Note that no bubble integrals involving different masses appear in one-loop diagrams contributing to the two-particle S-matrix with all massive external states.[22] The integrals which appear can be reduced in the same way as described above (using essentially the same identities). We shall treat IR-divergent integrals by introducing a small regulator mass. In the end it turns out that the massless modes give no contribution and thus could be truncated away from the beginning, i.e. the supercoset sigma model gives the full answer for the massive S-matrix even though it is not in general equivalent to the full superstring theory (at least not in the $AdS_2 \times S^2 \times T^6$ case where the supercoset model cannot be obtained by kappa symmetry gauge-fixing of the full 10d superstring action though it is a consistent classical truncation [3]).

This decoupling of the massless modes only holds in the regularization described above, i.e. is not true in general. For example, if one computes the S-matrix in the near-flat-space limit one gets the correct result by just using dimensional regularization but in that regularization the massless modes give a non-vanishing contribution (see, for example, [12]). If one used the regularization described above one would find again that they decouple, with the final result still being the same.

5 One-loop massive sector S-matrix

Having established the notation and the regularization scheme we now turn to the perturbative computation of the worldsheet S-matrix,

$$\mathbb{S} = \mathbb{1} + i\mathbb{T}, \qquad \mathbb{T} = \frac{1}{g}\mathbb{T}^{(0)} + \frac{1}{g^2}\mathbb{T}^{(1)} + \mathcal{O}(g^{-3}) \,, \tag{5.1}$$

---

[21]The integral $B^{02}(P)$ contains a divergence which happens to be a total derivative. In dimensional regularization this term gives no contribution but in a regularization which keeps surface terms it will contribute additional rational terms. This is the origin of the regularization ambiguity.

[22]Bubble integrals with one massive and one massless internal propagator appear in the calculation of the two-point function of massive fields in conformal gauge discussed in appendix C.2.

where the superscripts (0) and (1) denote the tree-level and one-loop contributions, respectively. The $\mathbb{T}$-matrix maps a two-particle in-state to a corresponding two-particle out-state

$$\mathbb{T}|A(p)B(q)\rangle = T_{AB}^{CD}|C(p)D(q)\rangle \,, \tag{5.2}$$

where the capital letters denote any type of bosonic or fermionic excitation. We will ignore the imaginary terms in T_{AB}^{CD} since they are completely determined in terms of tree-level amplitudes via the optical theorem and are not sensitive to regularization. Furthermore, for an integrable system in two dimensions the energy-momentum conservation implies that the outgoing momenta are at most a permutation of the incoming momenta p and q.

The specific in- and out-states that we will consider consist of massive bosonic and fermionic excitations. For the $n = 5$ or $n = 3$ theories where the worldsheet fields are complex, we will denote the two-particle asymptotic states as

$$|z_\pm^I(p)z_\pm^J(q)\rangle \,, \quad |z_\pm^I(p)y_\pm^J(q)\rangle \,, \quad |z_\pm^I(p)\chi_\pm^r(q)\rangle \,, \quad |y_\pm^I(p)\chi_\pm^r(q)\rangle \,, \quad |\chi_\pm^r(p)\chi_\pm^s(q)\rangle \,, \tag{5.3}$$

where $r, s = 1, \ldots, 4$ or $r, s = 1, 2$, $I, J = 1, 2$ or $I, J = 1$ and the $\pm$ subscript refers to the U(1) charge of a particle (see tables 2 and 3). For the $n = 2$ theory, on the other hand, we have real bosons and the relevant states will be denoted as

$$|x^k(p)x^l(q)\rangle \,, \quad |x^k(p)\chi_\pm^1(q)\rangle \,, \quad |\chi_\pm^1(p)\chi_\pm^1(q)\rangle \,, \qquad k, l = 1, 2 \,. \tag{5.4}$$

Having set up the notation let us now present the results of the computations. We will start with the $n = 5$ case where we will first compute the amplitudes directly, without implementing the wave function renormalization (1.2), and then show how the UV divergences cancel in the properly defined S-matrix elements (1.4).

5.1 $AdS_5 \times S^5$

Let us start with processes where we scatter z and y particles separately. Evaluating the amplitude, which is given by a sum of the topologies (4.2) and (4.3) we get

$$\mathbb{T}|z_\pm^I(p)z_\pm^I(q)\rangle = \ell_1^z|z_\pm^I(p)z_\pm^I(q)\rangle, \qquad \mathbb{T}|y_\pm^I(p)y_\pm^I(q)\rangle = \ell_1^y|y_\pm^I(p)y_\pm^I(q)\rangle \,, \tag{5.5}$$

$$\ell_1^z = -\frac{1}{g}l_1 + \frac{1}{g^2}\left(2\Theta_{HL} + \frac{1}{2\pi}\gamma(\epsilon)l_1\right), \qquad \ell_1^y = \frac{1}{g}l_1 + \frac{1}{g^2}\left(2\Theta_{HL} + \frac{1}{2\pi}\gamma(\epsilon)l_1\right), \tag{5.6}$$

where l_1 is the corresponding tree-level amplitude, Θ_{HL} is the one-loop contribution corresponding to the well known Hernandez-Lopez phase [30, 31] and

$$\gamma(\epsilon) = -\frac{2}{\epsilon} + \gamma_E - \log 4\pi \,. \tag{5.7}$$

The terms with $\gamma(\epsilon)$ are arising from the integral (1.3) evaluated in dimensional regularization. For the explicit representation of the HL phase term in our conventions see (B.2). As was mentioned above, we are ignoring imaginary terms in ℓ_1^z, ℓ_1^y.

Implementing the wave-function renormalization (1.2) we see that the above amplitudes become finite. At the same time, the scattering amplitude mixing equal numbers of

z and y particles also remains finite as the contributions from the wave-function renormalization cancel each other out. Indeed, we find

$$\mathbb{T}|z_{\pm}^{I}(p)y_{\pm}^{J}(q)\rangle = \ell_2|z_{\pm}^{I}(p)y_{\pm}^{J}(q)\rangle + \text{fermions}\,, \qquad \ell_2 = -\frac{1}{g}l_2 + \frac{1}{g^2}2\Theta_{HL}\,. \qquad (5.8)$$

For the scattering amplitudes involving two fermions in the final state we get

$$\mathbb{T}|z_{\pm}^{I}(p)z_{\mp}^{I}(q)\rangle = \sum_{r=1}^{4} \ell_{3,z}^{r}|\chi_{\pm}^{r}(p)\chi_{\mp}^{r}(q)\rangle + \dots\,, \qquad (5.9)$$

$$\mathbb{T}|y_{\pm}^{I}(p)y_{\mp}^{I}(q)\rangle = \sum_{r=1}^{4} \ell_{3,y}^{r}|\chi_{\pm}^{r}(p)\chi_{\mp}^{r}(q)\rangle + \dots\,, \qquad (5.10)$$

$$\ell_{3,z}^{r} = \left(-\frac{1}{g}l_3 + \frac{1}{g^2}\frac{1}{4\pi}\gamma(\epsilon)l_3\right)\delta_{I+2,r}\,, \qquad \ell_{3,y}^{r} = \left(\frac{1}{g}l_3 + \frac{1}{g^2}\frac{1}{4\pi}\gamma(\epsilon)l_3\right)\delta_{Ir}\,. \quad (5.11)$$

The fermions should not be renormalized (as implied by the off-shell finiteness of the two-point functions of fermions), and taking into account the wave-function renormalization of the bosons the corresponding S-matrix elements become finite.

In order to provide a further consistency check of our regularization method, let us consider a few more amplitudes. For example, for the diagonal scattering

$$\mathbb{T}|z_{\pm}^{I}(p)\chi_{\pm}^{r}(q)\rangle_{I+2=r} = \ell_4^{z\chi}|z_{\pm}^{I}(p)\chi_{\pm}^{r}(q)\rangle + \dots\,, \qquad \mathbb{T}|y_{\pm}^{I}(p)\chi_{\pm}^{r}(q)\rangle_{I=r} = \ell_4^{y\chi}|y_{\pm}^{I}(p)\chi_{\pm}^{r}(q)\rangle + \dots\,,$$

$$\mathbb{T}|z_{\pm}^{I}(p)\chi_{\pm}^{r}(q)\rangle_{I+2\neq r} = \ell_5^{z\chi}|z_{\pm}^{I}(p)\chi_{\pm}^{r}(q)\rangle + \dots\,, \qquad \mathbb{T}|y_{\pm}^{I}(p)\chi_{\pm}^{r}(q)\rangle_{I\neq r} = \ell_5^{y\chi}|y_{\pm}^{I}(p)\chi_{\pm}^{r}(q)\rangle + \dots\,,$$

we find

$$\ell_4^{z\chi} = -\frac{1}{g}l_4 + \frac{1}{g^2}\left(2\Theta_{HL} + \frac{1}{4\pi}\gamma(\epsilon)l_4\right)\,, \quad \ell_4^{y\chi} = \frac{1}{g}l_4 + \frac{1}{g^2}\left(2\Theta_{HL} + \frac{1}{4\pi}\gamma(\epsilon)l_4\right)\,, \quad (5.12)$$

$$\ell_5^{z\chi} = -\frac{1}{g}l_5 + \frac{1}{g^2}\left(2\Theta_{HL} + \frac{1}{4\pi}\gamma(\epsilon)l_5\right)\,, \quad \ell_5^{y\chi} = \frac{1}{g}l_5 + \frac{1}{g^2}\left(2\Theta_{HL} + \frac{1}{4\pi}\gamma(\epsilon)l_5\right)\,, \quad (5.13)$$

which again correspond to finite S-matrix elements after renormalization of only bosonic legs.

This also implies that all two-fermion scattering amplitudes should be finite at one-loop level. To check this explicitly the superstring action to sixth order in fermions is needed. In the $AdS_5 \times S^5$ the action is given by the supercoset construction to all orders in fermions (see e.g. [23]). After some work we indeed find a finite result

$$\mathbb{T}|\chi_{\pm}^{r}(p)\chi_{\pm}^{s}(q)\rangle = \ell_6^{rs}|\chi_{\pm}^{r}(p)\chi_{\pm}^{s}(q)\rangle + \dots\,, \quad \ell_6^{rs} = -\frac{1}{g}l_6(1-\delta_{rs}) + \frac{2}{g^2}\Theta_{HL}\,. \qquad (5.14)$$

Here $r, s = 1, 2$ or $3, 4$ and the Kronecker delta indicates that only scattering with different fermionic flavors have a non-zero tree-level term.[23]

To summarize, taking the wave-function renormalization (1.2) into account the one-loop contributions to the diagonal S-matrix elements are finite and completely captured by the HL phase term:

$$\ell_1^{z} = -\frac{1}{g}l_1 + \frac{2}{g^2}\Theta_{HL}, \qquad \ell_1^{y} = \frac{1}{g}l_1 + \frac{2}{g^2}\Theta_{HL}, \qquad\qquad \ell_2 = -\frac{1}{g}l_2 + \frac{2}{g^2}\Theta_{HL}$$

---

[23]For scattering processes with $r = 1, 2$ and $s = 3, 4$ the tree-level amplitudes vanish identically.

$$\ell_4^{z\chi} = -\frac{1}{g}l_4 + \frac{2}{g^2}\Theta_{HL}, \qquad \ell_4^{y\chi} = \frac{1}{g}l_4 + \frac{2}{g^2}\Theta_{HL}, \qquad\qquad \ell_5^{z\chi} = -\frac{1}{g}l_5 + \frac{2}{g^2}\Theta_{HL},$$

$$\ell_5^{y\chi} = \frac{1}{g}l_5 + \frac{2}{g^2}\Theta_{HL}, \qquad \ell_6^{rs} = -\frac{1}{g}l_6(1 - \delta_{rs}) + \frac{2}{g^2}\Theta_{HL}. \tag{5.15}$$

Additional imaginary parts of S-matrix elements, which as mentioned in the beginning of section 5 we ignored in our calculation, may be restored through the optical theorem. Also, the renormalized off-diagonal elements are

$$\ell_{3,z}^r = -\frac{1}{g}l_3\delta_{I+2,r}, \qquad \ell_{3,y}^r = \frac{1}{g}l_3\delta_{Ir}. \tag{5.16}$$

The off-diagonal amplitudes are finite and the one-loop contribution is purely imaginary, i.e. fully determined via unitarity by tree-level amplitudes. All the resulting amplitudes are in complete agreement with the predictions (see, e.g., [23])[24] coming from symmetries and integrability.

5.2 $AdS_3 \times S^3 \times T^4$

For $AdS_3 \times S^3 \times T^4$ we will for simplicity restrict consideration to purely bosonic in- and out-states and we will implement the wave-function renormalization (1.2) from the start. Here we have, in total, one transverse (complex) boson in AdS_3 and one in S^3. Looking at processes not mixing the two we get[25]

$$\mathbb{T}|z_\pm^1(p)z_\pm^1(q)\rangle = \ell_1^z|z_\pm^1(p)z_\pm^1(q)\rangle, \qquad \mathbb{T}|y_\pm^1(p)y_\pm^1(q)\rangle = \ell_1^y|y_\pm^1(p)y_\pm^1(q)\rangle,$$

$$\ell_1^z = -\frac{1}{g}l_1 + \frac{2}{g^2}\Theta_{\pm\pm}, \qquad\qquad \ell_1^y = \frac{1}{g}l_1 + \frac{2}{g^2}\Theta_{\pm\pm}, \tag{5.17}$$

where the phases $\Theta_{++} = \Theta_{--}$ and $\Theta_{+-} = \Theta_{-+}$ are the two BOSST [32–35] phases, see (B.4).

For scattering of bosonic particles with opposite U(1) charges we find

$$\mathbb{T}|z_\pm^1(p)z_\mp^1(q)\rangle = \ell_2^z|z_\pm^1(p)z_\mp^1(q)\rangle + \ldots, \qquad \mathbb{T}|y_\pm^1(p)y_\mp^1(q)\rangle = \ell_2^y|y_\pm^1(p)y_\mp^1(q)\rangle + \ldots$$

$$\ell_2^z = -\frac{1}{g}l_2 + \frac{2}{g^2}\Theta_{\pm\mp}, \qquad\qquad \ell_2^y = \frac{1}{g}l_2 + \frac{2}{g^2}\Theta_{\pm\mp}, \tag{5.18}$$

---

[24]Their $su(2|2)^2$-covariant fields are related to ours as follows:

$$Z^{\alpha\dot\alpha} = \frac{1}{\sqrt{2}}\begin{pmatrix} z_+^1 & z_-^2 \\ -z_+^2 & z_-^1 \end{pmatrix}, \qquad\qquad Y^{a\dot a} = \frac{1}{\sqrt{2}}\begin{pmatrix} y_+^1 & y_-^2 \\ -y_+^2 & y_-^1 \end{pmatrix},$$

$$\eta^{\alpha\dot\alpha} = \frac{1-i}{2}\begin{pmatrix} \chi_R^1 + \chi_L^1 & -\chi_R^2 - \chi_L^2 \\ \bar\chi_R^2 - \bar\chi_L^2 & \bar\chi_R^1 - \bar\chi_L^1 \end{pmatrix}, \qquad \theta^{a\dot a} = \frac{1-i}{2}\begin{pmatrix} \chi_R^3 + \chi_L^3 & \bar\chi_R^4 - \bar\chi_L^4 \\ -\chi_R^4 - \chi_L^4 & \bar\chi_R^3 - \bar\chi_L^3 \end{pmatrix},$$

as can be seen by matching the U(1)-charges and comparing the quadratic terms in the action. Note that the requirement that our fermions have a standard kinetic term breaks the $su(2|2)^2$-covariance and causes the S-matrix elements involving fermions to take a slightly different form then in [23].

[25]These amplitudes diverge before field renormalization.

which is again finite after the wave-function renormalization (1.2). Finally, for processes mixing the two bosonic coordinates we find[26]

$$\mathbb{T}|z_\pm^1(p)y_\mp^1(q)\rangle = \ell_3^{+-}|z_\pm^1(p)y_\mp^1(q)\rangle + \dots, \qquad \mathbb{T}|z_\pm^1(p)y_\pm^1(q)\rangle = \ell_3^{++}|z_\pm^1(p)y_\pm^1(q)\rangle + \dots$$

$$\ell_3^{+-} = -\frac{1}{g}l_3 + \frac{2}{g^2}\Theta_{\pm\pm}, \qquad\qquad \ell_3^{++} = -\frac{1}{g}l_3 + \frac{2}{g^2}\Theta_{\pm\mp}. \qquad (5.19)$$

This amplitude is finite even before using eq. (1.2), as expected from the fact that it mixes z and y particles.

5.3 $AdS_2 \times S^2 \times T^6$

The difference compared to the $n = 5, 3$ cases is that here the massive bosons, which we parameterize with two real coordinates x_1 and x_2, are neutral under the U(1) symmetries left after the light-cone gauge fixing.

For the amplitudes with bosonic in-states we find

$$\mathbb{T}|x^1(p)x^1(q)\rangle = \ell_1^{x_1}|x^1(p)x^1(q)\rangle + \ell_3^{x_1}|\chi_\pm^1(p)\chi_\mp^1(q)\rangle + \dots,$$
$$\mathbb{T}|x^2(p)x^2(q)\rangle = \ell_1^{x_2}|x^2(p)x^2(q)\rangle + \ell_3^{x_2}|\chi_\pm^1(p)\chi_\mp^1(q)\rangle + \dots,$$
$$\mathbb{T}|x^1(p)x^2(q)\rangle = \ell_2|x^1(p)x^2(q)\rangle + \ell_4|\chi_\pm^1(p)\chi_\mp^1(q)\rangle + \dots,$$

where

$$\ell_1^{x_1} = -\frac{1}{g}l_1 + \frac{4}{g^2}\Theta_{HL}, \quad \ell_1^{x_2} = \frac{1}{g}l_1 + \frac{4}{g^2}\Theta_{HL}, \qquad\qquad\qquad\qquad (5.20)$$

$$\ell_3^{x_1} = -\frac{1}{g}l_3, \qquad\qquad \ell_3^{x_2} = \frac{1}{g}l_3, \qquad\qquad \ell_2 = -\frac{1}{g}l_2 + \frac{4}{g^2}\Theta_{HL}, \quad \ell_4 = -\frac{1}{g}l_4. \quad (5.21)$$

Here we have already implemented the wave-function renormalization (1.2) (which, as was already mentioned earlier, differs by a factor of 2 from the $n = 5, 3$ cases).

The mixed $BF \to BF$ amplitudes are also finite after wave-function renormalization,

$$\mathbb{T}|x^1(p)\chi_\pm(q)\rangle = \ell_5^{x_1}|x^1(p)\chi_\pm(q)\rangle, \qquad\qquad \mathbb{T}|x^2(p)\chi_\pm(q)\rangle = \ell_5^{x_2}|x^2(p)\chi_\pm(q)\rangle,$$

$$\ell_5^{x_1} = -\frac{1}{g}l_5, \qquad\qquad\qquad\qquad \ell_5^{x_2} = \frac{1}{g}l_5. \qquad (5.22)$$

6 Conclusions

We have addressed the long standing question of how to properly compute the one-loop S-matrix of the $AdS_n \times S^n \times T^{10-2n}$ superstring around the BMN vacuum. By analyzing separately the one-loop 1-PI contribution to the two-particle scattering amplitude and the off-shell one-loop two-point functions of massive fields we demonstrated that the UV-divergences that appear should be interpreted as wave-function renormalization for the bosonic coordinates. Once this is taken into account the final expression for the one-loop S-matrix is UV finite.

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[26]Note that in our conventions y_+^1 and z_+^1 have the same sign of the charge which differs from the convention used in [34].

We have also outlined a regularization scheme which is consistent with the classical worldsheet symmetries. One-loop computations in this scheme fully reproduce all known results about the massive S-matrix predicted by symmetries and integrability. For the $n = 2, 3$ theories we found that the massless loop contributions to massive two-particle scattering amplitudes cancel out at one loop order. Thus, somewhat surprisingly, the massive sector S-matrix of the full superstring coincides with the one obtained from the $AdS_n \times S^n$ supercoset sigma-model. Our result lends support to the generalized unitarity-based prescription of [15] which also leads to a decoupling of massless modes at one loop for strings in $AdS_3 \times S^3 \times T^4$.

We initiated a comparative study of the light-cone and conformal gauge approaches to the one-loop S-matrix. While the former is well studied, the latter remains largely unexplored. A technical problem in conformal gauge is the presence of the unphysical massless longitudinal modes whose correct treatment remains to be understood. However, for the SU(2) sector of the S-matrix we found evidence that accounting for the massless modes should be equivalent to passing from the BDS S-matrix (with no phase) to the S-matrix dressed with the standard AFS/HL/BES phase.

In conformal gauge the one-loop two-point function for the bosons happens to receive a finite correction on-shell. This stands in contrast to the vanishing result in the light-cone gauge and suggests that the symmetries of the BMN vacuum have a different realization in the conformal gauge. For example, in conformal gauge the worldsheet energy is no longer related to the target space energy and thus to the spin chain magnon dispersion relation of the dual gauge theory. The two-dimensional symmetries preserved by the BMN solution may lead to an extension of the non-local symmetries generated by the Lax connection and may ultimately determine the exact worldsheet spectrum, perhaps along the lines of [36, 37].

One interesting extension of our work is to the two-loop order of the light-cone gauge-fixed superstring around the BMN vacuum. A first step in this direction is the computation of the two-loop correction to the two-point function. Apart from checking the strong coupling expansion of the magnon dispersion relation, this should give a valuable insight into the extension of our regularization procedure to higher loops. It should also shed light on the issue of (non)decoupling of massless modes at higher loops. Unitarity-based arguments suggest that the massless modes are no longer decoupled at three loops [14] in the S-matrix. Two-loop dispersion relation calculations in the near flat space limit [11] suggest that massless modes may not decouple already at the two-loop level.

It would also be very interesting to extend the analysis of this paper to the $AdS_3 \times S^3 \times T^4$ superstring with mixed NSNS and RR-flux [38–42]. In [14, 15] the one-loop dressing phase for this theory was obtained via generalized unitarity methods. It would be very interesting to reproduce this result from an explicit worldsheet calculation and thus justify in the mixed flux case the prescription for the treatment of the singular cuts.

Acknowledgments

A   Reduction of one-loop integrals

Using the identities (4.6) and (4.9) together with the assumption that we are allowed to shift the loop variable in $T^{rs}(P)$, we can rewrite the tadpole integrals as

$$T^{12}(P) = P_- T^{11}(0) + P_- P^2 T^{00}(0), \quad T^{21}(P) = P_+ T^{11}(0) + P_+ P^2 T^{00}(0),$$
$$T^{11}(P) = T^{11}(0) + P^2 T^{00}(0), \qquad T^{01}(P) = P_- T^{00}(0), \quad T^{10}(P) = P_+ T^{00}(0). \quad \text{(A.1)}$$

For the bubble integrals[27] with only left- or right-moving momenta in the numerator we get

$$B^{03}(P) = \frac{P_-^2}{2P_+}\left(P^2 - 3m^2\right) B^{00}(P), \qquad B^{30}(P) = \frac{P_+^2}{2P_-}\left(P^2 - 3m^2\right) B^{00}(P),$$

$$B^{02}(P) = -\frac{P_-}{P_+}\left(m^2 - \frac{1}{2}P^2\right) B^{00}(P), \qquad B^{20}(P) = -\frac{P_+}{P_-}\left(m^2 - \frac{1}{2}P^2\right) B^{00}(P),$$

$$B^{01}(P) = \frac{1}{2}P_- B^{00}(P), \qquad\qquad B^{10}(P) = \frac{1}{2}P_+ B^{00}(P). \quad \text{(A.2)}$$

Here we only recorded relations for the integrals that appear in the actual amplitudes (after using (4.6)).

B   Expressions appearing in the light-cone gauge S-matrix

Here we collect the explicit expressions for the amplitudes discussed in section 5. We will write some amplitudes in terms of $\omega_p = \sqrt{p^2 + 1}$ and p, while others are written in terms of right-moving momenta $p_- = \omega_p - p$.

B.1 $AdS_5 \times S^5$

For the tree-level amplitudes we have:

$$l_1 = \frac{1}{2}\frac{(p+q)^2}{\omega_q p - \omega_p q}, \qquad l_2 = \frac{1}{2}\left(\omega_q p + \omega_p q\right), \qquad l_3 = -\frac{1}{4}\frac{(1-p_-^2)(1-q_-^2)}{\sqrt{p_- q_-}(p_- + q_-)}, \quad \text{(B.1)}$$

$$l_4 = \frac{1}{4}\frac{(1-p_- q_-)(1-q_-^2)}{(p_- - q_-)q_-}, \quad l_5 = \frac{1}{4}\frac{(1+p_- q_-)(1-q_-^2)}{(p_- + q_-)q_-}, \quad l_6 = \frac{1}{2}\frac{(1-p_-^2)(1-q_-^2)}{p_-^2 - q_-^2}.$$

The one-loop Hernandez-Lopez phase term in our notation is

$$\Theta_{HL} = \frac{1}{16\pi}\frac{(1-p_-^2)^2(1-q_-^2)^2(p_-^2+q_-^2)}{p_- q_-(p_-^2 - q_-^2)^2}\log\frac{p_-}{q_-} - \frac{1}{16\pi}\frac{(1-p_-^2)^2(1-q_-^2)^2}{p_- q_-(p_-^2 - q_-^2)}. \quad \text{(B.2)}$$

---

[27]Note that for bubble-type integrals in the light-cone gauge, the two virtual particles always come with the same mass.

B.2 $AdS_3 \times S^3 \times T^4$

The tree-level amplitudes are

$$l_1 = \frac{1}{2}\frac{(p+q)^2}{\omega_q p - \omega_p q}, \qquad l_2 = \frac{1}{2}\frac{(p-q)^2}{\omega_q p - \omega_p q}, \qquad l_3 = \frac{1}{2}\left(\omega_p q + \omega_q p\right). \qquad \text{(B.3)}$$

The two one-loop phases, written in our notation, are

$$\Theta_{\pm\pm} = \frac{1}{32\pi}\frac{(1-p_-^2)^2(1-q_-^2)^2}{p_- q_-(p_- - q_-)^2}\log\frac{p_-}{q_-} - \frac{1}{64\pi}\frac{(p_- + q_-)(1-p_-^2)(1-q_-^2)(1-p_- q_-)^2}{p_-^2 q_-^2(p_- - q_-)}, \qquad \text{(B.4)}$$

$$\Theta_{\pm\mp} = \frac{1}{32\pi}\frac{(1-p_-^2)^2(1-q_-^2)^2}{p_- q_-(p_- + q_-)^2}\log\frac{p_-}{q_-} + \frac{1}{64\pi}\frac{(p_- - q_-)(1-p_-^2)(1-q_-^2)(1+p_- q_-)^2}{p_-^2 q_-^2(p_- + q_-)},$$

which satisfy

$$\Theta_{\pm\pm} + \Theta_{\pm\mp} = \Theta_{HL}. \qquad \text{(B.5)}$$

For the expressions relevant to the $AdS_2 \times S^2 \times T^6$ case we refer to [12].

C   Comments on near BMN S-matrix in conformal gauge

While the relation between worldsheet S matrix and gauge theory anomalous dimensions of "long" operators described by the asymptotic Bethe ansatz is best understood in a physical light-cone type "mixed" gauge adapted to the BMN vacuum (with p^+ or BMN charge being fixed in a uniform way) it is nevertheless interesting to explore if a similar relation may be formulated in the conformal gauge. There is a conceptual problem in establishing such a relation stemming from the fact that, in conformal gauge, the worldsheet theory has two unphysical (longitudinal) massless modes. Their correct treatment (should they be integrated out or should they be considered as external states of the S-matrix, etc.) remains to be understood. In this appendix we shall present results of some computations that may help clarify these issues.

Our tree-level S-matrix calculations below suggest that, at least in the SU(2) sector, the correct treatment of the massless modes should be equivalent to passing from the (strong coupling limit of the) "phaseless" BDS [43] S-matrix to the S-matrix dressed with the AFS/BES [44, 45] phase.

C.1   Tree-level bosonic S-matrix in the $AdS_n \times S^n \times T^{10-2n}$ theory

Let us start with fixing the conformal gauge in the string action (2.2)

$$\sqrt{-h}\,h^{ij} = \eta^{ij} \qquad \text{(C.1)}$$

and then expand the Lagrangian around the BMN solution $x^+ \equiv \frac{1}{2}(t+\varphi) = \tau$ [28] with $x^-, z^m, y^m = 0$. To quartic order the bosonic Lagrangian becomes the sum of three terms

$$\mathcal{L}_{B2} = \frac{1}{2}\left(\partial_i z_m \partial^i z^m + \partial_i y_m \partial^i y^m - z^2 - y^2\right) \qquad \text{(C.2)}$$

---

[28] As is well known, the $x^+ = \tau$ condition cannot be viewed as an analog of flat-space l.c. gauge that fixes remaining conformal reparametrizations as it does not solve the string equations for generic "transverse" string coordinates of $AdS_n \times S^n$ space.

$$\mathcal{L}_{B3} = -\partial_0 t \, z^m z_m - \partial_0 \phi \, y^m y_m \tag{C.3}$$

$$\mathcal{L}_{B4} = -\frac{1}{2}\partial_i t \partial^i t \, z^2 - \frac{1}{2}\partial_i \phi \partial^i \phi \, y^2 + \frac{1}{4}\left[z^2 \, \partial_i z_m \partial^i z^m - \partial_i y_m \partial^i y^m \, y^2 - (z^2)^2 + (y^2)^2 \right] \tag{C.4}$$

where $z^2 \equiv z^m z_m$, etc., and the index m runs over the transverse directions. We choose the fields to be real to interpolate easily between theories with different dimensions of $AdS \times S$.

Due to the presence of cubic interaction terms involving one massless longitudinal field, the t-channel contribution to the S matrix is singular on shell. We regularize this singularity as follows:

1. introduce a small regulating mass (as for one-loop IR-divergent integrals)

2. compute the off-shell four-point Green's function

3. put the Green's function on shell and amputate

4. take the regulating mass to zero

The result of this prescription is a finite tree-level S-matrix.

The "transverse" $SO(n-1) \times SO(n-1)$ symmetry of the Lagrangian as well as the decoupling of the AdS_n and S^n fluctuations in the conformal gauge require that the S-matrix takes the general form

$$\begin{aligned}
\mathbb{T}|z^m(p)z^n(q)\rangle &= \quad (A\delta_k^m\delta_l^n + B\delta_k^n\delta_l^m + C\delta^{mn}\delta_{kl})|z^k(p)z^l(q)\rangle \\
\mathbb{T}|y^m(p)y^n(q)\rangle &= -(A\delta_k^m\delta_l^n + B\delta_k^n\delta_l^m + C\delta^{mn}\delta_{kl})|y^k(p)y^l(q)\rangle \\
\mathbb{T}|y^m(p)z^n(q)\rangle &= 0 \, .
\end{aligned} \tag{C.5}$$

Using the above prescription for the massless modes and a relativistic normalization for the S-matrix the free coefficients in (C.5) are given by

$$A = 0 \, , \qquad B = -C = 4pq \, . \tag{C.6}$$

With a non-relativistic normalization and with a manifestly solved momentum-conservation constraint the above coefficients become

$$A = 0 \, , \qquad B = -C = \frac{4pq}{p\omega_q - q\omega_p} \, . \tag{C.7}$$

This S-matrix is, of course, consistent with the classical Yang-Baxter equation.

In the case of $AdS_5 \times S^5$, integrability together with the fact that $SO(4) \simeq SU(2) \otimes SU(2)$ require that

$$\mathbb{T} = \mathbb{1} \otimes \mathrm{T} + \mathrm{T} \otimes \mathbb{1} \, , \tag{C.8}$$

where $\mathbb{1}$ and T act on $SU(2)$ indices from the decomposition of the two $SO(4)$ factors as

$$\mathbb{1}_{ab}^{cd} = \delta_b^d \delta_a^c \, , \qquad P_{ab}^{cd} = \delta_b^c \delta_a^d \, . \tag{C.9}$$

It is not difficult to see that the non-zero entries of $\mathbb{T}$ may be written as

$$\mathbb{T}|z(p)z(q)\rangle = -\left[B\mathbb{1} \otimes \mathbb{1} - B(\mathbb{1} \otimes P + P \otimes \mathbb{1}) \right] |z(p)z(q)\rangle$$

$$\mathbb{T}|y(p)y(q)\rangle = \left[B\mathbb{1} \otimes \mathbb{1} - B(\mathbb{1} \otimes P + P \otimes \mathbb{1}) \right] |y(p)y(q)\rangle. \qquad \text{(C.10)}$$

This is indeed consistent with the factorized structure (C.8).

Longitudinal states appear to scatter trivially off the massive states. This may be understood in two steps. First, the cubic terms may be eliminated by a non-local field redefinition. While potentially worrisome, the effect of the non-locality is only to generate effective quartic interaction terms between massive fields which correspond to the Feynman graphs with exchange of longitudinal fields. The second step is to notice that momentum conservation implies that the S-matrix elements following from the quartic terms are proportional to the dispersion relation for the longitudinal fields and thus vanish on shell. Such trivial scattering of longitudinal modes may not be unexpected given that for massless fields it is notoriously difficult to define a consistent scattering theory that has a perturbative regime.

Clearly, the S-matrix (C.5) is different from the one obtained in the "light-cone" a-gauge [46]. While the latter has nontrivial $yz \to yz$ matrix elements, the former does not. Such matrix elements may be generated at loop level through fermion loops as well as loops of longitudinal modes. The non-zero matrix elements are also different; while the difference is proportional to the identity operator, $\mathbb{1} \otimes \mathbb{1}$, it is not only an overall phase as it affects differently the scattering of AdS and S fluctuations:

$$\delta\mathbb{T}_{z(p)z(q)\to z(p)z(q)} = \frac{1}{2} \left[(1-2a)(p\omega_q - q\omega_p) + \frac{p^2 + q^2}{p\omega_q - q\omega_p} \right] \mathbb{1} \otimes \mathbb{1}$$

$$\delta\mathbb{T}_{y(p)y(q)\to y(p)y(q)} = \frac{1}{2} \left[(1-2a)(p\omega_q - q\omega_p) - \frac{p^2 + q^2}{p\omega_q - q\omega_p} \right] \mathbb{1} \otimes \mathbb{1}. \qquad \text{(C.11)}$$

Through generalized unitarity tree-level differences imply [14] that the one-loop S matrix in conformal gauge is also different from the one-loop S matrix in the a-gauge.

It is interesting to note that, when restricted to the SU(2) sector, the S-matrix (C.5) is the same as the BDS S-matrix in the small momentum limit. It was suggested in [47, 48] that the dressing phase may be understood as a consequence of a nontrivial vacuum in the Bethe equations based on the BDS S-matrix. This may be viewed as a hint that the difference between the conformal gauge S-matrix and the light-cone gauge S-matrix from the perspective of the usual asymptotic Bethe ansatz may be due to a nontrivial choice of vacuum for the longitudinal excitations once a consistent scattering theory is defined for the latter. A somewhat similar suggestion was made for the non-transverse excitations of a principal chiral model on $\mathbb{R} \times S^3$ [49] and of some conformal sigma models [50]. In our case this interpretation is also supported by the fact that in the presence of the longitudinal fields the massive fields are potentially unstable, loosing energy by emitting low energy massless quanta.

C.2 One-loop bosonic dispersion relation

Apart from the S-matrix, the other essential ingredient of a Bethe ansatz is the exact dispersion relation for the elementary excitations. To one-loop order the quantum corrections to dispersion relation vanish in the $a = 1/2$ gauge. As discussed in the main text, computing the off-shell two-point functions leads to a nontrivial wave-function renormalization. It is interesting to carry out a similar study in the conformal gauge.

Let us compute the one-loop two-point function by directly expanding around the BMN vacuum. We will describe the calculation for the $AdS_5 \times S^5$ case and then comment on extension to lower-dimensional cases. Apart from the bosonic action to quartic order given in appendix C.1, we also need terms bilinear in fermions and up to quadratic order in bosons. They are obtained from the $AdS_5 \times S^5$ action in section 2 by imposing the κ-symmetry light-cone gauge $\Gamma^+\Theta = 0$.

There are in principle four graphs contributing to the two-point function of massive bosons: a bosonic bubble and a tadpole and also a fermionic bubble (which in our case vanishes identically) and a tadpole. The bosonic and fermionic contributions are separately divergent off shell, but the divergence is proportional to the classical equation of motion $(p_+p_- - 1)$ so they are finite on shell. There is a finite momentum dependent contribution to the two-point function which arises entirely from the the bosonic bubble graph:

$$i\Pi^{(1)} = 2\varepsilon \int \frac{d^2q}{(2\pi)^2} \frac{q_+^2 + q_-^2}{q^2((q+p)^2 - 1)} \tag{C.12}$$

where $\varepsilon = \pm 1$ for the transverse AdS and sphere fluctuations, respectively. These integrals, while logarithmically divergent by power counting, are finite in dimensional regularization. Evaluating them leads to[29]

$$\Pi^{(1)}(p_+, p_-) = \frac{\varepsilon}{4\pi} \frac{p_0^2 + p_1^2}{p_+p_-} \left(1 - \frac{1 - p_+p_-}{p_+p_-} \ln \frac{1 - p_+p_-}{p_+p_-} \right). \tag{C.13}$$

These expressions are non-vanishing on shell and they lead to a correction to the tree-level dispersion relation:

$$\omega^2 = 1 + p^2 + \frac{\varepsilon}{4\pi g}(1 + 2p^2). \tag{C.14}$$

The meaning of this correction and its effect on the symmetries of the S-matrix remain to be clarified.

To extend the above $AdS_5 \times S^5$ calculation to other $AdS_n \times S^n \times T^{10-2n}$ cases we notice that the only non-vanishing contribution comes from the bosonic bubble graph whose internal-line field content is uniquely fixed by the choice of the external field. Thus, we conclude that the same two-point function should appear in all other cases.

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