Global aspects of abelian duality in dimension three

Chris Beasley

Department of Mathematics, Northeastern University,
360 Huntington Ave, Boston, MA 02115, U.S.A.

E-mail: C.Beasley@neu.edu

ABSTRACT: In three dimensions, an abelian gauge field is related by duality to a free, periodic scalar field. Though usually considered on $\mathbb{R}^3$, this duality can be extended to a general three-manifold M, in which case topological features of M become important. Here I comment upon several of these features as related to the partition function on M. In a companion article, I discuss similarly the algebra of operators on a surface of genus g.

Contents

1 Introduction

In classical field theory, abelian duality amounts to a simple relation between the solutions of certain linear partial differential equations on $\mathbb{R}^n$ for $n \geq 2$. The most elementary example occurs in dimension two, for which one considers a harmonic function $\phi \in C^\infty(\mathbb{R}^2)$ satisfying the Laplace equation

$$\triangle\phi = d^\dagger d\phi = 0\,, \qquad d^\dagger = -\star d\star\,. \tag{1.1}$$

Here $\star$ is the Hodge star operator determined by the Euclidean metric on $\mathbb{R}^2$.

According to (1.1), the one-form $\star d\phi$ is closed. Since $\mathbb{R}^2$ has trivial cohomology in degree one, $\star d\phi$ is also exact. Hence one can associate to ϕ another function $\psi \in C^\infty(\mathbb{R}^2)$ via the duality relation

$$\star\, d\phi = d\psi\,. \tag{1.2}$$

The relation in (1.2) determines the function ψ up to the addition of a constant, and ψ is automatically harmonic by virtue of the identity

$$\triangle\psi = d^\dagger d\psi = \star d^2\phi = 0\,. \tag{1.3}$$

The classical duality in (1.2) thus relates one solution of the Laplace equation on $\mathbb{R}^2$ to another, distinct solution. As well-known, the dual harmonic functions ϕ and ψ can be combined as the real and imaginary parts of a single holomorphic function on $\mathbb{R}^2 \simeq \mathbb{C}$.

Similarly in dimension four, if A is a U(1)-connection on $\mathbb{R}^4$ which solves the source-free Maxwell equation

$$d \star F_A = 0, \qquad F_A = dA, \tag{1.4}$$

then A determines another U(1)-connection B up to gauge equivalence via

$$\star F_A = F_B. \tag{1.5}$$

By virtue of the Bianchi identity $dF_A = 0$, the curvature of the connection B also satisfies $d \star F_B = 0$, so the classical electric-magnetic duality in (1.5) relates distinct solutions of the source-free Maxwell equation on $\mathbb{R}^4$.

In this paper we are concerned with abelian duality in dimension three. In that case, classical abelian duality relates a U(1)-connection A which solves the Maxwell equation on $\mathbb{R}^3$ to a harmonic function ϕ on $\mathbb{R}^3$. By analogy to dimensions two and four, the classical duality relation in dimension three is

$$\star F_A = e^2 \, d\phi. \tag{1.6}$$

Unlike the preceding duality relations, the classical duality relation on $\mathbb{R}^3$ involves the electric coupling e^2, which appears in the Maxwell action

$$\begin{aligned}
\mathbf{I}(A) &= \frac{1}{4\pi e^2} \int_{\mathbb{R}^3} F_A \wedge \star F_A, \\
&= \frac{1}{8\pi e^2} \int_{\mathbb{R}^3} d^3 x \, \sqrt{g} \, F_{A,mn} F_A^{\ mn}, \qquad m, n = 1, 2, 3.
\end{aligned} \tag{1.7}$$

The factor of $1/4\pi$ in the first line of (1.7) is simply a notational convenience, which will eliminate other factors later. In the second line of (1.7), we rewrite the Maxwell action in components with respect to an arbitrary Riemannian metric g on $\mathbb{R}^3$, and we follow the standard Einstein convention in summing over the repeated indices m and n on the curvature F_A of the gauge field.

The appearance of the Maxwell coupling e^2 in the duality relation (1.6) is the first of several peculiarities specific to dimension three. Most famously, the duality relations in two and four dimensions are invariant under conformal transformations, which preserve both the one-form $\star d\phi$ on $\mathbb{R}^2$ in (1.2) and the two-form $\star F_A$ on $\mathbb{R}^4$ in (1.5). But in dimension three, if the metric g is scaled by a constant factor

$$g \longmapsto \Lambda^2 g, \qquad \Lambda \in \mathbb{R}_+, \tag{1.8}$$

the dual one-form $\star F_A$ in (1.6) scales non-trivially as well,

$$\star F_A \longmapsto \Lambda^{-1} \star F_A. \tag{1.9}$$

In components, $\star F_A$ is given by $\sqrt{g} \, \epsilon_{mnp} F^{mn} \, dx^p$, where ϵ_{mnp} is the anti-symmetric tensor on three indices, with fixed normalization $\epsilon_{123} = +1$. The scaling for $\star F_A$ in (1.9) follows from the combined scalings of $\sqrt{g}$ with weight Λ^3 and F^{mn} with weight Λ^{-4}. As a result, neither the duality relation in (1.6) nor the classical Maxwell action in (1.7) is invariant under scale, much less conformal, transformations of the metric on $\mathbb{R}^3$.

On the other hand, if the transformation of the metric in (1.8) is supplemented by a non-trivial scaling for the electric coupling e^2 itself,

$$e^2 \longmapsto \Lambda^{-1} e^2, \tag{1.10}$$

then the Maxwell action on $\mathbb{R}^3$ is invariant. Because $\star F_A$ and e^2 transform with identical weights, the abelian duality relation in (1.6) is also preserved under scaling. Conversely, the appearance of e^2 in the duality relation is dictated by invariance under the transformations in (1.8) and (1.10).

Each of the classical duality relations in (1.2), (1.5), and (1.6) extends to an equivalence of free quantum field theories defined on an arbitrary Riemannian manifold Σ, X, or M of corresponding dimension two, four, or three. Though the relevant pairs of quantum field theories are themselves trivial, the equivalence between them is generally non-trivial and may depend in interesting ways on the topology of the underlying manifold.

These topological issues are particularly sharp when M is a closed three-manifold. In that case, the global analysis of the classical Laplace equation on M is very different from the global analysis of the classical Maxwell equation on M. Solutions to the Laplace equation will be unique up to scale, but solutions to the Maxwell equation generically fall into continuous families, parametrized by the holonomies of the gauge field. So on a general three-manifold, there is no hope to interpret abelian duality classically, as a one-to-one correspondence (1.6) between solutions of the Laplace and Maxwell equations on M. Instead, abelian duality on M must be interpreted as an inherently quantum phenomenon.

A basic observable in any quantum field theory is the partition function, and as an initial question, one can ask how the partition function transforms under duality. Naively, one might expect the partition function to be invariant under duality, but famously in dimensions two and four, this is not so.

The most elegant statement [35] occurs for electric-magnetic duality of Maxwell theory on a four-manifold X. (See also [33] for related observations.) In this case, the Maxwell partition function Z_X depends upon both the Maxwell coupling e^2 and an angular parameter θ which enters the classical Lagrangian through the topological pairing

$$
\begin{aligned}
\mathbf{I}_\theta(A) &= \frac{i\,\theta}{8\pi^2} \int_X F_A \wedge F_A \,, \\
&= \frac{i\,\theta}{32\pi^2} \int_X d^4x \, \sqrt{g} \, \epsilon_{mnpq} \, F_A^{mn} \, F_A^{pq} \,.
\end{aligned} \tag{1.11}
$$

When X is a spin-manifold, the normalization in (1.11) ensures that θ has period 2π. Otherwise, θ has period 4π.

The angular parameter θ naturally complexifies the electric coupling e^2 via

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2} \,, \tag{1.12}$$

and electric-magnetic duality acts upon τ as a modular transformation $\tau \mapsto -1/\tau$. Moreover, as shown by direct computation in [35], the Maxwell partition function $Z_X(\tau)$ on X transforms under duality as a non-holomorphic modular form with weights $\frac{1}{4}(\chi - \sigma, \chi + \sigma)$,

$$Z_X(-1/\tau) = \tau^{\frac{1}{4}(\chi-\sigma)} \, \overline{\tau}^{\frac{1}{4}(\chi+\sigma)} \, Z_X(\tau) \,. \tag{1.13}$$

Here χ and σ are the respective Euler character and signature of the four-manifold X. The non-trivial transformation law for $Z_X(\tau)$ in (1.13) is kind of gravitational anomaly for duality, since both the Euler character and signature can be represented as the integrals of local densities constructed from the Riemann tensor on X. See also the discussion in section 3 of [32], where the modular anomaly in electric-magnetic duality was originally noted in the context of supersymmetric Yang-Mills theory.

Similarly on a Riemann surface Σ, the classic one-loop shift [3–5, 12, 28] in the dilaton under T-duality represents a comparable topological effect, depending again on the Euler character of Σ.

One motivation for the present work is to point out a modular property roughly analogous to (1.13) for the partition function of Maxwell theory on a closed, orientable three-manifold M.

Such modularity in three dimensions may sound surprising, because the partition function on M (as opposed to the partition function on X) can have no interesting dependence on the electric coupling e^2. A priori, the Maxwell partition function $Z_M(e^2, g)$ depends upon both the coupling e^2 and the Riemannian metric g on M. However, the scale transformations in (1.8) and (1.10) together preserve $Z_M(e^2, g)$ and can always be used to set $e^2 = 1$, so that any dependence on e^2 can be effectively absorbed into the dependence on the metric. In addition, the Euler character of any closed, orientable three-manifold vanishes, and there are no other local, generally covariant invariants of M that could appear in an anomaly such as (1.13).

The situation changes, though, as soon as we include additional parameters which play a role in three dimensions analogous to the role of the θ-angle in four dimensions. Very briefly, in dimension three the topological parameter $\zeta \in \mathcal{H}_{\mathbb{C}}^1(M)$ will be a complex harmonic one-form, which enters the classical gauge theory Lagrangian via the natural pairing

$$
\begin{aligned}
\mathbf{I}_\zeta(A) &= \frac{1}{2\pi i} \int_M \zeta \wedge F_A \,, \\
&= \frac{1}{2\pi i} \int_M d^3x \, \sqrt{g} \, \epsilon_{mnp} \, \zeta^p \, F_A^{mn}.
\end{aligned}
\tag{1.14}
$$

The partition function Z_M then depends upon ζ as a theta-function associated to the cohomology lattice of M, and abelian duality acts as a modular transformation on that theta-function. When $\zeta = 0$, the partition function is nonetheless invariant under duality, but in a fairly non-trivial way.

This observation appears at least implicitly in [2, 25], with which the present work has some overlap, but I believe it deserves further emphasis here. I also take the opportunity to clean up a few factors in [2], which otherwise detract from a very elegant analysis.

The plan of the paper. Very broadly, the purpose of this paper is to analyze the quantum analogue of the classical abelian duality relation in (1.6) when M is a general Riemannian three-manifold. Because the quantum field theories on both sides of the duality are free, this analysis is straightforward and can be carried out in an explicit fashion.

In fact, I will carry out the analysis two ways, working in both the Lagrangian and the Hamiltonian formalisms, since one learns different things from each. Here I focus on the Lagrangian perspective, and in a companion paper [1], I adopt the alternative Hamiltonian viewpoint.

In sections 2 and 3, I compute the respective partition functions for a periodic[1] scalar field and an abelian gauge field on the three-manifold M. Then in section 4, I perform a direct comparison of the resulting expressions for the partition function. As mentioned above, these expressions involve a novel theta-function attached to the three-manifold M, akin to the classical theta-function on the Jacobian variety of a Riemann surface. Duality acts by a modular transformation on the theta-function.

Both to orient the reader and for sake of completeness, I conclude in section 5 by reviewing the standard path integral explanation for abelian duality in three dimensions. A very nice exposition of the latter material appears in lecture 8 of [36], which I largely follow. I also discuss duality for three simple classes of operators (Wilson loops, vortex loops, and monopole operators in the language of Maxwell theory) whose commutator algebra on a Riemann surface of genus g will be analyzed in [1]. See also [18] for another recent approach to abelian duality, invoking the formalism of duality walls.

One coupling relevant in dimension three but with no equivalent in dimensions two and four is the Chern-Simons coupling, for which global issues feature prominently. In subsequent work, I apply ideas here and in [1] to clarify the meaning of abelian duality for Maxwell-Chern-Simons theory at level k.

2 Analysis of the abelian sigma model

We first compute the partition function for a free, periodic scalar field on M. Throughout this paper, M is a closed, oriented three-manifold, with Riemannian metric g. The most basic topological invariant of M is the first Betti number b_1, which is the dimension of the vector space $\mathcal{H}^1(M)$ of harmonic one-forms on M. As for the other Betti numbers, trivially $b_0 = b_3 = 1$, and $b_2 = b_1$ by Poincaré duality.

Unlike instances of topological quantum field theory, the abelian quantum field theories here will definitely depend upon the choice of the Riemannian metric g. The most elementary invariant of the metric on M is the total volume, parametrized in terms of an overall length scale ℓ,

$$\ell^3 = \int_M \mathrm{vol}_M\,, \qquad \mathrm{vol}_M = \star 1 \in \Omega^3(M)\,. \tag{2.1}$$

As we perform computations, we will wish to keep track of the dependence on both the length scale ℓ and the electric coupling e^2, which enters the fundamental duality relation in (1.6). This bookkeeping is easy, for under a scale transformation

$$g \longmapsto \Lambda^2\, g\,, \qquad \Lambda \in \mathbb{R}_+\,, \tag{2.2}$$

---

[1]The adjective "periodic" is traditional but possibly misleading. More precisely, the scalar field will be circle-valued.

the parameter ℓ naturally transforms as

$$\ell \longmapsto \Lambda\,\ell\,. \tag{2.3}$$

This transformation should be compared to the transformation in (1.10) of the electric coupling,

$$e^2 \longmapsto \Lambda^{-1}\,e^2\,. \tag{2.4}$$

From (2.3) and (2.4), we immediately see that the dimensionless combination $e^2\ell$ is invariant under an overall rescaling of the metric on M.

Because the abelian quantum field theories under consideration are free, they can always be defined so that the transformations in (2.2) and (2.4) preserve both the classical action and the quantum partition function on M. The two parameters e^2 and ℓ are then redundant, since either e^2 or ℓ can be scaled to unity with an appropriate choice of $\Lambda \in \mathbb{R}_+$. Nevertheless, I leave the dependence on both e^2 and ℓ explicit, and invariance under scaling will be a small check on our later formulas.

2.1 The classical sigma model

Classically, a periodic scalar field ϕ on M simply describes a map from M to the circle,

$$\phi : M \longrightarrow S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}\,. \tag{2.5}$$

As indicated on the right in (2.5), we interpret ϕ as an angular quantity, subject to the identification

$$\phi \sim \phi + 2\pi\,. \tag{2.6}$$

The assumption in (2.5) that ϕ is valued in S^1, as opposed to $\mathbb{R}$, has important global consequences.

Abstractly, a given choice for ϕ determines a point in the space $\mathcal{X}$ of all maps from M to S^1,

$$\mathcal{X} = \mathrm{Map}(M, S^1)\,. \tag{2.7}$$

In general, $\mathcal{X}$ is not connected, but rather decomposes into components labelled by the homotopy class of the map ϕ. By standard facts in topology (see for instance chapter 4.3 in [15]), homotopy classes of maps from M to S^1 are in one-to-one correspondence with cohomology classes in $H^1(M; \mathbb{Z})$. Under this correspondence, the cohomology class associated to a given map ϕ is the pullback to M under ϕ of a fixed generator for $H^1(S^1; \mathbb{Z}) \simeq \mathbb{Z}$. Abusing notation slightly, I write this pullback as

$$\left[\frac{d\phi}{2\pi}\right] \in H^1(M; \mathbb{Z})\,. \tag{2.8}$$

Throughout this paper, we will treat torsion in integral cohomology with care. By the Universal Coefficient Theorem, the cohomology group $H^1(M; \mathbb{Z})$ is generated freely over $\mathbb{Z}$, without torsion. Thus $H^1(M; \mathbb{Z})$ is a lattice with rank b_1,

$$\mathbb{L} \equiv H^1(M; \mathbb{Z}) \simeq \mathbb{Z}^{b_1}\,, \tag{2.9}$$

where the notation $\mathbb{L}$ merely serves as a convenient shorthand. As one might guess, the lattice $\mathbb{L}$ will play a prominent role in what follows.

Altogether, the space $\mathcal{X}$ in (2.7) decomposes into connected components labelled by a winding-number ω which is valued in the cohomology lattice $\mathbb{L}$,

$$\mathcal{X} = \bigsqcup_{\omega \in \mathbb{L}} \mathcal{X}_\omega \,. \tag{2.10}$$

Here $\mathcal{X}_\omega$ consists of those sigma model maps which satisfy

$$\mathcal{X}_\omega = \left\{ \phi : M \to S^1 \left| \left[\frac{d\phi}{2\pi} \right] = \omega \right. \right\} \,, \qquad \omega \in \mathbb{L} \,. \tag{2.11}$$

The free sigma model action for ϕ takes the standard form

$$
\begin{aligned}
\mathbf{I}_0(\phi) &= \frac{e^2}{4\pi} \int_M d\phi \wedge \star d\phi \,, \\
&= \frac{e^2}{4\pi} \int_M \sqrt{g}\, \partial_m \phi\, \partial^m \phi\, d^3 x \,, \qquad m = 1, 2, 3 \,.
\end{aligned}
\tag{2.12}
$$

In the second line of (2.12), we write the sigma model action in local coordinates on M, with the Einstein summation convention applied to the index 'm'. The factor of $1/4\pi$ in the normalization of $\mathbf{I}_0$ is again a numerical convenience.

The sigma model action $\mathbf{I}_0$ includes a prefactor which we will eventually identify with the electric coupling e^2 under duality. As for the discussion of the Maxwell action in section 1, the overall dependence on e^2 in (2.12) is fixed by invariance under the scale transformations in (2.2) and (2.4). Under the scaling of the metric, the field ϕ is necessarily invariant, since any non-trivial scaling of ϕ would be incompatible with the fixed angular identification in (2.6). Otherwise, due to its implicit metric dependence, the dual two-form $\star d\phi$ scales as $\star d\phi \mapsto \Lambda \star d\phi$.[2] Because the coupling e^2 scales inversely to $\star d\phi$, the sigma model action $\mathbf{I}_0$ is thus invariant.

As usual, the metric on M induces an inner-product on the space $\Omega^p(M)$ of smooth p-forms for each $p = 0, \ldots, 3$,

$$(\eta, \xi) = \int_M \eta \wedge \star \xi \,, \qquad \eta, \xi \in \Omega^p(M) \,. \tag{2.13}$$

In terms of the L^2 inner-product, the sigma model action can be abbreviated

$$\mathbf{I}_0(\phi) = \frac{e^2}{4\pi} (d\phi, d\phi) \,. \tag{2.14}$$

More or less immediately, the critical points of the free sigma model action in (2.14) are harmonic maps from M to S^1,

$$\delta \mathbf{I}_0(\phi_{\mathrm{cl}}) = 0 \quad \Longleftrightarrow \quad \triangle \phi_{\mathrm{cl}} = 0 \,. \tag{2.15}$$

---

[2]The scaling of the two-form $\star d\phi$ is perhaps most easily examined in local coordinates, where $\star d\phi \equiv \sqrt{g}\, \epsilon_{mnp}\, \partial^p \phi\, dx^m \wedge dx^n$. Under the scale transformation $g \mapsto \Lambda^2 g$, the volume factor $\sqrt{g}$ transforms as $\sqrt{g} \mapsto \Lambda^3 \sqrt{g}$. On the other hand, $\partial^p \phi = g^{pq}\, \partial_q \phi$ scales as $\partial^p \phi \mapsto \Lambda^{-2} \partial^p \phi$. So $\star d\phi$ scales in total as $\star d\phi \mapsto \Lambda \star d\phi$.

Because M is compact, any $\mathbb{R}$-valued harmonic function on M is constant and hence unique up to normalization. For S^1-valued harmonic functions as in (2.15), a roughly similar statement holds.

First, by classical Hodge theory, each integral cohomology class $\omega \in \mathbb{L}$ admits a unique harmonic representative with integral periods on M. Abusing notation slightly, I also use ω to denote the corresponding harmonic one-form, which depends upon the Riemannian metric on M. If $\phi_{\mathrm{cl}} \in \mathcal{X}_\omega$ is a circle-valued harmonic map with winding-number ω, then necessarily ϕ_{cl} is related to ω by

$$d\phi_{\mathrm{cl}} = 2\pi\omega\,, \qquad \omega \in \mathcal{H}^1(M)\,. \tag{2.16}$$

This condition implies both that ϕ_{cl} has winding-number ω and that ϕ_{cl} is harmonic, since

$$\triangle\phi_{\mathrm{cl}} = d^\dagger d\phi_{\mathrm{cl}} = 2\pi d^\dagger\omega = 0\,. \tag{2.17}$$

Given the integral harmonic form ω, the linear equation in (2.16) can always be solved and so determines ϕ_{cl} up to the addition of a constant. As a result, the moduli space of harmonic maps with winding-number ω is a copy of S^1.

A topological parameter for the sigma model. Given the decomposition for $\mathcal{X}$ in (2.10), we naturally extend the free sigma model action $\mathbf{I}_0(\phi)$ to include a topological term which is locally-constant on $\mathcal{X}$ and hence only sensitive to the winding-number ω.

The most obvious topological term for ϕ depends upon the choice of a de Rham cohomology class

$$\beta \in H^2(M;\mathbb{R}) \simeq \mathbb{R}^{b_1}, \tag{2.18}$$

with the pairing

$$\mathbf{I}_\beta(\phi) = \frac{1}{2\pi i} \int_M \beta \wedge d\phi\,. \tag{2.19}$$

Because β and $d\phi$ are both closed forms on the compact manifold M, the pairing in (2.19) depends only on the cohomology class of β and on the homotopy class of the map ϕ. In particular, $\mathbf{I}_\beta(\phi)$ is unchanged under any variation of ϕ, so the addition of $\mathbf{I}_\beta(\phi)$ to the sigma model action does not change the harmonic equation of motion for ϕ.

One might suppose that $\mathbf{I}_\beta(\phi)$ is the end of the story, since there are hardly any other topological couplings to write for the free sigma model. However, this paper is about duality, and the choice of β in (2.18) does not respect the fundamental duality on M — to wit, Poincaré duality.

The most elegant formulation of the scalar partition function on M occurs when we introduce a second cohomological parameter dual to β. The dual parameter α is a harmonic one-form on M,

$$\alpha \in \mathcal{H}^1(M) \simeq \mathbb{R}^{b_1}, \tag{2.20}$$

which couples linearly to the sigma model map ϕ via

$$\mathbf{I}_\alpha(\phi) = \frac{e^2}{2\pi} \int_M \star\alpha \wedge d\phi\,. \tag{2.21}$$

The choice of Riemannian metric on M enters both the definition of α as a harmonic one-form and the description of the coupling $\mathbf{I}_\alpha(\phi)$ in (2.21). However, precisely because $\star\alpha$ is closed, the value of $\mathbf{I}_\alpha(\phi)$ does not change under variations of ϕ, so $\mathbf{I}_\alpha(\phi)$ is also a locally-constant function on the space $\mathcal{X}$ of sigma model maps. Hence $\mathbf{I}_\alpha(\phi)$ does not alter the harmonic equation of motion for ϕ either. We include the prefactor of e^2 in (2.21) to ensure that $\mathbf{I}_\alpha(\phi)$ is invariant under the scale transformations in (2.2) and (2.4).

Including both the cohomological parameters α and β, the total sigma model action becomes

$$\begin{aligned} \mathbf{I}_{\text{tot}}(\phi) &= \mathbf{I}_0(\phi) + \mathbf{I}_\beta(\phi) + \mathbf{I}_\alpha(\phi)\,, \\ &= \mathbf{I}_0(\phi) + \frac{1}{2\pi i}\int_M (\beta + i\,e^2 \star \alpha) \wedge d\phi\,. \end{aligned} \qquad (2.22)$$

Though α and β enter the total action similarly, an asymmetry exists in our description of these parameters. The parameter α is a definite harmonic one-form on M, whereas β is any closed two-form representing the given cohomology class.

A loose theme running throughout this work and its companion [1] is the interpretation of abelian duality on M as a kind of quantum Poincaré duality, which here will exchange α and β. To make sense of this exchange, though, we will need to select a definite two-form on M to represent the cohomology class of β. Because $\alpha \in \mathcal{H}^1(M)$ is already harmonic, we also use the metric on M to determine a harmonic representative $\beta \in \mathcal{H}^2(M)$ for the cohomology class in (2.18).

With this choice, the sigma model action in (2.22) only depends on the parameters α and β in the holomorphic combination

$$\gamma = \beta + i\,e^2 \star \alpha \in \mathcal{H}^2_{\mathbb{C}}(M) \simeq \mathbb{C}^{b_1}, \qquad (2.23)$$

a complex harmonic two-form on M. By construction, γ is invariant under the combined scalings of the metric g and coupling e^2 in (2.2) and (2.4).

The total sigma model action can then be written concisely as

$$\mathbf{I}_{\text{tot}}(\phi) = \frac{e^2}{4\pi}(d\phi, d\phi) + \frac{1}{2\pi i}\langle \gamma, d\phi \rangle\,, \qquad (2.24)$$

where $(\,\cdot\,,\,\cdot\,)$ is the L^2 inner-product, and $\langle\,\cdot\,,\,\cdot\,\rangle$ is the canonical intersection pairing,

$$\langle \eta, \xi \rangle = \int_M \eta \wedge \xi\,, \qquad \eta\,,\xi \in \Omega^*(M)\,. \qquad (2.25)$$

2.2 Computing the partition function

We now evaluate the partition function for the periodic scalar field on M using the path integral presentation

$$Z_M(\gamma) = \sum_{\omega \in \mathbb{L}} \int_{\mathcal{X}_\omega} \mathcal{D}\phi\,\exp\!\big[-\mathbf{I}_{\text{tot}}(\phi)\big]\,. \qquad (2.26)$$

Here I indicate the explicit dependence of the partition function on the holomorphic parameter $\gamma \in \mathcal{H}^2_{\mathbb{C}}(M)$, and I leave implicit the combined dependence on the coupling e^2 and the Riemannian metric g.

Because the sigma model configuration space $\mathcal{X}$ decomposes into components labelled by the winding-number $\omega \in \mathbb{L}$, the sigma model path integral includes a sum over the cohomology lattice $\mathbb{L}$, followed by an integral over each component $\mathcal{X}_\omega \subset \mathcal{X}$. With the free sigma model action in (2.24), those integrals are all Gaussian and hold no particular mysteries. However, we would like to assign a definite normalization to $Z_M(\gamma)$, and for that goal, we must be careful about the normalization of the sigma model measure $\mathcal{D}\phi$ itself.

More about the sigma model measure. Although the configuration space $\mathcal{X} = \mathrm{Map}(M, S^1)$ decomposes into an infinite number of components $\mathcal{X}_\omega$ labelled by the winding-number ω, each component can be identified with the distinguished component $\mathcal{X}_0$, which consists of maps with trivial winding. To make the identification $\mathcal{X}_\omega \simeq \mathcal{X}_0$, we select a basepoint $\Phi_\omega \in \mathcal{X}_\omega$, corresponding to a fiducial map with winding-number ω. Given the fiducial map Φ_ω, any other map $\phi \in \mathcal{X}_\omega$ with the same winding-number can be written uniquely as a sum

$$\phi = \Phi_\omega + \psi \,, \qquad \psi \in \mathcal{X}_0 \,, \tag{2.27}$$

where $\psi : M \to S^1$ is a sigma model map with vanishing winding. The correspondence between ϕ and ψ in (2.27) provides the desired identification of $\mathcal{X}_\omega$ and $\mathcal{X}_0$. As an immediate corollary, if we wish to characterize the sigma model measure $\mathcal{D}\phi$ on $\mathcal{X}_\omega$, we need only characterize it on $\mathcal{X}_0$.

We will characterize the measure on $\mathcal{X}_0$ momentarily, but let us first make a definite choice for the fiducial map $\Phi_\omega \in \mathcal{X}_\omega$ in (2.27). Our choice will depend upon the Riemannian metric g on M, as well as the data of a point $p \in M$. Using the metric, we first impose the condition that $\Phi_\omega : M \to S^1$ be harmonic, or equivalently

$$d\Phi_\omega = 2\pi\omega \,, \qquad \omega \in \mathcal{H}^1(M) \,. \tag{2.28}$$

The condition in (2.28) determines Φ_ω up to the addition of a constant. To fix the constant, we next impose

$$\Phi_\omega(p) = 0 \bmod 2\pi \,, \qquad p \in M \,. \tag{2.29}$$

Together, the conditions in (2.28) and (2.29) determine the map Φ_ω uniquely.

At any given point $\phi \in \mathcal{X}_0$, the tangent space to $\mathcal{X}_0$ at ϕ is simply the space of real-valued functions $\Omega^0(M)$,

$$T_{[\phi]}\mathcal{X}_0 = \Omega^0(M) \,. \tag{2.30}$$

Indeed globally,

$$\mathcal{X}_0 = \Omega^0(M) \bmod 2\pi \,. \tag{2.31}$$

The metric on M immediately induces a metric on $\mathcal{X}_0$, given by

$$||\delta\phi||^2_{\mathcal{X}_0} = \frac{e^6}{(2\pi)^2} \int_M \delta\phi \wedge \star\delta\phi \,, \qquad \delta\phi \in \Omega^0(M) \,. \tag{2.32}$$

The appearance of the L^2-norm on $\Omega^0(M)$ should come as no surprise, but the coupling-dependent prefactor in (2.32) may be one. Under a scaling of the metric $g \mapsto \Lambda^2 g$, the

standard L^2-norm on $\Omega^0(M)$ scales according to the volume of M,

$$\int_M \delta\phi \wedge \star\delta\phi \longmapsto \Lambda^3 \int_M \delta\phi \wedge \star\delta\phi\,, \qquad \Lambda \in \mathbb{R}_+\,. \tag{2.33}$$

Since $e^2 \mapsto \Lambda^{-1}\, e^2$ according to (2.4), the prefactor of e^6 in $||\delta\phi||^2_{\mathcal{X}_0}$ ensures invariance of the metric on $\mathcal{X}_0$ under scaling. The remaining factors of 2π in (2.32) appear by convention.

Once $\mathcal{X}_0$ carries a Riemannian structure, we take $\mathcal{D}\phi$ to be the corresponding Riemannian measure. Under the identification $\mathcal{X}_\omega \simeq \mathcal{X}_0$, the sigma model measure then extends to all of the configuration space $\mathcal{X}$. Finally, by construction $\mathcal{D}\phi$ is invariant under translations by elements in $\Omega^0(M)$. Hence $\mathcal{D}\phi$ does not actually depend upon the particular choice of basepoint $\Phi_\omega \in \mathcal{X}_\omega$ specified in (2.28) and (2.29).

Though the description of $\mathcal{D}\phi$ in terms of the metric on $\mathcal{X}_0$ is purely formal, we will see later that this description, formal or no, allows us to perform a precise accounting under duality of all coupling-dependent factors in the partition function. This accounting clarifies the results in [2].

A sum over windings. To evaluate the partition function on M,

$$Z_M(\gamma) = \sum_{\omega \in \mathbb{L}} \int_{\mathcal{X}_\omega} \mathcal{D}\phi \, \exp\big[-\mathbf{I}_{\text{tot}}(\phi)\big]\,, \tag{2.34}$$

we first employ the componentwise identification $\mathcal{X}_\omega \simeq \mathcal{X}_0$ to rewrite the integration variable ϕ in (2.34) as the sum

$$\phi = \Phi_\omega + \psi\,, \qquad \Phi_\omega \in \mathcal{X}_\omega\,, \qquad \psi \in \mathcal{X}_0\,. \tag{2.35}$$

With the substitution in (2.35), the partition function can be computed in terms of a path integral over the distinguished component $\mathcal{X}_0$ alone,

$$Z_M(\gamma) = \sum_{\omega \in \mathbb{L}} \int_{\mathcal{X}_0} \mathcal{D}\psi \, \exp\big[-\mathbf{I}_{\text{tot}}(\Phi_\omega + \psi)\big]\,. \tag{2.36}$$

In general, when making changes of variables in the path integral, one must be careful about Jacobians, but according to our preceding description of $\mathcal{D}\phi$, the Jacobian for the substitution in (2.36) is unity.

In terms of the harmonic map Φ_ω and the homotopically-trivial map ψ, the sigma model action becomes

$$\mathbf{I}_{\text{tot}}(\Phi_\omega + \psi) = \frac{e^2}{4\pi}(d\Phi_\omega + d\psi, d\Phi_\omega + d\psi) + \frac{1}{2\pi i}\langle \gamma, d\Phi_\omega + d\psi\rangle\,, \tag{2.37}$$

or more explicitly,

$$\mathbf{I}_{\text{tot}}(\Phi_\omega + \psi) = \frac{e^2}{4\pi}\int_M (2\pi\omega + d\psi) \wedge \star(2\pi\omega + d\psi) + \frac{1}{2\pi i}\int_M \gamma \wedge (2\pi\omega + d\psi)\,. \tag{2.38}$$

In passing to (2.38), I recall that the fiducial map satisfies $d\Phi_\omega = 2\pi\omega$. Since both ω and γ are harmonic forms on M, the cross-terms in (2.38) which involve either ω or γ together

with $d\psi$ vanish, so that

$$
\begin{aligned}
\mathbf{I}_{\text{tot}}(\Phi_\omega + \psi) &= \frac{e^2}{4\pi} \int_M d\psi \wedge \star d\psi + \pi e^2 \int_M \omega \wedge \star \omega - i \int_M \omega \wedge \gamma \,, \\
&= \frac{e^2}{4\pi}(d\psi, d\psi) + \pi e^2 (\omega, \omega) - i\langle \omega, \gamma \rangle \,.
\end{aligned}
\tag{2.39}
$$

From the description of the sigma model action in (2.39), the partition function on M can be rewritten more explicitly as

$$
Z_M(\gamma) = \sum_{\omega \in \mathbb{L}} \int_{\mathcal{X}_0} \mathcal{D}\psi \, \exp\left[-\frac{e^2}{4\pi}(d\psi, d\psi) - \pi e^2(\omega, \, \omega) + i\langle \omega, \gamma \rangle \right].
\tag{2.40}
$$

Because the argument of the exponential in (2.40) is a sum of terms which depend separately on the variables $\omega \in \mathbb{L}$ and $\psi \in \mathcal{X}_0$, the partition function immediately factorizes,

$$
Z_M(\gamma) = \Delta_M \cdot \Theta_M(\gamma) \,,
\tag{2.41}
$$

where Δ_M is given by a Gaussian integral over the space $\mathcal{X}_0$ of homotopically-trivial maps $\psi : M \to S^1$,

$$
\Delta_M = \int_{\mathcal{X}_0} \mathcal{D}\psi \, \exp\left[-\frac{e^2}{4\pi}(d\psi, d\psi) \right],
\tag{2.42}
$$

and $\Theta_M(\gamma)$ is given by a discrete sum over the cohomology lattice $\mathbb{L} = H^1(M; \mathbb{Z})$,

$$
\Theta_M(\gamma) = \sum_{\omega \in \mathbb{L}} \exp\left[-\pi e^2(\omega, \, \omega) + i\langle \omega, \gamma \rangle \right].
\tag{2.43}
$$

Due to the Gaussian damping, the lattice sum which defines $\Theta_M(\gamma)$ in (2.43) is convergent for all $\gamma \in \mathcal{H}^2_{\mathbb{C}}(M)$.

From the perspective of duality, the more interesting term in the factorization (2.41) is $\Theta_M(\gamma)$, which carries the dependence on the holomorphic parameter γ and arises from the quantum sum over winding-sectors in the sigma model. Clearly $\Theta_M(\gamma)$ is a theta-function attached to the three-manifold M though the cohomology lattice $\mathbb{L} = H^1(M; \mathbb{Z})$, and wherever a theta-function appears, the modular group lurks. For the time being, though, I postpone discussion of $\Theta_M(\gamma)$ and its role in duality until section 4.

The normalization of $Z_M(\gamma)$. Finally, to fix the absolute normalization of the scalar partition function, we are left to compute the factor

$$
\Delta_M = \int_{\mathcal{X}_0} \mathcal{D}\psi \, \exp\left[-\frac{e^2}{4\pi}(d\psi, d\psi) \right],
\tag{2.44}
$$

depending on the coupling e^2 and the Riemannian metric on M. Because the path integral in (2.44) is Gaussian, the only trick will be to keep track of factors associated to the normalization of the path integral measure $\mathcal{D}\psi$.

As a first step, we expand ψ in an orthonormal basis for $\mathcal{X}_0 \simeq \Omega^0(M)$ mod 2π with respect to the metric in (2.32),

$$
\psi = \psi_0 \cdot \left[\frac{2\pi}{(e^2 \ell)^{3/2}} \right] + \sum_{\lambda > 0} \psi_\lambda \cdot \hat{f}_\lambda \,.
\tag{2.45}
$$

Here each $\hat{f}_\lambda$ is a normalized eigenfunction with eigenvalue λ for the scalar Laplacian $\triangle$ on M,

$$\triangle \hat{f}_\lambda = \lambda \, \hat{f}_\lambda \,, \qquad ||\hat{f}_\lambda||^2_{\mathcal{X}_0} = \frac{e^6}{(2\pi)^2} \int_M \hat{f}_\lambda^2 \, \mathrm{vol}_M = 1 \,, \tag{2.46}$$

and each spectral coefficient ψ_λ is valued in $\mathbb{R}$.

More important is the coefficient of the constant function

$$\hat{f}_0 = \left[\frac{2\pi}{(e^2 \ell)^{3/2}} \right] \,, \qquad ||\hat{f}_0||^2_{\mathcal{X}_0} = \frac{e^6}{(2\pi)^2} \int_M \hat{f}_0^2 \, \mathrm{vol}_M = 1 \,, \tag{2.47}$$

also chosen to have unit-norm in $\Omega^0(M)$. Because ψ satisfies the periodicity condition $\psi \sim \psi + 2\pi$, the coefficient ψ_0 of the constant mode $\hat{f}_0$ must have period

$$\psi_0 \sim \psi_0 + (e^2 \ell)^{3/2}. \tag{2.48}$$

Though we have fixed the periodicity of the sigma model field ψ to be independent of the coupling, the effective periodicity of the zero-mode ψ_0 does depend upon e^2 and diverges as $e^2 \to \infty$ with the volume of M held fixed.

In terms of the spectral decomposition (2.45) for ψ, the Gaussian path integral in (2.44) becomes

$$\Delta_M = \int_{\mathcal{X}_0} d\psi_0 \, \mathcal{D}\psi' \, \exp\left[- \frac{e^2}{4\pi} \, (\psi', \triangle \psi') \right] \,. \tag{2.49}$$

Here ψ' indicates the orthocomplement to the constant mode in $\Omega^0(M)$,

$$\psi' = \sum_{\lambda > 0} \psi_\lambda \cdot \hat{f}_\lambda \,, \qquad \psi_\lambda \in \mathbb{R} \,, \tag{2.50}$$

and at least formally, $\mathcal{D}\psi'$ is shorthand for the product measure

$$\mathcal{D}\psi' = \prod_{\lambda > 0} d\psi_\lambda \,. \tag{2.51}$$

As standard, in obtaining (2.49) we integrate by parts to produce the scalar Laplacian $\triangle = d^\dagger d$ acting on ψ' in the argument of the exponential.

Again with care for factors of e^2, we apply the expansion of ψ' in (2.50) to simplify that argument,

$$\begin{aligned} \frac{e^2}{4\pi}(\psi', \triangle \psi') &= \frac{e^2}{4\pi} \sum_{\lambda > 0} \lambda \, \psi_\lambda^2 \, (\hat{f}_\lambda, \hat{f}_\lambda) \,, \\ &= \frac{\pi}{e^4} \sum_{\lambda > 0} \lambda \, \psi_\lambda^2 \,. \end{aligned} \tag{2.52}$$

In passing to the second line of (2.52), we observe that the eigenfunction $\hat{f}_\lambda$ has L^2-norm $(\hat{f}_\lambda, \hat{f}_\lambda) = 4\pi^2/e^6$ according to (2.46).

So altogether,

$$\Delta_M = \int_{\mathcal{X}_0} d\psi_0 \, \mathcal{D}\psi' \, \exp\left[- \frac{\pi}{e^4} \sum_{\lambda > 0} \lambda \, \psi_\lambda^2 \right] \,, \tag{2.53}$$

or upon evaluating the Gaussian integrals over each spectral coefficient ψ_λ,

$$\Delta_M = (e^2\ell)^{3/2} \cdot \frac{1}{\sqrt{\det'(\triangle/e^4)}} \,. \tag{2.54}$$

The slightly tricky prefactor $(e^2\ell)^{3/2}$ arises from the integral over the zero-mode ψ_0 with the periodicity in (2.48), and $\det'$ indicates the determinant with kernel omitted. Some regularization method must be chosen to define the functional determinant in (2.54) as a real number, with zeta-function regularization being one possibility. See for instance [11, 23] for explicit calculations of such zeta-regularized determinants on lens spaces $M = S^3/\mathbb{Z}_k$, with the round metric inherited from S^3.

The expression for Δ_M in (2.54) makes manifest the fact that Δ_M is invariant under the simultaneous scalings of the metric and coupling in (2.2) and (2.4). In particular, one can easily check that the operator $\triangle/e^4$ is invariant under scaling, since $\triangle$ scales with Λ as $\triangle \mapsto \Lambda^{-2} \triangle$. However, the appearance of the coupling in the functional determinant $\det'(\triangle/e^4)$ is slightly awkward, and if one wishes, the dependence on e^2 in Δ_M can be made more explicit by pulling e^2 out from the determinant. The same issue arises in appendix A of [11], whose strategy of analysis we follow.

In zeta-regularization, the functional determinant is defined in terms of the zeta-function for the scalar Laplacian $\triangle$ on M,

$$\zeta_\triangle(s) = \sum_{\lambda > 0} \lambda^{-s}, \qquad s \in \mathbb{C} \,. \tag{2.55}$$

The sum over positive eigenvalues in (2.55) is convergent when the real part of s is sufficiently large, and $\zeta_\triangle(s)$ is defined for other values of s by analytic continuation. By standard manipulations, the functional determinant of $\triangle$ is defined in terms of the derivative of $\zeta_\triangle(s)$ at $s = 0$,

$$\det'(\triangle) = \exp\left[-\zeta'_\triangle(0)\right]. \tag{2.56}$$

Because we are interested in the determinant of the operator $\triangle/e^4$, we instead consider

$$\eta_\triangle(s) = \sum_{\lambda > 0} \left(\frac{\lambda}{e^4}\right)^{-s} = e^{4s} \cdot \zeta_\triangle(s) \,. \tag{2.57}$$

Then similarly,

$$\det'(\triangle/e^4) = \exp\left[-\eta'_\triangle(0)\right]. \tag{2.58}$$

On the other hand, directly from (2.57),

$$\eta'_\triangle(0) = \ln(e^4) \cdot \zeta_\triangle(0) + \zeta'_\triangle(0) \,, \tag{2.59}$$

from which we obtain the relation

$$\det'(\triangle/e^4) = e^{-4\zeta_\triangle(0)} \cdot \det'(\triangle) \,. \tag{2.60}$$

The value of $\zeta_\triangle(s)$ at $s = 0$ can be interpreted as a regularized dimension for the non-zero eigenspace of the operator $\triangle$. Very generally, if M is any compact manifold of odd

dimension, and $\triangle_p$ is the de Rham Laplacian acting on the space $\Omega^p(M)$ of p-forms, then the value of the associated zeta-function at $s = 0$ is given by

$$\zeta_{\triangle_p}(0) = -\dim \operatorname{Ker}\triangle_p. \tag{2.61}$$

See theorem 5.2 in [30] for a textbook proof of (2.61), which goes back to [20]. For us, the scalar Laplacian on M has a one-dimensional kernel, so $\zeta_{\triangle}(0) = -1$ in (2.60). We thence obtain

$$\det{}'(\triangle/e^4) = e^4 \cdot \det{}'(\triangle). \tag{2.62}$$

As a result, the normalization factor in (2.54) reduces to

$$\Delta_M = \frac{e\,\ell^{3/2}}{\sqrt{\det{}'(\triangle)}}. \tag{2.63}$$

The linear dependence of Δ_M on e also follows by well-known physical arguments involving the counting of zero-modes for the Laplacian.

The final result.   In summary, we have determined the scalar partition function on M to be

$$\begin{aligned}
Z_M(\boldsymbol{\gamma}) &= \Delta_M \cdot \Theta_M(\boldsymbol{\gamma}), \qquad \boldsymbol{\gamma} \in \mathcal{H}^2_{\mathbb{C}}(M) \simeq \mathbb{C}^{b_1}, \\
&= \frac{e\,\ell^{3/2}}{\sqrt{\det{}'(\triangle_0)}} \cdot \sum_{\omega \in \mathbb{L}} \exp\big[-\pi e^2(\omega,\,\omega) + i\langle\omega,\boldsymbol{\gamma}\rangle\big].
\end{aligned} \tag{2.64}$$

The subscript serves to emphasize that $\triangle_0$ is the scalar Laplacian, acting on forms of degree zero. We will similarly meet the de Rham Laplacian $\triangle_1$ for one-forms when we consider abelian gauge theory in section 3.

3   Analysis of the abelian gauge theory

Just as we computed the partition function for a periodic scalar field, we now compute the partition function for an abelian gauge field on the closed three-manifold M.

3.1   The classical Maxwell theory

Classically, the gauge field A is a connection on a fixed principal U(1)-bundle P over M,

$$\begin{array}{c} \mathrm{U}(1) \to P \\ \downarrow \\ M \end{array} . \tag{3.1}$$

The typical three-manifold admits many choices for the bundle P, whose topological type is characterized by the first Chern class

$$c_1(P) \in H^2(M;\mathbb{Z}). \tag{3.2}$$

Via (3.2), the possible types of U(1)-bundles on M are in one-to-one correspondence with elements of $H^2(M;\mathbb{Z})$.

In general, the abelian group $H^2(M; \mathbb{Z})$ contains torsion, and $c_1(P)$ may be a torsion class, of finite-order in $H^2(M; \mathbb{Z})$. Throughout this work, we will be fastidious about torsion, so we recall the exact sequence

$$0 \longrightarrow H^2(M; \mathbb{Z})_{\mathrm{tors}} \longrightarrow H^2(M; \mathbb{Z}) \longrightarrow H^2(M; \mathbb{Z})_{\mathrm{free}} \longrightarrow 0 \,. \tag{3.3}$$

Here $H^2(M; \mathbb{Z})_{\mathrm{tors}}$ is the subgroup of torsion classes, and $H^2(M; \mathbb{Z})_{\mathrm{free}}$ is the reduction of $H^2(M; \mathbb{Z})$ modulo torsion. Poincaré duality asserts that the lattice $\mathbb{L} = H^1(M; \mathbb{Z})$ is dual under the intersection pairing to the quotient lattice

$$\mathbb{L}^\vee = H^2(M; \mathbb{Z})_{\mathrm{free}} \simeq \mathbb{Z}^{b_1} \,. \tag{3.4}$$

The lattice $\mathbb{L}^\vee$ also embeds in the vector space $\mathcal{H}^2(M)$ as the set of harmonic two-forms with integral periods on M,

$$\mathbb{L}^\vee \subset \mathcal{H}^2(M) \simeq \mathbb{R}^{b_1} \,. \tag{3.5}$$

We will often assume implicitly the embedding in (3.5).

Though the exact sequence in (3.3) can be split, the sequence does not split in any natural way, so we should not think about $\mathbb{L}^\vee$ as a subgroup of $H^2(M; \mathbb{Z})$. However, we can always consider the reduction of any class in $H^2(M; \mathbb{Z})$ modulo torsion, to obtain a class valued in $\mathbb{L}^\vee$. Throughout this paper, we normalize the gauge field A so that the reduction of $c_1(P) \in H^2(M; \mathbb{Z})$ modulo torsion admits the de Rham representative

$$\left[\frac{F_A}{2\pi}\right] \in \mathbb{L}^\vee = H^2(M; \mathbb{Z})_{\mathrm{free}} \,. \tag{3.6}$$

As usual, $F_A = dA$ is the curvature of the connection.

With the normalization in (3.6), homotopically non-trivial gauge transformations act on A by shifts

$$A \longmapsto A + 2\pi\omega \,, \qquad \omega \in \mathbb{L} = H^1(M; \mathbb{Z}) \,. \tag{3.7}$$

Such shifts preserve the holonomy of A, as measured physically by the Wilson loop operator $\mathsf{W}_n(C)$ attached to an oriented closed curve $C \subset M$,

$$\mathsf{W}_n(C) = \exp\left[i\, n \oint_C A\right] \,, \qquad n \in \mathbb{Z} \,. \tag{3.8}$$

We have already seen that the lattice $\mathbb{L}$ plays an important role in characterizing the winding-number of the circle-valued map $\phi : M \to S^1$. The dual lattice $\mathbb{L}^\vee$ plays a similar role for Maxwell theory, since $\mathbb{L}^\vee$ determines the topology of the U(1)-bundle P, at least up to torsion. Given the canonical pairing between $\mathbb{L}$ and $\mathbb{L}^\vee$, one might wonder whether it is even necessary in the context of abelian duality to consider bundles for which $c_1(P)$ is torsion. As we will see in section 4, following the original observation in [2], a precise understanding of duality indeed requires that we consider all possibilities for $c_1(P)$ in $H^2(M; \mathbb{Z})$, including torsion classes.

Once the bundle P is fixed, we introduce the Maxwell action

$$\begin{aligned}
\mathbf{I}_0(A) &= \frac{1}{4\pi e^2} \int_M F_A \wedge \star F_A \,, \\
&= \frac{1}{8\pi e^2} \int_M \sqrt{g}\, F_{A,mn} F_A{}^{mn}\, d^3x \,, \qquad m, n = 1, 2, 3 \,.
\end{aligned} \tag{3.9}$$

Here e^2 is the electric coupling, and the factor of $1/4\pi$ is required to match our previous conventions for the scalar field. In the second line of (3.9), we recall the component expansion of the Maxwell action in local coordinates on M.

Critical points of $\mathbf{I}_0(A)$ correspond to connections A_{cl} on P whose curvatures satisfy

$$\delta\mathbf{I}_0(A_{\mathrm{cl}}) = 0 \qquad \Longleftrightarrow \qquad d^\dagger F_{A_{\mathrm{cl}}} = 0\,. \tag{3.10}$$

Trivially $dF_{A_{\mathrm{cl}}} = 0$, so any connection which solves the Maxwell equation on M has harmonic curvature

$$F_{A_{\mathrm{cl}}} \in \mathcal{H}^2(M)\,. \tag{3.11}$$

The integrality condition in (3.6) then determines $F_{A_{\mathrm{cl}}}$ uniquely according to the topological type of P. For future reference, we set

$$F_{A_{\mathrm{cl}}} = 2\pi\lambda\,, \qquad \lambda \in \mathbb{L}^\vee = H^2(M;\mathbb{Z})_{\mathrm{free}}\,. \tag{3.12}$$

Although flux quantization restricts the classical values for F_A to a discrete set, the moduli space of classical solutions to the Maxwell equation on M generally has positive dimension whenever $b_1 > 0$. Clearly, if A solves the Maxwell equation with harmonic curvature and $\eta \in \Omega^1(M)$ is any closed one-form, then $A + \eta$ also solves the Maxwell equation with the same curvature. Modulo gauge equivalence, the closed one-form η determines a point in the real torus

$$\mathscr{J}_M = H^1(M;\mathbb{R})/2\pi\mathbb{L} \simeq \mathrm{U}(1)^{b_1}\,, \tag{3.13}$$

where we have been careful to divide by the large gauge equivalences in (3.7).

For each $\mathrm{U}(1)$-bundle P over M, the moduli space of classical solutions to the Maxwell equation is simply a copy of the torus $\mathscr{J}_M$ in (3.13). As mentioned in the Introduction, this moduli space bears no resemblance to the moduli space of harmonic maps $\phi : M \to S^1$, which is instead a copy of S^1 for each winding-sector. Abelian duality on M must therefore involve a non-trivial quantum equivalence, even though the field theories involved are free. I will develop this theme further in [1], where I discuss the canonical quantization of these theories.

Adding topological couplings. Just as for the periodic scalar field in section 2, we now extend the classical Maxwell action to include additional couplings which will be topological in the sense that they do not change the classical Maxwell equation in (3.10).

The more obvious topological coupling depends upon the choice of a de Rham cohomology class

$$\alpha \in H^1(M;\mathbb{R}) \simeq \mathbb{R}^{b_1}\,, \tag{3.14}$$

for which we introduce the pairing

$$\mathbf{I}_\alpha(A) = -\frac{1}{2\pi i} \int_M \alpha \wedge F_A\,. \tag{3.15}$$

Because both α and F_A are closed forms, the pairing in (3.15) depends only on the respective cohomology classes of those forms. In particular, via the identification (3.6), the value of

$\mathbf{I}_\alpha(A)$ depends only on the image of the Chern class $c_1(P)$ in the lattice $\mathbb{L}^\vee$ and is insensitive to torsion. The sign in (3.15) is just a convention that will make the duality formulas in section 4 more elegant.

Dually, we also introduce a harmonic two-form β,

$$\beta \in \mathcal{H}^2(M) \simeq \mathbb{R}^{b_1}, \tag{3.16}$$

with coupling

$$\mathbf{I}_\beta(A) = \frac{1}{2\pi e^2} \int_M \star\beta \wedge F_A. \tag{3.17}$$

The harmonic condition ensures that $\star\beta$ is a closed one-form, so that the value of $\mathbf{I}_\beta(A)$ also depends only on the Chern class of the bundle P.

We will eventually identify α and β with the same parameters which we introduced for the periodic scalar field in section 2. However, the defining conditions on α and β are now reversed. In section 2, the one-form α was required to be harmonic and β was an arbitrary closed two-form, whereas here β is harmonic and α is an arbitrary closed one-form.

Including the parameters α and β, the Maxwell action on M becomes

$$\begin{aligned}
\mathbf{I}_{\text{tot}}(A) &= \mathbf{I}_0(A) + \mathbf{I}_\alpha(A) + \mathbf{I}_\beta(A), \\
&= \mathbf{I}_0(A) + \frac{1}{2\pi i} \int_M \left(-\alpha + \frac{i}{e^2} \star\beta \right) \wedge F_A.
\end{aligned} \tag{3.18}$$

Without loss, I select a harmonic representative for α and introduce the complex harmonic one-form appearing holomorphically in (3.18),

$$\zeta = -\alpha + \frac{i}{e^2} \star\beta \in \mathcal{H}^1_{\mathbb{C}}(M). \tag{3.19}$$

Just as for the harmonic two-form γ in (2.23), the harmonic one-form ζ is invariant under the combined scale transformations in (2.2) and (2.4). As one can easily check, the complex harmonic forms γ and ζ are related by

$$\gamma = -i\,e^2 \star\zeta \in \mathcal{H}^2_{\mathbb{C}}(M). \tag{3.20}$$

The total Maxwell action in (3.18) can then be written concisely in terms of the L^2 and intersection pairings,

$$\mathbf{I}_{\text{tot}}(A) = \frac{1}{4\pi e^2}(F_A, F_A) + \frac{1}{2\pi i}\langle \zeta, F_A \rangle. \tag{3.21}$$

Abelian duality at level k. As a special feature of abelian gauge theory in three dimensions, we can add to the Maxwell action (3.21) a Chern-Simons term proportional to

$$\mathsf{CS}(A) = \frac{1}{2\pi} \int_M A \wedge dA. \tag{3.22}$$

We follow the standard practice in writing the Chern-Simons functional with respect to a local trivialization for the bundle P. However, because A can be a connection on a non-trivial U(1)-bundle P over M, the global meaning of the trivialized form (3.22) of the Chern-Simons functional may be unclear.

For an alternative presentation, one can always choose a four-manifold X such that X bounds M and the U(1)-bundle P extends from M to X. The existence of X relies upon the vanishing of $H_3(B\mathrm{U}(1))$ and is discussed thoroughly in [9]. Given X, the Chern-Simons functional on M can be rewritten in the gauge-invariant fashion

$$\mathsf{CS}(A) = \frac{1}{2\pi} \int_X F_A \wedge F_A \,, \qquad \partial X = M \,. \tag{3.23}$$

Since $F_A/2\pi$ is an integral two-form on any closed manifold, and since the intersection pairing is also integral, the global expression for the Chern-Simons functional in (3.23) shows that the value of $\mathsf{CS}(A)$ is well-defined modulo 2π.

If we wish, we can then extend the Maxwell action (3.21) on M to a Maxwell-Chern-Simons action at level $k \in \mathbb{Z}$,

$$\mathbf{I}_{\mathrm{MCS}}(A) = \frac{1}{4\pi e^2}(F_A, F_A) + \frac{1}{2\pi i}\langle \zeta, F_A \rangle - i\,k\,\mathsf{CS}(A) \,. \tag{3.24}$$

Just as the parameter ζ is naturally related under duality to the parameter γ for the periodic scalar field ϕ, one can ask about the dual interpretation for the Chern-Simons level k. One standard answer to this question would be to say that the Chern-Simons level has no simple, local description in terms of the periodic scalar field. Strictly speaking, this answer is correct, but it is unsatisfying. Another standard answer, at least when $M = \mathbb{R}^3$, would be to say that Maxwell-Chern-Simons theory at level k is equivalent [8, 17] to the 'self-dual' model [31] of a massive, non-gauge-invariant Proca vector field with Chern-Simons term. Strictly speaking, this answer is also correct, but it is not correct for a general three-manifold.

I will discuss elsewhere a better, more global answer to the question "What is the dual of the Chern-Simons level?" The answer turns out to be most clear with the Hamiltonian formalism developed in [1]. For the present, I just set $k = 0$ and work with only the pure Maxwell theory on M.

3.2 Computing the partition function

The Maxwell partition function on M can be evaluated in a manner very similar to the evaluation of the scalar partition in section 2.2. So I will be relatively brief.

The Maxwell partition function admits the formal path integral presentation

$$Z_M^\vee(\zeta) = \sum_{c_1(P) \in H^2(M;\mathbb{Z})} \frac{1}{\mathrm{Vol}\big(\mathcal{G}(P)\big)} \int_{\mathcal{A}(P)} \mathcal{D}A \, \exp[-\mathbf{I}_{\mathrm{tot}}(A)] \,. \tag{3.25}$$

Evidently, the partition function involves both a sum over the topological type of the principal U(1)-bundle P as well as an integral over the affine space $\mathcal{A}(P)$ of all connections on P. Due to the gauge invariance of the Maxwell action, we divide the path integral by the volume of the group $\mathcal{G}(P)$ of gauge transformations on P. Geometrically, $\mathcal{G}(P)$ can be identified with the group of maps from M to U(1),

$$\mathcal{G}(P) = \mathrm{Map}\big(M, \mathrm{U}(1)\big) \,, \tag{3.26}$$

acting on P by bundle automorphisms.

Again, the most delicate aspect of our computation will be to fix the normalization of the partition function, for which we must be precise about the meaning of the measure $\mathcal{D}A$ on $\mathcal{A}(P)$.

More about the Maxwell measure. Let P_0 be the trivial U(1)-bundle over M. For every other bundle P, the space $\mathcal{A}(P)$ can be identified with $\mathcal{A}(P_0)$ as soon as we pick a basepoint in $\mathcal{A}(P)$, which will correspond geometrically to a fiducial connection on P. In close analogy to the choice of the fiducial harmonic map in section 2, we take the fiducial connection $\widehat{A}_P \in \mathcal{A}(P)$ to possess harmonic curvature

$$F_{\widehat{A}_P} = 2\pi\lambda, \qquad \lambda \in \mathbb{L}^\vee = H^2(M; \mathbb{Z})_{\text{free}}, \tag{3.27}$$

as well as trivial holonomy around a fixed set of curves $C \subset M$ which represent generators for $H_1(M; \mathbb{R})$.

The arbitrary connection $A \in \mathcal{A}(P)$ can then be expressed as a sum

$$A = \widehat{A}_P + \eta, \qquad \eta \in \mathcal{A}(P_0), \tag{3.28}$$

where η is a connection on the trivial bundle. The correspondence between A and η in (3.28) provides the requisite identification $\mathcal{A}(P) \simeq \mathcal{A}(P_0)$ for each U(1)-bundle P. Given this identification, we need only describe the measure $\mathcal{D}A$ for connections on the trivial bundle over M.

The fiducial connection on the trivial bundle P_0 is flat, from which we obtain a trivialization of P_0. We may thus regard connections on P_0 as ordinary one-forms on M. Following the same philosophy from section 2, we characterize the measure $\mathcal{D}A$ on $\mathcal{A}(P_0) \simeq \Omega^1(M)$ as the Riemannian measure induced from the L^2-norm

$$||\delta A||^2_{\mathcal{A}(P_0)} = \frac{e^2}{(2\pi)^2} \int_M \delta A \wedge \star \delta A, \qquad \delta A \in \Omega^1(M). \tag{3.29}$$

Like the corresponding expression in (2.32), the factor of e^2 in (3.29) is dictated by invariance under the scaling in (2.2) and (2.4), and the factors of 2π will prove to be a later numerical convenience. By construction, the measure $\mathcal{D}A$ is invariant under translations in $\mathcal{A}(P_0)$. Hence $\mathcal{D}A$ does not depend upon the choice of fiducial connection used to identify $\mathcal{A}(P_0) \simeq \Omega^1(M)$.

Lastly, to describe the volume factor appearing in (3.25), we must introduce a measure on the group $\mathcal{G}(P)$ of gauge transformations. As the group of maps to U(1), the Lie algebra of $\mathcal{G}(P)$ is simply the linear space $\Omega^0(M)$ of functions on M, with trivial Lie bracket,

$$\text{Lie}\left(\mathcal{G}(P)\right) = \Omega^0(M). \tag{3.30}$$

We have already introduced a suitable Riemannian metric on $\Omega^0(M)$ in (2.32). We extend this metric in a translation-invariant fashion over $\mathcal{G}(P)$, and we take $\text{Vol}(\mathcal{G}(P))$ to be the formal Riemannian volume. This volume is independent of the bundle P.

A sum over fluxes. Once we substitute for A as in (3.28), the Maxwell partition function can be rewritten as a path integral over connections on the trivial bundle P_0 alone,

$$Z_M^\vee(\zeta) = \sum_{c_1(P) \in H^2(M;\mathbb{Z})} \frac{1}{\text{Vol}(\mathcal{G}(P_0))} \int_{\mathcal{A}(P_0)} \mathcal{D}\eta \, \exp\left[-\mathbf{I}_{\text{tot}}(\widehat{A}_P + \eta)\right]. \quad (3.31)$$

In terms of the fiducial connection $\widehat{A}_P$ and the one-form η, the Maxwell action becomes

$$\mathbf{I}_{\text{tot}}(\widehat{A}_P + \eta) = \frac{1}{4\pi e^2}(F_{\widehat{A}_P} + d\eta, F_{\widehat{A}_P} + d\eta) + \frac{1}{2\pi i}(\zeta, F_{\widehat{A}_P} + d\eta), \quad (3.32)$$

or more explicitly,

$$\mathbf{I}_{\text{tot}}(\widehat{A}_P + \eta) = \frac{1}{4\pi e^2}\int_M (2\pi\lambda + d\eta) \wedge \star(2\pi\lambda + d\eta) + \frac{1}{2\pi i}\int_M \zeta \wedge (2\pi\lambda + d\eta). \quad (3.33)$$

In passing to (3.33), we recall the formula for the harmonic curvature $F_{\widehat{A}_P}$ in (3.27). Since λ and ζ are harmonic, all cross-terms which involve λ or ζ together with $d\eta$ vanish, and

$$\begin{aligned}
\mathbf{I}_{\text{tot}}(\widehat{A}_P + \eta) &= \frac{1}{4\pi e^2}\int_M d\eta \wedge \star d\eta + \frac{\pi}{e^2}\int_M \lambda \wedge \star\lambda - i\int_M \zeta \wedge \lambda, \\
&= \frac{1}{4\pi e^2}(d\eta, d\eta) + \frac{\pi}{e^2}(\lambda, \lambda) - i\langle\zeta, \lambda\rangle.
\end{aligned} \quad (3.34)$$

With this description for the Maxwell action, the partition function in (3.31) takes the more explicit form

$$Z_M^\vee(\zeta) = \text{Tor}_M \cdot \sum_{\lambda \in \mathbb{L}^\vee} \frac{1}{\text{Vol}(\mathcal{G}(P_0))} \int_{\mathcal{A}(P_0)} \mathcal{D}\eta \, \exp\left[-\frac{1}{4\pi e^2}(d\eta, d\eta) - \frac{\pi}{e^2}(\lambda, \lambda) + i\langle\zeta, \lambda\rangle\right]. \quad (3.35)$$

Here Tor_M is the number of elements in the torsion subgroup of $H^2(M;\mathbb{Z})$,

$$\text{Tor}_M = |H^2(M;\mathbb{Z})_{\text{tors}}|. \quad (3.36)$$

Since the Maxwell action is insensitive to torsion in $c_1(P)$, the sum over $H^2(M;\mathbb{Z})$ in (3.31) reduces to a sum over the quotient lattice $\mathbb{L}^\vee$ in (3.35).

Like the partition function (2.41) of the periodic scalar field, the Maxwell partition function also factorizes,

$$Z_M^\vee(\zeta) = \Delta_M^\vee \cdot \Theta_M^\vee(\zeta). \quad (3.37)$$

Here $\Delta_M^\vee$ is given by a path integral over the affine space $\mathcal{A}(P_0)$,

$$\Delta_M^\vee = \text{Tor}_M \cdot \frac{1}{\text{Vol}(\mathcal{G}(P_0))} \int_{\mathcal{A}(P_0)} \mathcal{D}\eta \, \exp\left[-\frac{1}{4\pi e^2}(d\eta, d\eta)\right], \quad (3.38)$$

and $\Theta_M^\vee(\zeta)$ is given by a sum over fluxes in the quotient lattice $\mathbb{L}^\vee = H^2(M;\mathbb{Z})_{\text{free}}$,

$$\Theta_M^\vee(\zeta) = \sum_{\lambda \in \mathbb{L}^\vee} \exp\left[-\frac{\pi}{e^2}(\lambda, \lambda) + i\langle\zeta, \lambda\rangle\right]. \quad (3.39)$$

The more interesting factor in the Maxwell partition function is $\Theta_M^\vee(\zeta)$, which is yet another theta-function attached to the three-manifold M. Including our previous work from section 2, we now have a dual pair of lattices $\mathbb{L}$ and $\mathbb{L}^\vee$, as well as a pair of theta-functions Θ_M and $\Theta_M^\vee$. As one might guess, and as we will demonstrate explicitly in section 4, Θ_M and $\Theta_M^\vee$ are related by a modular transformation. Before we discuss modular issues though, let us finish the computation of the Maxwell partition function on M.

The normalization of $Z_M^\vee(\zeta)$. To fix the absolute normalization of the Maxwell partition function, which will depend upon the electric coupling e^2 and the Riemannian metric g, we are left to evaluate the Gaussian path integral

$$\Delta_M^\vee = \text{Tor}_M \cdot \frac{1}{\text{Vol}(\mathcal{G})} \int_\mathcal{A} \mathcal{D}\eta \, \exp\left[-\frac{1}{4\pi e^2}(d\eta, d\eta)\right]. \tag{3.40}$$

Here I abbreviate $\mathcal{A} \equiv \mathcal{A}(P_0)$ and $\mathcal{G} \equiv \mathcal{G}(P_0)$, since we will only consider gauge theory on the trivial U(1)-bundle P_0 for the remainder of the discussion. Also, to orient the reader, I recall that the one-form η in (3.40) is effectively identified with the gauge field A after the bundle P_0 has been trivialized.

The computation of $\Delta_M^\vee$ is slightly more delicate than the analogous computation for the periodic scalar field, due to the gauge symmetry in the current problem. Because of the gauge symmetry, the argument of the exponential in (3.40) vanishes for any $\eta \in \Omega^1(M)$ of the form $\eta = d\varphi$, with $\varphi \in \Omega^0(M)$. Intrinsically, φ can be interpreted as element in the Lie algebra of the group $\mathcal{G}$, and we are simply observing that the Maxwell action is degenerate along orbits of $\mathcal{G}$.

To account for the degeneracy of the integrand in (3.40), we employ the standard BRST technique to fix the gauge symmetry. We cannot possibly fix the action for the full group $\mathcal{G}$ of all gauge transformations, since any gauge transformation generated by a constant function $\varphi_0 \in \mathbb{R}$ acts everywhere trivially on $\mathcal{A}$. Instead, we pick a point $p \in M$, and we consider the subgroup $\mathcal{G}_p \subset \mathcal{G}$ of gauge transformations which are *based* at p. An alternative treatment would involve the introduction of BRST ghosts-for-ghosts to deal with the constant gauge transformations, but I believe that working with the based gauge group is conceptually simpler for this example.

By definition, elements in $\mathcal{G}_p$ are gauge transformations which are the identity at the point p, and elements in the Lie algebra of $\mathcal{G}_p$ are functions $\varphi \in \Omega^0(M)$ which vanish at p,

$$\varphi(p) = 0, \qquad \varphi \in \text{Lie}(\mathcal{G}_p). \tag{3.41}$$

Due to the condition in (3.41), the identity is the only constant gauge transformation in $\mathcal{G}_p$, and the quotient of $\mathcal{G}$ by $\mathcal{G}_p$ is the group

$$\mathcal{G}/\mathcal{G}_p = \text{U}(1), \tag{3.42}$$

acting globally by constant gauge transformations on M. Rather than attempt to fix a gauge for $\mathcal{G}$, we instead fix a gauge for the slightly smaller, based group $\mathcal{G}_p$.

As usual in the BRST procedure, we introduce additional fields c, $\bar{c}$, and h, all valued in the Lie algebra of $\mathcal{G}_p$. Thus $(c, \bar{c}, h)$ are functions on M which vanish at p,

$$c(p) = \bar{c}(p) = h(p) = 0, \qquad c, \bar{c}, h \in \Omega^0(M). \tag{3.43}$$

By assumption, c and $\overline{c}$ are anti-commuting, Grassmann scalar fields, and h is a commuting scalar field. If one wishes, the vanishing constraint in (3.43) amounts to the insertion of a local operator $\mathcal{O}(p)$, whose role is to absorb the zero-modes of $(c, \overline{c}, h)$ which would otherwise be present in the BRST path integral.

To achieve the most elegant geometric formulation of the BRST procedure, I will depart somewhat from custom and introduce an extra bosonic field u, which will be an element in the based group $\mathcal{G}_p$. Equivalently, u is a sigma model map from M to U(1), satisfying

$$u : M \to \mathrm{U}(1) , \qquad u(p) = 1 . \tag{3.44}$$

Together, the pair (u, h) describes the cotangent bundle of the group $\mathcal{G}_p$,

$$T^* \mathcal{G}_p \simeq \mathcal{G}_p \times \mathrm{Lie}(\mathcal{G}_p) , \tag{3.45}$$

and the anti-commuting scalars $(c, \overline{c})$ can be interpreted as one-forms on $T^* \mathcal{G}_p$.[3]

The nilpotent BRST operator Q acts infinitesimally on the set of fields $(\eta, h, u, c, \overline{c})$ according to

$$\begin{aligned}
\delta\eta &= \frac{i}{2\pi}\, dc , & \delta c &= 0 , \\
\delta h &= 0 , & \delta\overline{c} &= h , \\
\delta u &= 0 .
\end{aligned} \tag{3.46}$$

Manifestly $Q^2 = 0$, and Q annihilates the Maxwell action in (3.40) by virtue of gauge invariance.

Using the BRST charge Q, we produce a gauge-fixing action $\mathbf{I}_{\mathrm{g.f.}}$ appropriate for harmonic gauge $d^\dagger\eta = 0$,

$$\mathbf{I}_{\mathrm{g.f.}} = \int_M \{Q, \mathbf{V}\} , \qquad \mathbf{V} = \overline{c} \wedge \star \left(\frac{e^6}{4\pi}\, h + i\, \frac{e^2}{2\pi}\, d^\dagger \eta^u \right) . \tag{3.47}$$

Here $\eta^u = \eta + i\, u^{-1} du$ is the image of the one-form η under a gauge transformation by u. The various factors of e^2 ensure invariance under the scaling in (2.2) and (2.4), and the factors of 2π are a numerical convenience, related to all the other factors of 2π that are floating around! Explicitly from (3.46),

$$\mathbf{I}_{\mathrm{g.f.}} = \int_M \left(\frac{e^6}{4\pi}\, h \wedge \star h + i\, \frac{e^2}{2\pi}\, h \wedge \star d^\dagger \eta^u + \frac{e^2}{(2\pi)^2}\, \overline{c} \wedge \star \triangle_0 c \right) , \tag{3.48}$$

where $\triangle_0 = d^\dagger d$ is the scalar Laplacian on M.

The essence of the BRST procedure amounts to an amusing way to rewrite unity,

$$1 = \int_{T^* \mathcal{G}_p} \mathcal{D}u\, \mathcal{D}h\, \mathcal{D}c\, \mathcal{D}\overline{c} \, \exp\!\left[-\,\mathbf{I}_{\mathrm{g.f.}} \right] . \tag{3.49}$$

A special feature of the path integral in (3.49) is the pairing of the bosonic measure $\mathcal{D}u\,\mathcal{D}h$ with the fermionic measure $\mathcal{D}c\,\mathcal{D}\overline{c}$. Each of $\mathcal{D}u$, $\mathcal{D}h$, $\mathcal{D}c$, and $\mathcal{D}\overline{c}$ can be defined once

---

[3]The bar on $\overline{c}$ does not indicate complex conjugation. The notation is traditional.

a metric on the Lie algebra of $\mathcal{G}_p$ is chosen. Provided that we make the same choice throughout, this choice does not matter, due to the familiar cancellation of Jacobians between bosons and fermions. But to make a definite choice, we use the scale-invariant version of the L^2-norm in (2.32).

Otherwise, the core of the BRST identity (3.49) is not so much the appearance of the constant '1' on the left-hand side of the identity as the independence of the right-hand side on the one-form η which enters the gauge-fixing action $\mathbf{I}_{\text{g.f.}}$ in (3.48). The latter property is really a property of harmonic gauge: for any one-form η, a gauge transformation by a unique $u \in \mathcal{G}_p$ exists so that $d^\dagger \eta^u = 0$. Given this statement, which follows from standard Hodge theory, the path integral over $\mathcal{G}_p$ washes out all dependence on η in the integrand of (3.49).

Using the BRST identity, we enlarge the path integral which describes $\Delta_M^\vee$ in (3.40) to a path integral over the product $\mathcal{A} \times T^* \mathcal{G}_p$,

$$\Delta_M^\vee = \text{Tor}_M \cdot \frac{1}{\text{Vol}(\mathcal{G})} \int_{\mathcal{A} \times T^* \mathcal{G}_p} \mathcal{D}\eta \, \mathcal{D}u \, \mathcal{D}h \, \mathcal{D}c \, \mathcal{D}\bar{c} \, \exp\left[-\frac{1}{4e^2}(d\eta, d\eta) \right]$$
$$\times \exp\left[-\frac{e^6}{4\pi}(h, h) - i\frac{e^2}{2\pi}(h, d^\dagger \eta^u) - \frac{e^2}{(2\pi)^2}(\bar{c}, \triangle_0 c) \right]. \tag{3.50}$$

The Gaussian integral over the auxiliary scalar h can be evaluated immediately, after which the normalization factor becomes

$$\Delta_M^\vee = \text{Tor}_M \cdot \frac{1}{\text{Vol}(\mathcal{G})} \int_{\mathcal{A} \times \mathcal{G}_p} \mathcal{D}\eta \, \mathcal{D}u \, \mathcal{D}c \, \mathcal{D}\bar{c} \, \exp\left[-\frac{1}{4\pi e^2}(d\eta, d\eta) \right]$$
$$\times \exp\left[-\frac{1}{4\pi e^2}(d^\dagger \eta^u, d^\dagger \eta^u) - \frac{e^2}{(2\pi)^2}(\bar{c}, \triangle_0 c) \right]. \tag{3.51}$$

To deal with the appearance of u in the integrand of (3.51), we note trivially

$$(d\eta, d\eta) = (d\eta^u, d\eta^u), \tag{3.52}$$

due to gauge-invariance of the Maxwell action. Gauge-invariance for the measure on $\mathcal{A}$ similarly implies $\mathcal{D}\eta = \mathcal{D}\eta^u$. As a result, η can be replaced by its gauge transform η^u in (3.51),

$$\Delta_M^\vee = \text{Tor}_M \cdot \frac{1}{\text{Vol}(\mathcal{G})} \int_{\mathcal{A} \times \mathcal{G}_p} \mathcal{D}u \, \mathcal{D}\eta^u \, \mathcal{D}c \, \mathcal{D}\bar{c} \, \exp\left[-\frac{1}{4\pi e^2}(d\eta^u, d\eta^u) \right]$$
$$\times \exp\left[-\frac{1}{4\pi e^2}(d^\dagger \eta^u, d^\dagger \eta^u) - \frac{e^2}{(2\pi)^2}(\bar{c}, \triangle_0 c) \right]. \tag{3.53}$$

After a change-of-variables from η^u back to η, the auxiliary field $u \in \mathcal{G}_p$ decouples from the integrand in (3.53), so that

$$\Delta_M^\vee = \text{Tor}_M \cdot \frac{1}{\text{Vol}(\mathcal{G})} \int_{\mathcal{A} \times \mathcal{G}_p} \mathcal{D}u \, \mathcal{D}\eta \, \mathcal{D}c \, \mathcal{D}\bar{c} \, \exp\left[-\frac{1}{4\pi e^2}(d\eta, d\eta) \right]$$
$$\times \exp\left[-\frac{1}{4\pi e^2}(d^\dagger \eta, d^\dagger \eta) - \frac{e^2}{(2\pi)^2}(\bar{c}, \triangle_0 c) \right]. \tag{3.54}$$

Since u appears nowhere in the integrand of (3.54), the path integral over u just contributes a factor of the group volume $\mathrm{Vol}(\mathcal{G}_p)$,

$$\Delta_M^\vee = \mathrm{Tor}_M \cdot \frac{\mathrm{Vol}(\mathcal{G}_p)}{\mathrm{Vol}(\mathcal{G})} \int_{\mathcal{A}} \mathcal{D}\eta \, \mathcal{D}c \, \mathcal{D}\overline{c} \, \exp\left[- \frac{1}{4\pi e^2}(\eta, \triangle_1 \eta) - \frac{e^2}{(2\pi)^2}(\overline{c}, \triangle_0 c) \right]. \tag{3.55}$$

In passing from (3.54) to (3.55), we also integrate by parts to produce the de Rham Laplacian $\triangle_1 = d^\dagger d + d d^\dagger$ acting on the one-form η.

Although both $\mathcal{G}_p$ and $\mathcal{G}$ have infinite dimension, the quotient $\mathcal{G}/\mathcal{G}_p = \mathrm{U}(1)$ has finite dimension, and the ratio of volumes in (3.55) is well-defined,

$$\frac{\mathrm{Vol}(\mathcal{G}_p)}{\mathrm{Vol}(\mathcal{G})} = \frac{1}{\mathrm{Vol}(\mathrm{U}(1))}. \tag{3.56}$$

Because $\mathrm{U}(1)$ acts by constant gauge transformations, $\mathrm{U}(1)$ is the stabilizer at all points in $\mathcal{A}$. As usual, the factor in (3.56) implies that the partition function on M is divided by the volume of the stabilizer. See for instance section 2.2 in [34] for a related discussion of the role of stabilizers in $\mathcal{G}$ and the normalization of the gauge theory partition function.

The Gaussian path integral over η, c, and $\overline{c}$ can be formally evaluated by expanding each field in an orthonormal basis of eigenmodes for the Laplacian, exactly as we did previously for the periodic scalar field in (2.53). With care for factors of e^2, one finds

$$\int_{\mathcal{A}} \mathcal{D}\eta \, \mathcal{D}c \, \mathcal{D}\overline{c} \, \exp\left[- \frac{1}{4\pi e^2}(\eta, \triangle_1 \eta) - \frac{e^2}{(2\pi)^2}(\overline{c}, \triangle_0 c) \right] = \frac{\mathrm{det}'(\triangle_0/e^4)}{\sqrt{\mathrm{det}'(\triangle_1/e^4)}} \cdot \mathrm{Vol}(\mathscr{J}_M). \tag{3.57}$$

The functional determinants of the respective scalar and vector Laplacians $\triangle_{0,1}$ arise from the Gaussian integrals over non-harmonic modes of $(\eta, c, \overline{c})$, and the volume of the torus $\mathscr{J}_M$ in (3.13) arises from the integral over the remaining harmonic modes of η. Exactly as in (2.54), the factor of $1/e^4$ in each functional determinant is required by invariance under the scaling in (2.2) and (2.4) and is a consequence of the coupling-dependence in the metrics on $\Omega^0(M)$ in (2.32) and $\Omega^1(M)$ in (3.29).

In total, the results in (3.55), (3.56), and (3.57) imply

$$\Delta_M^\vee = \mathrm{Tor}_M \cdot \frac{\mathrm{Vol}(\mathscr{J}_M)}{\mathrm{Vol}(\mathrm{U}(1))} \cdot \frac{\mathrm{det}'(\triangle_0/e^4)}{\sqrt{\mathrm{det}'(\triangle_1/e^4)}}. \tag{3.58}$$

Both the volume of $\mathrm{U}(1) \subset \mathcal{G}$ and the volume of $\mathscr{J}_M$ are to be evaluated using the metrics induced from the coupling-dependent L^2-norms in (2.32) and (3.29). With respect to (2.32), the norm-square of the constant function '1' is

$$||1||^2_{\Omega^0(M)} = \frac{(e^2 \ell)^3}{(2\pi)^2}, \tag{3.59}$$

from which we obtain

$$\mathrm{Vol}(\mathrm{U}(1)) = 2\pi \, ||1||_{\Omega^0(M)} = (e^2 \ell)^{3/2}. \tag{3.60}$$

The same factor appears in (2.54), for exactly the same reason.

To determine the volume of $\mathscr{J}_M$, we recall that $\mathscr{J}_M$ is concretely the quotient

$$\mathscr{J}_M = H^1(M; \mathbb{R})/2\pi \mathbb{L}, \qquad \mathbb{L} = H^1(M; \mathbb{Z}). \tag{3.61}$$

Let $\{\mathfrak{e}_1, \cdots, \mathfrak{e}_{b_1}\}$ be a basis of integral generators for $\mathbb{L}$,

$$\mathbb{L} \simeq \mathbb{Z}\mathfrak{e}_1 \oplus \cdots \oplus \mathbb{Z}\mathfrak{e}_{b_1}, \tag{3.62}$$

so that $\mathscr{J}_M$ becomes isomorphic to

$$\mathscr{J}_M \simeq \mathbb{R}^{b_1}/2\pi \mathbb{Z}^{b_1}. \tag{3.63}$$

Associated to the integral basis in (3.62) is the matrix of L^2 inner-products

$$\mathsf{Q}_{jk} = (\mathfrak{e}_j, \mathfrak{e}_k) = \int_M \mathfrak{e}_j \wedge \star \mathfrak{e}_k, \qquad j, k = 1, \ldots, b_1, \tag{3.64}$$

where we implicitly use the embedding $\mathbb{L} \subset \mathcal{H}^1(M)$ to identify the generators of $\mathbb{L}$ with harmonic one-forms on M. Manifestly, Q is a symmetric, positive-definite matrix, and basic linear algebra implies that the volume of $\mathscr{J}_M$ in (3.63) is proportional to the square-root of the determinant of Q,

$$\begin{aligned} \mathrm{Vol}(\mathscr{J}_M) &= (2\pi)^{b_1} \cdot \left(\frac{e}{2\pi}\right)^{b_1} \sqrt{\det \mathsf{Q}}, \\ &= e^{b_1} \sqrt{\det \mathsf{Q}}. \end{aligned} \tag{3.65}$$

The extra factor of $(e/2\pi)^{b_1}$ in the first line of (3.65) occurs due to the corresponding factor in the scale-invariant norm on $\Omega^1(M)$ in (3.29).

Finally, we extract factors of e^2 from the functional determinants in (3.58) using the zeta-function relation in (2.60). According to the general formula (2.61) for the value of the zeta-function at $s = 0$,

$$\zeta_{\triangle_0}(0) = -1, \qquad \zeta_{\triangle_1}(0) = -b_1, \tag{3.66}$$

so again,

$$\det'(\triangle_0/e^4) = e^4 \cdot \det'(\triangle_0), \qquad \det'(\triangle_1/e^4) = e^{4b_1} \cdot \det'(\triangle_1). \tag{3.67}$$

We use the formulas in (3.60), (3.65), and (3.67) to simplify our result in (3.58),

$$\Delta_M^\vee = \mathrm{Tor}_M \cdot \frac{e^{1-b_1}}{\ell^{3/2}} \sqrt{\det \mathsf{Q}} \cdot \frac{\det'(\triangle_0)}{\sqrt{\det'(\triangle_1)}}. \tag{3.68}$$

The overall dependence of $\Delta_M^\vee$ on the electric coupling as e^{1-b_1} can also be understood more physically (and perhaps more simply) by counting modes of the gauge field A modulo gauge equivalence. The latter perspective is taken for the computations in [2] and [35].

The final result. In summary, we have determined the Maxwell partition function on M to be

$$
\begin{aligned}
Z_M^\vee(\zeta) &= \Delta_M^\vee \cdot \Theta_M^\vee(\zeta), \qquad \zeta \in \mathcal{H}_{\mathbb{C}}^1(M), \\
&= \mathrm{Tor}_M \cdot \frac{e^{1-b_1}}{\ell^{3/2}} \sqrt{\det \mathsf{Q}} \cdot \frac{\det'(\triangle_0)}{\sqrt{\det'(\triangle_1)}} \cdot \sum_{\lambda \in \mathbb{L}^\vee} \exp\left[-\frac{\pi}{e^2}(\lambda, \lambda) + i\langle \zeta, \lambda \rangle \right].
\end{aligned}
\tag{3.69}
$$

As a reminder, Tor_M is the order of the torsion subgroup in $H^2(M; \mathbb{Z})$,

$$
\mathrm{Tor}_M = |H^2(M; \mathbb{Z})_{\mathrm{tors}}|,
\tag{3.70}
$$

and Q is the matrix of inner-products for an integral basis $\{\mathfrak{e}_1, \cdots, \mathfrak{e}_{b_1}\}$ of $H^1(M; \mathbb{Z})$,

$$
\mathsf{Q}_{jk} = (\mathfrak{e}_j, \mathfrak{e}_k) = \int_M \mathfrak{e}_j \wedge \star \mathfrak{e}_k, \qquad j, k = 1, \ldots, b_1.
\tag{3.71}
$$

4 Modularity, duality, and all that

Having evaluated the respective scalar and Maxwell partition functions, we now compare these results. Both $Z_M(\gamma)$ and $Z_M^\vee(\zeta)$ factorize,

$$
\begin{aligned}
Z_M(\gamma) &= \Delta_M \cdot \Theta_M(\gamma), \qquad \gamma \in \mathcal{H}_{\mathbb{C}}^2(M), \\
Z_M^\vee(\zeta) &= \Delta_M^\vee \cdot \Theta_M^\vee(\zeta), \qquad \zeta \in \mathcal{H}_{\mathbb{C}}^1(M),
\end{aligned}
\tag{4.1}
$$

and we will start by comparing the respective theta-functions $\Theta_M(\gamma)$ and $\Theta_M^\vee(\zeta)$ associated to the three-manifold M. See [22] or chapter 2 in [13] for an introduction to the geometry of theta-functions, the basics of which will be useful here.

4.1 A theta-function for three-manifolds

The hallmark of any theta-function is quasi-periodic behavior under integral shifts in the argument, and both $\Theta_M(\gamma)$ and $\Theta_M^\vee(\zeta)$ are easily seen to be quasi-periodic with respect to shifts in the variables γ and ζ.

For convenience, I focus on $\Theta_M(\gamma)$, given by the lattice sum

$$
\Theta_M(\gamma) = \sum_{\omega \in \mathbb{L}} \exp\left[-\pi e^2(\omega, \omega) + i\langle \omega, \gamma \rangle \right], \qquad \mathbb{L} = H^1(M; \mathbb{Z}).
\tag{4.2}
$$

Recall that $\mathbb{L}$ is dual to the quotient lattice $\mathbb{L}^\vee = H^2(M; \mathbb{Z})_{\mathrm{free}}$. If $\nu \in \mathbb{L}^\vee$ is such an integral two-form, then manifestly

$$
\Theta_M(\gamma + 2\pi\nu) = \Theta_M(\gamma), \qquad \nu \in \mathbb{L}^\vee.
\tag{4.3}
$$

Similarly, if $\mu \in \mathbb{L}$ is an integral one-form, then

$$
\Theta_M(\gamma + 2\pi i\, e^2 \star \mu) = \exp\left[\pi e^2(\mu, \mu) - i\langle \mu, \gamma \rangle \right] \cdot \Theta_M(\gamma), \qquad \mu \in \mathbb{L}.
\tag{4.4}
$$

– 27 –

The transformation formula in (4.4) follows by a standard calculation from the lattice sum in (4.2),

$$
\begin{aligned}
\Theta_M(\gamma + 2\pi i\,e^2 \star \mu) &= \sum_{\omega \in \mathbb{L}} \exp\big[- \pi e^2(\omega,\,\omega) + i\langle\omega,\gamma\rangle - 2\pi e^2(\omega,\mu)\big]\,, \\
&= \exp\big[\pi e^2(\mu,\mu)\big] \cdot \sum_{\omega \in \mathbb{L}} \exp\big[- \pi e^2(\omega + \mu,\,\omega + \mu) + i\langle\omega,\gamma\rangle\big]\,, \\
&= \exp\big[\pi e^2(\mu,\mu) - i\langle\mu,\gamma\rangle\big] \cdot \sum_{\omega' \in \mathbb{L}} \exp\big[- \pi e^2(\omega',\,\omega') + i\langle\omega',\gamma\rangle\big]\,, \\
&= \exp\big[\pi e^2(\mu,\mu) - i\,\langle\mu,\gamma\rangle\big] \cdot \Theta_M(\gamma)\,.
\end{aligned}
\tag{4.5}
$$

In passing from the second to the third line of (4.5), I shift the summand to $\omega' = \omega + \mu$, since $\mu \in \mathbb{L}$ is also integral.

Together, the transformation laws in (4.3) and (4.4) show that $\Theta_M(\gamma)$ is quasi-periodic with respect to the lattice

$$
2\pi\boldsymbol{\Lambda} \subset \mathcal{H}^2_{\mathbb{C}}(M) \simeq \mathbb{C}^{b_1}\,,
\tag{4.6}
$$

where

$$
\boldsymbol{\Lambda} = \mathbb{L}^{\vee} \oplus i\,e^2 \star \mathbb{L}\,.
\tag{4.7}
$$

Because the coupling e^2 appears in the definition of the complex lattice $\boldsymbol{\Lambda}$, the physical interpretations of the relations in (4.3) and (4.4) are very different. The periodicity of $\Theta_M(\gamma)$ under shifts in $\mathbb{L}^{\vee}$ is a classical property, visible already from the classical action (2.24) for the scalar field. Conversely, the quasi-periodicity of $\Theta_M(\gamma)$ under shifts in $i\,e^2 \star \mathbb{L}$ is a quantum effect, which relies upon the sum over winding-sectors in the scalar partition function.

The theta-function $\Theta_M(\gamma)$ definitely depends on the Riemannian structure on M, but this dependence occurs only through the matrix of L^2 inner-products in (3.64),

$$
\mathsf{Q}_{jk} = (\mathfrak{e}_j, \mathfrak{e}_k) = \int_M \mathfrak{e}_j \wedge \star\mathfrak{e}_k\,, \qquad j, k = 1, \ldots, b_1\,,
\tag{4.8}
$$

where we have selected an integral basis for $\mathbb{L} \simeq \mathbb{Z}\mathfrak{e}_1 \oplus \cdots \oplus \mathbb{Z}\mathfrak{e}_{b_1}$. In terms of the basis for $\mathbb{L}$ and the $b_1 \times b_1$ matrix Q, we can write $\Theta_M(\gamma)$ very concretely as a sum over a vector $\vec{n} \in \mathbb{Z}^{b_1}$ of integers,

$$
\Theta_M(\gamma) = \sum_{\vec{n} \in \mathbb{Z}^{b_1}} \exp\big[- \pi\,e^2\,\mathsf{Q}_{jk}\,n^j\,n^k + i\,\gamma_j\,n^j\big]\,.
\tag{4.9}
$$

In this expression, $\gamma_j \in \mathbb{C}$ for $j = 1, \ldots, b_1$ are the components of the complex two-form γ, expressed dually with respect to the basis for $\mathbb{L}$,

$$
\gamma_j = \langle\gamma, \mathfrak{e}_j\rangle = \int_M \gamma \wedge \mathfrak{e}_j\,.
\tag{4.10}
$$

For instance, if M has the rational homology of $S^1 \times S^2$, then $\Theta_M(\gamma)$ reduces to the classical Jacobi theta-function

$$
\Theta(z;\tau) = \sum_{n=-\infty}^{+\infty} \exp\big[\pi i\,n^2\,\tau + 2\pi i\,n\,z\big]\,,
\tag{4.11}
$$

evaluated at the purely-imaginary complex structure

$$\tau = i\,\frac{e^2\ell^3}{R^2} \in i\,\mathbb{R}\,, \qquad z = \frac{\gamma}{2\pi} \in \mathbb{C}\,. \tag{4.12}$$

Here R is a length scale naturally identified with the radius of S^1 when $M = S^1 \times S^2$. More generally, if we introduce the standard multi-variable extension of (4.11),

$$\Theta(\vec{z};\Omega) = \sum_{\vec{n}\in\mathbb{Z}^{b_1}} \exp\big[\pi\,i\,\Omega_{jk}\,n^j\,n^k + 2\pi i\,z_j\,n^j\big]\,, \qquad \vec{z}\in\mathbb{C}^{b_1}, \tag{4.13}$$

where Ω is a complex matrix with positive-definite imaginary part, then the geometric theta-function $\Theta_M(\gamma)$ in (4.9) agrees with the classical theta-function $\Theta(\vec{z};\Omega)$ under the assignments

$$\Omega = i\,e^2\,\mathsf{Q}\,, \qquad \vec{z} = \frac{\gamma}{2\pi}\,. \tag{4.14}$$

The modular transform of $\Theta_M(\gamma)$. Since $\Theta_M(\gamma)$ agrees with the classical Jacobi theta-function $\Theta(\vec{z};\Omega)$ when the period matrix Ω is imaginary, $\Theta_M(\gamma)$ also inherits the well-known modular properties of $\Theta(\vec{z};\Omega)$. A concise exposition of the latter can be found in chapter 2.5 of [22], whose notation I follow.

Not surprising for our discussion of abelian duality, the most important modular property will be the transformation of $\Theta_M(\gamma)$ under the analogue of the S-duality $\tau \mapsto -1/\tau$, acting here on the period matrix Ω by

$$S : \Omega \longmapsto -\Omega^{-1}. \tag{4.15}$$

If Ω is purely imaginary as in (4.14), then this feature is preserved under (4.15), so that S-duality also acts on the geometric theta-function $\Theta_M(\gamma)$ by the inversion

$$S : \mathsf{Q} \longmapsto e^{-4}\,\mathsf{Q}^{-1}. \tag{4.16}$$

At this stage, one could simply refer to the literature on theta-functions to determine the transformation of $\Theta_M(\gamma)$ under the operation in (4.16). However, for the convenience of the reader, I shall provide a brief derivation of the required transformation law.

As well-known, the transformation of the theta-function under the inversion in (4.15) or (4.16) can be understood as a consequence of Poisson resummation, which itself follows from the distributional identity

$$\sum_{n\in\mathbb{Z}} \delta(x-n) = \sum_{m\in\mathbb{Z}} e^{2\pi i m x}\,. \tag{4.17}$$

Applied to the concrete description of $\Theta_M(\gamma)$ in (4.9), this identity implies

$$\begin{aligned}
\Theta_M(\gamma) &= \sum_{\vec{n}\in\mathbb{Z}^{b_1}} \exp\big[-\pi\,e^2\,\mathsf{Q}_{jk}\,n^j\,n^k + i\,\gamma_j\,n^j\big]\,, \\
&= \sum_{\vec{n}\in\mathbb{Z}^{b_1}} \int_{\mathbb{R}^{b_1}} d^{b_1}x\;\delta(\vec{x}-\vec{n})\,\exp\big[-\pi\,e^2\,\mathsf{Q}_{jk}\,x^j\,x^k + i\,\gamma_j\,x^j\big]\,, \\
&= \sum_{\vec{m}\in\mathbb{Z}^{b_1}} \int_{\mathbb{R}^{b_1}} d^{b_1}x\;\exp\bigg[-\pi\,e^2\,\mathsf{Q}_{jk}\,x^j\,x^k + 2\pi i\bigg(m_j + \frac{\gamma_j}{2\pi}\bigg)x^j\bigg]\,.
\end{aligned} \tag{4.18}$$

We evaluate the Gaussian integral over $\vec{x} \in \mathbb{R}^{b_1}$ in the last line of (4.18) to obtain

$$\Theta_M(\gamma) = \sum_{\vec{m} \in \mathbb{Z}^{b_1}} \frac{1}{e^{b_1}\sqrt{\det \mathsf{Q}}} \, \exp\left[-\frac{\pi}{e^2}(\mathsf{Q}^{-1})^{jk}\left(m_j + \frac{\gamma_j}{2\pi}\right)\left(m_k + \frac{\gamma_k}{2\pi}\right)\right]. \qquad (4.19)$$

A more geometric interpretation for the right-hand side of (4.19) follows once we recognize Q^{-1} as the matrix whose elements encode the L^2 inner-products of the basis for $\mathbb{L}^\vee$ which is dual to the chosen basis for $\mathbb{L}$,

$$\mathbb{L}^\vee \simeq \mathbb{Z}\mathfrak{e}^{*1} \oplus \cdots \oplus \mathbb{Z}\mathfrak{e}^{*b_1}, \qquad \langle \mathfrak{e}^{*j}, \mathfrak{e}_k \rangle = \delta^j_k, \qquad (4.20)$$

so that

$$(\mathsf{Q}^{-1})^{jk} = (\mathfrak{e}^{*j}, \mathfrak{e}^{*k}) = \int_M \mathfrak{e}^{*j} \wedge \star\mathfrak{e}^{*k}, \qquad j, k = 1, \ldots, b_1. \qquad (4.21)$$

Here in (4.20) we introduce the Kronecker-delta, and we observe that integrality for the dual basis $\{\mathfrak{e}^{*1}, \cdots, \mathfrak{e}^{*b_1}\}$ of $\mathbb{L}^\vee$ is a non-trivial consequence of Poincaré duality on M. Otherwise, the interpretation for Q^{-1} in (4.21) derives from the tautological relation

$$\mathfrak{e}^{*j} = (\mathsf{Q}^{-1})^{jk}(\mathfrak{e}_k, \,\cdot\,) \in \mathbb{L}^\vee. \qquad (4.22)$$

As a result of (4.19) and (4.21), the geometric theta-function $\Theta_M(\gamma)$ on M can be written not only in terms of a sum over the lattice $\mathbb{L}$, but also in terms of a sum over the dual lattice $\mathbb{L}^\vee$,

$$\Theta_M(\gamma) = \frac{1}{e^{b_1}\sqrt{\det \mathsf{Q}}} \sum_{\lambda \in \mathbb{L}^\vee} \exp\left[-\frac{\pi}{e^2}\left(\lambda + \frac{\gamma}{2\pi}, \, \lambda + \frac{\gamma}{2\pi}\right)\right], \qquad (4.23)$$

where $(\,\cdot\,, \,\cdot\,)$ now indicates the L^2-norm on $\mathbb{L}^\vee \subset \mathcal{H}^2(M)$.

Of course, we also recall the description of the other theta-function $\Theta_M^\vee(\zeta)$ which enters the Maxwell partition function,

$$\Theta_M^\vee(\zeta) = \sum_{\lambda \in \mathbb{L}^\vee} \exp\left[-\frac{\pi}{e^2}(\lambda, \lambda) + i\,\langle \zeta, \lambda \rangle\right]. \qquad (4.24)$$

Comparing (4.23) and (4.24), we see that

$$\Theta_M(\gamma) = \frac{1}{e^{b_1}\sqrt{\det \mathsf{Q}}} \, \exp\left[-\frac{(\gamma, \gamma)}{4\pi e^2}\right] \cdot \Theta_M^\vee\left(\frac{i}{e^2} \star \gamma\right). \qquad (4.25)$$

The identification $\zeta = i \star \gamma/e^2$ in the argument of $\Theta_M^\vee$ agrees with our conventions for α and β in sections 2 and 3.

4.2 The role of torsion

The modular relation between Θ_M and $\Theta_M^\vee$ in (4.25) is the fundamental result which we need to compare the respective scalar and vector partition functions Z_M and $Z_M^\vee$ under duality. With the identification of parameters

$$\zeta = \frac{i}{e^2} \star \gamma, \qquad (4.26)$$

we compute the ratio

$$\frac{Z_M^\vee(\zeta)}{Z_M(\gamma)} = \frac{\Delta_M^\vee \cdot \Theta_M^\vee(\zeta)}{\Delta_M \cdot \Theta_M(\gamma)} = \frac{\Delta_M^\vee}{\Delta_M} \cdot e^{b_1} \sqrt{\det \mathsf{Q}} \cdot \exp\left[\frac{(\gamma,\gamma)}{4\pi e^2}\right]. \tag{4.27}$$

From (2.63) and (3.68), the ratio of the respective Gaussian factors Δ_M and $\Delta_M^\vee$ is

$$\frac{\Delta_M^\vee}{\Delta_M} = \mathrm{Tor}_M \cdot \frac{1}{e^{b_1}\ell^3} \sqrt{\det \mathsf{Q}} \cdot \frac{[\det'(\triangle_0)]^{3/2}}{[\det'(\triangle_1)]^{1/2}}. \tag{4.28}$$

Thus,

$$\frac{Z_M^\vee(\zeta)}{Z_M(\gamma)} = \mathrm{Tor}_M \cdot \frac{\det \mathsf{Q}}{\ell^3} \frac{[\det'(\triangle_0)]^{3/2}}{[\det'(\triangle_1)]^{1/2}} \cdot \exp\left[\frac{(\gamma,\gamma)}{4\pi e^2}\right]. \tag{4.29}$$

We now reach the most important question in the present paper.

When is the ratio of partition functions in (4.29) equal to one?

If $Z_M(\gamma)$ is to be equal to $Z_M^\vee(\zeta)$, then evidently $\gamma = \zeta = 0$ in (4.29). In that case, the expression on the right-hand side of (4.29) does not depend of the Maxwell coupling e^2, and the ratio reduces to the product

$$\frac{Z_M^\vee(0)}{Z_M(0)} = \mathrm{Tor}_M \cdot \tau_M, \qquad \tau_M = \frac{\det \mathsf{Q}}{\ell^3} \frac{[\det'(\triangle_0)]^{3/2}}{[\det'(\triangle_1)]^{1/2}}. \tag{4.30}$$

As before, Tor_M is the number of elements in the torsion subgroup $H^2(M;\mathbb{Z})_{\mathrm{tors}}$, an obvious topological invariant of M. So the remaining factor to examine is the mysterious ratio τ_M of functional determinants in (4.30)

Manifestly, τ_M depends only upon the Riemannian metric on M. By construction, both Z_M and $Z_M^\vee$ are invariant under the combined scalings in (2.2) and (2.4). Hence τ_M must also be preserved by the scale transformation $g \mapsto \Lambda^2 g$ of the metric in (2.2). This feature suggests that τ_M, like the quantity Tor_M in (4.30), could be a topological invariant of M. In fact, as I now explain, τ_M is precisely the Reidemeister torsion of the three-manifold, evaluated on an integral basis for the cohomology.

Reidemeister vs. Ray-Singer torsion. Before discussing τ_M, let me briefly recall a few facts about Reidemeister torsion. Nice expositions on Reidemeister torsion can be found in [10, 19, 24], and a prominent application of these ideas to gauge theory on a Riemann surface appears in [34]. Here I follow the presentation of Freed in [10], specialized to dimension three for concreteness.

The Reidemeister torsion is a combinatorial invariant of M, defined in terms of the chain complex $C_\bullet$ associated to a given triangulation (or cellular structure) on M,

$$C_\bullet : 0 \longrightarrow C_3 \xrightarrow{\partial} C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \longrightarrow 0. \tag{4.31}$$

We work with real coefficients throughout, and the homology of this chain complex is $H_*(M;\mathbb{R})$.

Each C_j for $j = 0, \ldots, 3$ is a finite-dimensional vector space with a distinguished set of generators, the elementary simplices in M. Because the vector space C_j has a basis,

C_j also has an associated metric, for which the generating simplices are orthonormal. Accompanying the metric on C_j is a volume form $\nu_j \in (\det C_j)^{-1}$. Here $\det C_j = \bigwedge^{\text{top}} C_j$ indicates the top exterior power. As standard in this business, we will not worry about orientations or signs; by convention, the torsion will be positive.

I first describe the torsion assuming the complex $C_\bullet$ to be acyclic, with trivial homology. Let $k_j = \dim \operatorname{Im} \partial \colon C_j \to C_{j-1}$, and pick an element $s_j \in \bigwedge^{k_j} C_j$ so that $\partial s_j \neq 0$. We now consider the following element in the alternating tensor product,

$$u = \bigotimes_{j=0}^{3} (\partial s_{j+1} \wedge s_j)^{(-1)^j} \in \bigotimes_{j=0}^{3} (\det C_j)^{(-1)^j}. \tag{4.32}$$

The element u is independent of the choices of the s_j, so we can define the torsion of the acyclic complex $C_\bullet$ as

$$\tau(C_\bullet) = u \otimes \bigotimes_{j=0}^{3} \nu_j^{(-1)^j} \in \mathbb{R}. \tag{4.33}$$

Of course, in the geometric situation $C_\bullet$ always has non-trivial homology, since $H_0(M) = H_3(M) = \mathbb{Z}$ for a closed, orientable three-manifold. To define $\tau(C_\bullet)$ more generally when $C_\bullet$ has homology, we split $C_\bullet$ as $C_\bullet = C'_\bullet \oplus C''_\bullet$, where $C'_\bullet$ is acyclic and the differential on $C''_\bullet$ vanishes, $\partial|_{C''_\bullet} = 0$. Hence C''_j is isomorphic to $H_j(M; \mathbb{R})$. We now apply the preceding construction to the acyclic summand $C'_\bullet$, with $k_j = \dim \operatorname{Im} \partial \colon C'_j \to C'_{j-1}$, $s_j \in \bigwedge^{k_j} C'_j$ satisfying $\partial s_j \neq 0$, and

$$u = \bigotimes_{j=0}^{3} (\partial s_{j+1} \wedge s_j)^{(-1)^j} \in \bigotimes_{j=0}^{3} (\det C'_j)^{(-1)^j}. \tag{4.34}$$

The torsion $\tau(C_\bullet)$ is then defined as the element

$$\tau(C_\bullet) = u \otimes \bigotimes_{j=0}^{3} \nu_j^{(-1)^j} \in \bigotimes_{j=0}^{3} \left[\det H_j(M; \mathbb{R}) \right]^{(-1)^{j+1}}. \tag{4.35}$$

Equivalently, $\tau(C_\bullet)$ lies in the dual of the determinant line

$$\det H_*(M; \mathbb{R}) = \bigotimes_{j=0}^{3} \left[\det H_j(M; \mathbb{R}) \right]^{(-1)^j}. \tag{4.36}$$

Finally, though we have defined the torsion for the chain complex $C_\bullet$ associated to a particular triangulation of M, one checks that (4.35) is invariant under any refinement of the triangulation, and hence $\tau_M = |\tau(C_\bullet)|$ defines a smooth invariant of M.

Let us be more explicit about where τ_M in (4.35) is valued. In dimension three, the dual of the determinant line is given (with the obvious abbreviations) by

$$\begin{aligned}
(\det H_*)^{-1} &= \left[\det H_0 \otimes (\det H_1)^{-1} \otimes \det H_2 \otimes (\det H_3)^{-1} \right]^{-1}, \\
&\simeq \left[\det H^0 \otimes (\det H^1)^{-1} \otimes \det H^2 \otimes (\det H^3)^{-1} \right],
\end{aligned} \tag{4.37}$$

where we apply the canonical duality between $H_*(M;\mathbb{R})$ and $H^*(M;\mathbb{R})$ in the second step. But Poincaré duality on M also implies the isomorphisms

$$\det H^2(M;\mathbb{R}) \simeq \left[\det H^1(M;\mathbb{R})\right]^{-1},$$
$$\det H^3(M;\mathbb{R}) \simeq \left[\det H^0(M;\mathbb{R})\right]^{-1}. \tag{4.38}$$

As a result, τ_M is valued in the one-dimensional vector space

$$\tau_M \in \left[\det H^0(M;\mathbb{R}) \otimes \left(\det H^1(M;\mathbb{R})\right)^{-1}\right]^{\otimes 2}. \tag{4.39}$$

To assign a value to τ_M as a real number, we evaluate τ_M in (4.39) on an integral basis for $H^1(M;\mathbb{R})$ and $H_0(M;\mathbb{R})$, the latter corresponding simply to the choice of a point $p \in M$. By the standard properties of the determinant, the result is independent of the choice of integral basis.

The description thus far of τ_M is combinatorial, depending upon the choice of a triangulation for M. This description is most useful for computations in examples. On the other hand, the quantity which actually appears in (4.30) is analytic in character,

$$\tau_M = \frac{\det \mathsf{Q}}{\ell^3} \frac{[\det'(\triangle_0)]^{3/2}}{[\det'(\triangle_1)]^{1/2}}. \tag{4.40}$$

Most famously, the ratio of functional determinants appearing in (4.40) is the Ray-Singer analytic[4] torsion [26, 27]

$$T_M = \frac{[\det'(\triangle_0)]^{3/2}}{[\det'(\triangle_1)]^{1/2}}, \tag{4.41}$$

and the equality between the combinatorial (4.35) and analytic (4.40) descriptions of τ_M is a consequence of the Cheeger-Müller theorem [6, 7, 21] relating Reidemeister to Ray-Singer torsion.

As the reader may note, the Reidemeister torsion τ_M and the Ray-Singer torsion T_M are not precisely equal in our situation, but instead obey

$$\tau_M = \frac{\det \mathsf{Q}}{\ell^3} \cdot T_M. \tag{4.42}$$

The correction factor $\det \mathsf{Q}/\ell^3$ is discussed in appendix B of [11] and arises due to the non-trivial homology of M. Very briefly, both τ_M and T_M are intrinsically valued in the dual of the determinant line $\det H_*(M;\mathbb{R})$, identified concretely in (4.39). To assign real values to τ_M and T_M, we must pick a basis for the cohomology on which we evaluate the torsions. For τ_M we naturally use an integral basis, and for T_M we use a basis which is orthonormal with respect to the L^2 inner-product. The correction factor in (4.42) is necessary to relate these different choices of basis.

Explicitly, let A_0 and A_1 be linear maps which express integral bases for $H^0(M;\mathbb{R})$ and $H^1(M;\mathbb{R})$ in terms of L^2-bases for the same spaces. Evaluating τ_M and T_M on the respective bases, we see that τ_M and T_M satisfy

$$\tau_M = \left(\frac{\det \mathsf{A}_1}{\det \mathsf{A}_0}\right)^2 T_M. \tag{4.43}$$

---

[4]Some authors define the analytic torsion as the logarithm of T_M, but I follow the convention already established for the Reidemeister torsion.

The square in (4.43) appears due to the corresponding square in the determinant line in (4.39).

The constant A_0 can be evaluated directly. The integral generator for $H^0(M;\mathbb{R})$ is the constant function 1, and the L^2-generator for $H^0(M;\mathbb{R})$ is the constant function $\ell^{-3/2}$ (where ℓ^3 is the volume of M), so

$$\mathsf{A}_0 = \ell^{3/2}. \tag{4.44}$$

As for A_1, we have already introduced integral generators $\{\mathfrak{e}_1, \ldots, \mathfrak{e}_{b_1}\}$ for $H^1(M;\mathbb{R})$ in (3.62). If $\{\omega_1, \ldots, \omega_{b_1}\}$ is a basis for $H^1(M;\mathbb{R})$ which is orthonormal with respect to the L^2 inner-product, then by definition

$$\mathfrak{e}_j = (\mathsf{A}_1^{j'})_j \, \omega_{j'}, \qquad j, j' = 1, \ldots, b_1, \tag{4.45}$$

and

$$\mathsf{Q}_{jk} = (\mathfrak{e}_j, \mathfrak{e}_k) = (\mathsf{A}_1^t \mathsf{A}_1)_{jk}. \tag{4.46}$$

Here A_1^t is the transpose of A_1. Together, the relations in (4.43), (4.44), and (4.46) produce the metric-dependent correction factor in (4.42).

Duality for the partition function. To summarize, the ratio of the Maxwell to scalar partition function on M is a topological invariant,

$$\frac{Z_M^\vee(0)}{Z_M(0)} = \mathrm{Tor}_M \cdot \tau_M, \qquad \mathrm{Tor}_M = |H^2(M;\mathbb{Z})_{\mathrm{tors}}|, \tag{4.47}$$

where Tor_M is the number of elements in $H^2(M;\mathbb{Z})_{\mathrm{tors}}$, and τ_M is the Reidemeister torsion evaluated with respect to an integral basis for the cohomology of M. Via its combinatorial definition, the Reidemeister torsion is eminently computable, and I claim

$$\tau_M = \frac{1}{\mathrm{Tor}_M}. \tag{4.48}$$

Compare also to theorem 2.39 in [24] and the surrounding discussion of the Alexander formula.[5] As a result, abelian duality for the partition function holds globally in the most naive sense, with

$$Z_M^\vee(0) = Z_M(0). \tag{4.49}$$

This result strengthens the conclusions in [2] beyond the case $b_1 = 0$.

When M is a rational homology sphere, with $b_1 = 0$, a short proof of the formula (4.48) for τ_M appears in [37], but the proof can be easily generalized to the arbitrary closed, orientable three-manifold. Such a manifold can always be given a cellular structure with exactly one 0-cell and one 3-cell, so that the chain complex $C_\bullet$ in (4.31) takes the concrete form

$$C_\bullet : 0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z}^N \xrightarrow{\partial} \mathbb{Z}^N \xrightarrow{0} \mathbb{Z} \longrightarrow 0. \tag{4.50}$$

Geometrically, a chain complex of the form in (4.50) is naturally associated to any Heegaard decomposition of $M = H_1 \cup H_2$ into handlebodies H_1 and H_2. Because

---

[5]I thank Maxim Braverman for pointing out this formula to me.

$H_0(M) = H_3(M) = \mathbb{Z}$, the indicated maps in (4.50) vanish, and Poincaré duality otherwise implies that the chain groups satisfy $C_1, C_2 \simeq \mathbb{Z}^N$ for some N. Once we select integral generators $\{v_1, \cdots, v_N\}$ and $\{w_1, \cdots, w_N\}$ for C_1 and C_2 to realize the isomorphism with $\mathbb{Z}^N$, the non-trivial differential ∂ in (4.50) can be identified concretely with a square, $N \times N$ integer matrix.

We first consider the simpler case that $b_1 = 0$. Necessarily, ∂ is injective and has full rank over $\mathbb{R}$. The first homology group $H_1(M) \simeq C_1/\mathrm{Im}(\partial)$ is purely torsion, and by the Universal Coefficient Theorem, $\mathrm{Tor}_M = |H_1(M)|$. Via standard arguments, the number of elements in the lattice quotient $C_1/\mathrm{Im}(\partial)$ is the volume of the simplex spanned by the images $\{\partial(w_1), \cdots, \partial(w_N)\}$ of the generators for C_2 under ∂. Concretely, this volume can be computed as the absolute value of the determinant of ∂ as an $N \times N$ matrix,

$$\mathrm{Tor}_M = |\det \partial|, \qquad b_1 = 0. \tag{4.51}$$

By comparison, to evaluate the Reidemeister torsion, we pick any point $p \in M$ to generate $H_0(M)$, and we use M itself to generate $H_3(M)$. Because we have only one 0-cell and one 3-cell, both of which represent the homology, C_0 and C_3 both contribute '1' to the formula for τ_M in (4.35). To treat C_1 and C_2, we trivialize the determinant lines $\det C_1, \det C_2 \simeq \mathbb{R}$ with the volume forms $\nu_1 = v_1 \wedge \cdots \wedge v_N$ and $\nu_2 = w_1 \wedge \cdots \wedge w_N$. For any choice of $s_2 \in \det C_2$, the formula for τ_M in (4.34) and (4.35) then reduces to the ratio

$$\tau_M = \left| \frac{s_2}{\partial s_2} \right| = \frac{1}{|\det \partial|}, \qquad s_2 \in \det C_2. \tag{4.52}$$

Hence $\tau_M = 1/\mathrm{Tor}_M$ if $b_1 = 0$.

When b_1 is non-vanishing, the ideas leading to (4.48) are much the same, except for the fact that the determinant of ∂ now vanishes. By assumption, $\partial : C_2 \to C_1$ has a non-trivial kernel, with $H_2(M) = \ker \partial$. Clearly $H_2(M) \simeq \mathbb{Z}^{b_1}$ is generated freely, and we assume without loss that the subset $\{w_1, \cdots, w_{b_1}\}$ generates the kernel of ∂ in C_2. Hence $\{w_1, \cdots, w_{b_1}\}$ represent integral generators for $H_2(M)$. On the other hand, $H_1(M) = C_1/\mathrm{Im}\,\partial$ may still contain a torsion subgroup $H_1(M)_{\mathrm{tors}} \simeq H^2(M; \mathbb{Z})_{\mathrm{tors}}$, with $H_1(M; \mathbb{R}) \simeq \mathbb{R}^{b_1}$. Again without loss, we assume that the generating subset $\{v_1, \cdots, v_{b_1}\}$ spans the cokernel of ∂ over $\mathbb{R}$. Thus $\{v_1, \cdots, v_{b_1}\}$ represent integral generators for $H_1(M)$ modulo torsion.

As before, the number of elements in $H_1(M)_{\mathrm{tors}}$ can be interpreted geometrically as the volume of the $(N - b_1)$-dimensional simplex in $\mathbb{R}^N$ spanned by the image of ∂ on the integral generators for C_2. To compute that volume, we extend ∂ linearly to a non-degenerate integral map $\widehat{\partial} : C_2 \to C_1$ via the assignments

$$\widehat{\partial}(w_1) = v_1, \qquad \widehat{\partial}(w_2) = v_2, \qquad \ldots, \qquad \widehat{\partial}(w_{b_1}) = v_{b_1}, \tag{4.53}$$

as well as $\widehat{\partial} = \partial$ for all other generators of C_2. By construction, the determinant of $\widehat{\partial}$ is non-vanishing and computes the requisite volume,

$$\mathrm{Tor}_M = |\det \widehat{\partial}|, \qquad b_1 \neq 0. \tag{4.54}$$

With our choices for homology generators, the description for τ_M in (4.34) and (4.35) similarly unravels to the ratio

$$\tau_M = \left| \frac{s_2 \wedge w_1 \wedge \cdots \wedge w_{b_1}}{\partial s_2 \wedge v_1 \wedge \cdots \wedge v_{b_1}} \right| = \frac{1}{|\det \widehat{\partial}|}, \qquad s_2 \in \det C_2'. \tag{4.55}$$

Here C_2' is the $(N - b_1)$-dimensional subspace of C_2 spanned by all generators other than $\{w_1, \cdots, w_{b_1}\}$, and s_2 is any non-vanishing element in the determinant line $\det C_2'$. Comparing (4.54) and (4.55), we obtain the promised reciprocal relation between the torsion invariants Tor_M and τ_M.

5 Path integral explanation

By somewhat laborious direct calculations, we have obtained the dual identity

$$Z_M^\vee(\zeta) = Z_M(\gamma) \cdot \exp\left[\frac{(\gamma, \gamma)}{4\pi e^2} \right], \qquad \zeta = \frac{i}{e^2} \star \gamma. \tag{5.1}$$

The relation between the scalar and the Maxwell partition functions can also be derived more economically by formal path integral manipulations of the sort in [28, 35]. This approach to abelian duality in three dimensions appears already in lecture 8 of [36], but I take the opportunity now to review it.

As one application, the path integral perspective on duality neatly explains the otherwise anomalous exponential factor involving γ in (5.1), which arose from the modular transformation of $\Theta_M(\gamma)$ in section 4. In preparation for the Hamiltonian analysis in [1], I conclude section 5 with a discussion of duality for three natural classes of operators on M.

5.1 Duality for the partition function

We start with the path integral which describes the scalar partition function on M,

$$Z_M(\gamma) = \sum_{\omega \in \mathbb{L}} \int_{\mathcal{X}_\omega} \mathcal{D}\phi \, \exp\left[-\mathbf{I}_{\mathrm{tot}}(\phi) \right], \tag{5.2}$$

where the sigma model action for $\phi : M \to S^1$ is given by

$$\mathbf{I}_{\mathrm{tot}}(\phi) = \frac{e^2}{4\pi}(d\phi, d\phi) + \frac{1}{2\pi i}\langle \gamma, d\phi \rangle, \qquad \gamma \in \mathcal{H}_{\mathbb{C}}^2(M). \tag{5.3}$$

To find a dual reformulation for the path integral, we enlarge the space of fields to include a connection B on the trivial U(1)-bundle P_0 over M. Under a homotopically-trivial gauge transformation, parametrized by a function $f \in \Omega^0(M)$, the pair (ϕ, B) transforms by

$$\phi \mapsto \phi + f, \qquad B \mapsto B - df. \tag{5.4}$$

As a result, the combination $d_B\phi = d\phi + B$ is gauge-invariant. For future reference, $\mathcal{G}_B$ will denote the group of gauge transformations acting on the pair $(\phi, B) \in \mathcal{X}_\omega \times \mathcal{A}(P_0)$.

Finally, the sigma model action in (5.3) can be promoted to a gauge-invariant action for the pair (ϕ, B) by substituting the covariant derivative $d_B\phi$ everywhere for $d\phi$,

$$\mathbf{I}_{\text{tot}}(\phi, B) = \frac{e^2}{4\pi}(d_B\phi, d_B\phi) + \frac{1}{2\pi i}\langle\gamma, d_B\phi\rangle, \qquad d_B\phi = d\phi + B. \tag{5.5}$$

The gauged sigma model for the pair (ϕ, B) with the classical action in (5.5) cannot be the whole story, because we must also incorporate the Maxwell gauge field A somehow. So we let P_λ be the U(1)-bundle with Chern class $\lambda \in H^2(M;\mathbb{Z})$, and we let A be a connection on P_λ. To couple A to the pair (ϕ, B), we consider the mixed Chern-Simons interaction

$$\mathsf{CS}(A, B) = \frac{1}{2\pi}\int_M F_A \wedge B, \qquad F_A = dA. \tag{5.6}$$

Manifestly, $\mathsf{CS}(A, B)$ is invariant under the group $\mathcal{G}_A$ of gauge transformations acting on A. In addition, $\mathsf{CS}(A, B)$ is invariant under homotopically-trivial elements of $\mathcal{G}_B$, and otherwise the value of $\mathsf{CS}(A, B)$ shifts by integral multiples of 2π under "large" gauge transformations in $\mathcal{G}_B$. Thus $\mathsf{CS}(A, B)$ is naturally valued in the circle,

$$\mathsf{CS}(A, B) \in \mathbb{R}/2\pi\mathbb{Z}. \tag{5.7}$$

The latter property is perhaps most transparent when $\mathsf{CS}(A, B)$ is considered via bordism. Let X be a four-manifold bounding M, over which A and B extend. Then alternatively,

$$\mathsf{CS}(A, B) = \frac{1}{2\pi}\int_X F_A \wedge F_B, \qquad M = \partial X. \tag{5.8}$$

Integrality of both $F_A/2\pi$ and $F_B/2\pi$ ensures that the intersection pairing in (5.8) is well-defined modulo 2π, regardless of the choice of X. We also see that the mixed Chern-Simons interaction in (5.6) occurs at level one, the minimum for gauge-invariance in the absence of additional geometric structure (e.g. a spin structure) on M.

We now couple the connection A to the pair (ϕ, B) through the classical action

$$\mathbf{I}_{\text{tot}}(\phi, A, B) = \frac{e^2}{4\pi}(d_B\phi, d_B\phi) + \frac{1}{2\pi i}\langle\gamma, d_B\phi\rangle - i\,\mathsf{CS}(A, B). \tag{5.9}$$

By construction, the exponential of the classical action in (5.9) is invariant under the product group $\mathcal{G}_A \times \mathcal{G}_B$, acting by separate gauge transformations on A and on the pair (ϕ, B). Associated to our classical action for (ϕ, A, B) is the generalized partition function

$$\widetilde{Z}_M(\gamma) = \sum_{(\omega,\lambda)\in\mathbb{L}\oplus H^2(M;\mathbb{Z})} \frac{1}{\text{Vol}(\mathcal{G}_A)} \frac{1}{\text{Vol}(\mathcal{G}_B)} \int_{\mathcal{X}_\omega \times \mathcal{A}(P_\lambda) \times \mathcal{A}(P_0)} \mathcal{D}\phi\,\mathcal{D}A\,\mathcal{D}B\,\exp\big[-\mathbf{I}_{\text{tot}}(\phi, A, B)\big]. \tag{5.10}$$

As indicated, the generalized partition function now involves a sum over all winding sectors for ϕ as well as a sum over all topological types for the U(1)-bundle on which A is a connection. We do not sum over the topological type of the bundle for the auxiliary connection B, though.

The generalized partition function in (5.10) can be studied in two ways.

First, since the connection A enters the classical action in (5.9) linearly through the Chern-Simons coupling $\mathsf{CS}(A, B)$, the path integral over A can be performed directly. As argued carefully in [37], the result is simply a delta-function that sets B to zero modulo gauge-equivalence,

$$\delta\big([B]\big) = \sum_{\lambda \in H^2(M;\mathbb{Z})} \frac{1}{\mathrm{Vol}(\mathcal{G}_A)} \int_{\mathcal{A}(P_\lambda)} \mathcal{D}A \, \exp\big[i \, \mathsf{CS}(A, B)\big] \,. \tag{5.11}$$

To explain this identity, we again decompose the arbitrary connection A on the bundle P_λ as a sum

$$A = \widehat{A}_\lambda + \eta \,, \tag{5.12}$$

where $\widehat{A}_\lambda$ is a fiducial connection with harmonic curvature $2\pi\lambda$, and η (like B) is a connection on the trivial bundle P_0. After we substitute for A in (5.12) and integrate by parts, the Chern-Simons pairing becomes

$$\mathsf{CS}(A, B) = \int_M \lambda \wedge B + \frac{1}{2\pi} \int_M \eta \wedge F_B \,, \qquad F_B = dB \,. \tag{5.13}$$

Up to normalization, the path integral over η produces a formal delta-function that sets $F_B = 0$. The remaining sum over $\lambda \in H^2(M;\mathbb{Z})$ produces a second delta-function that requires B to have trivial holonomy on M. Hence B is gauge-equivalent to zero. With a bit more work, one can verify [37] that the coefficient of the delta-function in (5.11) is precisely one, but I omit those details here.

After applying the identity in (5.11) to the generalized partition function in (5.10), we find

$$\widetilde{Z}_M(\gamma) = \sum_{\omega \in \mathbb{L}} \frac{1}{\mathrm{Vol}(\mathcal{G}_B)} \int_{\mathcal{X}_\omega \times \mathcal{A}(P_0)} \mathcal{D}\phi \, \mathcal{D}B \, \delta\big([B]\big) \, \exp\big[-\mathbf{I}_{\mathrm{tot}}(\phi, B)\big] \,. \tag{5.14}$$

Because B must be gauge-trivial due to the delta-function in the integrand of (5.14), we can set $B = 0$ by an appropriate gauge transformation. The path integral over B then contributes $\mathrm{Vol}(\mathcal{G}_B)$ to cancel the prefactor in (5.14), and we obtain the simple result

$$\widetilde{Z}_M(\gamma) = \sum_{\omega \in H^1(M;\mathbb{Z})} \int_{\mathcal{X}_\omega} \mathcal{D}\phi \, \exp\big[-\mathbf{I}_{\mathrm{tot}}(\phi)\big] = Z_M(\gamma) \,. \tag{5.15}$$

Thus $\widetilde{Z}_M(\gamma)$ agrees with the scalar partition function from section 2.

Alternatively, we return to the generalized partition function (5.10) and perform the respective path integrals over ϕ and B instead. Since ϕ is automatically gauge-trivial with respect to the action of $\mathcal{G}_B$, we set $\phi = 0$ by a gauge transformation and cancel the prefactor $1/\mathrm{Vol}(\mathcal{G}_B)$ to obtain

$$\widetilde{Z}_M(\gamma) = \sum_{\lambda \in H^2(M;\mathbb{Z})} \frac{1}{\mathrm{Vol}(\mathcal{G}_A)} \int_{\mathcal{A}(P_\lambda) \times \mathcal{A}(P_0)} \mathcal{D}A \, \mathcal{D}B \, \exp\big[-\mathbf{I}_{\mathrm{tot}}(A, B)\big] \,, \tag{5.16}$$

where

$$\mathbf{I}_{\mathrm{tot}}(A, B) = \frac{e^2}{4\pi}(B, B) + \frac{1}{2\pi i}\langle \gamma, B \rangle + \frac{1}{2\pi i}\langle F_A, B \rangle \tag{5.17}$$

The path integral over the auxiliary connection B in (5.16) and (5.17) is yet another Gaussian integral, of a much simpler form than the Gaussian integrals which we analyzed in sections 2 and 3. We immediately perform that integral to obtain a reformulation of $\widetilde{Z}_M(\gamma)$ involving only the Maxwell gauge field,

$$\widetilde{Z}_M(\gamma) = \sum_{\lambda \in H^2(M;\mathbb{Z})} \frac{1}{\text{Vol}(\mathcal{G}_A)} \int_{\mathcal{A}(P_\lambda)} \mathcal{D}A \, \exp\big[-\mathbf{I}_{\text{tot}}(A)\big] \,. \tag{5.18}$$

To determine the action for A, we substitute the classical value for $B = (i/e^2)(F_A + \gamma)$ into (5.17),

$$\begin{aligned}
\mathbf{I}_{\text{tot}}(A) &= \frac{1}{4\pi e^2}(F_A + \gamma,\, F_A + \gamma)\,, \qquad \zeta = \frac{i}{e^2} \star \gamma\,, \\
&= \frac{1}{4\pi e^2}(F_A, F_A) + \frac{1}{2\pi i}\langle \zeta, F_A \rangle + \frac{1}{4\pi e^2}(\gamma, \gamma)\,.
\end{aligned} \tag{5.19}$$

Comparing the classical action for A in (5.19) to the corresponding action (3.21) from section 3, we deduce

$$\widetilde{Z}_M(\gamma) = Z_M^\vee(\zeta) \cdot \exp\left[-\frac{1}{4\pi e^2}(\gamma, \gamma)\right]\,, \qquad \zeta = \frac{i}{e^2} \star \gamma\,. \tag{5.20}$$

Since $\widetilde{Z}_M$ is equal to Z_M, this relation reproduces (5.1).

5.2 Duality for operators

To conclude, let us review the dual descriptions for three natural classes of operators on the three-manifold M. For simplicity in the following, I set the cohomological parameters γ and ζ to zero.

Some local and non-local operators. Of the three operators that we consider, two are well-known: the vertex operator and the Wilson loop operator. The vertex operator is the local operator described classically in the sigma model by

$$\mathsf{V}_k(p) = \mathrm{e}^{ik\phi(p)}\,, \qquad k \in \mathbb{Z}\,, \tag{5.21}$$

for some point $p \in M$. The condition that $\mathsf{V}_k(p)$ be single-valued under the shift $\phi \mapsto \phi + 2\pi$ requires the parameter k to be an integer. Physically, k labels the charge of $\mathsf{V}_k(p)$ under the global U(1) symmetry which acts additively on ϕ by a constant shift,

$$\mathrm{U}(1) : \phi \longmapsto \phi + c\,, \qquad c \in \mathbb{R}/2\pi\mathbb{Z}\,. \tag{5.22}$$

In section 3, we have already introduced the Wilson loop operator $\mathsf{W}_n(C)$ attached to a closed, oriented curve C embedded in M,

$$\mathsf{W}_n(C) = \exp\left[i\,n \oint_C A\right]\,, \qquad n \in \mathbb{Z}\,. \tag{5.23}$$

If C is homologically non-trivial, the parameter n must be an integer to ensure that $\mathsf{W}_n(C)$ is invariant under "large", homotopically-nontrivial gauge transformations on M.

On the other hand, when C is trivial in $H_1(M;\mathbb{Z})$, the condition $n \in \mathbb{Z}$ can be relaxed. In the latter case, $C = \partial\Sigma$ is the boundary of a connected, oriented surface $\Sigma \subset M$, a so-called Seifert surface for the knot. See chapter 5 of [29] for a nice reference on Seifert surfaces. In terms of Σ, the Wilson loop operator can be rewritten as

$$\mathsf{W}_\nu\big(C,[\Sigma]\big) = \exp\left[i\,\nu \int_\Sigma F_A\right], \qquad \nu \in \mathbb{R}\,. \tag{5.24}$$

The expression for $\mathsf{W}_\nu(C,[\Sigma])$ in (5.24) is manifestly gauge-invariant for arbitrary real values of the charge ν, and since F_A is closed, the operator depends only on the relative homology class of the Seifert surface,

$$[\Sigma] \in H_2(M,C)\,. \tag{5.25}$$

The choice of $[\Sigma]$ is an extra discrete choice, necessary if we wish to extend the definition of the U(1) Wilson loop operator to non-integral charges.

As a special case, let us suppose that M is a rational homology sphere, with $b_1 = 0$ and hence $H_2(M) = 0$.[6] The relative exact sequence below,

$$\cdots \longrightarrow H_2(M) \longrightarrow H_2(M,C) \xrightarrow{\partial_*} H_1(C) \xrightarrow{\iota_*} H_1(M) \longrightarrow \cdots, \tag{5.26}$$

implies $H_2(M,C) \simeq H_1(C) = \mathbb{Z}$. By assumption, the image $\partial_*[\Sigma]$ generates $H_1(C)$, so $[\Sigma]$ is uniquely determined once the orientation of C is fixed. Thus when M is a rational homology sphere, the choice of Seifert surface can be omitted from our notation for the fractional Wilson loop operator, and we simply write

$$\mathsf{W}_\nu(C) = \exp\left[i\,\nu \int_\Sigma F_A\right], \qquad b_1 = 0\,. \tag{5.27}$$

Both the vertex operator $\mathsf{V}_k(p)$ and the Wilson loop operator $\mathsf{W}_n(C)$ depend upon the particular choices for the point $p \in M$ and the curve $C \subset M$. By contrast, the third operator $\mathsf{L}_\alpha(C)$ will be homological, depending only upon the class $[C] \in H_1(M)$ of the closed curve. In terms of the periodic scalar field ϕ,

$$\mathsf{L}_\alpha(C) = \exp\left[\frac{i\alpha}{2\pi} \oint_C d\phi\right], \qquad \alpha \in \mathbb{R}/2\pi\mathbb{Z}\,. \tag{5.28}$$

Because the periods of the one-form $d\phi$ are quantized in integral multiples of 2π, the expression on the right in (5.28) is invariant under a shift $\alpha \mapsto \alpha + 2\pi$. For this reason, α is best regarded as an angular parameter for the homological loop operator.

Vertex operators and monopoles. So far we have introduced three kinds of operators on M,

$$\mathsf{V}_k(p)\,, \qquad \mathsf{W}_n(C)\,, \qquad \mathsf{L}_\alpha(C)\,, \tag{5.29}$$

---

[6]For a compact orientable three-manifold, $H_2(M)$ is torsion-free. Thus vanishing of $b_1 = b_2$ implies the vanishing of $H_2(M)$.

labelled generally by parameters

$$k,\, n \in \mathbb{Z}\,, \qquad \alpha \in \mathbb{R}/2\pi\mathbb{Z}\,. \tag{5.30}$$

The vertex operator $\mathsf{V}_k(p)$ and the homological loop operator $\mathsf{L}_\alpha(C)$ are respectively specified in (5.21) and (5.28) as classical functionals of the scalar field ϕ, whereas the Wilson loop operator $\mathsf{W}_n(C)$ is a classical functional of the Maxwell gauge field A.

Duality between the scalar and the Maxwell field theories on M implies not only a relation between partition functions, but also a correspondence between operators in each theory. So how do we describe the vertex operator $\mathsf{V}_k(p)$ and the loop operator $\mathsf{L}_\alpha(C)$ dually in the language of Maxwell theory? And how do we describe the Wilson loop operator $\mathsf{W}_n(C)$ in terms of the periodic scalar field?

In answer to all three questions, the duals of $\mathsf{V}_k(p)$, $\mathsf{W}_n(C)$, and $\mathsf{L}_\alpha(C)$ will be operators of disorder-type [16], which create singularities in the dual classical field. To quickly explain both the notion of a disorder operator and its relevance for duality, let us derive the dual of the vertex operator $\mathsf{V}_k(p)$.

At first glance, one might be tempted to consider the (unnormalized) expectation value

$$\langle \mathsf{V}_k(p) \rangle = \int_{\mathcal{X}} \mathcal{D}\phi \, \mathsf{V}_k(p) \, \exp\big[-\mathbf{I}_{\text{tot}}(\phi)\big]\,. \tag{5.31}$$

Unless $k = 0$, in which case $\mathsf{V}_k(p)$ is the identity operator, $\mathsf{V}_k(p)$ transforms with charge k under the global U(1) symmetry in (5.22). Hence trivially

$$\langle \mathsf{V}_k(p) \rangle = 0\,, \qquad k \neq 0\,, \tag{5.32}$$

due to cancellations in the integral over the constant mode of ϕ. So we cannot learn much by thinking about the expectation value of $\mathsf{V}_k(p)$.

Instead, to discuss a non-trivial expectation value, we pick distinct points $p \neq q$ in M and consider vertex operators of opposite charge inserted at these points,

$$\langle \mathsf{V}_k(p)\,\mathsf{V}_{-k}(q) \rangle = \int_{\mathcal{X}} \mathcal{D}\phi \, \mathsf{V}_k(p)\,\mathsf{V}_{-k}(q) \, \exp\big[-\mathbf{I}_{\text{tot}}(\phi)\big]\,. \tag{5.33}$$

Because the expectation value in (5.33) is invariant under the global U(1) symmetry, the expectation value need not vanish, and we can meaningfully ask for the dual description of (5.33) in terms of the Maxwell gauge field A.

Just as for the analysis in section 5.1, the first step in dualizing the vertex operator path integral will be to promote the integrand in (5.33) to a functional of the pair (ϕ, B) which is invariant under the gauge transformation in (5.4). Since the vertex operators $\mathsf{V}_k(p)$ and $\mathsf{V}_{-k}(q)$ carry opposite charges, local gauge invariance can be achieved by introducing a Wilson line for the auxiliary gauge field B which runs between the vertex operators. Thus we choose an oriented curve Γ from q to p,

$$\partial\Gamma = p - q\,, \qquad \Gamma \subset M\,, \tag{5.34}$$

and we consider the expectation value of the gauge-invariant composite[7]

$$\left\langle \mathsf{V}_k(p) \exp\left[i\,k \int_\Gamma B\right] \mathsf{V}_{-k}(q) \right\rangle = \frac{1}{\mathrm{Vol}(\mathcal{G}_A)} \frac{1}{\mathrm{Vol}(\mathcal{G}_B)} \int_{\mathcal{X} \times \mathcal{A} \times \mathcal{A}(P_0)} \mathcal{D}\phi \, \mathcal{D}A \, \mathcal{D}B$$
$$\times \exp\left[i\,k\big(\phi(p) - \phi(q)\big) + i\,k \int_\Gamma B - \mathbf{I}_{\mathrm{tot}}(\phi, A, B)\right], \tag{5.35}$$

evaluated in the full theory of all three fields (ϕ, A, B) with the classical action $\mathbf{I}_{\mathrm{tot}}(\phi, A, B)$ in (5.9).

The Maxwell gauge field A still enters the integrand of (5.35) linearly through the Chern-Simons pairing $\mathsf{CS}(A, B)$. Thus the path integral over A again produces a delta-function for B with support on gauge-trivial field configurations. After we integrate over B using the delta-function, the extended path integral in (5.35) reduces to the path integral over ϕ alone in (5.33),

$$\langle \mathsf{V}_k(p) \, \mathsf{V}_{-k}(q) \rangle = \left\langle \mathsf{V}_k(p) \exp\left[i\,k \int_\Gamma B\right] \mathsf{V}_{-k}(q) \right\rangle. \tag{5.36}$$

As a corollary, the extended path integral in (5.35) does not depend upon the choice of the curve Γ from q to p.

Mimicking our previous analysis of the partition function, we alternately evaluate the path integral in (5.35) by using the local action of $\mathcal{G}_B$ to set $\phi = 0$, after which (5.35) reduces to a path integral involving only the gauge fields A and B,

$$\langle \mathsf{V}_k(p) \exp\left[i\,k \int_\Gamma B\right] \mathsf{V}_{-k}(q) \rangle = \frac{1}{\mathrm{Vol}(\mathcal{G}_A)} \int_{\mathcal{A} \times \mathcal{A}(P_0)} \mathcal{D}A \, \mathcal{D}B \, \exp\left[i\,k \int_\Gamma B - \mathbf{I}_{\mathrm{tot}}(A, B)\right], \tag{5.37}$$

where as in (5.19),

$$\mathbf{I}_{\mathrm{tot}}(A, B) = \frac{e^2}{4\pi}(B, B) + \frac{1}{2\pi i}\langle F_A, B \rangle. \tag{5.38}$$

To evaluate the Gaussian integral in (5.37) further, we introduce a two-form δ_Γ which has delta-function support along Γ and which represents the Poincaré dual of the curve,

$$\exp\left[i\,k \int_\Gamma B\right] = \exp\left[i\,k \int_M \delta_\Gamma \wedge B\right], \qquad \delta_\Gamma \in \Omega^2(M), \tag{5.39}$$

so that all terms in the argument of the exponential take the form of integrals over M. Because Γ is bounded by the points p and q, the two-form δ_Γ is not closed but rather satisfies the distributional identity

$$d\delta_\Gamma = -\delta_p + \delta_q, \qquad \delta_p, \delta_q \in \Omega^3(M). \tag{5.40}$$

By definition, δ_p and δ_q are three-forms with delta-function support at the points p and q. The identity in (5.40) is most easily deduced as a consequence of Stokes' theorem for the path Γ. For if $f \in \Omega^0(M)$ is any smooth function on M, then

$$f(p) - f(q) = \int_\Gamma df = \int_M \delta_\Gamma \wedge df = -\int_M d\delta_\Gamma \cdot f. \tag{5.41}$$

See for instance chapter 3 of [13] for more about distributional differential forms like our δ_Γ.

---

[7]We implicitly absorb the topological sums over the winding-number ω and the Chern class $c_1(P)$ into the definitions of the spaces $\mathcal{X} = \bigsqcup_\omega \mathcal{X}_\omega$ and $\mathcal{A} = \bigsqcup_{c_1(P)} \mathcal{A}(P)$.

Via the definition in (5.39), the Gaussian integral over A and B takes precisely the same form as the generalized partition function in (5.16) with singular $\gamma = 2\pi k\,\delta_\Gamma$,

$$
\begin{aligned}
&\left\langle \mathsf{V}_k(p)\,\exp\!\left[i\,k\int_\Gamma B\right]\mathsf{V}_{-k}(q)\right\rangle = \\
&\frac{1}{\mathrm{Vol}(\mathcal{G}_A)}\int_{\mathcal{A}\times\mathcal{A}(P_0)}\mathcal{D}A\,\mathcal{D}B\,\exp\!\left[-\frac{e^2}{4\pi}(B,B)+\frac{i}{2\pi}\langle F_A + 2\pi k\,\delta_\Gamma, B\rangle\right].
\end{aligned}
\tag{5.42}
$$

After performing the path integral over B, we obtain the desired reformulation

$$
\langle \mathsf{V}_k(p)\,\mathsf{V}_{-k}(q)\rangle = \frac{1}{\mathrm{Vol}(\mathcal{G}_A)}\int_{\mathcal{A}}\mathcal{D}A\,\exp\!\left[-\frac{1}{4\pi e^2}(\mathcal{F}_A,\,\mathcal{F}_A)\right],
\tag{5.43}
$$

where

$$
\mathcal{F}_A = F_A + 2\pi k\,\delta_\Gamma.
\tag{5.44}
$$

The interpretation of the modified Maxwell path integral in (5.43) is by now well understood. Due to the explicit delta-function in $\mathcal{F}_A$, the argument of the exponential diverges (and thus the integrand vanishes) unless F_A itself has the appropriate singularity along Γ to cancel the delta-function in $\mathcal{F}_A$,

$$
F_A = -2\pi k\,\delta_\Gamma + \cdots,
\tag{5.45}
$$

where the ellipses indicate regular terms in F_A. Thus, the insertion of the vertex operators $\mathsf{V}_k(p)$ and $\mathsf{V}_{-k}(q)$ in the scalar sigma model is interpreted dually as the instruction to perform the Maxwell path integral over connections with the specified singular behavior along Γ. Operators defined in this manner, as an instruction to perform the path integral over fields with given classical singularities, are said to be of disorder-type.

Because δ_Γ is not closed, the Bianchi identity for F_A is modified by the singularity in (5.45),

$$
dF_A = 2\pi k(\delta_p - \delta_q),
\tag{5.46}
$$

where we apply the Stokes' identity in (5.40). Physically, the new source terms in the Bianchi identity for F_A are interpreted as magnetic monopoles of charges $\pm k$ at the points p and q. Otherwise, so long as k is integral, the Dirac string singularity along the curve Γ is a gauge artifact.

In light of (5.46), we see that the operator $\mathsf{V}_k(p)$ itself is the monopole operator of charge k in the abelian gauge theory. By definition, the monopole operator of magnetic charge k is the local disorder operator which creates a curvature singularity in A at p of the form

$$
F_A = -\frac{k}{2}\,\star d\!\left(\frac{1}{r}\right),
\tag{5.47}
$$

where r is a local radial coordinate centered at p. With this singularity, the integral of F_A over any small sphere centered about p is equal to $2\pi k$, as required by the Bianchi identity in (5.46). Also, with the given local behavior in (5.47), F_A satisfies the classical source-free Maxwell equation $d\star F_A = 0$ on a punctured neighborhood of the point p.

Vortex loops and Wilson loops. The loop operators $\mathsf{L}_\alpha(C)$ and $\mathsf{W}_n(C)$ can be followed through the duality in much the same fashion as the vertex operator $\mathsf{V}_k(p)$. Very briefly, to dualize the homological loop operator $\mathsf{L}_\alpha(C)$ in (5.28), we consider its gauge-invariant extension in terms of the pair (ϕ, B),

$$\mathsf{L}_\alpha(C) = \exp\left[\frac{i\,\alpha}{2\pi} \oint_C d_B\phi\right] = \exp\left[\frac{i\,\alpha}{2\pi} \int_M \delta_C \wedge (d\phi + B)\right]. \tag{5.48}$$

Here δ_C is a two-form with delta-function support which represents the Poincaré dual of the closed curve $C \subset M$.

If we consider the expectation value of $\mathsf{L}_\alpha(C)$ in the extended theory of triples (ϕ, A, B) with total action (5.9), the path integral over A still provides a delta-function with support on gauge-trivial configurations for B. With this delta-function, B can then be gauged to zero to recover the expectation value for $\mathsf{L}_\alpha(C)$ in the theory of the periodic scalar field ϕ alone.

Alternatively, ϕ can be gauged to zero in the extended theory of triples (ϕ, A, B), after which we encounter a Gaussian integral over B taking precisely the same form as (5.42). Hence the operator $\mathsf{L}_\alpha(C)$ is interpreted in the dual Maxwell theory as a disorder operator which creates a curvature singularity along C,

$$F_A = -\alpha\,\delta_C + \cdots. \tag{5.49}$$

This curvature singularity looks very much like the preceding singularity (5.45) which we interpreted in terms of monopoles. However, C is now closed, without boundary, and α is not a multiple of 2π. As a result, the physical interpretation of (5.49) is different.

In a small tubular neighborhood of C, the singularity in (5.49) implies that the gauge field A behaves as

$$A = -\frac{\alpha}{2\pi}\, d\vartheta + \cdots, \tag{5.50}$$

where ϑ is an angular coordinate on the plane transverse to C, located at the origin. By Stokes' theorem, the angular form $d\vartheta$ satisfies $d(d\vartheta) = 2\pi\delta_C$, from which (5.49) follows.

Evidently, in the presence of the loop operator $\mathsf{L}_\alpha(C)$, the gauge field A has non-trivial monodromy $\Lambda = \exp(-i\,\alpha)$ about any small curve which links C. Of course, the value of the monodromy only depends upon the value of α modulo 2π. Physically, shifts in α by units of 2π can be accomplished by gauge transformations $u : M \to \mathrm{U}(1)$ which are themselves singular along C, of the local form

$$u = \mathrm{e}^{in\vartheta}, \qquad n \in \mathbb{Z}. \tag{5.51}$$

When α in (5.50) is an integral multiple of 2π, the singularity in A can be removed by such a gauge transformation, but not otherwise.

For gauge theories in four dimensions, the codimension-two singularity in (5.50) defines the basic Gukov-Witten [14] surface operator. Hence the loop operator $\mathsf{L}_\alpha(C)$ in three dimensions can be interpreted as the reduction of a surface operator from four dimensions. From the purely three-dimensional perspective, $\mathsf{L}_\alpha(C)$ can be considered as a kind of monodromy or vortex loop.

Reversing directions, we finally discuss the interpretation for the Wilson loop operator $\mathsf{W}_n(C)$ in terms of the periodic scalar field ϕ. For the time being, we do not make any assumption about the homology class of C, so the charge $n \in \mathbb{Z}$ must be an integer to maintain invariance under arbitrary gauge transformations.

In terms of the two-form δ_C with delta-function support, the abelian Wilson loop operator can be rewritten as

$$\mathsf{W}_n(C) = \exp\left[i\,n \oint_C A\right] = \exp\left[i\,n \int_M \delta_C \wedge A\right]. \tag{5.52}$$

The expectation value of $\mathsf{W}_n(C)$ can now be evaluated in the extended theory of triples (ϕ, A, B),

$$\begin{aligned}
\langle\mathsf{W}_n(C)\rangle &= \frac{1}{\mathrm{Vol}(\mathcal{G}_A)}\frac{1}{\mathrm{Vol}(\mathcal{G}_B)} \int_{\mathcal{X}\times\mathcal{A}\times\mathcal{A}(P_0)} \mathcal{D}\phi\,\mathcal{D}A\,\mathcal{D}B \\
&\times \exp\left[-\frac{e^2}{4\pi}(d_B\phi, d_B\phi) + \frac{i}{2\pi}\langle F_A, B\rangle + i\,n\langle\delta_C, A\rangle\right].
\end{aligned} \tag{5.53}$$

By gauging ϕ to zero and performing the Gaussian integral over B, one sees that the extended path integral in (5.53) describes the usual Wilson loop expectation value in Maxwell theory on M.

On the other hand, as also clear from (5.53), the gauge field A still enters the argument of the exponential linearly. Due to the new term involving δ_C, the path integral over A now produces a delta-function for B that enforces the condition

$$F_B = 2\pi n\,\delta_C\,, \tag{5.54}$$

and B has trivial holonomy otherwise.

At first glance, one might think that the curvature condition on B is vacuous, since we have already noted, in the discussion of the corresponding singularity for A, that the singularity in (5.54) can be removed by a gauge transformation of the local form in (5.51). However, we must remember that the group $\mathcal{G}_B$ acts simultaneously on both B and ϕ via (5.4), so if we perform a gauge transformation to remove the singularity in B, we will create a singularity in ϕ!

Specifically, once we perform the Wilson loop path integral over A in (5.53) and select a representative for the connection B satisfying (5.54), with trivial holonomies otherwise, we can rewrite the Wilson loop expectation value strictly in terms of ϕ,

$$\langle\mathsf{W}_n(C)\rangle = \int_{\mathcal{X}} \mathcal{D}\phi\,\exp\left[-\frac{e^2}{4\pi}(d_B\phi, d_B\phi)\right], \qquad d_B\phi \equiv d\phi + B\,. \tag{5.55}$$

Here B is now a background, spectator field, and we have fixed the action of $\mathcal{G}_B$ with our choice of representative connection.

If we wish to eliminate B entirely, we can introduce a new periodic scalar field $\widetilde{\phi}$, defined so that

$$d\widetilde{\phi} = d\phi + B\,, \tag{5.56}$$

after which

$$\langle \mathsf{W}_n(C) \rangle = \int_{\mathcal{X}} \mathcal{D}\widetilde{\phi} \, \exp\left[-\frac{e^2}{4\pi} (d\widetilde{\phi}, d\widetilde{\phi}) \right]. \tag{5.57}$$

Similar to (5.50), the background connection B behaves in a neighborhood of C as $B = n \, d\vartheta$. Thus, $\widetilde{\phi}$ must be related to ϕ near C by

$$\widetilde{\phi} = \phi + n \, \vartheta \,. \tag{5.58}$$

From (5.58) we see that $\widetilde{\phi}$ winds non-trivially around any small curve which encircles C. As a result, the Wilson loop operator $\mathsf{W}_n(C)$ is interpreted dually as the disorder operator which creates an additive monodromy in ϕ of n units about the meridian of C. From the dual perspective, the integrality of the charge n is necessary to ensure that ϕ is single-valued as a map from the knot complement $M^o = M - C$ to S^1.

When C is trivial in $H_1(M)$, we have noted that the parameter $n \in \mathbb{Z}$ in the Wilson loop operator can be extended to an arbitrary real number $\nu \in \mathbb{R}$, as appears in (5.24). By our preceding discussion, the operator $\mathsf{W}_\nu(C)$ then creates a fractional monodromy in ϕ.

To understand the fractional monodromy better, let us think about dualizing the Wilson loop operator in the form

$$\mathsf{W}_\nu(C) = \exp\left[i\,\nu \int_\Sigma F_A \right] = \exp\left[i\,\nu \int_M \delta_\Sigma \wedge F_A \right]. \tag{5.59}$$

As before, Σ is a Seifert surface bounding C. For convenience, I assume that M is a rational homology sphere, with $b_1 = 0$, so that the relative homology class of Σ is unique. Associated to Σ is the Poincaré dual one-form δ_Σ with delta-function support on Σ and satisfying the distributional identity $d\delta_\Sigma = \delta_C$.

By the same observations which we applied to (5.53), $\mathsf{W}_\nu(C)$ is described in terms of ϕ as the instruction to perform the path integral over ϕ in (5.55) with a background connection B which now satisfies

$$B = 2\pi\nu \, \delta_\Sigma \,. \tag{5.60}$$

Equivalently, we replace ϕ by a new field $\widetilde{\phi}$ so that $d\widetilde{\phi} = d\phi + B$ with the given B.

To characterize the local behavior of $\widetilde{\phi}$ near C, we assume that M is $\mathbb{R}^3$, with coordinates (x, y, z), and that the curve C extends upwards along the z-axis. We then take Σ to be the portion of the xz-plane with $x \geq 0$. Hence y is the local coordinate normal to Σ. In these local coordinates, the expression for B in (5.60) just becomes $B = 2\pi\nu \, H(x) \, \delta(y) \, dy$, where $H(x)$ is the Heaviside step-function.[8] See figure 1 for a sketch of the situation.

To obtain a corresponding local description for $\widetilde{\phi}$ near Σ, we integrate the relation $d\widetilde{\phi} = d\phi + B$ using our expression for B. In the region near Σ with $x > 0$, we find that ϕ is related to $\widetilde{\phi}$ by

$$\widetilde{\phi} = \phi + 2\pi\nu \, H(y), \qquad x > 0 \,. \tag{5.61}$$

The Heaviside function $H(y)$ arises from the integral of the delta-function $\delta(y)$. Thus, when C is null-homologous and the charge $\nu \in \mathbb{R}$ is fractional, the role of the Wilson loop

---

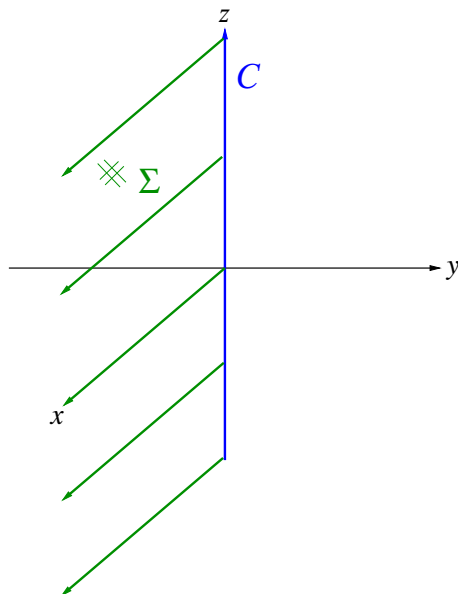[8]By definition $H(x) = 1$ for $x > 0$, and $H(x) = 0$ for $x < 0$.

Figure 1. Seifert surface Σ attached to C.

operator $\mathsf{W}_\nu(C)$ is dually to insert a discontinuity in the sigma model field ϕ transverse to the Seifert surface Σ. Physically, Σ can be interpreted as a kind of domain wall which is created by the fractional Wilson loop operator $\mathsf{W}_\nu(C)$.

Acknowledgments

References

[1] C. Beasley, *Abelian duality at higher genus*, *JHEP* **07** (2014) 157 [[arXiv:1405.2483](https://arxiv.org/abs/1405.2483)] [[INSPIRE](https://inspirehep.net)].

[2] B. Broda and G. Duniec, *Abelian duality in three-dimensions*, *Phys. Rev.* **D 70** (2004) 107702 [[hep-th/0210151](https://arxiv.org/abs/hep-th/0210151)] [[INSPIRE](https://inspirehep.net)].

[3] T.H. Buscher, *Quantum corrections and extended supersymmetry in new σ models*, *Phys. Lett.* **B 159** (1985) 127 [[INSPIRE](https://inspirehep.net)].

[4] T.H. Buscher, *A symmetry of the string background field equations*, *Phys. Lett.* **B 194** (1987) 59 [[INSPIRE](https://inspirehep.net)].

[5] T.H. Buscher, *Path integral derivation of quantum duality in nonlinear σ-models*, *Phys. Lett.* **B 201** (1988) 466 [[INSPIRE](https://inspirehep.net)].

[6] J. Cheeger, *Analytic torsion and Reidemeister torsion*,
    Proc. Nat. Acad. Sci. USA **74** (1977) 2651.

[7] J. Cheeger, *Analytic torsion and the heat equation*, *Ann. Math.* **109** (1979) 259.

[8] S. Deser and R. Jackiw, *"Self-duality" of topologically massive gauge theories*,
    Phys. Lett. **B 139** (1984) 371 [INSPIRE].

[9] R. Dijkgraaf and E. Witten, *Topological gauge theories and group cohomology*,
    Commun. Math. Phys. **129** (1990) 393 [INSPIRE].

[10] D. Freed, *Reidemeister torsion, spectral sequences, and Brieskorn spheres*,
    J. Reine Angew. Math. **429** (1992) 75.

[11] T. Friedmann and E. Witten, *Unification scale, proton decay and manifolds of G_2 holonomy*,
    Adv. Theor. Math. Phys. **7** (2003) 577 [hep-th/0211269] [INSPIRE].

[12] A. Giveon, M. Porrati and E. Rabinovici, *Target space duality in string theory*,
    Phys. Rept. **244** (1994) 77 [hep-th/9401139] [INSPIRE].

[13] P. Griffiths and J. Harris, *Principles of algebraic geometry*, John Wiley and Sons Inc.,
    New York U.S.A. (1978).

[14] S. Gukov and E. Witten, *Gauge theory, ramification, and the geometric Langlands program*,
    in *Current developments in mathematics*, D. Jerison et al. eds., Int. Press, Somerville U.S.A.
    (2008), pp. 35–180 [hep-th/0612073] [INSPIRE].

[15] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge U.K. (2002).

[16] G. 't Hooft, *On the phase transition towards permanent quark confinement*,
    Nucl. Phys. **B 138** (1978) 1 [INSPIRE].

[17] A. Karlhede, U. Lindström, M. Roček and P. van Nieuwenhuizen, *On 3D nonlinear
    vector-vector duality*, *Phys. Lett.* **B 186** (1987) 96 [INSPIRE].

[18] A. Kapustin and M. Tikhonov, *Abelian duality, walls and boundary conditions in diverse
    dimensions*, *JHEP* **11** (2009) 006 [arXiv:0904.0840] [INSPIRE].

[19] J. Milnor, *Whitehead torsion*, *Bull. Am. Math. Soc.* **72** (1966) 358.

[20] S. Minakshisundaram and A. Pleijel, *Some properties of the eigenfunctions of the Laplace
    operator on Riemannian manifolds*, *Can. J. Math.* **1** (1949) 242 [INSPIRE].

[21] W. Müller, *Analytic torsion and R-torsion of Riemannian manifolds*,
    Adv. Math. **28** (1978) 233.

[22] D. Mumford, *Tata lectures on theta, I*, Birkhäuser, Boston U.S.A. (1983).

[23] C. Nash and D.J. O'Connor, *Determinants of Laplacians, the Ray-Singer torsion on lens
    spaces and the Riemann zeta function*, *J. Math. Phys.* **36** (1995) 1462
    [*Erratum ibid.* **36** (1995) 4549] [hep-th/9212022] [INSPIRE].

[24] L. Nicolaescu, *The Reidemeister torsion of 3-manifolds*, De Gruyter Studies in Mathematics,
    Vol. 30, Walter De Gruyter Inc., Berlin Germany (2003).

[25] E.M. Prodanov and S. Sen, *Abelian duality*, *Phys. Rev.* **D 62** (2000) 045009
    [hep-th/9906143] [INSPIRE].

[26] D.B. Ray, *Reidemeister torsion and the Laplacian on lens spaces*, *Adv. Math.* **4** (1970) 109.

[27] D.B. Ray and I.M. Singer, *R-torsion and the Laplacian on Riemannian manifolds*, *Adv. Math.* **7** (1971) 145 [INSPIRE].

[28] M. Roček and E.P. Verlinde, *Duality, quotients and currents*, *Nucl. Phys.* **B 373** (1992) 630 [`hep-th/9110053`] [INSPIRE].

[29] D. Rolfsen, *Knots and links*, American Mathematical Society, Providence U.S.A. (1976).

[30] S. Rosenberg, *The Laplacian on a Riemannian manifold: an introduction to analysis on manifolds*, London Mathematical Society Student Texts, Vol. 31, Cambridge University Press, Cambridge U.K. (1997).

[31] P.K. Townsend, K. Pilch and P. van Nieuwenhuizen, *Self-duality in odd dimensions*, *Phys. Lett.* **B 136** (1984) 38 [*Addendum ibid.* **B 137** (1984) 443] [INSPIRE].

[32] C. Vafa and E. Witten, *A strong coupling test of S-duality*, *Nucl. Phys.* **B 431** (1994) 3 [`hep-th/9408074`] [INSPIRE].

[33] E.P. Verlinde, *Global aspects of electric-magnetic duality*, *Nucl. Phys.* **B 455** (1995) 211 [`hep-th/9506011`] [INSPIRE].

[34] E. Witten, *On quantum gauge theories in two dimensions*, *Commun. Math. Phys.* **141** (1991) 153 [INSPIRE].

[35] E. Witten, *On S-duality in Abelian gauge theory*, *Selecta Math.* **1** (1995) 383 [`hep-th/9505186`] [INSPIRE].

[36] E. Witten, *Dynamics of quantum field theory*, in *Quantum fields and strings: a course for mathematicians*, Vol. 2, P. Deligne et al. eds., American Mathematical Society, Providence U.S.A. (1999).

[37] E. Witten, $SL(2,\mathbb{Z})$ *action on three-dimensional conformal field theories with Abelian symmetry*, in *From fields to strings: circumnavigating theoretical physics*, Vol. 2, M. Shifman et al. eds., World Scientific, Singapore (2005), pp. 1173–1200 [`hep-th/0307041`] [INSPIRE].