

Restricted Schurs and correlators for $SO(N)$ and $Sp(N)$

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ABSTRACT: In a recent work, restricted Schur polynomials have been argued to form a complete orthogonal set of gauge invariant operators for the 1/4-BPS sector of free $\mathcal{N} = 4$ super Yang-Mills theory with an $SO(N)$ gauge group. In this work, we extend these results to the theory with an $Sp(N)$ gauge group. Using these operators, we develop techniques to compute correlation functions of any multi-trace operators with two scalar fields exactly in the free theory limit for both $SO(N)$ and $Sp(N)$.

KEYWORDS: Brane Dynamics in Gauge Theories, Gauge-gravity correspondence, AdS-CFT Correspondence

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1 Introduction

Restricted Schur polynomials have been argued to form a complete orthogonal set of gauge invariant operators for the 1/4-BPS sector of free $\mathcal{N} = 4$ super Yang-Mills theory with an $SO(N)$ gauge group [1]. For even N , [1] employed representation theory of the symmetric and hyper octahedral groups to construct a complete set of local operators depending on two scalar fields. Their two-point function was computed exactly and shown to be diagonal in the labels of these operators. More specifically, these operators were the generalisations of the so-called restricted Schur polynomials of the $U(N)$ gauge theory [2, 3] and [4].

Schur polynomials $\chi_R(Z)$ in the 1/2-BPS sector of the free $U(N)$ gauge theory were first studied in [5], in which these operators, labeled by irreducible representations (irreps) R of the symmetric and unitary groups, were shown to be an exactly orthogonal basis. Thus, these gauge invariant operators could be used as a basis to study the large N limit of operators whose dimension scales parametrically with N . The trace basis, for example, is no longer orthogonal in this case and computing non-planar corrections in correlation functions is a difficult task. When the Young diagram R labelling the Schur polynomial

has long columns, or long rows, ($O(N)$ boxes in each row/column) the operator is dual to a system of giant gravitons moving in the S^5 or AdS_5 [5–8] and [9].

By adding a different type of field to $\chi_R(Z)$, one can build the restricted Schur polynomial. A restricted Schur may simply be thought of as a linear combination all possible multi-matrix, multi-trace operators, where the sum is over permutations of the symmetric group. We can add different types of complex scalar fields, fermion fields or even gauge fields [10, 11] and [12]. For two scalar fields, the counting of restricted Schurs was shown to match the counting of states of the free theory in [13]. Their two-point function was computed exactly in [14] and was shown to be diagonal in the operator labels. [15] succeeded in writing any multi-trace operator involving two scalar fields as a linear combination of restricted Schurs. They also derived a product rule for these operators. This allowed the product of any two restricted Schurs to be written in terms of restricted Schurs and restricted Littlewood-Richardson coefficients.

At the level of summing Feynman diagrams, computing correlation functions of multi-trace multi-matrix operators is a difficult task at finite N . This is because the non-planar corrections are no longer suppressed and must be taken into account. The results of [14] and [15] transformed the problem of computing correlation functions of the multi-matrix, multi-trace operators into the problem of computing correlation functions of restricted Schur polynomials - something we can do exactly for any N .

Other orthogonal bases for the 1/4 and 1/8-BPS sectors of the free gauge theory has been proposed. [16] constructed operators from Z and Z^\dagger , while [17] constructed operators depending on X, Y and Z . At weak coupling some progress towards understanding the anomalous dimensions of an orthogonal basis of the free theory, which is manifestly covariant under the global symmetry, was achieved in [18].

Restricted Schur polynomials with n Z 's and m Y 's are labeled by three Young diagram labels, R, r, s , corresponding to irreducible representations of S_{n+m} and $S_n \times S_m$ respectively. When the number of Z 's and Y 's is $O(N)$, with $n \gg m$, these operators again have a D-brane interpretation in the string theory. For R having long columns (or rows), each with $O(N)$ boxes, the operator is dual to excited giant gravitons [2]. A system of excited giant gravitons can be thought of as a system of giants with strings attached. Amongst the other bases found for the 1/4-BPS sector it has been argued that restricted Schur polynomials is the most natural basis for studying open string dynamics of their dual D -brane states [15]. To this end, the one-loop dilation operator has been diagonalised and the spectrum of anomalous dimensions computed in [19–24]. The two-loop case was studied in [25]. Remarkably, for the non-planar limit studied in these works the spectrum was shown to be that of a system of decouple harmonic oscillators. This is evidence of integrability in these non-planar sectors of the gauge theory.

A similar program has been initiated for the gauge theory with an orthogonal gauge group in [26], and [27]. In the 1/2-BPS sector, Schur polynomials were constructed from the basic building block \boxplus which diagonalised the two-point function in the free theory limit. Indeed, orthogonality of Schur polynomials built from a single scalar field Z is a gauge group-independent property [28]. The counting of states, as given by the partition function, for $\text{SO}(4)$ and $\text{SO}(6)$, was shown to match the number of Schur polynomials that

could be defined. This basis was related to the trace basis and a product rule was derived for the Schurs. These results were then extended to the theory with symplectic gauge group, $\text{Sp}(N)$. $\text{Sp}(N)$ is related to $\text{SO}(N)$ by exchanging symmetrisations of irreps and replacing N by $-N$ [27, 29].

This study then progressed to the 1/4-BPS sector of the free $\text{SO}(N)$ gauge theory [1]. The counting of states, as given by the partition function, and the counting of restricted Schur polynomials was shown to agree by restricting to a particular class of Young diagram labels. R must have an even number of boxes in each row, while r and s are restricted to have an even number of boxes in each column. An explicit construction of these operators was given and their two-point function was evaluated exactly and shown to be diagonal.

Physics in the non-planar limit of $\text{SO}(N)$ and $\text{Sp}(N)$ gauge theory is different from that of $\text{U}(N)$. The $\text{SO}(N)$ and $\text{Sp}(N)$ gauge theories are matrix models with anti-symmetric and symplectic matrices respectively. When evaluating correlation functions, the leading non-planar correction comes from non-orientable Feynman diagrams with a single cross-cap - an effect not present in the $\text{U}(N)$ theory [27, 29, 30]. Furthermore, $\mathcal{N} = 4$ super Yang-Mills theory with an $\text{SO}(N)$ or $\text{Sp}(N)$ gauge group is dual to type IIB string theory with $\text{AdS}_5 \times \mathcal{RP}^5$ geometry. The Schur polynomials of [26, 27] and the restricted Schur polynomials of [1] may prove useful as a basis of operators to study non-planar limits of these gauge and dual string theories.

In this paper, we extend our results found in [1] for $\text{SO}(N)$ to $\text{Sp}(N)$. First, we express the free 1/4-BPS $\text{Sp}(N)$ partition function in terms the Littlewood-Richardson coefficients which count the number of $\text{Sp}(N)$ restricted Schurs. We then give a gauge invariant construction of these operators and evaluate their two-point function. As expected, the results are identical to those for $\text{SO}(N)$ except the Young diagrams are transposed and N is replaced by $-N$. We then relate the trace basis to the restricted Schur basis for both gauge groups. Lastly, we derive a product rule for our operators.

2 Recap of $\text{SO}(N)$

With a more convenient normalisation, the restricted Schurs defined in [1] were

$$O_{R(r,s)\alpha}^{\text{SO}(N)}(Z, Y) = \frac{1}{\sqrt{n!m!q!2^{2q}}} \sum_{\sigma \in S_{2q}} \chi_{R(r,s)\alpha}^{\text{SO}(N)}(\sigma) C_I^{4\nu} \sigma_J^I (Z^{\otimes 2n} \otimes Y^{\otimes 2m})^J \quad (2.1)$$

where we defined the $\text{SO}(N)$ restricted character to be

$$\chi_{R(r,s)\alpha}^{\text{SO}(N)}(\sigma) = \text{Tr}(\mathcal{O}_{R(r,s)\alpha}^{\text{SO}(N)} \Gamma_R(\sigma)), \quad \text{and} \quad \mathcal{O}_{R(r,s)\alpha}^{\text{SO}(N)} = |[S]\rangle\langle [A]_r, [A]_s, \beta| P_{R \rightarrow (r,s)\beta\alpha} \quad (2.2)$$

In (2.2), $\Gamma_R(\sigma)$ is the matrix representing the permutation $\sigma \in S_{2q}$ in irrep R . Furthermore, we have explicitly indicated which irreps subduce the two $[A]$'s. Irrep r of S_{2n} subduces irrep $[A]_r$ of $S_n[S_2]$, and irrep s of S_{2m} subduces irrep $[A]_s$ of $S_m[S_2]$. β still labels the particular copy of (r, s) subduced from irrep R of S_{2q} . Recall that the state $|[A]_r, [A]_s, \beta\rangle$ spans the 1-dimensional carrier space of the $S_n[S_2] \times S_m[S_2]$ irrep $([A]_r, [A]_s)\beta$, and is calculated as

the eigenvector of $P_{R \rightarrow ([A]_r, [A]_s)\beta}$ with eigenvalue 1. The intertwiner $P_{R \rightarrow (r,s)\alpha\beta}$ maps this state from one copy of (r, s) to another

$$P_{R \rightarrow (r,s)\alpha\beta} |[A]_r, [A]_s, \beta\rangle = |[A]_r, [A]_s, \alpha\rangle \quad (2.3)$$

Thus, we can write

$$\mathcal{O}_{R(r,s)\alpha}^{\text{SO}(N)} = |[S]\rangle \langle [A]_r, [A]_s, \alpha|. \quad (2.4)$$

For the normalisation in (2.1), the two-point function is

$$\langle \mathcal{O}_{R(r,s)\alpha}^{\text{SO}(N)}(Z, Y) \overline{\mathcal{O}}_{T(t,u)\beta}^{\text{SO}(N)}(Z, Y) \rangle = \delta_{RT} \delta_{rt} \delta_{su} \delta_{\alpha\beta} \frac{(2q)!}{d_R} \prod_{i \in \text{odd columns}} c_i \quad (2.5)$$

The tensor

$$C_I^{4\nu} = \delta_{k_1 k_2} \cdots \delta_{k_{2q-1} k_{2q}} (\sigma_{4\nu})_I^K = \delta_K (\sigma_{4\nu})_I^K \quad (2.6)$$

contracts the free indices in such a way as to produce a gauge invariant operator. In [1] we chose $\sigma_{4\nu} = (1, 2, 3, 4) \cdots (2q-3, 2q-2, 2q-1, 2q)$. This contracted indices 1 with 4, then 2 with 3, then 5 with 8, then 6 with 7, and so on. Finally, $2q-3$ is contracted with $2q$ and $2q-2$ is contracted with $2q-1$. The permutation $\sigma_{4\nu} = (1, 2)(3, 4) \cdots (2q-1, 2q)$, or simply the identity permutation, gives the same gauge invariant operator. For this permutation, indices 1 and 2 are contracted, 3 and 4 are contracted etc, until, finally, indices $2q-1$ and $2q$ are contracted. Thus, an equivalent definition for $\mathcal{O}_{R(r,s)\alpha}(Z, Y)$ is¹

$$\mathcal{O}_{R(r,s)\alpha}^{\text{SO}(N)}(Z, Y) = \frac{1}{\sqrt{n!m!q!2^{2q}}} \sum_{\sigma \in S_{2q}} \chi_{R(r,s)\alpha}^{\text{SO}(N)}(\sigma) \delta_I \sigma_J^I (Z^{\otimes 2n} \otimes Y^{\otimes 2m})^J \quad (2.7)$$

For the permutations $\sigma \in S_{2q}$, $\delta_I \sigma_J^I (Z^{\otimes 2n} \otimes Y^{\otimes 2m})^J$ gives all the possible multi-trace operators involving the two scalar fields Z and Y . For example, consider $q = 4$ with $n = m = 2$. There are only 4 multi-trace operators we can define. Here they are for 4 examples of σ

$$\sigma = (2, 4, 6, 8, 3) \quad \text{gives } \delta_I \sigma_J^I (Z^{\otimes 4} \otimes Y^{\otimes 4})^J = \text{Tr}(Z^2 Y^2) \quad (2.8)$$

$$\sigma = (2, 5)(3, 4, 5) \quad \text{gives } \delta_I \sigma_J^I (Z^{\otimes 4} \otimes Y^{\otimes 4})^J = \text{Tr}(ZY)^2 \quad (2.9)$$

$$\sigma = (1, 3, 2)(5, 8, 7) \quad \text{gives } \delta_I \sigma_J^I (Z^{\otimes 4} \otimes Y^{\otimes 4})^J = \text{Tr}(Z^2) \text{Tr}(Y^2) \quad (2.10)$$

$$\sigma = (1, 5, 2)(3, 4, 8, 6, 7) \quad \text{gives } \delta_I \sigma_J^I (Z^{\otimes 4} \otimes Y^{\otimes 4})^J = \text{Tr}(ZYZY) \quad (2.11)$$

3 Counting for $\text{Sp}(N)$

In this section, we will rewrite the partition function for the 1/4-BPS sector for free $\text{Sp}(N)$ gauge theory in terms Littlewood-Richardson coefficients, $g(r, s, R)$. For a given set of labels (r, s, R) , $g(r, s, R)$ counts the number of restricted Schur Polynomials with these labels.

Since our operators involve two scalar fields, the partition function will depend on two variables which we denote by t_1 and t_2 . Furthermore, the $\text{Sp}(N)$ partition function will also

¹The $|[S]\rangle$ in this operator is symmetric in boxes $1\&2, 3\&4 \cdots, 2q-1\&2q$.

depend on the character of $O \in \text{Sp}(N)$ in the adjoint representation. Denote the partition function for the 1/4-BPS sector of free $\text{Sp}(N)$ gauge theory by $G_{\text{Sp}(N)}(t_1, t_2)$. The form of $G_{\text{Sp}(N)}(t_1, t_2)$ is [31, 32],

$$G_{\text{Sp}(N)}(t_1, t_2) = \int_{O \in \text{Sp}(N)} [dO] e^{\sum_{m=1}^{\infty} \left(\frac{t_1^m + t_2^m}{m} \right) \chi_{\text{adj}}(O^m)} \quad (3.1)$$

where $\chi_{\text{adj}}(O)$ is the character of O in the adjoint representation of $\text{Sp}(N)$, and $[dO]$ is the $\text{Sp}(N)$ -invariant measure. We take $N = 2n$. In terms of the eigenvalues of O , the adjoint character and the integration measure are [33]

$$\chi_{\text{Sp}(N)}^{\text{adj}}(\mathbf{x}) = \sum_{1 \leq i < j \leq n} (x_i x_j + x_i^{-1} x_j + x_i x_j^{-1} + x_i^{-1} x_j^{-1}) + \sum_{i=1}^n (x_i^2 + x_i^{-2}) + n \quad (3.2)$$

and

$$\int_{O \in \text{Sp}(N)} [dO] f(\mathbf{x}) = \frac{(-1)^n}{2^n n!} \int_{T_n} \prod_{j=1}^n \frac{dx_j}{2\pi x_j} \prod_{j=1}^n (x_j - x_j^{-1})^2 \Delta(\mathbf{x} + \mathbf{x}^{-1})^2 f(\mathbf{x}) \quad (3.3)$$

where $T_n = S^1 \times S^1 \cdots \times S^1$ and $f(\mathbf{x}), \mathbf{x} = (x_1, x_2 \cdots x_n)$, is any symmetric function. Using (3.2) the exponential in (3.1), after some algebra, becomes

$$\begin{aligned} & e^{\sum_{m=1}^{\infty} \left(\frac{t_1^m + t_2^m}{m} \right) \chi_{\text{adj}}(O^m)} \\ &= e^{\sum_{m=1}^{\infty} \left(\frac{t_1^m + t_2^m}{m} \right) \sum_{1 \leq i < j \leq n} (x_i^m x_j^m + x_i^{-m} x_j^m + x_i^m x_j^{-m} + x_i^{-m} x_j^{-m}) + \sum_{i=1}^n (x_i^{2m} + x_i^{-2m}) + n} \\ &= \prod_{k=1}^2 \prod_{1 \leq i < j \leq n} \frac{1}{(1 - t_k x_i x_j)(1 - t_k x_i^{-1} x_j)(1 - t_k x_i x_j^{-1})(1 - t_k x_i^{-1} x_j^{-1})} \\ & \quad \times \frac{1}{(1 - t_k)^n} \prod_{1 \leq i \leq n} \frac{1}{(1 - t_k x_i^2)(1 - t_k x_i^{-2})} \end{aligned} \quad (3.4)$$

Changing variables

$$y_i = x_{\lceil i/2 \rceil} \quad \text{for odd } i \quad (3.5)$$

$$y_{2i} = x_i^{-1} \quad \text{for even } i \quad (3.6)$$

the exponential becomes

$$e^{\sum_{m=1}^{\infty} \left(\frac{t_1^m + t_2^m}{m} \right) \chi_{\text{adj}}(O^m)} = \prod_{k=1}^2 \prod_{1 \leq i \leq j \leq N} \frac{1}{1 - t_k y_i y_j} \quad (3.7)$$

$$= \prod_{k=1}^2 \left(\sum_{\lambda} s_{2\lambda}(\sqrt{t_k} y_1, \sqrt{t_k} y_2 \cdots \sqrt{t_k} y_N) \right) \quad (3.8)$$

$$= \sum_{\mu} \sum_{\lambda} (t_1)^{|\lambda|/2} (t_2)^{|\mu|/2} s_{2\lambda}(y_1, \cdots, y_N) s_{2\mu}(y_1, \cdots, y_N) \quad (3.9)$$

To expand the expression in (3.7) in terms of Schur functions, and thus proceed to (3.8), we used an identity found in [34]. Using the product rule for Schur polynomials, (3.9) may be written in terms of the Littlewood-Richardson coefficients

$$e^{\sum_{m=1}^{\infty} \left(\frac{t_1^m + t_2^m}{m} \right) \chi_{\text{adj}}(O^m)} = \sum_{\xi} \sum_{\mu} \sum_{\lambda} (t_1)^{|2\lambda|/2} (t_2)^{|2\mu|/2} g(2\lambda, 2\mu, \xi) s_{\xi}(y_1, \dots, y_N) \quad (3.10)$$

The partition function (3.1) becomes

$$G_{\text{Sp}(N)}(t_1, t_2) = \sum_{\xi} \sum_{\mu} \sum_{\lambda} (t_1)^{|2\lambda|/2} (t_2)^{|2\mu|/2} g(\lambda, \mu, \xi) \quad (3.11)$$

$$\times \frac{(-1)^n}{2^n n!} \int_{T_n} \prod_{j=1}^n \frac{dx_j}{2\pi x_j} \prod_{j=1}^n (x_j - x_j^{-1})^2 \Delta(x + x^{-1})^2 s_{\xi}(x_1, x_1^{-1}, \dots, x_n, x_n^{-1})$$

The integral in (3.11) is 1 for partitions ξ that have even multiplicity, i.e., an even number of boxes in each column, and 0 otherwise [35, 36]. Therefore we write ξ^2 for this partition and obtain

$$G_{\text{Sp}(N)}(t_1, t_2) = \sum_{\xi^2} \sum_{\mu} \sum_{\lambda} (t_1)^{|2\lambda|/2} (t_2)^{|2\mu|/2} g(2\lambda, 2\mu, \xi^2) \quad (3.12)$$

for the partition function. As we did for $\text{SO}(N)$, we succeeded in writing the partition function for $\text{Sp}(N)$ gauge theory in terms of the Littlewood-Richardson coefficients. Only partitions 2μ and 2λ , with an even number of boxes in each row, and ξ^2 , with an even number of boxes in each column, contribute to $G_{\text{Sp}(N)}$. The Littlewood-Richardson coefficients $g(2\lambda, 2\mu, \xi^2)$ count the number of restricted Schur polynomials that can be defined for labels $(2\lambda, 2\mu, \xi^2)$. Thus, the counting of restricted Schur polynomials for this class of Young diagram labels matches the counting of states in the free $\text{Sp}(N)$ gauge theory.

4 $\text{Sp}(N)$ restricted Schurs

4.1 Constructing the operators

We now give an explicit construction of restricted Schurs for $\text{Sp}(N)$ gauge theory. We continue to consider only $N = 2n$. The group $\text{Sp}(N)$ is the set of $N \times N$ matrices, S , satisfying

$$S^T J S = J, \quad J = \begin{pmatrix} 0 & \mathbb{I}_{N/2} \\ -\mathbb{I}_{N/2} & 0 \end{pmatrix}. \quad (4.1)$$

The matrix fields living in the adjoint representation of the $sp(N)$ algebra satisfy

$$Z^T J + J Z = 0, \quad \text{or} \quad Z^T = J Z J. \quad (4.2)$$

Our $\text{Sp}(N)$ restricted Schurs must match the counting found in (3.12). Since $(r, s)\alpha$ has an even number of boxes in each row, the irrep $([S]_r, [S]_s)\alpha$ of $S_n[S_2] \times S_m[S_2]$ may be subduced. To match the counting, we construct our operators using the state $|[S]_r, [S]_s, \alpha\rangle$.

Furthermore, because R has an even number of boxes in each column, the irrep $[A]$ of $S_q[S_2]$ may be subduced. This leads us to a natural definition for the $\text{Sp}(N)$ restricted characters

$$\chi_{R(r,s)\alpha}^{\text{Sp}(N)} = \text{Tr}(\mathcal{O}_{R(r,s)\alpha}^{\text{Sp}(N)} \Gamma_R(\sigma)) \quad (4.3)$$

where

$$\mathcal{O}_{R(r,s)\alpha}^{\text{Sp}(N)} = |[A]\rangle\langle [S]_r, [S]_s, \alpha| = |[A]\rangle\langle [S]_r, [S]_s, \beta| P_{R \rightarrow (r,s)\beta\alpha} \quad (4.4)$$

The $\text{Sp}(N)$ restricted character in (4.3) clearly has the property

$$\chi_{R(r,s)\alpha}^{\text{Sp}(N)}(\sigma) \text{sgn}(\xi) = \chi_{R(r,s)\alpha}^{\text{Sp}(N)}(\eta\sigma\xi) \quad \xi \in S_q[S_2], \eta \in S_n[S_2] \times S_m[S_2]. \quad (4.5)$$

This function, and its $\text{SO}(N)$ counterpart (2.2), resembles the bi-spherical functions discussed in [28, 37], where one of the hyperoctahedral groups in (4.5) is now a subgroup of the other. We construct the restricted Schurs to be invariant under sending $\sigma \rightarrow \eta\sigma\xi$. Define

$$\mathcal{O}_{R(r,s)\alpha}^{\text{Sp}(N)}(Z, Y) = \frac{1}{\sqrt{n!m!q!2^{2q}}} \sum_{\sigma \in S_{2q}} \chi_{R(r,s)\alpha}^{\text{Sp}(N)}(\sigma) J_I \sigma_J^I ((JZ)^{\otimes 2n} \otimes (JY)^{\otimes 2m})^J \quad (4.6)$$

where

$$J_I = J_{i_1 i_2} J_{i_3 i_4} \cdots J_{i_{2q-1} i_{2q}} \quad (4.7)$$

The quantity $J_I(\sigma) J_J^I ((JZ)^{\otimes 2n} \otimes (JY)^{\otimes 2m})^J$ indeed gives all the possible multitrace operators for n Z 's, m Y 's and for the permutations of S_{2q} . For example, for 2 Z 's and 2 Y 's we still have only 4 possible multi-trace operators,

$$\text{Tr}(Z^2 Y^2), \quad \text{Tr}(ZYZY), \quad \text{Tr}(Z^2)\text{Tr}(Y^2) \quad \text{and} \quad \text{Tr}(ZY)^2 \quad (4.8)$$

Indeed, $J_I(\sigma) J_J^I ((JZ)^{\otimes 4} \otimes (JY)^{\otimes 4})^J$ generate all of these for $\sigma \in S_8$. Next, we note that $J_I(\sigma) J_J^I ((JZ)^{\otimes 2n} \otimes (JY)^{\otimes 2m})^J$ has the desired symmetry property when $\sigma \rightarrow \eta\sigma\xi$. Since the J 's are anti-symmetric under transposition, ξ acting on J_I gives $\text{sgn}(\xi)J_I$, and since (JZ) and (JY) are symmetric under transposition, η leaves the tensor invariant. The state $|[A]\rangle$ is calculated as the eigenvector of

$$P_{[A]} = \frac{1}{q!2^q} \sum_{\xi \in S_q[S_2]} \text{sgn}(\xi) \Gamma_R(\xi). \quad (4.9)$$

with eigenvalue 1. The $|[S]_r, [S]_s, \alpha\rangle$ is calculated as the eigenvector of

$$P_{R \rightarrow ([S]_r, [S]_s)\alpha} = P_{R \rightarrow (r,s)\alpha\alpha} P_{[S], [S]}, \quad P_{[S], [S]} = \frac{1}{n!m!} \sum_{\eta \in S_n[S_2] \times S_m[S_2]} \Gamma_R(\eta). \quad (4.10)$$

with eigenvalue 1.

4.2 Two-point function

Now consider evaluating the two-point function of the operators in (4.6). First, it is straightforward to show that

$$\langle ((JZ)^{\otimes 2n} \otimes (JY)^{\otimes 2m})^J ((JZ)^{\otimes 2n} \otimes (JY)^{\otimes 2m})_L \rangle = \sum_{\eta \in S_n[S_2] \times S_m[S_2]} \eta_L^J \quad (4.11)$$

Using (4.11), the evaluation of the two-point function is analogous to that of the $\text{SO}(N)$ case [1]. Following the same steps, we arrive at

$$\langle O_{R(r,s)\alpha}^{\text{Sp}(N)} \overline{O}_{T(t,u)\beta}^{\text{Sp}(N)} \rangle = \delta_{RT} \delta_{rt} \delta_{su} \delta_{\alpha\beta} \frac{(2q)!}{d_R} \sum_{\psi \in \mathcal{B}_q} \text{Tr}(P_{[A]} \Gamma_R(\psi)) J_I J^M(\psi)_M^I \quad (4.12)$$

where \mathcal{B}_q is again the set of representatives of the coset $S_{2q}/S_q[S_2]$, chosen to be the permutations

$$\prod_{j=0}^{q-1} \prod_{i=1}^{2j+1} (i, 2j+1) \quad (4.13)$$

For the permutations in (4.13), $J_I J^M(\psi)_M^I$ gives exactly the same result as $\delta_I \delta^J(\psi)_J^I$. The identity permutation, for example, gives

$$J_{i_1 i_2} J_{i_3 i_4} \cdots J_{i_{2q-1} i_{2q}} J^{i_1 i_2} J^{i_3 i_4} \cdots J^{i_{2q-1} i_{2q}} = (J^T J)_{i_1}^{i_1} (J^T J)_{i_3}^{i_3} \cdots (J^T J)_{i_{2q-1}}^{i_{2q-1}} \quad (4.14)$$

$$= \delta_{i_1}^{i_1} \delta_{i_3}^{i_3} \cdots \delta_{i_{2q-1}}^{i_{2q-1}} \quad (4.15)$$

$$= N^q \quad (4.16)$$

A two-cycle $(i, 2j+1)$ returns N^{q-1} . For example, consider $q=4$ and $(3, 5)$. This gives

$$(J^T J)_{i_1}^{i_1} (J^T J J^T J)_{i_3}^{i_3} (J^T J)_{i_7}^{i_7} = \delta_{i_1}^{i_1} \delta_{i_3}^{i_3} \delta_{i_7}^{i_7} = N^3 \quad (4.17)$$

For (4, 7), we get

$$(J^T J)_{i_1}^{i_1} (J^2 J^2)_{i_3}^{i_3} (J^T J)_{i_5}^{i_5} = \delta_{i_1}^{i_1} \delta_{i_3}^{i_3} \delta_{i_7}^{i_7} = N^3 \quad (4.18)$$

For any $\psi \in \mathcal{B}_q$, $J_I J^M(\psi)_M^I$ always consists of products of $(J^T J)$ and an even number of J^2 's. If p is the number of two-cycles in ψ , then $J_I J^M(\psi)_M^I$ gives N^{q-p} . After summing over \mathcal{B}_q , equation (4.12) becomes

$$\langle O_{R(r,s)\alpha}^{\text{Sp}(N)} \overline{O}_{T(t,u)\beta}^{\text{Sp}(N)} \rangle = \delta_{RT} \delta_{rt} \delta_{su} \delta_{\alpha\beta} \frac{d_R}{(2q)!} \langle [A] | \prod_{j=0}^{q-1} (N + J_{2j+1}) | [A] \rangle \quad (4.19)$$

Instead of giving a product of weights coming from the odd columns in R , as it did for $\text{SO}(N)$, (4.19) now gives the product of weights coming from the odd rows of R . The two-point function for the $\text{Sp}(N)$ restricted Schurs is

$$\langle O_{R(r,s)\alpha}^{\text{Sp}(N)} \overline{O}_{T(t,u)\beta}^{\text{Sp}(N)} \rangle = \delta_{RT} \delta_{rt} \delta_{su} \delta_{\alpha\beta} \frac{(2q)!}{d_R} \prod_{i \in \text{odd rows in } R} c_i \quad (4.20)$$

This gives precisely the same result as the $\text{SO}(N)$ case, but with $N \rightarrow -N$, as expected. Thus, to calculate the two-point function of operators in $\text{Sp}(N)$, calculate instead the two-point function of operators in $\text{SO}(N)$, but with all Young diagrams conjugated (or transposed), and then replace N by $-N$ in the result. This gives the result for $\text{Sp}(N)$. As a check of these conclusions, we present two examples in appendix B.

5 Exact multi-trace correlators with two scalar fields

In this section, we express any multi-trace operator involving Z 's and Y 's as a linear combination of the $\text{SO}(N)$ and $\text{Sp}(N)$ restricted Schur polynomials, (2.1) and (4.6). Thereafter, we derive a product rule for these operators.

5.1 Relating trace operators to restricted Schurs

First, let's discuss $\text{SO}(N)$. Define the 'dual restricted character'

$$\chi_{\text{SO}(N)}^{R(r,s)\alpha}(\sigma) \equiv \frac{d_R \sqrt{n!m!q!2^{2q}}}{(2q)!} \text{Tr}(\mathcal{O}_{R(r,s)\alpha}^T \Gamma_R(\sigma^{-1})) \quad (5.1)$$

The aim is to use (5.1) to express any multi-trace operator in terms of the restricted Schurs in (2.1). The calculation is analogous to that in [15]. Calculate

$$\begin{aligned} \sum_{R,r,s,\alpha} \chi_{\text{SO}(N)}^{R(r,s)\alpha}(\sigma) \text{Tr}(\mathcal{O}_{R(r,s)\alpha} \Gamma_R(\tau)) \\ = \frac{\sqrt{n!m!q!2^{2q}}}{(2q)!} \sum_{R,r,s,\alpha} d_R \text{Tr}(\mathcal{O}_{R(r,s)\alpha}^T \Gamma_R(\sigma^{-1})) \text{Tr}(\mathcal{O}_{R(r,s)\alpha} \Gamma_R(\tau)) \end{aligned} \quad (5.2)$$

The Γ 's may be expanded in the R basis in the following way

$$\Gamma_R(\sigma) = \sum_{I,J} |R, I\rangle \langle R, J| \Gamma_R(\sigma)_{IJ} \quad (5.3)$$

where I labels a particular state in carrier space of irrep R . Using this, the trace $\text{Tr}(\mathcal{O}_{R(r,s)\alpha} \Gamma_R(\tau))$ may be written as

$$\text{Tr}(\mathcal{O}_{R(r,s)\alpha} \Gamma_R(\tau)) = \sum_K \langle R, K | \mathcal{O}_{R(r,s)\alpha} \Gamma_R(\tau) | R, K \rangle \quad (5.4)$$

$$= \sum_{K,L} \langle R, K | \mathcal{O}_{R(r,s)\alpha} | R, L \rangle \Gamma_R(\tau)_{LK} \quad (5.5)$$

Then, $\sum_{r,s,\alpha} \text{Tr}(\mathcal{O}_{R(r,s)\alpha}^T \Gamma_R(\sigma^{-1})) \text{Tr}(\mathcal{O}_{R(r,s)\alpha} \Gamma_R(\tau))$ in (5.2) becomes

$$\begin{aligned} \sum_{r,s,\alpha} \text{Tr}(\mathcal{O}_{R(r,s)\alpha}^T \Gamma_R(\sigma^{-1})) \text{Tr}(\mathcal{O}_{R(r,s)\alpha} \Gamma_R(\tau)) \\ = \sum_{r,s,\alpha} \left(\sum_{I,J} \langle R, I | \mathcal{O}_{R(r,s)\alpha}^T | R, J \rangle \Gamma_R(\sigma^{-1})_{JI} \right) \\ \times \left(\sum_{K,L} \langle R, K | \mathcal{O}_{R(r,s)\alpha} | R, L \rangle \Gamma_R(\tau)_{LK} \right) \\ = \sum_{r,s,\alpha} \sum_{I,J,K,L} \Gamma_R(\sigma^{-1})_{JI} \Gamma_R(\tau)_{LK} \\ \times \langle R, I | [A]_r, [A]_s, \alpha \rangle \langle [S] | R, J \rangle \langle R, K | [S] \rangle \langle [A]_r, [A]_s, \alpha | R, L \rangle \\ = \sum_{r,s,\alpha} \sum_{I,J,K,L} \Gamma_R(\sigma^{-1})_{JI} \Gamma_R(\tau)_{LK} \\ \times \langle R, K | [S] \rangle \langle [S] | R, J \rangle \langle R, I | [A]_r, [A]_s, \alpha \rangle \langle [A]_r, [A]_s, \alpha | R, L \rangle \end{aligned} \quad (5.6)$$

In the above equation, we recognise the following projectors

$$P_{[S]} = \frac{1}{q!2^q} \sum_{\xi \in S_q[S_2]} \Gamma_R(\xi) = |[S]\rangle\langle[S]| \quad (5.7)$$

$$P_{[A,A]} = \frac{1}{n!m!2^q} \sum_{\eta \in S_n[S_2] \times S_m[S_2]} \text{sgn}(\eta) \Gamma_R(\eta) \quad (5.8)$$

$$= \sum_{r,s,\alpha} |[A]_r, [A]_s, \alpha\rangle\langle[A]_r, [A]_s, \alpha| \quad (5.9)$$

In appendix A, we discuss going from (5.8) to (5.9) in more detail. Equation (5.6) then becomes

$$\begin{aligned} & \sum_{r,s,\alpha} \text{Tr}(\mathcal{O}_{R(r,s)\alpha}^T \Gamma_R(\sigma^{-1})) \text{Tr}(\mathcal{O}_{R(r,s)\alpha} \Gamma_R(\tau)) \\ &= \frac{1}{n!m!q!2^{2q}} \sum_{\xi \in S_q[S_2]} \sum_{\eta \in S_n[S_2] \times S_m[S_2]} \sum_{I,J,K,L} \\ & \quad \times \text{sgn}(\eta) \Gamma_R(\sigma^{-1})_{JI} \Gamma_R(\tau)_{LK} \Gamma_R(\xi)_{KJ} \Gamma_R(\eta)_{IL} \\ &= \frac{1}{n!m!q!2^{2q}} \sum_{\xi \in S_q[S_2]} \sum_{\eta \in S_n[S_2] \times S_m[S_2]} \text{sgn}(\eta) \chi_R(\sigma^{-1} \eta \tau \xi) \end{aligned} \quad (5.10)$$

Using (5.10), equation (5.2) becomes

$$\begin{aligned} & \sum_{R,r,s,\alpha} \chi_{\text{SO}(N)}^{R(r,s)\alpha}(\sigma) \text{Tr}(\mathcal{O}_{R(r,s)\alpha} \Gamma_R(\tau)) \\ &= \frac{1}{\sqrt{n!m!q!2^{2q}}} \sum_R \frac{d_R}{(2q)!} \sum_{\xi \in S_q[S_2]} \sum_{\eta \in S_n[S_2] \times S_m[S_2]} \text{sgn}(\eta) \chi_R(\sigma^{-1} \eta \tau \xi) \end{aligned} \quad (5.11)$$

In this expression, we are summing over all possible $R \vdash 2q$. In this sum, terms for which R does not have an even number of boxes in each row vanish.²

$$\begin{aligned} & \sum_{R,r,s,\alpha} \chi_{\text{SO}(N)}^{R(r,s)\alpha}(\sigma) \text{Tr}(\mathcal{O}_{R(r,s)\alpha} \Gamma_R(\tau)) \\ &= \frac{1}{\sqrt{n!m!q!2^{2q}}} \sum_{\xi \in S_q[S_2]} \sum_{\eta \in S_n[S_2] \times S_m[S_2]} \text{sgn}(\eta) \sum_R \frac{d_R}{(2q)!} \chi_R(\sigma^{-1} \eta \tau \xi) \\ &= \frac{1}{\sqrt{n!m!q!2^{2q}}} \sum_{\xi \in S_q[S_2]} \sum_{\eta \in S_n[S_2] \times S_m[S_2]} \text{sgn}(\eta) \delta(\sigma^{-1} \eta \tau \xi) \end{aligned} \quad (5.12)$$

where we used the well-known group theoretic result [38]

$$\sum_{R \vdash n} \frac{d_R}{n!} \chi_R(\sigma) = \delta(\sigma) \quad (5.13)$$

²This is a simple consequence of the fact that only R having even rows is capable of subducing the $S_q[S_2]$ irrep $[S]$.

5.2 A product rule

A product rule for operators $O_{R(r,s)\alpha}^{\text{SO}(N)}(Z, Y)$ in terms of restricted Littlewood-Richardson coefficients is easily derived. The basic idea is exactly the same as in [15]. We multiply two restricted Schurs, one having labels $R_1(r_1, s_1)\alpha_1$ and the other $R_2(r_2, s_2)\alpha_2$, to produce a linear combination of restricted Schurs with labels $R_1 + R_2, (r_1 + r_2, s_1 + s_2)\beta$. R_1 and R_2 are irreps of S_{2q_1} and S_{2q_2} respectively. (r_1, s_1) and (r_2, s_2) are irreps of $S_{2n_1} \times S_{2m_1}$ and $S_{2n_2} \times S_{2m_2}$ respectively, where $q_i = n_i + m_i$ ($i = 1, 2$). α_1 and α_2 are the multiplicity labels for the two respective subgroup irreps. Thus, $R_i(r_i, s_i)\alpha_i$ defines a restricted Schur having n_i Z 's and m_i Y 's. $R_1 + R_2$ is an irrep of $S_{2q_1+2q_2}$, $r_1 + r_2$ is an irrep of $S_{2n_1+2n_2}$ and $s_1 + s_2$ is an irrep of $S_{2m_1+2m_2}$. β simply labels the $(r_1 + r_2, s_1 + s_2)$ copy subduced from $R_1 + R_2$. The set of labels $\{R_1 + R_2, (r_1 + r_2, s_1 + s_2), \beta\}$ defines a restricted Schur having $n_1 + n_2$ Z 's and $m_1 + m_2$ Y 's. As in [15], it is convenient to streamline the notation. Thus, denote $\{i\} \equiv R_i(r_i, s_i)\alpha_i$ and $\{1 + 2\} \equiv \{R_1 + R_2, (r_1 + r_2, s_1 + s_2)\beta\}$. Also, write $n_{12} = n_1 + n_2$, $m_{12} = m_1 + m_2$ and $q_{12} = q_1 + q_2$. Thus, for example, (2.7) may be written as

$$O_{\{1+2\}}^{\text{SO}(N)}(Z, Y) = \frac{1}{\sqrt{n_{12}!m_{12}!q_{12}!2^{2q_{12}}}} \sum_{\rho \in S_{2q_{12}}} \chi_{\{1+2\}}^{\text{SO}(N)}(\rho) \delta_I \rho^I (Z^{\otimes 2n_{12}} \otimes Y^{\otimes 2m_{12}})^J \quad (5.20)$$

In the following derivation, we use the operators defined in (2.7) which are equivalent to the ones defined in (2.1). The factorisation we need in our product rule occurs naturally for the operators defined using the δ_I tensor, rather than the $C_I^{4\nu}$ tensor. The following derivation is for $\text{SO}(N)$. The derivation for $\text{Sp}(N)$ is completely analogous. Define the restricted Littlewood-Richardson coefficients to be

$$f_{\{1\},\{2\}}^{\{1+2\}} = \frac{1}{\sqrt{n_1!m_1!n_2!m_2!2^{2q_1+2q_2}}} \sum_{\sigma_1 \in S_{2q_1}} \sum_{\sigma_2 \in S_{2q_2}} \chi_{\{1\}}^{\text{SO}(N)}(\sigma_1) \chi_{\{2\}}^{\text{SO}(N)}(\sigma_2) \chi_{\text{SO}(N)}^{\{1+2\}}(\sigma_1 \cdot \sigma_2) \quad (5.21)$$

where $\chi_{\text{SO}(N)}^{\{1+2\}}$ is the dual restricted character defined in (5.1). We want to show that

$$\sum_{\{1+2\}} f_{\{1\},\{2\}}^{\{1+2\}} O_{\{1+2\}}^{\text{SO}(N)}(Z, Y). \quad (5.22)$$

is given by the product of two restricted Schurs, one labeled by $\{1\}$, and the other by $\{2\}$. To evaluate (5.22), we use equation (5.12) to write

$$\sum_{\{1+2\}} \chi_{\text{SO}(N)}^{\{1+2\}}(\sigma_1 \cdot \sigma_2) \chi_{\{1+2\}}^{\text{SO}(N)}(\rho) = \frac{1}{\sqrt{n_{12}!m_{12}!q_{12}!2^{2q_{12}}}} \sum_{\xi \in S_{q_{12}}[S_2]} \sum_{\eta \in S_{n_{12}}[S_2] \times S_{m_{12}}[S_2]} \text{sgn}(\eta) \times \delta(\sigma_1^{-1} \cdot \sigma_2^{-1} \eta \rho \xi) \quad (5.23)$$

Summing over ρ sets $\rho = \eta^{-1} \sigma_1 \sigma_2 \xi^{-1}$. As before, ξ acting on δ_I is invariant and η acting on the ZY tensor gives back an extra $\text{sgn}(\eta)$. Summing over ξ and η then cancels the

normalisation factor in (5.20), and thus we write

$$\begin{aligned} & \sum_{\{1+2\}} f_{\{1\},\{2\}}^{\{1+2\}} O_{\{1+2\}}^{\text{SO}(N)}(Z, Y) \\ &= \frac{1}{\sqrt{n_1!m_1!n_2!m_2!2^{2q_1+2q_2}}} \sum_{\sigma_1 \in S_{2q_1}} \sum_{\sigma_2 \in S_{2q_2}} \chi_{\{1\}}^{\text{SO}(N)}(\sigma_1) \chi_{\{2\}}^{\text{SO}(N)}(\sigma_2) \\ & \quad \delta_I(\sigma_1 \cdot \sigma_2)_J^I (Z^{\otimes 2n_{12}} \otimes Y^{\otimes 2m_{12}})^J \end{aligned} \quad (5.24)$$

By σ_1 , we mean the permutation that acts on the first $2n_1$ Z indices and first $2m_1$ Y indices. By σ_2 , we mean the permutation that acts on the second $2n_2$ Z indices and second $2m_2$ Y indices. The second line in (5.24) factors and we may write

$$\begin{aligned} & \sum_{\{1+2\}} f_{\{1\},\{2\}}^{\{1+2\}} O_{\{1+2\}}^{\text{SO}(N)}(Z, Y) \\ &= \frac{1}{\sqrt{n_1!m_1!q_1!2^{2q_1}}} \sum_{\sigma_1 \in S_{2q_1}} \chi_{\{1\}}^{\text{SO}(N)}(\sigma_1) \delta_{I_1}(\sigma_1)_{J_1}^{I_1} (Z^{\otimes 2n_1} \otimes Y^{\otimes 2m_1})^{J_1} \\ & \quad \times \frac{1}{\sqrt{n_2!m_2!q_2!2^{2q_2}}} \sum_{\sigma_2 \in S_{2q_2}} \chi_{\{2\}}^{\text{SO}(N)}(\sigma_2) \delta_{I_2}(\sigma_2)_{J_2}^{I_2} (Z^{\otimes 2n_2} \otimes Y^{\otimes 2m_2})^{J_2} \end{aligned} \quad (5.25)$$

Thus, we have achieved the desired result

$$\sum_{\{1+2\}} f_{\{1\},\{2\}}^{\{1+2\}} O_{\{1+2\}}^{\text{SO}(N)}(Z, Y) = O_{\{1\}}^{\text{SO}(N)}(Z, Y) O_{\{2\}}^{\text{SO}(N)}(Z, Y). \quad (5.26)$$

In appendix D, we check this product with a simple example.

6 Discussion

In this work, we have defined a basis for the 1/4-BPS sector of the free super Yang-Mills theory with a symplectic gauge group. These operators are very similar to those defined for the theory with orthogonal gauge group. The difference between the two cases is that the symmetrisations of the irreducible representations defining the operators have been exchanged, as expected. The two-point function for the symplectic gauge theory operators was related to its orthogonal gauge theory counterpart by replacing N by $-N$.

The results of this work make it possible to compute correlation functions of any kind of multi-matrix, multi-trace operators involving two scalar fields. In such a correlation function, each trace operator may be expressed in terms of restricted Schur polynomials. Using the product rule derived above, computing the correlation function of many restricted Schurs may be transformed into a simple two-point function computation, the formula for which is given in (2.5) and (4.20).

Studying the spectrum of anomalous dimensions of our restricted Schurs is an interesting problem, especially in the limit that these operators become dual to excited giant gravitons. Such a study may yield new insights into the non-perturbative physics of their

D-brane duals. Pursuing this direction may also allow us to make some concrete statements about whether or not integrability is preserved in non-planar limits of the $SO(N)$ or $Sp(N)$ gauge theory.

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A An expression for $P_{[A,A]}$

In this section, we try prove that

$$P_{[A,A]} = \frac{1}{n!m!2^q} \sum_{\eta \in S_n[S_2] \times S_m[S_2]} \text{sgn}(\eta) \Gamma_R(\eta) = \sum_{r,s,\nu} |[A]_r, [A]_s, \nu\rangle \langle [A]_r, [A]_s, \nu| \quad (\text{A.1})$$

Since $\eta_1 \cdot \eta_2 \in S_{2n} \times S_{2m}$, we can write [38]

$$\Gamma_R(\eta_1 \cdot \eta_2)_{IJ} = \sum_{r,s,\nu} \sum_{j_1 k_1 j_2 k_2} \langle R, I | R(r, s) \nu; j_1, j_2 \rangle \Gamma_r(\eta_1)_{j_1 k_1} \Gamma_s(\eta_2)_{j_2 k_2} \langle R(r, s) \nu; k_1, k_2 | R, J \rangle \quad (\text{A.2})$$

Writing out the IJ -th matrix element of $P_{[A,A]}$,

$$(P_{[A,A]})_{IJ} = \langle R, I | \left(\sum_{r,s,\nu} (P_{[A]_r})_{j_1 k_1} (P_{[A]_s})_{j_2 k_2} | R(r, s) \nu; j_1, j_2 \rangle \langle R(r, s) \nu; k_1, k_2 | \right) | R, J \rangle \quad (\text{A.3})$$

where we wrote

$$P_{[A]_r} = \frac{1}{n!2^n} \sum_{\eta_1 \in S_n[S_2]} \text{sgn}(\eta_1) \Gamma_r(\eta_1) \quad (\text{A.4})$$

with a similar expression for $P_{[A]_s}$. Since $P_{[A]_r}$ and $P_{[A]_s}$ are projectors, i.e., $P_{[A]_r}^2 = P_{[A]_r}$ (and similarly for $P_{[A]_s}$), this matrix element becomes

$$(P_{[A,A]})_{IJ} = \langle R, I | \left(\sum_{r,s,\nu} (P_{[A]_r})_{j_1 l_1} (P_{[A]_r})_{l_1 k_1} (P_{[A]_s})_{j_2 l_2} (P_{[A]_s})_{l_2 k_2} | R(r, s) \nu; j_1, j_2 \rangle \langle R(r, s) \nu; k_1, k_2 | \right) | R, J \rangle \quad (\text{A.5})$$

We then have

$$\begin{aligned} (P_{[A,A]})_{IJ} &= \langle R, I | \left(\sum_{r,s,\nu} \sum_{l_1 l_2} |[A]_r, [A]_s, \nu; l_1 l_2 \rangle \langle [A]_r, [A]_s, \nu | ; l_1 l_2 \right) | R, J \rangle \\ &= \sum_{r,s,\nu} \langle R, I | [A]_r, [A]_s, \nu \rangle \langle [A]_r, [A]_s, \nu | R, J \rangle \end{aligned} \quad (\text{A.6})$$

To get the last line, we used the fact that $([A], [A])$ is a 1-dimensional irrep.

B Sp(N) two-point function examples

First, consider for Sp(N)

$$R = \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \quad (r, s) = (\square, \square) \tag{B.1}$$

The restricted Schur (4.6) for this operator was calculated to be

$$O_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}}^{\text{Sp}(N)} = -\sqrt{6}\text{Tr}(ZY) \tag{B.2}$$

It's two-point function is

$$\langle O_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}}^{\text{Sp}(N)} \bar{O}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}}^{\text{Sp}(N)} \rangle = 12N(N+1). \tag{B.3}$$

Firstly, this agrees with (4.20) and secondly, we compare this to the SO(N) two-point function for

$$R_c = \begin{array}{|c|} \hline \square \\ \hline \end{array}, \quad (r_c, s_c) = \left(\begin{array}{|c|} \hline \square \\ \hline \end{array}, \begin{array}{|c|} \hline \square \\ \hline \end{array} \right) \tag{B.4}$$

The restricted Schur for SO(N), as given in equation (2.1), is (see [1])

$$O_{\begin{array}{|c|} \hline \square \\ \hline \end{array}}^{\text{SO}(N)} = \sqrt{6}\text{Tr}(ZY) \tag{B.5}$$

with two-point function

$$\langle O_{\begin{array}{|c|} \hline \square \\ \hline \end{array}}^{\text{SO}(N)} \bar{O}_{\begin{array}{|c|} \hline \square \\ \hline \end{array}}^{\text{SO}(N)} \rangle = 12N(N-1). \tag{B.6}$$

Clearly, sending N to $-N$ gives us the Sp(N) result (B.3).

Next, consider for Sp(N)

$$R = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array}, \quad (r, s) = (\square\square\square, \square\square\square) \tag{B.7}$$

The restricted Schur was calculated to be

$$O_{\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array}}^{\text{Sp}(N)} = 2\sqrt{30}\left(\text{Tr}(ZY)^2 + 2\text{Tr}(ZYZY)\right) \tag{B.8}$$

Its two-point function is

$$\langle O_{\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array}}^{\text{Sp}(N)} \bar{O}_{\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array}}^{\text{Sp}(N)} \rangle = 2880N(N+1)(N+2)(N+3), \tag{B.9}$$

agreeing with (4.20). We now compare this with the SO(N) two-point function for

$$R_c = \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \quad (r_c, s_c) = \left(\begin{array}{|c|} \hline \square \\ \hline \end{array}, \begin{array}{|c|} \hline \square \\ \hline \end{array} \right) \tag{B.10}$$

The restricted Schur (2.1) is

$$O_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}}^{\text{SO}(N)} = 2\sqrt{30}\left(\text{Tr}(ZY)^2 - 2\text{Tr}(ZYZY)\right) \tag{B.11}$$

with two-point function

$$\langle O_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}}^{\text{SO}(N)} \bar{O}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}}^{\text{SO}(N)} \rangle = 2880N(N-1)(N-2)(N-3). \tag{B.12}$$

Sending $N \rightarrow -N$, yields the Sp(N) result.

C Examples of multi-trace operators

In this appendix, we give two examples of our formula (5.16). Recall from [1], we calculated (with the normalisation of (2.1))

$$O_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}; \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}} = \sqrt{30} \left(2\text{Tr}(ZY)^2 + 2\text{Tr}(ZYZY) + \text{Tr}(Z^2)\text{Tr}(Y^2) + 4\text{Tr}(Z^2Y^2) \right) \quad (\text{C.1})$$

$$O_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}; \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} = \sqrt{6} \left(2\text{Tr}(ZY)^2 + 2\text{Tr}(ZYZY) + 3\text{Tr}(Z^2)\text{Tr}(Y^2) - 12\text{Tr}(Z^2Y^2) \right) \quad (\text{C.2})$$

$$O_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}; \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} = 2\sqrt{30} \left(\text{Tr}(ZY)^2 - 2\text{Tr}(ZYZY) \right) \quad (\text{C.3})$$

$$O_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}; \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} = 2\sqrt{6} \left(-\text{Tr}(ZY)^2 - \text{Tr}(ZYZY) + \text{Tr}(Z^2)\text{Tr}(Y^2) + \text{Tr}(Z^2Y^2) \right) \quad (\text{C.4})$$

Let $\sigma = (1, 3, 5)(4, 8, 6, 7)$. The dual restricted characters evaluated to

$$\chi_{\text{SO}(N)}^{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}; \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}}(\sigma) = \frac{1}{6\sqrt{30}} \quad (\text{C.5})$$

$$\chi_{\text{SO}(N)}^{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}; \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(\sigma) = \frac{1}{30\sqrt{6}} \quad (\text{C.6})$$

$$\chi_{\text{SO}(N)}^{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}; \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(\sigma) = \frac{1}{6\sqrt{30}} \quad (\text{C.7})$$

$$\chi_{\text{SO}(N)}^{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}; \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(\sigma) = -\frac{1}{15}\sqrt{\frac{2}{3}} \quad (\text{C.8})$$

Then adding the four operators with these coefficients, we found

$$\frac{1}{6\sqrt{30}} O_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}; \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}}^{\text{SO}(N)} + \frac{1}{30\sqrt{6}} O_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}; \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}^{\text{SO}(N)} + \frac{1}{6\sqrt{30}} O_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}; \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}^{\text{SO}(N)} - \frac{1}{15}\sqrt{\frac{2}{3}} O_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}; \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}^{\text{SO}(N)} = \text{Tr}(ZY)^2 \quad (\text{C.9})$$

We then calculated the right-hand-side of (5.16) and found

$$C_I^{4\nu}(\sigma)_J(Z^{\otimes 4} \otimes Y^{\otimes 4}) = \text{Tr}(ZY)^2 \quad (\text{C.10})$$

For one more example, Let $\sigma = (3, 4, 7, 8)$. The dual restricted characters evaluated to

$$\chi_{\text{SO}(N)}^{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}; \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}}(\sigma) = \frac{1}{12\sqrt{30}} \quad (\text{C.11})$$

$$\chi_{\text{SO}(N)}^{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}; \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(\sigma) = \frac{-1}{20\sqrt{6}} \quad (\text{C.12})$$

$$\chi_{\text{SO}(N)}^{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}; \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(\sigma) = 0 \quad (\text{C.13})$$

$$\chi_{\text{SO}(N)}^{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}; \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(\sigma) = \frac{1}{30\sqrt{6}} \quad (\text{C.14})$$

Adding the 4 Schurs with these coefficients, we found

$$\frac{1}{12\sqrt{30}} O_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}; \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}}^{\text{SO}(N)} - \frac{1}{20\sqrt{6}} O_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}; \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}^{\text{SO}(N)} + \frac{1}{30\sqrt{6}} O_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}; \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}^{\text{SO}(N)} = \text{Tr}(Z^2Y^2) \quad (\text{C.15})$$

We then calculated the right-hand-side of (5.16) and found

$$C_I^{4\nu}(\sigma)_J^I(Z^{\otimes 4} \otimes Y^{\otimes 4}) = \text{Tr}(Z^2 Y^2) \quad (\text{C.16})$$

D Example of the product rule

In this appendix, we give one simple example of our product rule in (5.26). We try evaluate the following product: $O_{\boxplus(\boxplus;\boxplus)}(Z, Y)O_{\boxplus(\boxplus;\boxplus)}(Z, Y)$. For the definition in (2.7),

$$O_{\boxplus(\boxplus;\boxplus)}(Z, Y) = -\sqrt{6}\text{Tr}(ZY) \quad (\text{D.1})$$

This means that

$$O_{\boxplus(\boxplus;\boxplus)}(Z, Y)O_{\boxplus(\boxplus;\boxplus)}(Z, Y) = 6\text{Tr}(ZY)^2 \quad (\text{D.2})$$

According to (5.26), this product should also be given as a sum over restricted Schurs, each multiplied by restricted Littlewood-Richardson coefficients, corresponding to the labels in (5.17). The operators for these labels evaluated to

$$O_{\boxplus\boxplus,(\boxplus;\boxplus)} = \sqrt{30}\left(2\text{Tr}(ZY)^2 + 2\text{Tr}(ZYZY) + \text{Tr}(Z^2)\text{Tr}(Y^2) + 4\text{Tr}(Z^2Y^2)\right) \quad (\text{D.3})$$

$$O_{\boxplus,(\boxplus;\boxplus)} = \sqrt{6}\left(2\text{Tr}(ZY)^2 + 2\text{Tr}(ZYZY) + 3\text{Tr}(Z^2)\text{Tr}(Y^2) - 12\text{Tr}(Z^2Y^2)\right) \quad (\text{D.4})$$

$$O_{\boxplus,(\boxplus;\boxplus)} = 2\sqrt{30}\left(\text{Tr}(ZY)^2 - 2\text{Tr}(ZYZY)\right) \quad (\text{D.5})$$

$$O_{\boxplus\boxplus,(\boxplus;\boxplus)} = -2\sqrt{6}\left(-\text{Tr}(ZY)^2 - \text{Tr}(ZYZY) + \text{Tr}(Z^2)\text{Tr}(Y^2) + \text{Tr}(Z^2Y^2)\right) \quad (\text{D.6})$$

Only the last operator differs from the operators defined in (2.1) and the difference is a minus sign. Here, $n_1 = m_1 = n_2 = m_2 = 1$, and $q_1 = q_2 = 2$. The restricted Littlewood-Richardson coefficients evaluated to

$$f_{\boxplus,(\boxplus;\boxplus);\boxplus,(\boxplus;\boxplus)}^{\boxplus\boxplus,(\boxplus;\boxplus)} = \frac{1}{\sqrt{30}} \quad (\text{D.7})$$

$$f_{\boxplus,(\boxplus;\boxplus);\boxplus,(\boxplus;\boxplus)}^{\boxplus,(\boxplus;\boxplus)} = \frac{1}{5\sqrt{6}} \quad (\text{D.8})$$

$$f_{\boxplus,(\boxplus;\boxplus);\boxplus,(\boxplus;\boxplus)}^{\boxplus,(\boxplus;\boxplus)} = \frac{1}{\sqrt{30}} \quad (\text{D.9})$$

$$f_{\boxplus,(\boxplus;\boxplus);\boxplus,(\boxplus;\boxplus)}^{\boxplus\boxplus,(\boxplus;\boxplus)} = \frac{2}{5}\sqrt{\frac{2}{3}} \quad (\text{D.10})$$

In the f 's above, there were two sums over the permutation group S_4 . The first sum was over permutations that permuted 1, 2, 5, 6 amongst themselves, and the second sum was over permutations that permuted 3, 4, 7, 8 amongst themselves. For the coefficients in (D.7) to (D.10) and operators (D.3) to (D.6), we found

$$\begin{aligned} & f_{\boxplus,(\boxplus;\boxplus);\boxplus,(\boxplus;\boxplus)}^{\boxplus\boxplus,(\boxplus;\boxplus)} O_{\boxplus\boxplus,(\boxplus;\boxplus)}(Y, Z) + f_{\boxplus,(\boxplus;\boxplus);\boxplus,(\boxplus;\boxplus)}^{\boxplus,(\boxplus;\boxplus)} O_{\boxplus,(\boxplus;\boxplus)} \\ & + f_{\boxplus,(\boxplus;\boxplus);\boxplus,(\boxplus;\boxplus)}^{\boxplus,(\boxplus;\boxplus)} O_{\boxplus,(\boxplus;\boxplus)} + f_{\boxplus,(\boxplus;\boxplus);\boxplus,(\boxplus;\boxplus)}^{\boxplus\boxplus,(\boxplus;\boxplus)} O_{\boxplus\boxplus,(\boxplus;\boxplus)} = 6\text{Tr}(ZY)^2 \end{aligned} \quad (\text{D.11})$$

precisely as expected.

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