Published for SISSA by Springer

Received: April 18, 2014
Accepted: July 21, 2014
Published: August 14, 2014

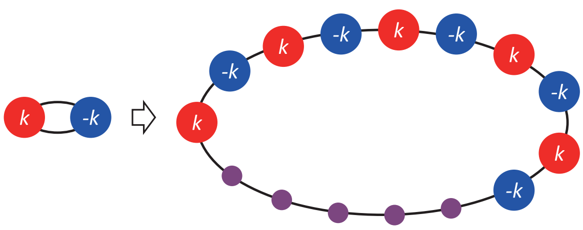
Instanton effects in orbifold ABJM theory

Masazumi Honda[a,b] **and Sanefumi Moriyama**[c,d,e]

[a] *Harish-Chandra Research Institute, Chhatnag Road, Jhusi, Allahabad 211019, India*
[b] *High Energy Accelerator Research Organization (KEK), Tsukuba, Ibaraki 305-0801, Japan*
[c] *Graduate School of Mathematics, Nagoya University, Nagoya 464-8602, Japan*
[d] *Kobayashi Maskawa Institute, Nagoya University, Nagoya 464-8602, Japan*
[e] *Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan*

E-mail: masazumihonda@hri.res.in, moriyama@math.nagoya-u.ac.jp

Abstract: We study the partition function of the orbifold ABJM theory on S^3, which is the $\mathcal{N} = 4$ necklace quiver Chern-Simons-matter theory with alternating levels, in the Fermi gas formalism. We find that the grand potential of the orbifold ABJM theory is expressed explicitly in terms of that of the ABJM theory. As shown previously, the ABJM grand potential consists of the naive but primary non-oscillatory term and the subsidiary infinitely-replicated oscillatory terms. We find that the subsidiary oscillatory terms of the ABJM theory actually give a non-oscillatory primary term of the orbifold ABJM theory. Also, interestingly, the perturbative part in the ABJM theory results in a novel instanton contribution in the orbifold theory. We also present a physical interpretation for the non-perturbative instanton effects.



Keywords: Supersymmetric gauge theory, Matrix Models, AdS-CFT Correspondence, M-Theory

ArXiv ePrint: 1404.0676

doi:10.1007/JHEP08(2014)091

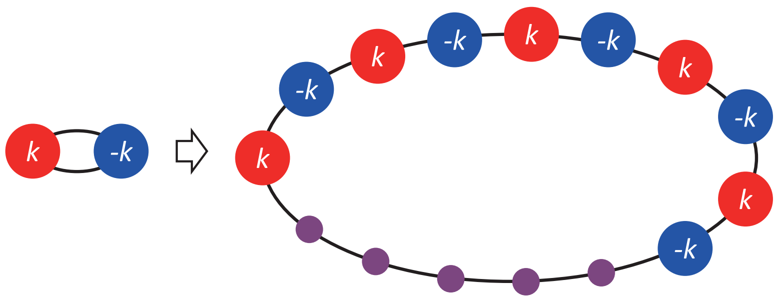
JHEP08(2014)091

Contents

1 Introduction and summary

Recently there has been a breakthrough in understanding M2-branes in M-theory. It was found by Aharony, Bergman, Jafferis and Maldacena [1] that the worldvolume theory of N multiple M2-branes on $\mathbb{C}^4/\mathbb{Z}_k$ is described by the $\mathcal{N}=6$ Chern-Simons-matter theory with gauge group $\mathrm{U}(N)\times\mathrm{U}(N)$ and levels k and $-k$. Prior to this important discovery, the program of finding the worldvolume theory of multiple M2-branes by supersymmetrizing the three-dimensional Chern-Simons theory dates back to the pioneering studies in [2]. Up to $\mathcal{N}=3$, supersymmetric Chern-Simons-matter theories were constructed for any gauge group and any representation [3–5]. For $\mathcal{N}=4$, one of the interesting realizations is the quiver gauge theory with gauge group being $[\mathrm{U}(N)\times\mathrm{U}(N)]^r$ ($r\in\mathbb{N}$) and levels k and $-k$ appearing alternatively [6, 7]. Especially, if we consider the case of $r=1$, the supersymmetry is enhanced to $\mathcal{N}=6$ and this is nothing but the ABJM theory. The quiver diagram of the ABJM theory is the Dynkin diagram of the affine Lie algebra $\widehat{A}_1$, while that of the $\mathcal{N}=4$ theory is the Dynkin diagram of $\widehat{A}_{2r-1}$ (see figure 1). Also, the gravity dual of the ABJM theory is $AdS_4\times S^7/\mathbb{Z}_k$, while that of the $\mathcal{N}=4$ theory is $AdS_4\times S^7/(\mathbb{Z}_r\times\mathbb{Z}_{kr})$. Since the gravity dual of this theory was identified to be the orbifold of the ABJM theory [8–11], let us call this theory orbifold ABJM theory.

For the ABJM case, the partition function and the vacuum expectation values of the supersymmetric Wilson loops were extensively studied in [12–30]. First, using the localization technique [31], the infinite-dimensional path integral used in defining the expectation



(a) (b)

Figure 1. The quiver diagrams of the ABJM theory (a) and the orbifold ABJM theory (b). The orbifold ABJM theory is the Chern-Simons-matter theory with gauge group $[\mathrm{U}(N) \times \mathrm{U}(N)]^r$ and alternating levels k and $-k$. Each node represents the $\mathrm{U}(N)$ vector multiplet, while each line represents the bi-fundamental hypermultiplet.

values of supersymmetric quantities in general $\mathcal{N} \geq 2$ supersymmetric gauge theories on S^3 is reduced to a finite-dimensional matrix integral [12, 13, 32–34]. After the standard matrix model analyses in the 't Hooft limit [12–17], the partition function for a general $\mathcal{N} \geq 3$ necklace quiver Chern-Simons-matter theory was rewritten into that of an ideal Fermi gas system [19], which is more suitable to access the M-theory regime (see also [18, 35]). This formalism further enables us to continue to study instanton effects in the ABJM theory [22–26, 28–30], where an infinite cancellation of divergences between worldsheet instantons and membrane instantons was found [24]. Finally it turned out [28] that the instanton effects in the ABJM theory are described by certain limits of the refined topological string on the dual geometry, local $\mathbb{P}^1 \times \mathbb{P}^1$.

The aim of this paper is to study how the interesting instanton calculus of the supersymmetric quantities in the ABJM theory is generalized to a larger class of theories. Particularly, as a first step, we shall study the partition function of the $\mathcal{N} = 4$ orbifold ABJM theory on S^3, which is expected to be the simplest one compared with other $\mathcal{N} = 3$ Chern-Simons-matter theories. For the studies in the M-theory limit, of other gauge groups or other quiver diagrams which are expected to have interesting M-theory interpretations, see e.g. [36–44] and see [45–48] for those in the 't Hooft limit.

Before explaining our work, let us briefly overview the ABJM partition function. After the study of the large N behavior in the 't Hooft limit in a seminal paper [14], it was found in [17] that all the perturbative corrections are summed up into the Airy function

$$Z_1(N) \sim C^{-1/3}\,\mathrm{Ai}\big[C^{-1/3}N\big] = \int_{-\infty i}^{\infty i} \frac{d\mu}{2\pi i} e^{\frac{C}{3}\mu^3 - \mu N}, \tag{1.1}$$

with a coefficient C. This integral expression is reminiscent of the statistical mechanics. Namely, if we consider the grand potential

$$e^{\widehat{J}_1(\mu)} = \sum_{N=0}^{\infty} e^{\mu N} Z_1(N), \tag{1.2}$$

JHEP08(2014)091

by regarding N, the rank of the gauge group, as the number of particles and introducing a chemical potential μ, we find that the inverse transformation given by

$$Z_1(N) = \int_{-\pi i}^{\pi i} \frac{d\mu}{2\pi i} e^{\widehat{J}_1(\mu)-\mu N}, \tag{1.3}$$

looks very similar to the expression of the Airy function (1.1). Hence, we are led to the grand canonical ensemble naturally.

However, note that there are also some discrepancies. First, the grand potential defined in (1.2) is invariant under the shift of μ by $2\pi i$, while the cubic polynomial in (1.1) is of course not. Secondly, the integration domain is $[-\pi i, \pi i)$ in the inverse transformation (1.3), while it is the whole imaginary axis for the Airy function (1.1).

These two discrepancies can be resolved simultaneously [24]. Namely, to restore the $2\pi i$ shift symmetry, let us consider a quantity $J_1(\mu)$ and express the total grand potential $\widehat{J}_1(\mu)$ by infinite replicas of it,

$$e^{\widehat{J}_1(\mu)} = \sum_{n=-\infty}^{\infty} e^{J_1(\mu+2\pi i n)}. \tag{1.4}$$

If we use this quantity $J_1(\mu)$, we can extend the integral domain to the whole imaginary axis by substituting (1.4) into (1.3) and connecting various intervals of integral domains $[-\pi i + 2\pi i n, \pi i + 2\pi i n)$ with different n,

$$Z_1(N) = \int_{-\infty i}^{\infty i} \frac{d\mu}{2\pi i} e^{J_1(\mu)-\mu N}. \tag{1.5}$$

Note that this argument should be handled with care. We have implicitly assumed the analyticity of $J_1(\mu)$, though generally it may contain branch cuts,[1] which invalidate the above argument of substituting complex-valued chemical potentials into $J_1(\mu)$ in (1.4). This assumption is supported by our numerical studies later in section 6.

If we expand the total grand potential into[2]

$$\widehat{J}_1(\mu) = J_1(\mu) + \widetilde{J}_1(\mu), \tag{1.6}$$

we find that the $n \neq 0$ terms

$$\widetilde{J}_1(\mu) = \log\left[1 + \frac{1}{e^{J_1(\mu)}} \sum_{n\neq 0} e^{J_1(\mu+2\pi i n)}\right], \tag{1.7}$$

give an oscillatory behavior [24]. On the contrary, the $n = 0$ term $J_1(\mu)$ does not contain any oscillations depending on μ. Even though the integration domain in (1.5) is different from the original inverse transformation (1.3), the extra oscillation $\widetilde{J}_1(\mu)$ in (1.7) is completely determined by the quantity $J_1(\mu)$. Hence, we consider the $n = 0$ term $J_1(\mu)$ as a

[1] Indeed the result of small k expansion [19] contains a branch cut in the e^μ-plane at $(-\infty, -4]$.

[2] For suitability and simplicity, we have changed the notation slightly from section 3 in [24]: $[\widehat{J}(\mu)]^{\text{here}} = [J(\mu)]^{\text{HMO}}$, $[J(\mu)]^{\text{here}} = [J^{\text{naive}}(\mu)]^{\text{HMO}}$, $[\widetilde{J}(\mu)]^{\text{here}} = [J^{\text{osc}}(\mu)]^{\text{HMO}}$.

JHEP08(2014)091

naive but primary term while regard the extra $n \neq 0$ terms contributing to the oscillatory behavior $\widetilde{J}_1(\mu)$ as subsidiary. Note that if we just wanted to restore the shift symmetry $\mu \sim \mu + 2\pi i$, at the first sight, we could define the naive term with $n \neq 0$ as well, though this is not the case. The $n = 0$ term is characterized by the property that it does not contain any complex phases depending on μ.

The relation to the statistical mechanics was observed and fully incardinated in [19] by identifying the partition function of the ABJM theory as that of an ideal Fermi gas system and expressing the grand potential in terms of the Fredholm determinant using the density matrix ρ_1 of the Fermi gas system,

$$e^{\widehat{J}_1(\mu)} = \det(1 + e^{\mu}\rho_1). \tag{1.8}$$

Then, combining the results from the topological string theory [13, 14, 16, 17, 24, 28], the 't Hooft expansion [14, 16, 17], the WKB expansion of the Fermi gas system [19, 25], numerical studies [20, 24, 26] and the infinite divergence cancellation between the worldsheet instantons and the membrane instantons [24–26, 28], we finally end up with an exact expression, where $J_1(\mu)$ consists of the perturbative part $J_1^{\mathrm{pert}}(\mu)$ and the non-perturbative part $J_1^{\mathrm{np}}(\mu)$,

$$J_1(\mu) = J_1^{\mathrm{pert}}(\mu) + J_1^{\mathrm{np}}(\mu), \tag{1.9}$$

where each part is given explicitly by

$$J_1^{\mathrm{pert}}(\mu) = \frac{C}{3}\mu^3 + B\mu + A, \quad J_1^{\mathrm{np}}(\mu) = \sum_{\substack{\ell,m=0 \\ (\ell,m)\neq(0,0)}}^{\infty} f_{\ell,m}(\mu) \exp\left[-\left(2\ell + \frac{4m}{k}\right)\mu\right]. \tag{1.10}$$

The perturbative coefficients C, B and A are constants depending only on k, while the non-perturbative coefficient $f_{\ell,m}(\mu)$ is a polynomial of μ, whose explicit form can be found, for example, in [28]. The exponentially suppressed corrections $e^{-2\mu}$ and $e^{-\frac{4\mu}{k}}$ correspond to D2-branes wrapping a Lagrangian submanifold $\mathbb{RP}^3$ in $\mathbb{CP}^3$ and fundamental strings wrapping a holomorphic cycle $\mathbb{CP}^1$ in it, respectively [16, 19, 49], where $\mathbb{CP}^3$ is obtained from $S^7/\mathbb{Z}_k$ in the type IIA string theory limit $k \to \infty$.

Let us summarize our main result in this paper. Here we study the partition function $Z_r(N)$ of the orbifold ABJM theory on S^3, which, as explained above, is realized by the necklace quiver $[\mathrm{U}(N) \times \mathrm{U}(N)]^r$ with the alternating levels k and $-k$ and is expected to be dual to M-theory on $AdS_4 \times S^7/(\mathbb{Z}_r \times \mathbb{Z}_{kr})$. Similarly to the ABJM matrix model, let us introduce the grand potential

$$e^{\widehat{J}_r(\mu)} = \sum_{N=0}^{\infty} e^{\mu N} Z_r(N). \tag{1.11}$$

Again, in order to preserve the $2\pi i$ periodicity of μ, we expect the grand potential $\widehat{J}_r(\mu)$ to be expressed as

$$e^{\widehat{J}_r(\mu)} = \sum_{n=-\infty}^{\infty} e^{J_r(\mu + 2\pi i n)}, \tag{1.12}$$

and we shall concentrate on the primary non-oscillatory term $J_r(\mu)$. As in the ABJM case, we can rewrite this as the Fredholm determinant [19],

$$e^{\widehat{J}_r(\mu)} = \det(1 + e^{\mu}\rho_r), \tag{1.13}$$

with a density matrix ρ_r. As we will see later in section 3, one can show that ρ_r is given by the r-th power of the ABJM density matrix ρ_1, namely,

$$\rho_r = \rho_1^r. \tag{1.14}$$

This structure leads us to the following decomposition

$$e^{\widehat{J}_r(\mu)} = \prod_{j=-\frac{r-1}{2}}^{\frac{r-1}{2}} e^{\widehat{J}_1\left(\frac{\mu+2\pi ij}{r}\right)}, \tag{1.15}$$

where the index j in the product runs with step 1. Thus, in terms of the primary grand potentials in (1.4) and (1.12), the relation (1.15) is translated into

$$\sum_{n=-\infty}^{\infty} e^{J_r(\mu+2\pi in)} = \prod_{j=-\frac{r-1}{2}}^{\frac{r-1}{2}} \sum_{n_j=-\infty}^{\infty} e^{J_1\left(\frac{\mu+2\pi ij}{r}+2\pi in_j\right)}. \tag{1.16}$$

We would like to extract $J_r(\mu)$ out of this expression. In section 4, we will prove that the explicit form of $J_r(\mu)$ is given by

$$e^{J_r(\mu)} = \sum_{\sum_j n_j=0} \prod_{j=-\frac{r-1}{2}}^{\frac{r-1}{2}} e^{J_1\left(\frac{\mu+2\pi ij}{r}+2\pi in_j\right)}, \tag{1.17}$$

where the summation symbol denotes that the summation of r variables $\{n_j\}_{j=-\frac{r-1}{2}}^{\frac{r-1}{2}}$ over all integers is performed with the constraint $\sum_j n_j = 0$. In the original ABJM theory, we have noticed that the $n \neq 0$ replicas play only subsidiary roles. This identity shows that the subsidiary oscillatory replicas of the ABJM grand potential also contribute to the non-oscillatory primary term of $\widehat{J}_r(\mu)$, as long as the combination of the replicas satisfies the condition $\sum_j n_j = 0$.

Let us draw the physical implications of this result in section 5. After substituting the ABJM grand potential (1.10) into the grand potential of the orbifold theory (1.17), we find that the perturbative part of the grand potential $J_r(\mu)$ is given by[3]

$$J_r^{\text{pert}}(\mu) = \frac{C}{3r^2}\mu^3 + \left(B - \frac{\pi^2 C(r^2-1)}{3r^2}\right)\mu + rA, \tag{1.18}$$

while for the non-perturbative corrections the grand potential $J_r(\mu)$ contains the following three types of non-perturbative instanton effects,

$$\exp\left(-\frac{2\mu}{r}\right), \quad \exp\left(-\frac{4\mu}{kr}\right), \quad \exp\left(-\frac{4\mu}{kr^2}\right). \tag{1.19}$$

[3]It was found in [19] that the grand potential of a general $\mathcal{N} \geq 3$ necklace quiver Chern-Simons-matter theory is given by a cubic polynomial. Our result (1.18) agrees with this general claim.

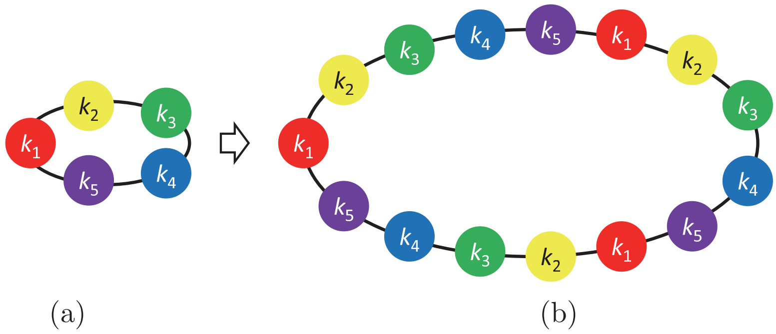


Figure 2. A general quiver Chern-Simons-matter theory (b) constructed repetitively from a more fundamental one (a) with levels satisfying $\sum_i k_i = 0$.

It is surprising to find that the last non-perturbative term in (1.19) originally comes from the perturbative part,

$$\exp\Bigl(-\frac{2\pi^2 C\mu}{r^2}\Bigr), \tag{1.20}$$

which reduces to the last one in (1.19) after we plug in the perturbative coefficient $C = 2/(\pi^2 k)$ for the ABJM matrix model.

Let us note that, in the language of the canonical partition function, the perturbative part is given by

$$Z_r^{\mathrm{pert}}(N) = \Bigl(\frac{C}{r^2}\Bigr)^{-1/3} e^{rA}\,\mathrm{Ai}\Bigl[\Bigl(\frac{C}{r^2}\Bigr)^{-1/3}\Bigl(N - B + \frac{\pi^2 C(r^2-1)}{3r^2}\Bigr)\Bigr], \tag{1.21}$$

while the non-perturbative effects are described by

$$\exp\bigl(-\pi\sqrt{2kN}\bigr), \quad \exp\Bigl(-2\pi\sqrt{\frac{2N}{k}}\Bigr), \quad \exp\Bigl(-\frac{2\pi}{r}\sqrt{\frac{2N}{k}}\Bigr), \tag{1.22}$$

in the large N limit. As we will see later, both the perturbative part and the non-perturbative effects match well with the gravity dual. Especially, these non-perturbative instanton corrections correspond to D2-branes wrapping $\mathbb{RP}^3/\mathbb{Z}_r$, fundamental strings wrapping $\mathbb{CP}^1$ and $\mathbb{CP}^1/\mathbb{Z}_r$, respectively. Note that the last instanton effect arises from a new cycle compared with the original ABJM theory.

Although we start with the ABJM theory and consider its orbifold theory, we stress that our computation is applicable to a more general setup. Namely, even if we start with a more general $\mathcal{N} = 3$ necklace quiver Chern-Simons-matter theory having the same expression of the grand potential (1.10) with different coefficients and consider its cousin with repetitive levels, the expression for the grand potential (1.17), the perturbative sum (1.18) and the consequent instanton effect (1.20) are still valid. For example, if we have the grand potential for the necklace quiver $\widehat{A}_2$ with levels (k_1, k_2, k_3) $(k_1 + k_2 + k_3 = 0)$, we can easily find the grand potential for the necklace quiver $\widehat{A}_{3r-1}$ with the repetitive levels $(k_1, k_2, k_3, k_1, k_2, \cdots, k_3)$. See figure 2 for another case of $r = 3$, constructed out of the $\widehat{A}_4$ quiver Chern-Simons-matter theory.

In the following section, we briefly review the gravity dual and discuss various possible instanton effects from it. Then, we explain the relation (1.14) of the density matrices between the orbifold ABJM theory and the original theory in section 3, and show the relation (1.17) between the grand potentials in section 4. In section 5 we proceed to study the physical implications. After presenting a few examples of our result in section 6, we conclude with some further directions in the final section.

2 Gravity dual

Before starting our study of the partition function of the $\mathcal{N}=4$ necklace quiver Chern-Simons-matter theory with the alternating levels, we shall first review its gravity dual [1] in this section. We also argue that we expect three types of instanton effects from the gravity dual. It has been expected that this $\mathcal{N}=4$ theory describes the low-energy effective theory of M2-branes on $\mathbb{C}^4 \ni (z_1, z_2, z_3, z_4)$ divided by the following three orbifold actions [8–10],

$$\begin{aligned} \phi_A &: (z_1, z_2, z_3, z_4) \mapsto (e^{\frac{2\pi i}{r}} z_1, e^{-\frac{2\pi i}{r}} z_2, z_3, z_4), \\ \phi_B &: (z_1, z_2, z_3, z_4) \mapsto (z_1, z_2, e^{\frac{2\pi i}{r}} z_3, e^{-\frac{2\pi i}{r}} z_4), \\ \phi_C &: (z_1, z_2, z_3, z_4) \mapsto (e^{\frac{2\pi i}{kr}} z_1, e^{-\frac{2\pi i}{kr}} z_2, e^{\frac{2\pi i}{kr}} z_3, e^{-\frac{2\pi i}{kr}} z_4). \end{aligned} \tag{2.1}$$

Since ϕ_B is not an independent action due to the relation $\phi_A \phi_B = \phi_C^k$, the moduli space is $\mathbb{C}^4/(\mathbb{Z}_r \times \mathbb{Z}_{kr})$. Thus the gravity dual background of this theory is described by

$$ds_{11}^2 = \frac{R^2}{4} ds_{AdS_4}^2 + R^2 ds_{S^7/(\mathbb{Z}_r \times \mathbb{Z}_{kr})}^2, \tag{2.2}$$

where the radius R is given by

$$R = (32\pi^2 kr^2 N)^{\frac{1}{6}} l_p, \tag{2.3}$$

with the Planck length l_p. If we identify the direction of M-theory circle φ with that orbifolded by $\mathbb{Z}_{kr}$,

$$ds_{S^7/(\mathbb{Z}_r \times \mathbb{Z}_{kr})}^2 = \frac{1}{(kr)^2} (d\varphi + \cdots)^2 + ds_{\mathbb{CP}^3/\mathbb{Z}_r}^2, \tag{2.4}$$

the radius of the M-theory circle in the unit of the Planck length l_p, the radius of the covering space $\mathbb{CP}^3$ in the unit of the type IIA string length l_s and the string coupling are given respectively by

$$\frac{R_{11}}{l_p} = \frac{R}{krl_p} = \left(\frac{32\pi^2 N}{k^5 r^4}\right)^{\frac{1}{6}}, \quad \frac{R_{\mathbb{CP}^3}}{l_s} = \frac{R}{l_s} = \left(\frac{32\pi^2 N}{k}\right)^{\frac{1}{4}}, \quad g_s^2 = \left(\frac{32\pi^2 N}{k^5 r^4}\right)^{\frac{1}{2}}. \tag{2.5}$$

Therefore, the eleven-dimensional supergravity picture is valid for $k^5 r^4 \ll N$, while the type IIA supergravity is good for $k \ll N \ll k^5 r^4$.

In the eleven-dimensional supergravity on $AdS_4 \times X^7$ with the boundary S^3, the free energy obeys the famous $N^{3/2}$-law [50] given by (see e.g. [51] for a derivation)

$$\log Z_{\text{supergravity}} = -\sqrt{\frac{2\pi^6}{27\,\text{vol}(X^7)}} N^{3/2}, \tag{2.6}$$

which relates the free energy of the current theory to that of the ABJM theory

$$\log Z_{S^7/(\mathbb{Z}_r\times\mathbb{Z}_{kr})} = r\log Z_{S^7/\mathbb{Z}_k}. \tag{2.7}$$

There are several non-trivial cycles in this geometry $\mathbb{CP}^3/\mathbb{Z}_r$. If we consider the subspace $z_1 = z_2 = 0$, we find that the only independent action is ϕ_C. Since the direction of $\mathbb{Z}_{kr}$ is identified as the M-theory circle, we do not have any further orbifold actions. Hence, the subspace is a holomorphic cycle $\mathbb{CP}^1$. If we consider the subspace $z_1 = z_3 = 0$, both of the actions ϕ_A and ϕ_C remain and we should consider the subspace as $\mathbb{CP}^1/\mathbb{Z}_r$. Similarly, we have a Lagrangian submanifold $\mathbb{RP}^3/\mathbb{Z}_r$.

Since D2-branes can wrap $\mathbb{RP}^3/\mathbb{Z}_r$, we expect the D2-brane instanton effects,

$$\exp\Bigl(-T_{\mathrm{D2}}\mathrm{Vol}(\mathbb{RP}^3/\mathbb{Z}_r)\Bigr) = \exp\biggl(-\frac{1}{(2\pi)^2 l_s^3 g_s}\cdot\frac{\pi^2 R^3_{\mathbb{CP}^3}}{r}\biggr) = \exp\Bigl(-\pi\sqrt{2kN}\Bigr). \tag{2.8}$$

Also, fundamental strings can wrap $\mathbb{CP}^1$ or $\mathbb{CP}^1/\mathbb{Z}_r$ in this geometry and we expect two kinds of worldsheet instanton effects given by

$$\begin{aligned}\exp\Bigl(-T_{\mathrm{F1}}\mathrm{Vol}(\mathbb{CP}^1)\Bigr) &= \exp\biggl(-\frac{1}{2\pi l_s^2}\cdot\pi R^2_{\mathbb{CP}^3}\biggr) = \exp\biggl(-2\pi\sqrt{\frac{2N}{k}}\biggr),\\ \exp\Bigl(-T_{\mathrm{F1}}\mathrm{Vol}(\mathbb{CP}^1/\mathbb{Z}_r)\Bigr) &= \exp\biggl(-\frac{2\pi}{r}\sqrt{\frac{2N}{k}}\biggr).\end{aligned} \tag{2.9}$$

In section 5, we shall reproduce these expected results of the perturbative part (2.7) and the non-perturbative part (2.8) and (2.9). Before that in the subsequent sections, we shall first justify (1.14) and (1.17). Since the gravity dual of this $\mathcal{N} = 4$ theory is the orbifold of the ABJM theory, hereafter we shall call the corresponding quiver Chern-Simons-matter theory the orbifold ABJM theory.

3 Orbifold ABJM theory as a Fermi gas

In this section we shall show the relation between the density matrix of the orbifold ABJM theory and that of the original theory (1.14), which enables us to express the total grand potential of the orbifold ABJM theory in terms of that of the original theory (1.15). By using the localization method [32–34], the partition function of the orbifold ABJM theory on S^3 is given by ($\mu_i^{(r+1)} = \mu_i^{(1)}$)

$$Z_r(N) = \frac{1}{(N!)^{2r}}\int\prod_{a=1}^{r}\prod_{i=1}^{N}D\mu_i^{(a)}D\nu_i^{(a)}\prod_{a=1}^{r}\frac{\prod_{i<j}\bigl(2\sinh\frac{\mu_i^{(a)}-\mu_j^{(a)}}{2}\bigr)^2\prod_{i<j}\bigl(2\sinh\frac{\nu_i^{(a)}-\nu_j^{(a)}}{2}\bigr)^2}{\prod_{i,j}2\cosh\frac{\mu_i^{(a+1)}-\nu_j^{(a)}}{2}\cdot 2\cosh\frac{\nu_j^{(a)}-\mu_i^{(a)}}{2}}, \tag{3.1}$$

where we have introduced the same notation as in [29, 52],

$$D\mu_i^{(a)} = \frac{d\mu_i^{(a)}}{2\pi}e^{\frac{ik}{4\pi}(\mu_i^{(a)})^2},\quad D\nu_i^{(a)} = \frac{d\nu_i^{(a)}}{2\pi}e^{-\frac{ik}{4\pi}(\nu_i^{(a)})^2}. \tag{3.2}$$

Let us rewrite this into the Fermi gas formalism as in the ABJM case [19, 29]. Using the Cauchy determinant formula

$$\frac{\prod_{i<j} 2\sinh\frac{x_i-x_j}{2}\cdot 2\sinh\frac{y_i-y_j}{2}}{\prod_{i,j} 2\cosh\frac{x_i-y_j}{2}} = \det_{i,j}\frac{1}{2\cosh\frac{x_i-y_j}{2}}, \tag{3.3}$$

we find that the partition function can be rewritten into

$$Z_r(N) = \frac{1}{(N!)^{2r}}\int\prod_{a=1}^{r}\prod_{i=1}^{N} D\mu_i^{(a)} D\nu_i^{(a)}\prod_{a=1}^{r}\det_{i,j}\frac{1}{2\cosh\frac{\mu_i^{(a+1)}-\nu_j^{(a)}}{2}}\det_{j,i}\frac{1}{2\cosh\frac{\nu_j^{(a)}-\mu_i^{(a)}}{2}}. \tag{3.4}$$

If we expand the determinants and trivialize the permutations except the one over $\mu_i^{(1)}$, we arrive at the following ideal Fermi gas representation

$$Z_r(N) = \frac{1}{N!}\int\prod_{i=1}^{N} D\mu_i^{(1)}\sum_{\sigma\in S_N}(-1)^\sigma\prod_i \rho_r(\mu_{\sigma(i)}^{(1)},\mu_i^{(1)}), \tag{3.5}$$

where ρ_r is the density matrix defined by

$$\rho_r(\mu_j^{(1)},\mu_i^{(1)}) = \int\prod_{a=2}^{r} D\mu^{(a)}\prod_{a=1}^{r} D\nu^{(a)}\prod_{a=1}^{r}\frac{1}{2\cosh\frac{\mu^{(a+1)}-\nu^{(a)}}{2}}\frac{1}{2\cosh\frac{\nu^{(a)}-\mu^{(a)}}{2}}\bigg|_{\substack{\mu^{(r+1)}\to\mu_j^{(1)}\\ \mu^{(1)}\to\mu_i^{(1)}}}. \tag{3.6}$$

Then, it is more convenient to move to the grand canonical formalism,

$$e^{\widehat{J}_r(\mu)} = \sum_{N=0}^{\infty} Z_r(N)e^{\mu N} = \det(1+e^\mu\rho_r). \tag{3.7}$$

In the ABJM case ($r=1$), the density matrix (3.6) is reduced to

$$\rho_1(\mu_j,\mu_i) = \int D\nu\frac{1}{2\cosh\frac{\mu_j-\nu}{2}}\frac{1}{2\cosh\frac{\nu-\mu_i}{2}}. \tag{3.8}$$

If we define the matrix multiplication among ρ_1's with $D\mu$ (3.2), then we easily see that two density matrices are related by

$$\rho_r = \rho_1^r. \tag{3.9}$$

Hence in the present case, the total grand potential $\widehat{J}_r(\mu)$ can be rewritten as

$$e^{\widehat{J}_r(\mu)} = \det(1+e^\mu\rho_1^r) = \prod_{j=-\frac{r-1}{2}}^{\frac{r-1}{2}}\det(1+e^{\frac{2\pi ij}{r}}e^{\frac{\mu}{r}}\rho_1) = \prod_{j=-\frac{r-1}{2}}^{\frac{r-1}{2}} e^{\widehat{J}_1(\frac{\mu+2\pi ij}{r})}. \tag{3.10}$$

4 Derivation of the grand potential

In this section we shall justify our expression of the grand potential (1.17). Namely, we shall prove that

- after summing over the replicas, the total grand potential of the orbifold theory reproduces the product of the total grand potential of the original theory (3.10), i.e., the grand potential of the orbifold theory (1.17) satisfies the relation (1.16), and
- the $\sum_j n_j = 0$ term does not contain oscillatory behaviors.

Namely, besides the condition (1.16), as we have noted below (1.7), the property that characterizes the naive primary term is that it does not contain any oscillatory behavior in μ. Hence, we should also confirm this property.

We shall prove these two facts in the following two subsections.

4.1 Summing over replicas

Let us show that (1.17) satisfies the relation (1.16), namely,

$$\sum_{n=-\infty}^{\infty}\left[\sum_{\sum_j n_j=0}\prod_{j=-\frac{r-1}{2}}^{\frac{r-1}{2}} e^{J_1(\frac{\mu+2\pi ij}{r}+2\pi in_j)}\Big|_{\mu\to\mu+2\pi in}\right]=\prod_{j=-\frac{r-1}{2}}^{\frac{r-1}{2}}\sum_{n_j=-\infty}^{\infty} e^{J_1(\frac{\mu+2\pi ij}{r}+2\pi in_j)}. \tag{4.1}$$

For this purpose first let us redefine the variables in the summation or the product as

$$j' \equiv j+n \pmod r, \quad n'_j = n_j + \frac{j+n-j'}{r}, \tag{4.2}$$

such that j' runs over the same range as j, namely from $-(r-1)/2$ to $(r-1)/2$ with step 1. Then, we find that the argument of J_1 becomes

$$J_1\left(\frac{\mu+2\pi ij}{r}+2\pi in_j\right)\Big|_{\mu\to\mu+2\pi in} = J_1\left(\frac{\mu+2\pi ij'}{r}+2\pi in'_j\right). \tag{4.3}$$

Since we have only shifted j by n in (4.2), it is clear that j' runs over the same values as j exactly once,

$$\prod_{j=-\frac{r-1}{2}}^{\frac{r-1}{2}}\cdots = \prod_{j'=-\frac{r-1}{2}}^{\frac{r-1}{2}}\cdots . \tag{4.4}$$

Note also from (4.2) that n'_j are all integers and the constraint $\sum_j n_j = 0$ is translated into $\sum n'_j = n$. Hence the constraint in the summation is lifted,

$$\sum_{n=-\infty}^{\infty}\sum_{\sum_j n_j=0}\cdots = \sum_{n=-\infty}^{\infty}\sum_{\sum_j n'_j=n}\cdots = \sum_{n'_j=-\infty}^{\infty}\cdots . \tag{4.5}$$

After removing the primes, this is nothing but the right-hand-side of (4.1).

4.2 No oscillations

Next let us check that $J_r(\mu)$ does not give any oscillatory behavior. Namely, we see that there is no imaginary μ dependence in the exponents.

Let us first rewrite (1.17) as

$$J_r(\mu)=\sum_{j=-\frac{r-1}{2}}^{\frac{r-1}{2}}J_1\left(\frac{\mu+2\pi ij}{r}\right)+\log\left[1+\sum_{\substack{\sum_j n_j=0\\(\exists j)(n_j\neq 0)}}\prod_{j=-\frac{r-1}{2}}^{\frac{r-1}{2}}\frac{e^{J_1(\frac{\mu+2\pi ij}{r}+2\pi in_j)}}{e^{J_1(\frac{\mu+2\pi ij}{r})}}\right]. \tag{4.6}$$

The first term coming from the sector with $n_j=0$ for all j apparently contains no oscillatory terms. For the exponents in the parenthesis coming from the sector with $n_j\neq 0$ for some j, it is useful to introduce a polynomial $g_{\ell,m}(\mu)$ determined explicitly by $f_{\ell,m}(\mu)$ in (1.10) and express the grand potential as

$$e^{J_1(\mu)}=e^{J_1^{\rm pert}(\mu)}\left[1+\sum_{\substack{\ell,m=0\\(\ell,m)\neq(0,0)}}^{\infty}g_{\ell,m}(\mu)e^{-(2\ell+\frac{4m}{k})\mu}\right]. \tag{4.7}$$

Then, the exponents become

$$\prod_{j=-\frac{r-1}{2}}^{\frac{r-1}{2}}\frac{e^{J_1(\frac{\mu+2\pi ij}{r}+2\pi in_j)}}{e^{J_1(\frac{\mu+2\pi ij}{r})}}=\left[\prod_{j=-\frac{r-1}{2}}^{\frac{r-1}{2}}\frac{e^{J_1^{\rm pert}(\frac{\mu+2\pi ij}{r}+2\pi in_j)}}{e^{J_1^{\rm pert}(\frac{\mu+2\pi ij}{r})}}\right]\left[1+\sum_{\substack{\ell,m=0\\(\ell,m)\neq(0,0)}}^{\infty}h_{\ell,m}(\mu;\{n_j\})e^{-(2\ell+\frac{4m}{k})\frac{\mu}{r}}\right], \tag{4.8}$$

for a polynomial $h_{\ell,m}(\mu;\{n_j\})$, which also depends on n_j. If we substitute the perturbative part (1.10) into the first factor, we find

$$\begin{aligned}&J_1^{\rm pert}\left(\frac{\mu+2\pi ij}{r}+2\pi in_j\right)-J_1^{\rm pert}\left(\frac{\mu+2\pi ij}{r}\right)\\&=\frac{2\pi iCn_j}{r^2}\mu^2-\frac{(2\pi)^2C}{r}n_j\left(n_j+\frac{2j}{r}\right)\mu+2\pi in_j\left(B-\frac{(2\pi)^2C}{3}\left(n_j^2+3\frac{n_jj}{r}+3\frac{j^2}{r^2}\right)\right).\end{aligned} \tag{4.9}$$

After summing over j and using the condition $\sum_j n_j=0$, we find the imaginary quadratic term vanishing, while the linear term is real. Hence, we find that there is no oscillatory contribution.

5 Physical implications

In the previous section, we have justified our proposal (1.17). In the argument of no oscillations, we have fully utilized the relation (4.9). Here we shall see that actually this relation has further physical implications on the perturbative part and the non-perturbative instanton corrections.

5.1 Perturbative part

Let us look more carefully into the relation (4.9) in the previous section. Since $2j$ satisfies $-(r-1)\le 2j\le r-1$, it is not difficult to find that the coefficient of the linear term satisfies

$$n_j\left(n_j+\frac{2j}{r}\right)\ge 0, \tag{5.1}$$

where the equality holds only when $n_j = 0$. Therefore, sectors with $n_j \neq 0$ for some j always contain an exponentially decaying factor and do not contribute to the perturbative part of the orbifold theory. Namely, the perturbative part has only contributions from the sector with $n_j = 0$ for all j,

$$J_r^{\rm pert}(\mu) = \sum_{j=-\frac{r-1}{2}}^{\frac{r-1}{2}} J_1^{\rm pert}\left(\frac{\mu + 2\pi i j}{r}\right). \tag{5.2}$$

After plugging in the expression (1.10), we arrive at the expressions for $J_r^{\rm pert}(\mu)$ (1.18) and $Z_r^{\rm pert}(N)$ (1.21). In the large N limit with k and r fixed, the partition function $Z_r^{\rm pert}(N)$ is expanded as

$$\log Z_r^{\rm pert}(N) = -\frac{2}{3} r C^{-1/2} N^{3/2} + r C^{-1/2}\left(B - \frac{\pi^2 C(r^2-1)}{3r^2}\right) N^{1/2} - \frac{1}{4}\log N + \mathcal{O}(1). \tag{5.3}$$

The first term reproduces the result (2.7) of the classical eleven-dimensional supergravity. The logarithmic behavior in the third term also agrees with the 1-loop supergravity analysis in [53].

5.2 Instanton corrections

As we have seen in section 2, we expect the three kinds of the instanton effects (2.8) and (2.9) from the gravity dual. Here we discuss that these instanton effects can be naturally understood as exponentially suppressed corrections in the grand potential $J_r(\mu)$.

First it is obvious that we have the following two corrections

$$\exp\left(-\frac{2\mu}{r}\right), \quad \exp\left(-\frac{4\mu}{kr}\right), \tag{5.4}$$

which come from substituting $\mu/r + \cdots$ into $J_1(\mu)$ as in (4.6). Besides it, from the linear term in (4.9) we find another kind of exponentially suppressed correction

$$\exp\left(-n\frac{2\pi^2 C\mu}{r^2}\right), \tag{5.5}$$

with the positive integer n given by

$$n = 2\sum_{j=-\frac{r-1}{2}}^{\frac{r-1}{2}} n_j(n_j r + 2j). \tag{5.6}$$

After plugging in the ABJM value $C = 2/\pi^2 k$ and setting the instanton number n to be 1, this becomes

$$\exp\left(-\frac{4\mu}{kr^2}\right). \tag{5.7}$$

Using (4.8), we find that the sectors with $n_j \neq 0$ for some j can be expressed as

$$\sum_{\substack{\sum_j n_j = 0 \\ (\exists j)(n_j \neq 0)}} \prod_{j=-\frac{r-1}{2}}^{\frac{r-1}{2}} \frac{e^{J_1(\frac{\mu+2\pi i j}{r} + 2\pi i n_j)}}{e^{J_1(\frac{\mu+2\pi i j}{r})}} = \sum_{\substack{\ell,m,n=0 \\ n\neq 0}}^{\infty} g_{\ell,m,n}(\mu) e^{-(\frac{2\ell}{r} + \frac{4m}{kr} + \frac{4n}{kr^2})\mu}, \tag{5.8}$$

generally with a polynomial $g_{\ell,m,n}(\mu)$. Thus, finally we conclude that the non-perturbative part of $J_r(\mu)$ takes the following form

$$J_r^{\rm np}(\mu) = \sum_{\substack{\ell,m,n=0\\(\ell,m,n)\neq(0,0,0)}}^{\infty} f_{\ell,m,n}(\mu)\exp\left[-\left(\frac{2\ell}{r}+\frac{4m}{kr}+\frac{4n}{kr^2}\right)\mu\right], \tag{5.9}$$

with a polynomial $f_{\ell,m,n}(\mu)$.

For comparison with our result in (2.8) and (2.9), let us return to the canonical partition function. In the large N limit, the integration over the chemical potential μ is dominated by the saddle point μ_* related to N by

$$\left.\frac{\partial J_r(\mu)}{\partial\mu}\right|_{\mu=\mu_*} = N, \quad \mu_* = \pi r\sqrt{\frac{kN}{2}}. \tag{5.10}$$

Hence, the exponentially suppressed corrections (5.4) and (5.7) on the gauge theory side are translated to (1.22)

$$\exp\left(-\pi\sqrt{2kN}\right), \quad \exp\left(-2\pi\sqrt{\frac{2N}{k}}\right), \quad \exp\left(-\frac{2\pi}{r}\sqrt{\frac{2N}{k}}\right),$$

which are exactly the same instanton effects found from the gravity side in (2.8) and (2.9). For finite N, these instanton effects contribute to $Z_r(N)$ by a superposition of

$$\mathrm{Ai}\left[\left(\frac{C}{r^2}\right)^{-1/3}\left(N-B+\frac{\pi^2C(r^2-1)}{3r^2}+\frac{2\ell}{r}+\frac{4m}{kr}+\frac{4n}{kr^2}\right)\right] \tag{5.11}$$

and their derivatives.

Note that unlike the original instanton effects, n obeys an interesting selection rule and only takes a discrete set of integers (5.6). For example, n always has to be an even number and the smallest instanton number is $n=4$ which comes from the combination of n_j with its non-zero components given by

$$n_{-\frac{r-1}{2}} = 1, \quad n_{\frac{r-1}{2}} = -1. \tag{5.12}$$

6 Examples

Let us use our formula (1.17) to write down several terms explicitly. Note that once we have a general formula, all these results can be obtained very easily from [24, 26]. One purpose of this section is to see a rough structure, which would be useful in the future study of more general $\mathcal{N}=3$ Chern-Simons-matter theories. In the following we will show the non-perturbative part of the grand potential $J_{r,k}^{\rm np}(\mu)$ for the case of $k=2$. The reason we choose this case is because of its novel behavior, though we can study for any r and any k.

$$\begin{aligned}J_{2,2}^{\rm np}(\mu) &= \left[-\frac{2\mu^2+2\mu+2}{\pi^2}+2\right]e^{-\mu} + \left[-\frac{52\mu^2+2\mu+9}{4\pi^2}+18\right]e^{-2\mu}\\ &+\left[-\frac{368\mu^2-304\mu/3+308/9}{3\pi^2}+\frac{608}{3}\right]e^{-3\mu}\end{aligned}$$

$$+\left[-\frac{2701\mu^2-13949\mu/12+11291/48}{2\pi^2}+2514\right]e^{-4\mu}+\mathcal{O}(e^{-5\mu}), \tag{6.1}$$

$$J^{\rm np}_{3,2}(\mu)=\left[-\frac{4(4\mu+3)}{3\sqrt{3}\pi}+\frac{16}{9}\right]e^{-\frac{2}{3}\mu}+e^{-\frac{8}{9}\mu}+\left[-\frac{104\mu+3}{3\sqrt{3}\pi}-\frac{104}{9}\right]e^{-\frac{4}{3}\mu}$$
$$+\left[\frac{4(4\mu+3)}{\sqrt{3}\pi}+\frac{16}{3}\right]e^{-\frac{14}{9}\mu}+\mathcal{O}(e^{-\frac{16}{9}\mu}), \tag{6.2}$$

$$J^{\rm np}_{4,2}(\mu)=\left[-\frac{2\mu+2}{\pi}+1\right]e^{-\frac{1}{2}\mu}+\left[\frac{13\mu^2+\mu+9}{2\pi^2}+\frac{8(\mu+1)}{\pi}-43\right]e^{-\mu}$$
$$+\left[\frac{32(\mu+1)^2}{\pi^2}+\frac{344\mu-376/3}{3\pi}-\frac{152}{3}\right]e^{-\frac{3}{2}\mu}$$
$$+\left[\frac{256(\mu+1)^3}{3\pi^3}-\frac{2957\mu^2-10877\mu/6+14363/12}{4\pi^2}-\frac{2720\mu+2528/3}{3\pi}+5754\right]e^{-2\mu}$$
$$+\mathcal{O}(e^{-\frac{5}{2}\mu}). \tag{6.3}$$

Note that for the case of $r=2$ the coefficients are very similar to the $(k,M)=(4,1)$ case of the ABJ theory [52], which is the $\mathcal{N}=6$ Chern-Simons-matter theory with gauge group $\mathrm{U}(N)\times\mathrm{U}(N+M)$ and levels k and $-k$ [54, 55]. For the case of $r=3$, the novel instanton term $e^{-\frac{16\mu}{kr^2}}$ discussed previously does not mix with other contributions. For the case of $r=4$, the coefficient polynomials of instantons are not necessarily quadratic any more.

Forgetting about the result (1.17), we have also checked (6.1)–(6.3) numerically by the same method used in [24]. Namely, using the exact values of the partition function $Z_r(N)$ in the orbifold ABJM theory obtained from those in the original theory with the relation (1.14), we try to find out the exact coefficients of the primary grand potential $J_r(\mu)$ numerically using the inverse transformation

$$Z_r(N)=\int_{-\infty i}^{\infty i}\frac{d\mu}{2\pi i}e^{J_r(\mu)-\mu N}. \tag{6.4}$$

We find that the coefficients match well with those in (6.1)–(6.3) within about 1% errors. Since we have less exact values (with the number divided by r) and relatively milder decaying factors in the instanton effects (especially for the $r=3$ case due to the intermediate new instanton effect (5.7)), it is difficult to check with high precision to high instanton corrections. However, we believe that at least it is safe to claim that we have detected the appearance of the new instanton numerically.[4]

7 Discussions

We have studied the partition function of the orbifold ABJM theory via the grand canonical formalism. We have found the explicit formula (1.17) for the grand potential in terms of the grand potential of the ABJM theory. It is surprising to find that the subsidiary oscillatory terms of the original theory lead to the primary non-oscillatory term of the orbifold theory and the perturbative part in the original theory results in the new non-perturbative instanton effect (1.20) in the orbifold theory. We have identified this instanton

[4]We are grateful to Y. Hatsuda for valuable discussions.

effect as the string worldsheet wrapping the cycle $\mathbb{CP}^1/\mathbb{Z}_r$ in $\mathbb{C}^4/(\mathbb{Z}_r \times \mathbb{Z}_{kr})$. Let us discuss some future directions.

As we have explained in section 1, there are several methods to study the matrix model. We would like to detect the new instanton effect from complementary methods. For example, since the density matrix of the orbifold theory is related directly to that of the ABJM theory (1.14), it is great to see how the grand potential is also derived from the exact quantization as in [30]. Also, we expect that the new instanton effects can be reproduced from the genus 1 analysis in the 't Hooft limit as in [14].

It has been found that the new instanton effect obeys the interesting selection rule (5.6). It is important to understand the origin of the selection rule both from the direct field theoretical study and from the gravity dual [8–10].

The result that the perturbative part in the original theory results in the non-perturbative effect in the orbifold theory (1.20) may look less surprising in view of the following interpretation from the gravity dual.[5] Since this non-perturbative effect corresponds to the string worldsheet wrapping the cycle $\mathbb{CP}^1/\mathbb{Z}_r$ which is not present in the original $AdS_4 \times S^7/\mathbb{Z}_k$ background, they can only have a perturbative origin. We hope to understand it from the field theoretical study as well.

In the context of resurgence theory, it was found [58–60] that the perturbative asymptotic expansion contains the information of the non-perturbative effects. In extended supersymmetric theories [61], even though there are no ambiguities in the asymptotic expansion, our result still shows that the perturbative coefficient appears in the non-perturbative effects (1.20). It would be great to understand more extensively also from the viewpoint of the resurgence theory.

We have often substituted the complex values of the chemical potential μ into the primary non-oscillatory grand potential of the ABJM theory as in (1.17). However, the original expression is only literally valid for large real chemical potential μ. This is allowed only when they are in the same Stokes sector. Our numerical check in section 6 is a crucial support for this assumption, though we hope to study the analytical structure of the grand potential carefully.

Our result on the orbifold ABJM theory (1.17) has been obtained based on the result of the ABJM theory. We hope that there is a more direct expression with topological invariants on a certain dual geometry of the orbifold ABJM theory.

As we have noted in section 1, our expression of the grand potential (1.17) including the perturbative sum (1.18) and the new instanton effect (1.20) are very general. Namely, they are applicable to any necklace quiver Chern-Simons-matter theory as long as the quiver diagram is repetitive as in figure 2. We hope to study more general theories using these results. For the ABJ theory [54, 55], in the Fermi gas formalism of [56, 57], the grand potential is expressed in a similar manner. This means that our result is also applicable to these cases, though this fact is not so obvious from the formalism of [52].

The meaning of various sectors with different combinations of n_j is not very clear. In some sense, the sector with $n_j = 0$ for all j is similar to the untwisted sector in the string

[5] We are grateful to the referee for valuable comments.

orbifold theory, while the others are similar to the twisted ones. We hope to clarify the physical meaning of these sectors.

Acknowledgments

We are grateful to S. Hirano, Y. Imamura, J. Källén, M. Mariño, K. Ohta, K. Okuyama, J. Park, M. Shigemori, A. Tanaka, M. Unsal, S. Yamaguchi and especially Y. Hatsuda for valuable discussions. M. H. would like to thank Kavli IPMU and S. M. would like to thank KEK for warm hospitality. We would also like to thank the participants of the JSPS/RFBR conference at Biwako for important feedbacks, where part of this work was presented by S. M.

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JHEP08(2014)091

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