

Non-equilibrium dynamics and AdS_4 Robinson-Trautman

Ioannis Bakas^a and Kostas Skenderis^b

^a*Department of Physics, School of Applied Mathematics and Physical Sciences
National Technical University, 15780 Athens, Greece*

^b*Mathematical Sciences and STAG Research Center, University of Southampton
Highfield, Southampton SO17 1BJ, United Kingdom*

E-mail: bakas@mail.ntua.gr, k.skenderis@soton.ac.uk

ABSTRACT: The Robinson-Trautman space-times provide solutions of Einstein's equations with negative cosmological constant, which settle to AdS_4 Schwarzschild black hole at late times. Via gauge/gravity duality they should describe a system out of equilibrium that evolves towards thermalization. We show that the area of the past apparent horizon of these space-times satisfies a generalized Penrose inequality and we formulate as well as provide evidence for a suitable generalization of Thorne's hoop conjecture. We also compute the holographic energy-momentum tensor and deduce its late time behavior. It turns out that the complete non-equilibrium process on the boundary is governed by Calabi's flow on S^2 . Upon linearization, only special modes that arise as supersymmetric zero energy states of an associated supersymmetric quantum mechanics problem contribute to the solution. We find that each pole of radiation has an effective viscosity given by the eigenvalues of the Laplace operator on S^2 and there is an apparent violation of the KSS bound on η/s for the low lying harmonics of large AdS_4 black holes. These modes, however, do not satisfy Dirichlet boundary conditions, they are out-going and they do not appear to have a Kruskal extension across the future horizon \mathcal{H}^+ .

KEYWORDS: Gauge-gravity correspondence, Black Holes

ARXIV EPRINT: [1404.4824](https://arxiv.org/abs/1404.4824)

Contents

1	Introduction	1
2	Spherical gravitational waves	4
2.1	Robinson-Trautman metrics	5
2.2	Linearization of the solutions	6
2.3	Calabi flow on S^2	10
2.4	Late-time behavior of solutions	11
3	Global aspects	13
3.1	Kruskal extension	13
3.2	Past apparent horizon	16
3.3	Penrose inequality	19
3.4	Thorne's hoop conjecture	21
4	Holographic aspects	24
4.1	Energy-momentum tensor	24
4.2	Linearization of T_{ab}	27
4.3	Effective viscosity	28
4.4	Late time expansion of T_{ab}	31
4.5	Entropy production	33
5	Conclusions and discussion	34
A	Derivation of Robinson-Trautman equation	36
B	Large and small AdS_4 black holes	38
C	Polar perturbations of spherical black holes	39

1 Introduction

An important question in physics is to understand the evolution of out-of-equilibrium quantum systems. Gauge/gravity duality provides a new tool to study non-equilibrium dynamics, mapping this problem to classical gravity in AdS space. More precisely, thermal states of the dual theory are mapped to AdS black holes and the process of thermalization is mapped to black hole formation. Close to thermal equilibrium, the long wave-length, late time behavior of QFT is described by hydrodynamics and, furthermore, this regime has a corresponding bulk description in terms of solutions constructed in a gradient expansion. The complete description of non-equilibrium phenomena require global time-dependent

solutions approaching the (appropriate) AdS black hole at late times. With this in mind, much effort has been devoted in recent years in obtaining numerical solutions that address the problem in general terms (see, for instance, [1] and references therein). Here, we aim to discuss certain aspects of non-equilibrium dynamics using analytic solutions. In particular, we focus on a class of gravitational solutions provided by the Robinson-Trautman metrics, which may be regarded as non-linear version of a specific type of perturbations of AdS_4 Schwarzschild black holes.

Linear perturbations of the Schwarzschild black hole (with or without cosmological constant) have been a topic of immense study over the years, reducing their description to an effective Schrödinger problem. What is perhaps less known is the remarkable fact that the effective quantum mechanics of all four-dimensional black holes is supersymmetric in that the Hamiltonian can be written as the square of a supercharge. However, the boundary conditions imposed by the black hole make the Hamiltonian non-Hermitian and its spectrum is not even real. The allowed energy levels provide the quasi-normal modes of the equilibrium state, which turns out to be stable under all kind of perturbations. Under appropriate boundary conditions, the effective supersymmetry pairs the non-zero modes: parity even (polar) perturbations are paired with parity odd (axial) perturbation and, as usual with supersymmetry, the zero modes are special satisfying a first order equation. These supersymmetric zero energy modes, which are called algebraically special modes and they correspond to purely dissipative perturbations of black holes, are the focus of our work. They can be lifted to exact solutions of the full non-linear Einstein equations, giving rise to the class of Robinson-Trautman metrics. In that respect, our discussion extends known results to space-times with non-zero cosmological constant.

When the cosmological constant is zero, the solutions we are considering represent isolated gravitationally radiating systems. The system relaxes to equilibrium by radiating the excess energy, which escapes to null infinity, and settles to a spherically symmetric configuration provided by the Schwarzschild solution. One can quantify the energy that is radiated away by computing the corresponding Bondi mass, [2]: as energy escapes to infinity, the Bondi mass decreases monotonically (in retarded time) and becomes equal to the Schwarzschild mass after infinitely long time. The space-time has a naked singularity in the past, reflecting the fact that the solution does not include the sources generating the excitation. There is, however, a past apparent horizon which may be regarded as a surface that surrounds the isolated system. When the cosmological constant is negative, there are new issues because the asymptotic structure of space-time is different. Since asymptotically AdS space-times do not have null infinity, the radiation cannot escape. Instead, it is absorbed at the boundary modifying the boundary metric.

In AdS/CFT correspondence, the boundary conditions represent sources for dual operators and the boundary metric is the source for the energy-momentum tensor of the dual theory. The energy-momentum tensor is one of main observables that one wishes to extract and use it to study the approach to thermal equilibrium. The boundary geometry of AdS_4 Robinson-Trautman space-times is topologically $\mathbb{R} \times S^2$, with \mathbb{R} representing time, and the complete non-equilibrium process is governed by the Calabi flow on S^2 . This flow is a well-known geometric evolution equation in mathematics describing volume preserving

deformations of Kähler manifolds (S^2 in our case) given by a fourth order non-linear diffusion process. Starting from a general initial metric on S^2 , the flow continuously deforms the metric transforming it to the constant curvature metric after infinitely long time. The Robinson-Trautman solution provides a space-time interpretation of the two-dimensional Calabi flow with retarded time playing the role of the deformation parameter. While other geometric evolution equations, such as the normalized Ricci-flow are also related to linear perturbations of large AdS_4 black holes, as will be seen later, only the Calabi flow is known to be associated with exact non-linear solutions of Einstein equations for all values of the cosmological constant.

In the context of gauge/gravity duality, one expects to find that sufficiently close to equilibrium the energy-momentum tensor takes the form of dissipative hydrodynamics. We will indeed find that this is the case. One should note, however, that the underlying physics is very different than in other discussions that appeared in the literature so far. In most works, the following problem is posed: one creates a non-equilibrium state by coupling the dual QFT to an external source for a finite time interval. After switching off the interaction the system loses energy to the black hole and equilibrates. The physical mechanism for dissipation is black hole absorption. In our case, however, there is only out-going radiation and nothing is absorbed by the black hole; instead, the dissipation is due to the external couplings of the system. The fact that the radiation is out-going means that the perturbation ought to vanish at the horizon, as it does. Expressing the out-going modes in terms of an in-coming basis, one finds that these modes are not smooth at the horizon, in general, and for sufficiently low multi-poles of radiation there is no Kruskal extension at all. While this is in principle a major concern, one may view the solutions as representing the gravitational field outside an isolated compact object losing its asymmetry via gravitational radiation until it relaxes to a spherical symmetric state. In that picture, the complete solution should be described by a matter-filled interior solution joined to the Robinson-Trautman solution, which is now covering only the exterior region, but it remains an open problem how precisely to implement it.

In standard treatments of holography, the linearized perturbations around a given solution satisfy Dirichlet boundary conditions and in-coming boundary conditions at the horizon yield two-point functions in the dual QFT. From the low energy limit of those correlation functions one subsequently extracts the transport coefficients via the Kubo formula. In our case, the algebraically special modes satisfy mixed boundary conditions, due to the first order equation they obey, which in turn imply unusual boundary conditions for the metric and, moreover, these modes are purely out-going. Thus, a clean holographic interpretation of these modes is somewhat challenging. We find that the effective viscosity depends on the eigenvalues of the Laplacian on S^2 (related to the pole of radiation we are considering) and for sufficiently low harmonics of large AdS_4 black holes there is a violation of the KSS bound on η/s . It should be said right away that non-universality and violation of the KSS bound have also been observed in higher derivative theories of gravity [3, 4] as well as in two-derivative gravity coupled to matter in the presence of anisotropy [5, 6] (see, for instance, [7] for a topical review and further references), but the underlying physics is different in those studies compared to ours.

The Bondi mass, which is naturally defined at null infinity of asymptotically flat space-times, describes the total mass of the system at some given instant of (retarded) time. For AdS_4 Robinson-Trautman there is no null infinity, but one can still define by analogy a “Bondi mass” that decreases monotonically under the evolution and eventually becomes equal to the Schwarzschild mass, as in the asymptotically flat case. Moreover, we will show that it satisfies a Penrose inequality, which is a central condition for the formation of horizons in general relativity. It turns out that the Bondi mass is greater than a specific function of the area of the past apparent horizon, hereby extending known results for the Robinson-Trautman metrics with $\Lambda = 0$. A closely related condition for the formation of horizons is Thorne’s hoop conjecture stating that the mass M gets compacted into a region whose circumference is bounded in all directions. We will also provide evidence for a suitable version of this conjecture that holds for Robinson-Trautman space-times with negative cosmological constant, using the Bondi mass and some appropriately chosen circumference to surround its past apparent horizon. The two inequalities are independent from each other, in general, though one follows from the other under some special conditions that are also discussed in the text.

This paper is organized as follows. In section 2, we introduce the Robinson-Trautman solutions, as spherical gravitational waves, and show that they are non-linear versions of special “supersymmetric” perturbations of AdS_4 Schwarzschild metric. We also present their relation to Calabi flow and extract their late time behavior. In section 3, we discuss a number of global issues, focusing on the (lack of) Kruskal extension and the existence, uniqueness and explicit construction of the past apparent horizon. We prove a Penrose inequality for the Bondi mass and formulate as well as provide evidence for Thorne’s hoop conjecture in the presence of negative cosmological constant. In section 4, we consider holographic aspects of the solutions by computing the corresponding energy-momentum tensor. We determine its late time behavior and compute the effective viscosity of the system. Here, we also define an entropy current of the boundary fluid and show by explicit computation in late time expansion that there is entropy production upon thermalization. In section 5, we present our conclusions and outline some open problems. We also collect a number of useful results in three appendices. In appendix A, we review the derivation of the Robinson-Trautman solution for the benefit of the readers. In appendix B, we review the relevant aspects of large and small AdS_4 black holes and collect some useful formulae that are implicitly used in the text. Finally, in appendix C, we review the theory of polar perturbations of spherical black holes.

In all expressions appearing in the text, Newton’s constant G is normalized to 1, whereas the gravitational coupling constant $8\pi G$ is denoted by κ^2 , as usual.

2 Spherical gravitational waves

The Robinson-Trautman metrics provide an exact class of radiative solutions of Einstein equations, which are available for all values of the cosmological constant Λ , including, in particular, $\Lambda < 0$. This class exists in four space-time dimensions and has been thought to describe the effect of spherical gravitational waves emitted by bounded sources. Although

such metrics do not capture the general features of gravitational radiation, they are often interesting to consider in detail at linear and non-linear level. Higher dimensional generalizations of Robinson-Trautman metrics have also been considered in the literature, but they do not seem to support non-trivial radiative solutions. Here, we summarize some basic facts about them that will be used later.

2.1 Robinson-Trautman metrics

The Robinson-Trautman metrics are singled out in four space-time dimensions by their unique geometric property to admit a geodesic null congruence with zero shear and twist and non-vanishing divergence. Thus, the metric has the following general form, [8, 9] (but see also the textbooks [10–12] and some early references on the subject [13–15]),

$$ds^2 = 2r^2 e^{\Phi(z, \bar{z}; u)} dz d\bar{z} - 2du dr - F(r, u, z, \bar{z}) du^2. \quad (2.1)$$

The variable r is a radial coordinate in space and u is a *retarded time* coordinate. Closed surfaces with constant r and u represent distorted two-dimensional spheres in a system of conformally flat (Kähler) coordinates (z, \bar{z}) .

Using Einstein equations, in the presence of a cosmological constant Λ , the form of the front factor F is uniquely determined in terms of Φ as

$$F = r \partial_u \Phi - \Delta \Phi - \frac{2m}{r} - \frac{\Lambda}{3} r^2, \quad (2.2)$$

where

$$\Delta = e^{-\Phi} \partial_z \partial_{\bar{z}} = \frac{1}{2} \nabla^2 \quad (2.3)$$

denotes the Laplacian on the distorted 2-spheres. The cosmological constant term in F accounts for the appropriate asymptotic behavior of the metric as $r \rightarrow \infty$. The parameter m is an integration constant, which is taken to be positive and it is related to the physical mass of the configuration. The remaining Einstein equations are completely solved, provided that $\Phi(z, \bar{z}; u)$ satisfies the following parabolic fourth order non-linear differential equation on S^2 ,

$$3m \partial_u \Phi + \Delta \Delta \Phi = 0, \quad (2.4)$$

which is called Robinson-Trautman equation. Note that this equation is insensitive to the presence of Λ . The essential steps of the derivation are summarized for completeness in appendix A.

General u -dependent solutions are thought to describe spherical gravitational waves, in a sense that will be made more precise later, in the full non-linear regime of Einstein equations. Although equation (2.4) has been studied extensively in physics as well as in mathematics (for it arises in a seemingly unrelated problem in geometric analysis, namely the Calabi flow on S^2 , as well be seen later in more detail), hardly any non-trivial explicit solutions exist to this day. However, some strong qualitative results are available in the literature regarding the long time existence of the general solution. For appropriate (smooth) initial data, the u -dependent solutions converge towards a fixed point, as $u \rightarrow \infty$, [15–19], which exhibits stability against small perturbations.

The fixed point is the u -independent solution of the Robinson-Trautman equation associated to the round metric of radius 1 on S^2 with conformal factor

$$e^{\Phi_0} = \frac{1}{(1 + z\bar{z}/2)^2}. \tag{2.5}$$

The corresponding four-dimensional space-time metric reads

$$ds^2 = \frac{2r^2}{(1 + z\bar{z}/2)^2} dzd\bar{z} - 2dudr - \left(1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2\right) du^2 \tag{2.6}$$

and it coincides with the Schwarzschild solution in the Eddington-Filkenstein frame in the presence of cosmological constant Λ . Indeed, by introducing the change of coordinates

$$u = t - r_*, \quad z = \sqrt{2} \cot \frac{\theta}{2} e^{i\phi}, \tag{2.7}$$

where r_* is the so called tortoise coordinate of the corresponding Schwarzschild metric, which is detailed in appendix B together with other useful definitions, we arrive at the standard form of the solution

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \tag{2.8}$$

with the required profile function

$$f(r) = 1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2. \tag{2.9}$$

2.2 Linearization of the solutions

The Robinson-Trautman equation can be regarded as non-linear diffusion process (of fourth order) on S^2 , which has the topology of the black hole horizon. Since small curvature perturbations of the round sphere correspond to linear perturbations of the Schwarzschild metric as the configuration tends towards equilibrium as $u \rightarrow \infty$, it is appropriate to parametrize the fluctuating two-dimensional metric as

$$ds_2^2 = [1 + \epsilon_l(u)P_l(\cos \theta)] (d\theta^2 + \sin^2 \theta d\phi^2) \tag{2.10}$$

with small parameters $\epsilon_l(u)$. Here, we only consider axially symmetric deformations, without loss of generality, using Legendre polynomials $P_l(\xi)$ with $l \geq 2$. The cases $l = 0$ and $l = 1$ are omitted, since we are paying attention to quadrupole radiation terms or higher on physical grounds. This parametrization preserves the area A of S^2 to linear order in ϵ_l , since for all $l \geq 1$ we have

$$\delta_\epsilon A = \frac{(-1)^l}{2} V_0 \epsilon_l(u) \int_{-1}^{+1} d\xi P_l(\xi) = 0. \tag{2.11}$$

To work out the form of the Robinson-Trautman equation to linear order, we set for convenience

$$K(\theta, u) = \epsilon_l(u)P_l(\cos \theta). \tag{2.12}$$

A simple calculation shows that

$$3m\partial_u K = -\Delta_0[(\Delta_0 + 1)K], \quad (2.13)$$

where Δ_0 denotes the Laplacian of the round S^2 whose eigenfunctions $P_l(\cos\theta)$ have eigenvalues $-l(l+1)/2$; likewise, the corresponding eigenvalues of the operator $\Delta_0 + 1$ are $-(l-1)(l+2)/2$. This immediately yields the following evolution for $\epsilon_l(u)$, [13],

$$\epsilon_l(u) = \epsilon_l(0) \exp\left(-\frac{u}{12m}(l-1)l(l+1)(l+2)\right) \equiv \epsilon_l(0)e^{-i\omega_s u}, \quad (2.14)$$

where ω_s denote collectively the set of purely imaginary frequencies (damping modes)

$$\omega_s = -i\frac{(l-1)l(l+1)(l+2)}{12m} \quad (2.15)$$

with $l \geq 2$. The spectrum of ω_s does not depend on the cosmological constant Λ , as for the Robinson-Trautman equation.

The result, which approximates well the asymptotic behavior of the full non-linear evolution as $u \rightarrow \infty$, shows that all curvature perturbations are damped exponentially fast until the configuration finally takes its canonical round metric form. All such linear solutions represent different modes of gravitational radiation in multi-pole expansion, when elevated to the four-dimensional space-time metric. Higher order poles of radiation are damped faster than the lower ones, as expected on physical grounds. In more detail, using small u -dependent perturbations of the form (2.10), the Robinson-Trautman line element takes the following form in the vicinity of the Schwarzschild solution,

$$ds^2 = -\left(1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2 + r\partial_u K - (\Delta_0 + 1)K\right) du^2 - 2dudr + r^2(1+K)(d\theta^2 + \sin^2\theta d\phi^2), \quad (2.16)$$

choosing to work with spherical coordinates (θ, ϕ) and using $K(\theta, u)$ as defined above. Then, taking into account the dependence of ϵ_l upon u and the action of Δ_0 on K , it turns out that

$$r\partial_u K - (\Delta_0 + 1)K = \frac{1}{2}(l-1)(l+2)\left(1 - \frac{r}{6m}l(l+1)\right)K. \quad (2.17)$$

Changing variables (u, r) to (t, r) by $u = t - r_*$, so that direct comparison can be subsequently made with the canonical form of gravitational perturbations of black holes, it follows that the underlying metric takes the form

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + \delta g_{\mu\nu} \quad (2.18)$$

around the static Schwarzschild metric, which is assigned the superscript (0). The perturbations are tabulated in terms of the (t, r, θ, ϕ) components, as

$$\delta g_{\mu\nu} = \begin{pmatrix} f(r)H(r) & -H(r) & 0 & 0 \\ -H(r) & H(r)/f(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix} e^{-i\omega_s(t-r_*)} P_l(\cos\theta), \quad (2.19)$$

where

$$H(r) = \frac{i\omega_s}{f(r)} \left(r - \frac{6m}{l(l+1)} \right). \quad (2.20)$$

The integration constant $\epsilon_l(0)$ represents the freedom to scale the form of the linear perturbations $\delta g_{\mu\nu}$ by an overall factor.

These perturbations are the so called algebraically special modes of the black hole, studied by Chandrasekhar [20] and connected to Robinson-Trautman metrics, upon linearization, for $\Lambda = 0$ [21] (but see also [22] for earlier work on the subject). Here, we are simply extending all that to $\Lambda \neq 0$, as in [23]. These modes are special in many respects as we will discuss shortly and the preceding analysis shows that the class of Robinson-Trautman metrics are formed by a consistent non-linear superposition of them. Note that it does not seem possible to truncate the Robinson-Trautman equation to solutions that encompass a finite number of algebraically special modes, for, otherwise, we would have simple models of gravitational radiation in our disposal (though at late times this looks plausible). More general radiative metrics are formed by non-linear superposition of all quasi-normal modes, not just the algebraically special ones, and hence they are notoriously more difficult to study, though physically more interesting.

The algebraically special modes are special case of the polar perturbations of spherical black holes. The theory of perturbations of spherical black holes is a well studied subject [24–30]. We now briefly summarize some of the relevant aspects of this theory; more details can be found in appendix C. The perturbations can be split into parity even (polar) and parity odd (axial) perturbations. It turns out that the study of these perturbations can be reduced to the effective Schrödinger problems

$$H_{\pm} \Psi_{\pm}(r_{\star}) = E \Psi_{\pm}(r_{\star}), \quad (2.21)$$

where

$$H_{\pm} = -\frac{d^2}{dr_{\star}^2} + W^2 \pm \frac{dW}{dr_{\star}}. \quad (2.22)$$

The plus sign corresponds to polar perturbations and the minus to axial perturbations. The function W is given by

$$W(r) = \frac{6mf(r)}{r[(l-1)(l+2)r + 6m]} + i\omega_s \quad (2.23)$$

and

$$E = \omega^2 - \omega_s^2. \quad (2.24)$$

Remarkably, despite the fact that the space-time itself has no supersymmetry and there are not even fermions, the effective Schrödinger problems can be combined as partners into supersymmetric quantum mechanics with W playing the role of the superpotential. Indeed, as discussed in appendix C, defining $\mathcal{H} = \text{diag}(H_+, H_-)$ and $\Psi = (\Psi_+, \Psi_-)^T$, one finds that the two Schrödinger problems combine into one

$$\mathcal{H}\Psi = E\Psi \quad (2.25)$$

and the Hamiltonian is a square of a supercharge, i.e., $\mathcal{H} = \{\mathcal{Q}, \mathcal{Q}^\dagger\}$, where \mathcal{Q} is defined as in equation (C.19). Due to boundary conditions, the Hamiltonian is only formally Hermitian and the spectrum is not even real.

The algebraically special modes are the zero energy states of this problem and they are annihilated by the supercharge \mathcal{Q} . Using equations (C.19) and (C.16), we find that these perturbations satisfy the first order equation,

$$\mathcal{Q}\Psi_+^{(0)}(r_\star) = \left(-\frac{d}{dr_\star} + W(r_\star)\right)\Psi_+^{(0)}(r_\star) = 0. \quad (2.26)$$

This equation can be integrated to yield the corresponding wave-function

$$\Psi_+^{(0)}(r_\star) = C \frac{r}{(l-1)(l+2)r + 6m} \exp(i\omega_s r_\star), \quad (2.27)$$

where C is an integration constant, which, for later convenience, is chosen to be

$$C = \frac{6m}{l(l+1)}. \quad (2.28)$$

Then, in the notation of appendix C, we find that the system of equations for the metric coefficients of polar perturbations admit the following simple solution at $\omega = \omega_s$,

$$H_0(r) = H_2(r) = -H_1(r) = i \frac{\omega_s}{f(r)} \left(r - \frac{6m}{l(l+1)}\right) K(r) \quad (2.29)$$

and

$$K(r) = \exp(i\omega_s r_\star) = \exp\left(\frac{r_\star}{12m}(l-1)l(l+1)(l+2)\right) \quad (2.30)$$

that yield precisely $\delta g_{\mu\nu}$ for the linearization problem of the Robinson-Trautman metrics around the Schwarzschild solution, as given above, for all values of the cosmological constant.

Note that $\Psi_+^{(0)}(r_\star)$ vanishes on the black hole horizon, irrespective of Λ , complying with the physical boundary condition that nothing can come out directly of a black hole. As $r \rightarrow \infty$, $\Psi_+^{(0)}(r_\star)$ blows up exponentially when $\Lambda = 0$, but when $\Lambda < 0$, which is the case of primary interest here, it reaches a finite value. Thus, for AdS_4 black holes, all algebraically special modes have normalizable wave functions, i.e.,

$$\int_{-\infty}^0 dr_\star |\Psi_+^{(0)}(r_\star)|^2 < \infty. \quad (2.31)$$

Furthermore, the algebraically special modes satisfy *mixed boundary conditions* at $r = \infty$. Indeed, using (2.26) and evaluating $W(r_\star)$ at $r = \infty$ (equivalently at $r_\star = 0$) we find

$$\frac{d}{dr_\star}\Psi_+^{(0)}(r_\star)|_{r_\star=0} = \left(i\omega_s - \frac{2m\Lambda}{(l-1)(l+2)}\right)\Psi_+^{(0)}(r_\star=0). \quad (2.32)$$

These non-standard boundary conditions are responsible for some peculiar properties of the solution that will be derived later.

2.3 Calabi flow on S^2

The Robinson-Trautman equation also arises in mathematics and coincides with the so called Calabi flow on S^2 , as was first pointed out by Tod [32], which describes certain type of geometric deformations of the metric

$$ds_2^2 = 2e^{\Phi(z, \bar{z}; u)} dz d\bar{z} . \tag{2.33}$$

Recall that the Calabi flow is generally defined for metrics $g_{a\bar{b}}$ on Kähler manifolds M of arbitrary dimension with local coordinates (z^a, \bar{z}^a) and assumes the form [33, 34] (but see also [35, 36])

$$\partial_u g_{a\bar{b}} = \frac{\partial^2 R}{\partial z^a \partial \bar{z}^b} \tag{2.34}$$

in terms of the Ricci curvature scalar R of the metric. As such, it provides volume preserving deformations within a given Kähler class of the metric. Specializing to S^2 , we have $R = -2\Delta\Phi$ and the identification with the Robinson-Trautman equation follows immediately, setting $3m = 2$ in appropriate units.

The Calabi flow is a parabolic equation of fourth order, and, as such, it is difficult to study by standard techniques that are mainly applicable to second order equations. By considering appropriate (smooth) initial data $g_{a\bar{b}}(z, \bar{z}; 0)$, the flow deforms the metric towards its canonical form, which is typically represented by a constant curvature metric (provided that the Kähler manifold admits one). The quadratic curvature functional on M , also known as Calabi functional,

$$\mathcal{C} = \int_M R^2, \tag{2.35}$$

where R the curvature scalar, decreases monotonically along that flow, reaching its extremum at the fixed point (also called *extremal metric* on M). The fixed point is the familiar constant curvature metric when $M \simeq S^2$. Many aspects of the Robinson-Trautman metrics are derived from the mathematical theory of Calabi flows. The results are qualitatively similar to the heat equation on S^2 , thinking of the Ricci scalar curvature R as temperature; in that case, the average temperature (viz. the Euler number) remains constant throughout the evolution, whereas the average temperature square (viz. the quadratic curvature functional) decreases monotonically until thermal equilibrium is reached at the fixed point, where the temperature is constant everywhere. This framework has also been used more generally, as tool, to explore the geometrization of Kähler manifolds.

The Calabi flow is one of many geometric evolution equations studied in mathematics. For later use, we introduce here another geometric evolution equation, known as Ricci flow, which is of second order and deforms the metric on a Riemannian manifold by the Ricci curvature tensor. In that case, the volume of space is not preserved under the Ricci flow, but it is always possible to define a variant of it, known as normalized Ricci flow, which is volume preserving and will be used for comparison with Calabi flow. The Ricci flow for the class of conformally flat metrics on S^2 , (2.33), takes the following form

$$\partial_u \Phi = \Delta \Phi, \tag{2.36}$$

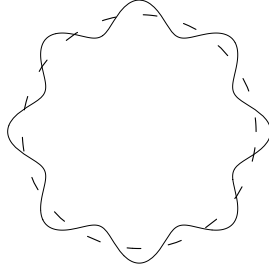


Figure 1. Curvature perturbations of the round sphere.

whereas the corresponding normalized Ricci flow on S^2 with fixed area 4π is given by

$$\partial_u \Phi = \Delta \Phi + 1 . \quad (2.37)$$

The constant curvature metric provides the fixed point for both Calabi and normalized Ricci flow equations on S^2 . In either case, the canonical metric is reached from a given initial data after sufficiently long time.

It is instructive to examine the spectrum of linear perturbations around the equilibrium state, using axially symmetric deformations of the round sphere parametrized by $\epsilon_l(u)P_l(\cos \theta)$, as in the linear approximation (2.10) of the Robinson-Trautman equation. It can be easily verified that as $u \rightarrow \infty$, the metric perturbations decay as follows, under the normalized Ricci flow,

$$\epsilon_l(u) = \epsilon_l(0) \exp \left(-\frac{u}{2}(l-1)(l+2) \right) \equiv \epsilon_l(0)e^{-i\Omega_s} , \quad (2.38)$$

with spectrum of purely imaginary frequencies (damping modes) for all $l \geq 2$

$$\Omega_s \sim -i \frac{(l-1)(l+2)}{2} . \quad (2.39)$$

Ω_s is unique up to a universal factor that depends on the physical scale of the problem. Later, in section 4, the scale of the Ricci flow will be identified with $3r_h/2$ to match the values of some purely dissipative modes in the hydrodynamic description of very large AdS_4 black holes, whereas the scale of u of the Calabi flow is provided by $3m$. The dependence of Ω_s upon l , being quadratic versus the quartic dependence of ω_s for the Calabi flow, reflects the order of the corresponding geometric evolution equations.

A schematic representation of curvature perturbations on S^2 is depicted in figure1. All modes of perturbation dissipate faster under Calabi flow when the radius of the sphere is sufficiently small. For spheres of large radius, however, the situation is partially reversed, as perturbations with sufficiently small l dissipate faster under the normalized Ricci flow, whereas modes with large enough l are still dissipating faster under Calabi flow.

2.4 Late-time behavior of solutions

Next, we describe the late time behavior of solutions to the Robinson-Trautman (Calabi flow) equation including non-linear effects. Let us parametrize the conformal factor of the

S^2 line element as

$$e^{\Phi(z, \bar{z}; u)} = \frac{1}{\sigma^2(z, \bar{z}; u) (1 + z\bar{z}/2)^2}, \quad (2.40)$$

so that $\sigma(z, \bar{z}; u)$ describes the deviations away from the static black hole solution. First, it was established that for any sufficiently smooth initial data $\sigma(z, \bar{z}; u_0)$ solutions of the Robinson-Trautman equation exist for all $u \geq u_0$, [16, 17]; the short time existence of solutions was established earlier using the parabolic nature of the equation in u , [15]. Second, it was found that there is an asymptotic expansion of $\sigma(z, \bar{z}; u)$ for large u of the following form, [37],

$$\sigma(z, \bar{z}; u) = 1 + \sum_{p \geq 1, q \geq 0} \sigma_{p,q}(z, \bar{z}) u^q e^{-2pu/m}, \quad (2.41)$$

where $\sigma_{p,q}(z, \bar{z})$ are appropriately chosen smooth functions on S^2 with $\sigma_{0,0}$ normalized to 1 without loss of generality. Thus, by exponential damping, the Robinson-Trautman metrics approach asymptotically fast the Schwarzschild solution, $\sigma = 1$, as $u \rightarrow \infty$. These results are quite general and they are valid irrespective of Λ for the Robinson-Trautman equation is insensitive to the presence of cosmological constant.

The above series expansion of $\sigma(z, \bar{z}; u)$ captures the linear as well as the non-linear effects of gravitational radiation at late retarded times. Linear effects are solely described by the system of algebraically special modes, which take the values

$$i\omega_s = \frac{2}{m}, \quad \frac{10}{m}, \quad \frac{30}{m}, \quad \dots \quad (2.42)$$

for $l = 2, 3, 4, \dots$, respectively. They only contribute to the terms $\sigma_{1,0}, \sigma_{5,0}, \sigma_{15,0}, \dots$, which in this case are described by the corresponding spherical harmonics on S^2 , depending on l . Other coefficients in the series are naturally related to non-linear gravitational effects; they include contributions to all terms with $p > 1$ as well as to terms with $q \neq 0$.

Although some coefficients $\sigma_{p,q}(z, \bar{z})$ may vanish, it has already been established that there exist generic solutions with non-vanishing coefficient $\sigma_{15,1}(z, \bar{z})$ - the first possible such term with $q \neq 0$ - so that the asymptotic expansion assumes the following form,

$$\begin{aligned} \sigma(z, \bar{z}; u) = & 1 + \sigma_{1,0}(z, \bar{z})e^{-2u/m} + \sigma_{2,0}(z, \bar{z})e^{-4u/m} \dots + \sigma_{14,0}(z, \bar{z})e^{-28u/m} \\ & + [\sigma_{15,0}(z, \bar{z}) + \sigma_{15,1}(z, \bar{z})u]e^{-30u/m} + \mathcal{O}(e^{-32u/m}). \end{aligned} \quad (2.43)$$

Of course, there might be higher order terms $p \geq 16$ with $q \neq 0$, but their effect is subdominant.

Following Chruściel and collaborators, [37], we consider axially symmetric deformations, so that all coefficients depend on the angle θ and not ϕ , and further impose *antipodal symmetry* for simplicity. It implies that only even values of l contribute to the general solution. Then, in that case, the first few coefficients in the series turn out to be

$$\sigma_{1,0}(x) = a \left(x^2 - \frac{1}{3} \right), \quad (2.44)$$

$$\sigma_{2,0}(x) = -a^2 \left(\frac{23}{78}x^4 - \frac{47}{39}x^2 + \frac{49}{234} \right), \quad (2.45)$$

$$\sigma_{3,0}(x) = a^3 \left(\frac{997}{5226}x^6 - \frac{36697}{47034}x^4 + \frac{25309}{15678}x^2 - \frac{8899}{47034} \right) \quad (2.46)$$

and so on, where $x = \cos \theta$. The functions above are uniquely determined and they can be rewritten in terms of the even Legendre polynomials $P_2(x) = (3x^2 - 1)/2$, $P_4(x) = (35x^4 - 30x^2 + 3)/8$ and so on.

The first term $\sigma_{1,0}(x)$ represents the quadrupole gravitational radiation, as in the linear approximation, and is set equal to $(2a/3)P_2(x)$ using an arbitrary constant a . All other terms up to $\sigma_{5,0}(x)$ represent non-linear effects of gravitational radiation. One could add to $\sigma_{5,0}(x)$ a term proportional to $P_3(x)$, with a new coefficient independent of a , which represents the effect of eighth-pole radiation in the linear approximation and shows up at order $\exp(-10u/m)$. However, such term will be excluded, if antipodal symmetry is imposed. Higher order terms up to $\sigma_{15,0}(x)$ are also uniquely determined in terms of a and they also represent non-linear effects of quadrupole radiation. One can also add to $\sigma_{15,0}(x)$ a term proportional to $P_4(x)$ that represents the effect of sixteenth-pole radiation in the linear approximation and shows up at order $\exp(-30u/m)$. Such a term is compatible with the antipodal symmetry. There is an additional non-linear effect that appears to this order attributed to the presence of $\sigma_{15,1}(x)$, which is relatively very small but definitely not zero; it was determined numerically in terms of a to be $\sigma_{15,1}(x) \simeq (0.2155750672866a)^{15} P_4(x)$, [37]. Compared to the typical coefficients of $\sigma_{p,0}(x)$ with $1 \leq p \leq 15$, $\sigma_{15,1}(x)$ is smaller by nine orders of magnitude. The pattern repeats itself for higher values of p although it has not been studied exhaustively by numerical methods to the best of our knowledge.

The late time expansion of solutions to the Robinson-Trautman equation has implications for the global structure of the corresponding space-times when considering the Kruskal extension across the horizon \mathcal{H}^+ . It will also be used to provide the late time expansion of the associated holographic energy-momentum tensor that is constructed later in section 4.

3 Global aspects

The global structure of Robinson-Trautman space-time is sensitive to the cosmological constant Λ (see [37] for $\Lambda = 0$, [38] for $\Lambda > 0$, [39] for $\Lambda < 0$). For $\Lambda < 0$, it also turns out to depend on the relative size of AdS_4 black holes. This might look surprising at first sight for the the Robinson-Trautman equation depends only on m and not on Λ .

3.1 Kruskal extension

Let us introduce Kruskal-type coordinates in the bulk space-time, which are generally defined as

$$\tilde{u} = -e^{-u/2\delta_h}, \quad \tilde{v} = e^{v/2\delta_h}, \tag{3.1}$$

using the retarded and advanced time coordinates, $u = t - r_\star$ and $v = t + r_\star$, respectively. The parameter δ_h is defined for all values of Λ , which is taken here to be non-positive, as

$$\delta_h = \frac{r_h^2}{2(3m - r_h)}, \tag{3.2}$$

using the radius r_h of the horizon of the static solution, and it reduces to the familiar choice $\delta_h = 2m$ when $\Lambda = 0$; otherwise, it varies in the interval $2m > \delta_h > 0$ as Λ changes from 0 to $-\infty$, keeping m fixed. The surface gravity κ_{gr} and the temperature T of the equilibrium black hole state are related to δ_h as follows,

$$\kappa_{gr} = \frac{f'(r_h)}{2} = 2\pi T = \frac{1}{2\delta_h} . \tag{3.3}$$

In these coordinates, the line element of the Robinson-Trautman metric becomes

$$ds^2 = 2r^2 e^\Phi dzd\bar{z} - 4\delta_h^2 f(r) e^{-r_*/\delta_h} d\tilde{u}d\tilde{v} - 4\delta_h^2 (F - f(r)) \frac{d\tilde{u}^2}{\tilde{u}^2} , \tag{3.4}$$

where

$$F - f(r) = r\partial_u \Phi - \Delta \Phi - 1 . \tag{3.5}$$

The static AdS_4 black hole has $F = f(r)$ and always admits smooth extension across the event horizon. Time-dependent solutions, however, have $F \neq f(r)$ and the regularity of the Kruskal extension is questionable. Are the extensions through the null hypersurface \mathcal{H}^+ given by $u = \infty$ (equivalently $\tilde{u} = 0$) smooth or not? If not, it will imply, in particular, that an observer can determine by local measurements whether he/she has crossed the event horizon.

It turns out that the asymptotic series expansion of the function $\sigma(z, \bar{z}; u)$, which describes the deviation of the metric from the canonical form, as $\Phi = \Phi_0 - 2\log\sigma$, takes the following form in terms of the Kruskal coordinate \tilde{u} , [37],

$$\begin{aligned} \sigma(z, \bar{z}; \tilde{u}) = & 1 + \sigma_{1,0}(z, \bar{z})(-\tilde{u})^{4\delta_h/m} + \sigma_{2,0}(z, \bar{z})(-\tilde{u})^{8\delta_h/m} + \dots + \sigma_{14,0}(z, \bar{z})(-\tilde{u})^{56\delta_h/m} \\ & + [\sigma_{15,0}(z, \bar{z}) - 2\delta_h\sigma_{15,1}(z, \bar{z})\log|\tilde{u}|](-\tilde{u})^{60\delta_h/m} + \mathcal{O}\left((-\tilde{u})^{64\delta_h/m}\right) . \end{aligned} \tag{3.6}$$

Due to the presence of the $\log|\tilde{u}|$ term, the function σ is not smooth at $\tilde{u} = 0$ but it is at most $C^{[a]}$ -differentiable with $[a]$ being the integer part of

$$a = \frac{60\delta_h}{m} - 1 . \tag{3.7}$$

In turn, it implies that the Robinson-Trautman metric, which is now written in the form (3.4), is at most $C^{[a]-2}$ -differentiable at $\tilde{u} = 0$, because the coefficient in front of the $d\tilde{u}^2$ term contains an additional factor $1/\tilde{u}^2$. As will be seen shortly, the regularity of the metric may be even lower, depending on the size of δ_h . As a result, the extension of the Robinson-Trautman metric across the horizon \mathcal{H}^+ can not be smooth in general. The degree of differentiability depends crucially on Λ . When $\Lambda = 0$, $\delta_h = 2m$ and $a = 119$, i.e., the full four-dimensional metric is only C^{117} -differentiable and not C^∞ , [37]. When $\Lambda < 0$, which is the case of primary interest here, the degree of differentiability decreases simply because $\delta_h < 2m$. In fact, the parameters m and Λ can be easily arranged so that the metric is not even C^1 ; it may also very well be that $\partial\sigma/\partial\tilde{u}$ diverges at \mathcal{H}^+ .

Actually, there is a striking difference between $\Lambda = 0$ and $\Lambda < 0$ black holes, which is already seen at the linear level. Retaining only the contribution of the algebraically

special modes in the asymptotic expansion of $\sigma(z, \bar{z}; \tilde{u})$, one observes that $F - f(r)$ is of order $\mathcal{O}(\tilde{u}^{4\delta_h/m})$ as \tilde{u} approaches 0, receiving the most dominant contribution from the quadrupole radiation term. As a result, the overall coefficient of the $d\tilde{u}^2$ term of the metric is of order $\mathcal{O}(\tilde{u}^{(4\delta_h/m)-2})$. Thus, for $\Lambda = 0$ black holes, this coefficient is of order $\mathcal{O}(\tilde{u}^6)$ and the extension through the horizon appears to be smooth at the linear level, as noticed in reference [32]; of course, we know better now that the smoothness breaks down due to non-linear effects attributed to the presence of the non-vanishing term $\sigma_{15,1}(z, \bar{z})$, [37]. On the other hand, for $\Lambda < 0$ black holes, it may happen that $(4\delta_h/m) - 2$ is negative, in which case there are divergences at \mathcal{H}^+ already appearing at the linear level, [39]. For example, one may consider large AdS_4 black holes with $m > r_h$, [40], which actually have $4\delta_h/m < 1$, as can be easily verified. By the same token, small AdS black holes have $4\delta_h/m > 1$ and in fact the lower bound exceeds the value 2 for sufficiently small (i.e., not of intermediate size) black holes. For the relevant definitions of large versus small black holes we refer the reader to appendix B.

Thus, the Kruskal extension of the Robinson-Trautman metric breaks down completely for large AdS_4 black holes because the coefficient of $d\tilde{u}^2$ becomes singular at $\tilde{u} = 0$. This is induced perturbatively by the first algebraically special mode (associated to quadrupole radiation) and signals an instability towards the formation of a null singularity at \mathcal{H}^+ . It is intriguing that this perturbative instability appears only for large black holes. Note that in the original work [39] the authors did not describe the result in thermodynamic terms, distinguishing large from small AdS black holes, [40].

More generally, within the linear approximation, each algebraically special mode gives a contribution to the $d\tilde{u}^2$ term of the Kruskal extension of the metric of order $\mathcal{O}(\tilde{u}^{2i\omega_s\delta_h-2})$, since

$$F - f(r) = \frac{(l-1)(l+2)}{2} \left(1 - \frac{l(l+1)}{6m}r\right) P_l(\cos\theta) (-\tilde{u})^{2i\omega_s\delta_h}. \quad (3.8)$$

Thus, for large black holes and sufficiently low l satisfying the bound¹

$$\frac{m}{r_h} > \frac{l(l+1)}{6}, \quad (3.9)$$

we have the relation

$$2i\omega_s\delta_h = \frac{(l-1)l(l+1)(l+2)r_h^2}{12m(3m-r_h)} < 1 \quad (3.10)$$

generalizing the effect of the quadrupole term to all other potentially dangerous terms arising in the multi-pole expansion. For higher values of l we have $2i\omega_s\delta_h > 1$, but the lower bound may still not exceed the value 2. It can also be verified that large black holes with sufficiently large values of l , so that

$$\sqrt{2} \frac{m}{r_h} < \frac{l(l+1)}{6}, \quad (3.11)$$

always satisfy the higher bound $2i\omega_s\delta_h > 2$. It is rather odd behavior of large AdS_4 black holes, which are favored thermodynamically, [40]. Sufficiently small black holes, on the other hand, satisfy the bound $2i\omega_s\delta_h > 2$ for all values of l .

¹Note that the (tt) and (rr) components of the algebraically special perturbations $\delta g_{\mu\nu}$ of large AdS_4 black holes become negative close to r_h for precisely those values of l (see equations (2.19) and (2.20)).

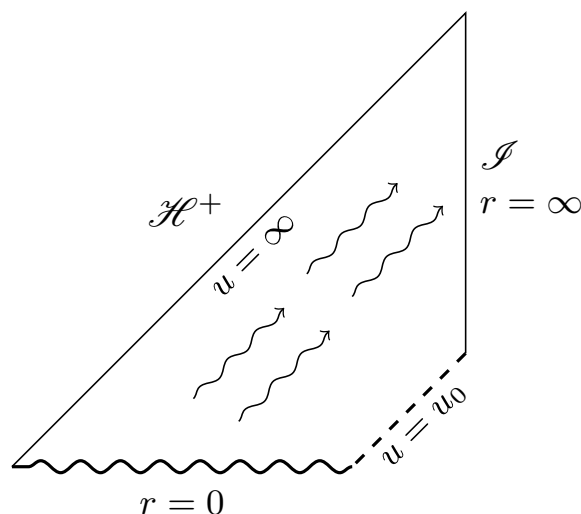


Figure 2. Penrose diagram of Robinson-Trautman space-times with $\Lambda < 0$.

One should note, however, that the curvature invariants are smooth at \mathcal{H}^+ . In fact, all invariants formed by powers of the Riemann tensor (without taking covariant derivatives) have the same value as the invariants evaluated on the Schwarzschild solution; this was shown in reference [16, 17] when $\Lambda = 0$ and we checked it for the Kretschmann invariant when $\Lambda < 0$. On the other hand, the authors of reference [37] argued that at least one component of the Riemann tensor (or any power of covariant derivatives acting on it) will blow up at \mathcal{H}^+ in any coordinate system. This indicates that there is a null singularity at the future horizon.

Summarizing, the Kruskal extension of the Robinson-Trautman metric breaks down completely for AdS_4 black holes when the coefficient of the $d\tilde{u}^2$ term becomes singular at $\tilde{u} = 0$. The radiative solutions settle down to AdS_4 Schwarzschild black hole at late retarded times, and, sometimes, depending on l , they can be extended across the horizon to include the black hole interiors. In those cases, the interior of a static black hole can be jointed to an external Robinson-Trautman space-time across \mathcal{H}^+ located at $\tilde{u} = 0$, but the extension is not smooth. The Penrose diagram in figure2 does not include the extension across the horizon. The situation resembles the behavior of out-going modes of a scalar field in AdS_4 black hole background, which are not smooth near the future horizon, as pointed out in reference [41]. Fortunately, the smoothness of \mathcal{S} at $r = \infty$ is not affected at all by possible discontinuities across the horizon.

3.2 Past apparent horizon

The Robinson-Trautman space-time exhibits a past apparent horizon. The construction we present here generalizes previously known results for $\Lambda = 0$, [32], to the case $\Lambda < 0$.

Let us define a 2-surface Σ in space-time by the following embedding relations,

$$u = \text{const.}, \quad r = U(z, \bar{z}) . \tag{3.12}$$

The unit tangent vectors to Σ with (u, r, z, \bar{z}) components are

$$m^\mu = \frac{1}{re^{\Phi/2}} (0, \bar{\partial}U, 0, 1), \quad \bar{m}^\mu = \frac{1}{re^{\Phi/2}} (0, \partial U, 1, 0) \quad (3.13)$$

and the induced metric on Σ is

$$g_{z\bar{z}} = r^2 e^\Phi |_{r=U, u=\text{const.}} = U^2 e^\Phi |_{u=\text{const.}} . \quad (3.14)$$

There is also a complementary set of two null vectors that are both orthogonal to those tangent vectors,

$$n^\mu = \left(1, -\frac{F}{2} + \frac{\partial U \bar{\partial} U}{r^2 e^\Phi}, \frac{\bar{\partial} U}{r^2 e^\Phi}, \frac{\partial U}{r^2 e^\Phi} \right), \quad (3.15)$$

$$l^\mu = (0, 1, 0, 0) . \quad (3.16)$$

All together they form a null tetrad $(m^\mu, \bar{m}^\mu, l^\mu, n^\nu)$, which is used in the Newman-Penrose formalism to cast the inverse Robinson-Trautman metric into the simple form

$$g^{\mu\nu} = m^\mu \bar{m}^\nu + \bar{m}^\mu m^\nu - l^\mu n^\nu - n^\mu l^\nu . \quad (3.17)$$

Next, using these definitions, we compute the convergence of the two null vectors and find, respectively,

$$m^\mu \bar{m}^\nu n_{(\mu;\nu)} = \frac{1}{2U} \left(2\Delta(\log U) + \Delta\Phi + \frac{2m}{U} + \frac{\Lambda}{3} U^2 \right), \quad (3.18)$$

$$m^\mu \bar{m}^\nu l_{(\mu;\nu)} = \frac{1}{r} . \quad (3.19)$$

The geodesic null congruence of Robinson-Trautman metrics is generated by the vector field $l_\mu = (-1, 0, 0, 0)$, which has zero shear and twist by definition. Setting the convergence of the null vector field n^μ equal to zero,

$$2\Delta(\log U) + \Delta\Phi + \frac{2m}{U} + \frac{\Lambda}{3} U^2 = 0, \quad (3.20)$$

makes Σ a marginally past trapped 2-surface and provides the differential equation that $\Upsilon(z, \bar{z})$ has to satisfy at each instant $u = \text{const.}$ Condition (3.20) is the generalization of the so called Penrose-Tod equation to all values of Λ .

It turns out that Σ is the unique marginally trapped 2-surface on the hypersurface of constant u and it is the outer boundary of past-trapped surfaces on this hypersurface, thus making Σ the past apparent horizon of space-time. The proof is outlined below in three steps, generalizing Tod's proof, [32], to $\Lambda \leq 0$. The case $\Lambda > 0$ is different and it will not be addressed here.

First, one has to establish the existence of one *strictly positive solution* of the generalized Penrose-Tod equation (3.20). For $\Lambda = 0$, the proof of existence follows from a theorem of Aubin, [42], as outlined in reference [32]. For $\Lambda < 0$, the proof of existence is based on the so called sub- and super-solution method adapted to Riemannian manifolds. It is outlined in the more recent papers [43, 44] that construct the past apparent horizon as

we do here. Second, we note that the uniqueness of the construction follows immediately from equation (3.20) provided that at least a *strictly positive solution* of this differential equation exists, as it does. Indeed, if U_1 and U_2 are two such distinct solutions, their ratio $R = U_1/U_2$ will satisfy the equation

$$2\Delta(\log R) + \frac{2m}{U_1}(1 - R) - \frac{\Lambda}{3}U_2^2(1 - R^2) = 0 . \quad (3.21)$$

Multiplying by $1 - R$ and integrating over the sphere with metric $ds_2^2 = 2(\exp \Phi)dzd\bar{z}$, one obtains the relation

$$2 \int_{S^2} (1 - R)\Delta(\log R) + 2m \int_{S^2} \frac{(1 - R)^2}{U_1} + \left(-\frac{\Lambda}{3}\right) \int_{S^2} U_2^2(1 + R)(1 - R)^2 = 0 . \quad (3.22)$$

The first term is non-negative as can be easily seen after integration by parts,

$$2 \int_{S^2} (1 - R)\Delta(\log R) = \int_{S^2} \frac{|\nabla R|^2}{R} . \quad (3.23)$$

The other two terms are also non-negative when $-\Lambda$ is non-negative, since U_1 and U_2 are assumed to be strictly positive. Thus, we necessarily have $R = 1$, i.e., $U_1 = U_2$, establishing the uniqueness of the solution when $\Lambda \leq 0$. Clearly, this argument does not apply when $\Lambda > 0$. Finally, as in Tod [32], we may express any other 2-surface $\Sigma_{(\lambda)}$ on the hypersurface of constant u as $r = \lambda U$, where λ is a strictly positive function. The convergence of the null vector $n_{(\lambda)}^\mu$ associated to the 2-surface $\Sigma_{(\lambda)}$ turns out to be

$$m_{(\lambda)}^\mu \bar{m}_{(\lambda)}^\nu n_{(\mu;\nu)}^{(\lambda)} = \frac{1}{2\lambda U} \left[2\Delta\lambda + \frac{2m}{\lambda U}(1 - \lambda) + \left(-\frac{\Lambda}{3}\right) U^2(1 - \lambda^2) \right]. \quad (3.24)$$

All three terms on the right hand side are negative for $\Lambda \leq 0$ when $\lambda > 1$, i.e., when $\Sigma_{(\lambda)}$ extends to larger values of r than Σ itself. Then, it follows that Σ is the outermost boundary of past-trapped surfaces on the hypersurface of constant u , completing the proof.

The line element of Σ takes the following form

$$ds^2(\Sigma) = \left(\frac{U}{\sigma}\right)^2 (d\theta^2 + \sin^2\theta d\phi^2) , \quad (3.25)$$

where σ is given by $\Phi = \Phi_0 - 2\log\sigma$. The area of Σ is determined by taking the integral over the unit round sphere,

$$\text{Area}(\Sigma) = \int_{S^2} d\mu_0 \left(\frac{U}{\sigma}\right)^2 , \quad (3.26)$$

where $d\mu_0 = \sin\theta d\theta d\phi$. Clearly, $\text{Area}(\Sigma)$ depends on u .

Using the late time expansion of solutions to the Robinson-Trautman equation, one can determine the late time expansion of the function U , and, hence, of the area of Σ . Using the first few terms of the late time expansion of axisymmetric deformations of S^2 characterized by

$$\sigma(\theta; u) = 1 + \frac{a}{3}(3x^2 - 1)e^{-2u/m} - \frac{a^2}{234}(69x^4 - 282x^2 + 49)e^{-4u/m} + \dots \quad (3.27)$$

with $x = \cos \theta$, we find

$$\begin{aligned} \frac{U}{\sigma}(\theta; u) = & r_h - \frac{amr_h}{2r_h + 3m}(3x^2 - 1)e^{-2u/m} + \frac{a^2mr_h}{78(2r_h + 3m)^2(3r_h + m)} \times \\ & \left((909m^2 + 2967mr_h + 716r_h^2)x^4 - 6 \frac{657m^3 + 2601m^2r_h + 2046mr_h^2 + 472r_h^3}{2r_h + 3m}x^2 \right. \\ & \left. + \frac{2025m^4 + 7470m^3r_h + 4275m^2r_h^2 + 142mr_h^3 - 408r_h^4}{(2r_h + 3m)(3m - r_h)} \right) e^{-4u/m} + \dots \end{aligned} \quad (3.28)$$

The result is complicated and becomes even more so to order $\mathcal{O}(e^{-6u/m})$. However, the final result for the area takes very simple form

$$\text{Area}(\Sigma) = 4\pi r_h^2 \left[1 + \frac{16a^2mr_h^2}{15(3m - r_h)(2r_h + 3m)^2} e^{-4u/m} + \mathcal{O}(e^{-6u/m}) \right], \quad (3.29)$$

providing the leading behavior of its u -dependence at very late retarded times for all $\Lambda \leq 0$. $\text{Area}(\Sigma)$ receives no corrections at the linearized level, since the integral of the Legendre polynomials vanishes. Also, the leading contribution of non-linear effects comes with positive sign which indicates that the area decreases at later retarded times, irrespective of the size of the black hole (recall that in all cases r_h cannot exceed $2m$).

An interesting open question is whether the area of the past apparent horizon decreases monotonically in u to all orders in the expansion, thus providing a new entropy functional for the Calabi flow on S^2 . Some results already exist in this direction for $\Lambda = 0$, [45], and we expect them to generalize to other values of Λ . We hope to return to this problem elsewhere with more details.

3.3 Penrose inequality

A natural quantity in Robinson-Trautman space-times is provided by the so called Bondi mass, [2], which is defined as follows

$$\mathcal{M}_{\text{Bondi}} = \frac{m}{4\pi} \int_{S^2} d\mu_0 \frac{1}{\sigma^3}, \quad (3.30)$$

taking the integral with respect to the unit round metric on S^2 . It incorporates the effect of gravitational radiation in space-time.

The Bondi mass is a function of the retarded time u , enjoying two important properties. First, $\mathcal{M}_{\text{Bondi}} \geq m$, which follows immediately from Holder's inequality

$$\left(\int_{S^2} d\mu_0 \frac{1}{\sigma^3} \right)^{2/3} \left(\int_{S^2} d\mu_0 \right)^{1/3} \geq \int_{S^2} d\mu_0 \frac{1}{\sigma^2}, \quad (3.31)$$

using also the fact that the evolution of the metric

$$ds^2(S) = \frac{1}{\sigma^2} (d\theta^2 + \sin^2 \theta d\phi^2), \quad (3.32)$$

preserves the volume of S^2 , i.e.,

$$\int_{S^2} d\mu_0 \frac{1}{\sigma^2} = 4\pi. \quad (3.33)$$

Second, using the Robinson-Trautman equation it follows that [18, 19]

$$\frac{d}{du} \mathcal{M}_{\text{Bondi}} \leq 0 . \tag{3.34}$$

$\mathcal{M}_{\text{Bondi}}$ decreases monotonically along the flow, reaching its minimal value m at $u = \infty$.

Using the late time expansion of the solutions to the Robinson-Trautman equation, as before, we find the following leading behavior of $\mathcal{M}_{\text{Bondi}}$,

$$\mathcal{M}_{\text{Bondi}} = m \left[1 + \frac{2a^2}{15} e^{-4u/m} + \mathcal{O} \left(e^{-6u/m} \right) \right] . \tag{3.35}$$

The Bondi mass equals m at the linearized level, since the integral of the Legendre polynomials vanishes, as before. Also, the leading contribution of non-linear effects comes with positive sign so that $\mathcal{M}_{\text{Bondi}}$ decreases at later retarded times, in agreement with the general relation (3.34).

The Bondi mass is subsequently used to formulate and test the validity of Penrose inequality [46] in Robinson-Trautman space-time. In fact, we will prove the following generalized version of Penrose inequality in the presence of cosmological constant $\Lambda \leq 0$, as first stated by Gibbons [47] (but see also reference [48] for an overview of the subject),

$$16\pi \mathcal{M}^2 \geq \text{Area}(\Sigma) \left(1 - \frac{\Lambda}{3} \frac{\text{Area}(\Sigma)}{4\pi} \right)^2 , \tag{3.36}$$

using the area of the past apparent horizon Σ and letting

$$\mathcal{M} = \mathcal{M}_{\text{Bondi}} . \tag{3.37}$$

We admit, however, that we do not yet have complete justification for choosing the same $\mathcal{M}_{\text{Bondi}}$ for all $\Lambda \leq 0$ other than it works for Robinson-Trautman space-times.

It is convenient for our purpose to rewrite the differential equation for Σ in the form

$$\frac{2m}{\sigma^3} = \frac{U}{\sigma} \left[1 - 2\Delta_0 \left(\log \frac{U}{\sigma} \right) \right] - \frac{\Lambda}{3} \left(\frac{U}{\sigma} \right)^3 , \tag{3.38}$$

where $\Delta_0 = e^{-\Phi_0} \partial \bar{\partial}$ provides the Laplacian on the unit round sphere. Upon integration over the unit sphere with measure $d\mu_0$, we obtain

$$8\pi \mathcal{M}_{\text{Bondi}} = \int_{S^2} d\mu_0 \frac{U}{\sigma} \left[1 - 2\Delta_0 \left(\log \frac{U}{\sigma} \right) \right] - \frac{\Lambda}{3} \int_{S^2} d\mu_0 \left(\frac{U}{\sigma} \right)^3 . \tag{3.39}$$

Since for a function f on the unit sphere we have the relation $2f\Delta_0(\log f) = 2\Delta_0 f - |\nabla f|^2/f$, we arrive to the following identity

$$\int_{S^2} d\mu_0 \left(f + \frac{|\nabla f|^2}{f} \right) = \int_{S^2} d\mu_0 f (1 - 2\Delta_0(\log f)) , \tag{3.40}$$

dropping the contribution of $\Delta_0 f$ to the integral (it is a total derivative term on the round S^2). Furthermore, as was shown by Tod, [49], using a combination of Sobolev and

Holder inequalities, one has the following relation for all functions f , provided that they are nowhere vanishing,

$$\int_{S^2} d\mu_0 \left(f + \frac{|\nabla f|^2}{f} \right) \geq \left(4\pi \int_{S^2} d\mu_0 f^2 \right)^{1/2}. \quad (3.41)$$

To prove Penrose inequality for Robinson-Trautman space-times, it suffices to choose

$$f = \frac{U}{\sigma} \quad (3.42)$$

so that the first integral on the right hand-side of equation (3.39) is bounded from below by the square root of the area functional, as

$$\int_{S^2} d\mu_0 \frac{U}{\sigma} \left[1 - 2\Delta_0 \left(\log \frac{U}{\sigma} \right) \right] \geq \left(4\pi \int_{S^2} d\mu_0 \left(\frac{U}{\sigma} \right)^2 \right)^{1/2} \equiv \sqrt{4\pi \text{Area}(\Sigma)}. \quad (3.43)$$

The second integral on the right hand-side of equation (3.39) is also bounded from below, as can be easily seen using Holder's inequality

$$\left(\int_{S^2} d\mu_0 \left(\frac{U}{\sigma} \right)^3 \right)^{2/3} \left(\int_{S^2} d\mu_0 \right)^{1/3} \geq \int_{S^2} d\mu_0 \left(\frac{U}{\sigma} \right)^2. \quad (3.44)$$

Then, for $\Lambda < 0$ we have the inequality

$$-\frac{\Lambda}{3} \int_{S^2} d\mu_0 \left(\frac{U}{\sigma} \right)^3 \geq -\frac{\Lambda}{3} 4\pi \left(\sqrt{\frac{\text{Area}(\Sigma)}{4\pi}} \right)^3, \quad (3.45)$$

which when combined with (3.43) yields the Penrose inequality in the form (3.36) with $\mathcal{M} = \mathcal{M}_{\text{Bondi}}$. This derivation extends the proof of Penrose inequality for Robinson-Trautman space-times with $\Lambda = 0$, as described in reference [49], to the more general case $\Lambda \leq 0$.

As consistency check, one may use the late time expansion of $\mathcal{M}_{\text{Bondi}}$ and $\text{Area}(\Sigma)$ obtained above to verify the validity of Penrose's inequality for AdS_4 Robinson-Trautman space-times up to order $\exp(-4u/m)$. To zeroth order we get the identity $m = m$ and to order $\exp(-4u/m)$ we get the inequality

$$m \geq \frac{4mr_h^2}{(2r_h + 3m)^2}, \quad (3.46)$$

which is certainly true for black holes of all sizes. Similar results hold to higher order.

3.4 Thorne's hoop conjecture

Another mass inequality in space-times containing a black hole is provided by the so called hoop conjecture due to Thorne, [50], stating that

$$4\pi\mathcal{M} \geq C \quad (3.47)$$

for appropriately defined mass \mathcal{M} and circumference C . We expect that the hoop conjecture admits a suitable generalization in the presence of negative cosmological constant. We propose that for AdS_4 Robinson-Trautman space-times the following inequality holds

$$4\pi\mathcal{M}_{\text{Bondi}} \geq C(\Sigma) \left(1 - \frac{\Lambda}{3} \left(\frac{C(\Sigma)}{2\pi} \right)^2 \right), \quad (3.48)$$

where $C(\Sigma)$ is the length of a hoop around the past apparent horizon Σ . For definiteness, one may adopt Gibbons' proposal [51, 52] for the optimal hoop (but see also reference [53] for some important related work), choosing

$$C(\Sigma) = \beta(\Sigma) \quad (3.49)$$

given by the so called Birkhoff's invariant [54] of Σ . Recall that $\beta(\Sigma)$ is a geometric invariant providing the minimum length of a hoop (closed inextensible string) one can slip over a spherical two-dimensional surface Σ . There always exists a closed geodesic on Σ - though not necessarily the shortest - with length equal to $\beta(\Sigma)$. However, $\beta(\Sigma)$ encodes the notion of "least circumference" in all directions.

In general, it is not possible to construct $\beta(\Sigma)$ analytically. Instead, we begin by considering the length of the equatorial and meridional (polar) geodesic curves, using for simplicity axisymmetric metrics on Σ . The line element (3.25) yields, respectively,

$$l_e(\Sigma) = 2\pi \frac{U}{\sigma}(\theta = \pi/2), \quad l_p(\Sigma) = 2 \int_0^\pi d\theta \frac{U}{\sigma}(\theta), \quad (3.50)$$

where the factor of 2π in $l_e(\Sigma)$ arises from integration over the angle ϕ . We further set

$$C(\Sigma) = \min(l_e(\Sigma), l_p(\Sigma)). \quad (3.51)$$

Using the late time behavior of the solutions of the Robinson-Trautman equation, in particular the expansion (3.28) for U/σ , we find the following result as $u \rightarrow \infty$,

$$l_e(\Sigma) = 2\pi r_h \left[1 + \frac{am}{2r_h + 3m} e^{-2u/m} + \mathcal{O}(e^{-4u/m}) \right] \quad (3.52)$$

and

$$l_p(\Sigma) = 2\pi r_h \left[1 - \frac{am}{2(2r_h + 3m)} e^{-2u/m} + \mathcal{O}(e^{-4u/m}) \right]. \quad (3.53)$$

The leading order corrections to l_e and l_p are provided by the linear quadrupole radiation terms. The coefficient a is an arbitrary constant of either sign that determines the size of l_e relative to l_p . In either case, we have to leading order,

$$C(\Sigma) = \min(l_e(\Sigma), l_p(\Sigma)) < 2\pi r_h. \quad (3.54)$$

At the same time, as noted earlier, $\mathcal{M}_{\text{Bondi}}$ does not receive any corrections to order $\exp(-2u/m)$ and therefore inequality (3.48) is obviously valid. Of course, there are corrections to $l_e(\Sigma)$ and $l_p(\Sigma)$ appearing to order $\exp(-4u/m)$, as in the Bondi mass, but their

effect is subdominant and cannot change the result. Note that $C(\Sigma)$ increases with u at late times.

For higher poles of gravitational radiation, the spherical part of the Robinson-Trautman metric assumes the form (2.10),

$$ds_2^2 = [1 + \epsilon_l(0)P_l(\cos\theta)e^{-i\omega_s u}](d\theta^2 + \sin^2\theta d\phi^2), \tag{3.55}$$

to linear approximation. Such terms become relevant when all other subleading poles of radiation are absent from the solution. The corresponding conformal factor of the past apparent horizon has

$$\frac{U}{\sigma} = r_h \left(1 + \frac{3\epsilon_l(0)m}{(l-1)(l+2)r_h + 6m} P_l(\cos\theta)e^{-i\omega_s u} \right) \tag{3.56}$$

for all $l \geq 2$ (we set $\epsilon_2(0) = -4a/3$ to compare with the previous discussion). We can still use (3.51) to define a hoop $C(\Sigma)$, as for the quadrupole radiation. The equatorial and meridional closed geodesic curves, $l_e(\Sigma)$ and $l_p(\Sigma)$, are still given by equation (3.50). The integral of $P_l(\cos\theta)$ showing up in $l_p(\Sigma)$ is always positive. Note, however, that the sign of $P_l(0)$ appearing in $l_e(\Sigma)$ depends upon l : it is negative for $l = 4k - 2$, positive for $l = 4k$ and it vanishes for odd l . Consequently, $C(\Sigma)$ is definitely less than $2\pi r_h$ for $l = 4k - 2$, as for $l = 2$, and less or equal to $2\pi r_h$ for odd l . For $l = 4k$, comparison of $C(\Sigma)$ to $2\pi r_h$ depends on the sign of the arbitrary constant $\epsilon_l(0)$, making the definition (3.51) of $C(\Sigma)$ inappropriate. Then, in that case, one has to consider lassoing Σ sideways, trying to find better ways to satisfy the inequality (3.48), which to linear order reads

$$C(\Sigma) \left(1 - \frac{\Lambda}{3} \left(\frac{C(\Sigma)}{2\pi} \right)^2 \right) \leq 2\pi r_h \left(1 - \frac{\Lambda}{3} r_h^2 \right) = 4\pi m. \tag{3.57}$$

We will not pursue this calculation here, since there is a systematic way to prove the inequality for $l = 4k$, as for $l = 4k - 2$, at linear as well as non-linear level, as will be seen next. The Birkhoff length need not be equal to any one of the choices described above.

For Robinson-Trautman space-times that are axisymmetric and also admit antipodal symmetry, so that only terms with even l appear in the linearized approximation, it is possible to give a general proof of the hoop conjecture (3.48) for appropriate choice of $C(\Sigma)$. The proof puts in context the previous discussion and it is applicable to the full non-linear regime of Einstein equations. In that case, Σ also has antipodal symmetry by construction and, thus, its area is bounded from below

$$\text{Area}(\Sigma) \geq \frac{l^2(\Sigma)}{\pi} \tag{3.58}$$

in terms of the length $l(\Sigma)$ of the shortest non-trivial closed geodesic on Σ (the area and length are both computed with respect to a given metric on Σ). This is a rather old mathematical result due to Pu, [55], which is applicable to all metrics on 2-spheres with antipodal symmetry. Then, the hoop inequality (3.48) follows immediately from the generalized form of Penrose inequality (3.36) that was proved before, choosing $C(\Sigma) = l(\Sigma)$.

As such, it generalizes the relation between Penrose and Thorne inequalities described in [51, 52], but now in the presence of cosmological constant $\Lambda \leq 0$. Gibbon’s version of the hoop conjecture provides a more stringent lower bound, since the Birkhoff invariant is not necessarily the length of the shortest closed geodesic, i.e., $\beta(\Sigma) \geq l(\Sigma)$, but we do not yet have a general proof of it for Robinson-Trautman space-times.

A related interesting open question is whether the length of the hoop, being $l(\Sigma)$ or $\beta(\Sigma)$, exhibits monotonicity under Calabi flow that governs the evolution of the past apparent horizon.

4 Holographic aspects

In this section, we briefly review the construction of the energy-momentum tensor for AdS gravity, based on holography [56–65], and apply it to the general class of Robinson-Trautman metrics with negative cosmological constant. Then, the resulting energy-momentum tensor is specialized to the Schwarzschild solution and its perturbations with respect to the algebraically special modes and determine their effective viscosity. The complete expression also includes non-linear effects of gravitational radiation, which can be described systematically at late times.

4.1 Energy-momentum tensor

The AdS_4 Robinson-Trautman solution is an example of an asymptotically locally AdS space-time.² For these space-times the metric near the conformal boundary \mathcal{I} takes the Fefferman-Graham form

$$ds^2 = -\frac{3}{\Lambda} \left[\frac{d\varrho^2}{\varrho^2} + \frac{1}{\varrho^2} (g_{(0)ab}(x) + \varrho^2 g_{(2)ab}(x) + \varrho^3 g_{(3)ab}(x) + \dots) dx^a dx^b \right], \quad (4.1)$$

where $\sqrt{-3/\Lambda}$ is the AdS radius and $\varrho = 0$ is the location of the conformal boundary. If the conformal boundary is topologically $\mathbb{R} \times S^2$ and $g_{(0)}$ is conformally flat then the space-time is asymptotically AdS, otherwise it is only asymptotically locally AdS. Asymptotically locally AdS space-times come equipped with a conserved symmetric tensor, T_{ab} , which in even dimensions is also traceless, i.e.,

$$\nabla^b T_{ab} = 0, \quad T_a^a = 0, \quad (4.2)$$

where ∇^a is the covariant derivative associated to $g_{(0)ab}$. This tensor can be extracted from the asymptotics of the solution, [62],

$$T_{ab} = -\frac{3}{2\kappa^2} \left(-\frac{3}{\Lambda} \right) g_{(3)ab} \quad (4.3)$$

²We refer the reader to section 3 of the original work [64] for the precise definition. Note, however, that asymptotically locally AdS space-times were called asymptotically AdS in that reference. Here, as in most of the literature, we reserve the terminology asymptotically AdS only for space-times that look like AdS close to the boundary.

and is called the holographic energy-momentum tensor because it also represents the expectation value of the energy-momentum tensor in the dual QFT, [59–62, 64], as

$$\langle T_{ab} \rangle = \frac{2}{\sqrt{-\det g_{(0)}}} \frac{\delta S_{\text{ren}}}{\delta g_{(0)}^{ab}}. \quad (4.4)$$

S_{ren} is the on-shell gravitational action supplemented by covariant counter-terms to remove the infinite volume divergences, [59–62]. It provides a well defined prescription for implementing the Brown-York proposal for quasi-local energy, [66], without using a reference space-time. One can show that if the space-time possesses asymptotic Killing vectors, there will be conserved charges, which one can compute using T_{ab} , [65] (but see also references [61, 63]). Specializing to asymptotically AdS space-times, which is the only case that was discussed in all generality in earlier works, one recovers previous prescriptions for the computation of conserved charges.

Thus, in order to compute the holographic energy-momentum tensor, it suffices to change to Fefferman-Graham coordinates near the conformal boundary and then extract the coefficient $g_{(3)}$. The Fefferman-Graham coordinates are Gaussian normal coordinates centered at the conformal boundary. To reach these coordinates we need to ensure that $g_{\varrho\varrho} = 1/\varrho^2$ and the off-diagonal terms between ϱ and x^a are zero up to sufficiently high order. In practice, we first change variables $u = t - r_*$, introducing explicit r -dependence into the function $\Phi(z, \bar{z}; u)$ so that the evolution is taken with respect to the real time t rather than the retarded time u ; of course, we have $\partial_u \Phi = \partial_t \Phi$. Since the conformal boundary is located at $r_* = 0$, we are led to perform the following change of variables,

$$r_* \rightarrow \varrho - \frac{1}{2}(\partial_t \hat{\Phi})\varrho^2 + \left[\frac{\Lambda}{9} + \frac{3}{16}(\partial_t \hat{\Phi})^2 + \frac{1}{4}\partial_t^2 \hat{\Phi} + \frac{\Lambda}{12}\hat{\Delta}\hat{\Phi} \right]\varrho^3 - \frac{1}{48} \left[3(\partial_t \hat{\Phi})^3 + 8\partial_t \hat{\Phi}(3 + \partial_t^2 \hat{\Phi}) + 4(2m + \partial_t^3 \hat{\Phi}) + 4(3\hat{\Delta}\hat{\Phi}\partial_t \hat{\Phi} + 2\partial_t(\hat{\Delta}\hat{\Phi})) \right]\varrho^4 + \mathcal{O}(\varrho^5), \quad (4.5)$$

$$t \rightarrow t - \frac{1}{2}(\partial_t \hat{\Phi})\varrho^2 + \frac{1}{6} \left[\frac{2\Lambda}{3} + (\partial_t \hat{\Phi})^2 + \partial_t^2 \hat{\Phi} + \frac{2\Lambda}{3}\hat{\Delta}\hat{\Phi} \right]\varrho^3 - \frac{1}{32} \left[2(\partial_t \hat{\Phi})^3 + \partial_t \hat{\Phi} \left(\frac{16\Lambda}{3} + 5\partial_t^2 \hat{\Phi} \right) + 2\partial_t^3 \hat{\Phi} + \frac{2\Lambda}{3}(4\hat{\Delta}\hat{\Phi}\partial_t \hat{\Phi} + 3\partial_t(\hat{\Delta}\hat{\Phi})) \right]\varrho^4 + \mathcal{O}(\varrho^5), \quad (4.6)$$

$$z \rightarrow z + e^{-\hat{\Phi}}\partial_{\bar{z}} \left[-\frac{\Lambda}{18}(\partial_t \hat{\Phi})\varrho^3 + \frac{\Lambda}{48} \left(\partial_t^2 \hat{\Phi} - \frac{1}{4}(\partial_t \hat{\Phi})^2 + \frac{\Lambda}{3}\hat{\Delta}\hat{\Phi} \right)\varrho^4 \right] + \mathcal{O}(\varrho^5), \quad (4.7)$$

$$\bar{z} \rightarrow \bar{z} + e^{-\hat{\Phi}}\partial_z \left[-\frac{\Lambda}{18}(\partial_t \hat{\Phi})\varrho^3 + \frac{\Lambda}{48} \left(\partial_t^2 \hat{\Phi} - \frac{1}{4}(\partial_t \hat{\Phi})^2 + \frac{\Lambda}{3}\hat{\Delta}\hat{\Phi} \right)\varrho^4 \right] + \mathcal{O}(\varrho^5), \quad (4.8)$$

where $\hat{\Phi}$ is the boundary value of Φ ,

$$\hat{\Phi}(z, \bar{z}; t) = \lim_{r_* \rightarrow 0} \Phi(z, \bar{z}; u). \quad (4.9)$$

Note that $\hat{\Phi}$ has the same functional form as Φ , since $u = t - r_*$, and it satisfies the fourth-order diffusion equation in real time,

$$3m\partial_t \hat{\Phi} + \hat{\Delta}\hat{\Delta}\hat{\Phi} = 0, \quad (4.10)$$

where $\hat{\Delta}$ is the Laplacian on the spherical spatial slices of the three-dimensional boundary \mathcal{I} ,

$$\hat{\Delta} = e^{-\hat{\Phi}} \partial_z \partial_{\bar{z}} . \tag{4.11}$$

After this change of variables, the space-time metric takes the form (4.1) with the metric components $g_{(0)}, g_{(2)}$ and $g_{(3)}$ given by

$$\begin{aligned} ds_{(0)}^2 &= -dt^2 - \frac{6}{\Lambda} e^{\hat{\Phi}} dz d\bar{z} \\ ds_{(2)}^2 &= \frac{1}{2} \left[\frac{1}{4} (\partial_t \hat{\Phi})^2 + \partial_t^2 \hat{\Phi} - \frac{\Lambda}{3} \hat{\Delta} \hat{\Phi} \right] dt^2 + \left[\frac{3}{4\Lambda} (\partial_t \hat{\Phi})^2 + \hat{\Delta} \hat{\Phi} \right] e^{\hat{\Phi}} dz d\bar{z} + [(\partial_z \partial_t \hat{\Phi}) dt dz]_{\text{c.c.}} \\ ds_{(3)}^2 &= -\frac{4m\Lambda^2}{27} dt^2 + \frac{4m\Lambda}{9} e^{\hat{\Phi}} dz d\bar{z} - \frac{1}{3} \left[\frac{2\Lambda}{3} \partial_z (\hat{\Delta} \hat{\Phi}) dt dz + \partial_t \left(\frac{1}{2} (\partial_z \hat{\Phi})^2 - \partial_z^2 \hat{\Phi} \right) dz^2 \right]_{\text{c.c.}} \end{aligned} \tag{4.12}$$

where c.c. means that one has to add the complex conjugate to the corresponding bracketed terms.

The metric $g_{(0)}$ is the representative of the boundary conformal structure. Thus, we set

$$ds_{\mathcal{I}}^2 = -dt^2 - \frac{6}{\Lambda} e^{\hat{\Phi}} dz d\bar{z} . \tag{4.13}$$

To check whether this metric is conformally flat, we compute its Cotton tensor. Recall that the Cotton tensor of a three-dimensional metric γ_{ab} is defined as follows,

$$\begin{aligned} C^{ab} &= \frac{1}{2\sqrt{-\det\gamma}} \left(\epsilon^{acd} \nabla_c R^b{}_d + \epsilon^{bcd} \nabla_c R^a{}_d \right) \\ &= \frac{\epsilon^{acd}}{\sqrt{-\det\gamma}} \nabla_c \left(R^b{}_d - \frac{1}{4} \delta^b{}_d R \right) , \end{aligned} \tag{4.14}$$

letting $\epsilon^{tz\bar{z}} = i$ (equivalently, $\epsilon^{t\theta\phi} = 1$ in spherical coordinates). It is a symmetric and traceless tensor that is covariantly conserved identically, without employing the classical equations of motion. The density $\sqrt{\det\gamma} C^a{}_b$ remains invariant under local conformal changes of the metric γ_{ab} and it vanishes if and only if the metric is conformally flat. In our case, it takes the form

$$C_{zz} = -\frac{i}{4} \partial_t \left((\partial_z \hat{\Phi})^2 - 2\partial_z^2 \hat{\Phi} \right) , \quad C_{tz} = -i \frac{\Lambda}{6} \partial_z (\hat{\Delta} \hat{\Phi}) \tag{4.15}$$

and

$$C_{\bar{z}\bar{z}} = -\bar{C}_{zz} , \quad C_{t\bar{z}} = -\bar{C}_{tz} , \tag{4.16}$$

whereas all other components vanish identically. Thus, $g_{(0)}$ is not conformally flat for general Robinson-Trautman solutions; these solutions are indeed only asymptotically locally AdS.

The asymptotic analysis of reference [62] implies that the $g_{(2)}$ should be given by

$$g_{(2)ab} = -\mathcal{R}_{ab} + \frac{1}{4} \mathcal{R} g_{(0)ab} , \tag{4.17}$$

where $\mathcal{R}_{ij}, \mathcal{R}$ are the Ricci and scalar curvatures of $g_{(0)}$. Indeed, using

$$\begin{aligned} \mathcal{R}_{tt} &= -\partial_t^2 \hat{\Phi} - \frac{1}{2}(\partial_t \hat{\Phi})^2, & \mathcal{R}_{z\bar{z}} &= -\frac{1}{2}e^{\hat{\Phi}} \left(\frac{3}{\Lambda} \left(\partial_t^2 \hat{\Phi} + (\partial_t \hat{\Phi})^2 \right) + 2\hat{\Delta} \hat{\Phi} \right), \\ \mathcal{R}_{tz} &= -\frac{1}{2}\partial_t \partial_z \hat{\Phi}, & \mathcal{R}_{t\bar{z}} &= -\frac{1}{2}\partial_t \partial_{\bar{z}} \hat{\Phi}, & \mathcal{R} &= 2\partial_t^2 \hat{\Phi} + \frac{3}{2}(\partial_t \hat{\Phi})^2 + \frac{2\Lambda}{3}\hat{\Delta} \hat{\Phi}, \end{aligned} \quad (4.18)$$

one can check that this is the case.

Finally, the energy-momentum tensor follows from $g_{(3)}$ via equation (4.3). We obtain

$$\kappa^2 T_{tt} = -\frac{2m\Lambda}{3}, \quad \kappa^2 T_{tz} = -\frac{1}{2}\partial_z(\hat{\Delta} \hat{\Phi}), \quad (4.19)$$

$$\kappa^2 T_{z\bar{z}} = me^{\hat{\Phi}}, \quad \kappa^2 T_{zz} = -\frac{3}{4\Lambda}\partial_t \left((\partial_z \hat{\Phi})^2 - 2\partial_z^2 \hat{\Phi} \right), \quad (4.20)$$

whereas

$$T_{t\bar{z}} = \bar{T}_{tz}, \quad T_{\bar{z}\bar{z}} = \bar{T}_{zz}. \quad (4.21)$$

Note that the energy-momentum tensor is traceless and conserved using the classical equations of motion, $T^a_a = 0$ and $\nabla^a T_{ab} = 0$, as required on general grounds.

Note also the following relations between the components of the energy-momentum and the Cotton tensor,

$$C_{zz} = i\frac{\Lambda}{3}\kappa^2 T_{zz}, \quad C_{\bar{z}\bar{z}} = -i\frac{\Lambda}{3}\kappa^2 T_{\bar{z}\bar{z}}, \quad (4.22)$$

$$C_{tz} = i\frac{\Lambda}{3}\kappa^2 T_{tz}, \quad C_{t\bar{z}} = -i\frac{\Lambda}{3}\kappa^2 T_{t\bar{z}}. \quad (4.23)$$

Despite appearances, all components of the energy-momentum as well as the Cotton tensor are real when written in spherical coordinates (θ, ϕ) .

4.2 Linearization of T_{ab}

We first apply the formulae to the simple example of static AdS_4 Schwarzschild solution that serves as reference frame to study the effect of linear (as well as non-linear) perturbations. In this case, the three-dimensional metric on \mathcal{S} describes an Einstein universe, which is conformally flat, written in spherical coordinates as

$$ds_{\mathcal{S}}^2 = -dt^2 - \frac{3}{\Lambda}(d\theta^2 + \sin^2 \theta d\phi^2). \quad (4.24)$$

The renormalized energy-momentum tensor has the following non-vanishing components

$$\kappa^2 T_{tt}^{(0)} = -\frac{2m\Lambda}{3}, \quad \kappa^2 T_{\theta\theta}^{(0)} = m, \quad \kappa^2 T_{\phi\phi}^{(0)} = m \sin^2 \theta, \quad (4.25)$$

reproducing the expressions already known in the literature. The superscript (0) is used for reference to the static background.

Next, we consider the algebraically special modes with purely imaginary frequencies

$$\omega_s = -i \frac{(l-1)l(l+1)(l+2)}{12m} \quad (4.26)$$

that arise to linear order as perturbations of AdS_4 Schwarzschild metric and compute their contribution to the energy-momentum tensor. Assuming, for simplicity, that the perturbations are axially symmetric, so that

$$\hat{\Phi}(\theta; t) = e^{-i\omega_s t} P_l(\cos\theta) + 4\log\left(\sin\frac{\theta}{2}\right), \quad (4.27)$$

the three-dimensional metric on \mathcal{S} takes the form

$$ds_{\mathcal{S}}^2 = -dt^2 - \frac{3}{\Lambda}[1 + e^{-i\omega_s t} P_l(\cos\theta)](d\theta^2 + \sin^2\theta d\phi^2), \quad (4.28)$$

whereas T_{ab} is

$$T_{ab} = T_{ab}^{(0)} + \delta T_{ab}. \quad (4.29)$$

We have $\delta T_{t\phi} = 0 = \delta T_{\theta\phi}$, as consequence of axial symmetry, and $\delta T_{tt} = 0$. The remaining components turn out to be

$$\begin{aligned} \kappa^2 \delta T_{\theta\theta} = m \left(1 + \frac{3i\omega_s}{4m\Lambda} l(l+1) \right) e^{-i\omega_s t} P_l(\cos\theta) + \\ \frac{3i\omega_s}{2\Lambda} e^{-i\omega_s t} \cot\theta \partial_{\theta} P_l(\cos\theta) \end{aligned} \quad (4.30)$$

$$\begin{aligned} \kappa^2 \delta T_{\phi\phi} = m \left(1 - \frac{3i\omega_s}{4m\Lambda} l(l+1) \right) e^{-i\omega_s t} \sin^2\theta P_l(\cos\theta) - \\ \frac{3i\omega_s}{2\Lambda} e^{-i\omega_s t} \sin\theta \cos\theta \partial_{\theta} P_l(\cos\theta) \end{aligned} \quad (4.31)$$

$$\kappa^2 \delta T_{t\theta} = \frac{1}{4}(l-1)(l+2)e^{-i\omega_s t} \partial_{\theta} P_l(\cos\theta). \quad (4.32)$$

4.3 Effective viscosity

We are now in position to examine whether the algebraically special models admit an effective hydrodynamic representation.

Given a conserved energy-momentum tensor with non-negative energy density, i.e., assuming $T_{ab}u^a u^b \geq 0$ for all time-like vectors, it is always possible to solve the eigenvalue problem

$$T_{ab}u^b = -\rho u_a \quad (4.33)$$

in terms of a unique time-like vector u^a that is normalized as $u^a u_a = -1$. This vector defines a frame, usually called the energy or Landau frame, which we adopt in the following. The energy-momentum tensor of a fluid admits the decomposition

$$T^{ab} = \rho u^a u^b + p\Delta^{ab} + \Pi^{ab}, \quad (4.34)$$

where

$$\Delta^{ab} = u^a u^b + g^{ab} \quad (4.35)$$

and ρ , p are the energy density and pressure, respectively, in the local rest frame. For conformal fluids in three space-time dimensions ρ and p obey the equation of state $\rho = 2p$.

Π^{ab} is a transverse tensor, $u_a \Pi^{ab} = 0$, that describes the viscous part of the energy-momentum tensor of a fluid and it admits an expansion in derivatives of u^a ,

$$\Pi^{ab} = \Pi_{(1)}^{ab} + \Pi_{(2)}^{ab} + \dots \quad (4.36)$$

First order hydrodynamics is concerned with the structure of $\Pi_{(1)}^{ab}$ and is well studied (see, for instance, the classic reference [67]). For notational purposes, we use the bracketed tensor associated to any second rank tensor A^{ab} in three dimensions,

$$A^{<ab>} = \frac{1}{2} \left(\Delta^{ac} \Delta^{bd} (A_{cd} + A_{dc}) - \Delta^{ab} \Delta^{cd} A_{cd} \right), \quad (4.37)$$

which is transverse, $u_a A^{<ab>} = 0$, and traceless, $g_{ab} A^{<ab>} = 0$. Then, the viscosity tensor has the following general form in relativistic first order hydrodynamics,

$$\Pi_{(1)}^{ab} = -\eta \sigma^{ab} - \zeta \Delta^{ab} (\nabla_c u^c), \quad (4.38)$$

where

$$\sigma^{ab} = 2 \nabla^{<a} u^{b>} \quad (4.39)$$

expresses the symmetric, transverse and traceless part of Π^{ab} up to first derivatives in u^a . The coefficients η and ζ depend, in general, on ρ and they are called shear and bulk viscosity, respectively. Conformal fluids have $\zeta = 0$, the speed of sound in three dimensions is $1/\sqrt{2}$ and η is non-zero. This is precisely the case we are considering here.

The energy-momentum tensor associated to the static AdS_4 black hole represents a perfect fluid with velocity vector $u_a = (-1, 0, 0)$ and energy density

$$\kappa^2 \rho = -\frac{2m\Lambda}{3}. \quad (4.40)$$

Switching on perturbations, due to algebraically special modes, yield a velocity vector in the energy frame with components

$$u_t = -1, \quad u_\phi = 0, \quad (4.41)$$

as before, whereas u_θ changes to

$$u_\theta = \frac{1}{4m\Lambda} (l-1)(l+2) e^{-i\omega_s t} \partial_\theta P_l(\cos \theta). \quad (4.42)$$

The energy density ρ is not affected by the perturbations. Then, it is straightforward to compute the deviation from the perfect fluid form. The corresponding viscosity tensor has components

$$\kappa^2 \Pi_{\theta\theta}^{(1)} = \frac{1}{16m\Lambda} (l-1)l(l+1)(l+2) e^{-i\omega_s t} [l(l+1)P_l(\cos \theta) + 2\cot\theta \partial_\theta P_l(\cos \theta)], \quad (4.43)$$

whereas

$$\Pi_{\phi\phi}^{(1)} = -\sin^2 \theta \Pi_{\theta\theta}^{(1)}. \quad (4.44)$$

All other components vanish identically.

The shear viscosity associated to the l -th algebraically special mode turns out to be

$$\kappa^2 \eta = \frac{1}{4} l(l+1), \tag{4.45}$$

as can be easily verified. The dependence of η upon l implies that the ratio of shear viscosity to the entropy density of an AdS_4 black hole,

$$\frac{\eta}{s} = \frac{4}{r_h^2} \left(-\frac{3}{\Lambda} \right) \eta = \frac{l(l+1)}{8\pi} \frac{r_h}{2m - r_h}, \tag{4.46}$$

also depends upon l and thus η/s differs from the celebrated KSS value $1/4\pi$, [68–70]. In fact, large black holes with sufficiently low l , so that

$$\frac{m}{r_h} > \frac{l(l+1) + 2}{4}, \tag{4.47}$$

appear to violate the KSS bound, since

$$\frac{\eta}{s} < \frac{1}{4\pi}. \tag{4.48}$$

Large black holes with higher values of l satisfy $\eta/s \geq 1/4\pi$. On the other hand, small black holes satisfy the inequality $\eta/s > 1/4\pi$ for all values of l .

In AdS/CFT correspondence, linear perturbations around AdS black black satisfying Dirichlet boundary conditions at the conformal boundary and incoming boundary conditions at the horizon compute retarded thermal correlators. From those one can compute the shear viscosity via the Kubo formula. For such fluctuations there is a holographic derivation of the KSS bound. To obtain the correct physical interpretation of the results for the algebraically special modes one would need to understand first what exactly are these fluctuation computing in the dual QFT.

Note also that there is some correlation between the global structure of Robinson-Trautman space-times and the violation of the KSS bound by the algebraically special modes. As explained before, the algebraically special perturbations of large AdS_4 black holes do not have a Kruskal extension across the horizon for sufficiently small values of l . In fact, for the values of l constrained by inequality (4.47) we have $2i\omega_s \delta_h < 1$, in the notation of section 3.1, and this leads to divergencies on the null hypersurface \mathcal{H}^+ that are correlated with the result $\eta/s < 1/4\pi$. The divergencies on \mathcal{H}^+ wash away for very high values of l , since $2i\omega_s \delta_h > 2$, in which case $\eta/s > 1/4\pi$. However, there are intermediate values of l that lead to divergencies on \mathcal{H}^+ , though milder, but they are still capable to produce $\eta/s > 1/4\pi$.

We can rewrite the result for the ratio η/s in simple form that is applicable to very large AdS_4 black holes. It has been observed in the literature, as result of numerical investigations, that very large AdS_4 black holes exhibit another set of purely dissipative modes with frequencies

$$\Omega_s = -i \frac{(l-1)(l+2)}{3r_h} \tag{4.49}$$

for all $l \geq 2$, [71, 72] (but see also reference [73] for a derivation). They correspond to *axial* perturbations, satisfying perfectly reflecting Dirichlet boundary conditions.³ Also, they are true hydrodynamic modes of very large AdS_4 black holes, saturating the KSS bound $\eta/s = 1/4\pi$, [74]. Then, using Ω_s , the ratio of shear viscosity to the entropy density of the algebraically special modes ω_s of very large (flat) black holes takes the following form

$$\frac{\eta}{s} = \frac{1}{4\pi} \frac{\omega_s}{\Omega_s}. \tag{4.50}$$

The deviations from the KSS bound depend on the size of ω_s relative to Ω_s for any given l . Actually, as pointed out in section 2, the values Ω_s provide the characteristic inverse time scale, call it τ_R , for the decay of linear perturbations of a round sphere under the normalized Ricci flow in the multi-pole expansion.⁴ Likewise, ω_s provide the characteristic inverse time scale, call it τ_C , for the decay of linear perturbations of a round sphere under the Calabi flow, and, therefore,

$$\frac{\omega_s}{\Omega_s} = \frac{\tau_R}{\tau_C}. \tag{4.51}$$

This ratio is used to compare how fast the curvature perturbations dissipate under the two curvature flows. For large black holes it becomes less than 1 for low lying l .

We emphasize that the late time asymptotic expansion of the energy-momentum tensor is not the same as the derivative expansion of the viscosity tensor Π^{ab} . Recall that first order hydrodynamics captures effects to first order in the derivatives while higher order hydrodynamics associated to $\Pi_{(n)}^{ab}$ terms captures terms of order n in the derivatives. Each of $\Pi_{(n)}^{ab}$ can be expanded in a basis of independent tensors and the coefficients of these tensors defined the n -th order transport coefficients. When we evaluate Π^{ab} using the late time expansion of the metric higher order $\Pi_{(n)}^{ab}$ may become degenerate with $\Pi_{(1)}^{ab}$ and the effective viscosity computed via the late time expansion would then be equal to the true viscosity plus a combinations of higher order transport coefficients. Thus, the hydrodynamic representation of the linear gravitational perturbations at late times is not in general sufficient in order to extract the true shear viscosity of the system.

4.4 Late time expansion of T_{ab}

The late time expansion of solutions to Robinson-Trautman equation, as described in section 2.4, can be used to obtain a similar expansion for the energy-momentum tensor and the boundary metric. In principle, one can compute recursively all higher order corrections to the energy-momentum tensor away from the static black hole solution, going beyond the linear approximation, but the expansion will be truncated here to order $\exp(-4m/t)$ for

³Obviously, the same set of modes also arise for polar perturbations satisfying mixed boundary conditions that are supersymmetric partner to those particular axial perturbations of very large AdS_4 black holes.

⁴Note, however, that unlike the Robinson-Trautman metric, which provides a non-linear extension of the algebraically special modes, there is no such extension known for the Ω_s modes. An interesting question is whether the normalized Ricci flow can be embedded into Einstein equations with cosmological constant $\Lambda < 0$, thus providing a new class of exact radiative metrics that settle down to very large black holes (at least up to $1/r_h^2$ corrections). If so, there will be an analytic explanation for the spectrum Ω_s .

illustration. The result captures the effect of the most dominant non-linear terms beyond the linear quadrupole approximation. We have

$$T_{ab} = T_{ab}^{(0)} - \frac{4a}{3} \delta T_{ab}^{(1)} + \frac{8a^2}{39} \delta T_{ab}^{(2)}, \quad (4.52)$$

where $\delta T_{ab}^{(1)}$ is the contribution of the lowest lying algebraically special mode, $l = 2$, which is of order $\exp(-2t/m)$, and $\delta T_{ab}^{(2)}$ denotes the first non-linear corrections of order $\exp(-4t/m)$.

Keeping a arbitrary and using spherical coordinates we have the following non-vanishing quadrupole perturbations, specializing our previous results to $l = 2$,

$$\kappa^2 \delta T_{\theta\theta}^{(1)} = \frac{m}{2} \left(3 \cos^2 \theta - 1 - \frac{9}{m^2 \Lambda} \sin^2 \theta \right) e^{-2t/m}, \quad (4.53)$$

$$\kappa^2 \delta T_{\phi\phi}^{(1)} = \frac{m}{2} \left(3 \cos^2 \theta - 1 + \frac{9}{m^2 \Lambda} \sin^2 \theta \right) \sin^2 \theta e^{-2t/m}, \quad (4.54)$$

$$\kappa^2 \delta T_{t\theta}^{(1)} = -3 \sin \theta \cos \theta e^{-2t/m}. \quad (4.55)$$

The first non-linear terms contributing to the renormalized energy-momentum tensor are

$$\begin{aligned} \kappa^2 \delta T_{\theta\theta}^{(2)} &= \frac{m}{2} \left(35 \cos^4 \theta - 43 \cos^2 \theta + \frac{22}{3} \right) e^{-4t/m} + \\ &\quad \frac{18}{m\Lambda} (9 \cos^4 \theta - 14 \cos^2 \theta + 5) e^{-4t/m}, \end{aligned} \quad (4.56)$$

$$\begin{aligned} \kappa^2 \delta T_{\phi\phi}^{(2)} &= \frac{m}{2} \left(35 \cos^4 \theta - 43 \cos^2 \theta + \frac{22}{3} \right) \sin^2 \theta e^{-4t/m} - \\ &\quad \frac{18}{m\Lambda} (9 \cos^4 \theta - 14 \cos^2 \theta + 5) \sin^2 \theta e^{-4t/m}, \end{aligned} \quad (4.57)$$

$$\kappa^2 \delta T_{t\theta}^{(2)} = 2 \sin \theta \cos \theta (21 \sin^2 \theta + 1) e^{-4t/m}. \quad (4.58)$$

Since axial symmetry has been used in the calculation, the $(t\phi)$ and $(\theta\phi)$ components of $T_{ab}^{(2)}$ are zero. Also, $\delta T_{tt}^{(2)}$ vanishes. As for the three-dimensional metric on \mathcal{S} , it takes the following form, up to this order,

$$\begin{aligned} ds_{\mathcal{S}}^2 &= -dt^2 - \frac{3}{\Lambda} \left[1 - \frac{2a}{3} (3 \cos^2 \theta - 1) e^{-2t/m} + \right. \\ &\quad \left. \frac{4a^2}{39} \left(35 \cos^4 \theta - 43 \cos^2 \theta + \frac{22}{3} \right) e^{-4t/m} \right] (d\theta^2 + \sin^2 \theta d\phi^2). \end{aligned} \quad (4.59)$$

Subleading non-linear corrections of order $\exp(-6m/t)$ or higher are more cumbersome to compute explicitly.

The Calabi flow on S^2 incorporates all higher viscosity terms in closed geometric form. One should note that these higher order corrections need not capture purely hydrodynamic modes.

4.5 Entropy production

In this subsection we discuss entropy production as the system approaches equilibrium. As is well known [67], a dissipative system with positive shear viscosity has an entropy current with non-negative divergence given by

$$s^a = s u^a \tag{4.60}$$

where s is the local entropy density.

In the hydrodynamic regime the system is in local thermal equilibrium and the local proper energy ρ , the pressure p , the local temperature \mathcal{T} and the local entropy density s satisfy the standard thermodynamic relations,

$$\rho + p = \mathcal{T} s, \quad d\rho = \mathcal{T} ds, \quad dp = s d\mathcal{T} . \tag{4.61}$$

Since we have a conformal fluid, the thermodynamic relations easily integrate to yield

$$s = \gamma \mathcal{T}^2, \quad \rho = 2p = \frac{2\gamma}{3} \mathcal{T}^3 . \tag{4.62}$$

The bulk solutions need to approach large AdS black holes at late times, for which $r_h \gg L = \sqrt{-3/\Lambda}$, in order to be in the hydrodynamic regime [75]. In this limit, the equilibrium values of ρ , p , \mathcal{T} and s are

$$\rho_0 = \frac{1}{\kappa^2} r_h^3 \left(-\frac{\Lambda}{3}\right)^2 = 2p_0, \quad \mathcal{T}_0 = \frac{3}{4\pi} r_h \left(-\frac{\Lambda}{3}\right) \quad s_0 = \frac{1}{4} r_h^2 \left(-\frac{\Lambda}{3}\right), \tag{4.63}$$

fixing $\gamma = -4\pi^2/3\Lambda$. In this limit, we also have $r_h^3 = -6m/\Lambda$, as discussed in appendix B. To work out the leading order corrections to the equilibrium values (4.63), we set

$$\rho = \rho_0 + \rho_1, \quad p = p_0 + p_1, \quad s = s_0 + s_1, \quad \mathcal{T} = \mathcal{T}_0 + \mathcal{T}_1 . \tag{4.64}$$

A short computation based on (4.61) shows that s_1 and \mathcal{T}_1 are determined in terms of ρ_1 , as

$$s_1 = \frac{\rho_1}{\mathcal{T}_0}, \quad \frac{\mathcal{T}_1}{\mathcal{T}_0} = \frac{1}{3} \frac{\rho_1}{\rho_0}, \tag{4.65}$$

whereas the correction to the pressure is obtained using $p = \rho/2$.

Using the late time expansion of T_{ab} found in section 4.4 together with the boundary metric (4.59), we solve $T_{ab} u^b = -\rho u_a$ to find,

$$\rho_1 = -\frac{16a^2}{3m\kappa^2} \sin^2 \theta \cos^2 \theta e^{-4t/m} \tag{4.66}$$

The components of the normalized time-like unit vector u_a are given to the same order by

$$\begin{aligned} u_t &= -1 + \frac{8a^2}{3m^2\Lambda} \sin^2 \theta \cos^2 \theta e^{-4t/m}, \\ u_\theta &= \frac{4a}{m\Lambda} \sin \theta \cos \theta e^{-2t/m} + \frac{8a^2}{39m\Lambda} \sin \theta \cos \theta \left(2 + 42 \sin^2 \theta - \frac{39}{m^2\Lambda} \sin^2 \theta \right) e^{-4t/m}, \\ u_\phi &= 0 \end{aligned} \tag{4.67}$$

and it turns out that there is a non-vanishing divergence

$$\nabla_a u^a = \frac{8a^2}{3m^3\Lambda} \sin^2 \theta (9 \cos^2 \theta - 1) e^{-4t/m}. \quad (4.68)$$

Actually, these particular results hold for all value of r_h , large or small compared to the AdS radius L , but they will be used next only in the limit of large r_h .

Using the equilibrium temperature of large AdS black holes \mathcal{T}_0 , it follows from (4.66) that

$$s_1 = \frac{\rho_1}{\mathcal{T}_0} = -\frac{16a^2}{9r_h^4} \left(-\frac{3}{\Lambda}\right)^2 \sin^2 \theta \cos^2 \theta e^{-4t/m}. \quad (4.69)$$

We observe that s_1 is negative and so the entropy s increases and becomes equal to that of the Schwarzschild solution in the limit $t \rightarrow \infty$, showing that there is entropy production in the problem at hand. We also compute the divergence of the entropy current, keeping leading order terms in the large r_h limit, and find

$$\nabla_a s^a = \frac{\partial s_1}{\partial t} + s_0 \nabla_a u^a = \frac{16a^2}{9r_h^7} \left(-\frac{3}{\Lambda}\right)^3 \sin^4 \theta e^{-4t/m}. \quad (4.70)$$

Thus, the entropy current has a non-negative divergence. Finally, one may check, using equations (4.43)–(4.44), that the divergence of the entropy current can be cast in the form

$$\nabla_a s^a = \frac{\eta}{2\mathcal{T}_0} \sigma_{ab} \sigma^{ab}, \quad (4.71)$$

as required on general grounds. This last step makes use of the form for σ_{ab} for quadrupole radiation to linear order, which was determined earlier in section 4.3,

$$\kappa^2 \eta \sigma_{\theta\theta} = -\frac{6a}{m\Lambda} \sin^2 \theta e^{-2t/m}, \quad \sigma_{\phi\phi} = -\sin^2 \theta \sigma_{\theta\theta} \quad (4.72)$$

setting $\kappa^2 \eta = 3/2$ (for $l = 2$) and taking proper account of the normalization of $\delta T_{ab}^{(1)}$ (factor of $-4a/3$) appearing in the late time expansion (4.52). The expressions for σ_{ab} shown in (4.72) are valid for black holes of all sizes, but we are applying them here only to the case of large r_h , $r_h \gg \sqrt{-3/\Lambda}$ letting $m = -\Lambda r_h^3/6$.

We have obtained the entropy current using local thermodynamics. One may wonder whether this current can also be obtained directly from a bulk computation, as in [76]. In a time dependent context one expects the entropy current to be associated with a future apparent horizon [77]. Future apparent horizons lie behind the event horizons and in our case a bulk surface whose area would give the corrected entropy would also have to lie behind the horizon at some $r < r_h$. However, as we discussed earlier, the Robinson-Trautman solutions do not extend smoothly across $r = r_h$ and, thus, one does not expect to be able to geometrize this entropy current.

5 Conclusions and discussion

We discussed various aspects of the AdS_4 Robinson-Trautman space-times. This particular class of solutions may be thought to describe the gravitational field outside a compact source

that radiates away its asymmetry and settles to a spherically symmetric configuration, which is the AdS_4 Schwarzschild solution. The corresponding asymptotically flat solution is well-studied and its global properties are well-known. In this paper we extended the analysis to the case of negative cosmological constant and also discussed holographic aspects of the solutions.

There is a number of similarities and differences between Robinson-Trautman metrics with zero and negative cosmological constant Λ . In either case there is a naked singularity in the past, as well as a past apparent horizon, which can be regarded as a surface surrounding the compact source. The evolution is governed by the Calabi flow on S^2 , which is a geometric evolution equation that describes certain volume preserving deformations of S^2 , and it is independent of Λ . At late times, the metric on S^2 approaches the constant curvature metric and the four-dimensional Robinson-Trautman solution becomes the Schwarzschild solution. More precise, as the retarded time u tends to infinity, the future horizon \mathcal{H}^+ of Schwarzschild space-time is reached. The deviation from Schwarzschild metric at linear order is described by algebraically special modes. Remarkably, these modes are supersymmetric zero energy states of the effective quantum mechanics problem for the linear perturbations of the Schwarzschild solution.

The algebraically special modes describe out-going gravitational radiation and they vanish at \mathcal{H}^+ . One may ask whether these modes, as well as the Robinson-Trautman solution, admit a Kruskal extension across the horizon. The result depends on the cosmological constant. When $\Lambda = 0$, all individual modes have a smooth extension, but this is severely affected by non-linear effects. When $\Lambda < 0$, the low lying multi-pole modes of large AdS_4 Schwarzschild black holes do not appear to have a Kruskal extension at all. The curvature invariants are smooth at \mathcal{H}^+ , but it is expected that at least one of the component of (covariant derivatives of) the curvature tensor blows up at \mathcal{H}^+ in all smooth coordinate systems that extend through \mathcal{H}^+ . This indicates that there is a null singularity at the future horizon.

We also considered the Bondi mass, which is a characteristic physical quantity of radiative solutions that decreases monotonically at late times. We proved that it satisfies a Penrose inequality and formulated and provided evidence for a version of the hoop conjecture for $\Lambda < 0$. While the Bondi mass is a natural concept in asymptotically flat gravity, measuring the mass at null infinity, its meaning is less clear with AdS asymptotics, since there is no null infinity in that case. In this paper, we assigned a Bondi mass to the AdS_4 Robinson-Trautman space-times in exact analogy with the $\Lambda = 0$ case. It would be interesting to understand whether the analogue of Bondi mass can be defined in general asymptotically locally AdS space-times from first principles.

The space-time has a holographic interpretation when the cosmological constant is negative. In that context, the relaxation to AdS_4 Schwarzschild describes the approach to equilibrium in the dual QFT. One of the basic observables is the holographic energy-momentum tensor T_{ab} . It encodes the ADM conserved charges from the bulk perspective and it computes the expectation value of the energy and momentum in the system from the dual QFT perspective. Near thermal equilibrium, T_{ab} takes the form of dissipative hydrodynamics. It turns out that the effective shear viscosity is not universal and for low

multi-poles the KSS bound is violated. The modes, however, are out-going rather than incoming at the horizon, they do not satisfy Dirichlet boundary conditions at the conformal boundary and they do not have a Kruskal extension across the future horizon. In that case, dissipation is due to the coupling to external sources. Thus, the physics of the system is very different from those studied in the literature (it is well known that the KSS bound can be violated by perturbations that become singular at the horizon, but the nature of the singularity is different here).

It would be interesting to understand better the gauge/gravity duality of these solutions, focusing on the meaning of the unusual boundary condition imposed by the algebraically special modes. It would also be interesting to understand the meaning of the Bondi mass from the perspective of the dual QFT as well as the boundary manifestation of the Penrose inequality and the hoop conjecture. Finally, it would be interesting to investigate whether one can produce a smooth metric by joining the Robinson-Trautman solution to matter-filled interior. This could lead to the exciting possibility of describing holographically compact sources. We hope to return to these problems elsewhere.

Note added: while this paper was in final states we received [78] which contains related material. In particular, the authors analyze the time-reversed Robinson-Trautman solution and present a thorough analysis of hydrodynamics. Their results for the energy-momentum tensor of Robinson-Trautman and the boundary Cotton tensor (after taken into account the time reversal) are identical to ours.

Acknowledgments

This research of I.B. is partially supported and implemented under the ARISTEIA action of the operational programme for education and long life learning and is co-funded by the European Union (European Social Fund) and National Resources of Greece. K.S. acknowledges support from a grant of the John Templeton Foundation. The opinions expressed in this publication are those of the authors and do not necessarily reflect the views of the John Templeton Foundation. Preliminary versions of this work were presented in a number of workshops, including the “New frontiers in dynamical gravity” held 24-28 of March 2014 at Cambridge, UK.

A Derivation of Robinson-Trautman equation

The Robinson-Trautman metric (1.1) with front factor F given by (1.2) involves a parameter m , which, in general, can be taken to depend upon u . Then, the Robinson-Trautman equation assumes the following general form, irrespective of the cosmological constant Λ , which can be positive, negative or zero,

$$3m(u)\partial_u\Phi + \Delta\Delta\Phi + 2\partial_u m(u) = 0 . \tag{A.1}$$

The derivation is done by successive integration of Einstein equations $R_{\mu\nu} = \Lambda g_{\mu\nu}$ after computing the components of the Ricci curvature tensor,

$$R_{uz} = \frac{1}{2}\partial_z(\partial_r F - \partial_u \Phi), \quad (\text{A.2})$$

$$R_{u\bar{z}} = \frac{1}{2}\partial_{\bar{z}}(\partial_r F - \partial_u \Phi), \quad (\text{A.3})$$

$$R_{z\bar{z}} = 2re^\Phi(\partial_u \Phi) - \partial_z \partial_{\bar{z}} \Phi - e^\Phi \partial_r(rF), \quad (\text{A.4})$$

$$R_{ur} = \frac{1}{2}\partial_r^2 F + \frac{1}{r}(\partial_r F - \partial_u \Phi), \quad (\text{A.5})$$

$$R_{uu} = -\partial_u^2 \Phi - \frac{1}{2}(\partial_u \Phi)^2 + \frac{1}{2}F \left(\partial_r^2 F + \frac{2}{r}\partial_r F \right) - \frac{1}{2}(\partial_r F)(\partial_u \Phi) + \frac{1}{r}\partial_u F + \frac{1}{r^2}\Delta F, \quad (\text{A.6})$$

whereas all other components vanish identically.

First, we integrate the (uz) and $(u\bar{z})$ components of Einstein equations and obtain that $\partial_r F - \partial_u \Phi$ is independent of z and \bar{z} . Thus, we have

$$\partial_r F - \partial_u \Phi = \lambda(r, u), \quad (\text{A.7})$$

where $\lambda(r, u)$ will be determined shortly. Next, we integrate the $(z\bar{z})$ component of Einstein equations and use the previous result to determine F as follows,

$$F = r(\partial_u \Phi) - \Delta \Phi - r\lambda(r, u) - \Lambda r^2. \quad (\text{A.8})$$

Taking the above into account, the (ur) component of Einstein equations yields a differential equation for $\lambda(r, u)$,

$$\partial_r^2(r\lambda(r, u)) = \frac{2}{r}\lambda(r, u), \quad (\text{A.9})$$

whose general solution is

$$\lambda(r, u) = \alpha(u)r + \frac{\beta(u)}{r^2}. \quad (\text{A.10})$$

Finally, using the (uu) component of Einstein equations, we arrive after some manipulation to the following equation,

$$\left[\partial_u \alpha(u) + (\partial_u \Phi) \left(\Lambda + \frac{3}{2}\alpha(u) \right) \right] r + \left[\partial_u \beta(u) + \Delta \Delta \Phi + \frac{3}{2}\beta(u)(\partial_u \Phi) \right] \frac{1}{r^2} = 0, \quad (\text{A.11})$$

which is valid for all r . Thus, the coefficients of the r and $1/r^2$ should vanish separately and one obtains, respectively, the system of equations

$$\partial_u \alpha(u) + (\partial_u \Phi) \left(\Lambda + \frac{3}{2}\alpha(u) \right) = 0, \quad (\text{A.12})$$

$$\partial_u \beta(u) + \Delta \Delta \Phi + \frac{3}{2}\beta(u)(\partial_u \Phi) = 0. \quad (\text{A.13})$$

Since Φ depends upon z and \bar{z} , apart from u , the first equation implies automatically that

$$\alpha(u) = -\frac{2}{3}\Lambda, \quad (\text{A.14})$$

whereas the second equation, setting

$$\beta(u) = 2m(u), \tag{A.15}$$

implies the Robinson-Trautman equation (A.1) with u -dependent mass term m , in general.

Note, however, that the Robinson-Trautman line element, as we have it so far, is form-invariant under the following class of coordinate transformations,

$$u' = \rho(u), \quad r' = \frac{r}{\partial_u \rho(u)}, \quad z' = f(z), \quad \bar{z}' = \bar{f}(\bar{z}) \tag{A.16}$$

provided that

$$e^{\Phi'(z', \bar{z}'; u')} = \frac{(\partial_u \rho(u))^2}{|\partial f|^2} e^{\Phi(z, \bar{z}; u)} \tag{A.17}$$

and

$$m'(u') = \frac{m(u)}{(\partial_u \rho(u))^3}. \tag{A.18}$$

Thus, by appropriate choice of $\rho(u)$ one can fix $m(u)$ to a constant and transform the RT equation to the familiar form. This analysis also shows how $\Phi(z, \bar{z}; u)$ should transform under changes of z and \bar{z} coordinates.

B Large and small AdS_4 black holes

Einstein equations in four space-time dimensions with cosmological constant Λ admit the Schwarzschild metric as spherically symmetric static solution,

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \tag{B.1}$$

with profile function

$$f(r) = 1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2 \tag{B.2}$$

that gives the appropriate asymptotic behavior, as $r \rightarrow \infty$, for all values of Λ .

Here, we mainly consider AdS_4 Schwarzschild metric, having $\Lambda < 0$. The Schwarzschild radius of AdS_4 black holes is provided by the real root of $f(r) = 0$ occurring at

$$r_h = \frac{1}{\sqrt{-\Lambda}} \left[\left(\sqrt{1 - 9m^2\Lambda} + 3m\sqrt{-\Lambda} \right)^{1/3} - \left(\sqrt{1 - 9m^2\Lambda} - 3m\sqrt{-\Lambda} \right)^{1/3} \right]. \tag{B.3}$$

Thus, the black hole radius takes values $0 < r_h < 2m$ depending on the size of Λ . When Λ approaches zero, r_h tends to $2m$, whereas for $\Lambda \ll 0$, r_h comes close to 0.

It is also useful to introduce the *tortoise* coordinate r_* , which is defined by

$$dr_* = \frac{dr}{f(r)}. \tag{B.4}$$

When $\Lambda = 0$, r_* ranges from $-\infty$ to $+\infty$ as r ranges from the black hole horizon located at $r = r_h$ to infinity. But when $\Lambda < 0$, which is the case of interest here, r_* ranges from $-\infty$

up to some finite value that can be set equal to zero by appropriate choice of integration constant. For AdS_4 black holes, in particular, we have explicitly

$$r_\star = \frac{r_h}{4(r_h - 3m)} \left(r_h \log \frac{(2r + r_h)^2 + a^2}{4(r - r_h)^2} + 2a \frac{r_h - 6m}{r_h + 6m} \left[\arctan \frac{2r + r_h}{a} - \frac{\pi}{2} \right] \right), \quad (\text{B.5})$$

setting for notational convenience

$$a = \sqrt{-\frac{3}{\Lambda} \left(1 + \frac{6m}{r_h} \right)}. \quad (\text{B.6})$$

AdS black holes come in different sizes and their thermodynamic properties depend crucially on the magnitude of r_h relative to the AdS radius $L = \sqrt{-3/\Lambda}$, [40]. Large black holes have $r_h > L$ and become the dominant configurations at high temperatures, whereas small black holes have $r_h < L$ and they are always unstable decaying either to pure thermal radiation or to black holes with larger mass. In general, we have the following relation among the parameters of the AdS_4 Schwarzschild background

$$m - r_h = \frac{1}{2L^2} r_h (r_h^2 - L^2). \quad (\text{B.7})$$

Thus, large black holes have $r_h < m$, whereas small black holes have $r_h > m$.

Finally, we recall that very large black holes are naturally associated to the limit $r_h \rightarrow \infty$, in which case $f(r)$ is simply replaced by

$$f(r) = -\frac{2m}{r} - \frac{\Lambda}{3} r^2, \quad (\text{B.8})$$

dropping the constant term. Then, the black holes become essentially flat and their horizon is related to the other parameters by the simple expression

$$r_h^3 = -\frac{6m}{\Lambda}. \quad (\text{B.9})$$

C Polar perturbations of spherical black holes

This is a particular class of metric perturbations of Schwarzschild solution (in the presence of cosmological constant), which have the parity of Legendre polynomial $P_l(\cos\theta)$, i.e., $(-1)^l$. For simplicity, we only consider axisymmetric perturbations, which are parametrized by four arbitrary radial functions and they assume the general form

$$\delta g_{\mu\nu} = \begin{pmatrix} f(r)H_0(r) & H_1(r) & 0 & 0 \\ H_1(r) & H_2(r)/f(r) & 0 & 0 \\ 0 & 0 & r^2K(r) & 0 \\ 0 & 0 & 0 & r^2K(r)\sin^2\theta \end{pmatrix} e^{-i\omega t} P_l(\cos\theta). \quad (\text{C.1})$$

They correspond to the so called sound modes in the dictionary of AdS/CFT correspondence. As such, they should be contrasted to the axial perturbations of spherical black holes, also called shear modes in the context of AdS/CFT, which form a complementary set of perturbations with opposite parity, i.e., $(-1)^{l+1}$, but their explicit form is not needed for the purposes of the present work. We refer the reader to the original papers [24, 25], the textbook [26], the review article [27] as well as the more recent papers [28–30] for generalizations in the presence of cosmological constant.

We require $l \geq 2$ to describe physical modes of gravitational radiation in the linearized approximation (recall that there is no dipole radiation in general relativity). The study of polar perturbations $\delta R_{\mu\nu} = \Lambda \delta g_{\mu\nu}$ about the Schwarzschild background yields

$$H_0(r) = H_2(r) . \tag{C.2}$$

Tedious computation also shows that the (tr) ($r\theta$) and $(t\theta)$ components of the perturbation yield the following equations, respectively,

$$rK'(r) + \left(1 - \frac{rf'(r)}{2f(r)}\right) K(r) - H_0(r) - i\frac{l(l+1)}{2\omega r} H_1(r) = 0, \tag{C.3}$$

$$(f(r)H_0(r))' - f(r)K'(r) + i\omega H_1(r) = 0, \tag{C.4}$$

$$(f(r)H_1(r))' + i\omega (H_0(r) + K(r)) = 0 . \tag{C.5}$$

Together they form a coupled system of first order differential equations for the three unknown functions $H_0(r)$, $H_1(r)$ and $K(r)$. The other components of the perturbation either yield second order equations or else $\delta R_{\mu\nu}$ vanishes identically. Note, however, that there is an additional algebraic condition among the three radial functions

$$\begin{aligned} (2f(r) - rf'(r) - l(l+1)) H_0(r) + \frac{i}{2\omega} (4\omega^2 r - l(l+1)f'(r)) H_1(r) = \\ \left(2f(r) + rf'(r) - l(l+1) + 2\Lambda r^2 + \frac{r^2}{2f(r)} (4\omega^2 + f'^2(r))\right) K(r), \end{aligned} \tag{C.6}$$

which follows from consistency of the various second order equations with the first order system above; it can also be viewed as integral of the first order system above.

Following Zerilli, [25], we define the following function,

$$\Psi_+(r) = \frac{r^2}{(l-1)(l+2)r + 6m} \left(K(r) - i\frac{f(r)}{\omega r} H_1(r) \right), \tag{C.7}$$

which turns out to satisfy an effective Schrödinger equation with respect to the tortoise coordinate r_* ,

$$\left(-\frac{d^2}{dr_*^2} + V_+(r) \right) \Psi_+(r) = \omega^2 \Psi_+(r), \tag{C.8}$$

where the potential is only determined implicitly in terms of r_* via

$$\begin{aligned} V_+(r) = \frac{f(r)}{[(l-1)(l+2)r + 6m]^2} \left(l(l+1)(l-1)^2(l+2)^2 - 24m^2\Lambda \right. \\ \left. + \frac{6m}{r}(l-1)^2(l+2)^2 + \frac{36m^2}{r^2}(l-1)(l+2) + \frac{72m^3}{r^3} \right). \end{aligned} \tag{C.9}$$

The solutions determine the quasi-normal mode spectrum ω of the polar perturbations under appropriate boundary conditions and lead to expressions for the three unknown radial functions described above.

It turns out that the Zerilli potential admits the following representation

$$V_+(r) = W^2(r) + \frac{dW(r)}{dr_\star} + \omega_s^2 \tag{C.10}$$

in terms of the following (real) function

$$W(r) = \frac{6mf(r)}{r[(l-1)(l+2)r+6m]} + i\omega_s, \tag{C.11}$$

where

$$\omega_s = -\frac{i}{12m}(l-1)l(l+1)(l+2). \tag{C.12}$$

There is parallel story to be told about the parity odd (axial) perturbations, but we skip details that are not relevant to the present work. In that case, the important thing is that there is an effective Schrödinger problem for the perturbations with corresponding potential, called Regge-Wheeler potential, [24], which is also written in terms of W as

$$V_-(r) = W^2(r) - \frac{dW(r)}{dr_\star} + \omega_s^2. \tag{C.13}$$

Thus, black hole perturbations in four space-time dimensions are neatly described by two complementary Schrödinger problems, for all Λ ,

$$H_\pm \Psi_\pm = E \Psi_\pm, \tag{C.14}$$

where the Hamiltonian are given by

$$H_\pm = -\frac{d^2}{dr_\star^2} + W^2 \pm \frac{dW}{dr_\star}. \tag{C.15}$$

The energy $E = \omega^2 - \omega_s^2$ is not bounded from below by zero (in fact, it is not even real, in general) due to the particular boundary conditions imposed on the wave functions at the black hole horizon.

Remarkably, the system (C.14) has the structure of supersymmetric quantum mechanics (see, for instance, [31]). Introducing the first order operators (analogue of annihilation and creation operators of a harmonic oscillator)

$$Q = -\frac{d}{dr_\star} + W(r_\star), \quad Q^\dagger = \frac{d}{dr_\star} + W(r_\star), \tag{C.16}$$

we have a pair of Hamiltonians

$$H_+ = Q^\dagger Q = -\frac{d^2}{dr_\star^2} + W^2 + \frac{dW}{dr_\star}, \quad H_- = Q Q^\dagger = -\frac{d^2}{dr_\star^2} + W^2 - \frac{dW}{dr_\star} \tag{C.17}$$

corresponding to the two Schrödinger problems (C.14). They give rise to a supersymmetry algebra,

$$\{Q, Q^\dagger\} = \mathcal{H}, \quad \{Q, Q\} = 0 = \{Q^\dagger, Q^\dagger\}, \tag{C.18}$$

in terms of supercharges

$$\mathcal{Q} = \begin{pmatrix} 0 & 0 \\ Q & 0 \end{pmatrix}, \quad \mathcal{Q}^\dagger = \begin{pmatrix} 0 & Q^\dagger \\ 0 & 0 \end{pmatrix} \tag{C.19}$$

and

$$\mathcal{H} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}. \tag{C.20}$$

Defining

$$\Psi = \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix}, \tag{C.21}$$

the two Schrödinger problems (C.14) take the form

$$\mathcal{H}\Psi = E\Psi. \tag{C.22}$$

The Hamiltonian is only formally Hermitian because of the physical boundary conditions that give rise to the quasi-normal mode spectrum of the black hole.

The algebraically special modes, which are discussed in the text, are special case of polar perturbations, satisfying the first order equation

$$Q\Psi(r_\star) = \left(-\frac{d}{dr_\star} + W(r_\star)\right)\Psi(r_\star) = 0. \tag{C.23}$$

As such, they are supersymmetric zero energy states of H_+ having no partner in the complementary set of axial perturbations.

It is a mystery up to this day why the two complementary sets of perturbations of four-dimensional black holes form supersymmetric partner potentials although the space-time itself has no supersymmetry. This observation was first made by Chandrasekhar for $\Lambda = 0$ [26] (but see also earlier references therein) and subsequently generalized to $\Lambda \neq 0$ (see, for instance, [30]).

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] P.M. Chesler and L.G. Yaffe, *Numerical solution of gravitational dynamics in asymptotically anti-de Sitter spacetimes*, *JHEP* **07** (2014) 086 [[arXiv:1309.1439](https://arxiv.org/abs/1309.1439)] [[INSPIRE](https://inspirehep.net/literature/117700)].
- [2] H. Bondi, M.G.J. van der Burg and A.W.K. Metzner, *Gravitational waves in general relativity. 7. Waves from axisymmetric isolated systems*, *Proc. Roy. Soc. Lond.* **A 269** (1962) 21 [[INSPIRE](https://inspirehep.net/literature/117700)].
- [3] M. Brigante, H. Liu, R.C. Myers, S. Shenker and S. Yaida, *Viscosity Bound Violation in Higher Derivative Gravity*, *Phys. Rev.* **D 77** (2008) 126006 [[arXiv:0712.0805](https://arxiv.org/abs/0712.0805)] [[INSPIRE](https://inspirehep.net/literature/15213)].

- [4] Y. Kats and P. Petrov, *Effect of curvature squared corrections in AdS on the viscosity of the dual gauge theory*, *JHEP* **01** (2009) 044 [[arXiv:0712.0743](#)] [[INSPIRE](#)].
- [5] J. Erdmenger, P. Kerner and H. Zeller, *Non-universal shear viscosity from Einstein gravity*, *Phys. Lett. B* **699** (2011) 301 [[arXiv:1011.5912](#)] [[INSPIRE](#)].
- [6] A. Rebhan and D. Steineder, *Violation of the Holographic Viscosity Bound in a Strongly Coupled Anisotropic Plasma*, *Phys. Rev. Lett.* **108** (2012) 021601 [[arXiv:1110.6825](#)] [[INSPIRE](#)].
- [7] S. Cremonini, *The Shear Viscosity to Entropy Ratio: A Status Report*, *Mod. Phys. Lett. B* **25** (2011) 1867 [[arXiv:1108.0677](#)] [[INSPIRE](#)].
- [8] I. Robinson and A. Trautman, *Spherical Gravitational Waves*, *Phys. Rev. Lett.* **4** (1960) 431 [[INSPIRE](#)].
- [9] I. Robinson and A. Trautman, *Some spherical gravitational waves in general relativity*, *Proc. Roy. Soc. Lond. A* **265** (1962) 463 [[INSPIRE](#)].
- [10] D. Kramer, H. Stephani, E. Herlt and M. MacCallum, *Exact Solutions of Einstein's Field Equations*, Cambridge University Press, Cambridge (1980).
- [11] M. Carmeli, *Group Theory and General Relativity*, McGraw Hill, New York (1977).
- [12] J.B. Griffiths and J. Podolsky, *Exact Space-Times in Einstein's General Relativity*, Cambridge University Press, Cambridge (2009).
- [13] J. Foster and E.-T. Newman, *Note on the Robinson-Trautman solutions*, *J. Math. Phys.* **8** (1967) 189.
- [14] B. Lukacs, Z. Perjés, J. Porter and A. Sebestyén, *Lyapunov functional approach to radiative metrics*, *Gen. Rel. Grav.* **16** (1984) 691.
- [15] B.G. Schmidt, *Existence of solutions of the Robinson-Trautman equation and spatial infinity*, *Gen. Rel. Grav.* **20** (1988) 65.
- [16] A. Rendall, *Existence and asymptotic properties of global solutions of the Robinson-Trautman equation*, *Class. Quant. Grav.* **5** (1988) 1339.
- [17] D. Singleton, *On global existence and convergence of vacuum Robinson-Trautman solutions*, *Class. Quant. Grav.* **7** (1990) 1333.
- [18] P. Chruściel, *Semiglobal existence and convergence of solutions of the Robinson-Trautman (two-dimensional Calabi) equation*, *Commun. Math. Phys.* **137** (1991) 289 [[INSPIRE](#)].
- [19] P.T. Chruściel, M.A.H. MacCallum and D.B. Singleton, *Gravitational waves in general relativity: 14. Bondi expansions and the polyhomogeneity of Scri*, *Proc. Roy. Soc. Lond. A* **436** (1992) 299 [[gr-qc/9305021](#)] [[INSPIRE](#)].
- [20] S. Chandrasekhar, *On algebraically special perturbations of black holes*, *Proc. Roy. Soc. Lond. A* **392** (1984) 1.
- [21] G.-Y. Qi and B.F. Schutz, *Robinson-Trautman equations and Chandrasekhar's special perturbation of the Schwarzschild metric*, *Gen. Rel. Grav.* **25** (1993) 1185 [[INSPIRE](#)].
- [22] W.E. Couch and E.T. Newman, *Algebraically special perturbations of the Schwarzschild metric*, *J. Math. Phys.* **14** (1973) 285 [[INSPIRE](#)].
- [23] O.J.C. Dias and H.S. Reall, *Algebraically special perturbations of the Schwarzschild solution in higher dimensions*, *Class. Quant. Grav.* **30** (2013) 095003 [[arXiv:1301.7068](#)] [[INSPIRE](#)].

- [24] T. Regge and J.A. Wheeler, *Stability of a Schwarzschild singularity*, *Phys. Rev.* **108** (1957) 1063 [INSPIRE].
- [25] F.J. Zerilli, *Effective potential for even parity Regge-Wheeler gravitational perturbation equations*, *Phys. Rev. Lett.* **24** (1970) 737 [INSPIRE].
- [26] S. Chandrasekhar, *The Mathematical Theory of Black Holes*, Oxford University Press, Oxford (1983).
- [27] K.D. Kokkotas and B.G. Schmidt, *Quasinormal modes of stars and black holes*, *Living Rev. Rel.* **2** (1999) 2 [gr-qc/9909058] [INSPIRE].
- [28] V. Cardoso and J.P.S. Lemos, *Quasinormal modes of Schwarzschild anti-de Sitter black holes: Electromagnetic and gravitational perturbations*, *Phys. Rev. D* **64** (2001) 084017 [gr-qc/0105103] [INSPIRE].
- [29] I.G. Moss and J.P. Norman, *Gravitational quasinormal modes for anti-de Sitter black holes*, *Class. Quant. Grav.* **19** (2002) 2323 [gr-qc/0201016] [INSPIRE].
- [30] I. Bakas, *Energy-momentum/Cotton tensor duality for AdS₄ black holes*, *JHEP* **01** (2009) 003 [arXiv:0809.4852] [INSPIRE].
- [31] F. Cooper, A. Khare and U. Sukhatme, *Supersymmetry and quantum mechanics*, *Phys. Rept.* **251** (1995) 267 [hep-th/9405029] [INSPIRE].
- [32] K.P. Tod, *Analogues of the past horizon in the Robinson-Trautman metrics*, *Class. Quant. Grav.* **6** (1989) 1159.
- [33] E. Calabi, *Extremal Kähler metrics*, in *Seminar on Differential Geometry*, S.-T. Yau ed., Annals of Mathematics Studies, **102** Princeton University Press, Princeton (1982).
- [34] E. Calabi, *Extremal Kähler metrics II*, in *Differential Geometry and Complex Analysis*, I. Chavel and H. Farkas eds., Springer-Verlag, Berlin (1985).
- [35] A. Futaki, *Kähler-Einstein Metrics and Integral Invariants*, Lecture Notes in Mathematics, vol. 1314, Springer-Verlag, Berlin (1988).
- [36] G. Tian, *Canonical Metrics in Kähler Geometry*, Lectures in Mathematics, Birkhäuser, Basel (2002).
- [37] P.T. Chruściel and D.B. Singleton, *Nonsmoothness of event horizons of Robinson-Trautman black holes*, *Commun. Math. Phys.* **147** (1992) 137 [INSPIRE].
- [38] J. Bicak and J. Podolsky, *Cosmic no hair conjecture and black hole formation: An Exact model with gravitational radiation*, *Phys. Rev. D* **52** (1995) 887 [INSPIRE].
- [39] J. Bicak and J. Podolsky, *Global structure of Robinson-Trautman radiative space-times with a cosmological constant*, *Phys. Rev. D* **55** (1997) 1985 [gr-qc/9901018] [INSPIRE].
- [40] S.W. Hawking and D.N. Page, *Thermodynamics of Black Holes in anti-de Sitter Space*, *Commun. Math. Phys.* **87** (1983) 577 [INSPIRE].
- [41] G.T. Horowitz and V.E. Hubeny, *Quasinormal modes of AdS black holes and the approach to thermal equilibrium*, *Phys. Rev. D* **62** (2000) 024027 [hep-th/9909056] [INSPIRE].
- [42] T. Aubin, *Non-linear Analysis on Manifolds. Monge-Ampère Equations*, Springer-Verlag, Berlin (1982).
- [43] O. Svitek, *Apparent horizons in D-dimensional Robinson-Trautman spacetime*, *AIP Conf. Proc.* **1122** (2009) 404 [arXiv:0812.3345] [INSPIRE].

- [44] J. Podolsky and O. Svitek, *Past horizons in Robinson-Trautman spacetimes with a cosmological constant*, *Phys. Rev. D* **80** (2009) 124042 [[arXiv:0911.5317](#)] [[INSPIRE](#)].
- [45] E.W.M. Chow and A.W.C. Lun, *Apparent horizons in vacuum Robinson-Trautman space-times*, [gr-qc/9503065](#) [[INSPIRE](#)].
- [46] R. Penrose, *Gravitational collapse: The role of general relativity*, *Riv. Nuovo Cim.* **1** (1969) 252 [[INSPIRE](#)].
- [47] G.W. Gibbons, *Some comments on gravitational entropy and the inverse mean curvature flow*, *Class. Quant. Grav.* **16** (1999) 1677 [[hep-th/9809167](#)] [[INSPIRE](#)].
- [48] H.L. Bray and P.T. Chrusciel, *The Penrose inequality*, [gr-qc/0312047](#) [[INSPIRE](#)].
- [49] K.P. Tod, *More on Penrose's quasi-local mass*. *Class. Quant. Grav.* **3** (1986) 1169.
- [50] K.S. Thorne, *Non-spherical gravitational collapse: A short review*, in *Magic Without Magic*, J. Klauder ed., Freeman, San Francisco (1972).
- [51] G.W. Gibbons, *Birkhoff's invariant and Thorne's Hoop Conjecture*, [arXiv:0903.1580](#) [[INSPIRE](#)].
- [52] M. Cvetič, G.W. Gibbons, C.N. Pope, G.W. Gibbons and C.N. Pope, *More about Birkhoff's Invariant and Thorne's Hoop Conjecture for Horizons*, *Class. Quant. Grav.* **28** (2011) 195001 [[arXiv:1104.4504](#)] [[INSPIRE](#)].
- [53] K.P. Tod, *The hoop conjecture and the Gibbons-Hawking construction of trapped surfaces*, *Class. Quant. Grav.* **9** (1992) 1581.
- [54] G.D. Birkhoff, *Dynamical systems with two degrees of freedom*, *Trans. Amer. Math. Soc.* **18** (1917) 199.
- [55] P.M. Pu, *Some inequalities in certain non-orientable Riemannian manifolds*, *Pacific J. Math.* **2** (1952) 55.
- [56] J.M. Maldacena, *The Large- N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [[hep-th/9711200](#)] [[INSPIRE](#)].
- [57] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys. Lett. B* **428** (1998) 105 [[hep-th/9802109](#)] [[INSPIRE](#)].
- [58] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [[hep-th/9802150](#)] [[INSPIRE](#)].
- [59] M. Henningson and K. Skenderis, *The Holographic Weyl anomaly*, *JHEP* **07** (1998) 023 [[hep-th/9806087](#)] [[INSPIRE](#)].
- [60] M. Henningson and K. Skenderis, *Holography and the Weyl anomaly*, *Fortsch. Phys.* **48** (2000) 125 [[hep-th/9812032](#)] [[INSPIRE](#)].
- [61] V. Balasubramanian and P. Kraus, *A Stress tensor for Anti-de Sitter gravity*, *Commun. Math. Phys.* **208** (1999) 413 [[hep-th/9902121](#)] [[INSPIRE](#)].
- [62] S. de Haro, S.N. Solodukhin and K. Skenderis, *Holographic reconstruction of space-time and renormalization in the AdS/CFT correspondence*, *Commun. Math. Phys.* **217** (2001) 595 [[hep-th/0002230](#)] [[INSPIRE](#)].
- [63] K. Skenderis, *Asymptotically Anti-de Sitter space-times and their stress energy tensor*, *Int. J. Mod. Phys. A* **16** (2001) 740 [[hep-th/0010138](#)] [[INSPIRE](#)].

- [64] K. Skenderis, *Lecture notes on holographic renormalization*, *Class. Quant. Grav.* **19** (2002) 5849 [[hep-th/0209067](#)] [[INSPIRE](#)].
- [65] I. Papadimitriou and K. Skenderis, *Thermodynamics of asymptotically locally AdS spacetimes*, *JHEP* **08** (2005) 004 [[hep-th/0505190](#)] [[INSPIRE](#)].
- [66] J.D. Brown and J.W. York, *Quasilocal energy and conserved charges derived from the gravitational action*, *Phys. Rev. D* **47** (1993) 1407 [[gr-qc/9209012](#)] [[INSPIRE](#)].
- [67] L.D. Landau and E.M. Lifshitz, *Fluid Mechanics*, second edition, Pergamon Press, New York (1987).
- [68] P.K. Kovtun, D.T. Son and A.O. Starinets, *Viscosity in strongly interacting quantum field theories from black hole physics*, *Phys. Rev. Lett.* **94** (2005) 111601 [[hep-th/0405231](#)] [[INSPIRE](#)].
- [69] P.K. Kovtun and A.O. Starinets, *Quasinormal modes and holography*, *Phys. Rev. D* **72** (2005) 086009 [[hep-th/0506184](#)] [[INSPIRE](#)].
- [70] D.T. Son and A.O. Starinets, *Viscosity, Black Holes and Quantum Field Theory*, *Ann. Rev. Nucl. Part. Sci.* **57** (2007) 95 [[arXiv:0704.0240](#)] [[INSPIRE](#)].
- [71] E. Berti and K.D. Kokkotas, *Quasinormal modes of Reissner-Nordstrom-anti-de Sitter black holes: Scalar, electromagnetic and gravitational perturbations*, *Phys. Rev. D* **67** (2003) 064020 [[gr-qc/0301052](#)] [[INSPIRE](#)].
- [72] V. Cardoso, R. Konoplya and J.P.S. Lemos, *Quasinormal frequencies of Schwarzschild black holes in anti-de Sitter space-times: A Complete study on the asymptotic behavior*, *Phys. Rev. D* **68** (2003) 044024 [[gr-qc/0305037](#)] [[INSPIRE](#)].
- [73] A.S. Miranda and V.T. Zanchin, *Quasinormal modes of plane-symmetric anti-de Sitter black holes: A Complete analysis of the gravitational perturbations*, *Phys. Rev. D* **73** (2006) 064034 [[gr-qc/0510066](#)] [[INSPIRE](#)].
- [74] G. Michalogiorgakis and S.S. Pufu, *Low-lying gravitational modes in the scalar sector of the global AdS₄ black hole*, *JHEP* **02** (2007) 023 [[hep-th/0612065](#)] [[INSPIRE](#)].
- [75] S. Bhattacharyya, S. Lahiri, R. Loganayagam and S. Minwalla, *Large rotating AdS black holes from fluid mechanics*, *JHEP* **09** (2008) 054 [[arXiv:0708.1770](#)] [[INSPIRE](#)].
- [76] S. Bhattacharyya, V.E. Hubeny, R. Loganayagam, G. Mandal, S. Minwalla et al., *Local Fluid Dynamical Entropy from Gravity*, *JHEP* **06** (2008) 055 [[arXiv:0803.2526](#)] [[INSPIRE](#)].
- [77] P.M. Chesler and L.G. Yaffe, *Horizon formation and far-from-equilibrium isotropization in supersymmetric Yang-Mills plasma*, *Phys. Rev. Lett.* **102** (2009) 211601 [[arXiv:0812.2053](#)] [[INSPIRE](#)].
- [78] G.B. de Freitas and H.S. Reall, *Algebraically special solutions in AdS/CFT*, *JHEP* **06** (2014) 148 [[arXiv:1403.3537](#)] [[INSPIRE](#)].