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$\mathcal{N} = 1$ vacua in exceptional generalized geometry

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Abstract: We study $\mathcal{N} = 1$ Minkowski vacua in compactifications of type II string theory in the language of exceptional generalized geometry (EGG). We find the differential equations governing the EGG analogues of the pure spinors of generalized complex geometry, namely the structures which parameterize the vector and hypermultiplet moduli spaces of the effective four-dimensional $\mathcal{N} = 2$ supergravity obtained after compactification. In order to do so, we identify a twisted differential operator that contains NS and RR fluxes and transforms covariantly under the U-duality group, $E_{7(7)}$. We show that the conditions for $\mathcal{N} = 1$ vacua correspond to a subset of the structures being closed under the twisted derivative.

Keywords: Flux compactifications, Superstring Vacua, Differential and Algebraic Geometry

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1 Introduction

Since the seminal paper of Candelas, Horowitz, Strominger and Witten [1], the geometrical perspective in compactifications of string theory from ten to four dimensions had great insights. Supersymmetry conditions have been shown to constrain the allowed internal manifolds to certain specific classes. When there are no fluxes, the internal spaces should be Calabi-Yau. Such manifolds satisfy an *algebraic* condition, namely the existence of a globally defined, nowhere vanishing, internal spinor, and a *differential* one, that the spinor is covariantly constant. The algebraic condition is necessary in order to recover a supersymmetric ($\mathcal{N} = 2$) effective *theory* in four dimensions, while the differential one is required in order to have supersymmetric *vacua*. In the presence of fluxes, the algebraic condition stays intact (i.e., in order to have $\mathcal{N} = 2$ supersymmetry off-shell, a globally defined internal spinor is needed), but the differential one becomes more intricate.

The role of fluxes in string theory, combined with the warped nature of the compactification, has become of primary interest mainly for the possibility of fixing moduli and providing a hierarchy of scales [2]. This motivated the search for a geometric description of backgrounds with fluxes, which was very much guided by the framework of generalized geometry developed by Hitchin [3–6]. In rough terms, generalized complex geometry is complex geometry applied to the generalized tangent bundle of the space, consisting of the sum of tangent and cotangent bundles. The parameters encoding the symmetries of the metric plus the B-field, namely diffeomorphisms plus gauge transformations of B, live on this bundle. This formulation has therefore a natural action of T-duality, which exchanges these two. On the generalized tangent bundle one can define (generalized) almost complex structures, and study their integrability (integrable generalized complex structures allow to integrate the one-forms dZ^i and find global complex coordinates). Generalized almost complex structures are in one-to-one correspondence with pure spinors, which are built by tensoring the internal spinor with itself and with its charge conjugate. Spinors on the generalized tangent bundle are isomorphic to sums of forms on the cotangent bundle, and the integrability condition for the structure is nicely recast into closure of the pure spinor under the exterior derivative twisted by H.[1]

Generalized complex geometry was used in [7, 8] to characterize $\mathcal{N} = 1$ vacua. In analogy with the fluxless case, off-shell supersymmetry requires an algebraic condition, namely the existence of two pure spinors on the generalized tangent bundle. $\mathcal{N} = 1$ vacua require one of the pure spinors to be closed (and therefore the generalized almost complex structure associated to be integrable), while RR fluxes act as a defect for integrability of the other structure. In order to geometrize the RR fields as well, and give a purely algebraic geometrical characterization of the vacua (which would allow, for example, to study their deformations, i.e. their moduli spaces, in a model-independent manner), one needs to extend the generalized tangent bundle such that it includes the extra symmetries corresponding to gauge transformations of the RR fields. Such extension has been worked out in [9, 10], and was termed exceptional generalized geometry, alluding to the exceptional

[1] Integrability condition is actually weaker, it requires $(d - H\wedge)\Phi = X\Phi$ for some generalized tangent vector X, where Φ is the pure spinor corresponding to the generalized almost complex structure.

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groups arising in U-duality. In this paper we study compactifications of type II down to four-dimensions, where the relevant group is $E_{7(7)}$.

The algebraic conditions to have off-shell $\mathcal{N} = 2$ supersymmetry in four-dimensions have been worked out in [11]. Very much in analogy to the generalized complex geometric case, they require the existence of two algebraic structures on the exceptional generalized tangent bundle (in fact one of them, rather than a single structure, is actually a triplet, satisfying an $\mathrm{SU}(2)_R$ algebra), which are built by tensoring the internal spinors. The $\mathrm{SU}(2)_R$-singlet structure, that we call L, describes the vector multiplet moduli space, while the triplet of structures (named K_a) describes the hypermultiplets. The $\mathcal{N} = 1$ preserved supersymmetry breaks the $\mathrm{SU}(2)_R$ into $\mathrm{U}(1)_R$, selecting a vector r^a along this U(1), and a complex orthogonal vector z^a. The complex combination $z^a K_a$ describes the $\mathcal{N} = 1$ chiral multiplets contained in the hypermultiplets.[2]

In this paper we obtain the differential conditions on the algebraic structures L, K_a required by $\mathcal{N} = 1$ on-shell supersymmetry.[3] The first step is to identify the appropriate twisted derivative that generalizes $d - H\wedge$ to include the RR fluxes, or in other words to identify the right generalized connection. Such connection is obtained as in standard differential geometry by the operation $g^{-1}Dg$, where g are the E_7-adjoint elements corresponding to the shift symmetries (the so-called "B- and C-transforms"), and the derivative operator D is embedded in the fundamental representation of E_7 [11]. The key point is that this connection, which a priori transforms as a generic tensor product of adjoint and fundamental representations, should only belong to a particular irreducible representation in this tensor product, which in the case at hand is the **912**. Having identified the appropriate connection, we rewrite supersymmetry conditions in terms of closure of the structures. The equations we get are given in (5.12)–(5.16). We find that $\mathcal{N} = 1$ supersymmetry requires on one hand closure of L, as conjectured in [11]. On the other hand, the components of the twisted derivative of $r^a K_a$ with an even number of internal indices have to vanish, while those with an odd number are proportional to derivatives of the warp factor. A similar thing happens with $z^a K_a$, except that this time closure occurs upon projecting onto the holomorphic sub-bundle defined by L.

The paper is organized as follows: in section 2 we review the basic features of generalized geometry and its extension achieved by exceptional generalized geometry. In section 3 we present the relevant algebraic structures for compactifications with off-shell $\mathcal{N} = 2$ supersymmetry. In section 4 we review the constrains on the (traditional and generalized complex) structures imposed by on-shell supersymmetry. In section 5 we study supersymmetric vacua in the framework of exceptional generalized geometry. In particular, we introduce the twisted derivative operator in 5.1, we present the $\mathcal{N} = 1$ equations in 5.2, and finally in section 5.3 we outline the proof that supersymmetry requires those equations. Various technical details, as well as the full derivation of the equations, are left to appendices A to G.

[2] The vectors r^a and z^a are also used to identify respectively the $\mathcal{N} = 1$ D term and superpotential out of the triplet of Killing prepotentials in $\mathcal{N} = 2$ theories.

[3] Steps in this direction were done in [11] (see also in [10] for the M-theory case), where a set of natural E_7-covariant equations was conjectured to describe $\mathcal{N} = 1$ vacua. While the spinor components of such equations reproduce those of [8] and are therefore true conditions for susy vacua, other components failed to reproduce supersymmetry conditions.

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2 Generalized geometry

2.1 Generalized Complex Geometry

In this section we present the basic concepts of Generalized Complex Geometry (GCG) in six-dimensions (we will restrict to the six-dimensional case, though most of what follows can be generalized to any dimension), which will be used as a mathematical tool for describing flux vacua.

In Generalized (Complex) Geometry, the algebraic structures are not defined on the usual tangent bundle TM but on $TM \oplus T^*M$, on which there is a natural metric η

$$\eta = \begin{pmatrix} 0 & 1_6 \\ 1_6 & 0 \end{pmatrix} . \tag{2.1}$$

Following the language of usual complex geometry, a generalized almost complex structure (GACS for short) $\mathcal{J}$ is a map from $TM \oplus T^*M$ to itself such that it satisfies the hermiticity condition ($\mathcal{J}^t \eta \mathcal{J} = \eta$) and $\mathcal{J}^2 = -1_{12}$. One can define projectors $\Pi_\pm$ for the complexified generalized tangent bundle as

$$\Pi_\pm = \frac{1}{2}(1_{12} \mp i\mathcal{J}) \tag{2.2}$$

which can be used to define a maximally isotropic sub-bundle (six-dimensional) of $TM \oplus T^*M$ as the i-eigenbundle of $\mathcal{J}$

$$L_{\mathcal{J}} = \{x + \xi \in TM \oplus T^*M \big| \Pi_+(x+\xi) = x + \xi\} . \tag{2.3}$$

There is a one-to-one correspondence between a GACS and a "pure spinor" Φ of $O(6,6)$. A spinor is said to be pure if its annihilator space

$$L_\Phi = \{x + \xi \in TM \oplus T^*M \big| (x+\xi) \cdot \Phi = 0\} \tag{2.4}$$

is maximal (here $\cdot$ refers to the Clifford action $X \cdot \Phi = X_A \Gamma^A \Phi$). The one-to-one correspondence is then[4]

$$\mathcal{J} \leftrightarrow \Phi, \text{ if } L_{\mathcal{J}} = L_\Phi . \tag{2.5}$$

One can construct the GACS from the spinor by

$$\mathcal{J}^{\pm A}{}_B \;=\; i\,\frac{\langle \bar\Phi^\pm, \Gamma^A{}_B \Phi^\pm \rangle}{\langle \Phi^\pm, \bar\Phi^\pm \rangle} , \tag{2.6}$$

Weyl pure spinors of $O(6,6)$ can be built by tensoring two $O(6)$ spinors (η^1, η^2) as follows

$$\Phi^+ = e^{-\phi} \eta^1_+ \eta^{2\dagger}_+ , \qquad \Phi^- = e^{-\phi} \eta^1_+ \eta^{2\dagger}_- \tag{2.7}$$

where the plus and minus refers to chirality, and ϕ is the dilaton, which defines the isomorphism between the spinor bundle and the bundle of forms. Using Fierz identities, these can be expanded as

$$\eta^1_\pm \eta^{2\dagger}_\pm = \frac{1}{8} \sum_{k=0}^{6} \frac{1}{k!} (\eta^{2\dagger}_\pm \gamma_{m_k \ldots i_1} \eta^1_\pm) \gamma^{i_1 \ldots m_k} . \tag{2.8}$$

[4]The correspondence is actually one-to-many as the norm of the spinor is unfixed.

Using the isomorphism between the spinor bundle and the bundle of differential forms (often referred to as Clifford map):

$$A_{m_1\dots m_k}\gamma^{m_1\dots m_k} \longleftrightarrow A_{m_1\dots m_k} dx^{m_1}\wedge\cdots\wedge dx^{m_k} \tag{2.9}$$

the spinor bilinears (2.8) can be mapped to a sum of forms. Under this isomorphism, the inner product of spinors $\Phi\chi$ is mapped to the following action on forms, called the Mukai pairing

$$\langle\Phi,\chi\rangle = (\Phi\wedge s(\chi))_6, \qquad \text{where } s(\chi) = (-)^{\mathrm{Int}[n/2]}\chi \tag{2.10}$$

and the subindex 6 means the six-form part of the wedge product.

For Weyl spinors, the corresponding forms are only even (odd) for a positive (negative) chirality $\mathrm{O}(6,6)$ spinor. In the special case where $\eta^1 = \eta^2 \equiv \eta$, familiar from Calabi-Yau compactifications, we get

$$\Phi^+ = e^{-\phi}e^{-iJ}\,, \qquad \Phi^- = -ie^{-\phi}\Omega \tag{2.11}$$

where J, Ω are respectively the symplectic and complex structures of the manifold (more details are given in section 4.1.1).

Pure spinors can be "rotated" by means of $\mathrm{O}(6,6)$ transformations. Of particular interest is the nilpotent subgroup of $\mathrm{O}(6,6)$ defined by the generator

$$\mathcal{B} = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}, \tag{2.12}$$

with B an antisymmetric 6×6 matrix, or equivalently a two-form. On spinors, it amounts to the exponential action

$$\Phi^\pm \to e^{-B}\Phi^\pm \equiv \Phi_D^\pm \tag{2.13}$$

where on the polyform associated to the spinor, the action is $e^{-B}\Phi = (1 - B\wedge + \frac{1}{2}B\wedge B\wedge + \dots)\Phi$. We will refer to Φ as naked pure spinor, while Φ_D will be called dressed pure spinor. The pair (Φ_D^+, Φ_D^-) defines a positive definite metric on the generalized tangent space, which in turn defines a positive metric and a two-form (the B field) on the six-dimensional manifold.

2.2 Exceptional generalized geometry

Exceptional generalized geometry (EGG) [9, 10] is an extension of the $\mathrm{O}(6,6)$ (T-duality) covariant formalism of generalized geometry to an $E_{7(7)}$ (U-duality) covariant one, such that the RR fields are incorporated into the geometry.

We saw in the previous section that there is a particular $\mathrm{O}(6,6)$ adjoint action (2.12) corresponding to shifts of the B-field. In EGG, shifts of the B-field as well as shifts of the sum of internal RR fields $C^- = C_1 + C_3 + C_5$,[5] which transforms as a chiral $\mathrm{O}(6,6)$ spinor, correspond to particular E_7 adjoint actions. To form a set of gauge fields that is closed

[5] In this paper we will concentrate on type IIA, but most of the statements can be easily changed to type IIB by switching chiralities.

under U-duality, we also have to consider the shift of the six-form dual to B_2, which we will call $\tilde{B}$.[6]

Decomposing the adjoint **133** representation of $E_{7(7)}$ under $\mathrm{O}(6,6)\times\mathrm{SL}(2,\mathbb{R})$, we have

$$\begin{aligned} \mathbf{133} &= (\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{66}) + (\mathbf{2},\mathbf{32}') \\ \mu &= ({\mu^i}_j\,,\ {\mu^A}_B\,,\ \mu^{i-}) \end{aligned} \tag{2.14}$$

where $i=1,2$ is a doublet index of $\mathrm{SL}(2,\mathbb{R})$, raised and lowered with ϵ_{ij}, and the $\mathrm{O}(6,6)$ fundamental indices $A,B=1,\ldots,12$ are raised and lowered with the metric η in (2.1). The B-transform action (2.12) is part of ${\mu^A}_B$, while the C-transformations are naturally embedded in one of the two $\mathbf{32}'$ representations. Let us call v^i the $\mathrm{SL}(2,\mathbb{R})$ vector pointing in the direction of the C-field, which we can take without loss of generality to be

$$v^i = (1,0)\,. \tag{2.15}$$

The GL(6) assignments of the different components shown in appendix C, indicate that the shift symmetries are given by the following sum of generators

$$\left(\tilde{B}v^i v_j, \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}, v^i C^-\right) \equiv A \tag{2.16}$$

where $v_i=\epsilon_{ij}v^j$. Using (A.4) it is not hard to show that given this embedding we recover the right commutation relations

$$\left[B+\tilde{B}+C^-, B'+\tilde{B}'+C^{-\prime}\right] = 2\langle C^-, C^{-\prime}\rangle + B\wedge C^{-\prime} - B'\wedge C^-\,, \tag{2.17}$$

where the first term on the rhs is a six-form and therefore corresponds to a $\tilde{B}$ transformation and the other two, to an RR shift.

The fundamental **56** representation of E_7 decomposes under $\mathrm{O}(6,6)\times\mathrm{SL}(2,\mathbb{R})$ as

$$\begin{aligned} \mathbf{56} &= (\mathbf{2},\mathbf{12}) + (\mathbf{1},\mathbf{32}) \\ \nu &= (\nu^{iA}, \nu^+)\,. \end{aligned} \tag{2.18}$$

It combines all the gauge transformations: vectors plus one-forms correspond to diffeomorphisms and gauge transformations of the B-field. Their $\mathrm{SL}(2,\mathbb{R})$ duals[7] are gauge transformations of B_6 (given by a five-form, or analogously a vector) and diffeomorphisms for the dual vielbein (sourced by KK monopoles), given by a one-form tensored a six-form. Gauge transformations of the RR fields combine forming again a spinor representation, this time with positive chirality. The generalized tangent bundle $T\oplus T^*$ is therefore extended to the exceptional tangent bundle (EGT) E

$$E = TM\oplus T^*M\oplus\Lambda^5T^*M\oplus(T^*M\otimes\Lambda^6T^*M)\oplus\Lambda^{\mathrm{even}}T^*M\,. \tag{2.19}$$

[6] Equivalently these are shifts of the dual axion $B_{\mu\nu}$.

[7] The $\mathrm{SL}(2,\mathbb{R})$ here is the "heterotic S-duality", where the complex field that transforms by fractional linear transformations is $S=\tilde{B}+i.e.^{-2\phi}$. For the connection between this and type IIB S-duality, see [12].

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In what follows, we will mostly use the decomposition of E_7 under $SL(8,\mathbb{R})$. The fundamental representation decomposes as

$$\begin{aligned} \mathbf{56} &= \mathbf{28} + \mathbf{28}' \\ \nu &= (\nu^{ab}, \tilde{\nu}_{ab}) \end{aligned} \tag{2.20}$$

where $a, b = 1, \ldots, 8$ and $\nu_{ab} = -\nu_{ba}$. The adjoint decomposes as

$$\begin{aligned} \mathbf{133} &= \mathbf{63} + \mathbf{70} \\ \mu &= (\mu^a{}_b, \mu_{abcd}) \end{aligned} \tag{2.21}$$

where $\mu^a{}_a = 0$ and μ_{abcd} is fully antisymmetric.

In order to identify the embedding of the gauge fields (2.16) in $SL(8,\mathbb{R})$, we use the $GL(6,\mathbb{R})$ properties of the different components of the adjoint representation given in (C.4). We get[8]

$$A = \left(e^{2\phi}\tilde{B}v^i v_j - v^i e^{\phi} C_m + e^{\phi}(*C_5)^m v_i \, , -\frac{1}{2} e^{\phi} C_{mnp} v_i - \frac{1}{2} B_{mn} \epsilon_{ij} \right) , \tag{2.22}$$

or in other words

$$\begin{aligned} A^1{}_2 &= -e^{2\phi}\tilde{B} \, , \qquad A^1{}_m = -e^{\phi} C_m \, , \quad A^m{}_2 = -e^{\phi}(*C_5)^m \\ A_{mnp2} &= \frac{1}{2} e^{\phi} C_{mnp} \, , \quad A_{mn12} = -\frac{1}{2} B_{mn} \end{aligned} \tag{2.23}$$

where the factors and signs are chosen in order to match the supergravity conventions. Here and in the following, $*$ refers to a six-dimensional Hodge dual, while we use $\star$ for the eight-dimensional one.

3 $E_{7(7)}$ algebraic structures

In this section we present the algebraic structures in E_7 constructed in [11] that play the role of the $O(6,6)$ pure spinors $\Phi^{\pm}$. We start by building the analogous of the naked pure spinors, and then discuss their orbits under the action of the gauge fields A in (2.16), (2.22).

Spinors transform under the maximal compact subgroup of the duality group. In the GCG case, this subgroup is $O(6) \times O(6)$, which acts on the pair (η^1, η^2). In EGG, the relevant group is $SU(8)$. We can combine the two ten-dimensional supersymmetry parameters such that the $SU(8)$ transformation of their internal piece is manifest. The most general ten-dimensional spinor ansatz relevant to four-dimensional $\mathcal{N} = 2$ theories is

$$\begin{pmatrix} \epsilon^1 \\ \epsilon^2 \end{pmatrix} = \zeta_-^1 \otimes \theta^1 + \zeta_-^2 \otimes \theta^2 + \text{c.c.} \tag{3.1}$$

[8]To avoid introducing new notation, we are using the same as in (2.16), in particular $v_i \equiv \epsilon_{ij} v^j$, although indices in $SL(8,\mathbb{R})$ are raised and lowered with the metric $\hat{g}$ given in (C.2).

where $\zeta_-^{1,2}$ are four-dimensional spinors of negative chirality, and $\theta^{1,2}$ are never parallel. In this paper we will be dealing with equations for $\mathcal{N}=1$ vacua, where there is a relation between ζ^1 and ζ^2. In that case, we can use the special parameterization

$$\theta^1 = \begin{pmatrix} \eta_+^1 \\ 0 \end{pmatrix}, \qquad \theta^2 = \begin{pmatrix} 0 \\ \eta_-^2 \end{pmatrix}. \tag{3.2}$$

A nowhere vanishing spinor θ defines an SU(7) $\subset$ SU(8) structure. The pair (θ^1, θ^2) defines an SU(6) structure.[9] We can take the SU(4) spinors to be normalized to 1. In that case the SU(8) spinors are orthonormal, namely

$$\bar{\theta}_I\, \theta^J = \delta_I{}^J\,. \tag{3.3}$$

where $I=1,2$ is a fundamental $\mathrm{SU}(2)_R$ index (for conventions on the conjugate spinors, see appendix B). The two spinors can be combined into the following $\mathrm{SU}(2)_R$ singlet and triplet combinations

$$L = e^{-\phi}\epsilon_{IJ}\theta^I\theta^J\,, \qquad K_a = \frac{1}{2}e^{-\phi}\sigma_{aI}{}^J\theta^I\bar{\theta}_J\,, \qquad K_0 = \frac{1}{2}e^{-\phi}\delta_I{}^J\theta^I\bar{\theta}_J\,, \tag{3.4}$$

where we have introduced K_0 for future convenience. The triplet K_a satisfies the $su(2)$ algebra with a scaling given by the dilaton, i.e.

$$[K_a, K_b] = 2ie^{-\phi}\epsilon_{abc}K_c \tag{3.5}$$

L and K_a are the E_7 structures that play the role of the generalized almost complex structures Φ^+ and Φ^-. They belong respectively to the **28** and **63** representations of SU(8), which are in turn part of the **56** and **133** representations of E_7. Using the decompositions $\mathbf{56} = \mathbf{28} + \overline{\mathbf{28}}$ and $\mathbf{133} = \mathbf{63} + \mathbf{35} + \overline{\mathbf{35}}$ shown in (B.3) and (B.4), they read

$$L = \left(e^{-\phi}\epsilon_{IJ}\theta^{I\alpha}\theta^{J\beta}, e^{-\phi}\epsilon_{IJ}\theta^{I*}_\alpha\theta^{J*}_\beta\right) \qquad K_a = \left(e^{-\phi}\frac{1}{2}\sigma_{aI}{}^J\theta^{I\alpha}\bar{\theta}_{J\beta}, 0, 0\right). \tag{3.6}$$

To make contact with the pure spinors of GCG, we note that using the parameterization (3.2), we get

$$L = \begin{pmatrix} 0 & \Phi^+ \\ -s(\bar{\Phi}^+) & 0 \end{pmatrix} \tag{3.7}$$

where the operation s is introduced in (2.10).

Using (3.2), we get for $K_\pm = K_1 \pm iK_2$

$$K_+ = \begin{pmatrix} 0 & \Phi^- \\ 0 & 0 \end{pmatrix}, \qquad K_- = \begin{pmatrix} 0 & 0 \\ -s(\bar{\Phi}^-) & 0 \end{pmatrix}, \tag{3.8}$$

[9] Note that an SU(6) structure can be built out of a single globally defined internal spinor η, taking $\eta^1 = \eta^2 = \eta$.

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while for K_3 we get

$$K_3 = \begin{pmatrix} \Phi_1^+ & 0 \\ 0 & -\bar{\Phi}_2^+ \end{pmatrix}$$

where we have defined

$$\Phi_1^+ = e^{-\phi}\eta_+^1\eta_+^{1\dagger}\,, \qquad \Phi_2^+ = e^{-\phi}\eta_+^2\eta_+^{2\dagger}\,, \tag{3.9}$$

We see that L contains the pure spinor Φ^+, which spans the vector multiplets in type IIA (see (2.11)), while K_+ is built from the pure spinor Φ^-, which is part of the hypermultiplets. K_3 contains on the contrary the even-form bilinears of the same SU(4) spinor, or in other terms the symplectic structures defined by each spinor (see (2.11)).

To get the SL(8) components of L and K_a, we use (B.8). Using the decomposition of the gamma matrices given in (B.9), we get that the only non-zero components of L and K_a are

$$\begin{aligned} L: &\qquad L^{12}, L^{mn} \\ K_1, K_2: &\qquad {K_2}^{m1}, {K_2}^{m2}, {K_2}^{mnp1}, {K_2}^{mnp2} \\ K_0, K_3: &\qquad {K_3}^{mn}, {K_3}^{12}, {K_3}^{mnpq}, {K_3}^{mn12} \end{aligned} \tag{3.10}$$

where L^{12} and L^{mn} involve the zero and two-form pieces of Φ^+, K_+^{mi}, K_+^{mnpi} contain the one and three-form pieces of Φ^+ (where the difference between the two SL(2) components is a different GL(6) weight), while K_3 contains the different components of Φ_1^+ and Φ_2^+.

In an analogous way as for the pure spinors, the structures L and K_a can be dressed by the action of the gauge fields B, $\tilde{B}$ and C^- in (2.16), (2.23), i.e. we define

$$L_D = e^C e^{\tilde{B}} e^{-B} L\,, \qquad K_{aD} = e^C e^{\tilde{B}} e^{-B} K_a\,. \tag{3.11}$$

In the GCG case, the B-field twisted pure spinors span the orbit $\frac{\mathrm{O}(6,6)}{\mathrm{SU}(3,3)} \times \mathbb{R}^+$, where $\mathrm{SU}(3,3)$ is the stabilizer of the pure spinor and the $\mathbb{R}^+$ factor corresponds to the norm. Quotienting by the $\mathbb{C}^*$ action $\Phi_D \to c\Phi_D$, we get the space $\frac{\mathrm{O}(6,6)}{\mathrm{U}(3,3)}$ which is local Special Kähler. Similarly, our EGG structures L_D and K_{aD} span orbits in E_7 which are respectively Special Kähler and Quaternionic-Kähler. As shown in [11], the structure L_D is stabilized by $E_{6(2)}$, and the corresponding local Special Kähler space is $\frac{E_7}{E_{6(2)}} \times \mathrm{U}(1)$. The triplet K_{aD} is stabilized by an $SO^*(12)$ subgroup of E_7, and the corresponding orbit is the quaternionic space $\frac{E_7}{SO^*(12)\times \mathrm{SU}(2)}$, where the SU(2) factor corresponds to rotations of the triplet. The $SO^*(12)$ and $E_{6(2)}$ structures intersect on an SU(6) structure if L and K_a satisfy the compatibility condition

$$L\,K_a|_{\mathbf{56}} = 0\,, \tag{3.12}$$

where we have to apply the projection on the **56** on the product $\mathbf{56} \times \mathbf{133}$. This condition is automatically satisfied for the structures (3.4) built as spinor bilinears.

4 String vacua and integrability conditions

In the previous sections we have presented the relevant algebraic structures that are used to describe an off-shell $\mathcal{N} = 2$ four-dimensional effective action. We now turn to the differential conditions imposed by requiring on-shell supersymmetry, or in other words, by demanding that the vacua are supersymmetric. As we will show, these translate into integrability of some of the algebraic structures.

4.1 Warm up: fluxless case

It will be useful for the following to recall the conditions for supersymmetric vacua in the absence of fluxes. We start by reviewing the integrability conditions in ordinary complex geometry, and then re-express them in the language of GCG.

4.1.1 Conditions for the structures on TM

In the absence of fluxes, inserting the $\mathcal{N} = 2$ spinor ansatz (3.2) in the supersymmetry condition $\delta\psi_m = 0$ (see (F.2)), we get

$$\nabla_m \theta^I = 0 \,. \tag{4.1}$$

When there is only one globally defined spinor η, we take $\eta^1 = \eta^2 \equiv \eta$, and (4.1) reduces to the familiar Calabi-Yau condition

$$\nabla_m \eta = 0 \,, \tag{4.2}$$

which implies that the SU(3) structure defined by η is integrable, or in other words that the manifold has SU(3) holonomy [13]. The holonomy is defined as the group generated by parallel transporting an arbitrary spinor around a closed loop. Riemaniann geometries can be classified by specifying the holonomy of the Levi-Civita connection. A general Riemaniann six-dimensional space has holonomy SO(6) $\simeq$ SU(4). However if the manifold admits one (or more) Killing spinors, the group is reduced: it lies within the stabilizer group. In six dimensions, the existence of a globally defined, nowhere vanishing, covariantly constant spinor implies that the holonomy is reduced to SU(3) $\subset$ SU(4).

Integrability of an SU(3) structure can also be recast in terms of integrability of two seemingly very different algebraic structures that intersect on an SU(3), namely a complex and a symplectic one. The existence of a globally defined nowhere vanishing spinor is equivalent to the existence of an almost symplectic 2-form J (which defines an almost symplectic Sp$(6,\mathbb{R})$ structure) and a 3-form Ω (which defines an almost complex GL$(3,\mathbb{C})$ structure). These two structures intersect on an SU(3). If the structures are integrable, i.e. if they satisfy

$$dJ = 0 \,, \qquad d\Omega = \xi \wedge \Omega \,, \tag{4.3}$$

for any one-form ξ, one can define local complex and local symplectic coordinates which can be "integrated" (i.e. there exist local complex coordinates z^i and symplectic ones $(x^i, y^{\hat{\imath}})$ $(i, \hat{\imath} = 1, 2, 3)$ such that the local complex and symplectic one forms dz^i, $(dx^i, dy^{\hat{\imath}})$ are indeed their differentials). If additionally $\xi = 0$, then the canonical bundle is holomorphically trivial and the manifold is Calabi-Yau. Since J and Ω can be written as bilinears of the spinor η, the supersymmetry requirement (4.2) is equivalent to the conditions (4.3) and the additional requirement $\xi = 0$.

Note that for an almost complex structure, there are many equivalent ways to check its integrability. Instead of the second requirement in (4.3), one can find conditions on the corresponding map $I : TM \to TM$.[10] The almost complex structure I is integrable if the i-eigenbundle is closed under the Lie bracket, i.e. iff

$$\pi_\mp[\pi_\pm x, \pi_\pm y] = 0, \quad \forall x, y \in TM \qquad \text{where } \pi_\pm = \frac{1}{2}(1 \mp iI) \tag{4.4}$$

and [,] denotes the Lie bracket. As we will see, either requirement (4.3) and (4.4) will have its analogue in generalized complex geometry. In exceptional generalized geometry, we will only deal with conditions of the form (4.3).

4.1.2 Conditions for the structures on $TM \oplus T^*M$

As shown in section 2.1, almost complex and symplectic structures on the tangent bundle are expressed on the same footing in terms of generalized almost complex structures on $TM \oplus T^*M$. Furthermore, a generic GACS reduces on the tangent bundle to a structure that is locally a product of lower dimensional complex and symplectic structures.

As in the case of ordinary complex structures, eq. (4.4), a GACS is integrable if its i-eigenbundle is closed under an extension of the Lie bracket to $T \oplus T^*$, i.e. $\mathcal{J}$ is integrable iff

$$\Pi_\mp[\Pi_\pm(X), \Pi_\pm(Y)]_C = 0, \quad \forall X, Y \in TM \oplus T^*M \tag{4.5}$$

where the projectors $\Pi_\pm$ are defined in (2.2) and the bracket is the Courant bracket

$$[x + \xi, y + \eta]_C = [x, y] + \mathcal{L}_x\eta - \mathcal{L}_y\xi - \frac{1}{2}d(i_x\eta - i_y\xi) \tag{4.6}$$

with $\mathcal{L}$ the Lie derivative. Again, in a similar fashion to ordinary complex structures, the integrability condition (4.5) is equivalent to requiring that the pure spinor Φ associated to $\mathcal{J}$ satisfies

$$d\Phi = X \cdot \Phi \tag{4.7}$$

for some generalized vector $X = x + \xi$, and where $\cdot$ is the Clifford product, whose action on forms is

$$X \cdot \Phi = \iota_x\Phi + \xi \wedge \Phi\,. \tag{4.8}$$

The $\mathcal{N} = 2$ supersymmetry requirement (4.1) that arises in the absence of fluxes, translates into

$$d\Phi^+ = 0\,, \qquad d\Phi^- = 0\,, \tag{4.9}$$

which means that both GACS are integrable (and both canonical bundles are trivial), or in other words that the SU(3) × SU(3) structure is integrable. In the case $\eta^1 = \eta^2 = \eta$,

[10] Similarly to the case of GACS, there is a one-to-one (or rather many-to-one (see footnote 4)) correspondence between a 3-form $\Omega = dz^1 \wedge dz^2 \wedge dz^3$ and a map I satisfying $I^2 = -1$ such that the i-eigenbundle of I is generated by the dual vectors ∂_{z^i}.

this reduces to the Calabi-Yau conditions (4.3) with $\xi = 0$. Manifolds satisfying (4.9) have been termed "generalized Calabi-Yau *metric* geometries" in [6].[11] They are more general than Calabi-Yau's in the sense that the pure spinors need not be purely complex or pure symplectic, as happens when $\eta^1 = \eta^2$, but can correspond to (integrable) hybrid complex-symplectic structures.

4.2 Flux case in CGC

In this section we review the results of [14] (in the language of GCG, as in [15]) and [8] where the conditions for respectively $\mathcal{N} = 2$ supersymmetry with NS flux only, and $\mathcal{N} = 1$ with NS and RR fluxes were found.

4.2.1 Vacua with NS fluxes

In section 2.1 we saw how GCG incorporates the B-field, in particular by means of the B-twisted pure spinors (2.13). When B is not globally well-defined, i.e. when NS fluxes are switched on, the B-twisted pure spinors are not global sections of $TM \oplus T^*M$, but they are rather sections of a particular fibration of T^*M over TM involving the B-field. For reasons that will become clear later, in this paper we choose the alternative "untwisted picture" as in [6], where pure spinors are naked (or dressed by just a closed B field), and the H-flux is introduced explicitly in, e.g. the integrability conditions.[12]

A closed B field is an automorphism of the Courant bracket, while in the presence of $H = dB$ flux, there is an extra term

$$[e^{-B}(x+\xi), e^{-B}(y+\eta)]_C = e^{-B}[x+\xi, y+\eta]_C + e^{-B}\iota_x\iota_y H \tag{4.10}$$

where the action of B is $e^{-B}(x+\xi) = x + \xi - \iota_x B$. The H-twisted Courant bracket is defined by adding this last term to (4.6).

If a GACS is "twisted integrable", then the corresponding pure spinor satisfies

$$d_H\Phi = X \cdot \Phi \tag{4.11}$$

where the H-twisted differential is

$$d_H \equiv d - H\wedge\,. \tag{4.12}$$

Note the equivalence between the twisted and untwisted picture. If a naked pure spinor is twisted closed, then the dressed pure spinor is closed under the ordinary exterior derivative, i.e.

$$0 = d_H\Phi = (d - dB\wedge)\Phi = e^B d(e^{-B}\Phi) = e^B d\Phi_D\,. \tag{4.13}$$

[11] Note the addition of the word "metric", to distinguish them from the generalized Calabi-Yau manifolds defined in [3–5] that require closure of only one pure spinor, and will play a main role in the next sections.

[12] We use the terming "twisted picture" to refer to the scenario where pure spinors are dressed by the (non-closed) B-field, and the integrability conditions are given in terms of the ordinary exterior derivative (or equivalently the ordinary Courant bracket (4.6)), as in [3–5], while in the "untwisted picture" of [6], the spinors are untwisted (or just twisted by a closed B), while the H-flux appears explicitly in the differential or in the bracket. The two pictures are equivalent, and depending on the situation one can be more convenient than the other.

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This shows how to construct the twisted exterior derivative from the ordinary one, and the action of the B-field

$$d_H = e^B d e^{-B} \tag{4.14}$$

which will be extended in section (5.1) to include the RR fluxes.

Supersymmetry conditions in the presence of H-flux amount precisely to H-twisting the generalized Calabi-Yau metric condition (4.9). More precisely, vacua preserving four-dimensional $\mathcal{N}=2$ supersymmetry in the presence of NS fluxes should satisfy [15]

$$d_H \Phi^+ = 0\,, \qquad d_H \Phi^- = 0\,, \tag{4.15}$$

i.e. they require H-twisted generalized Calabi-Yau metric structures.

Vacua with $\mathcal{N}=1$ supersymmetry in the presence of NS fluxes were obtained in [16, 17], and reinterpreted in the language of G-structures in [19–21]. They read

$$\begin{aligned} d_H(e^{-\phi}\Phi^-) &= 0\,, \\ d(e^{-\phi}\Phi^+) &= i e^{-2\phi} * H \end{aligned} \tag{4.16}$$

where $\Phi^\pm$ are those for an SU(3) structure, (2.11). Note that in the second equation H does not enter as a twisting in the standard way, and therefore the even pure spinor is not twisted integrable. It would be interesting to get the right GCG description of $\mathcal{N}=1$ vacua with NS fluxes.

4.2.2 Vacua with NS and RR fluxes

Compactifications on Minkowski space preserving $\mathcal{N}=1$ supersymmetry in the presence of NS and RR fluxes require the spacetime to be a warped product, i.e.

$$ds^2 = e^{2A}\eta_{\mu\nu}dx^\mu dx^\nu + ds_6^2\,. \tag{4.17}$$

The preserved spinor can be parameterized within the $\mathcal{N}=2$ spinor ansatz (3.2) by a doublet $n_I = (a, \bar{b})$ such that the supersymmetry preserved is given by $\epsilon = n_I \epsilon^I$, i.e.

$$\epsilon = \xi_- \otimes \theta + c.c.\,, \qquad \text{with } \theta = \begin{pmatrix} a\eta^1_+ \\ \bar{b}\eta^2_- \end{pmatrix}\,, \tag{4.18}$$

and we take $|\eta^1|^2 = |\eta^2|^2 = 1$ (while $|a|$ and $|b|$ are related to the warp factor, as we will see). The vector n_I distinguishes a $\mathrm{U}(1)_R \subset \mathrm{SU}(2)_R$ such that any triplet can be written in terms of a U(1) complex doublet and a U(1) singlet by means of the vectors

$$\begin{aligned} (z^+, z^-, z^3) &= n_I(\sigma^a)^{IJ} n_J = (a^2, -\bar{b}^2, -2a\bar{b})\,, \\ (r^+, r^-, r^3) &= n_I(\sigma^a)^I{}_J \bar{n}^J = (ab, \bar{a}\bar{b}, |a|^2 - |b|^2)\,. \end{aligned} \tag{4.19}$$

Using these vectors, one can extract respectively an $\mathcal{N}=1$ superpotential and D-term from the triplet of Killing prepotentials $\mathcal{P}_a$ that give the potential in the $\mathcal{N}=2$ theory, by

$$\mathcal{W} = z^a \mathcal{P}_a\,, \qquad \mathcal{D} = r^a \mathcal{P}_a\,. \tag{4.20}$$

For type IIA compactifications, the triplet $\mathcal{P}_a$ reads [22, 23]

$$\mathcal{P}_+ = \langle \Phi^+, d_H \Phi^- \rangle\,, \quad \mathcal{P}_- = \langle \Phi^+, d_H \bar{\Phi}^- \rangle\,, \quad \mathcal{P}_3 = -\langle \Phi^+, F^+ \rangle\,. \tag{4.21}$$

The conditions for flux vacua have been obtained in the language of GCG either using the ten-dimensional gravitino and dilatino variations [8], or by extremizing the superpotential of the four-dimensional $\mathcal{N}=1$ theory and setting the D-term to zero [24, 25]. For the case $|a| = |b|$, which arises when sources are present, they read

$$d_H(e^{2A}\Phi'^+) = 0 \tag{4.22}$$

$$d_H(e^{A}\mathrm{Re}\Phi'^-) = 0 \tag{4.23}$$

$$d_H(e^{3A}\mathrm{Im}\Phi'^-) = e^{4A} * s(F^+) \tag{4.24}$$

where

$$\Phi'^+ = 2a\bar{b}\,\Phi^+\,, \quad \Phi'^- = 2ab\,\Phi^-\,. \tag{4.25}$$

Finally, $N=1$ supersymmetry requires

$$|a|^2 + |b|^2 = e^A\,. \tag{4.26}$$

Conditions (4.22)–(4.24) can be understood as coming from F and D-term equations. Equation (4.23) corresponds to imposing $\mathcal{D}=0$, while (4.22) and (4.24) come respectively from variations of the superpotential with respect to Φ^- and Φ^+.

The susy condition (4.22) says that the GACS corresponding $\mathcal{J}^+$ is twisted integrable, and furthermore that the canonical bundle is trivial, and therefore the required manifold is a twisted Generalized Calabi-Yau (see footnote 11). The other GACS appearing in (4.23)–(4.24) is "half integrable", i.e. its real part is closed, while the non-integrability of the imaginary part is due to the RR fluxes. In the EGG formulation, RR fluxes are also encoded in the twisting of the differential operator, and therefore we expect to rephrase these equations purely in terms of integrability of the structures defined on the EGT space. Note that in the limit of RR fluxes going to zero, eqs. (4.22)–(4.24) for $\mathcal{N}=1$ vacua reduce to (4.15) (for $F=0$, (4.22)–(4.24) imply $A=0$), i.e. $F\to 0$ is a singular limit of (4.22) where supersymmetry is enhanced to $\mathcal{N}=2$.

On top of supersymmetry conditions (4.22)–(4.24), the fluxes must satisfy the Bianchi identities

$$dH = 0\,, \quad d_H F = 0 \tag{4.27}$$

in the absence of sources, while in the presence of D-branes or orientifold planes, the right hand sides get modified by the appropriate charge densities.

5 Flux vacua in exceptional generalized geometry

In this section we discuss the conditions for $\mathcal{N}=1$ vacua in the language of EGG. The putative conditions for supersymmetric vacua come from variations of the E_7-covariant expression for the triplet of Killing prepotentials [11]

$$\mathcal{P}_a = \mathcal{S}(L_D, DK_{aD}) = \mathcal{S}(L, e^B e^{-\tilde{B}} e^{-C} D e^C e^{\tilde{B}} e^{-B} K_a)\,. \tag{5.1}$$

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Here $\mathcal{S}$ is the symplectic invariant on the **56** whose decomposition in terms of $\mathrm{O}(6,6)\times\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(8,\mathbb{R})$ are given respectively in (A.1) and (A.10). The derivative D is an element in the **56**, whose $\mathrm{O}(6,6)\times\mathrm{SL}(2,\mathbb{R})$ decomposition is

$$D=(D^{iA},D^{+})=(v^{i}\nabla^{A},0)\,,\qquad \text{where }\nabla^{A}=(0,\nabla_{m})\,, \tag{5.2}$$

while in $\mathrm{SL}(8,\mathbb{R})$ we have

$$D=(D^{ab},\tilde{D}_{ab})=(0,v_{i}\nabla_{m})\,. \tag{5.3}$$

(where we are using again $v_i=\epsilon_{ij}v^j=(0,-1)$), DK_a in (5.1) is an element in the $\mathbf{56}\times\mathbf{133}$, which is projected to the **56** by the symplectic product. In the second equality in (5.1) we have used the E_7 invariance of the symplectic product to untwist the structures L_D and K_{aD} and express the Killing prepotentials in terms of naked structures, and a twisted derivative. We will now see how to properly define this twisted derivative, needed to get the equations for vacua.

5.1 Twisted derivative and generalized connection

For the gauge fields A and the derivative operator $D^{\mathcal{A}}$, $\mathcal{A}=1,\dots,56$, one can define a connection $\phi^{\mathcal{A}\mathcal{B}}{}_{\mathcal{C}}\in\mathbf{56}\times\mathbf{133}$ by the following twisting of the Levi-Civita one

$$(e^{B}e^{-\tilde{B}}e^{-C})^{\mathcal{B}}{}_{\mathcal{D}}D^{\mathcal{A}}(e^{C}e^{\tilde{B}}e^{-B})^{\mathcal{D}}{}_{\mathcal{C}}\equiv D^{\mathcal{A}}\delta^{\mathcal{B}}{}_{\mathcal{C}}+\phi^{\mathcal{A}\mathcal{B}}{}_{\mathcal{C}}\,. \tag{5.4}$$

The connection ϕ contains derivatives of the gauge fields. The key point is that in the tensor product

$$\mathbf{56}\times\mathbf{133}=\mathbf{56}+\mathbf{912}+\mathbf{6480} \tag{5.5}$$

only the terms in the **912** representation involve exterior derivatives of the gauge potentials [12], while the other representations contain non-gauge invariant terms (like divergences of potentials). We therefore define the twisted derivative as

$$\mathcal{D}=D+\mathcal{F}\,,\quad \text{where }\mathcal{F}=e^{B}e^{-\tilde{B}}e^{-C}D\,e^{C}e^{\tilde{B}}e^{-B}|_{\mathbf{912}}\,. \tag{5.6}$$

The fact that the fluxes lie purely in the **912** is consistent with the supersymmetry requirement that the embedding tensor of the resulting four-dimensional gauge supergravity be in the **912** [26].

The **912** decomposes in the following $\mathrm{O}(6,6)\times\mathrm{SL}(2,\mathbb{R})$ representations

$$\mathcal{F}=(\mathcal{F}^{iA},\mathcal{F}^{i}{}_{j}{}^{+},\mathcal{F}^{A-},\mathcal{F}^{iABC})$$
$$\mathbf{912}=(\mathbf{2},\mathbf{12})+(\mathbf{3},\mathbf{32})+(\mathbf{1},\mathbf{352})+(\mathbf{2},\mathbf{220})$$

where $\Gamma_A\mathcal{F}^{A-}=0$ and $\mathcal{F}^{iABC}$ is fully antisymmetric in ABC. The only nonzero components of the connection (5.6) are (see appendix D for details)

$$\mathcal{F}^{1}{}_{2}{}^{+}=-F^{+}\,,\qquad \mathcal{F}^{1}{}_{mnp}=-H_{mnp}\,, \tag{5.7}$$

where $F^{+}=e^{B}dC^{-}$.

In the $SL(8,\mathbb{R})$ decomposition, the generalized connection decomposes in the following representations

$$\begin{aligned} \mathbf{912} &= \mathbf{36} + \mathbf{420} + \mathbf{36}' + \mathbf{420}' \\ \mathcal{F} &= (\mathcal{F}^{ab}, \mathcal{F}^{abc}{}_d, \tilde{\mathcal{F}}_{ab}, \tilde{\mathcal{F}}_{abc}{}^d) \end{aligned} \tag{5.8}$$

where $\mathcal{F}^{ba} = \mathcal{F}^{ab}$ and $\mathcal{F}^{abc}{}_c = 0$ and similarly for the objects with a tilde. The NS and RR fluxes give the following non-zero components

$$\begin{aligned} &\mathcal{F}^{11} = e^{\phi} F_0 \,, && \mathcal{F}^{mnp}{}_2 = -\frac{1}{2}(*H)^{mnp} \,, && \mathcal{F}^{mn1}{}_2 = -e^{\phi}\,\frac{1}{2}(*F_4)^{mn} \\ &\tilde{\mathcal{F}}_{22} = e^{\phi} *F_6 \,, && \tilde{\mathcal{F}}_{mn2}{}^1 = -e^{\phi}\,\frac{1}{2} F_{mn} \,. \end{aligned} \tag{5.9}$$

In applying the twisted derivative to the algebraic structures L and K, the following tensor products appear

$$\begin{aligned} \mathcal{D}L = \quad & DL \quad + \quad \mathcal{F}L \,, & \qquad \mathcal{D}K = \quad & DK \quad + \quad \mathcal{F}K \\ & \mathbf{56}\times\mathbf{56} \;+\; \mathbf{912}\times\mathbf{56} & & \mathbf{56}\times\mathbf{133} \;+\; \mathbf{912}\times\mathbf{133} \end{aligned}$$

If we think of the vacua equations as coming from variations of the Killing prepotentials (5.1), out of these tensor products of representations, the equations should lie in the **133** representation for $\mathcal{D}L$, and in the **56** in $\mathcal{D}K$. We give in (E.1)–(E.16) the full expression for the twisted derivative of an element in the **56** and an element in the **133**. In section 5.3 we rewrite the only components that are non-zero in the case of $\mathcal{N}=1$ vacua, i.e. for L and K whose only non-zero components are those in (3.10).

5.2 Equations for $\mathcal{N}=1$ vacua

By following the same reasoning that leads from the superpotential to the equations for $\mathcal{N}=1$ vacua in the GCG case, a set of three equations were conjectured in [11] to be the EGG analogue of (4.22)–(4.24). While the spinor component in the $O(6,6)\times SL(2,\mathbb{R})$ decomposition of each equation reproduced the GCG equations (4.22)–(4.24), other representations did not work. Here, we show that the conjectured equations do work if we introduce two modifications: first, instead of using dressed bispinors and an untwisted derivative, we use undressed bispinors and a twisted derivative, appropriately projected onto the **912**. The projection to the **912** gets rid of the non gauge invariant terms (proportional, for example, to $\nabla^m C$) arising in the vector parts of the equations conjectured in [11]. Second, we add a right hand side to the equations with a single internal index, proportional to the derivative of the warp factor or the dilaton.[13]

The equations are written in terms of L and K_a using the following parameterisation for the spinors

$$\theta^1 = \begin{pmatrix} a\eta^1_+ \\ 0 \end{pmatrix}, \qquad \theta^2 = \begin{pmatrix} 0 \\ \bar{b}\eta^2_- \end{pmatrix} \tag{5.10}$$

[13] Except for a single equation in (5.16), where the right hand side contains a term proportional to the NS-flux H, that does not seem to be proportional to a derivative of the dilaton or warp factor.

With this parameterisation, the combinations that are relevant for $\mathcal{N}=1$ supersymmetry are

$$\begin{aligned} L' &\equiv e^{2A} L\,, \\ K'_1 &\equiv e^{A} r^a K_a = e^A K_1\,, \\ K'_+ &\equiv e^{3A} z^a K_a = e^{3A}(K_3 + iK_2)\,. \end{aligned} \tag{5.11}$$

In the language of EGG, $\mathcal{N}=1$ supersymmetry requires for L',

$$\mathcal{D}L'\big|_{\mathbf{133}} = 0\,, \tag{5.12}$$

for $\mathcal{D}K'_1|_{\mathbf{56}}$,[14]

$$\begin{aligned} (\mathcal{D}K'_1)^{mn} &= 0, & \widetilde{(\mathcal{D}K'_1)}_{mn} &= 0\,, \\ (\mathcal{D}K'_1)^{12} &= 0, & \widetilde{(\mathcal{D}K'_1)}_{12} &= 0\,, \\ (\mathcal{D}K'_1)^{m2} &= 0, & \widetilde{(\mathcal{D}K'_1)}_{m1} &= 0\,, \end{aligned} \tag{5.13}$$

and for $\mathcal{D}K'_+|_{\mathbf{56}}$

$$\begin{aligned} (\mathcal{D}K'_+)_{mn} - i\widetilde{(\mathcal{D}K'_+)}_{mn} &= 0\,, \\ (\mathcal{D}K'_+)_{12} - i\widetilde{(\mathcal{D}K'_+)}_{12} &= 0\,, \\ (\mathcal{D}K'_+)^{m2} &= 0\,. \end{aligned} \tag{5.14}$$

The remaining components of $\mathcal{D}K$ (all with one internal index) are proportional to derivatives of the dilaton and warp factor as follows

$$(\mathcal{D}K'_1)^{m1} = 4e^{-2A}\partial_p A K'_+{}^{mp}, \quad \widetilde{(\mathcal{D}K'_1)}_{m2} = -4e^{-2A}\partial_p A\,(2K'_+{}^{p}{}_{m12} + i\delta^p_m K'_+{}^{1}{}_2), \tag{5.15}$$

$$(\mathcal{D}(e^{-\phi}K'_+))^{m1} = -4ie^{-\phi}g^{mp}\partial_p A K'_+{}^{1}{}_2\,, \; (\mathcal{D}(\widetilde{e^{2A-\phi}}K'_+))_{m2} = -e^{2A-\phi}H_{mpq}K'_+{}^{12pq} \tag{5.16}$$

$$(\mathcal{D}(\widetilde{e^{-4A+\phi}}K'_+))_{m1} = 0\,.$$

The equations for L, K'_3 and K'_+ in (5.12)–(5.14) are respectively the EGG version of (4.22), (4.23) and (4.24). The vectorial equations are a combination of (4.22)–(4.24) plus (4.26). Note that the symmetry group under which these equations are covariant is $\mathrm{GL}(6,\mathbb{R}) \subset \mathrm{SL}(8,\mathbb{R})$.

5.3 From SUSY conditions to EGG equations

We will sketch here the proof that $\mathcal{N}=1$ supersymmetry requires (5.12)–(5.14) and leave the details, as well the proof of eqs. (5.15), (5.16), to appendix G.

[14] We are using the notation in (2.20), where a tilde denotes the component in the $\mathbf{28}'$ representation

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Using (3.10) in (E.1)–(E.8), we get that the only nontrivial components of eq. (5.12) are

$$(\mathcal{D}L')^1{}_2 = -e^{\phi}[iF_0 + (*F_6)]L'^{12} + \frac{e^{\phi}}{2}[F_{mn} + i(*F_4)_{mn}]L'^{mn}\,, \tag{5.17}$$

$$(\mathcal{D}L')^1{}_m = -\nabla_m L'^{12} \tag{5.18}$$

$$(\mathcal{D}L')^m{}_2 = -\nabla_p L'^{mp} + \frac{i}{2}(*H)^{mnp}L'_{np} \tag{5.19}$$

$$(\mathcal{D}L')_{mnp2} = \frac{3i}{2}\nabla_{[m}L'_{np]} + \frac{1}{2}H_{mnp}L'^{12}\,, \tag{5.20}$$

where we used (B.7), while for K'_1 we get

$$(\mathcal{D}K'_1)^{mn} = -2\nabla_p K_1'^{mnp2} + (*H)^{mnp}K_1'^2{}_p \tag{5.21}$$

$$(\widetilde{\mathcal{D}K'_1})_{mn} = -2\nabla_{[m}K_1'^2{}_{n]} \tag{5.22}$$

$$(\widetilde{\mathcal{D}K'_1})_{12} = -\nabla_n K_1'^n{}_1 - \frac{1}{3}H_{npq}K_1'^{2npq} \tag{5.23}$$

$$(\mathcal{D}K'_1)^{m1} = e^{\phi}F_0 K_1'^m{}_1 - e^{\phi}(*F_4)^{mn}K_1'^2{}_n - e^{\phi}F_{np}K_1'^{2npm} \tag{5.24}$$

$$(\widetilde{\mathcal{D}K'_1})_{m2} = -e^{\phi}{*F_6}K_1'^2{}_m - e^{\phi}F_{mn}K_1'^n{}_1 + e^{\phi}(*F_4)^{np}K'_{1\,1npm} \tag{5.25}$$

and for K'_+

$$(\mathcal{D}K'_+)^{mn} = -2\nabla_p K_+'^{\,mnp2} + (*H)^{mnp}K_+'^{\,2}{}_p + e^{\phi}(*F_4)^{mn}K_+'^{\,2}{}_1 \tag{5.26}$$

$$(\widetilde{\mathcal{D}K'_+})_{mn} = -2\nabla_{[m}K_+'^{\,2}{}_{n]} + e^{\phi}F_{mn}K_+'^{\,2}{}_1 \tag{5.27}$$

$$(\mathcal{D}K'_+)^{m1} = 2\nabla_p K_+'^{\,mp12} + e^{\phi}F_0 K_+'^{\,m}{}_1 - e^{\phi}(*F_4)^{mn}K_+'^{\,2}{}_n - e^{\phi}F_{np}K_+'^{\,2npm} \tag{5.28}$$

$$(\widetilde{\mathcal{D}K'_+})_{m1} = -\nabla_m K_+'^{\,2}{}_1 \tag{5.29}$$

$$(\widetilde{\mathcal{D}K'_+})_{m2} = -\nabla_p K_+'^{\,p}{}_m - H_{mpq}K_+'^{\,pq12} - e^{\phi}{*F_6}K_+'^{\,2}{}_m - e^{\phi}F_{mp}K_+'^{\,p}{}_1 + e^{\phi}(*F_4)^{pq}K'_{+1pqm} \tag{5.30}$$

$$(\mathcal{D}K'_+)^{12} = -e^{\phi}F_0 K_+'^{\,2}{}_1 \tag{5.31}$$

$$(\widetilde{\mathcal{D}K'_+})_{12} = -\nabla_n K_+'^{\,n}{}_1 - \frac{1}{3}H_{npq}K_+'^{\,2npq} - e^{\phi}{*F_6}K_+'^{\,2}{}_1 \tag{5.32}$$

where we should keep in mind that the components of K_+ with an odd (even) number of internal indices are proportional to K_2 (K_3) (see (3.10)).

We now show that supersymmetry requires (5.12), in particular the components appearing in (5.17) and (5.18). The proof for the rest of the components is in appendix G.1.

It is not hard to show that exactly the same combination of RR fluxes appearing on the right hand side of (5.17) is obtained by multiplying eq. (G.5), coming from the external gravitino variation, by Γ^2, and tracing over the spinor indices, namely

$$0 = \sqrt{2}\,\mathrm{Tr}\left(i\Gamma^2\Delta_e\pi'\right) = -e^{\phi}[iF_0 + (*F_6)]L'^{12} + \frac{e^{\phi}}{2}\left[F_{mn} + i(*F_4)_{mn}\right]L'^{mn} = (\mathcal{D}L')^1{}_2$$

where in the second equality the term proportional to the derivative of the warp factor goes away by symmetry, and we have used (B.7) to relate the SU(8) and SL(8) components of L. Supersymmetry requires therefore $(\mathcal{D}L')^1{}_2 = 0$.

For the equations that involve a covariant derivative of L^{ab}, we use (G.1) coming from the internal gravitino variation, multiplied by Γ^{ab} and we trace over the spinor indices (see eq. (B.7)). For $ab = 12$, for example, this gives

$$0 = \frac{\sqrt{2}}{4}\text{Tr}\,\left(\Gamma^{12}\Delta_m L'\right) = \nabla_m L'^{12} - \partial_m(2A-\phi)L'^{12} - \frac{i}{4}H_{mnp}L'^{np} + \frac{e^{\phi}}{8}[F_{pq} + i(*F_4)_{pq}]\pi'^{2pq}{}_m$$

where π' is defined in (G.2) and (G.3). Now we use eqs. (G.4) and (G.6) multiplied by Γ_m and traced over the spinor indices to cancel the terms containing derivatives of the dilaton and warp factor. In doing this, the term involving H and F fluxes completely cancel, i.e.

$$\begin{aligned} 0 &= \frac{\sqrt{2}}{4}\text{Tr}\,\left(\Gamma^{12}\Delta_m L' + i\Gamma_m(-2\Delta_e L' + \Delta_d L')\right) \\ &= \nabla_m L'^{12} \\ &= (\mathcal{D}L')^1{}_m\,. \end{aligned}$$

We show in appendix G.1 how supersymmetry requires the remaining equations, (5.19) and (5.20), to vanish.

The equations for K work similarly. For example, to show that (5.21) should vanish, we use (G.11) coming from internal gravitino, in the following way

$$\begin{aligned} 0 =& -\frac{i}{4}\text{Tr}\,\left[\Gamma^{mnp2}(e^A\Delta_p K_1)\right] \\ =& -2e^{A-\phi}\nabla_p(e^{\phi}K_1{}^{mnp2}) + \frac{1}{2}H^{mnp}K'^{\,1}_1{}_p + \frac{3}{2}(*H)^{mnp}K'^{\,2}_1{}_p \\ & -2e^{-2A+\phi}F_0 K'^{mn12}_+ - e^{-2A+\phi}F^{[m|p}K_{+p}{}^{|n]}\,. \end{aligned} \tag{5.33}$$

We combine this with external gravitino equations (G.14), (G.16) and dilatino equations (G.15), (G.17) to get (see more details in appendix G.2)

$$\begin{aligned} 0 =& -\frac{i}{4}\,\text{Tr}\,\left[\Gamma^{mnp2}(e^A\Delta_p K_1) + \{\Gamma^{mn1}, \Delta_e K'_1 - \Delta_d K'_1\}\right] \\ =& -2\nabla_p K_1'^{\,mnp2} + (*H)^{mnp}K'^{\,2}_1{}_p \\ =&(\mathcal{D}K'_1)^{mn} \end{aligned} \tag{5.34}$$

where we have used the notation in (G.28).

We give the details about the rest of the components of the twisted derivative of K'_1 and K'_+ in appendix G.2.

We will now connect the equations found to their generalized complex geometric counterparts, eqs. (4.22)–(4.24) and (4.26). Eqs. (5.18)–(5.20)) reduce to (4.22). The right hand side of eq. (5.17) is proportional to $\langle F, \Phi^+\rangle$, which can be seen to vanish by wedging (4.22) with C^- (this means that actually (5.17) can be derived from (5.18)–(5.20)). The mn and 12 components of the EGG equations for K'_1 and K'_+ combine to build up respectively (4.23) and (4.24). Interestingly, eq. (4.26), which is not part of the pure spinor equations but has to be added by hand in the GCG language, becomes one of the EGG equations, namely the one on the second line of (5.16). This can be seen by using (5.29) and the fact that $K'^{\,2}_{+\;1} = K'^{\,2}_{3\;1} = -\frac{i}{4}e^{3A-\phi}(|\eta_1|^2+|\eta_2|^2)$ The other vectorial components of $\mathcal{D}K$ involve for example terms of the form $\langle F, \Gamma^A\Phi^-\rangle$, which making use of (4.22)–(4.24), can be shown to be proportional to derivatives of the warp factor.

Since (4.22)–(4.26) were shown in [27] to be equivalent to supersymmetry conditions, we conclude that the EGG equations (5.12)–(5.16) are completely equivalent to requiring $\mathcal{N}=1$ supersymmetry, i.e., supersymmetry requires (5.12)–(5.16), and (5.12)–(5.16) implies supersymmetry.

As mentioned in section 3, L defines an $E_{6(2)}$ structure in E_7. We have shown here that $\mathcal{N}=1$ supersymmetry requires this structure to be twisted closed, upon projection to the **133**. It would be very nice to show that this is equivalent to the structure being integrable.[15] For constant warp factor and dilaton, also K'_1 is twisted closed. Most of the components of K'_+ are also twisted closed after projection onto holmorphic indices in the **56**. The vectorial components of $\mathcal{D}K$ are proportional to derivatives of the warp factor and dilaton, except the second equation in (5.16), which does not seem to be expressible in terms of such derivatives.

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A $E_{7(7)}$ basics and tensor products of representations

$E_{7(7)}$ can be defined as the subgroup of $\mathrm{Sp}(56,\mathbb{R})$ which in addition to preserve the symplectic structure $\mathcal{S}(\lambda,\lambda')$, preserves also a totally symmetric quartic invariant. We exploit the decomposition of $E_{7(7)}$ representations under two subgroups

1. $\mathrm{SL}(2,\mathbb{R})\times \mathrm{O}(6,6)$ is the physical subgroup appearing as the factorization of ("heterotic") S-duality and the T-duality group that emerges in the framework of generalized geometry

2. $\mathrm{SL}(8,\mathbb{R})$. This subgroup contains the product $\mathrm{SL}(2,\mathbb{R})\times \mathrm{GL}(6,\mathbb{R})$, and allows to make contact with $\mathrm{SU}(8)/\mathbb{Z}_2$, the maximal compact subgroup of $E_{7(7)}$. The latter is the group under which the spinors transform, and therefore the natural language to formulate supersymmetry via the Killing spinor equations.

[15] Unlike the case of generalized complex structures, even if there is an exceptional Courant bracket [10], there is no known correspondence between the differential conditions on the structure and closure of a subset (defined by the structure) of the exceptional generalized tangent bundle under the exceptional Courant bracket.

A.1 $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{O}(6,6)$

The fundamental **56** representation decomposes as

$$\nu = (\nu^{iA}, \nu^{+})$$
$$\mathbf{56} = (\mathbf{2},\mathbf{12}) + (\mathbf{1},\mathbf{32})$$

For the adjoint **133** of E_7 we have

$$\mu = (\mu^{i}{}_{j}, \mu^{A}{}_{B}, \mu^{i-})$$
$$\mathbf{133} = (\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{66}) + (\mathbf{2},\mathbf{32'})$$

where $\mu^{i}{}_{i} = 0$ and $\mu^{AB} = \mu^{A}{}_{C}\, \eta^{CB}$ is antisymmetric. The **912** decomposes as

$$\phi = (\phi^{iA}, \phi^{i}{}_{j}{}^{+}, \phi^{A-}, \phi^{iABC})$$
$$\mathbf{912} = (\mathbf{2},\mathbf{12}) + (\mathbf{3},\mathbf{32}) + (\mathbf{1},\mathbf{352}) + (\mathbf{2},\mathbf{220})$$

where $\Gamma_A \Phi^{A-} = 0$ and ϕ^{iABC} is fully antisymmetric in ABC.

There are various tensor products projected on some particular representation that are used throughout the paper. These are:

$\mathbf{56} \times \mathbf{56}\big|_{\mathbf{1}}$ (i.e. the symplectic invariant)

$$\mathcal{S}(\nu, \hat{\nu}) = \epsilon_{ij}\eta_{AB}\nu^{iA}\hat{\nu}^{jB} + \langle \nu^{+}, \hat{\nu}^{+}\rangle \tag{A.1}$$

$\mathbf{56} \times \mathbf{56}\big|_{\mathbf{133}}$

$$\begin{aligned}
(\nu\cdot\hat{\nu})^{i}{}_{j} &= 2\epsilon_{jk}\eta_{AB}\nu^{iA}\hat{\nu}^{kB} \\
(\nu\cdot\hat{\nu})^{A}{}_{B} &= 2\epsilon_{ij}(\nu^{iA}\hat{\nu}^{j}{}_{B} + \hat{\nu}^{iA}\nu^{j}{}_{B}) + \langle \nu^{+}, \Gamma^{A}{}_{B}\hat{\nu}^{+}\rangle \\
(\nu\cdot\hat{\nu})^{i-} &= \nu^{iA}\Gamma_{A}\hat{\nu}^{+} + \hat{\nu}^{iA}\Gamma_{A}\nu^{+};
\end{aligned} \tag{A.2}$$

$\mathbf{56} \times \mathbf{133}\big|_{\mathbf{56}}$

$$\begin{aligned}
(\nu\cdot\mu)^{iA} &= \mu^{i}{}_{j}\nu^{jA} + \mu^{A}{}_{B}\nu^{iB} + \langle \mu^{i-}, \Gamma^{A}\nu^{+}\rangle \\
(\nu\cdot\mu)^{+} &= \frac{1}{4}\mu_{AB}\Gamma^{AB}\nu^{+} + \epsilon_{ij}\nu^{iA}\Gamma_{A}\mu^{j-}\ ;
\end{aligned} \tag{A.3}$$

the adjoint action on the adjoint, i.e. $\mathbf{133} \times \mathbf{133}\big|_{\mathbf{133}}$;

$$\begin{aligned}
(\mu\cdot\hat{\mu})^{i}{}_{j} &= \hat{\mu}^{i}{}_{k}\mu^{k}{}_{j} - \mu^{i}{}_{k}\hat{\mu}^{k}{}_{j} + \epsilon_{jk}(\langle \hat{\mu}^{i-}, \mu^{k-}\rangle - \langle \mu^{i-}, \hat{\mu}^{k-}\rangle) \\
(\mu\cdot\hat{\mu})^{A}{}_{B} &= \hat{\mu}^{A}{}_{C}\mu^{C}{}_{B} - \mu^{A}{}_{C}\hat{\mu}^{C}{}_{B} + \epsilon_{ij}\langle \hat{\mu}^{i-}, \Gamma^{A}{}_{B}\mu^{j-}\rangle \\
(\mu\cdot\hat{\mu})^{i-} &= \hat{\mu}^{i}{}_{j}\mu^{j-} - \mu^{i}{}_{j}\hat{\mu}^{j-} + \frac{1}{4}\hat{\mu}_{AB}\Gamma^{AB}\mu^{i-} - \frac{1}{4}\mu_{AB}\Gamma^{AB}\hat{\mu}^{i-}
\end{aligned} \tag{A.4}$$

and $\mathbf{56} \times \mathbf{133}\big|_{\mathbf{912}}$

$$\begin{aligned}
(\nu\cdot\mu)^{iA} &= \mu^{i}{}_{j}\nu^{jA} + \mu^{A}{}_{B}\nu^{iB} + \langle \nu^{+}, \Gamma^{A}\mu^{i-}\rangle \\
(\nu\cdot\mu)^{i}{}_{j}{}^{+} &= \mu^{i}{}_{j}\nu^{+} - \epsilon_{jk}\nu^{(i|A}\Gamma_{A}\mu^{k)-} \\
(\nu\cdot\mu)^{A-} &= -\mu^{A}{}_{B}\Gamma^{B}\nu^{+} + \frac{1}{10}\mu_{BC}\Gamma^{ABC}\nu^{+} + \epsilon_{ij}\nu^{iA}\mu^{j-} - \frac{1}{11}\epsilon_{ij}\nu^{iB}\Gamma_{B}{}^{A}\mu^{j-} \\
(\nu\cdot\mu)^{iABC} &= 3\nu^{i[A}\mu^{BC]} + \langle \nu^{+}, \Gamma^{ABC}\mu^{i-}\rangle\,.
\end{aligned} \tag{A.5}$$

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A.2 SL(8, ℝ)

The decomposition of the E_7 representations we use in terms of SL(8, ℝ) are the following. For the fundamental **56** we have

$$\begin{aligned} \nu &= (\nu^{ab}, \tilde{\nu}_{ab}) \\ \mathbf{56} &= \mathbf{28} + \mathbf{28}' \,. \end{aligned} \tag{A.6}$$

with $\nu^{ba} = -\nu^{ab}$.
The adjoint **133** decomposes as

$$\begin{aligned} \mu &= (\mu^a{}_b, \mu_{abcd}) \\ \mathbf{133} &= \mathbf{63} + \mathbf{70} \end{aligned} \tag{A.7}$$

with $\mu^a{}_a = 0$, and μ_{abcd} fully antisymmetric.
For the **912** we have

$$\begin{aligned} \phi &= (\phi^{ab}, \phi^{abc}{}_d, \tilde{\phi}_{ab}, \tilde{\phi}_{abc}{}^d) \\ \mathbf{912} &= \mathbf{36} + \mathbf{420} + \mathbf{36}' + \mathbf{420}' \end{aligned} \tag{A.8}$$

with $\phi^{ab} = \phi^{ba}$, $\phi^{abc}{}_d = \phi^{[abc]}{}_d$ and $\phi^{abc}{}_c = 0$ and similarly for the tided objects.
The SL(8, ℝ) decomposition of the tensor products is the following.
The adjoint action on the fundamental, $\mathbf{56} \times \mathbf{133}\big|_{\mathbf{56}}$ is.[16]

$$\begin{aligned} (\nu \cdot \mu)^{ab} &= \mu^a{}_c \nu^{cb} + \mu^b{}_c \nu^{ac} + \star\mu^{abcd} \tilde{\nu}_{cd} \\ (\nu \cdot \mu)_{ab} &= -\mu^c{}_a \tilde{\nu}_{cb} - \mu^c{}_b \tilde{\nu}_{ac} - \mu_{abcd} \nu^{cd} \end{aligned} \tag{A.9}$$

The symplectic invariant $\mathbf{56} \times \mathbf{56}\big|_{\mathbf{1}}$ reads

$$\mathcal{S}(\nu, \hat{\nu}) = \nu^{ab} \tilde{\hat{\nu}}_{ab} - \tilde{\nu}_{ab} \hat{\nu}^{ab} \tag{A.10}$$

The $\mathbf{56} \times \mathbf{56}\big|_{\mathbf{133}}$ reads

$$\begin{aligned} (\nu \cdot \hat{\nu})^a{}_b &= \left(\nu^{ca} \tilde{\hat{\nu}}_{cb} - \frac{1}{8} \delta^a{}_b \nu^{cd} \tilde{\hat{\nu}}_{cd} \right) + \left(\hat{\nu}^{ca} \tilde{\nu}_{cb} - \frac{1}{8} \delta^a{}_b \hat{\nu}^{cd} \tilde{\nu}_{cd} \right) \\ (\nu \cdot \hat{\nu})_{abcd} &= -3 \left(\tilde{\nu}_{[ab} \tilde{\hat{\nu}}_{cd]} + \frac{1}{4!} \epsilon_{abcdefgh} \nu^{ef} \hat{\nu}^{gh} \right) \end{aligned} \tag{A.11}$$

where $\star\mu$ is the 8-dimensional Hodge dual, while the adjoint action on the adjoint $\mathbf{133} \times \mathbf{133}\big|_{\mathbf{133}}$ gives

$$\begin{aligned} (\mu \cdot \hat{\mu})^a{}_b &= (\mu^a{}_c \hat{\mu}^c{}_b - \hat{\mu}^a{}_c \mu^c{}_b) - \frac{1}{3} (\star\mu^{acde} \hat{\mu}_{bcde} - \star\hat{\mu}^{acde} \mu_{bcde}) \\ (\mu \cdot \hat{\mu})_{abcd} &= 4 (\mu^e{}_{[a} \hat{\mu}_{bcd]e} - \hat{\mu}^e{}_{[a} \mu_{bcd]e}) \end{aligned} \tag{A.12}$$

[16]Note tht this convention differs by a sign in the $\star\mu$ term than the one used in [10, 28]. This choice is correlated with the choice in (E.17), and affects a few signs in the equations that follow (those in the terms involving $\star\mu$).

The $\mathbf{56} \times \mathbf{133}\big|_{\mathbf{912}}$ is

$$
\begin{aligned}
(\nu \cdot \mu)^{ab} &= (\nu^{ac} \mu^b{}_c + \nu^{bc} \mu^a{}_c) \\
(\nu \cdot \mu)_{ab} &= -(\tilde{\nu}_{ac} \mu^c{}_b + \tilde{\nu}_{bc} \mu^c{}_a) \\
(\nu \cdot \mu)^{abc}{}_d &= -3 \left(\nu^{[ab} \mu^{c]}{}_b - \frac{1}{3} \nu^{e[a} \mu^b{}_e \delta^{c]}{}_d \right) + 2 \left(\tilde{\nu}_{ed} \star \mu^{abce} + \frac{1}{2} \tilde{\nu}_{ef} \star \mu^{ef[ab} \delta^{c]}{}_d \right) \\
(\nu \cdot \mu)_{abc}{}^d &= -3 \left(\tilde{\nu}_{[ab} \mu^d{}_{c]} - \frac{1}{3} \tilde{\nu}_{e[a} \mu^e{}_b \delta^d{}_{c]} \right) + 2 \left(\nu^{ed} \mu_{abce} + \frac{1}{2} \nu^{ef} \mu_{ef[ab} \delta^d{}_{c]} \right)
\end{aligned}
\tag{A.13}
$$

The $\mathbf{912} \times \mathbf{56}\big|_{\mathbf{133}}$ gives

$$
\begin{aligned}
(\phi \cdot \nu)^a{}_b &= (\nu^{ca} \tilde{\phi}_{cb} + \tilde{\nu}_{cb} \phi^{ca}) + (\tilde{\nu}_{cd} \phi^{cda}{}_b - \nu^{cd} \tilde{\phi}_{cdb}{}^a) \\
(\phi \cdot \nu)_{abcd} &= -4 \left(\tilde{\phi}_{[abc}{}^e \tilde{\nu}_{d]e} - \frac{1}{4!} \epsilon_{abcdm_1m_2m_3m_4} \phi^{m_1m_2m_3}{}_e \nu^{m_4e} \right)
\end{aligned}
\tag{A.14}
$$

and finally $\mathbf{912} \times \mathbf{133}\big|_{\mathbf{56}}$ is

$$
\begin{aligned}
(\phi \cdot \mu)^{ab} &= -(\phi^{ac} \mu^b{}_c - \phi^{bc} \mu^a{}_c) - 2 \phi^{abc}{}_d \mu^d{}_c \\
&\quad + \frac{2}{3} (\tilde{\phi}_{m_1m_2m_3}{}^a \star \mu^{m_1m_2m_3b} - \tilde{\phi}_{m_1m_2m_3}{}^b \star \mu^{m_1m_2m_3a})
\end{aligned}
\tag{A.15}
$$

$$
\begin{aligned}
(\phi \cdot \mu)_{ab} &= (\tilde{\phi}_{ac} \mu^c{}_b - \tilde{\phi}_{bc} \mu^c{}_a) - 2 \tilde{\phi}_{abc}{}^d \mu^c{}_d \\
&\quad - \frac{2}{3} (\phi^{m_1m_2m_3}{}_b \, \mu_{m_1m_2m_3a} - \phi^{m_1m_2m_3}{}_a \, \mu_{m_1m_2m_3b})
\end{aligned}
\tag{A.16}
$$

B SU(8) and SU(4) × SU(2) conventions

The spinor θ^α transforms in the fundamental representation of SU(8). The standard intertwining relations

$$
\Gamma_M^\dagger = A \Gamma_M A^{-1}, \qquad \Gamma_M^T = C^{-1} \Gamma_M C, \qquad (\Gamma_M)^* = -D^{-1} \Gamma_M D \tag{B.1}
$$

allow to define the conjugate spinors

$$
\bar{\theta} = \theta^\dagger A \,, \quad \theta^t = C \theta^T \,, \quad \theta^c = D \theta^* \,. \tag{B.2}
$$

Under SU(8), the **56** decomposes according to

$$
\begin{aligned}
\nu &= (\nu^{\alpha\beta}, \bar{\nu}_{\alpha\beta}) \\
\mathbf{56} &= \mathbf{28} + \bar{\mathbf{28}}
\end{aligned}
\tag{B.3}
$$

while for the adjoint **133** we have

$$
\begin{aligned}
\mu &= (\mu^\alpha{}_\beta, \mu^{\alpha\beta\gamma\delta}, \bar{\mu}_{\alpha\beta\gamma\delta}) \\
\mathbf{133} &= \mathbf{63} + \mathbf{35} + \bar{\mathbf{35}} \,.
\end{aligned}
\tag{B.4}
$$

where $\mu^\alpha{}_\alpha = 0$ and $\bar\mu_{\alpha\beta\gamma\delta} = \star\mu_{\alpha\beta\gamma\delta}$. Note that these are very similar to the SL$(8,\mathbb{R})$ decompositions (A.6), (A.7). To go from one to the other, we use for the **56** [10]

$$\nu^{ab} = \frac{\sqrt{2}}{8}(\nu^{\alpha\beta} + \bar\nu^{\alpha\beta})\Gamma^{ab}{}_{\beta\alpha}\,, \tag{B.5}$$

$$\tilde\nu_{ab} = -\frac{\sqrt{2}}{8}i(\nu^{\alpha\beta} - \bar\nu^{\alpha\beta})\Gamma^{ab}{}_{\beta\alpha}\,. \tag{B.6}$$

In the main text we use a complex **28** object, defined from its real pieces $\lambda^{ab}, \tilde\lambda_{ab}$ in the obvious way

$$L^{ab} = \lambda^{ab} + i\tilde\lambda^{ab} = \frac{\sqrt{2}}{4}L^{\alpha\beta}\Gamma^{ab}{}_{\beta\alpha} \tag{B.7}$$

From the **63** adjoint representation of SU(8) (i.e. taking $\mu_{\alpha\beta\gamma\delta} = 0$) one recovers the following SL$(8,\mathbb{R})$ components

$$\begin{aligned} \mu_{ab} &= -\frac{1}{4}\mu^\alpha{}_\beta \Gamma_{ab}{}^\beta{}_\alpha \\ \mu_{abcd} &= \frac{i}{8}\mu^\alpha{}_\beta \Gamma_{abcd}{}^\beta{}_\alpha \end{aligned} \tag{B.8}$$

where $\mu_{ba} = -\mu_{ab}$ and $\star\mu_{abcd} = -\mu_{abcd}$ (the symmetric and self-dual pieces are obtained from the **70** representation $\mu^{\alpha\beta\gamma\delta}$) and $\mu_{ab} = g_{ac}\mu^c{}_b$ (at this point there is a metric since SL$(8) \cap$ SU$(8) =$ SO(8)).

When breaking SU$(8) \to$ SU$(4) \times$ SU(2), the spinor index decomposes in a pair of indices $\alpha = \hat\alpha I$, where $\hat\alpha$ is an SU(4) spinor index. For the *Cliff*$(8,0)$ gamma matrices, we have used the following basis in terms of *Cliff*$(6,0)$ and Pauli sigma-matrices

$$\begin{aligned} \Gamma^{m\alpha}{}_\beta &= \gamma^m \otimes \sigma_3 \\ \Gamma^{1\alpha}{}_\beta &= \mathbb{I}_6 \otimes \sigma_1 \\ \Gamma^{2\alpha}{}_\beta &= \mathbb{I}_6 \otimes \sigma_2\,. \end{aligned} \tag{B.9}$$

The intertwiners A, C, D also split into *Cliff*$(6)\otimes$*Cliff*(2) intertwiners. In particular, C splits as

$$C = \hat C \otimes c \tag{B.10}$$

where $\hat C$ is the intertwiner

$$\gamma^{mT} = -\hat C^{-1}\gamma^m \hat C\,. \tag{B.11}$$

We get that

$$C_{\alpha\beta} = \hat C \otimes \sigma_1 \tag{B.12}$$

We will use a basis for the *Cliff*$(6,0)$ gamma matrices in which $\hat A = \hat C = \hat D = \mathbb{I}$, and therefore the SU(4) conjugate spinors are just

$$\bar\eta = \eta^\dagger\,, \quad \eta^t = \eta^T\,, \quad \eta^c = \eta^* \tag{B.13}$$

and $\eta_- \equiv \eta_+^*$. In this basis, the SU(8) spinors in (3.2) have conjugates

$$\theta^{1t} = (0, \eta_-^{1T}) \tag{B.14}$$

$$\bar\theta_1 = \theta^{1\dagger} = (\eta_+^{1\dagger}, 0)\,. \tag{B.15}$$

C GL(6, ℝ) embedding in SL(8, ℝ)

The GL(6, ℝ) weights of the different O(6, 6) × SL(2, ℝ) representations is worked out in [11]. It turns out that the two components of an SL(2, ℝ) doublet have different GL(6, ℝ) weights. To find the GL(6, ℝ) weight in the SL(8, ℝ) decomposition, we use that SL(8, ℝ) ⊃ SL(2, ℝ) × GL(6, ℝ) ⊂ O(6, 6) × SL(2, ℝ), where the common GL(6, ℝ) piece corresponds to the diffeomorphisms. Decomposing $a = (m, i)$ with $m = 1, \ldots, 6$ a GL(6) index and $i = 1, 2$ an SL(2) index, the embedding of SL(2, ℝ) × GL(6, ℝ) ⊂ SL(8, ℝ) is the following

$$
\begin{aligned}
M^a{}_b &= \begin{pmatrix} (\det a)^{-1/4} a^m{}_n & 0 \\ 0 & (\det a)^{1/4} \begin{pmatrix} (\det a)^{-1/2} e^{\phi} & 0 \\ 0 & (\det a)^{1/2} e^{-\phi} \end{pmatrix} \end{pmatrix} \\
&= \begin{pmatrix} (\det a)^{-1/4} a^m{}_n & 0 & 0 \\ 0 & (\det a)^{-1/4} e^{\phi} & 0 \\ 0 & 0 & (\det a)^{3/4} e^{-\phi} \end{pmatrix}
\end{aligned}
\tag{C.1}
$$

where $M \in \mathrm{SL}(8,\mathbb{R}), a \in \mathrm{GL}(6,\mathbb{R})$, and we have added explicit factors of the dilaton that are needed in order to get the right transformation properties of the connection. Since a six-form transforms by a factor $(\det g)^{1/2}$ (or equivalently $1/\det a$), we can write the 8-dimensional metric as

$$
\hat{g}_{ab} = \begin{pmatrix} (\det g)^{-1/4} g_{mn} & 0 & 0 \\ 0 & (\det g)^{-1/4} e^{-2\phi} & 0 \\ 0 & 0 & (\det g)^{3/4} e^{2\phi} \end{pmatrix}
\tag{C.2}
$$

The different SL(8, ℝ) components of **56** representation $\nu = (\nu^{ab}, \tilde{\nu}_{ab})$ transform therefore according to

$$
\begin{aligned}
\tilde{\nu}_{mn} &\in \left(\Lambda^6 T^* M\right)^{-1/2} \otimes \Lambda^2 T^* M\,, & \nu^{mn} &\in \left(\Lambda^6 T^* M\right)^{-1/2} \otimes \Lambda^4 T^* M \\
\tilde{\nu}_{1m} &\in \mathcal{L} \otimes \left(\Lambda^6 T^* M\right)^{-1/2} \otimes T^* M\,, & \nu^{1m} &\in \mathcal{L}^{-1} \otimes \left(\Lambda^6 T^* M\right)^{-1/2} \otimes \Lambda^5 T^* M \\
\tilde{\nu}_{2m} &\in \mathcal{L}^{-1} \otimes \left(\Lambda^6 T^* M\right)^{-1/2} \otimes (T^* M \otimes \Lambda^6 T^* M)\,, & \nu^{2m} &\in \mathcal{L} \otimes \left(\Lambda^6 T^* M\right)^{-1/2} \otimes TM \\
\tilde{\nu}_{12} &\in \left(\Lambda^6 T^* M\right)^{-1/2} \otimes \Lambda^6 T^* M\,, & \nu^{12} &\in \left(\Lambda^6 T^* M\right)^{-1/2}
\end{aligned}
\tag{C.3}
$$

where we have introduced a trivial real line bundle $\mathcal{L}$ with sections $e^{-\phi} \in \mathcal{L}$ to account for factors of the dilaton. The adjoint $\mu = (\mu^a{}_b, \mu_{abcd})$ has the following GL(6, ℝ) and dilaton assignments

$$
\begin{aligned}
\mu^1{}_1 = -\mu^2{}_2 &\in \mathbb{R}\,, & \mu^1{}_2 &\in \mathcal{L}^{-2} \otimes \Lambda^6 T^* M\,, & \mu^2{}_1 &\in \mathcal{L}^2 \otimes \Lambda^6 TM\,, \\
\mu^m{}_n &\in TM \otimes T^* M\,, & \mu^1{}_m &\in \mathcal{L}^{-1} \otimes T^* M\,, & \mu^2{}_m &\in \mathcal{L} \otimes \Lambda^5 TM\,, \\
\mu^m{}_1 &\in \mathcal{L} \otimes TM\,, & \mu^m{}_2 &\in \mathcal{L}^{-1} \otimes \Lambda^5 T^* M\,, & \mu_{mnpq} &\in \Lambda^2 TM\,, \\
\mu_{mnp1} &= \mathcal{L} \otimes \Lambda^3 TM\,, & \mu_{mnp2} &= \mathcal{L}^{-1} \otimes \Lambda^3 T^* M\,, & \mu_{mn12} &\in \Lambda^2 T^* M
\end{aligned}
\tag{C.4}
$$

Finally, the **912** multiplied by $\mathcal{L}\otimes\left(\Lambda^6T^*M\right)^{-1/2}$ (a T-duality invariant factor), transforms as

$$\begin{aligned}
\phi^{11} &\in \mathcal{L}^{-1}\otimes\mathbb{R}\,, & \phi'_{11} &\in \mathcal{L}^{3}\otimes\Lambda^6TM\\
\phi^{12} &\in \Lambda^6TM\,, & \phi'_{12} &\in \mathbb{R}\\
\phi^{22} &\in \mathcal{L}^{3}\otimes(\Lambda^6TM)^2\,, & \phi'_{22} &\in \mathcal{L}^{-1}\otimes\Lambda^6T^*M\\
\phi^{mnp}{}_q &\in \Lambda^3TM\otimes T^*M\,, & \phi'_{mnp}{}^q &\in \Lambda^3TM\otimes TM\\
\phi^{mnp}{}_1 &\in \mathcal{L}^{2}\otimes\Lambda^3TM\,, & \phi'_{mnp}{}^1 &\in \Lambda^3TM\\
\phi^{mnp}{}_2 &\in \Lambda^3T^*M\,, & \phi'_{mnp}{}^2 &\in \mathcal{L}^{2}\otimes\Lambda^3TM\otimes\Lambda^6TM\\
\phi^{mn1}{}_2 &\in \mathcal{L}^{-1}\otimes\Lambda^4T^*M\,, & \phi'_{mn1}{}^2 &\in \mathcal{L}^{3}\otimes\Lambda^4TM\otimes\Lambda^6TM\\
\phi^{mn2}{}_1 &\in \mathcal{L}^{3}\otimes\Lambda^2TM\otimes\Lambda^6TM\,, & \phi'_{mn2}{}^1 &\in \mathcal{L}^{-1}\otimes\Lambda^2T^*M
\end{aligned} \tag{C.5}$$

D Computing the twisted derivative

We show in the following how to obtain the connection from twisting the Levi-Civita covariant derivative (5.2) by the gauge fields B, $\tilde{B}$ and C^- in the **133** representation. Using the Hadamard formula we get for any element A in the adjoint

$$e^{-A}\nabla e^{A} = \nabla + \nabla A + \frac{1}{2}[\nabla A, A] + \frac{1}{6}[[\nabla A, A], A] + \dots$$

Using (2.17) we get in the $\mathrm{O}(6,6)\times\mathrm{SL}(2,\mathbb{R})$ decomposition

$$\begin{aligned}
(e^{B}e^{-\tilde{B}}e^{-C}\nabla e^{C}e^{\tilde{B}}e^{-B})^i{}_j &= \delta^i{}_j\nabla + v^iv_j\nabla\tilde{B} + v^iv_j\langle\nabla C^-, C^-\rangle\,,\\
(e^{B}e^{-\tilde{B}}e^{-C}\nabla e^{C}e^{\tilde{B}}e^{-B})^B{}_C &= \delta^B{}_C\nabla - \nabla B^B{}_C\,,\\
(e^{B}e^{-\tilde{B}}e^{-C}\nabla e^{C}e^{\tilde{B}}e^{-B})^{i-} &= v^i(e^B\nabla C^-)\,.
\end{aligned} \tag{D.1}$$

We now promote the Levi-Civita connection ∇ to an element carrying a fundamental **56** index, as in (5.2): $D^{\mathcal{A}} = (v^i\nabla^A, 0)$ and $\nabla^A = (0, \nabla_m)$. Finally, we project to the **912** representation using the tensor product $\mathbf{56}\times\mathbf{133}\big|_{\mathbf{912}}$ for the subgroup $\mathrm{SL}(2,\mathbb{R})\times\mathrm{O}(6,6)$ given in (A.5). We recover the simple result

$$\mathcal{F}^1{}_2{}^+ = -F^+\,, \qquad \mathcal{F}^1{}_{mnp} = -H_{mnp}\,, \tag{D.2}$$

where $F^+ = e^BdC^-$, and all the other components are zero.

One can alternatively express the connection in terms of the $\mathrm{SL}(8,\mathbb{R})$ subgroup. The derivative $D^{\mathcal{A}}$ is given in this case by

$$D_{m2} = -D_{2m} = \nabla_m\,, \tag{D.3}$$

while all other components are zero. Applying this to the gauge fields in (2.23), and projecting onto the **912** using (A.13), we find the following non-vanishing components

$$\mathcal{F}^{mnp}{}_2 = -\frac{1}{2}(*H)^{mnp}\,, \quad \mathcal{F}^{mn1}{}_2 = -\frac{e^\phi}{2}(*F_4)^{mn}\,, \quad \tilde{\mathcal{F}}_{mn2}{}^1 = -\frac{e^\phi}{2}F_{mp}\,, \quad \tilde{\mathcal{F}}_{22} = e^\phi*F_6\,. \tag{D.4}$$

Notice that the mass parameter $F_{(0)}$ cannot be obtained this way, and should be added by hand. Using (C.5), we note that the component ϕ^{11} transforms as a scalar, and we therefore assign

$$\mathcal{F}^{11} = e^{\phi} F_0 \,. \tag{D.5}$$

E Twisted derivative of L and K

Inserting the SL(8, ℝ) decomposition of the derivative and of the fluxes given respectively in (5.3) and (5.9), and the corresponding SL(8, ℝ) components of the tensor products given in (A.11) and (A.14), we get the following expressions for the twisted derivative of $\lambda = (\lambda^{ab}, \tilde{\lambda}_{ab})$, projected onto the **133**

$$(\mathcal{D}\lambda)^1{}_1 = -\frac{1}{4}\nabla_p \lambda^{p2} \tag{E.1}$$

$$(\mathcal{D}\lambda)^2{}_2 = \frac{3}{4}\nabla_m \lambda^{m2} \tag{E.2}$$

$$(\mathcal{D}\lambda)^1{}_2 = -\nabla_m \lambda^{1m} - e^{\phi}(*F_6)\lambda^{12} + e^{\phi}F_0\tilde{\lambda}_{12} + \frac{e^{\phi}}{2}F_{mn}\lambda^{mn} - \frac{e^{\phi}}{2}(*F_4)^{np}\tilde{\lambda}_{np} \tag{E.3}$$

$$(\mathcal{D}\lambda)^m{}_2 = -\nabla_p \lambda^{mp} - \frac{1}{2}(*H)^{mnp}\tilde{\lambda}_{np} - e^{\phi}(*F_6)\lambda^{m2} - e^{\phi}(*F_4)^{mn}\tilde{\lambda}_{n1} \tag{E.4}$$

$$(\mathcal{D}\lambda)^1{}_m = \nabla_m \lambda^{12} + e^{\phi}F_0\tilde{\lambda}_{1m} + e^{\phi}F_{mn}\lambda^{n2} \tag{E.5}$$

$$(\mathcal{D}\lambda)^n{}_m = \nabla_m \lambda^{n2} - \frac{1}{4}g^n{}_m \nabla_p \lambda^{p2} \tag{E.6}$$

$$(\mathcal{D}\lambda)_{mnp2} = -\frac{3}{2}\nabla_{[m}\tilde{\lambda}_{np]} + \frac{1}{2}H_{mnp}\lambda^{12} - \frac{3}{2}e^{\phi}F_{[mn|}\tilde{\lambda}_{|p]1} - \frac{e^{\phi}}{2}F_{mnpq}\lambda^{2q} \tag{E.7}$$

$$(\mathcal{D}\lambda)_{mn12} = -\nabla_{[m}\tilde{\lambda}_{n]1} + \frac{1}{2}H_{mnp}\lambda^{p2} \,. \tag{E.8}$$

To get the twisted derivative of K projected on the **56**, we use the tensor products (A.9) and (A.15). We find

$$(\mathcal{D}K)^{mn} = -2\nabla_p K^{mnp2} + (*H)^{mnp}K^2{}_p + e^{\phi}(*F_4)^{mn}K^2{}_1 \tag{E.9}$$

$$\widetilde{(\mathcal{D}K)}_{mn} = -2\nabla_{[m}K^2{}_{n]} + e^{\phi}F_{mn}K^2{}_1 \tag{E.10}$$

$$(\mathcal{D}K)^{m1} = 2\nabla_p K^{mp12} + e^{\phi}F_0 K^m{}_1 - e^{\phi}(*F_4)^{mn}K^2{}_n - e^{\phi}F_{np}K^{2npm} \tag{E.11}$$

$$\widetilde{(\mathcal{D}K)}_{m1} = -\nabla_m K^2{}_1 \tag{E.12}$$

$$(\mathcal{D}K)^{m2} = 0 \tag{E.13}$$

$$\widetilde{(\mathcal{D}K)}_{m2} = -\nabla_p K^p{}_m - H_{mpq}K^{pq12} - e^{\phi}(*F_6)K^2{}_m - e^{\phi}F_{mp}K^p{}_1 + e^{\phi}(*F_4)^{pq}K_{1pqm} \tag{E.14}$$

$$(\mathcal{D}K)^{12} = -e^{\phi}F_0 K^2{}_1 \tag{E.15}$$

$$\widetilde{(\mathcal{D}K)}_{12} = -\nabla_n K^n{}_1 - \frac{1}{3}H_{npq}K^{2npq} - e^{\phi}(*F_6)K^2{}_1 \tag{E.16}$$

where we have used that

$$\star K^{abcd} = -K^{abcd} \tag{E.17}$$

which is a consequence of fact that K is purely in the **63** of SU(8).

F Supersymmetric variations for the $\mathcal{N}=1$ spinor ansatz

The supersymmetry transformations of the fermionic fields of type IIA read, in the democratic formulation [29]

$$\delta\psi_M = \nabla_M \epsilon + \frac{1}{4} \not{H}_M \mathcal{P}\epsilon + \frac{1}{16} e^{\phi} \sum_n \not{F}_n^{(10)} \Gamma_M \mathcal{P}_n \epsilon\,, \tag{F.1}$$

$$\delta\lambda = \left(\not{\partial}\phi + \frac{1}{2}\not{H}\mathcal{P}\right)\epsilon + \frac{1}{8} e^{\phi} \sum_n (5-n) \not{F}_n^{(10)} \mathcal{P}_n \epsilon\,. \tag{F.2}$$

where $\mathcal{P} = -\sigma^3$ and $\mathcal{P}_n = (-\sigma_3)^{n/2}\sigma_1$. We use the standard decomposition of ten-dimensional gamma matrices

$$\gamma_\mu^{(10)} = \gamma_\mu \otimes 1\,, \quad \gamma_m^{(10)} = \gamma_5 \otimes \gamma_m\,, \tag{F.3}$$

the Poincare invariant ansatz for the RR fluxes

$$F_{2n}^{(10)} = F_{2n} + \mathrm{vol}_4 \wedge \tilde{F}_{2n-4} \quad \text{where } \tilde{F}_{2n-4} = (-1)^{Int[n]} *_6 F_{10-2n} \tag{F.4}$$

and we notice that, according to (B.9), $\mathcal{P} = i\Gamma^{12}$, $\mathcal{P}_0 = \mathcal{P}_4 = \Gamma^1$, $\mathcal{P}_2 = \mathcal{P}_6 = -i\Gamma^2$, $\gamma^m \mathcal{P}_0 = -i\Gamma^{2m}$ and $\gamma^m \mathcal{P}_2 = \Gamma^{1m}$.[17]

We use the $\mathcal{N}=1$ spinor ansatz (4.18), parameterised using two internal spinors, namely $\theta = \theta^1 + \theta^2$, where θ^1, θ^2 given in (5.10), we get from the internal components of the gravitino variation that $\mathcal{N}=1$ supersymmetry requires

$$\delta\psi_m = 0 \;\Leftrightarrow\; \nabla_m \theta^1 + \frac{i}{8} H_{mnp} \Gamma^{np12} \theta^1 - \frac{e^{\phi}}{8} \not{F}_{\rm i} \Gamma_m \theta^2 = 0\,, \tag{F.5}$$

and the same exchaging $1 \leftrightarrow 2$, where we have defined

$$\not{F}_i = -i \not{F}_h \Gamma^2 + \not{F}_a \Gamma^1 \tag{F.6}$$

in terms of the "hermitean" and "antihermitean" pieces of F, namely

$$F_h = \frac{1}{2}(F + s(F)) = F_0 + F_4\,, \quad F_a = \frac{1}{2}(F - s(F)) = F_2 + F_6 \tag{F.7}$$

and finally

$$\not{F}_{(n)} = \frac{1}{n!} F_{i_1 \dots i_n} \Gamma^{i_1 \dots i_n}\,. \tag{F.8}$$

We will also need the equations involving $\bar{\theta}$, which is

$$\nabla_m \bar{\theta}^1 - \frac{i}{8} H_{mnp} \bar{\theta}^1 \Gamma^{np12} + \frac{e^{\phi}}{8} \bar{\theta}^2 \Gamma_m \not{F}_{\rm i} = 0\,, \tag{F.9}$$

From the external gravitino variation, we get that $\mathcal{N}=1$ vacua should satisfy

$$\delta\psi_\mu = 0 \;\Leftrightarrow\; i \not{\partial}_{\rm e} A\, \theta^1 + \frac{e^{\phi}}{4} \not{F}_{\rm e} \theta^2 = 0, \tag{F.10}$$

[17] To avoid clustering of determinant factors, in this section we use the basis for Cliff(8) gamma matrices in (B.9) without the determinant factors.

and similarly exchanging 1 and 2, where

$$\not{F}_{\rm e} = \not{F}_h \Gamma^1 - i \not{F}_a \Gamma^2 \tag{F.11}$$

and

$$\not{\partial}_{\rm e} A = \partial_m A \, \Gamma^{m12} \,. \tag{F.12}$$

The hermitean conjugate equation reads

$$i \, \bar{\theta}^1 \not{\partial}_{\rm e} A + \frac{e^\phi}{4} \bar{\theta}^2 \not{F}_{\rm e} = 0, \tag{F.13}$$

From the dilatino variation, we get

$$\delta\lambda = 0 \; \Leftrightarrow i \not{\partial}_{\rm e} \phi \, \theta^1 + \frac{1}{12} H_{mnp} \Gamma^{mnp} \theta^1 + \frac{e^\phi}{4} \not{F}_{\rm d} \theta^2 = 0 \tag{F.14}$$

where we have defined

$$\not{F}_{\rm d} = (5-n) \not{F}_{\rm e} \,. \tag{F.15}$$

The hermitean conjugate equation reads

$$i \bar{\theta}^1 \not{\partial}_{\rm e} \phi - \frac{1}{12} H_{mnp} \bar{\theta}^1 \Gamma^{mnp} + \frac{e^\phi}{4} \bar{\theta}^2 \not{F}_{\rm d} = 0 \tag{F.16}$$

G $\mathcal{D}L$ and $\mathcal{D}K$ versus $\mathcal{N} = 1$ supersymmetry

G.1 $\mathcal{D}L$

Multiplying eq. (F.5) (coming from the internal gravitino variation) for the covariant derivative of θ^1 (θ^2), on the right by $e^{2A-\phi}\theta^2$ ($e^{2A-\phi}\theta^1$), and substracting the two equations, we get the following equation for the covariant derivative of L'

$$(\Delta_m L')^{\alpha\beta} \equiv \nabla_m L'^{\alpha\beta} - \partial_m (2A-\phi) L'^{\alpha\beta} + \frac{1}{4} (i H_{mnp} \Gamma^{np12} L')^{\alpha\beta} - \frac{e^\phi}{4} (\not{F}_{\rm i} \Gamma_m \pi')^{\alpha\beta} = 0 \,. \tag{G.1}$$

where we have defined

$$\pi'^{\alpha\beta} \equiv e^{2A-\phi} (\theta^2 \theta^2 - \theta^1 \theta^1)^{\alpha\beta} \equiv e^{2A-\phi} \pi^{\alpha\beta} \,. \tag{G.2}$$

We will also need the SL(8) object π^{abcd}, which we define to be

$$\pi'^{abcd} = \frac{\sqrt{2}}{4} \pi'^{\alpha\beta} \Gamma^{abcd}{}_{\beta\alpha} \tag{G.3}$$

Multiplying (F.10) (coming from external gravitino variation on θ^1) by θ^2, and substracting to the equation with θ^1 and θ^2 exchanged, we get the following equation

$$(\Delta_e L)^{\alpha\beta} \equiv i \partial_m A \, (\Gamma^{m12} L)^{\alpha\beta} + \frac{e^\phi}{4} (\not{F}_{\rm e} \pi)^{\alpha\beta} = 0 \,. \tag{G.4}$$

If instead we multiply (F.10) by θ^1 and substract the corresponding equation for θ^2 multiplied by θ^2, we get

$$(\Delta_e\pi)^{\alpha\beta} \equiv i\partial_m A\,(\Gamma^{m12}\pi)^{\alpha\beta} + \frac{e^\phi}{4}(\not{F}_{\rm e}L)^{\alpha\beta} = 0\,. \tag{G.5}$$

Doing the same on the dilatino (F.14) we get

$$(\Delta_d L)^{\alpha\beta} \equiv i\partial_m\phi\,(\Gamma^{m12}L)^{\alpha\beta} + \frac{1}{12}H_{mnp}(\Gamma^{mnp}L)^{\alpha\beta} + \frac{e^\phi}{4}(\not{F}_{\rm d}\pi)^{\alpha\beta} = 0\,, \tag{G.6}$$

and a similar equations with L and π exchanged, that will not be used.

We show here how supersymmetry requires equations (5.18)–(5.20) to vanish. For each of them, we use (G.1) plus l_e times (G.4) and l_d times (G.6), and take in the one to last step

$$l_e = -2\,, \quad l_d = 1\,. \tag{G.7}$$

We show that susy requires eq. (5.18) to vanish by

$$\begin{aligned}
0 &= \frac{\sqrt{2}}{4}\text{Tr}\,\left(\Gamma^{12}\Delta_m L' + i\Gamma_m(l_e\Delta_e + l_d\Delta_d)L'\right)\\
&= \nabla_m L'^{12} - \partial_m(2A-\phi)L'^{12} - \partial_m(l_eA + l_d\phi)L'^{12} + \frac{i}{4}(-1+l_d)H_{mpq}L'^{pq}\\
&\quad - \frac{e^\phi}{8}[F_{pq}(-1+l_e+3l_d) - i(*F_4)(1+l_e+l_d)]\pi'^{2pq}{}_m\\
&= \nabla_m L'^{12}\\
&= (\mathcal{D}L')^1{}_m\,,
\end{aligned} \tag{G.8}$$

To get (5.19) we do

$$\begin{aligned}
0 &= \frac{\sqrt{2}}{4}\text{Tr}\,\left(-\Gamma^{mn}\Delta_n L' + i\Gamma^{m12}(l_d\Delta_d L' + l_e\Delta_e L')\right)\\
&= -\nabla_p L'^{mp} + \partial_n(2A-\phi)L'^{mn} + \partial_n(l_eA + l_d\phi)L'^{mn} + \frac{i}{4}(3-l_d)(*H)^{mpq}L'_{pq}\\
&\quad - \frac{e^\phi}{8}[F_{pq}(-1+l_e+3l_d) - i(*F_4)_{pq}(1+l_e+l_d)]\pi'^{1pq}{}_m\\
&= -\nabla_p L'^{mp} + \frac{i}{2}(*H)^{mnp}L'_{np}\\
&= (\mathcal{D}L')^m{}_2\,,
\end{aligned}$$

while for (5.20) we use

$$
\begin{aligned}
0 &= \frac{\sqrt{2}}{8}\mathrm{Tr}\,\left(3i\Gamma_{[mn|}\Delta_{p]}L' - \Gamma_{mnp12}(l_d\Delta_d L' + l_e\Delta_e L')\right) \\
&= \frac{3i}{2}\nabla_{[m}L'_{np]} - \frac{3}{2}i\partial_{[m}(2A-\phi)L'_{|np]} - \frac{3}{2}i\partial_{[m|}(l_eA + l_d\phi)L'_{|np]} \\
&\quad + \frac{1}{4}(3-l_d)H_{mnp}L'_{12} + \frac{3}{4}(-1+l_d)(*H)_{[mn|q}L'^{q}{}_{|p]} \\
&\quad + \frac{e^\phi}{8}[F_0(-3+l_e+5l_d) - i(*F_6)(3+l_e-l_d)]\pi'_{2mnp} \\
&\quad + 3\frac{e^\phi}{8}[iF_{[m|q}(-1+l_e+3l_d) + (*F_4)_{[m|q}(1+l_e+l_d)]\pi'^{1q}{}_{|np]} \\
&= \frac{3i}{2}\nabla_{[m}L'_{np]} + \frac{1}{2}H_{mnp}L'^{12} \\
&= (\mathcal{D}L')_{mnp2}\,.
\end{aligned}
$$

G.2 $\mathcal{D}K$

We define the following quantities

$$
K'_0 = e^A K_0\,, \qquad K'_1 = e^A K_1\,, \qquad K'_2 = e^{3A}K_2\,, \qquad K'_3 = e^{3A}K_3\,. \tag{G.9}
$$

Combining (F.5) multiplied by $\bar{\theta}$ with (F.9) multiplied by θ, we obtain

$$
\Delta_m K_0 \equiv e^{-\phi}\nabla_m(e^\phi K_0)^\alpha{}_\beta + \frac{i}{8}H_{mnp}[\Gamma^{np12}K'_0 - K'_0\Gamma^{np12}]^\alpha{}_\beta - \frac{e^\phi}{8}[\not{F}_i\Gamma_m K'_1 - K'_1\Gamma_m\not{F}_i]^\alpha{}_\beta = 0 \tag{G.10}
$$

$$
\Delta_m K_1 \equiv e^{-\phi}\nabla_m(e^\phi K_1)^\alpha{}_\beta + \frac{i}{8}H_{mnp}[\Gamma^{np12}K'_1 - K'_1\Gamma^{np12}]^\alpha{}_\beta - \frac{e^\phi}{8}[\not{F}_i\Gamma_m K'_0 - K'_0\Gamma_m\not{F}_i]^\alpha{}_\beta = 0 \tag{G.11}
$$

$$
\Delta_m K_2 \equiv e^{-\phi}\nabla_m(e^\phi K_2)^\alpha{}_\beta + \frac{i}{8}H_{mnp}[\Gamma^{np12}K'_2 - K'_2\Gamma^{np12}]^\alpha{}_\beta - i\frac{e^\phi}{8}[\not{F}_i\Gamma_m K'_3 + K'_3\Gamma_m\not{F}_i]^\alpha{}_\beta = 0 \tag{G.12}
$$

$$
\Delta_m K_3 \equiv e^{-\phi}\nabla_m(e^\phi K_3)^\alpha{}_\beta + \frac{i}{8}H_{mnp}[\Gamma^{np12}K'_3 - K'_3\Gamma^{np12}]^\alpha{}_\beta + i\frac{e^\phi}{8}[\not{F}_i\Gamma_m K'_2 + K'_2\Gamma_m\not{F}_i]^\alpha{}_\beta = 0 \tag{G.13}
$$

where the factors of the dilaton inside the covariant derivatives are there to cancel the explicit dilaton dependence of K (see (3.6)).

Multiplying the external gavitino or dilatino equation, eqs. (F.10) and (F.14) by $\bar{\theta}^2$ on the right, and adding it to the same equation with θ^1 and θ^2 exchanged, we get

$$
(\Delta_e K_1)^\alpha{}_\beta \equiv i\partial_m A[\Gamma^{m12}K_1]^\alpha{}_\beta + \frac{e^\phi}{4}[\not{F}_e K_0]^\alpha{}_\beta = 0\,, \tag{G.14}
$$

$$
(\Delta_d K_1)^\alpha{}_\beta \equiv i\partial_m\phi[\Gamma^{m12}K_1]^\alpha{}_\beta + \frac{1}{12}H_{mpq}[\Gamma^{mpq}K_1]^\alpha{}_\beta + \frac{e^\phi}{4}[\not{F}_d K_0]^\alpha{}_\beta = 0\,. \tag{G.15}
$$

We can also use the complex conjugate equations (F.13), (F.16) multiplied on the left by θ^2. This gives

$$(K_1\Delta_e)^\alpha{}_\beta \equiv i\partial_m A[K_1\Gamma^{m12}]^\alpha{}_\beta + \frac{e^\phi}{4}[K_0\not{F}_e]^\alpha{}_\beta = 0\,, \tag{G.16}$$

$$(K_1\Delta_d)^\alpha{}_\beta \equiv i\partial_m \phi[K_1\Gamma^{m12}]^\alpha{}_\beta - \frac{1}{12}H_{mpq}[K_1\Gamma^{mpq}]^\alpha{}_\beta + \frac{e^\phi}{4}[K_0\not{F}_d]^\alpha{}_\beta = 0 \tag{G.17}$$

We will also need the corresponding equations mixing K_3 and K_2

$$(\Delta_e K_3)^\alpha{}_\beta \equiv i\partial_m A[\Gamma^{m12}K_3]^\alpha{}_\beta - i\frac{e^\phi}{4}[\not{F}_e K_2]^\alpha{}_\beta = 0 \tag{G.18}$$

$$(K_3\Delta_e)^\alpha{}_\beta \equiv i\partial_m A[K_3\Gamma^{m12}]^\alpha{}_\beta + i\frac{e^\phi}{4}[K_2\not{F}_e]^\alpha{}_\beta = 0 \tag{G.19}$$

$$(\Delta_d K_3)^\alpha{}_\beta \equiv i\partial_m \phi[\Gamma^{m12}K_3]^\alpha{}_\beta + \frac{1}{12}H_{mpq}[\Gamma^{mpq}K_3]^\alpha{}_\beta - i\frac{e^\phi}{4}[\not{F}_d K_2]^\alpha{}_\beta = 0 \tag{G.20}$$

$$(K_3\Delta_d)^\alpha{}_\beta \equiv i\partial_m \phi[K_3\Gamma^{m12}]^\alpha{}_\beta - \frac{1}{12}H_{mpq}[K_3\Gamma^{mpq}]^\alpha{}_\beta + i\frac{e^\phi}{4}[K_2\not{F}_d]^\alpha{}_\beta = 0 \tag{G.21}$$

$$(\Delta_e K_2)^\alpha{}_\beta \equiv i\partial_m A[\Gamma^{m12}K_2]^\alpha{}_\beta + i\frac{e^\phi}{4}[\not{F}_e K_3]^\alpha{}_\beta = 0 \tag{G.22}$$

$$(K_2\Delta_e)^\alpha{}_\beta \equiv i\partial_m A[K_2\Gamma^{m12}]^\alpha{}_\beta - i\frac{e^\phi}{4}[K_3\not{F}_e]^\alpha{}_\beta = 0 \tag{G.23}$$

$$(\Delta_d K_2)^\alpha{}_\beta \equiv i\partial_m \phi[\Gamma^{m12}K_1]^\alpha{}_\beta + \frac{1}{12}H_{mpq}[\Gamma^{mpq}K_1]^\alpha{}_\beta + i\frac{e^\phi}{4}[\not{F}_d K_3]^\alpha{}_\beta = 0 \tag{G.24}$$

$$(K_2\Delta_d)^\alpha{}_\beta \equiv i\partial_m \phi[K_1\Gamma^{m12}]^\alpha{}_\beta - \frac{1}{12}H_{mpq}[K_1\Gamma^{mpq}]^\alpha{}_\beta - i\frac{e^\phi}{4}[K_3\not{F}_d]^\alpha{}_\beta = 0 \tag{G.25}$$

and the following ones involving K_0 and K_1

$$(\Delta_e K_0)^\alpha{}_\beta \equiv i\partial_m A[\Gamma^{m12}K_0]^\alpha{}_\beta + \frac{e^\phi}{4}[\not{F}_e K_1]^\alpha{}_\beta = 0\,, \tag{G.26}$$

$$(K_0\Delta_e)^\alpha{}_\beta \equiv i\partial_m A[K_0\Gamma^{m12}]^\alpha{}_\beta + \frac{e^\phi}{4}[K_1\not{F}_e]^\alpha{}_\beta = 0 \tag{G.27}$$

Given a generic K and product of gamma matrices $\Gamma^{a_1\dots a_i}$ we will make use of the following type of combinations

$$\mathrm{Tr}\,([\Gamma^{a_1\dots a_i},\Delta_d]K) \equiv \mathrm{Tr}\,((\Gamma^{a_1\dots a_i}\Delta_d - \Delta_d\Gamma^{a_1\dots a_i})K) = \mathrm{Tr}\,(\Gamma^{a_1\dots a_i}\Delta_d K - K\Delta_d\Gamma^{a_1\dots a_i})\,. \tag{G.28}$$

and similarly for the anticommutator.

G.2.1 $\mathcal{D}K_1'$

We want to show that susy requires (5.13) and (5.15). We recall that as shown in (3.10), K_1 has only nonzero components with an odd number of internal indices.

The idea is to reconstruct the twisted derivative of the corresponding K' appearing in each of the equations by summing an equation coming from internal gravitino (which gives a covariant derivative of K with no dilaton or warp factors) together with equations coming from external gravitino plus dilatino, which contribute the required derivatives of dilaton and warp factor.

We start by showing that susy requires (5.21) to vanish. We use the following combination of equations: (G.11) coming from internal gravitino, (G.14) and (G.16) from external gravitino, and (G.15), (G.17) from dilatino (the last four multiplied by arbitrary coefficients n_e and n_d, that will be set to $n_e = 1, n_d = -1$).

$$\begin{aligned}
0 &= -\frac{i}{4}\mathrm{Tr}\,\left(\Gamma^{mnp2}(e^A\Delta_p K_1) + \{\Gamma^{mn1}, (n_e\Delta_e + n_d\Delta_d)\}K_1'\right) \\
&= -2e^{A-\phi}\nabla_p(e^\phi K_1{}^{mnp2}) - 2\partial_p(n_eA + n_d\phi)K_1'{}^{mnp2} \\
&\quad + \frac{1}{2}(1+n_d)H^{mn}{}_pK_1'{}^{1p} + \frac{1}{2}(3+n_d)(*H)^{mn}{}_pK_1'{}^2{}_p \\
&\quad - \frac{1}{2}e^{-2A+\phi}F_0(4+n_e+5n_d)K_+'{}^{mn12} - \frac{1}{4}e^{-2A+\phi}(*F_4)_{pq}(n_e+n_d)K_+'{}^{pqmn} \\
&\quad - \frac{1}{2}e^{-2A+\phi}F^{[m|p}(2+n_e+3n_d)K_+'{}_p{}^{|n]} \\
&= -2\nabla_pK_1'{}^{mnp2} + (*H)^{mnp}K_1'{}^2{}_p \\
&= (\mathcal{D}K_1')^{mn}
\end{aligned} \tag{G.29}$$

where in the third equality we have used the values $n_e = 1, n_d = -1$.

To show that (5.22) vanishes, we use

$$\begin{aligned}
0 &= -\frac{1}{4}\mathrm{Tr}\,\left(2\Gamma^2{}_{[m}(e^A\Delta_{n]}K_1) - i[\Gamma^{mn1}, n_e\Delta_e + n_d\Delta_d]K_1'\right) \\
&= -2e^{A-\phi}\nabla_{[m}(e^\phi K_1{}^2{}_{n]}) - 2\partial_{[m}(n_eA + n_d\phi)K_1'{}^2{}_{n]} \\
&\quad - H_{pq[m}K'_1{}^{1pq}{}_{n]}(1+n_d) + \frac{1}{2}e^{-2A+\phi} * F_6\,(-2+n_e-n_d)K_{+mn}'{}^{12} \\
&\quad + \frac{1}{4}e^{-2A+\phi}F_{pq}(2+n_e+3n_d)K_+'{}^{pq}{}_{mn} + \frac{1}{2}e^{-2A+\phi}(*F_4)_{[m|}{}^p(n_e+n_d)K_{+p|n]}' \\
&= -2\nabla_{[m}K_1'{}^2{}_{n]} \\
&= (\widetilde{\mathcal{D}K_1'})_{mn}
\end{aligned} \tag{G.30}$$

where we have chosen again $n_e = 1, n_d = -1$.

To show that (5.23) vanishes, we use

$$0 = -\frac{i}{4}\mathrm{Tr}\,\left(i\Gamma^n{}_1(e^A\Delta_nK_1) + \Gamma^2(n_d\Delta_d + n_e\Delta_e)K_1'\right) \tag{G.31}$$

$$\begin{aligned}
&= -e^{A-\phi}\nabla_n(e^\phi K_1{}^n{}_1) - \partial_p(n_eA + n_d\phi)K_1'{}^p{}_1 - \frac{1}{6}H_{pqr}(3+n_d)K_1'{}^{2pqr} \\
&\quad + \frac{1}{4}e^{-2A+\phi}\Big[F_{pq}(2+n_e+3n_d) + i(*F_4)_{pq}(n_e+n_d)\Big]K_+'{}^{pq12} \\
&= -\nabla_nK_1'{}^n{}_1 - \frac{1}{3}H_{pqr}K_1'{}^{2pqr} \\
&= (\widetilde{\mathcal{D}K_1'})_{12}
\end{aligned} \tag{G.32}$$

where we have used again $n_e = 1, n_d = -1$.

For the vectorial equation (5.24), we use (G.13) and (G.19) to get

$$\begin{aligned} 0 &= \text{Tr}\left(-e^A\Delta_m K_3 + K_0'\Delta_e\Gamma^m\right) \\ &= -4e^A\partial_p A K_3^{mp} + e^\phi F_0 K_1'^{\,m}{}_1 - e^\phi(*F_4)^{mn}K_1'^{\,2}{}_n - e^\phi F_{np}K_1'^{\,2npm} \\ &= -4e^A\partial_p A K_3^{mp} + (\mathcal{D}K_1')^{m1} \end{aligned} \tag{G.33}$$

where we have used $K_2 = K_1\Gamma^{12}$ and $K_0 = -iK_3\Gamma^{12}$, and in the last line we have used (5.24). For the last equation (5.25) we use (G.10) and (G.18)

$$\begin{aligned} 0 &= \text{Tr}\left(e^A\Delta_m K_0 + iK_3'\Delta_e\Gamma^m\right) \\ &= 4ie^{A-\phi}\nabla_m(e^\phi K_3{}^1{}_2) - 8e^A\partial_p A K_{3m}{}^{p12} - e^\phi{*F_6}K_1'^{\,2}{}_m - e^\phi F_{mn}K_1'^{\,n}{}_1 + e^\phi(*F_4)^{np}K'_{1\,1npm} \\ &= 4ie^A\partial_m A\, K_3{}^1{}_2 - 8e^A\partial_p A K_{3m}{}^{p12} + (\widetilde{\mathcal{D}K_1'})_{m2} \end{aligned} \tag{G.34}$$

where in the second equality we have used again $K_0 = -iK_3\Gamma^{12}$, and in the third equality we have used (5.16) (which will be shown to hold below).

G.2.2 $\mathcal{D}K'_+$

The other set of equations involves

$$K'_+ = K'_3 + iK'_2 = e^{3A}(K_3 + iK_2)\,. \tag{G.35}$$

From (3.10), we see that K_+ with an odd number of internal indices is proportional to iK_2, while for an even number of internal indices, K_+ is proportional to K_3.

To show the first equation in (5.14), we use (G.12), (G.22) and (G.24) to get

$$\begin{aligned} 0 &= \frac{1}{4}\text{Tr}\left(\Gamma^{mnp2}(e^{3A}\Delta_p K_2) + i\Gamma^{mn1}(n_e\Delta_e K_2' + n_d\Delta_d K_2')\right) \\ &= -2e^{3A-\phi}\nabla_p(e^\phi K_+{}^{mnp2}) - 2\partial_p(n_eA + n_d\phi)K_+'{}^{mnp2} + 2i\partial_{[m}(n_eA + n_d\phi)K'^{2}_{+n]} \\ &\quad + \frac{1}{2}(1+n_d)H_{mnp}K_+'^{1p} + in_dH_{pq[m}K_+'^{1pq}{}_{|n]} + \frac{1}{2}(3+n_d)(*H)^{mnp}K'^{2}_{+p} \\ &\quad + \frac{e^\phi}{4}\left(F_0(n_e+5n_d) - i(*F_6)(-4+n_e-n_d)\right)K_+'^{mn} \\ &\quad + \frac{e^\phi}{4}\left(iF^{mn}(4+n_e+3n_d) - (*F_4)^{mn}(n_e+n_d)\right)K_+'^{12} \\ &\quad + \frac{e^\phi}{8}\left(i(*F_2)^{mn}{}_{pq}(n_e+3n_d) - F^{mn}{}_{pq}(n_e+n_d)\right)K_+'^{pq} \\ &\quad + e^\phi\left(F^{[m|}{}_p(n+3n_d) - i(*F_4)^{[m|}{}_p(-2+n_e+n_d)\right)K_+'^{p12|n]}\,. \end{aligned} \tag{G.36}$$

and

$$0 = \frac{1}{4}\mathrm{Tr}\,\left(2i\Gamma_{2[n}(e^{3A}\Delta_{m]}K_2) - \Gamma_{mn1}(n_e\Delta_e K_2' + n_d\Delta_d K_2')\right) \tag{G.37}$$

$$= -2e^{3A-\phi}\nabla_{[m}(e^{\phi}K_+{}^2{}_{n]}) - 2i\partial_p(n_eA + n_d\phi)K_+'^{mnp2} - 2\partial_{[m}(n_eA + n_d\phi)K'^2_{+|n]}$$
$$+ i\frac{n_d}{2}H_{mnp}K_+'^{1p} - (1+n_d)H_{pq[m}K'^{1pq}{}_{|n]} + i\frac{n_d}{2}(*H)^{mnp}K'^2_{+p}$$
$$+ \frac{e^{\phi}}{4}\left(iF_0(2+n_e+5n_d) + (*F_6)(n_e-n_d)\right)K'_{+mn}$$
$$- \frac{e^{\phi}}{4}\left(F_{mn}(n_e+3n_d) + i(*F_4)_{mn}(2+n_e+n_d)\right)K_+'^{12}$$
$$- \frac{e^{\phi}}{8}\left((*F_2)_{mnpq}(n_e+3n_d) + iF_{mnpq}(-2+n_e+n_d)\right)K'_+{}^{pq}$$
$$+ e^{\phi}\left(iF_{[m|p}(n_e+3n_d) + (*F_4)_{[m|p}(n_e+n_d)\right)K_+{}'^{p12}{}_{|n]}\,.$$

Note that in the NS sector K_+ reduces to K_2, while in the RR sector it is proportional to K_3. We combine these two, choosing $n_e = \frac{3}{2}, n_d = -\frac{1}{2}$, and we get

$$0 = (\mathrm{G.36}) - i\ (\mathrm{G.37})$$
$$= -2\nabla_p K'_+{}^{mnp2} + 2i\nabla_{[m}K'_+{}^2{}_{n]} + (*H)_{mnp}K'_+{}^2{}_p - e^{\phi}(*F_4 - iF_2)_{mn}K'_+{}^{12}$$
$$= (\mathcal{D}K'_+)_{mn} - i(\widetilde{\mathcal{D}K'_+})_{mn}.$$

For the 12 components we use

$$0 = \frac{1}{4}\mathrm{Tr}\,\left(i\Gamma^{n1}(e^{3A}\Delta_n K_2) - i\Gamma^2(n_e\Delta_e K_2' + n_d\Delta_d K_2')\right)$$
$$= ie^{3A-\phi}\nabla_n(e^{\phi}K_+^{n1}) + i\partial_n(n_eA + n_d\phi)K_+'^{n1} + \frac{i}{2}(1+\frac{n_d}{3})H_{pqr}K_+'^{2pqr}$$
$$+ \frac{e^{\phi}}{4}\left(-F_0(6+n_e+5n_d) + i * F_6(n_e-n_d)\right)K_+'^{12}$$
$$- \frac{e^{\phi}}{8}\left(iF_{mn}(n_e+3n_d) - (*F_4)_{mn}(-2+n_e+n_d)\right)K_+'^{mn}$$
$$= i\nabla_n K_+'^{n1} + \frac{i}{3}H_{pqr}K_+'^{2pqr} + e^{\phi}(-F_0 + i * F_6)K_+'^{12}$$
$$= (\mathcal{D}K'_+)_{12} - i(\widetilde{\mathcal{D}K'_+})_{12}\,. \tag{G.38}$$

where we have chosen $n_e = 3, n_d = -1$.

We are left with the vectorial components. The last equation in (5.14) is trivial (see (E.13)). To show the $m1$ component, we use

$$0 = -\frac{1}{4}\mathrm{Tr}\,\left(\Gamma^{12}(e^{3A}\Delta_m K_3) + in_e\{\Delta_e,\Gamma_m\}K_3'\right)$$
$$= e^{3A-\phi}\nabla_m(e^{\phi}K_+^{12}) - n_e\partial_m AK_+'^{12} + i\frac{e^{\phi}}{4}(n_e-1)[-F_0K'_{+m1} + (*F_4)_{mp}K_+'^{2p} + F_{pq}K_+'^{2pq}{}_m]$$
$$= (\widetilde{\mathcal{D}K'_+})_{m1} - \partial_m(4A-\phi)K_+'^{12}$$

where we have taken $n_e = 1$.

For the $(\widetilde{\mathcal{D}K'})_{m2}$ equation, we first note that supersymmetry requires their RR pieces to vanish by itself, namely

$$0 = \text{Tr}\,\left(\Delta_m K_3'\right) = e^{\phi}\left((*F_6)(K_+')_{m2} + F_{mp}K_+'^{1p} + (*F_4)^{pq}(K_+')_{1pqm}\right) = \mathcal{F}_{RR}\big|_{m2}\,,$$

while in the $m1$ equation, the RR piece is proportional to a derivative of the warp factor, i.e.

$$\begin{aligned}0 &= \text{Tr}\,\left(e^{3A}\Delta^m K_0\right) = 4ie^{3A}\nabla_m K_+{}^1{}_2 + e^{\phi}\left(F_0 K_+'{}^{m1} - (*F_4)^{mp}K_+'{}^2{}_p - F_{pq}K_+'^{2pqm}\right)\\ &= 4i\partial_m A K_+'{}^1{}_2 + \mathcal{F}_{RR}\big|^{m1}\,.\end{aligned}$$

Then we use

$$\begin{aligned}0 &= \frac{1}{4}\text{Tr}\,\left(i\Gamma^{mp12}(e^{3A}\Delta_p K_3) + [\Gamma^m, n_e\Delta_e + n_d\Delta_d]K_3'\right) + \mathcal{F}_{RR}\big|^{m1} + 4i\partial_m A K_+'{}^1{}_2\\ &= +2e^{3A}\nabla_p K_+^{mp12} + 2\partial_p(n_e A + n_d\phi)K_+'^{mp12} - \frac{1}{4}(n_d+2)H^m{}_{pq}K_+'^{mp12}\\ &\quad + i\frac{e^{\phi}}{4}\Big[(*F_6)(5-n_e+n_d)K_+'^{m1} + F_{mp}(3+3n_d+n_e)K_+'^{2p} + (*F_4)_{pq}(-1+n_e+n_d)K_+'^{2pqm}\Big]\\ &\quad + \mathcal{F}_{RR}\big|^{m1} + 4i\partial_m K_+'{}^1{}_2\\ &= (\mathcal{D}K_+')^{m1} - 2\partial_p\phi K_+'^{mp12} + 4i\partial_m A K_+'{}^1{}_2\end{aligned}$$

where in the last equality we have chosen $n_e = 3, n_d = -2$. For the $m2$ component, we use

$$\begin{aligned}0 &= \frac{1}{4}\text{Tr}\,\left(\Gamma^p{}_m e^{3A}\Delta_p K_3 - i[\Gamma_{m12}, n_e\Delta_e + n_d\Delta_d]K_3'\right) + \mathcal{F}_{RR}\big|_{m2}\\ &= -e^{3A}\nabla_p K_+{}^p{}_m - \partial_p(n_e A + n_d\phi)K_+'^p{}_m - \frac{1}{2}H_{mpq}K_+'^{pq12}(2+n_d)\\ &\quad + i\frac{e^{\phi}}{4}\Big[F_0(5+n_e+5n_d)K_{+m2}' + F_{pq}(1+n_e+3n_d)K_+'^{1pq}{}_m + (*F_{(4)})_{mp}(-3+n_e+n_d)K_+'^{1p}\Big]\\ &\quad + \mathcal{F}_{RR}\big|_{m2}\\ &= (\widetilde{\mathcal{D}K_+'})_{m2} - \partial_p(2A-\phi)K_+'^p{}_m + H_{mpq}K_+'^{pq12}\end{aligned}$$

where here we have inserted $n_e = 5, n_d = -2$.

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