Published for SISSA by Springer

Received: April 20, 2023
Accepted: June 28, 2023
Published: July 28, 2023

Localization on $AdS_3 \times S^2$. Part I. The 4d/5d connection in off-shell Euclidean supergravity

Axel Ciceri,[a] **Imtak Jeon**[b,c] **and Sameer Murthy**[a]

[a] *Department of Mathematics, King's College London, The Strand, London WC2R 2LS, U.K.*
[b] *Asia Pacific Center for Theoretical Physics, Postech, Pohang 37673, Korea*
[c] *Department of Physics, Postech, Pohang 37673, Korea*

E-mail: axel.ciceri@kcl.ac.uk, imtakjeon@gmail.com, sameer.murthy@kcl.ac.uk

Abstract: We begin to develop the formalism of localization for the functional integral of supergravity on $AdS_3 \times S^2$. We show how the condition of supersymmetry in the Euclidean $\mathbb{H}^3/\mathbb{Z} \times S^2$ geometry naturally leads to a twist of the S^2 around the time direction of AdS_3. The twist gives us a five-dimensional Euclidean supergravity background dual to the elliptic genus of (0, 4) $SCFT_2$ at the semiclassical level. On this background we set up the off-shell BPS equations for one of the Killing spinors, such that the functional integral of five-dimensional Euclidean supergravity on $\mathbb{H}^3/\mathbb{Z} \times S^2$ localizes to its space of solutions. We obtain a class of solutions to these equations by lifting known off-shell BPS solutions of 4-dimensional supergravity on $AdS_2 \times S^2$. In order to do this consistently, we construct and use a Euclidean version of the off-shell 4d/5d lift of arXiv:1112.5371, which could be of independent interest.

Keywords: AdS-CFT Correspondence, Black Holes in String Theory, Supersymmetric Gauge Theory

ArXiv ePrint: 2301.08084

https://doi.org/10.1007/JHEP07(2023)218

Contents

1 Introduction

In this paper we take the first steps to develop the formalism of localization in supergravity on asymptotic $\mathrm{AdS}_3 \times \mathrm{S}^2$. Our main physical motivation is to study an exact sector of $\mathrm{AdS}_3/\mathrm{CFT}_2$ holography. By exact, we mean the inclusion of all quantum effects in supergravity or, equivalently, all finite-charge corrections to the results derived in the limit of infinite central charge of CFT_2. Our eventual goal is to calculate observables protected by supersymmetry in the bulk gravitational theory using localization applied to functional integrals in supergravity. The same approach has yielded rich results in theories on AdS_2 (times S^2 or S^3) culminating in the calculation of the exact quantum entropy of black holes in four dimensions [1–4] and five-dimensions [5]. The idea is to extend such calculations to higher dimensional AdS spaces.

JHEP07(2023)218

In this paper we focus on the $\mathrm{AdS}_3/\mathrm{CFT}_2$ context and consider the bulk AdS_3 calculation of the elliptic genus of the dual SCFT_2. Many of the intermediate calculations are motivated by the embedding of this problem in string theory, wherein two-dimensional SCFTs are realized as the low-energy theory on the world-volume of effective strings. Some canonical examples are the D1/D5 system in IIB string theory wrapped on $T^4/K3$ [6, 7], $M5$-brane bound states in M-theory/CY_3 [8], and D3 branes wrapping curves on the base of elliptically fibered CY_3 in F-theory [9]. All these SCFTs have at least $(0,4)$ supersymmetry in two dimensions with a corresponding SU(2) R-symmetry. Focussing on the MSW theory [8], the gravitational dual in the generic theory has the form $\mathrm{AdS}_3 \times \mathrm{S}^2 \times \mathrm{CY}_3$.[1,2]

The low energy theory is summarized by five-dimensional supergravity with asymptotic $\mathrm{AdS}_3 \times \mathrm{S}^2$ boundary conditions, which is the theory that we study in this paper in the off-shell conformal formalism. The classical Lorentzian $\mathrm{AdS}_3 \times \mathrm{S}^2$ theory in this formalism has been studied in a series of nice papers [11–13]. Our eventual goal is to calculate functional integrals for quantum observables using localization in the corresponding Euclidean supergravity theory defined on global $\mathrm{AdS}_3 \times \mathrm{S}^2$ with a periodic Euclidean time coordinate, i.e., the manifold $\mathbb{H}^3/\mathbb{Z} \times \mathrm{S}^2$. The three-dimensional part of the manifold has the topology of a solid torus, and we often refer to it as such.

The localization problem in supergravity is substantially more difficult compared to its counterpart in quantum field theories. To begin with, one defines a rigid supercharge in the gravitational theory, using the background field method in theories with soft gauge algebras as applied to supergravity [14, 15]. At a practical level the problem reduces to finding all bosonic gravitational configurations that admit a Killing spinor whose asymptotic limit is one of the supercharges of $\mathrm{AdS}_3 \times \mathrm{S}^2$. Secondly, one finds all matter configurations invariant under this supercharge. Thirdly, one evaluates the supergravity action at a generic point in this manifold. Finally one calculates the one-loop determinant of the non-BPS fluctuations around the localization manifold.

In this paper we address the first two questions whose solutions comprise the so-called localization locus. We work in the context of 5d off-shell $\mathcal{N}=1$ supergravity coupled to an arbitrary number of vector multiplets. Our idea is to use the 4d/5d lift [16], which relates solutions of off-shell 4d supergravity to those of off-shell 5d supergravity compactified on a circle.[3] The localization manifold in 4d $\mathcal{N}=2$ supergravity on asymptotically $\mathrm{AdS}_2\times\ \mathrm{S}^2$ has been completely determined, and we can lift those solutions to $\mathrm{AdS}_3 \times \mathrm{S}^2$. Although this is not guaranteed to produce all supersymmetric solutions, it should give all solutions that are independent of the circle of compactification. Similar ideas have been used successfully to make progress in localization on $\mathrm{AdS}_2\times\ \mathrm{S}^3$ theories in [5, 18, 19].

As it turns out, implementing this idea is not quite straightforward. Firstly, the 4d/5d map in [16] is given for Lorentzian backgrounds while we need it for Euclidean backgrounds.

[1]One has $\mathrm{AdS}_3 \times \mathrm{S}^3 \times T^4/K3$ in the D1/D5 system and $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{CY}_3$ in the F-theory case [10]. In either of these cases, we consider the $\mathrm{SU}(2) \times \widetilde{\mathrm{SU}(2)}$ action on S^3, and we think of the S^2 with $\widetilde{\mathrm{SU}(2)}$ action as being embedded in the S^3.

[2]One could also consider black strings in AdS_5 which needs a different treatment using gauged supergravity which we will not consider here.

[3]This is different from the 4d/5d lift of [17] which involves a lift on a Taub-NUT space.

JHEP07(2023)218

To this end, we first work out a consistent set of supersymmetry transformations in the five-dimensional Euclidean supergravity theory. In four dimensions we use the Euclidean supergravity discussed in [15, 20–23]. We then map off-shell BPS solutions of the 4d theory to off-shell BPS solutions of the 5d theory. Here there is an additional problem, namely that the 4d Euclidean theory carries a redundancy of allowed reality conditions which has no counterpart in the 5d Euclidean theory. We show that this redundancy can be absorbed in a parameter whose role is to implement the symmetry breaking $\mathrm{SO}(1,1)_R \to \mathbb{I}$.[4] The second problem has to do with the global identifications of the background that we are interested in, i.e. $\mathbb{H}^3/\mathbb{Z} \times \mathrm{S}^2$, which is not a Kaluza-Klein lift of Euclidean $\mathrm{AdS}_2 \times \mathrm{S}^2$. The Kaluza-Klein condition was used in [16] for the off-shell 4d/5d lift and, indeed, general off-shell configurations do not consistently lift from Euclidean $\mathrm{AdS}_2 \times \mathrm{S}^2$ to $\mathbb{H}^3/\mathbb{Z} \times \mathrm{S}^2$. Nevertheless, the class of off-shell solutions relevant for the 4d black hole problem can be lifted to the supersymmetric $\mathbb{H}_3/\mathbb{Z}$, due to their enhanced rotational symmetry.

In this manner we obtain an adaptation of the 4d/5d lift relevant for the Euclidean $\mathrm{AdS}_3/\mathrm{CFT}_2$ problem, whose details we work out in section 4. The final part of the paper, in section 5, is an application of this lift to find a class of off-shell solutions in the AdS_3 theory which contribute to the elliptic genus problem in the boundary SCFT_2. In a sequel to this paper we study the subsequent steps in the localization problem, in particular the 5d supergravity action and its associated boundary terms.

There is another subtlety that appears in the Euclidean functional integral realization of the supersymmetric observables in the $\mathrm{AdS}_3/\mathrm{CFT}_2$ context even before beginning the 4d/5d lift. Consider a 2d $(0,4)$ superconformal field theory SCFT_2 on the boundary torus with complex structure τ. The elliptic genus can be defined as a trace over the Hilbert space on S^1 in the Ramond sector with an insertion of $(-1)^F$, which can be calculated by reducing to the free theory or by using index theorems [24–26]. In the functional integral formalism, one correspondingly chooses periodic boundary conditions for the fermions around both cycles of the torus. This leads to a constant Killing spinor on the torus, which is used to localize the SCFT_2 functional integral [27]. Now consider the calculation of the same functional integral in the bulk theory. The bosonic vacuum configuration is that of thermal AdS_3. One of the circles (the space circle in the thermal AdS_3) is contractible and therefore the fermions should have half-integer momentum around that circle at infinity. Demanding that a spinor obeys the Killing spinor equation forces its momenta around the two cycles to be equal, so that it has fixed non-zero momentum also along the non-contractible (time) direction. On the other hand, the non-contractible direction has periodicity $\mathrm{Im}(\tau)$ which is arbitrary, and therefore the above spinor with non-zero momentum is not well-defined on the torus. This problem can be resolved by turning on a twist of the S^2 around the non-contractible circle in AdS_3, which allows for spinors which are constant in time and therefore well-defined. As we discuss in section 3, this allows us to set up a supersymmetric background of the form $\mathbb{H}^3/\mathbb{Z}\times \mathrm{S}^2$.[5] The bulk calculation is the NS-sector calculation of the elliptic genus, which is equivalent to the boundary Ramond

[4]This parameter is the Euclidean analog of the parameter that enforces $\mathrm{SO}(2)_R \to \mathbb{I}$ in [16].

[5]A related supersymmetric set-up has been discussed in the literature in the context of supersymmetric black holes in AdS space [28] and, in particular, for BTZ black holes in [29].

JHEP07(2023)218

sector trace by a spectral flow in the $(0, 4)$ algebra. We show how the twist reduces the final asymptotic algebra to be a sub-algebra of the Brown-Henneaux-Coussaert [30, 31] $(0, 4)$ algebra on AdS_3.

We note that our study of localization of supergravity on $\mathrm{AdS}_3 \times \mathrm{S}^2$ has similarities with localization of 3d supersymmetric gauge theories on AdS_3, which has been studied in [32]. Indeed the analysis of asymptotic boundary conditions on the gauge fields is very similar in both cases. The main difference and new challenge (apart from the two extra dimensions) is that the metric also fluctuates in our analysis, as can be seen in our localization solutions. We also note that the idea of exact holography has also been explored in the literature in contexts different to the one that we study here. This includes Koszul duality [33], topological string theory [34], and classical (large-N) gravitational theories [35]. It would be interesting to explore possible connections with these approaches.

The plan of this paper is as follows. In section 2 we discuss five-dimensional supergravity coupled to matter multiplets. We then describe the classical $\mathrm{AdS}_3 \times \mathrm{S}^2$ solution in Lorentzian signature, and the Killing spinors and superalgebra on this background. In section 3 we present the Euclidean $\mathbb{H}^3/\mathbb{Z} \times \mathrm{S}^2$ solution, the Killing spinors and superalgebra on this background, and the relation of the path integral to the trace definition of the elliptic genus. In section 4 we present the off-shell 4d/5d map modified to the Euclidean signature and present the lift of $\mathrm{AdS}_2 \times \mathrm{S}^2$ to $\mathbb{H}^3/\mathbb{Z} \times \mathrm{S}^2$. In section 5 we apply our formalism to lift localization solutions from $\mathrm{AdS}_2 \times \mathrm{S}^2$ localization solutions on $\mathbb{H}^3/\mathbb{Z} \times \mathrm{S}^2$. In various appendices, we present the spinor and Clifford algebra conventions, the Killing spinors, Killing vectors, and Lie brackets of global $\mathrm{AdS}_3 \times \mathrm{S}^2$, and the four- and five-dimensional Euclidean supersymmetry transformations in the respective conformal supergravities.

2 Background and set-up of the problem

In this section we discuss five-dimensional supergravity coupled to matter multiplets. We show that the Euclidean theory is obtained by redefinitions of fields of the Lorentzian theory that follow simply from the Wick rotation. We review the elements of the supergravity theory including the supermultiplets and the supersymmetry transformations. We then describe the classical $\mathrm{AdS}_3 \times \mathrm{S}^2$ solution in Lorentzian signature, and present the Killing spinors on this background and the consequent superalgebra.

2.1 Off-shell 5d supergravity

Off-shell supergravity in the superconformal formalism in Lorentzian signature in various dimensions has been known for many decades (see the book [36]). Euclidean supergravity, on the other hand, is a less-studied subject and few references exist (e.g. [20–23]). In these references the method of time-like reduction from a higher-dimensional Lorentzian theory is used to systematically construct the Euclidean-signature theory. In this section we discuss the formalism of 5d conformal supergravity with $\mathcal{N} = 1$ (minimal) supersymmetry, i.e., 8 supercharges. The Lorentzian theory was constructed in [37–39], and in [40–42], and it is reviewed in the more recent [16, 43] whose conventions we follow. One potential systematic approach to construct the Euclidean theory would be to perform a timelike reduction on a 6d theory. However, we use a less formal approach here: we start from a Wick rotation and make an appropriate set of transformations on all the fields of the Lorentzian theory so that we obtain a consistent 5d Euclidean theory.[6]

The starting point is the usual Wick rotation $t = -\mathrm{i}t_{\mathrm{E}}$ relating the Lorentzian and Euclidean time coordinates. This is followed by the appropriate transformations of all tensors for this coordinate change. Accordingly, the time directional gamma matrices are related by $\gamma_t = \mathrm{i}\gamma_{t_{\mathrm{E}}}$. Changing the signature of spacetime by this Wick rotation, in general, demands changing the nature of irreducible spinors. For instance, in four dimensions, while the Majorana representation of irreducible spinors is allowed in the Lorentzian theory, it is not the case in the Euclidean theory. Therefore, we need an appropriate field redefinition of spinors. This can be achieved in a simple manner by going to the symplectic-Majorana basis which exists in both the Lorentzian and Euclidean theory. In this basis, the charge conjugation matrix is the same in both the theories. Therefore the Euclidean action follows from the Lorentzian action in this basis by simply implementing the Wick rotation (See appendix B.4 of [15] for details of this map in four dimensions). Under the Wick rotation, the Lagrangian density is unchanged, i.e., $\mathcal{L}_{\text{Lor.}} = \mathcal{L}_{\text{Eucl.}}$, and so the action of the Lorentizan theory and Euclidean theory are related by $\mathrm{i}S_{\text{Lor.}} = S_{\text{Eucl.}}$. Our conventions are such that the Euclidean action is to be negative definite when we impose well defined path integral.

The reality property of the fermionic fields in the Lorentzian and Euclidean theories are different. In the Lorentzian theory, an $\mathrm{SU}(2)_R$ spinor doublet ψ^i with $i = 1, 2$ follows a symplectic-Majorana condition, as reviewed in (A.5),

$$(\psi^i)^\dagger \gamma_{\hat{t}} = \varepsilon_{ij} (\psi^j)^T C \,, \tag{2.1}$$

where C is the unique choice of the charge conjugation matrix in five dimensions (this is more generally true in odd dimensions). Compatibility with supersymmetry leads to the standard reality property on fluctuating bosons, in which the gauge fields and the metric are real. In the Euclidean theory the allowed reality condition for the spinors

$$(\psi^i)^\dagger = \varepsilon_{ij} (\psi^j)^T C \tag{2.2}$$

is not compatible with supersymmetry if we impose the usual reality conditions for fluctuating bosons. Therefore, we treat ψ^1 and ψ^2 as two independent Dirac spinors, formally doubling the fermionic degrees of freedom and then choosing a middle-dimensional contour in the functional integral, following the standard treatment of fermions in the Euclidean theory. This allows us to impose the standard reality conditions on bosonic fluctuations, so that the Euclidean action is negative-definite and the functional integral is well-defined at the perturbative level.[7]

[6] Indeed, such an approach also works successfully in four dimensions and the result agrees with the timelike reduction [15].

[7] Note that we still allow reality conditions on the background values of the bosonic fields which are different from the usual Lorentzian ones: these can be imposed on on-shell fields (e.g. A_0 or g_{0i} are typically imaginary as dictated by the Wick rotation), or on off-shell BPS fluctuations (e.g. the localization manifold on AdS_2 involves scalar fields with $\bar{X} \neq X^*$ [1, 44, 45]).

Weyl	$E_M{}^A, \Psi^i_M, b_M, V_{M,i}{}^j, T_{MN}, D, \chi^i$
Vector	$\sigma^I, W^I_M, \Omega^{Ii}, Y^I_{ij}$
Hyper	$A_i{}^\alpha, \zeta^\alpha$
SUSY parameters	ϵ^i, η^i

Table 1. Independent fields of the supersymmetric multiplets and Q, S-supersymmetry parameters in five-dimensional $\mathcal{N} = 1$ conformal supergravity.

Supermultiplets: for the $\mathcal{N} = 1$ 5d conformal supergravity theory, we follow the conventions of [43]. We consider the Weyl multiplet, which couples to N_v number of vector multiplets as well as a single hyper multiplet. Independent fields in each multiplet is summarized in table 1.

The Weyl multiplet consists of the gauge fields corresponding to all the symmetry generators of $\mathcal{N} = 1$ superconformal algebra $\{P^A, M^{AB}, D, K^A, Q^i, S^i, V_j{}^i\}$. Among all the gauge fields, the gauge fields associated with $\{M^{AB}, K^A, S^i\}$ are composite, i.e. they are expressed in terms of other gauge fields. The independent gauge fields in Weyl multiplet are the vielbein $E_M{}^A$, dilatation gauge field b_M, gaugino ψ^i_M, and the $\mathrm{SU}(2)_R$ gauge field $V_{Mj}{}^i$. For the Weyl multiplet to be off-shell supermultiplet, it includes auxiliary two-form tensor T_{AB}, auxiliary fermion χ^i, and auxiliary scalar D. Hence the independent fields of the Weyl multiplet are summarized as

$$\text{Weyl:} \quad \{E_M{}^A\,,\, \Psi^i_M\,,\, b_M\,,\, V_{M,i}{}^j\,;\, T_{MN}\,,\, \chi^i\,,\, D\}\,. \tag{2.3}$$

Here, the indices $\{A, B, \cdots\}$, $\{M, N, \cdots\}$, $\{i, j, \cdots\}$ are five-dimensional flat tangent space, curved spacetime, and SU(2) fundamental indices, respectively, which are summarized in appendix A. In the Weyl multiplet fields, we use the special conformal symmetry (that acts only on b_M) to gauge-fix $b_M = 0$, so that from here on this field will not appear. We consider N_v vector multiplets labeled by I, each of which consists of

$$\text{Vector:} \quad \{\sigma^I\,,\, W^I_M\,,\, \Omega^{I\,i}\,,\, Y^I_{ij}\}\,, \qquad I = 1\,, 2\,, \cdots\,, N_\mathrm{v}\,. \tag{2.4}$$

They corresponds to a scalar, a U(1) gauge field, gaugini, and an auxiliary symmetric SU(2) triplet. The i, j indices are raised and lowered using the SU(2) symplectic metric ε, where, explicitely, $\varepsilon_{12} = \varepsilon^{12} = 1$. In particular, we have $Y_{ij} = \varepsilon_{ik}\,\varepsilon_{j\ell}\,Y^{k\ell}$. We finally consider a single hypermultiplet, which consists of

$$\text{Hyper:} \quad \{A_i{}^\alpha\,,\, \zeta^\alpha\}, \tag{2.5}$$

corresponding to the hyper scalar, and the hyper fermion, where $\alpha = 1, 2$. There is no known off-shell Lorentz-covariant hypermultiplet with finite number of fields. In the backgrounds that we consider below, the hypermultiplet turns out to take its on-shell value, and we can therefore integrate it out at the semi-classical level. For the full localization calculation, one could construct an off-shell hypermultiplet for one complex supercharge (see e.g. [46, 47]). One of the N_v vector multiplets and the single hypermultiplet act as

JHEP07(2023)218

the two compensators to be added to the Weyl multiplet to form a 5d $\mathcal{N} = 1$ Poincaré supergravity multiplet.

Supersymmetry algebra: the supersymmetry transformations of the various spinor fields under the Q and S supersymmetry transformations are given in (B.1). Two Q-supersymmetry transformations, parametrized by spinors ϵ_1 and ϵ_2 respectively, close onto the bosonic symmetries of the theory as

$$[\delta_Q(\epsilon_1)\,, \delta_Q(\epsilon_2)] = \delta_{\text{gct}}(\xi^\mu) + \delta_M(\lambda) + \delta_S(\eta) + \delta_K(\Lambda_K) \tag{2.6}$$

where δ_{gct} are the general coordinate transformations, δ_M is a local Lorentz transformation, δ_S is a conformal supersymmetry transformation, and δ_K is special conformal transformation. The relevant parameters to this paper are

$$\begin{aligned} \xi^\mu &= 2\bar{\epsilon}_{2i}\gamma^\mu\epsilon_1^i\,, \\ \lambda^{AB} &= -\xi^\mu\omega_\mu{}^{AB} + \frac{\mathrm{i}}{2}T^{CD}\bar{\epsilon}_{2i}(6\gamma^{[A}\gamma_{CD}\gamma^{B]} - \gamma^{AB}\gamma_{CD} - \gamma_{CD}\gamma^{AB})\epsilon_1^i\,. \end{aligned} \tag{2.7}$$

Action: the bosonic Lagrangian at two-derivative level is

$$L_{\text{bulk}} = E\,\left(\mathcal{L}_{\text{V}} + \mathcal{L}_{\text{VW}} + \mathcal{L}_{\text{H}} + \mathcal{L}_{\text{HW}} + \mathcal{L}_{\text{CS}}\right)\,, \tag{2.8}$$

where $E \equiv \det(E_M{}^A)$, $\mathcal{L}_{\text{V}}$ contains purely vector multiplet terms, $\mathcal{L}_{\text{VW}}$ contains mixing between vector and Weyl, $\mathcal{L}_{\text{H}}$ is the kinetic hyper scalar piece, $\mathcal{L}_{\text{HW}}$ contains coupling of hyper to Weyl, and $\mathcal{L}_{\text{CS}}$ is the five-dimensional Chern-Simons action:

$$\begin{aligned} \mathcal{L}_{\text{V}} &= \frac{1}{2}\,c_{IJK}\,\sigma^I\,\left(\frac{1}{2}\,D^M\,\sigma^J\,D_M\,\sigma^K + \frac{1}{4}\,F^J_{MN}\,F^{MN\,K} - 3\sigma^J\,F^K_{MN}\,T^{MN} - Y^J_{ij}\,Y^{K\,ij}\right)\,, \\ \mathcal{L}_{\text{VW}} &= -C(\sigma)\left(\frac{1}{8}\,R - 4D - \frac{39}{2}\,T^2\right)\,, \\ \mathcal{L}_{\text{H}} &= -\frac{1}{2}\Omega_{\alpha\beta}\varepsilon^{ij}D_M\,A_i{}^\alpha\,D^M\,A_j{}^\beta\,, \\ \mathcal{L}_{\text{HW}} &= \chi\,\left(\frac{3}{16}\,R + 2D + \frac{3}{4}\,T^2\right)\,, \\ \mathcal{L}_{\text{CS}} &= -\frac{i}{48\,E}\varepsilon^{MNOPQ}c_{IJK}W^I_M F^J_{NO}F^K_{PQ}\,. \end{aligned} \tag{2.9}$$

In the Chern-Simons Lagrangian $\mathcal{L}_{\text{CS}}$, the object ε^{MNOPQ} is a fully antisymmetric tensor density taking values in $\{-1,\,0,\,1\}$. The scalar norms appearing in $\mathcal{L}_{\text{VW}}$ and $\mathcal{L}_{\text{HW}}$ are:

$$C(\sigma) \coloneqq \frac{1}{6}c_{IJK}\,\sigma^I\sigma^J\sigma^K\,, \tag{2.10}$$

$$\chi \coloneqq \frac{1}{2}\,\Omega_{\alpha\beta}\,\varepsilon^{ij}A_i{}^\alpha A_j{}^\beta\,. \tag{2.11}$$

The action of the theory is

$$S_{\text{bulk}} = \frac{1}{8\pi^2}\int_{\mathcal{M}} d^5x\,L_{\text{bulk}}\,, \tag{2.12}$$

for an appropriate coordinate chart on the 5d manifold $\mathcal{M}$.

JHEP07(2023)218

2.2 Global Lorentzian $\mathrm{AdS}_3 \times \mathrm{S}^2$

We consider the fully supersymmetric $\mathrm{AdS}_3 \times \mathrm{S}^2$ solution of the Lorentzian supergravity described above, corresponding to the near-horizon geometry of the half-BPS magnetic black string [48]. The metric in Lorentzian signature is

$$ds^2 = 4\ell^2 \left(-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\psi^2\right) + \ell^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{2.13}$$

where the coordinates of the AdS_3 have the ranges $\rho \in [0, \infty)$, $\psi \in [0, 2\pi]$, $t \in (-\infty, \infty)$ and the angles on the S^2 have ranges $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$. The radii of the AdS_3 and the S^2 are (2ℓ) and ℓ respectively, where this relative factor of 2 is determined by supersymmetry. Note that in the off-shell theory, ℓ is free and parametrizes the dilatations of the theory, while in the on-shell theory (where dilatations are broken) it is determined by the magnetic charges of the solution via the D-gauge condition. These magnetic charges p^I enter the solution through the vector multiplet. The non-trivial fields of the vector multiplet are:

$$\sigma^I = -\frac{p^I}{\ell}\,, \qquad F^I_{\theta\phi} = p^I \sin\theta\,. \tag{2.14}$$

Note that the solution does not have electric flux, which allows us to turn on flat gauge connections on the AdS_3. This aspect will become relevant in the following. The five-dimensional Newton's constant is $G_5 = 6\pi\ell^3/p^3$ and the three-dimensional Newton's constant is obtained by setting $G_5 = \mathrm{Area}_{S^2} \times G_3 = 4\pi\ell^2 G_3$.

In the off-shell formalism of section 2.1, one requires additional auxiliary fields. In the Weyl multiplet, the non-trivial fields are:

$$T_{\theta\phi} = -\frac{\ell}{4}\sin\theta\,. \tag{2.15}$$

In the compensating hypermultiplet, the BPS equation is solved by

$$A_i{}^\alpha = c_i{}^\alpha\,, \tag{2.16}$$

where the constants $c_i{}^\alpha$ are determined in terms of the charge p^I by the field equation for the auxiliary field D to be

$$\Omega_{\alpha\beta}\,\varepsilon^{ij} c_i{}^\alpha c_j{}^\beta = \frac{2}{3\ell^3}\, c_{IJK}\, p^I p^J p^K\,. \tag{2.17}$$

In this paper, we fix an explicit choice for the $c_i{}^\alpha$ as

$$c_1{}^2 = c_2{}^1 = 0\,, \qquad c_1{}^1 = c_2{}^2 = \sqrt{\frac{p^3}{3\ell^3}}\,. \tag{2.18}$$

2.3 Supersymmetry algebra in Lorentzian $\mathrm{AdS}_3 \times \mathrm{S}^2$

Killing spinors. The Q- and S- supersymmetry parameters, ϵ^i and η^i respectively, that are preserved by the bosonic fields of the global $\mathrm{AdS}_3 \times S^2$ background are determined by setting the variation of gravitino and the variation the auxiliary fermion in (B.1) to zero. These two equations are, respectively,

$$0 = 2\,\mathcal{D}_M\epsilon^i + \frac{\mathrm{i}}{2}T_{AB}(3\gamma^{AB}\gamma_M - \gamma_M\gamma^{AB})\epsilon^i - \mathrm{i}\gamma_M\eta^i\,, \tag{2.19}$$

$$\begin{aligned}0 = &\frac{1}{2}\epsilon^i D + \frac{1}{64}R_{MNj}{}^i(V)\gamma^{MN}\epsilon^j + \frac{3}{64}\mathrm{i}(3\gamma^{AB}\not{D} + \not{D}\gamma^{AB})T_{AB}\epsilon^i \\ &-\frac{3}{16}T_{AB}T_{CD}\gamma^{ABCD}\epsilon^i + \frac{3}{16}T_{AB}\gamma^{AB}\eta^i\,.\end{aligned} \tag{2.20}$$

On our bosonic background, the second equation (2.20) immediately determines the S-supersymmetry spinor as $\eta^i = 0$. The first equation is referred to as the Killing spinor equation. We analyze its solutions in appendix C and summarize the results below.

The complex basis of the Killing spinor on $\mathrm{AdS}_3\times\mathrm{S}^2$ is given by the following four Killing spinors,

$$\begin{aligned}\epsilon_+{}^+ &= \sqrt{\frac{\ell}{2}}\,\epsilon^+_{\mathrm{AdS}_3}\otimes\epsilon^+_{\mathrm{S}^2}\,, &\qquad \epsilon_+{}^- &= \sqrt{\frac{\ell}{2}}\,\epsilon^+_{\mathrm{AdS}_3}\otimes\epsilon^-_{\mathrm{S}^2}\,,\\ \epsilon_-{}^+ &= \sqrt{\frac{\ell}{2}}\,\epsilon^-_{\mathrm{AdS}_3}\otimes\epsilon^+_{\mathrm{S}^2}\,, &\qquad \epsilon_-{}^- &= \sqrt{\frac{\ell}{2}}\,\epsilon^-_{\mathrm{AdS}_3}\otimes\epsilon^-_{\mathrm{S}^2}\,,\end{aligned} \tag{2.21}$$

with

$$\begin{aligned}\epsilon^+_{\mathrm{AdS}_3} &= \mathrm{e}^{\frac{\mathrm{i}}{2}(t+\psi)}\begin{pmatrix}\cosh\frac{\varrho}{2}\\ -\sinh\frac{\varrho}{2}\end{pmatrix}, &\qquad \epsilon^-_{\mathrm{AdS}_3} &= \mathrm{e}^{-\frac{\mathrm{i}}{2}(t+\psi)}\begin{pmatrix}-\sinh\frac{\varrho}{2}\\ \cosh\frac{\varrho}{2}\end{pmatrix},\\ \epsilon^+_{\mathrm{S}^2} &= \mathrm{e}^{\frac{\mathrm{i}}{2}\phi}\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix}, &\qquad \epsilon^-_{\mathrm{S}^2} &= \mathrm{e}^{-\frac{\mathrm{i}}{2}\phi}\begin{pmatrix}-\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{pmatrix}.\end{aligned} \tag{2.22}$$

These 4 Killing spinors organize themselves into the 8 pairs of symplectic Majorana spinors

$$\begin{aligned}&\epsilon^i_{(1)} = (-\mathrm{i}\epsilon_+{}^+, \epsilon_-{}^-)\,, \quad \epsilon^i_{(2)} = (\epsilon_+{}^+, -\mathrm{i}\epsilon_-{}^-)\,, \quad \epsilon^i_{(3)} = -(\epsilon_-{}^-, \mathrm{i}\epsilon_+{}^+)\,, \quad \epsilon^i_{(4)} = -(\mathrm{i}\epsilon_-{}^-, \epsilon_+{}^+)\,,\\ &\tilde\epsilon^i_{(1)} = (\epsilon_+{}^-, \mathrm{i}\epsilon_-{}^+)\,, \quad \tilde\epsilon^i_{(2)} = (\mathrm{i}\epsilon_+{}^-, \epsilon_-{}^+)\,, \quad \tilde\epsilon^i_{(3)} = (-\mathrm{i}\epsilon_-{}^+, \epsilon_+{}^-)\,, \quad \tilde\epsilon^i_{(4)} = (\epsilon_-{}^+, -\mathrm{i}\epsilon_+{}^-)\,,\end{aligned} \tag{2.23}$$

to form the 8 real basis of the Killing spinor on $\mathrm{AdS}_3\times\mathrm{S}^2$. Each pair satisfies the following symplectic Majorana condition (A.5) appropriate to the 5d Lorentzian theory, i.e. $(\epsilon^i)^\dagger\gamma_{\hat t} = \varepsilon_{ij}(\epsilon^j)^T\mathcal{C}$ in the conventions of appendix C.

Superconformal algebra: let us denote

$$\mathcal{Q}_a = \delta(\epsilon^i_{(a)})\,, \qquad \widetilde{\mathcal{Q}}_a = \delta(\tilde\epsilon^i_{(a)})\,, \qquad a = 1\,,2\,,3\,,4\,, \tag{2.24}$$

with the Grassmann even Killing spinors ϵ^i. Then,

$$\{\mathcal{Q}_a\,,\mathcal{Q}_b\} = -2\mathrm{i}\delta_{ab}(L_0 - J^3)\,, \qquad \{\widetilde{\mathcal{Q}}_a\,,\widetilde{\mathcal{Q}}_b\} = -2\mathrm{i}\delta_{ab}(L_0 + J^3)\,, \tag{2.25}$$

$$\{\mathcal{Q}_a\,,\widetilde{\mathcal{Q}}_b\} = \begin{pmatrix}-2\mathrm{i}J^2 & 2\mathrm{i}J^1 & -(L_+ - L_-) & \mathrm{i}(L_+ + L_-)\\ -2\mathrm{i}J^1 & -2\mathrm{i}J^2 & -\mathrm{i}(L_+ + L_-) & -(L_+ - L_-)\\ L_+ - L_- & \mathrm{i}(L_+ + L_-) & -2\mathrm{i}J^2 & -2\mathrm{i}J^1\\ -\mathrm{i}(L_+ + L_-) & L_+ - L_- & 2\mathrm{i}J^1 & -2\mathrm{i}J^2\end{pmatrix}, \tag{2.26}$$

where SL$(2,R)$ generators $L_0, L_\pm$ and SO(3) generators $J^{\boldsymbol{a}}$ satisfy

$$[L_+,\, L_-] = -2L_0\,, \qquad [L_0\,, L_\pm] = \pm L_\pm\,, \qquad [J^{\boldsymbol{a}}\,, J^{\boldsymbol{b}}] = \mathrm{i}\epsilon^{\boldsymbol{abc}} J^{\boldsymbol{c}}\,. \tag{2.27}$$

Their explicit representation on the $\mathrm{AdS}_3 \times \mathrm{S}^2$ is given in appendix D.

Let us define the supercharges $G^{i\alpha}_\gamma$

$$\begin{aligned}
G^{++}_+ &\equiv \frac{\mathrm{i}\mathcal{Q}_1 + \mathcal{Q}_2}{2}\,, \quad G^{+-}_- \equiv \frac{-\mathcal{Q}_3 + \mathrm{i}\mathcal{Q}_4}{2}\,, \quad G^{+-}_+ \equiv \frac{\widetilde{\mathcal{Q}}_1 - \mathrm{i}\widetilde{\mathcal{Q}}_2}{2}\,, \quad G^{++}_- \equiv \frac{\mathrm{i}\widetilde{\mathcal{Q}}_3 + \widetilde{\mathcal{Q}}_4}{2}\,,\\
G^{--}_- &\equiv \frac{\mathcal{Q}_1 + \mathrm{i}\mathcal{Q}_2}{2}\,, \quad G^{-+}_+ \equiv \frac{\mathrm{i}\mathcal{Q}_3 - \mathcal{Q}_4}{2}\,, \quad G^{-+}_- \equiv \frac{-\mathrm{i}\widetilde{\mathcal{Q}}_1 + \widetilde{\mathcal{Q}}_2}{2}\,, \quad G^{--}_+ \equiv \frac{\widetilde{\mathcal{Q}}_3 + \mathrm{i}\widetilde{\mathcal{Q}}_4}{2}\,,
\end{aligned} \tag{2.28}$$

where γ is the sign of the L_0 eigenvalue, i is the outer automorphism from the SU(2) R-symmetry of the supergravity, and α is the SU(2) R-symmetry index corresponding to isometries of the S^2. Then, we obtain the non-trivial commutation relations:

$$\{G^{+\alpha}_\pm\,, G^{-\beta}_\mp\} = \epsilon^{\alpha\beta} L_0 \pm (\epsilon\boldsymbol{\tau_a})^{\beta\alpha} J^{\boldsymbol{a}}\,, \qquad \{G^{+\alpha}_\pm\,, G^{-\beta}_\pm\} = \mp\mathrm{i}\epsilon^{\alpha\beta} L_\pm\,, \tag{2.29}$$

and

$$\begin{aligned}
\Big[L_0\,, G^{i\alpha}_\pm\Big] &= \pm\frac{1}{2}\, G^{i\alpha}_\pm\,, \qquad \Big[L_\pm\,, G^{i\alpha}_\mp\Big] = -\mathrm{i}\, G^{i\alpha}_\pm\,,\\
\Big[J^3\,, G^{i\pm}_\gamma\Big] &= \pm\frac{1}{2}\, G^{i\pm}_\gamma\,, \qquad \Big[J^\pm\,, G^{i\mp}_\gamma\Big] = G^{i\pm}_\gamma\,,
\end{aligned} \tag{2.30}$$

where $J^\pm \equiv J^1 \pm \mathrm{i}J^2$. The algebra (2.27), (2.29), (2.30) is the global part $su(1,1|2)$ of the NS-sector chiral $\mathcal{N}=4$ superconformal algebra. Denoting the super Virasoro charges as $\mathcal{L}_n$, $n \in \mathbb{Z}$ and $\mathcal{G}^\alpha_{\dot{A},r}$, $r \in \mathbb{Z} + \frac{1}{2}$, $\dot{A} = (+,-)$, the embedding into the $\mathcal{N}=4$ superconformal algebra as presented e.g. in [49] is given by $L_\pm = \mp\mathrm{i}\mathcal{L}_{\mp 1}$, $L_0 = \mathcal{L}_0$, $G^{\pm\alpha}_\pm = \pm\mathcal{G}^\alpha_{\mp,\mp 1/2}$, $G^{\mp\alpha}_\pm = \pm\mathcal{G}^\alpha_{\pm,\mp 1/2}$, and the $su(2)$ zero-modes are unchanged.

3 Supersymmetric $\mathbb{H}^3/\mathbb{Z} \times \mathrm{S}^2$ and twisting

In this section we move from the Lorentzian $\mathrm{AdS}_3 \times \mathrm{S}^2$ configuration to the Euclidean $\mathbb{H}^3/\mathbb{Z} \times \mathrm{S}^2$ geometry. We begin, in the first subsection, by reviewing the thermal AdS_3 ($=\mathbb{H}^3/\mathbb{Z}$) geometry and some aspects of the AdS/CFT correspondence in this set up. In the second subsection we move to the supersymmetric version of the thermal geometry which requires a non-trivial twist. In the third subsection we discuss the Hamiltonian trace interpretation of the functional integral on this twisted configuration, and discuss how this is related to the elliptic genus in the semi-classical limit.

3.1 Thermal compactification of AdS_3

In this subsection we review the general set-up of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ dictionary in the context of our problem, following the treatment of [50]. We begin with the three-dimensional pure Einstein-Hilbert action with a cosmological constant:

$$S_{\text{grav}} = \frac{1}{16\pi G_3}\int d^3x\sqrt{g}\left(R - \frac{1}{2\ell^2}\right)\,. \tag{3.1}$$

AdS_3 is a solution of the equations of motion of this theory with constant negative curvature. The Wick rotation $t = -\mathrm{i}t_{\mathrm{E}}$ leads to the Euclidean ($\mathbb{H}^3$) metric

$$ds^2 = 4\ell^2 \left(\cosh^2\rho \, dt_{\mathrm{E}}{}^2 + d\rho^2 + \sinh^2\rho \, d\psi^2\right) . \tag{3.2}$$

When the range of t_{E} is $(-\infty, \infty)$ we obtain the cylinder geometry corresponding to the global solution. Thermal AdS_3 corresponds to the quotient $\mathbb{H}^3/\mathbb{Z}$, obtained by imposing the periodicities

$$(t_{\mathrm{E}} \,, \psi) \sim (t_{\mathrm{E}} + 2\pi\tau_2 \,, \psi + 2\pi\tau_1) \sim (t_{\mathrm{E}} \,, \psi + 2\pi) \,, \tag{3.3}$$

so that the geometry is that of a solid torus. Physically, this corresponds to setting the chemical potential conjugate to angular momentum (i.e. the angular velocity) to $2\pi\tau_1$ and the chemical potential conjugate to energy (i.e. the inverse temperature) to $2\pi\tau_2$.

It is convenient to introduce the coordinates $z \equiv \psi + \mathrm{i}\, t_{\mathrm{E}}$, $\bar{z} \equiv \psi \; - \mathrm{i}\, t_{\mathrm{E}}$. Upon taking the large-$\rho$ expansion of the transverse metric tensor $g_{\alpha\beta}$, where $x^\alpha = (z, \bar{z})$, one obtains the Fefferman-Graham form

$$g_{\alpha\beta} = \mathrm{e}^{2\rho} g^{(0)}_{\alpha\beta} + g^{(2)}_{\alpha\beta} + \cdots . \tag{3.4}$$

The conformal boundary metric $g^{(0)}_{\alpha\beta}$ is

$$g^{(0)}_{\alpha\beta} = \ell^2 \, dz \, d\bar{z} \,, \tag{3.5}$$

with the identifications

$$(z, \bar{z}) \sim (z + 2\pi, \bar{z} + 2\pi) \sim (z + 2\pi\tau, \bar{z} + 2\pi\bar{\tau}) \,, \tag{3.6}$$

consistent with the interpretation that the boundary CFT lives on the flat torus with modular parameter τ.

Boundary conditions on the gauge fields. More generally, we include constant chemical potentials μ^I for a number of conserved U(1) charges $q_I = \int J_I$ where J_I are the corresponding conserved currents in the boundary CFT. The partition function of such a CFT is

$$\mathrm{Tr}_{\mathcal{H}_{\mathrm{CFT}}} \, \mathrm{e}^{-\beta H + \mu^I q_I} \, . \tag{3.7}$$

The dual gravitational theory (3.1) includes the same number of U(1) gauge fields[8] W^I. The most relevant term governing their dynamics at low energies is given by the Chern-Simons action

$$-\frac{\mathrm{i}}{8\pi} k_{IJ} \int W^I \wedge dW^J = -\frac{\mathrm{i}}{8\pi} k_{IJ} \int d^3x \, \varepsilon^{\mu\nu\lambda} \, W^I_\mu \, \partial_\nu W^J_\lambda \, . \tag{3.8}$$

(Here, unlike in the rest of the paper, we employ the indices μ, $\nu \cdots$ to denote the 3d coordinates $x^\mu = (\rho, x^\alpha)$.) In the gauge $W^I_\rho = 0$, the gauge fields admit a large-ρ expansion analogous to (3.4) as:

$$W^I_\alpha = W^{I\,(0)}_\alpha + \mathrm{e}^{-2\rho} \, W^{I\,(2)}_\alpha + \cdots . \tag{3.9}$$

The asymptotic equations of motion imply that $W^{I\,(0)}_\alpha$ is flat.

[8]In this subsection, W^I and all other fields are three-dimensional.

As is well-known, the fact that the CS term has a first order kinetic term so that the two legs $W^I_{z,\bar{z}}$ form canonical pairs in the Hamiltonian theory [51]. One should therefore impose Dirichlet boundary conditions on only one of the legs:

$$\delta W_z^{I\,(0)} = 0\,, \qquad W_{\bar{z}}^{I\,(0)} \text{ not fixed.} \tag{3.10}$$

Now, in accord with the bulk/boundary correspondence, the boundary source μ^I must be identified with the asymptotic value of the gauge field $W_z^{I\,(0)}$. Since the ψ-cycle is contractible, any smooth configuration must have $W^I_\psi = 0$ at the origin. The saddle-point configurations have flat gauge fields due to the equations of motion, and therefore obey

$$W^I_z = -W^I_{\bar{z}} = -\mathrm{i}\,\mu^I\,. \tag{3.11}$$

The AdS/CFT correspondence states that the trace (3.7) equals the following bulk functional integral, up to a Casimir-energy-like term,

$$Z_{\mathrm{AdS}}(\tau,\,\mu) = \int D\phi_{\mathrm{grav}}\; \mathrm{e}^{-S_{\mathrm{ren}}(\tau,\,\mu)}\,, \tag{3.12}$$

where ϕ_{grav} denotes the gravitational fields of the theory, and

$$S_{\mathrm{ren}} \;\equiv\; S_{\mathrm{bulk}} + S_{\mathrm{bdry}} \tag{3.13}$$

is the renormalized action of the gravitational theory. Here, S_{bulk} is the bulk Euclidean action of the Einstein-Hilbert-matter theory and S_{bdry} is the boundary action required to make the total action finite and well-defined under our choice of boundary conditions. In particular, it includes the Gibbons-Hawking boundary term, and a Chern-Simons boundary term given by

$$-\,\frac{\mathrm{i}}{8\pi}\,k_{IJ}\int dz\,d\bar{z}\,\Big[W^I_z\,W^J_{\bar{z}}\Big]_{\mathrm{bdry}}\,. \tag{3.14}$$

This last term is required to ensure the consistency of the variational principle of the gauge fields with the boundary conditions (3.10).

The semi-classical contribution: at leading order, the partition function (3.12) is given by the value of S_{ren} on the thermal AdS_3 configuration described above. The value of this action is [50]

$$S_{\mathrm{ren}}(\tau,\,\mu) = -\pi\tau_2 k - \pi\tau_2\,k_{IJ}\,\mu^I\mu^J\,, \tag{3.15}$$

where $6k = \frac{3(2\ell)}{2G_3}$ is the Brown-Henneaux central charge of the gravitational theory for the AdS_3 space (3.2), and k_{IJ} is the level of the Chern-Simons term (3.8). Here, note that the boundary U(1) current obtained from (3.8), (3.14) is right-moving. The choice of opposite relative sign between (3.8) and (3.14) leads to the opposite chirality.

In the context of the five-dimensional theory of the previous section, recall from the discussion around (2.14) that $G_5 = 6\pi\ell^3/p^3$, so that

$$6k = \frac{3(2\ell)}{2G_3} = \frac{12\pi\ell^3}{G_5} = 2p^3\,. \tag{3.16}$$

To identify k_{IJ}, we reduce the five-dimensional Chern-Simons boundary term in (3.42) onto the S^2, and compare the resulting action with the three-dimensional Chern-Simons boundary term (3.14) on thermal AdS_3. This leads to:

$$k_{IJ} = \frac{2}{3} c_{IJK}\, p^K \,. \tag{3.17}$$

3.2 Twisted background and superalgebra

Now we consider the supersymmetric theory on the 5d geometry Under the Wick rotation $t = -\mathrm{i}t_\mathrm{E}$, the Lorentzian metric (2.13) rotates to that of Euclidean $\mathbb{H}^3 \times S^2$. The non-trivial fields in the Weyl multiplet are:

$$ds^2 = 4\ell^2 \left(\cosh^2\rho\, dt_\mathrm{E}^2 + d\rho^2 + \sinh^2\rho\, d\psi^2\right) + \ell^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \tag{3.18}$$

$$T_{\theta\phi} = -\frac{\ell}{4} \sin\theta \,. \tag{3.19}$$

If the Euclidean time coordinate t_E runs from $(-\infty, \infty)$, the topology is that of a solid cylinder times a sphere, which we call the Euclidean cylinder frame. Although the Killing spinor equations (2.19) and (2.20) are formally solved by the same set of eight spinors (2.23) in this background, these spinors are no longer well-defined because they diverge at the ends of the Euclidean cylinder. The solution to this problem involves compactifying the Euclidean time on a circle and simultaneously rotating the S^2 as we go around the time circle. This twisted quotient makes for a well-defined background, as we now describe.

We start from the configuration (3.18) describing an infinite solid cylinder (times a sphere), and make the following identifications,

$$(t_\mathrm{E}\,, \psi\,, \phi) \;\sim\; (t_\mathrm{E}\,, \psi + 2\pi\,, \phi) \;\sim\; (t_\mathrm{E} + 2\pi\tau_2\,, \psi + 2\pi\tau_1\,, \phi + \mathrm{i}2\pi\tau_2\Omega)\,. \tag{3.20}$$

Equivalently, we can define a new set of "twisted" coordinates,

$$t'_\mathrm{E} = t_\mathrm{E}\,, \qquad \phi' \;\equiv\; \phi - \mathrm{i}\Omega t_\mathrm{E}\,, \tag{3.21}$$

which have the identification

$$(t'_\mathrm{E}\,, \psi\,, \phi') \;\sim\; (t'_\mathrm{E}\,, \psi + 2\pi\,, \phi') \;\sim\; (t'_\mathrm{E} + 2\pi\tau_2\,, \psi + 2\pi\tau_1\,, \phi')\,. \tag{3.22}$$

We denote the corresponding complex coordinates as $z' = \psi + \mathrm{i}t'_\mathrm{E}$, $\bar{z}' = \psi - \mathrm{i}t'_\mathrm{E}$, identified as $(z', \bar{z}') \;\sim\; (z' + 2\pi\tau, \bar{z}' + 2\pi\bar{\tau})$.

In the twisted frame, the on-shell background configuration is

$$\begin{aligned} ds^2 &= 4\ell^2 \Big(\cosh^2\rho\, dt'^2_\mathrm{E} + d\rho^2 + \sinh^2\rho\, d\psi^2\Big) + \ell^2\Big(d\theta^2 + \sin^2\theta \,(d\phi' + \mathrm{i}\Omega dt'_\mathrm{E})^2\Big)\,, \\ T_{\theta\phi'} &= -\frac{\ell}{4}\sin\theta\,, \qquad T_{\theta t'_\mathrm{E}} = -\mathrm{i}\frac{\ell}{4}\Omega\sin\theta\,, \\ \sigma^I &= -\frac{p^I}{\ell}\,, \qquad W^I_{t_\mathrm{E}'} = 2\mu^I - \mathrm{i}\Omega\, p^I\cos\theta\,, \qquad W^I_{\phi'} = -p^I\cos\theta\,, \\ A_1{}^1 &= A_2{}^2 = \sqrt{\frac{p^3}{3\ell^3}}\,. \end{aligned} \tag{3.23}$$

JHEP07(2023)218

The S^2 in (3.23) is fibered over the time circle of AdS_3, and we refer to this configuration as the *twisted torus* background. We also note that in the expression for $W^I_{t_{\mathrm{E}}'}$ we have introduced an arbitrary constant μ^I which is allowed by the supersymmetry and equations of motion, which we will interpret as the source of a U(1) current in the boundary CFT. The BPS equations may also allow W^I_ψ to take a constant value, but this constant is forced to be zero due to the contractibility of the ψ-cycle.

To see that the twisted torus background (3.23) has well-defined supersymmetry, we solve the Killing spinor equation from the variation of gravitino (B.1), which is rewritten now as

$$0 = 2\mathcal{D}_M \varepsilon^i - \frac{\mathrm{i}}{4\ell}(3\gamma^{\hat\theta\hat\phi}\gamma_M - \gamma_M\gamma^{\hat\theta\hat\phi})\,\varepsilon^i\,. \tag{3.24}$$

Here we use the following gamma matrices in the Euclidean theory, which follow from the Wick rotation,

$$\gamma_{\hat t_{\mathrm{E}}} = \boldsymbol{\sigma}_3\otimes\boldsymbol{\tau}_3\,,\quad \gamma_{\hat\rho} = \boldsymbol{\sigma}_1\otimes\boldsymbol{\tau}_3\,,\quad \gamma_{\hat\psi} = \boldsymbol{\sigma}_2\otimes\boldsymbol{\tau}_3\,,\quad \gamma_{\hat\theta} = \mathbb{I}\otimes\boldsymbol{\tau}_1\,,\quad \gamma_{\hat\phi} = \mathbb{I}\otimes\boldsymbol{\tau}_2\,, \tag{3.25}$$

where we relate the σ_3 with the Lorentzian gamma matrix $\boldsymbol{\sigma}_0$ in (C.4) by $\boldsymbol{\sigma}_3 \equiv -\mathrm{i}\boldsymbol{\sigma}_0$. We will take the representation $(\boldsymbol{\sigma}_3,\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) = (-\boldsymbol{\tau}_3,\boldsymbol{\tau}_1,\boldsymbol{\tau}_2)$ with the Pauli sigma matrix $\boldsymbol{\tau}_a$. Note that unlike the case of global $\mathrm{AdS}_3\times$ S^2 in the subsection 2.3, the Killing spinor equation (3.24) does not split into the equations of AdS_3 and S^2. This is because we have the following spin connections

$$\omega^{12}_{t_{\mathrm{E}}'} = -\sinh\rho\,,\quad \omega^{45}_{t_{\mathrm{E}}'} = \mathrm{i}\Omega\cos\theta\,,\quad \omega^{23}_{\psi} = \cosh\rho\,,\quad \omega^{45}_{\phi'} = \cos\theta\,, \tag{3.26}$$

where there is mixing between AdS_3 and S^2 directions through the non-zero twisting parameter Ω.

The solution of Killing spinors can be easily found by following the twisting construction. It is clear that the Euclidean continuation of the set of 8 Lorentzian Killing spinors (2.21), (2.23), followed by the coordinate transformation (3.21) obeys the new Killing spinor equation. Upon setting the parameter

$$\Omega = 1 + \mathrm{i}\frac{\tau_1}{\tau_2}\,, \tag{3.27}$$

the following[9] 4 of the original 8 Killing spinors

$$\varepsilon^i_{(1)} = (-\mathrm{i}\varepsilon_+^{\ +},\varepsilon_-^{\ -})\,,\quad \varepsilon^i_{(2)} = (\varepsilon_+^{\ +},-\mathrm{i}\varepsilon_-^{\ -})\,,\quad \varepsilon^i_{(3)} = -(\varepsilon_-^{\ -},\mathrm{i}\varepsilon_+^{\ +})\,,\quad \varepsilon^i_{(4)} = -(\mathrm{i}\varepsilon_-^{\ -},\varepsilon_+^{\ +})\,, \tag{3.28}$$

where

$$\begin{aligned}
\varepsilon_+^{\ +} &= \sqrt{\frac{\ell}{2}}\,\mathrm{e}^{\frac12(1-\Omega)t_{\mathrm{E}}' + \frac{\mathrm{i}}{2}(\psi+\phi')}\begin{pmatrix}\cosh\frac{\rho}{2}\\ -\sinh\frac{\rho}{2}\end{pmatrix}\otimes\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix},\\
\varepsilon_-^{\ -} &= \sqrt{\frac{\ell}{2}}\,\mathrm{e}^{-\frac12(1-\Omega)t_{\mathrm{E}}' - \frac{\mathrm{i}}{2}(\psi+\phi')}\begin{pmatrix}-\sinh\frac{\rho}{2}\\ \cosh\frac{\rho}{2}\end{pmatrix}\otimes\begin{pmatrix}-\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{pmatrix},
\end{aligned} \tag{3.29}$$

[9]The choice $\Omega = -1 - \mathrm{i}\frac{\tau_1}{\tau_2}$ also gives rise to a different 4 set of Killing spinors.

JHEP07(2023)218

respect the periodicity (3.22) (they are periodic around the non-conctractible circle and anti-periodic around the contractible circle).

One could also directly solve the new Killing spinor equations (3.24). The only non-trivial change compared to the untwisted case is the equation in the $t'_{\rm E}$ direction,

$$0 = (2\partial_{t'_{\rm E}} - \omega_{t'_{\rm E}}^{12}\gamma_{12} - \omega_{t'_{\rm E}}^{34}\gamma_{34})\,\varepsilon_\pm{}^{\pm} - \frac{\rm i}{2\ell}E_{t'_{\rm E}}{}^{1}\gamma_{45}\gamma_1\,\varepsilon_\pm{}^{\pm} - \frac{\rm i}{\ell}E_{t'_{\rm E}}{}^{5}(\gamma_{45}\gamma_5)\,\varepsilon_\pm{}^{\pm}\,. \tag{3.30}$$

Comparing to the equation (3.24) that would be written in untwisted frame, The difference with the equation in the untwisted frame is that $2\partial_{t'_{\rm E}}$ acting on the Killing spinor (3.29) gives $\pm(1-\Omega)$ instead of ±1. Also, the third and the last terms are new. By the projection property along S^2 direction of the Killing spinor $(1\otimes {\rm e}^{-{\rm i}\tau_2\theta}\tau_3)\varepsilon_\pm{}^{\pm} = \pm\varepsilon_\pm{}^{\pm}$, one can check that the effect of the third and the last term indeed cancels the contribution of Ω in the time-derivative acting on the Killing spinor.

Supersymmetry algebra: the supercharges $\mathcal{Q}_a = \delta(\varepsilon^i_{(a)})$, with the Killing spinors $\varepsilon^i_{(a)}$, $a=1,2,3,4$ defined in (3.28), obey

$$\{\mathcal{Q}_a\,,\mathcal{Q}_b\} = -2{\rm i}\delta_{ab}(L_0 - J^3)\,,\qquad [L_0 - J^3\,,\mathcal{Q}_a] = 0\,. \tag{3.31}$$

Consider the following four supercharges $G^{i\alpha}_\gamma$,

$$G^{++}_+ \equiv \frac{{\rm i}\mathcal{Q}_1 + \mathcal{Q}_2}{2}\,,\quad G^{--}_- \equiv \frac{\mathcal{Q}_1 + {\rm i}\mathcal{Q}_2}{2}\,,\quad G^{+-}_- \equiv \frac{-\mathcal{Q}_3 + {\rm i}\mathcal{Q}_4}{2}\,,\quad G^{-+}_+ \equiv \frac{{\rm i}\mathcal{Q}_3 - \mathcal{Q}_4}{2}\,, \tag{3.32}$$

where γ is the sign of the L_0 eigenvalue, i is the doublet index under the outer automorphism coming from the SU(2) R-symmetry of the supergravity, and α is the doublet index under the SU(2) R-symmetry arising from the isometry of the S^2. They are charged under the bosonic generators L_0 and J^3 as

$$\left[L_0\,,G^{i\pm}_\pm\right] = \pm\frac{1}{2}\,G^{i\pm}_\pm\,,\qquad \left[J^3\,,G^{i\pm}_\pm\right] = \pm\frac{1}{2}\,G^{i\pm}_\pm\,, \tag{3.33}$$

so that

$$[L_0 - J^3\,,G^{i\pm}_\pm] = 0\,, \tag{3.34}$$

and they obey the anticommutation relations

$$\{G^{+\pm}_\pm\,,G^{-\mp}_\mp\} = \pm\left(L_0 - J^3\right)\,,\qquad \{G^{+\pm}_\pm\,,G^{-\pm}_\pm\} = 0\,. \tag{3.35}$$

The above algebra (3.34), (3.35) forms a subalgebra of the global part of $\mathcal{N}=4$ superconformal algebra in the NS sector given in section 2.3. Note that the subalgebra can also be thought of as the spectral flow,[10] with parameter $\eta = 1/2$, to the following Ramond sector zero-modes as $L_0 - J^3 + c/24 \mapsto \mathcal{L}^R_0$, $G^{\pm\mp}_\mp \mapsto \mp\mathcal{G}^\mp_{\mp,0}$, $G^{\pm\pm}_\pm \mapsto \pm\mathcal{G}^\pm_{\mp,0}$.

[10] The spectral flow is taken on the charges with algebra in [49] as $\mathcal{L}_n \mapsto \mathcal{L}_n + 2\eta J^3_n + \eta^2\frac{c}{6}\delta_{n,0}$, $J^3_m \mapsto J^3_m + \eta\frac{c}{6}\delta_{n,0}$, $J^\pm_m \mapsto J^\pm_{m\pm2\eta}$, $\mathcal{G}^\pm_{\dot{A},r} \mapsto \mathcal{G}^\pm_{\dot{A},r\pm\eta}$.

3.3 The trace interpretation and the semiclassical limit

We now interpret the functional integral on $\mathbb{H}^3/\mathbb{Z}\times \mathrm{S}^2$ as a trace on the Hilbert space of the boundary CFT. First, a small change of *conventions* in this section: we consider $(0,4)$ SCFT_2 on the boundary, where supersymmetry algebra acts on the right-moving sector, whose generators we denote with a bar e.g. $\overline{L}_0$ and $\overline{J}^3$. This means we have to flip the convention (regarding the bars) of the previous subsection so that we have the following representation in the twisted coordinates,

$$L_0 = \mathrm{i}\frac{1}{2}(\mathrm{i}\partial_{t'_\mathrm{E}} - \partial_\psi + \Omega\,\partial_{\phi'})\,, \qquad \overline{L}_0 = \mathrm{i}\frac{1}{2}(\mathrm{i}\partial_{t'_\mathrm{E}} + \partial_\psi + \Omega\,\partial_{\phi'})\,, \qquad \overline{J}^3 = \mathrm{i}\partial_{\phi'}\,, \tag{3.36}$$

with $\Omega = 1 + \mathrm{i}\tau_1/\tau_2$. Now consider the gravitational functional integral Z corresponding to the partition function on the twisted torus (3.22), (3.23). Bosonic fields are periodic around both the cycles of the torus, while fermionic fields are periodic around the t'_E-circle (which has periodicity $2\pi\tau_2$) and are anti-periodic around the contractible ψ circle (which has periodicity 2π). In addition, we have the chemical potential $2\pi\mathrm{i}\tau_1$ for the angular momentum ∂_ψ around the AdS_3, and the chemical potentials μ^I coupling to U(1) current(s) q_I. By the usual interpretation of the functional integral, we have

$$\begin{aligned}
\mathrm{e}^{C(\tau,\mu)}\,Z(\tau,\mu) &= \mathrm{Tr}_\mathrm{NS}\,(-1)^F\,\exp\big(2\pi\tau_2\,\partial_{t'_\mathrm{E}} + 2\pi\tau_1\,\partial_\psi + \mu^I q_I\big)\,,\\
&= \mathrm{Tr}_\mathrm{NS}\,(-1)^F\,\exp\big(-2\pi\tau_2\,(L_0 + \overline{L}_0 - \Omega\overline{J}^3) + 2\pi\mathrm{i}\tau_1\,(L_0 - \overline{L}_0) + \mu^I q_I\big)\,,\\
&= \mathrm{Tr}_\mathrm{NS}\,(-1)^F\,q^{L_0}\,\overline{q}^{\overline{L}_0 - \overline{J}^3}\,\mathrm{e}^{\mu^I q_I}\,,
\end{aligned}\tag{3.37}$$

with $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$, $\overline{q} = \mathrm{e}^{-2\pi\mathrm{i}\overline{\tau}}$, and $\tau = \tau_1 + \mathrm{i}\tau_2$. The Casimir-energy-type term $C(\tau,\mu)$ arises while relating the functional integral to the Hamiltonian trace [50].

We recognize the right-hand side of (3.37) as the elliptic genus. Indeed, from the anticommutator (3.35) we see that the above trace can be written as

$$\mathrm{e}^{C(\tau,\mu)}\,Z(\tau,\mu) = \mathrm{Tr}_\mathrm{NS}\,(-1)^F\,q^{L_0}\,\overline{q}^{\mathrm{i}\mathcal{Q}^2}\,\mathrm{e}^{\mu^I q_I}\,, \tag{3.38}$$

where we have chosen one supercharge $\mathcal{Q} = \frac{1}{\sqrt{2}}\mathcal{Q}_1 = \frac{1}{\sqrt{2}}(G_-^{--} - \mathrm{i}G_+^{++})$ as in (3.31) and (3.32). The pairing of all non-BPS modes with respect to the supercharge $\mathcal{Q}$ enforces that the elliptic genus is a holomorphic function of τ. Note that there is no regularization scheme for the functional integral in which the pre-factor $C(\tau,\mu)$ respects modular invariance and holomorphy. In particular, if one chooses the pre-factor $C(\tau,\mu)$ to respect modular invariance (and the gauge invariance associated to μ), it suffers from a holomorphic anomaly and cannot be purely holomorphic in τ [52].

On-shell action. Now that we have set up the twisted torus background, we can evaluate the functional integral as explained in section 3.1 for the 3d untwisted theory. We have

$$Z_\mathrm{AdS}(\tau,\,\mu) = \int D\phi_\mathrm{grav}\;\mathrm{e}^{-S_\mathrm{ren}(\tau,\,\mu)}\,, \qquad S_\mathrm{ren} \,\equiv\, S_\mathrm{bulk} + S_\mathrm{bdry}\,, \tag{3.39}$$

as an integral over all field configurations ϕ_grav of the 5d supergravity theory, with the renormalized action S_ren of the gravitational theory.

JHEP07(2023)218

The bulk supergravity action (2.12) evaluated on the twisted torus (3.23) is

$$\begin{aligned} S_{\text{bulk}}(\tau_2,p,\mu) &= \frac{1}{8\pi^2}\int_0^{\rho_0} d\rho \int_0^{2\pi} d\psi \int_0^{\pi} d\theta \int_0^{2\pi} d\phi' \int_0^{2\pi\tau_2} dt'_{\text{E}}\, L_{\text{bulk}} \\ &= -\frac{\pi\tau_2}{3}p^3 + \mathrm{O}(e^{2\rho_0})\,. \end{aligned} \tag{3.40}$$

The second term on the right-hand side denotes terms in the bulk action that diverge when the radial cutoff $\rho_0 \to \infty$, and is absorbed by standard boundary terms.

The boundary terms in the action of the gauge fields behave essentially in the same way as in the untwisted theory, but with slightly different details. In the cylinder frame (3.18), the gauge fields $W_{z,\bar{z}}$ on the AdS_3 factor have the boundary conditions (3.10), while the components $W_{\theta,\phi}$ on the S^2 are fixed at the boundary. Twisting these boundary conditions using (3.21) gives the boundary conditions for the gauge fields on the twisted torus:

$$\delta W^{I\,(0)}_{z'} = 0\,, \qquad W^{I\,(0)}_{\bar{z}'} \text{ not fixed}, \qquad \delta W^{I\,(0)}_{\theta,\phi'} = 0\,, \tag{3.41}$$

where the (0) indicates the boundary values in the large-ρ expansion as in (3.9). The Chern-Simons boundary action consistent with these boundary conditions is:

$$S^{\text{bdry}}_{\text{CS}} = -c_{IJK}\frac{\mathrm{i}p^I}{48\pi^2}\int_{\partial\mathcal{M}} dz' d\bar{z}' d\theta d\phi' \,\sin\theta \left[\left(W^J_{z'} - \frac{1}{2}\Omega W^J_{\phi'}\right) W^K_{\bar{z}'}\right]_{\text{bdry}}\,, \tag{3.42}$$

which on the twisted torus (3.23) evaluates to[11]

$$S^{\text{bdry}}_{\text{CS}} = -\frac{2\pi\tau_2}{3}c_{IJK}\,\mu^I\mu^J p^K\,. \tag{3.43}$$

The boundary terms in the gravitational sector are more subtle. Recall that in the standard treatment of AdS_3 gravity, see e.g. [50], the variation of the boundary metric is fixed, but not that of its normal derivative. As is well known, one requires the addition of a Gibbons-Hawking boundary term, in order for the variational principle to be well-defined. In addition, one requires additional local counterterms on the boundary to cancel the divergences arising from the bulk action as in (3.40) as well as from the Gibbons-Hawking term. These considerations are systematically summarized in the procedure of holographic renormalization [53]. As it turns out, in the localization of the path integral for the elliptic genus, we need to impose slightly different boundary conditions on the metric compared to the standard ones, and correspondingly we have a different structure of boundary terms. However, these differences are only relevant when the metric goes off-shell, and do not change the on-shell background that we have discussed so far. We postpone the details to a follow-up publication. Thus the value of the full renormalized action on the twisted background (3.23) is

$$S_{\text{ren}} = -\pi k\tau_2 - \pi\tau_2\, k_{IJ}\,\mu^I\mu^J\,. \tag{3.44}$$

Note that the twisting procedure only affects global properties and does not change the Newton's constant. Therefore the central charge continues to be $c = 6k = 2p^3$ as in (3.16).

[11]Evaluating actions of this type is more conviently done by transforming back from the $(z',\,\bar{z}')$ to the $(\psi,\,t'_{\text{E}})$ coordinates where the ranges are as in (3.40).

Similarly, the level k_{IJ} of the boundary current algebra also does not change. To see this, note that the relation between the twisted and cylinder-frame fields is:

$$W_{z'} = W_z + \frac{1}{2}\Omega\, W^I_\phi(\theta)\,, \qquad W_{\bar{z}'} = W_{\bar{z}} - \frac{1}{2}\Omega\, W^I_\phi(\theta)\,, \qquad W_{\phi'} = W_\phi(\theta)\,, \tag{3.45}$$

where $W_{z,\bar{z}}$ only depends on the AdS$_3$ coordinates $(\rho, z, \bar{z}) = (\rho, z', \bar{z}')$ while $W^I_\phi = -p^I \cos\theta$. Substituting this into (3.42) gives:

$$S^{\text{bdry}}_{\text{CS}} = -c_{IJK}\frac{\mathrm{i}p^I}{12\pi}\int_{\partial\mathcal{M}} dz' d\bar{z}' \Big[W^J_z W^K_{\bar{z}}\Big]_{\text{bdry}}\,, \tag{3.46}$$

which is the same as the 3d Chern-Simons boundary term (3.14), since the integration ranges of $(z', \bar{z}')$ are the same as for $(z, \bar{z})$. This shows that $k_{IJ} = \frac{2}{3}c_{IJK}\, p^K$ as in the untwisted theory (3.17).

4 The Euclidean 4d/5d lift

In this section we present a formalism to obtain off-shell localization solutions in 5d supergravity by lifting the localization manifold around Euclidean AdS$_2 \times$ S^2. In particular, this allows us to obtain localization solutions around the supersymmetric twisted torus $\mathbb{H}^3/\mathbb{Z} \times$ S^2 background presented in (3.23).

We briefly recall the first step of the localization problem that the formalism addresses. We define the localization supercharge $\mathcal{Q} = \frac{1}{\sqrt{2}}\mathcal{Q}_1 = \frac{1}{\sqrt{2}}\delta(\varepsilon^i_{(1)})$ where the Killing spinor $\varepsilon^i_{(1)}$ is given in (3.28). (Equivalently, $\mathcal{Q} = \frac{1}{\sqrt{2}}(G^{--}_{-} - \mathrm{i}G^{++}_{+})$ in terms of the super-Virasoro generators.) It obeys the algebra

$$\mathcal{Q}^2 = -\mathrm{i}(\overline{L}_0 - \overline{J}^3)\,, \qquad [\overline{L}_0 - \overline{J}^3, \mathcal{Q}] = 0\,. \tag{4.1}$$

We would like to study the space of solutions to the BPS equations given by setting the supersymmetry variations generated by $\mathcal{Q}$ of all the fermions (B.1) to zero.

The BPS equations form a system of matrix-valued partial differential equations in terms of the bosonic fields of the theory. One systematic approach to solve them, assuming no fermionic backgrounds, begins by forming various Killing spinor bilinears [54, 55]. The BPS equations may then be expressed as a set of coupled first-order equations for these tensor fields, which describe the bosonic background of the solution. This approach was used in [19, 44] to solve the off-shell problem in the AdS$_2 \times$ S^2 (and S^3) background. The general solutions to the resulting equations are, however, typically difficult to obtain, and we do not solve this problem of general classification in this paper. Instead, we leverage what is already known about the localization solutions in 4d supergravity around the Euclidean AdS$_2 \times$ S^2 background [1, 44, 45], by lifting them to five dimensions. This involves the KK lift of AdS$_2 \times$ S^2 to AdS$_3 \times$ S^2, which we describe in section 4.1. Note, however, that while the 4d localization manifold has been determined completely, there may be additional solutions in 5d that *do* depend on the KK direction, and that will therefore not emerge from the lift. We postpone the discussion of such solutions to future work.

JHEP07(2023)218

To lift the 4d localization solutions, we use the idea of the 4d/5d off-shell connection of [16]. However, as mentioned in the introduction, implementing this idea is not straightforward for the following reasons. Firstly, while the formalism in [16] was developed for Lorentzian supergravities, our 4d/5d connection needs to be adapted to accommodate the Euclidean supergravities in both four and five dimensions. A subtlety here, as we will shortly see, is that the 4d Euclidean theory has a redundancy in the choice of reality conditions and correspondingly a redundancy of $\mathrm{AdS}_2 \times \mathrm{S}^2$ backgrounds, which has no counterpart in the 5d theory. Secondly, recall that the 4d/5d lift produces a five-dimensional background in the Kaluza-Klein anstaz and so, in order to reach the five-dimensional theory on the supersymmetric twisted torus $\mathbb{H}^3/\mathbb{Z} \times \mathrm{S}^2$ from the four-dimensional theory on $\mathrm{AdS}_2 \times \mathrm{S}^2$, we require a mapping of the twisted torus (3.23) into the Kaluza-Klein frame of $\mathrm{AdS}_3 \times \mathrm{S}^2$. In section 4.1 we present the mapping from the Kaluza-Klein frame to the cylinder frame. The twisted frame can then easily be mapped to the cylinder frame (3.18) by the local coordinate transformation (3.21). In section 4.2 we review the 4d Euclidean supergravity and the $\mathrm{AdS}_2 \times \mathrm{S}^2$ background. In section 4.3, we present our construction of the Euclidean 4d/5d off-shell lift. Further, we show that the redundancy of the 4d theory mentioned above can be absorbed into the mapping parameter. We conclude the section by presenting the steps of lifting the 4d off-shell solutions to the 5d twisted torus.

4.1 The Kaluza-Klein coordinate frame

In this subsection we map the cylinder frame to the Kaluza-Klein frame. This mapping requires the local coordinate transformations as well as local Lorentz transformations. After presenting the general mechanism, we find the specific coordinate and Lorentz transformations, and the resulting background configuration and supercharges for $\mathrm{AdS}_3\times\, \mathrm{S}^2$ in the Kaluza-Klein frame.

The general mechanism is as follows. Let $\{\dot{M}, \dot{N}, \cdots\}$ and $\{\dot{A}, \dot{B}, \cdots\}$ be the spacetime and tangent indices, respectively, in this Kaluza-Klein frame. The vielbein in the Euclidean cylinder frame $E_M{}^A$ maps to the vielbein in the KK frame $\dot{E}_{\dot{N}}{}^{\dot{A}}$ under a diffeomorphism together with some local rotation $L_A{}^{\dot{A}}$ which acts on the frame as [56]

$$E_M{}^A(x) = \frac{\partial \dot{x}^{\dot{N}}}{\partial x^M} \dot{E}_{\dot{N}}{}^{\dot{A}}(\dot{x}) L^{-1A}_{\dot{A}}\,. \tag{4.2}$$

Correspondingly, the spin connection transforms as

$$\omega_{MA}{}^B = \frac{\partial \dot{x}^{\dot{N}}}{\partial x^M} \left(L_A{}^{\dot{A}} \dot{\omega}_{\dot{N}\dot{A}}{}^{\dot{B}} L^{-1B}_{\dot{B}} + (\partial_N L_A{}^{\dot{A}})\, L^{-1B}_{\dot{A}} \right)\,. \tag{4.3}$$

Likewise, the remaining non-trivial background fields and the Killing spinors are mapped into the KK frame using the same diffeomorphism and local rotation $L_A{}^{\dot{A}}$, and a corresponding spinor rotation $\mathcal{L}$, as

$$T_{AB} = L_A{}^{\dot{A}} L_B{}^{\dot{B}} \dot{T}_{\dot{A}\dot{B}}\,, \qquad F_{AB} = L_A{}^{\dot{A}} L_B{}^{\dot{B}} \dot{F}_{\dot{A}\dot{B}}\,, \qquad \epsilon^i = \mathcal{L}\, \dot{\varepsilon}^j\,, \tag{4.4}$$

where $L_A{}^{\dot{A}}$ and $\mathcal{L}$ are related such that the gamma matrix is preserved:

$$L_A{}^{\dot{B}}\, \mathcal{L}\, \gamma_{\dot{B}}\, \mathcal{L}^{-1} = \gamma_A\,. \tag{4.5}$$

The diffeomorphism and local rotation in (4.2) should be chosen such that the vielbein in KK frame $\dot{E}_{\dot{N}}{}^{\dot{A}}$ has the following reduction ansatz. Decomposing the KK frame coordinate as $x^{\dot{M}} = \{x^\mu,\, x^5\}$ and $x^{\dot{A}} = \{x^a, 5\}$, the reduction ansatz of the vielbein is

$$\dot{E}_{\dot{M}}{}^{\dot{A}} = \begin{pmatrix} e_\mu{}^a & B_\mu \phi^{-1} \\ 0 & \phi^{-1} \end{pmatrix}, \qquad \dot{E}_{\dot{A}}{}^{\dot{M}} = \begin{pmatrix} e_a{}^\mu & -e_a{}^\mu B_\mu \\ 0 & \phi \end{pmatrix}, \tag{4.6}$$

where all the fields in the KK frame are independent of the compactified x^5 coordinate. Note that the KK ansatz (4.6) breaks the 5d diffeomorphisms to 4d diffeomorphisms and a $\mathrm{U}(1)_{\text{gauge}}$, and breaks the 5d local rotation symmetry $O(5)$ to $O(4) \times \mathbb{Z}_2$. Using the $\mathbb{Z}_2$ we can fix the ϕ to have a fixed sign, say, positive. The vielbein (4.6) is equivalent to the following metric in the KK frame (with $x^5 \sim x^5 + 2\pi$),

$$\dot{G}_{\dot{M}\dot{N}}\, dx^{\dot{M}} dx^{\dot{N}} = g_{\mu\nu}\, dx^\mu dx^\nu + \phi^{-2}(dx^5 + B_\mu\, dx^\mu)^2 \,. \tag{4.7}$$

We see from the (4.6) and (4.7) that the five-dimensional vielbein $\dot{E}_{\dot{N}}{}^{\dot{A}}$ or metric $\dot{G}_{\dot{M}\dot{N}}$ are related to the four-dimensional veilbein $e_\mu{}^a$ or metric $g_{\mu\nu}$, a gauge field B_μ and a scalar ϕ. The reduction ansatz leads to the following reduction of the spin connection as

$$\dot{\omega}_{\dot{A}}{}^{bc} = \begin{pmatrix} \omega_a{}^{bc} \\ \frac{1}{2}\phi^{-1} F(B)^{bc} \end{pmatrix}, \qquad \dot{\omega}_{\dot{A}}{}^{b5} = \begin{pmatrix} -\frac{1}{2}\phi^{-1} F(B)_a{}^b \\ -\phi^{-1} D^b \phi \end{pmatrix}. \tag{4.8}$$

Here we see that the gauge field B_μ appears through its field strength $F(B)_{ab}$ in four dimensions.

Now we find the coordinate transformations and the local rotation in (4.2) that map the cylinder frame background in (3.18) to fit into the KK frame ansatz (4.6) and (4.7). The cylinder frame coordinates x^M and the KK frame coordinates $\dot{x}^{\dot{M}}$

$$x^M = (\rho\,, \psi\,, \theta\,, \phi\,, t_{\mathrm{E}})\,, \quad \dot{x}^{\dot{M}} = (\eta\,, \chi\,, \theta\,, \phi\,, x^5)\,, \tag{4.9}$$

are related as

$$(\,\rho, \psi, t_{\mathrm{E}}) = \left(\frac{\eta}{2}, \chi + \mathrm{i}\frac{x^5}{2}, \frac{x^5}{2}\right) \quad \Leftrightarrow \quad (\eta\,, \chi\,, x^5) = (2\rho\,, \psi - \mathrm{i}t_{\mathrm{E}}\,, 2t_{\mathrm{E}})\,, \tag{4.10}$$

with the coordinates (θ, ϕ) remaining the same. Note that the global conditions on the periodicities are not respected by this map (e.g. x^5 is compact whereas t_{E} is not). The corresponding local rotation matrix $L_A{}^{\dot{A}}$ is given as a rotation in the 2–5 plane (along $\hat{\psi}$ and $\hat{t}_{\mathrm{E}}$ direction) with angle $\omega = -\mathrm{i}\eta/2$:

$$L_1{}^{\dot{1}} = L_3{}^{\dot{3}} = L_4{}^{\dot{4}} = 1\,, \qquad L_2{}^{\dot{2}} = L_5{}^{\dot{5}} = \cosh\frac{\eta}{2}\,, \qquad L_2{}^{\dot{5}} = -L_5{}^{\dot{2}} = \mathrm{i}\sinh\frac{\eta}{2}\,. \tag{4.11}$$

In the exponential form, we have $L_A{}^{\dot{A}} = (\mathrm{e}^{\Omega})_A{}^{\dot{A}}$, where the $2-5$ component of the matrix in the exponent is $\Omega_{25} = -\Omega_{52} = -\omega = \mathrm{i}\eta/2$.[12] By the relation (4.5), the corresponding

[12]The rotation with angle ω is $\exp\left(\omega\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix}$. Here, we take $\omega = -\mathrm{i}\eta/2$ for the rotation in 2–5 plane. Note that the angle is imaginary, because the coordinate x^5 is Euclidean time.

spinor rotation is

$$\mathcal{L} = \exp\left(\frac{1}{4}\Omega_{AB}\gamma^{AB}\right) = \exp\left(\frac{\mathrm{i}}{4}\eta\,\gamma_{25}\right) = \begin{pmatrix} \cosh\frac{\eta}{4} & -\sinh\frac{\eta}{4} \\ -\sinh\frac{\eta}{4} & \cosh\frac{\eta}{4} \end{pmatrix} \otimes \mathbb{I}_2\,. \tag{4.12}$$

We note that although the spin connection in the cylinder frame has zero component for ${\omega_{\hat\rho}}^{\hat\psi\hat t_{\mathrm{E}}}$, as can be seen in (C.2), the corresponding spin connection of KK frame mapped by (4.3), ${\dot\omega_{\hat\eta}}^{\hat\psi\hat t_{\mathrm{E}}}(={\dot\omega_{\dot 1}}^{\dot 2\dot 5})$, is non-zero due to the contribution of the Lorentz transformation matrix in the second term of (4.3). According to (4.8), this non-zero component gives the non-zero value of the electric flux along AdS_2. This explains why there is electric flux on AdS_2 even though the AdS_3 does not have any electric flux.

We now summarize the on-shell supersymmetric field configuration in the KK coordinates. By using the transformations (4.10), (4.11), (4.12) on the background Weyl multiplet as in (3.18) and matter multiplets as in (2.14), (2.16), we obtain the following configuration:

$$\begin{aligned} ds_5{}^2 &= \ell^2\left(d\eta^2 + \sinh^2\eta\, d\chi^2 + d\theta^2 + \sin^2\theta\, d\phi^2\right) + \ell^2\left(dx^5 + \mathrm{i}(\cosh\eta - 1)\,d\chi\right)^2\,, \\ \dot T_{34} &= -\frac{1}{4\ell}\,, \\ \dot\sigma^I &= -\frac{p^I}{\ell}\,, \qquad\qquad \dot F^I_{\theta\phi} = p^I\sin\theta\,, \qquad\qquad \dot W^I_{x^5} = \mu^I\,, \\ \dot A_1{}^1 &= \dot A_2{}^2 = \sqrt{\frac{p^3}{3\ell^3}}\,. \end{aligned} \tag{4.13}$$

We note that the background geometry has an S^1 fibration over the four-dimensional base, which is Euclidean $\mathrm{AdS}_2\times\mathrm{S}^2$. The angular coordinate χ of Euclidean AdS_2 has periodicity 2π. By comparing the metric with the KK ansatz (4.7), we identify the following values for the KK one-form and scalar:

$$B = \mathrm{i}(\cosh\eta - 1)\,d\chi\,, \qquad \phi = \ell^{-1}\,. \tag{4.14}$$

The background configuration given in (4.13) has well-defined supersymmetry. To see that, we look for the Killing spinors. By the Eulidean continuation of the Lorentzian Killing spinors (2.23) followed by the coordinate transformation (4.10) and the Lorentz transformation (4.12) we obtain

$$\begin{aligned} \dot\varepsilon^i_{(1)} &= \left(-\mathrm{i}\dot\varepsilon_+{}^+\,, \dot\varepsilon_-{}^-\right), & \dot\varepsilon^i_{(2)} &= \left(\dot\varepsilon_+{}^+\,, -\mathrm{i}\dot\varepsilon_-{}^-\right), \\ \dot\varepsilon^i_{(3)} &= \left(-\dot\varepsilon_-{}^-\,, -\mathrm{i}\dot\varepsilon_+{}^+\right), & \dot\varepsilon^i_{(4)} &= \left(-\mathrm{i}\dot\varepsilon_-{}^-\,, -\dot\varepsilon_+{}^+\right), \\ \dot{\tilde\varepsilon}^i_{(1)} &= \left(\dot\varepsilon_+{}^-\,, \mathrm{i}\dot\varepsilon_-{}^+\right), & \dot{\tilde\varepsilon}^i_{(2)} &= \left(\mathrm{i}\dot\varepsilon_+{}^-\,, \dot\varepsilon_-{}^+\right), \\ \dot{\tilde\varepsilon}^i_{(3)} &= \left(-\mathrm{i}\dot\varepsilon_-{}^+\,, \dot\varepsilon_+{}^-\right), & \dot{\tilde\varepsilon}^i_{(4)} &= \left(\dot\varepsilon_-{}^+\,, -\mathrm{i}\dot\varepsilon_+{}^-\right), \end{aligned} \tag{4.15}$$

where the spinors $\dot\varepsilon_\pm{}^\pm$ and $\dot\varepsilon_\pm{}^\mp$ are

$$\begin{aligned} \dot\varepsilon_+{}^+ &= \sqrt{\frac{\ell}{2}}\,\mathrm{e}^{\frac{\mathrm{i}}{2}(\chi+\phi)}\begin{pmatrix}\cosh\frac{\eta}{2}\\ -\sinh\frac{\eta}{2}\end{pmatrix}\otimes\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix}, & \dot\varepsilon_+{}^- &= \sqrt{\frac{\ell}{2}}\,\mathrm{e}^{\frac{\mathrm{i}}{2}(\chi-\phi)}\begin{pmatrix}\cosh\frac{\eta}{2}\\ -\sinh\frac{\eta}{2}\end{pmatrix}\otimes\begin{pmatrix}-\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{pmatrix}, \\ \dot\varepsilon_-{}^+ &= \sqrt{\frac{\ell}{2}}\,\mathrm{e}^{-\frac{\mathrm{i}}{2}(\chi-\phi)}\begin{pmatrix}-\sinh\frac{\eta}{2}\\ \cosh\frac{\eta}{2}\end{pmatrix}\otimes\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix}, & \dot\varepsilon_-{}^- &= \sqrt{\frac{\ell}{2}}\,\mathrm{e}^{-\frac{\mathrm{i}}{2}(\chi+\phi)}\begin{pmatrix}-\sinh\frac{\eta}{2}\\ \cosh\frac{\eta}{2}\end{pmatrix}\otimes\begin{pmatrix}-\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{pmatrix}. \end{aligned} \tag{4.16}$$

JHEP07(2023)218

Note that they are well-defined with respect to the global structure of the geometry (4.7) because they do not depend on the x^5 direction (the spinors above are in fact precisely the four-dimensional Killing spinors on $\mathrm{AdS}_2\times\,\mathrm{S}^2$, as we spell out in appendix E). Note also that, as in the twisted torus frame, we cannot impose any reality conditions on the Euclidean spinors. This is because although they formally satisfy $(\varepsilon^i)^\dagger \mathrm{i}\gamma_5 = \varepsilon_{ij}(\varepsilon^j)^T C$, which is the same symplectic-Majorana condition of the Lorentzian theory (A.5), this condition is not compatible with the local rotation of the Euclidean theory.

4.2 4d Euclidean supergravity and $\mathrm{AdS}_2\times\mathrm{S}^2$ background

The Kaluza-Klein formalism described in the previous subsection naturally connects the 5d supergravity on the $\mathrm{AdS}_3\times\mathrm{S}^2$ background in KK coordinates given in (4.13) to the 4d supergravity on an $\mathrm{AdS}_2\times\mathrm{S}^2$ background. In this subsection, we review the 4d Euclidean conformal supergravity and the $\mathrm{AdS}_2\times\mathrm{S}^2$ background in more detail. In the 4d Euclidean theory, there is a one-parameter redundancy for describing this background that comes from the possible choice of reality condition for the fermions.

4d $\mathcal{N}=2$ supergravity: for the 4d $\mathcal{N}=2$ Euclidean conformal supergravity, we consider the Weyl multiplet, coupled to $N_\mathrm{v}+1$ vector multiplets and one hypermultiplet. One of the vector multiplets and the single hypermultiplet act as the compensators to consistently gauge-fix the dilatations of the off-shell theory. The fields of the Weyl multiplet are

$$\{e_\mu{}^a,\, \psi_a^i,\, A_\mu^D,\, A_\mu^R,\, \mathcal{V}_\mu{}^i{}_j,\, T_{ab}^{\pm},\, \mathcal{D},\, \chi_{4d}^i\}\,, \tag{4.17}$$

corresponding, respectively, to the vielbein, gravitino, dilatations gauge field, the $\mathrm{SO}(1,1)_R$ and $\mathrm{SU}(2)_R$ gauge fields,[13] auxiliary self-dual/anti-self-dual two-form, auxiliary scalar, and the auxiliary fermion. As in the five-dimensional case, we fix $A_\mu^D=0$ using the K-gauge. The fields of the $N_\mathrm{v}+1$ vector multiplets are

$$\{X^\mathcal{I},\, \overline{X}^\mathcal{I},\, A_\mu^\mathcal{I},\, \lambda^{\mathcal{I}\,i},\, \mathcal{Y}^{\mathcal{I}\,ij}\}, \qquad \mathcal{I}=0,\cdots,N_\mathrm{v}\,, \tag{4.18}$$

corresponding to the complex scalar and its conjugate, the U(1) gauge field, the gaugino, and the auxiliary SU(2) triplet. Finally, the hypermultiplet consists of scalars and fermions,

$$\{\mathcal{A}_i{}^\alpha,\, \zeta_{4d}^\alpha\}\,. \tag{4.19}$$

The supersymmetry transformations on the spinor fields ψ_a^i, $\lambda^{\mathcal{I}\,i}$, ζ_{4d}^α are presented in (E.1), following the conventions of [15].

The 4d $\mathcal{N}=2$ supergravity is governed by the prepotential $F(X)$ which is homogeneous of degree 2. Here, we choose the prepotential as [16]

$$F(X) = -\frac{1}{12}\, c_{IJK}\frac{X^I X^J X^K}{X^0}\,, \tag{4.20}$$

(the sum running over $I=1,\ldots N_\mathrm{v}$), such that the vector multiplet sector of the 4d theory matches that of the 5d theory described in the section 2.1, according to the 4d/5d map that we will present shortly in subsection 4.3.

[13] The R-symmetry group of the Euclidean theory is $\mathrm{SU}(2)\times\mathrm{SO}(1,1)$ compared to $\mathrm{SU}(2)\times\mathrm{U}(1)$ in the Lorentzian case.

Reality conditions: note that in the Euclidean theory the fields $X^{\mathcal{I}}$ and $\overline{X}^{\mathcal{I}}$ — and, more generally, fields related by complex conjugation in the Lorentzian theory (e.g. T^{+}_{ab} and T^{-}_{ab}) — are independent in the Euclidean theory. In order to preserve the number of degrees of freedom, we should impose reality conditions in the Euclidean theory. This may be done by imposing an appropriate reality condition on the spinors and using supersymmetry. The minimal spinors in the $\mathcal{N}=2$ four-dimensional Euclidean theory can be chosen to obey the symplectic-Majorana condition. We note that there are actually an infinite number of such consistent conditions which, for any spinor ψ^i, are parametrized by a real number α as

$$(\psi^i)^\dagger \mathrm{e}^{\mathrm{i}\alpha\gamma_5} = \epsilon_{ij}(\psi^i)^T C\,, \qquad \alpha\in\mathbb{R}\,. \tag{4.21}$$

This infinite choice stems from the fact that the chiral and anti-chiral spinors are independent in Euclidean 4d, and the symplectic-Majorana condition for the chiral and anti-chiral spinors can be imposed with relatively different phases. Two natural examples are:

$$\alpha=\pi/2: \quad (\psi^i)^\dagger \mathrm{i}\gamma_5 = \epsilon_{ij}(\psi^j)^T C\,, \qquad \alpha=0: \quad (\psi^i)^\dagger = \epsilon_{ij}(\psi^j)^T C\,. \tag{4.22}$$

A spinor satisfying the general reality condition (4.21) (which we denote by $\psi^i(\alpha)$) is related to spinors satisfying (4.22)

$$\psi^i(\alpha) = \mathrm{e}^{\frac{\mathrm{i}}{2}\gamma_5\left(\alpha-\frac{\pi}{2}\right)}\psi^i(\pi/2) = \mathrm{e}^{\frac{\mathrm{i}}{2}\alpha\gamma_5}\psi^i(0)\,. \tag{4.23}$$

Now, if we impose one such condition on all the spinors of the theory (including the Killing spinors), then the consistency of the supersymmetry transformations under this condition fixes specific reality conditions on the bosonic fields. For the two examples above we have, respectively, the following conditions for the relevant bosonic fields:

$$\begin{aligned} \alpha=\pi/2: \qquad & (T^{\pm}_{ab})^* = T^{\pm}_{ab}\,, \quad (X^{\mathcal{I}})^* = X^{\mathcal{I}}\,, \quad (\overline{X}^{\mathcal{I}})^* = \overline{X}^{\mathcal{I}}\,, \\ \alpha=0: \qquad & (T^{\pm}_{ab})^* = -T^{\pm}_{ab}\,, \quad (X^{\mathcal{I}})^* = -X^{\mathcal{I}}\,, \quad (\overline{X}^{\mathcal{I}})^* = -\overline{X}^{\mathcal{I}}\,. \end{aligned} \tag{4.24}$$

However, note that imposing either reality condition in (4.24) causes the wrong sign for the kinetic terms of the action, making path integral ill-defined. In fact, this is the case for all bosonic reality conditions implied from supersymmetry by (4.21). As was discussed in section 2.1, the resolution is to impose the standard reality condition on the bosonic fluctuations, e.g. $(\delta X^{\mathcal{I}})^* = \delta\overline{X}^{\mathcal{I}}$, so that path integral is well-defined, and to treat the fermion fluctuations ψ^1 and ψ^2 as being independent. For the background, however, the effect of the choice for α still remains: there is a one-parameter family of Killing spinors that satisfy the reality condition (4.21), and the supersymmetric bosonic background has a corresponding dependence on the choice of α as we will shortly see below.

4d $\mathbf{AdS_2\times S^2}$ background: here we present the Euclidean $\mathrm{AdS}_2\times\mathrm{S}^2$ background, including the complete Weyl multiplet and matter multiplets. This solution can be obtained by Wick rotation of the Lorentzian $\mathrm{AdS}_2\times\mathrm{S}^2$ solution, which carries both electric and magnetic charges $\left(q_{\mathcal{I}},\, p^{\mathcal{I}}\right)$. The non-trivial fields are:

$$
\begin{aligned}
& ds_4{}^2 = g_{\mu\nu}\, dx^\mu dx^\nu = \ell^2 \left(d\eta^2 + \sinh^2\eta\, d\chi^2 + d\theta^2 + \sin^2\theta d\phi^2\right)\,, \\
& T^-_{12} = -\mathrm{i}\,\omega\,, \qquad T^+_{12} = -\mathrm{i}\,\overline{\omega}\,, \qquad T^-_{34} = \mathrm{i}\,\omega\,, \qquad T^+_{34} = -\mathrm{i}\,\overline{\omega}\,, \\
& A^{\mathcal{I}} = -\mathrm{i}\, e^{\mathcal{I}}(\cosh\eta - 1)\, d\chi - p^{\mathcal{I}}\cos\theta\, d\phi\,, \\
& X^{\mathcal{I}} = \frac{\omega}{8}(e^{\mathcal{I}} + \mathrm{i}p^{\mathcal{I}})\,, \qquad \overline{X}^{\mathcal{I}} = \frac{\overline{\omega}}{8}(e^{\mathcal{I}} - \mathrm{i}p^{\mathcal{I}})\,, \\
& \mathcal{A}^\alpha_i = a_i{}^\alpha = \text{constant}\,.
\end{aligned}
\tag{4.25}
$$

By the field equation for the auxiliary scalar $\mathcal{D}$, the $a_i{}^\alpha$ are constrained to obey:

$$\Omega_{\alpha\beta}\, \varepsilon^{ij} a_i{}^\alpha a_j{}^\beta = -4\mathrm{i}(F_{\mathcal{I}}\overline{X}^{\mathcal{I}} - \overline{F}_{\mathcal{I}} X^{\mathcal{I}})\,. \tag{4.26}$$

By the attractor equations [57], the electric field $e^{\mathcal{I}}$ is related to the electric charge $q_{\mathcal{I}}$ as

$$4\mathrm{i}\left(\overline{\omega}^{-1}\frac{\partial \overline{F}(\overline{X})}{\partial \overline{X}^{\mathcal{I}}} - \omega^{-1}\frac{\partial F(X)}{\partial X^{\mathcal{I}}}\right) = q_{\mathcal{I}}\,, \tag{4.27}$$

and the two independent complex parameters ω and $\bar{\omega}$ (unlike in the Lorentzian theory, they are not complex conjugate to each other) are related to the length scale of the metric ℓ as

$$\ell^2 = \frac{16}{\omega\overline{\omega}}\,, \tag{4.28}$$

which indeed scales consistently with Weyl weight (-2) and $\mathrm{SO}(1,1)_R$ weight 0. Since the two complex parameters ω and $\bar{\omega}$ carry opposite charges under the $\mathrm{SO}(1,1)_R$ gauge symmetry, we can set their magnitude to be same:

$$|\omega| = |\overline{\omega}| = 4/\ell\,. \tag{4.29}$$

Note that to match our 5d set-up, we uphold the dilatational symmetry, which is manifested here in the form of an arbitrary value for ℓ (one may break the symmetry by fixing ℓ to 1 for instance, as in [1]). The relation (4.28), (4.29) indicates that ω and $\bar{\omega}$ are now formally conjugate to each other so that we can rewrite them using the following parametrization:

$$\omega(\alpha) = \frac{4}{\ell}e^{\mathrm{i}\alpha}\,, \qquad \overline{\omega}(\alpha) = \frac{4}{\ell}e^{-\mathrm{i}\alpha}\,, \qquad \alpha \in \mathbb{R}\,. \tag{4.30}$$

Unlike in the 4d Lorentzian theory, where the phase α is fixed by the $\mathrm{U}(1)_R$ gauge symmetry, in the Euclidean theory it remains as a free parameter. It is, in fact, precisely the parameter that determines the choice of reality condition for the spinors as in (4.21), i.e. the background described in (4.25) with generic α as in (4.30) preserves the supersymmetries generated by Killing spinors obeying the reality condition (4.21). For the case of $\alpha = \pi/2$ the 8 pairs of Killing spinors are presented in (E.22) and the Killing spinors for a generic α can be read off from (4.23). Note that the Killing spinors in (E.22) are exactly same Killing spinors as those of the 5d KK frame given in (4.15).

Now, by comparing the 4d background (4.25) to the 5d KK-frame background (4.13), it is clear that the $\mathrm{AdS}_2 \times \mathrm{S}^2$ metric in the former is the reduction of the $\mathrm{AdS}_3 \times \mathrm{S}^2$ metric in the latter, as mentioned in section 4.1. However, it is not yet clear how the 4d/5d background values of the other fields are related (beyond just the metric), and how off-shell fluctuations are connected. In the next subsection, we will elucidate these points by describing the full off-shell map between the Euclidean 4d and 5d supergravity. Using this map, we will explicitly present how the 4d/5d backgrounds are mapped.

4.3 The off-shell Euclidean 4d/5d lift

In this subsection, we describe the off-shell connection between the 4d Euclidean and 5d Euclidean theory. We present how the $\mathrm{AdS}_2 \times \mathrm{S}^2$ on-shell background in (4.25) maps to the $\mathrm{AdS}_3 \times \mathrm{S}^2$ on-shell background in KK frame (4.13). This involves a choice of the relevant parameters of the 4d background, specifically (e^0, p^0) in (4.25), and depending on the choice of parameter ω and $\bar{\omega}$ (4.25), a proper mapping parameter is determined. We then show how to reach the 5d twisted torus background. We end the section with the steps to lift off-shell localization solutions to the 5d twisted torus.

To obtain the Euclidean 4d/5d connection, we use the Lorentzian 4d/5d relations of [16] and map the two theories to their consistent Euclidean counterparts. Getting the Euclidean 5d theory by the Wick rotation is straightforward, as explained in section 2.1. We follow the conventions of the 4d Euclidean theory in [15]. Equivalently, one can start from the relations between the 5d Lorentzian and 4d Euclidean theories of [23], and Wick rotate the 5d theory. The map obtained in this approach differs from ours only in the way that the conventions of the 4d Euclidean theory of [23] differ from those of the 4d Euclidean theory of [15].[14]

Under Kaluza-Klein reduction of the 5d conformal supergravity to 4d, the vector multiplets $I = 1, \ldots, N_{\mathrm{v}}$ reduce to the corresponding 4d matter vector multiplets $\mathcal{I} = 1, \ldots, N_{\mathrm{v}}$, and the Weyl multiplet reduced to the 4d Weyl multiplet and the additional Kaluza-Klein vector multiplet $\mathcal{I} = 0$.

One can expect that the Kaluza-Klein scalar ϕ associated with the 5d metric (4.6) falls into the scalar in the 4d Kaluza-Klein vector multiplet. However, directly performing this reduction only gives one real scalar degree-of-freedom, while there should be two real degree-of-freedom for the scalars of the vector multiplet. Additionally, the 4d $\mathrm{SO}(1,1)_R$ symmetry factor is not realized in any of the multiplets. To recover the missing scalar d.o.f., an additional field φ is introduced [16, 23] to define the two 4d scalars in the KK vector multiplet as

$$X^0 = -\frac{\mathrm{i}}{2}\mathrm{e}^{-\varphi}\phi\,, \qquad \overline{X}^0 = \frac{\mathrm{i}}{2}\mathrm{e}^{\varphi}\phi\,. \tag{4.31}$$

The field φ transforms locally under $\mathrm{SO}(1,1)_R$ as

$$\varphi \to \varphi + \Lambda^0\,, \tag{4.32}$$

where Λ^0 is real. One can then consistently couple φ to the remaining 4d fields, so that the $\mathrm{SO}(1,1)_R$ of the 4d theory is realized.

[14] The mapping between these conventions is also presented in [15].

We now present the explicit 4d/5d mappings, up to quadratic order in the fermions, keeping the general φ dependence. The 4d Weyl multiplet is related to the 5d Weyl multiplet as:

$$
\begin{aligned}
e_\mu{}^a &= \dot{E}_\mu{}^a\,, &(4.33)\\
\psi_a^i &= e^{-\frac{1}{2}\varphi\gamma_5}\dot{\Psi}_a{}^i\,, &(4.34)\\
A_a^R &= -6\mathrm{i}\dot{T}_{a5} + e_a{}^\mu\partial_\mu\varphi &(4.35)\\
\mathcal{V}_{aj}^i &= \dot{V}_{a\,j}{}^i\,, &(4.36)\\
T_{ab}^{4d\pm} &= e^{\pm\varphi}(24\,\dot{T}_{ab}^{5d} + \mathrm{i}\phi^{-1}\varepsilon_{abcd}F(B)^{cd})^{\pm}\,, &(4.37)\\
\mathcal{D} &= 4\dot{D} + \frac{1}{4}\phi^{-1}e^{a\mu}D_\mu(e_a{}^\nu D_\nu\phi) + \frac{3}{32}\phi^{-2}F(B)^{ab}F(B)_{ab} &(4.38)\\
&\quad + \frac{3}{2}\dot{T}_{\dot{A}\dot{B}}\dot{T}^{\dot{A}\dot{B}} + \frac{1}{4}\phi^2\dot{V}_{x^5 i}{}^j\dot{V}_{x^5 j}{}^i\,, \\
\chi_{4d}^i &= 8\dot{\chi}^i + \frac{1}{48}\gamma^{ab}F(B)_{ab}\dot{\Psi}_{x^5}^i - \frac{3\mathrm{i}}{4}\phi\dot{T}_{ab}\gamma^5\gamma^{ab}\dot{\Psi}_{x^5}^i &(4.39)\\
&\quad + \frac{1}{4}\phi^{-1}\gamma_5\not{\mathcal{D}}(\phi^2\dot{\Psi}_{x^5}^i) - \frac{1}{2}\phi^2 V_{x^5 j}{}^i\dot{\Psi}_{x^5}^j - \frac{9}{4}\mathrm{i}\phi\dot{T}_{a5}\gamma^a\dot{\Psi}_{x^5}^i\,,
\end{aligned}
$$

where ε_{abcd} is the four-dimensional Levi-Civita symbol, and the 4d conformally invariant D'Alembertian is $e^{a\mu}D_\mu(e_a{}^\nu D_\nu\phi) = (\mathcal{D}^a\mathcal{D}_a + \frac{1}{6}R)\phi$, where R is the 4d Ricci scalar and $\mathcal{D}_a$ is the 4d Lorentz and R-symmetry covariant derivative. The 4d supersymmetry parameters are given in terms of the 5d supersymmetry parameters and 5d Weyl multiplet fields as

$$
\begin{aligned}
\varepsilon_{4d}^i &= \mathrm{e}^{-\frac{1}{2}\varphi\gamma_5}\dot{\varepsilon}^i\,, &(4.40)\\
\eta_{4d}^i &= -\mathrm{i}\gamma_5\mathrm{e}^{\frac{1}{2}\varphi\gamma_5}\left(\dot{\eta}^i - 2\dot{T}_{a5}\gamma^a\gamma^5\dot{\varepsilon}^i + \frac{\mathrm{i}}{8}\phi^{-1}\gamma_5(F(B)_{ab} - 4\mathrm{i}\phi\dot{T}_{ab}\gamma_5)\gamma^{ab}\dot{\varepsilon}^i\right)\,. &(4.41)
\end{aligned}
$$

Moving on to the vector multiplets, the 4d KK vector multiplet fields in terms of the 5d Weyl multiplet are:

$$
\begin{aligned}
X^0 &= -\frac{\mathrm{i}}{2}e^{-\varphi}\phi\,, \qquad \overline{X}^0 = \frac{\mathrm{i}}{2}e^{\varphi}\phi\,, &(4.42)\\
A_a^0 &= e_a{}^\mu B_\mu\,, &(4.43)\\
\lambda^{0\,i} &= e^{-\frac{1}{2}\varphi\gamma_5}\dot{\Psi}_5{}^i\phi\,, &(4.44)\\
\mathcal{Y}^{0\,i}{}_j &= \phi\,\dot{V}_{5\,j}{}^i\,, &(4.45)
\end{aligned}
$$

and the 4d matter vector multiplet fields in terms of the 5d vector multiplet fields are:

$$
\begin{aligned}
X^I &= \frac{1}{2}e^{-\varphi}(\sigma^I + \mathrm{i}\dot{W}_5^I)\,, \qquad \overline{X}^I = \frac{1}{2}e^{\varphi}(\sigma^I - \mathrm{i}\dot{W}_5^I)\,, &(4.46)\\
A_a^I &= \dot{W}_a^I\,, &(4.47)\\
\lambda^{I\,i} &= e^{-\frac{1}{2}\varphi\gamma_5}\left(\dot{\Omega}^{I\,i} - \dot{W}_5^I\dot{\Psi}_5{}^i\right)\,, &(4.48)\\
\mathcal{Y}^{I\,i}{}_j &= -2\left(Y^{I\,i}{}_j + \frac{1}{2}\dot{W}_5^I\dot{V}_{5\,j}{}^i\right)\,. &(4.49)
\end{aligned}
$$

Finally, the 4d hypermultiplet in terms of the 5d hypermultiplet is

$$\mathcal{A}_i{}^\alpha = \phi^{-1/2} \dot{A}_i{}^\alpha \,. \tag{4.50}$$

Using the above maps, the 4d supersymmetry transformation is obtained from the 5d supersymmetry transformation together with a 5d local rotation,

$$\delta^{4d} = \delta^{5d} + \delta_M(\varepsilon) \,, \qquad \varepsilon_{5a} = -\varepsilon_{a5} = \bar{\dot{\varepsilon}}_i \gamma_a \Psi_5^i \,, \tag{4.51}$$

where the rotation parameter ε_{AB} is chosen to fix the gauge $\dot{E}_{x^5}{}^a = \dot{E}_5{}^\mu = 0$. We also need the supersymmetry transformation rule of φ,

$$\delta^{5d}\varphi = \bar{\dot{\varepsilon}}_i \dot{\Psi}_5^i \,. \tag{4.52}$$

For the purpose of lifting the 4d configuration to 5d, we use the inverse map, namely the 5d fields in terms of the 4d fields. The 5d Weyl multiplet fields are given in terms of the 4d Weyl multiplet and 4d KK multiplet as:

$$\dot{E}_\mu{}^a = e_\mu{}^a \,, \qquad \dot{E}_\mu{}^5 = \phi^{-1} B_\mu \,, \qquad \dot{E}_{x^5}{}^5 = \phi^{-1} \,, \tag{4.53}$$

$$\dot{\Psi}_a^i = e^{\frac{1}{2}\varphi\gamma_5} \psi_a^i \,, \qquad \dot{\Psi}_5{}^i = \phi^{-1} e^{\frac{1}{2}\varphi\gamma_5} \lambda^{0\,i} \,, \tag{4.54}$$

$$\dot{T}_{ab} = \frac{1}{24}\left(e^{-\varphi} T_{ab}^+ + e^{\varphi} T_{ab}^- - \mathrm{i}\phi^{-1} \varepsilon_{abcd} F(B)^{cd} \right) , \tag{4.55}$$

$$\dot{T}_{a5} = \frac{\mathrm{i}}{6}\left(A_a^R - e_a{}^\mu \partial_\mu \varphi \right) , \tag{4.56}$$

$$\dot{V}_{a\,j}{}^i = \mathcal{V}_a{}^i{}_j \,, \qquad \dot{V}_{5\,j}{}^i = \phi^{-1} \mathcal{Y}^{0\,i}{}_j \,, \tag{4.57}$$

$$\begin{aligned} \dot{D} = \frac{1}{4}\Big(\mathcal{D} &- \frac{1}{4}\phi^{-1} e^{a\mu} D_\mu (e_a{}^\nu D_\nu \phi) - \frac{3}{32}\phi^{-2} F(B)^{ab} F(B)_{ab} \\ &- \frac{3}{2}\dot{T}_{\dot{A}\dot{B}} \dot{T}^{\dot{A}\dot{B}} - \frac{1}{4}\phi^2 \dot{V}_{x^5 i}{}^j \dot{V}_{x^5 j}{}^i \Big) , \end{aligned} \tag{4.58}$$

where

$$\phi = 2\mathrm{i} e^{\varphi} X^0 = -2\mathrm{i} e^{-\varphi} \overline{X}^0 \,, \qquad B_\mu = A_\mu^0 \,. \tag{4.59}$$

The 5d supersymmetry parameters are:

$$\dot{\varepsilon}^i = \mathrm{e}^{\frac{1}{2}\varphi\gamma_5} \varepsilon_{4d}^i \,, \tag{4.60}$$

$$\dot{\eta}^i = \gamma_5 \left(\mathrm{i} e^{-\frac{1}{2}\varphi\gamma_5} \eta_{4d}^i + 2\dot{T}_{a5} \gamma^a \dot{\varepsilon}^i - \frac{\mathrm{i}}{8}\phi^{-1} (F(B)_{ab} - 4\mathrm{i}\phi \dot{T}_{ab}\gamma_5) \gamma^{ab} \dot{\varepsilon}^i \right) . \tag{4.61}$$

The 5d vector multiplet is given in term of the 4d vector multiplet as:

$$\dot{\sigma}^I = e^{\varphi} X^I + e^{-\varphi} \overline{X}^I \,, \tag{4.62}$$

$$\dot{W}_a^I = A_a^I \,, \qquad \dot{W}_5^I = -\mathrm{i}\left(e^{\varphi} X^I - e^{-\varphi} \overline{X}^I \right) , \tag{4.63}$$

$$\dot{\Omega}^{I\,i} = e^{\frac{1}{2}\varphi\gamma_5} \lambda^{i\;I} + \dot{W}_5^I \dot{\Psi}_5{}^i , \tag{4.64}$$

$$\dot{Y}^{I\,i}{}_j = -\frac{1}{2}\mathcal{Y}^{I\,i}{}_j - \frac{1}{2}\dot{W}_5^I \dot{V}_{5\,j}{}^i \,. \tag{4.65}$$

The 5d hyper scalar given in terms of the 4d hypermultiplet is

$$\dot{A}_i{}^\alpha = \phi^{1/2} \mathcal{A}_i{}^\alpha \,. \tag{4.66}$$

JHEP07(2023)218

Mapping 4d/5d classical backgrounds: by the above 4d/5d map, the relation between the 4d $\mathrm{AdS}_2 \times \mathrm{S}^2$ backgrounds (4.25) and 5d $\mathrm{AdS}_3 \times \mathrm{S}^2$ background in (4.13) in KK coordinates becomes more manifest. One important subtlety is about the choice of φ in (4.59). In the case of the Lorentzian 4d/5d connection, φ is just a $\mathrm{U}(1)_R$ gauge parameter that fixes the gauge-redundant phase of X^0 and $\overline{X}^0$, making the ϕ automatically real. However, in the Euclidean case, the 4d theory has an $\mathrm{SO}(1,1)_R$ gauge symmetry instead of $\mathrm{U}(1)_R$, whereas the background values for X^0 and $\overline{X}^0$ have a relative phase coming from the choice of the parameter ω and $\bar{\omega}$ and value of the charge e^0 and p^0. Therefore, unlike in the Lorentzian case, the value of φ is not a 'gauge fixing' to kill the phase of X^0 and $\overline{X}^0$, but rather a 'choice' to cancel the phase of X^0 and $\overline{X}^0$. (By the $\mathrm{SO}(1,1)_R$ gauge redundancy and by the rule (4.32), we shift the φ to set the magnitude of X^0 and $\overline{X}^0$ to be same.)

Recalling the background value of X^0 and $\overline{X}^0$ as given in (4.25), where the ω and $\bar{\omega}$ are parametrized by α as in (4.30), the value of the mapping parameter φ is determined to be

$$\varphi^{\pm}(\alpha\,, e^0\,, p^0) = -\mathrm{i}\alpha \pm \mathrm{i}\frac{\pi}{2} - \mathrm{i}\arctan\left(\frac{p^0}{e^0}\right)\,, \tag{4.67}$$

by the condition that ϕ be real. There remains an ambiguity of $\pm\pi/2$ that is related to an overall sign choice for ϕ. We now consider specific examples for two distinct choices of (e^0, p^0), keeping the choice of α to be generic. These are:

$$\begin{aligned}
&(1) \qquad (e^0,\, p^0) = (e^0,\, 0)\,, \qquad \varphi_1^{\pm}(\alpha) = -\mathrm{i}\alpha \pm \mathrm{i}\pi/2 \qquad\quad\;\; \Rightarrow \quad \phi = \mp\frac{e^0}{\ell}\,,\\
&(2) \qquad (e^0,\, p^0) = (0,\, p^0)\,, \qquad \varphi_2^{\pm}(\alpha) = -\mathrm{i}(\alpha+\pi/2) \pm \mathrm{i}\pi/2 \quad \Rightarrow \quad \phi = \mp\frac{p^0}{\ell}\,.
\end{aligned} \tag{4.68}$$

Here we see that, by the mapping parameter $\varphi^{\pm}$, the background value of the lifted 5d field ϕ is indeed real, but there is dependence on the choice $\pm$. We note that for both cases in (4.68) and, more generally, with any choice (4.67), all the lifted 5d fields are independent of the choice of phase $\omega \equiv \exp(\mathrm{i}\alpha)$ in the 4d background (4.25).

The resulting 5d background fields are listed in table 2. The 4d configuration with $(e^0,\, p^0,\, \varphi) = (e^0,\, 0,\, \varphi_1^{\pm})$ as in (1) lifts to an $\mathrm{AdS}_3 \times \mathrm{S}^2$ background, while the one with $(e^0,\, p^0,\, \varphi) = (0,\, p^0,\, \varphi_2^{\pm})$ as in (2) lifts to an $\mathrm{AdS}_2 \times \mathrm{S}^3$ background. For the latter case, the localization solutions were studied in [19]. In both cases, the choice of the sign in $\varphi^{\pm}$ gives the opposite sign for the background values of ϕ, $\dot{T}_{\dot{A}\dot{B}}, \dot{\sigma}$ and hyper norm $\dot{\chi}$. At the level of the Killing spinor equation (that we review in appendix C), choosing either sign gives a set of Killing spinors corresponding, respectively, to the right- or left-moving supercharges in terms of the 2d chiral $\mathcal{N} = 4$ super algebra.

Now, for our problem, the full specification of parameters to lift the Euclidean $\mathrm{AdS}_2 \times \mathrm{S}^2$ backgrounds (4.25) to the 5d KK frame (4.13) is

$$(e^0,\, p^0,\, \varphi) = (-1,\, 0,\, \varphi_1^{+})\,, \tag{4.69}$$

with identification $e^I = \mu^I$ and φ_1^{+} given in (4.68). To relate Euclidean $\mathrm{AdS}_2 \times \mathrm{S}^2$ to the twisted torus (3.23), this lift is then followed by the following steps: taking the lifted 5d KK

	$\varphi = \varphi_1^{\pm}$	$\varphi = \varphi_2^{\pm}$	
$(e^0,\, p^0) = (e^0,\, 0)$	$ds^2 = \mathrm{AdS}_3 \times \mathrm{S}^2$ $\dot{T}_{34} = \mp 1/(4\ell)$ $\dot{\sigma}^I = \mp p^I/\ell,$ $\dot{F}^I_{34} = p^I/\ell^2$ $\dot{A}_{1,2}{}^{1,2} = \sqrt{\pm \frac{p^3}{3\ell^3}}$ $S_{\text{bulk}} = \frac{p^3}{12 e^0}$	$ds^2 = \mathrm{AdS}_2 \times \mathrm{S}^3$ $\dot{T}_{12} = \mp \mathrm{i}/(4\ell)$ $\dot{\sigma}^I = \pm e^I/\ell,$ $\dot{F}^I_{12} = -\mathrm{i} e^I/\ell^2$ $\dot{A}_{1,2}{}^{1,2} = \sqrt{\mp \frac{e^3}{3\ell^3}}$ $S_{\text{bulk}} = \frac{e^3}{12 p^0}$	$(e^0,\, p^0) = (0,\, p^0)$

Table 2. The non-trivial 5d fields obtained by lifting the 4d backgrounds (4.25) with (ω, ϖ) as given in (4.30) and with different choices for (e^0, p^0) and φ. For the choice of (e^0, p^0) on the left and right panel, the 4d hyper scalar that is lifted is determined by the $\mathcal{D}$-field equation constraint (4.26) as ${a_1}^1 = {a_2}^2 = 1/\ell\sqrt{-p^3/3e^0}$ and ${a_1}^1 = {a_2}^2 = 1/\ell\sqrt{e^3/3p^0}$ respectively. We also include the value for the finite piece of the bulk action (2.12). The field configurations on the right entry are solutions corresponding to the near-horizon of the supersymmetric Euclidean 5d black hole. The field configurations on the left entry are the Euclidean $\mathrm{AdS}_3 \times \mathrm{S}^2$ solutions.

frame background (4.13) with Q- Killing spinors (4.15), one applies the local coordinate transformations (4.10), (3.21), the spinor Lorentz rotation in (4.4) with (4.12), and finally one imposes the periodicity conditions (3.22) with Ω given in (3.27). In this procedure, only four of the eight Q- Killing spinors mapped from (4.15) are well-defined on the twisted torus, as expected.

Mapping 4d localization solution to the 5d twisted torus frame: having identified the relevant 4d background, together with the correct mapping parameter (4.69) that relates it to the 5d twisted torus background (3.23), we now want to map the *off-shell* localization solution of 4d supergravity on that background to the 5d localization solution around the twisted torus background. The strategy for this mapping follows the same steps as the mapping of the backgrounds presented above. Here, we assume that phase factors in the quantum fluctuation of the scalars X^0 and $\overline{X}^0$ are appropriately cancelled by a fluctuating value of φ around its value in (4.69), such that it makes the quantum fluctuation of the 5d field ϕ real.[15] It will turn out that for our off-shell localization solution, we can use the same value of φ as was chosen in (4.69).

Here, we summarize the steps as follows:

1. Start with the 4d localization manifold whose background is the Euclidean $\mathrm{AdS}_2 \times \mathrm{S}^2$ solution (4.25) with $(e^0, p^0) = (-1, 0)$. Since the result does not depend on the choice of α in (4.30), without loss of generality we take $\alpha = \pi/2$ for convenience.

2. Apply the 4d/5d lift with the mapping parameter $\varphi = \varphi_1^+(\pi/2) = 0$ to obtain 5d localization solutions in the KK frame (4.13) of Euclidean $\mathrm{AdS}_3 \times \mathrm{S}^2$.

[15] Since we choose the reality condition for the fluctuation of X^0 and $\overline{X}^0$ to be complex conjugate to each other, as explained after (4.24), and since this condition is the same as the condition in the Lorentzian theory, it appears there may be some $\mathrm{U}(1)_R$ gauge symmetry hidden in the fluctuating field, and it may justify our assumption.

3. Transform these localization solutions to the twisted torus frame by applying the local coordinate maps (4.10), (3.21), the spinor Lorentz rotation in (4.4) with (4.12), and finally imposing the periodicity conditions (3.20) with Ω given in (3.27).

Note that a consistent lift to the twisted torus requires that the lifted solutions respect the periodicities (3.20). As an example of an inconsistent lift, consider a scalar field fluctuation on $\mathrm{AdS}_2 \times \mathrm{S}^2$ with non-zero momentum on χ, which therefore has 2π-periodicity in χ. Recalling that $\chi = \psi - \mathrm{i}t_{\mathrm{E}}$, we see that such a mode, lifted to 5d, does not respect the second periodicity condition in (3.20). As we discuss in the next section, the fields in the four-dimensional localization manifold depend only the radial coordinate $\eta = 2\rho$ and therefore lift consistently to the 5d twisted torus.

5 The lift of localization solutions on $\mathrm{AdS}_2 \times \mathrm{S}^2$ to $\mathbb{H}^3/\mathbb{Z} \times \mathrm{S}^2$

In this section, we apply the lifting procedure to obtain localization solutions around the supersymmetric $\mathbb{H}^3/\mathbb{Z} \times \mathrm{S}^2$ background. We find a set of solutions to the BPS equations parametrized by $N_{\mathrm{v}}+1$ real coordinates $C^{\mathcal{I}}$, $\mathcal{I} = 0, \ldots, N_{\mathrm{v}}$. These coordinates are inherited from the 4d localization manifold, where each $C^{\mathcal{I}}$ parametrizes the off-shell solution for the $\mathcal{I}^{\text{th}}$ vector multiplet. In the 4d $\mathrm{AdS}_2 \times \mathrm{S}^2$ problem, the boundary conditions fix all the fields to their attractor values at infinity. The localization solution consists of the scalar fields $X^{\mathcal{I}}$ going off-shell in the interior, with a radially-decaying shape that is fixed by supersymmetry. The parameter $C^{\mathcal{I}}$ labels the size of deviation at the origin. In 5d, the C^I, $I = 1, \cdots, N_{\mathrm{v}}$ parametrize the size of the off-shell solution in the vector multiplet, and C^0 parametrizes a certain excitation of the Weyl multiplet. Here, we have an $\mathrm{AdS}_3 \times \mathrm{S}^2$ background, where one leg of the gauge field ($W_{z'}$) is fixed at infinity to its on-shell value while the other ($W_{\overline{z}'}$) is free to fluctuate, as we described in section 3.3. The parameter C^I labels the deviation of both $W^I_{z'}$ and $W^I_{\overline{z}'}$ from their on-shell value at the origin as well as the boundary fluctuation of $W^I_{\overline{z}'}$. The precise solutions are presented in (5.6)–(5.9) for the Weyl multiplet, and in (5.13)–(5.16) for the vector multiplets. The hypermultiplet also fluctuates, and the solution is given in (5.17).

4d localization solutions: the most general solution in 4d around the $\mathrm{AdS}_2 \times \mathrm{S}^2$ background is parametrized by one real parameter in each vector multiplet and one real parameter in the Weyl multiplet, before fixing the gauge for local scale transformations [44]. The gauge can be chosen so that there is no off-shell fluctuations in the Weyl multiplet [1]. The off-shell solution in the vector multiplets takes the following form:

$$X^{\mathcal{I}} = \frac{\mathrm{i}}{2\ell}\left(e^{\mathcal{I}} + \mathrm{i}\,p^{\mathcal{I}} + \frac{C^{\mathcal{I}}}{\cosh\eta}\right), \quad \overline{X}^{\mathcal{I}} = -\frac{i}{2\ell}\left(e^{\mathcal{I}} - \mathrm{i}\,p^{\mathcal{I}} + \frac{C^{\mathcal{I}}}{\cosh\eta}\right), \tag{5.1}$$

$$A^{\mathcal{I}} = -\mathrm{i}\,e^{\mathcal{I}}(\cosh\eta - 1)\,d\chi - p^{\mathcal{I}}\cos\theta\,d\phi\,, \tag{5.2}$$

$$\mathcal{Y}^{\mathcal{I}\,1}_{1} = \mathcal{Y}^{\mathcal{I}}_{12} = \frac{-C^{\mathcal{I}}}{\ell^2\cosh^2\eta}\,, \tag{5.3}$$

where we use $(\omega(\pi/2), \overline{\omega}(\pi/2)) = (4\mathrm{i}/\ell, -4\mathrm{i}/\ell)$. The $C^{\mathcal{I}}$ parametrize the off-shell fluctuations around the background (4.25).

JHEP07(2023)218

Lift to the Weyl multiplet: for the lift to the Weyl multiplet, the relevant fields of the 4d localization solution (5.1) are those of the KK vector multiplet $\mathcal{I}=0$. Using (4.59), we first obtain the off-shell values for the KK scalar and one-form:

$$\phi = \frac{1}{\ell}\left(1 - \frac{C^0}{\cosh\eta}\right), \qquad B_\chi = \mathrm{i}(\cosh\eta - 1)\,. \tag{5.4}$$

It is useful to define the function

$$\phi(x) \coloneqq 1 - \frac{C^0}{\cosh x}\,. \tag{5.5}$$

Now, using the lifting equations (4.53)–(4.58) with $(e^0,\, p^0) = (-1,\, 0)$ and $\varphi = 0$, we obtain the full Weyl multiplet configuration in the KK frame. After applying the coordinate maps (4.10) and (3.21) to the twisted torus frame, the non-trival fields are:

$$E_{M'}{}^{A} = \ell\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & \sinh\rho\left(1+\frac{1-C^0}{\phi(2\rho)}\right) & 0 & 0 & \mathrm{i}\cosh\rho\left(1-\frac{1-C^0}{\phi(2\rho)}\right) \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \sin\theta & 0 \\ 0 & -\mathrm{i}\sinh\rho\left(1-\frac{1+C^0}{\phi(2\rho)}\right) & 0 & \mathrm{i}\Omega\sin\theta & \cosh\rho\left(1+\frac{1+C^0}{\phi(2\rho)}\right) \end{pmatrix}, \tag{5.6}$$

$$T_{\theta\phi'} = \ell\sin\theta\frac{1}{12}\left(\frac{1}{\phi(2\rho)} - 4\right), \qquad T_{\theta t_{\mathrm{E}}'} = \mathrm{i}\ell\sin\theta\frac{\Omega}{12}\left(\frac{1}{\phi(2\rho)} - 4\right), \tag{5.7}$$

$$V_\psi = \mathrm{i}\left(1 - \frac{1}{\cosh 2\rho}\right)\frac{\phi(2\rho)-1}{\phi^2(2\rho)}\boldsymbol{\tau}_3\,, \qquad V_{t_{\mathrm{E}}'} = \left(1 + \frac{1}{\cosh 2\rho}\right)\frac{\phi(2\rho)-1}{\phi^2(2\rho)}\boldsymbol{\tau}_3\,, \tag{5.8}$$

$$D = \frac{1}{8\ell^2\phi(2\rho)^2}\left((1-\phi(2\rho))\frac{\sinh^2 2\rho}{\cosh^2 2\rho} - \frac{2}{3}(1-\phi(2\rho))^2\right). \tag{5.9}$$

The line-element corresponding to the vielbein (5.6) is:

$$\begin{aligned} ds^2 &= 4\ell^2 d\rho^2 + 4\ell^2\left(\sinh^2\rho - \sinh^4\rho\left(\frac{1}{\phi^2(2\rho)} - 1\right)\right)d\psi^2 \\ &\quad + \ell^2\left(d\theta^2 + \sin^2\theta\,(d\phi' + \mathrm{i}\Omega dt_{\mathrm{E}}')^2\right) + 2\mathrm{i}\ell^2\sinh^2 2\rho\left(\frac{1}{\phi^2(2\rho)} - 1\right)d\psi dt_{\mathrm{E}}' \\ &\quad + 4\ell^2\left(\cosh^2\rho + \cosh^4\rho\left(\frac{1}{\phi^2(2\rho)} - 1\right)\right)dt_{\mathrm{E}}'^{\,2}\,. \end{aligned} \tag{5.10}$$

Here, recall from section 3.2 that $\Omega = 1 + \mathrm{i}\tau_1/\tau_2$ in the twisted torus frame.

It remains to apply the lift to the Q- and S-Killing spinors. In principle, off-shell fluctuations in the bosonic fields of the Weyl multiplet may induce off-shell fluctuations in the 5d Killing spinors such that the BPS equations of the multiplet remain solved. Note however that the 4d Weyl multiplet in the 4d localization solution does not fluctuate, and so the 4d Q- and S- Killing spinors that we lift are just those of the 4d background, namely the eight spinors $\varepsilon^i_{4d}(\pi/2)$, given explicitly in (E.22), and $\eta^i_{4d}(\pi/2) = 0$ (recall we have fixed $\alpha = \pi/2$). Further note that the lifting equation (4.60) for the 5d Q- spinors only involves the 4d Q- spinors (which are on-shell). We conclude that the lift of the Q-spinors is unchanged from the on-shell case, i.e. we obtain, in the twisted torus frame, the four well-defined on-shell Q-spinors $\varepsilon_{(a)}$, $a = 1, 2, 3, 4$, as given in (3.28). In contrast, the lifting equation (4.61) of the S- spinors η_{4d} involves bosonic 5d fields which do fluctuate. The 5d S- spinors, which are zero on-shell, therefore acquire a non-zero value off-shell. In the twisted frame, we obtain four well-defined S- spinors $\eta_{(a)}$, associated with the four Q-spinors $\varepsilon_{(a)}$. The one associated to the localization supercharge $\varepsilon_{(1)}$ has value

$$\eta^1_{(1)} = -\frac{\frac{C^0}{\cosh(2\rho)}\, \mathrm{e}^{\frac{\mathrm{i}}{2}(\psi+\phi'+\mathrm{i}(\Omega-1)t'_{\mathrm{E}})}}{3\sqrt{2}\ell\phi(2\rho)} \begin{pmatrix} \cos\frac{\theta}{2}\cosh\frac{\rho}{2} \\ -\sin\frac{\theta}{2}\cosh\frac{\rho}{2} \\ -\cos\frac{\theta}{2}\sinh\frac{\rho}{2} \\ \sin\frac{\theta}{2}\sinh\frac{\rho}{2} \end{pmatrix}, \tag{5.11}$$

$$\eta^2_{(1)} = -\frac{\mathrm{i}\frac{C^0}{\cosh(2\rho)}\, \mathrm{e}^{-\frac{\mathrm{i}}{2}(\psi+\phi'+\mathrm{i}(\Omega-1)t'_{\mathrm{E}})}}{3\sqrt{2}\ell\,\phi(2\rho)} \begin{pmatrix} \sin\frac{\theta}{2}\sinh\frac{\rho}{2} \\ \cos\frac{\theta}{2}\sinh\frac{\rho}{2} \\ -\sin\frac{\theta}{2}\cosh\frac{\rho}{2} \\ -\cos\frac{\theta}{2}\cosh\frac{\rho}{2} \end{pmatrix}. \tag{5.12}$$

Lift of the vector multiplet: the relevant 4d fields are those of (5.1) with $\mathcal{I} = I$. Using the lifting equations (4.62)–(4.65) followed by the coordinate transformations (4.10) and (3.21), we obtain the following non-trivial fields of the vector multiplet configuration in the twisted torus frame:

$$\sigma^I = -\frac{p^I}{\ell}, \qquad W^I_{\phi'} = -p^I\cos\theta\,, \tag{5.13}$$

$$W^I_{\psi} = \frac{2\mathrm{i}\left(C^I/\mu^I + C^0\right)\frac{\sinh^2(\rho)}{\cosh 2\rho}}{\phi(2\rho)}\mu^I\,, \tag{5.14}$$

$$W^I_{t'_{\mathrm{E}}} = -\mathrm{i}p^I\Omega\cos\theta + \frac{\frac{C^I/\mu^I - C^0}{\cosh 2\rho} + C^I/\mu^I + C^0 + 2}{\phi(2\rho)}\mu^I\,, \tag{5.15}$$

$$Y^I_{12} = \frac{1}{2\ell^2\phi(2\rho)}\frac{C^I/\mu^I + C^0}{\cosh^2 2\rho}\mu^I\,. \tag{5.16}$$

Lift of the hypermultiplet: finally, the lift for the hypermultiplet (4.66) gives the following non-trivial components for the off-shell hyper scalar:

$${A_1}^1 = {A_2}^2 = \left(\frac{\phi(2\rho)}{\ell}\right)^{1/2}\sqrt{\frac{p^3}{3\ell^3}}\,. \tag{5.17}$$

To summarize, the field configuration of the Weyl multiplet (5.6)–(5.9), the vector multiplet (5.13)–(5.16), and the hypermultiplet (5.17) are the 5d localization solutions. These configurations are off-shell fixed-points of the variations generated by the supercharge $\mathcal{Q}$ given in (4.1), around the supersymmetric $\mathbb{H}^3/\mathbb{Z}\times\mathrm{S}^2$ given in (3.23).

Index	Range	Description
$M,\, N,\, \cdots$	$(\rho,\, \psi,\, \theta,\, \phi,\, t_{\rm E})$	5d cylinder coordinates
$M',\, N',\, \cdots$	$(\rho,\, \psi,\, \theta,\, \phi',\, t'_{\rm E})$	5d twisted torus coordinates
$A,\, B,\, \cdots$	$(\hat\rho,\, \hat\psi,\, \hat\theta,\, \hat\phi,\, \hat{t_{\rm E}})$	tangent frame for cylinder and twisted torus coordinates
$\dot M,\, \dot N,\, \cdots$	$(\eta,\, \chi,\, \theta,\, \phi,\, x^5)$	5d Kaluza-Klein coordinates
$\dot A,\, \dot B,\, \cdots$	$(1,\, 2,\, 3,\, 4,\, 5)$	tangent frame for Kaluza-Klein coordinates
$\mu,\, \nu,\, \cdots$	$(\eta,\, \chi,\, \theta,\, \phi)$	Euclidean $\mathrm{AdS}_2 \times \mathrm{S}^2$ coordinates
$a,\, b,\, \cdots$	$(1,2,3,4)$	tangent frame for Euclidean $\mathrm{AdS}_2 \times \mathrm{S}^2$ coordinates
$\alpha,\, \beta,\, \cdots$	$(z, \bar z)$	thermal AdS_3 boundary coordinates
$i,\, j,\, \cdots$	$(1,\, 2)$ or $(+,\, -)$	Fundamental SU(2)

Table 3. Summary of index notation.

Acknowledgments

We would like to thank Satoshi Nawata and Valentin Reys for useful discussions and comments. This work is supported by the ERC Consolidator Grant N. 681908, “Quantum black holes: A microscopic window into the microstructure of gravity”, and by the STFC grant ST/P000258/1. A.C. is supported by the STFC studentship ST/S505468/1. IJ is supported by an appointment to the JRG Program at the APCTP through the Science and Technology Promotion Fund and Lottery Fund of the Korean Government, by the Korean Local Governments — Gyeongsangbuk-do Province and Pohang City, and by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2021R1F1A1048531).

A Notations and conventions

We summarize the various index notations in table 3.

Spinors and gamma matrices: we denote a basis for the d-dimensional Clifford algebra as

$$\left\{\Gamma = \mathbb{I},\, \gamma^{A_1},\, \gamma^{A_1A_2}, \cdots \gamma^{A_1A_2\cdots A_d}\right\}\,, \tag{A.1}$$

where:

$$\gamma^{A_1\cdots A_k} = \gamma^{[A_1}\cdots\gamma^{A_k]}\,. \tag{A.2}$$

In five dimension with Lorentzian signature, a consistent choice of gamma matrix satisfies the following relations:

$$\begin{aligned}
&\gamma_A^\dagger = -A\gamma_A A^{-1}\,, && A = \gamma_0\,, && A^\dagger = A^{-1} = -\gamma_0\,, && \\
&\gamma_A^T = \mathcal{C}\gamma_A\mathcal{C}^{-1}\,, && \mathcal{C}^T = -\mathcal{C}\,, && \mathcal{C}^\dagger = \mathcal{C}^{-1}\,, && \\
&\gamma_A^* = -B\gamma_A B^{-1}\,, && B^T = \mathcal{C}A^{-1}\,, && B^\dagger = B^{-1}\,, && B^*B = -1\,.
\end{aligned} \tag{A.3}$$

This is followed by the property, regarding the charge conjugation matrix $\mathcal{C}$,

$$(\mathcal{C}\Gamma^{(r)})^T = -(-)^{r(r-1)/2}\mathcal{C}\Gamma^{(r)} \,, \tag{A.4}$$

where $\Gamma^{(r)}$ is a matrix of the set (A.1) with rank r. Due to the property of the charge conjugation matrix, we can use the spinor representation satisfying the symplectic-Majorana condition

$$\bar{\psi}_i = (\psi^i)^\dagger \gamma_0 \,, \tag{A.5}$$

where i is an $\mathrm{SU}(2)_R$ index, and where $\bar{\psi}_i$ is the symplectic-Majorana conjugate of ψ^i, defined as

$$\bar{\psi}_i \coloneqq \varepsilon_{ij}(\psi^j)^T \mathcal{C} \,, \tag{A.6}$$

with ε_{ij} being the SU(2) symplectic metric $\varepsilon_{12} = -\varepsilon_{21} = 1$.

The five-dimensional Euclidean case is obtained by the Wick rotation of time direction x^0, using the redefinition: $x^0 = -\mathrm{i}x^5$. This consistently redefines the 0^{th} gamma matrix as the 5-th directional one as $\gamma_0 = \mathrm{i}\gamma_5$. The relations on the Lorentizan gamma matrices (A.3) then become, for the Euclidean case:

$$\begin{aligned} &\gamma_A^\dagger = \gamma_A \\ &\gamma_A^* = \gamma_A^T = \mathcal{C}\gamma_A\mathcal{C}^{-1} \,, \qquad \mathcal{C}^\dagger = \mathcal{C}^{-1} \,, \qquad \mathcal{C}^T = -\mathcal{C} \;\Leftrightarrow\; \mathcal{C}^*\mathcal{C} = -1 \,, \end{aligned} \tag{A.7}$$

with charge-conjugation matrix property:

$$(\mathcal{C}\Gamma^{(r)})^T = -(-)^{r(r-1)/2}\mathcal{C}\Gamma^{(r)} \,. \tag{A.8}$$

In the main text, we often consider Lorentz scalars of the type

$$\bar{\lambda}_i \, \Gamma^{(r)} \epsilon^j \,. \tag{A.9}$$

For two Grassman even spinors ϵ^i, λ^j, the property (A.8) leads to the following Majorana-flip relations:

$$\bar{\lambda}_i \, \Gamma^{(r)} \epsilon^j = (-)^{r(r-1)/2} \left(\delta_i^j \, \bar{\epsilon}_k \, \Gamma^{(r)} \lambda^k - \bar{\epsilon}_i \, \Gamma^{(r)} \lambda^j \right) \,. \tag{A.10}$$

Note some useful consequences of (A.10) for $\lambda = \epsilon$:

$$\bar{\epsilon}_i \, \epsilon^j \;=\; \frac{1}{2} \left(\bar{\epsilon}_k \, \epsilon^k \right) \, \delta_i^j \,, \tag{A.11}$$

$$\bar{\epsilon}_i \, \gamma^A \epsilon^j \;=\; \frac{1}{2} \left(\bar{\epsilon}_k \, \gamma^A \epsilon^k \right) \, \delta_i^j \,, \tag{A.12}$$

$$\bar{\epsilon}_k \, \gamma^{AB} \epsilon^k \;=\; 0 \,, \qquad \bar{\epsilon}_k \, \gamma^{ABC} \epsilon^k = 0 \,, \tag{A.13}$$

where we used $r = 0, 1, 2, 3$ respectively. The spinors in the Euclidean theory can also be chosen to be symplectic-Majorana, but differently from (A.5), satisfying

$$\bar{\psi}_i = (\psi^i)^\dagger \,, \tag{A.14}$$

with the same definition of the symmplectic Majorana conjugate $\overline{\psi}_i$ as (A.6). However, we note that, as is commented in the begining of section 2.1, we does not impose (A.14) for quantum theory.

B Supersymmetry transformations in Euclidean 5d supergravity

Up to higher order in fermions, the infinitesimal Q- and S-supersymmetry transformations on the spinor fields of the theory are as follows:

$$\begin{aligned}
\delta\Psi^i_M &= 2\mathcal{D}_M\epsilon^i + \frac{i}{2}T_{AB}(3\gamma^{AB}\gamma_M - \gamma_M\gamma^{AB})\epsilon^i - \mathrm{i}\gamma_M\eta^i\,,\\
\delta\chi^i &= \frac{1}{2}\epsilon^i D + \frac{1}{64}R_{MNj}{}^i(V)\gamma^{MN}\epsilon^j + \frac{3\mathrm{i}}{64}(3\gamma^{AB}\gamma^C + \gamma^C\gamma^{AB})\epsilon^i D_C T_{AB}\\
&\quad - \frac{3}{16}T_{AB}T_{CD}\gamma^{ABCD}\epsilon^i + \frac{3}{16}T_{AB}\gamma^{AB}\eta^i\,,\\
\delta\Omega^i &= -\frac{1}{2}(F_{AB} - 4\sigma T_{AB})\gamma^{AB}\epsilon^i - \mathrm{i}\gamma^A\epsilon^i D_A\sigma - 2\varepsilon_{jk}Y^{ij}\epsilon^k + \sigma\eta^i\,,\\
\delta\zeta^\alpha &= -\mathrm{i}\gamma^A\epsilon^i D_A A_i{}^\alpha + \frac{3}{2}A_i{}^\alpha\eta^i\,.
\end{aligned} \tag{B.1}$$

where the curvature $R_{MNi}{}^j(V)$ is given by:

$$R_{MNi}{}^j(V) = 2\,\partial_{[M}V_{N]}{}_i{}^j - 2V_{[M}{}_i{}^k V_{N]}{}_k{}^j\,. \tag{B.2}$$

The relevant supercovariant derivatives acting on each field are covariant with respect to all bosonic gauge symmetries except conformal boosts:

$$\begin{aligned}
D_M\,\epsilon^i &= \left(\partial_M - \frac{1}{4}\,\omega_M{}^{AB}\,\gamma_{AB} + \frac{1}{2}b_M\right)\epsilon^i + \frac{1}{2}V_{Mj}{}^i\,\epsilon^j\,,\\
D_M T_{AB} &= (\partial_M - b_M)T_{AB} - \omega_{MA}{}^C T_{CB} - \omega_{MB}{}^C T_{AC}\,,\\
D_M\,\sigma^I &= (\partial_M - b_M)\,\sigma^I\,,\\
D_M\,A_i{}^\alpha &= \left(\partial_M - \frac{3}{2}b_M\right)A_i{}^\alpha - \frac{1}{2}V_{Mi}{}^j A_j{}^\alpha\,.
\end{aligned} \tag{B.3}$$

C Killing spinors on AdS_3 and S^2

In this appendix, we present solution of the Killing spinor equation (2.19) on the $\mathrm{AdS}_3\times\mathrm{S}^2$ background given in (2.13) and (2.15). Here, let us decompose the spacetime and local indices into those for $3+2$ dimensions as $M = \{\mu, \mathrm{m}\}$ and $A = \{a, \mathrm{a}\}$. Then the Killing spinor equation (2.19) splits as

$$\mathcal{D}_\mu\epsilon^i = s\frac{\mathrm{i}}{4\ell}\gamma^{\hat\theta\hat\phi}\gamma_\mu\epsilon^i\,, \qquad \mathcal{D}_\mathrm{m}\epsilon^i = s\frac{\mathrm{i}}{2\ell}\gamma^{\hat\theta\hat\phi}\gamma_\mathrm{m}\epsilon^i\,, \tag{C.1}$$

where we inserted the sign factor $s = \pm1$ to keep track of the choice of the background value of T_{MN}; $s = +1$ is for our background value of T_{MN} in (2.15), and $s = -1$ is for another background value by changing $T_{MN} \to -T_{MN}$ from the (2.15) (which involves changing $\sigma \to -\sigma$ from (2.14) by the BPS equation of vector multiplet). Note that, since the background metric (2.13) is direct product of 3 and 2 dimensions, the spin connection is also well separated as $-\frac{1}{4}\omega_\mu^{AB}\gamma_{AB} = -\frac{1}{4}\omega_\mu^{ab}\gamma_{ab}$ and $-\frac{1}{4}\omega_\mathrm{m}^{AB}\gamma_{AB} = -\frac{1}{4}\omega_\mathrm{m}^{\mathrm{ab}}\gamma_{\mathrm{ab}}$. This can be seen explicitly by noting that the non-zero spin connection components are

$$\omega_t^{\hat t\hat\rho} = -\sinh\rho\,, \quad \omega_\psi^{\hat\rho\hat\psi} = \cosh\rho\,, \quad \omega_\phi^{\hat\theta\hat\phi} = \cos\theta\,. \tag{C.2}$$

JHEP07(2023)218

We now decompose the spinor as

$$\epsilon^i = \epsilon^i_{AdS_3} \otimes \epsilon^i_{S^2}\,, \tag{C.3}$$

and take the following decomposition for the gamma matrices

$$\gamma_{\hat{t}} = \boldsymbol{\sigma}_0 \otimes \boldsymbol{\tau}_3\,, \quad \gamma_{\hat{\rho}} = \boldsymbol{\sigma}_1 \otimes \boldsymbol{\tau}_3\,, \quad \gamma_{\hat{\psi}} = \boldsymbol{\sigma}_2 \otimes \boldsymbol{\tau}_3\,, \quad \gamma_{\hat{\theta}} = \mathbb{I} \otimes \boldsymbol{\tau}_1\,, \quad \gamma_{\hat{\phi}} = \mathbb{I} \otimes \boldsymbol{\tau}_2\,, \tag{C.4}$$

where $\boldsymbol{\tau}_a$, $a = 1,2,3$, denotes the Pauli sigma matrix and $\boldsymbol{\sigma}_a$ with $a = 0,1,2$ denotes the 3 dimensional gamma matrix. Here we choose $\boldsymbol{\sigma}_0 = -\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2$ such that $\gamma_{\hat{t}\hat{\rho}\hat{\psi}\hat{\theta}\hat{\phi}} = \mathrm{i}$ for our convention. The charge conjugation matrix can also be set to

$$\mathcal{C} = -\mathrm{i}\boldsymbol{\sigma}_2 \otimes \boldsymbol{\tau}_1\,, \tag{C.5}$$

such that the gamma matrix relation (A.3) is satisfied. With this splitting of spinors and gamma matrices, we arrive at the Killing spinor equations for AdS_3 and S^2 with radii 2ℓ and ℓ respectively:

$$0 = \left(\mathcal{D}_\mu \epsilon^i_{AdS_3} + s\frac{1}{4\ell}\boldsymbol{\sigma}_\mu \epsilon^i_{AdS_3}\right) \otimes \epsilon^i_{S^2}\,, \quad 0 = \epsilon^i_{AdS_3} \otimes \left(\mathcal{D}_\mathrm{m}\epsilon^i_{S^2} + s\frac{1}{2\ell}\boldsymbol{\tau}_3\boldsymbol{\tau}_\mathrm{m}\epsilon^i_{S^2}\right)\,. \tag{C.6}$$

The general solutions of these equations are well known [58], and the solutions are given by

$$\epsilon_{AdS_3} = \mathrm{e}^{-s\frac{1}{2}\boldsymbol{\sigma}_1\rho}\mathrm{e}^{-s\frac{1}{2}\boldsymbol{\sigma}_0 t}\mathrm{e}^{\frac{1}{2}\boldsymbol{\sigma}_{12}\psi}A\,, \tag{C.7}$$

$$\epsilon_{S^2} = \mathrm{e}^{-s\mathrm{i}\frac{1}{2}\boldsymbol{\tau}_2\theta}\mathrm{e}^{\mathrm{i}\frac{1}{2}\boldsymbol{\tau}_3\phi}B\,, \tag{C.8}$$

where A and B are constant two-component complex spinors.

Let us write down the Killing spinor explicitly. We set the sign factor $s = 1$, denote the chiral and anti-chiral component of the constant spinors as $A_\pm$ and $B_\pm$, and choose the 3 dimensional gamma matrix representation as

$$\boldsymbol{\sigma}_a = (-\mathrm{i}\,\boldsymbol{\tau}_3\,, \boldsymbol{\tau}_1\,, \boldsymbol{\tau}_2)\,. \tag{C.9}$$

Then we can rewrite the solutions as

$$\epsilon_{\mathrm{AdS}_3} = A_+\epsilon^+_{\mathrm{AdS}} + A_-\epsilon^-_{\mathrm{AdS}}\,, \qquad \epsilon_{\mathrm{S}^2} = B_+\epsilon^+_{\mathrm{S}^2} + B_-\epsilon^-_{\mathrm{S}^2}\,, \tag{C.10}$$

where

$$\epsilon^+_{\mathrm{AdS}_3} = \mathrm{e}^{\frac{\mathrm{i}}{2}(t+\psi)}\begin{pmatrix}\cosh\frac{\rho}{2} \\ -\sinh\frac{\rho}{2}\end{pmatrix}\,, \qquad \epsilon^-_{\mathrm{AdS}_3} = \mathrm{e}^{-\frac{\mathrm{i}}{2}(t+\psi)}\begin{pmatrix}-\sinh\frac{\rho}{2} \\ \cosh\frac{\rho}{2}\end{pmatrix}\,, \tag{C.11}$$

$$\epsilon^+_{\mathrm{S}^2} = \mathrm{e}^{\frac{\mathrm{i}}{2}\phi}\begin{pmatrix}\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\end{pmatrix}\,, \qquad \epsilon^-_{\mathrm{S}^2} = \mathrm{e}^{-\frac{\mathrm{i}}{2}\phi}\begin{pmatrix}-\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2}\end{pmatrix}\,. \tag{C.12}$$

By direct product of the Killing spinors (C.11) and those of (C.12), we obtain four complex basis of Killing spinors as (2.21), or 8 pairs of symplectic Majorana spinors as in (2.23).

Note that the effect of the different sign s is to flip the sign of both ρ and t in the Killing spinors. We also note that in odd dimensions there are two inequivalent representations of gamma matrix. For instance, we can also choose $\boldsymbol{\sigma}_a = (+\mathrm{i}\boldsymbol{\tau}_3, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$ instead of (C.9). Then this is equivalent to the changing the sign of t in the Killing spinors.

JHEP07(2023)218

D Global symmetry generators of AdS_3 and S^2

The global AdS_3 geometry, in coordinates given in (2.13), has isometries generated by the following Killing vectors,

$$\begin{aligned}
\bar{\ell}_- &= \frac{1}{2}\left[\tanh\rho\, \mathrm{e}^{-\mathrm{i}(t-\psi)}\partial_t - \coth\rho\, \mathrm{e}^{-\mathrm{i}(t-\psi)}\partial_\psi + \mathrm{i}\mathrm{e}^{-\mathrm{i}(t-\psi)}\partial_\rho\right] ,\\
\bar{\ell}_0 &= -\frac{\mathrm{i}}{2}(\partial_t - \partial_\psi) ,\\
\bar{\ell}_+ &= -\frac{1}{2}\left[\tanh\rho\, \mathrm{e}^{\mathrm{i}(t-\psi)}\partial_t - \coth\rho\, \mathrm{e}^{\mathrm{i}(t-\psi)}\partial_\psi - \mathrm{i}\mathrm{e}^{\mathrm{i}(t-\psi)}\partial_\rho\right] ,\\
\ell_- &= \frac{1}{2}\left[\tanh\rho\, \mathrm{e}^{-\mathrm{i}(t+\psi)}\partial_t + \coth\rho\, \mathrm{e}^{-\mathrm{i}(t+\psi)}\partial_\psi + \mathrm{i}\mathrm{e}^{-\mathrm{i}(t+\psi)}\partial_\rho\right] ,\\
\ell_0 &= -\frac{\mathrm{i}}{2}(\partial_t + \partial_\psi) ,\\
\ell_+ &= -\frac{1}{2}\left[\tanh\rho\, \mathrm{e}^{\mathrm{i}(t+\psi)}\partial_t + \coth\rho\, \mathrm{e}^{\mathrm{i}(t+\psi)}\partial_\psi - \mathrm{i}\mathrm{e}^{\mathrm{i}(t+\psi)}\partial_\rho\right] .
\end{aligned} \tag{D.1}$$

They form the $\mathrm{SL}(2,\mathbb{R})_L \times \mathrm{SL}(2,\mathbb{R})_R$ algebra through the Lie bracket:

$$\begin{aligned}
\left[\bar{\ell}_0\,,\bar{\ell}_\pm\right]_{\mathrm{Lie}} &= \pm\bar{\ell}_\pm\,, \qquad \left[\bar{\ell}_+\,,\bar{\ell}_-\right]_{\mathrm{Lie}} = -2\bar{\ell}_0\,,\\
\left[\ell_0\,,\ell_\pm\right]_{\mathrm{Lie}} &= \pm\ell_\pm\,, \qquad \left[\ell_+\,,\ell_-\right]_{\mathrm{Lie}} = -2\ell_0\,.
\end{aligned} \tag{D.2}$$

The S^2 geometry, in coordinates given in (2.13), has the isometries generated by the following Killing vectors,

$$\begin{aligned}
j_1 &= \mathrm{i}(\sin\phi\partial_\theta + \cos\phi\cot\theta\partial_\phi) ,\\
j_2 &= -\mathrm{i}(\cos\phi\partial_\theta - \sin\phi\cot\theta\partial_\phi) ,\\
j_3 &= -\mathrm{i}\partial_\phi ,
\end{aligned} \tag{D.3}$$

which satisfy the SO(3) algebra

$$[j_i\,,j_j]_{\mathrm{Lie}} = \mathrm{i}\epsilon_{ijk}j_k\,. \tag{D.4}$$

The bosonic sector of the supersymmetry algebra of $\mathrm{AdS}_3\times\mathrm{S}^2$ presented in section 2.3, contains $\mathrm{SL}(2,\mathbb{R})_R$ and the SO(3) symmetry generators, acting on all the fields of 5d supergravity. Their representations as variations on fields are given by the combination of the differential operators presented in (D.1), (D.3) with the corresponding local Lorentz transformation given as follows:[16]

$$\begin{aligned}
J^1 &= -j^1 + \frac{\mathrm{i}}{2}\delta_M(\lambda_{2\tilde{1}})\,, & J^2 &= -j^2 + \frac{\mathrm{i}}{2}\delta_M(\lambda_{1\tilde{1}})\,, & J^3 &= -j^3\,,\\
L_+ &= -\ell_+ + \frac{1}{2}\delta_M(\mathrm{i}\lambda_{4\tilde{1}} + \lambda_{3\tilde{1}})\,, & L_- &= -\ell_- + \frac{1}{2}\delta_M(\mathrm{i}\lambda_{4\tilde{1}} - \lambda_{3\tilde{1}})\,, & L_0 &= -\ell_0\,.
\end{aligned} \tag{D.5}$$

[16]The negative sign in front of j^a, $\ell_{\pm,0}$ appears from the change in representation as differential operators on functions to variational action on fields [36].

4d Weyl	$e_\mu{}^a$, ψ^i_a, A^D_μ, A^R_μ, $\mathcal{V}_\mu{}^i{}_j$, $T^\pm_{ab}$, $\mathcal{D}$, χ^i_{4d}
4d Vector	$X^\mathcal{I}, \overline{X}^\mathcal{I}, A^\mathcal{I}_\mu, \mathcal{Y}^\mathcal{I}_{ij}, \lambda^{\mathcal{I}\,i}$
4d Hyper	$\mathcal{A}_i{}^\alpha$, ζ^α_{4d}
4d SUSY parameters	ϵ^i_{4d}, η^i_{4d}

Table 4. Independent fields of the supersymmetric multiplets and Q, S-supersymmetry parameters in four-dimensional $\mathcal{N}=2$ conformal supergravity.

Here, $\delta_M(\hat{\lambda}_{a\tilde{b}})$ is the local Lorentz transformation in the $\{\mathcal{Q}_a, \tilde{\mathcal{Q}}_b\}$ algebra, as it appears in (2.6), with field dependent parameters $(\lambda_{a\tilde{b}})_{AB}$.[17] On the background (2.13)–(2.16), their values are

$$\begin{aligned}
&(\lambda_{1\tilde{1}})_{\hat{\theta}\hat{\phi}} = 2\frac{\sin\phi}{\sin\theta}\,, && (\lambda_{2\tilde{1}})_{\hat{\theta}\hat{\phi}} = 2\frac{\cos\phi}{\sin\theta}\,, && \\
&(\lambda_{3\tilde{1}})_{\hat{t}\hat{\rho}} = \frac{\cos(t+\psi)}{\cosh\rho}\,, && (\lambda_{3\tilde{1}})_{\hat{t}\hat{\psi}} = -\sin(t+\psi)\,, && (\lambda_{3\tilde{1}})_{\hat{\rho}\hat{\psi}} = -\frac{\cos(t+\psi)}{\sinh\rho} \\
&(\lambda_{4\tilde{1}})_{\hat{t}\hat{\rho}} = \frac{\sin(t+\psi)}{\cosh\rho}\,, && (\lambda_{4\tilde{1}})_{\hat{t}\hat{\psi}} = \cos(t+\psi)\,, && (\lambda_{4\tilde{1}})_{\hat{\rho}\hat{\psi}} = -\frac{\sin(t+\psi)}{\sinh\rho}\,.
\end{aligned} \tag{D.6}$$

E Euclidean 4d supersymmetry and $\mathrm{AdS}_2 \times \mathrm{S}^2$

In this appendix, we present the supersymmetry transformation of the fermions in Euclidean 4d conformal supergravity, following the convention of [15], and setting all fermions to zero. The field content in Euclidean 4d superconformal gravity is given in table 4. We also present the Euclidean $\mathrm{AdS}_2 \times \mathrm{S}^2$ background and its Killing spinors. All fields appearing in this section refer to four-dimensional ones, so we omit the 4d subscripts.

Euclidean 4d supersymmetry transformations: the Q and S-supersymmetry transformations of the fermionic fields are

$$\begin{aligned}
\delta\psi^i_\mu &= 2D_\mu\varepsilon^i + \mathrm{i}\frac{1}{16}\gamma_{ab}(T^{ab+}+T^{ab-})\gamma_\mu\varepsilon^i + \gamma_\mu\gamma_5\eta^i\,, \\
\delta\chi^i &= \frac{\mathrm{i}}{24}\gamma_{ab}\not{D}(T^{ab+}+T^{ab-})\varepsilon^i + \frac{1}{6}\widehat{R}(\mathcal{V})^i{}_{j\mu\nu}\gamma^{\mu\nu}\varepsilon^j - \frac{1}{3}\widehat{R}(A^R)_{\mu\nu}\gamma^{\mu\nu}\gamma_5\varepsilon^i \\
&\quad + \mathcal{D}\,\varepsilon^i + \mathrm{i}\frac{1}{24}(T^+_{ab}+T^-_{ab})\gamma^{ab}\gamma_5\eta^i\,, \\
\delta\lambda^i_+ &= -2\mathrm{i}\gamma^a D_a X\varepsilon^i_- - \frac{1}{2}\mathcal{F}_{ab}\gamma^{ab}\varepsilon^i_+ + \mathcal{Y}^{ij}\varepsilon_{jk}\varepsilon^k_+ + 2\mathrm{i}X\eta^i_+\,, \\
\delta\lambda^i_- &= -2\mathrm{i}\gamma^a D_a \overline{X}\varepsilon^i_+ - \frac{1}{2}\mathcal{F}_{ab}\gamma^{ab}\varepsilon^i_- + \mathcal{Y}^{ij}\varepsilon_{jk}\varepsilon^k_- - 2\mathrm{i}\overline{X}\eta^i_-\,, \\
\delta\zeta^\alpha &= \not{D}\mathcal{A}_i{}^\alpha\varepsilon^i - \mathcal{A}_i{}^\alpha\gamma_5\eta^i\,,
\end{aligned} \tag{E.1}$$

[17] The $\delta_M((\lambda_{a\tilde{b}})_{AB})$ acts on a spinor ψ as $\frac{1}{4}(\lambda_{a\tilde{b}})_{AB}\gamma^{AB}\psi$, and on a vector V^A as $(\lambda_{a\tilde{b}})^A{}_B V^B$.

where:

$$\mathcal{F}_{\mu\nu} = F_{\mu\nu} - \left(\frac{1}{4}\overline{X}\,T^-_{\mu\nu} + \frac{1}{4}X\,T^+_{\mu\nu}\right) . \tag{E.2}$$

The covariant derivatives are:

$$D_\mu\varepsilon^i = \left(\partial_\mu - \frac{1}{4}\omega_{\mu ab}\gamma^{ab} + \frac{1}{2}A^D_\mu + \frac{1}{2}A^R_\mu\gamma_5\right)\varepsilon^i + \frac{1}{2}\mathcal{V}_\mu{}^i{}_j\varepsilon^j , \tag{E.3}$$

$$D_\mu X = (\partial_\mu - A^D_\mu + A^R_\mu)X , \tag{E.4}$$

$$D_\mu\overline{X} = (\partial_\mu - A^D_\mu - A^R_\mu)\overline{X} , \tag{E.5}$$

$$D_\mu\mathcal{A}_i{}^\alpha = (\partial_\mu\mathcal{A}_i{}^\alpha - b_\mu)\,\mathcal{A}_i{}^\alpha + \frac{1}{2}\mathcal{V}_\mu{}^j{}_i\mathcal{A}_j{}^\alpha , \tag{E.6}$$

and the curvatures are:

$$\begin{aligned} \widehat{R}_{\mu\nu}(A^R) &= 2\partial_{[\mu}A^R_{\nu]} , \\ \widehat{R}_{\mu\nu}(\mathcal{V})^i{}_j &= 2\partial_{[\mu}\mathcal{V}_{\nu]}{}^i{}_j + \mathcal{V}_{[\mu}{}^i{}_k\mathcal{V}_{\nu]}{}^k{}_j . \end{aligned} \tag{E.7}$$

Supersymmetric $\mathbf{AdS_2 \times S^2}$ background and Killing spinors: Recall the fully supersymmetric, Euclidean $\mathrm{AdS}_2 \times \mathrm{S}^2$ solution of the 4d theory considered in (4.25):

$$ds^2 = \ell^2\left[d\eta^2 + \sinh^2\eta\, d\chi^2 + d\theta^2 + \sin^2\theta\, d\phi^2\right] , \tag{E.8}$$

$$F^{\mathcal{I}}_{12} = -i\frac{e^{\mathcal{I}}}{\ell^2} , \qquad F^{\mathcal{I}}_{34} = \frac{p^{\mathcal{I}}}{\ell^2} , \qquad \leftrightarrow A^{\mathcal{I}} = -\mathrm{i}e^{\mathcal{I}}(\cosh\eta - 1)d\chi - p^{\mathcal{I}}\cos\theta d\phi \tag{E.9}$$

$$X^{\mathcal{I}} = \frac{\omega}{8}(e^{\mathcal{I}} + ip^{\mathcal{I}}) , \quad \bar{X}^{\mathcal{I}} = \frac{\bar{\omega}}{8}(e^{\mathcal{I}} - ip^{\mathcal{I}}) , \qquad \mathcal{I} = 0, 1, \cdots, N_{\mathrm{v}} \tag{E.10}$$

$$T^-_{12} = -\mathrm{i}\omega , \qquad T^-_{34} = \mathrm{i}\omega , \qquad T^+_{12} = -\mathrm{i}\bar{\omega} , \qquad T^+_{34} = -\mathrm{i}\bar{\omega} . \tag{E.11}$$

Here, ℓ is the radius of AdS_2 and S^2, and ω, $\overline{\omega}$ are two independent complex constants satisfying

$$\ell^2 = \frac{16}{\bar{\omega}\omega} . \tag{E.12}$$

As discussed in section 4.2, we may pick the $\mathrm{SO}(1,1)_R$ gauge (4.29) such that (E.12) implies the following parametrization:

$$\omega(\alpha) = \frac{4}{\ell}e^{\mathrm{i}\alpha} , \qquad \overline{\omega}(\alpha) = \frac{4}{\ell}e^{-\mathrm{i}\alpha} , \qquad \alpha \in \mathbb{R} . \tag{E.13}$$

Here, we choose $\alpha = \pi/2$ and derive the corresponding Killing spinors.

We express the $\mathrm{AdS}_2 \times \mathrm{S}^2$ metric above in vielbein form:

$$e^1 = \ell\, d\eta , \quad e^2 = \ell\sinh\eta\, d\chi , \quad e^3 = \ell d\theta , \quad e^4 = \ell\sin\theta d\phi . \tag{E.14}$$

We also choose the following gamma matrix representation, where $\boldsymbol{\tau}_a$ and σ_a, $a = 1, 2, 3$ are the Pauli matrices

$$\gamma_1 = \boldsymbol{\tau}_1 \otimes \sigma_3 , \quad \gamma_2 = \boldsymbol{\tau}_2 \otimes \sigma_3 , \quad \gamma_3 = \mathbb{I}_2 \otimes \sigma_1 , \quad \gamma_4 = \mathbb{I}_2 \otimes \sigma_2 , \quad \gamma_5 = \gamma_{1234} = -\boldsymbol{\tau}_3 \otimes \sigma_3 . \tag{E.15}$$

With this representation, the four-dimensional Killing spinor equation, given in (E.1) as

$$\mathcal{D}_\mu \varepsilon = -\frac{\mathrm{i}}{32}(T^+_{ab} + T^-_{ab})\gamma_{ab}\gamma_\mu \varepsilon = -\frac{1}{2\ell}(\mathbb{I}_2 \times \sigma_3)\gamma_\mu \varepsilon\,, \tag{E.16}$$

splits into the Killing spinor equations of AdS_2 and S^2. Indeed, decomposing the spinor $\varepsilon = \varepsilon_{\mathrm{AdS}_2} \otimes \varepsilon_{\mathrm{S}^2}$, one obtains the AdS_2 part as

$$(\partial_\mu + \omega_\mu)\varepsilon_{\mathrm{AdS}_2} = -\frac{1}{2}\boldsymbol{\tau}_\mu\, \varepsilon_{\mathrm{AdS}_2}\,, \quad \omega_\chi = -\frac{\mathrm{i}}{2}\cosh\eta\, \boldsymbol{\tau}_3\,, \tag{E.17}$$

and the S^2 as

$$(\partial_\mu + \omega_\mu)\varepsilon_{\mathrm{S}^2} = -\frac{1}{2}\sigma_3\sigma_\mu\varepsilon_{\mathrm{S}^2}\,, \quad \omega_\phi = -\frac{\mathrm{i}}{2}\cos\theta\, \sigma_3 \tag{E.18}$$

The Killing spinors for AdS_2 and S^2 are given by

$$\varepsilon^+_{\mathrm{AdS}_2} = e^{\frac{\mathrm{i}}{2}\chi}\begin{pmatrix} -\cosh\frac{\eta}{2} \\ \sinh\frac{\eta}{2}\end{pmatrix}\,, \qquad \varepsilon^-_{\mathrm{AdS}_2} = e^{-\frac{\mathrm{i}}{2}\chi}\begin{pmatrix} \sinh\frac{\eta}{2} \\ -\cosh\frac{\eta}{2}\end{pmatrix}\,, \tag{E.19}$$

and

$$\varepsilon^+_{\mathrm{S}^2} = e^{\frac{\mathrm{i}}{2}\phi}\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\end{pmatrix}\,, \qquad \varepsilon^-_{\mathrm{S}^2} = e^{-\frac{\mathrm{i}}{2}\phi}\begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2}\end{pmatrix}\,. \tag{E.20}$$

Taking the direct product of the spinors (E.17) on AdS_2 with the spinors (E.18) on S^2, we obtain the following complex basis of Killing spinors on $\mathrm{AdS}_2 \times \mathrm{S}^2$:

$$\begin{aligned} \dot\varepsilon_+{}^+ &= \sqrt{\frac{\ell}{2}}\,\varepsilon^+_{\mathrm{AdS}_2} \otimes \varepsilon^+_{\mathrm{S}^2}\,, & \dot\varepsilon_+{}^- &= \sqrt{\frac{\ell}{2}}\,\varepsilon^+_{\mathrm{AdS}_2} \otimes \varepsilon^-_{\mathrm{S}^2}\,, \\ \dot\varepsilon_-{}^+ &= \sqrt{\frac{\ell}{2}}\,\varepsilon^-_{\mathrm{AdS}_2} \otimes \varepsilon^+_{\mathrm{S}^2}\,, & \dot\varepsilon_-{}^- &= \sqrt{\frac{\ell}{2}}\,\varepsilon^-_{\mathrm{AdS}_2} \otimes \varepsilon^-_{\mathrm{S}^2}\,. \end{aligned} \tag{E.21}$$

Note that, these spinors are identical to the Killing spinors on the Kaluza-Klein frame of $\mathrm{AdS}_3 \times \mathrm{S}^2$, given in (4.16). The spinors (E.21) organize themselves to form the following 8 real set of basis for Killing spinors on $\mathrm{AdS}_2 \times \mathrm{S}^2$,

$$\begin{aligned} \dot\varepsilon^i_{(1)} &= (-\mathrm{i}\dot\varepsilon_+{}^+\,, \dot\varepsilon_-{}^-)\,, & \dot\varepsilon^i_{(2)} &= (\dot\varepsilon_+{}^+\,, -\mathrm{i}\dot\varepsilon_-{}^-)\,, \\ \dot\varepsilon^i_{(3)} &= (-\dot\varepsilon_-{}^-\,, -\mathrm{i}\dot\varepsilon_+{}^+)\,, & \dot\varepsilon^i_{(4)} &= (-\mathrm{i}\dot\varepsilon_-{}^-\,, -\dot\varepsilon_+{}^+)\,, \\ \dot{\dot\varepsilon}^i_{(1)} &= (\dot\varepsilon_+{}^-\,, \mathrm{i}\dot\varepsilon_-{}^+)\,, & \dot{\dot\varepsilon}^i_{(2)} &= (\mathrm{i}\dot\varepsilon_+{}^-\,, \dot\varepsilon_-{}^+)\,, \\ \dot{\dot\varepsilon}^i_{(3)} &= (-\mathrm{i}\dot\varepsilon_-{}^+\,, \dot\varepsilon_+{}^-)\,, & \dot{\dot\varepsilon}^i_{(4)} &= (\dot\varepsilon_-{}^+\,, -\mathrm{i}\dot\varepsilon_+{}^-)\,, \end{aligned} \tag{E.22}$$

which is the same basis as for the 5d KK-frame (4.15). The spinors in (E.22) satisfy

$$(\varepsilon^i)^\dagger \mathrm{i}\gamma_5 = \epsilon_{ij}(\varepsilon^j)^T C\,. \tag{E.23}$$

which is indeed the reality condition, given in (4.22), for $\alpha = \pi/2$.

References

[1] A. Dabholkar, J. Gomes and S. Murthy, *Quantum black holes, localization and the topological string*, *JHEP* **06** (2011) 019 [arXiv:1012.0265] [INSPIRE].

[2] A. Dabholkar, J. Gomes and S. Murthy, *Localization & Exact Holography*, *JHEP* **04** (2013) 062 [arXiv:1111.1161] [INSPIRE].

[3] A. Dabholkar, J. Gomes and S. Murthy, *Nonperturbative black hole entropy and Kloosterman sums*, *JHEP* **03** (2015) 074 [arXiv:1404.0033] [INSPIRE].

[4] L.V. Iliesiu, S. Murthy and G.J. Turiaci, *Black hole microstate counting from the gravitational path integral*, arXiv:2209.13602 [INSPIRE].

[5] R.K. Gupta, S. Murthy and M. Sahni, *Quantum entropy of BMPV black holes and the topological M-theory conjecture*, *JHEP* **06** (2022) 053 [arXiv:2104.02634] [INSPIRE].

[6] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, *Phys. Lett. B* **379** (1996) 99 [hep-th/9601029] [INSPIRE].

[7] J.M. Maldacena, G.W. Moore and A. Strominger, *Counting BPS black holes in toroidal Type II string theory*, hep-th/9903163 [INSPIRE].

[8] J.M. Maldacena, A. Strominger and E. Witten, *Black hole entropy in M theory*, *JHEP* **12** (1997) 002 [hep-th/9711053] [INSPIRE].

[9] C. Vafa, *Black holes and Calabi-Yau threefolds*, *Adv. Theor. Math. Phys.* **2** (1998) 207 [hep-th/9711067] [INSPIRE].

[10] B. Haghighat, S. Murthy, C. Vafa and S. Vandoren, *F-Theory, Spinning Black Holes and Multi-string Branches*, *JHEP* **01** (2016) 009 [arXiv:1509.00455] [INSPIRE].

[11] A. Castro, J.L. Davis, P. Kraus and F. Larsen, 5*D attractors with higher derivatives*, *JHEP* **04** (2007) 091 [hep-th/0702072] [INSPIRE].

[12] A. Castro, J.L. Davis, P. Kraus and F. Larsen, 5*D Black Holes and Strings with Higher Derivatives*, *JHEP* **06** (2007) 007 [hep-th/0703087] [INSPIRE].

[13] A. Castro, J.L. Davis, P. Kraus and F. Larsen, *String Theory Effects on Five-Dimensional Black Hole Physics*, *Int. J. Mod. Phys. A* **23** (2008) 613 [arXiv:0801.1863] [INSPIRE].

[14] B. de Wit, S. Murthy and V. Reys, *BRST quantization and equivariant cohomology: localization with asymptotic boundaries*, *JHEP* **09** (2018) 084 [arXiv:1806.03690] [INSPIRE].

[15] I. Jeon and S. Murthy, *Twisting and localization in supergravity: equivariant cohomology of BPS black holes*, *JHEP* **03** (2019) 140 [arXiv:1806.04479] [INSPIRE].

[16] N. Banerjee, B. de Wit and S. Katmadas, *The Off-Shell* 4*D*/5*D Connection*, *JHEP* **03** (2012) 061 [arXiv:1112.5371] [INSPIRE].

[17] D. Gaiotto, A. Strominger and X. Yin, *New connections between* 4*D and* 5*D black holes*, *JHEP* **02** (2006) 024 [hep-th/0503217] [INSPIRE].

JHEP07(2023)218

[18] J. Gomes, *Quantum entropy and exact 4d/5d connection*, JHEP **01** (2015) 109 [arXiv:1305.2849] [INSPIRE].

[19] R.K. Gupta, S. Murthy and M. Sahni, *On the localization manifold of* 5d *supersymmetric spinning black holes*, JHEP **10** (2019) 172 [arXiv:1904.08876] [INSPIRE].

[20] V. Cortes, C. Mayer, T. Mohaupt and F. Saueressig, *Special geometry of Euclidean supersymmetry. Part 1. Vector multiplets*, JHEP **03** (2004) 028 [hep-th/0312001] [INSPIRE].

[21] V. Cortes, C. Mayer, T. Mohaupt and F. Saueressig, *Special geometry of Euclidean supersymmetry. Part II. Hypermultiplets and the c-map*, JHEP **06** (2005) 025 [hep-th/0503094] [INSPIRE].

[22] V. Cortes and T. Mohaupt, *Special Geometry of Euclidean Supersymmetry. Part III. The Local r-map, instantons and black holes*, JHEP **07** (2009) 066 [arXiv:0905.2844] [INSPIRE].

[23] B. de Wit and V. Reys, *Euclidean supergravity*, JHEP **12** (2017) 011 [arXiv:1706.04973] [INSPIRE].

[24] E. Witten, *Elliptic Genera and Quantum Field Theory*, Commun. Math. Phys. **109** (1987) 525 [INSPIRE].

[25] A.N. Schellekens and N.P. Warner, *Anomalies, Characters and Strings*, Nucl. Phys. B **287** (1987) 317 [INSPIRE].

[26] O. Alvarez, T.P. Killingback, M.L. Mangano and P. Windey, *String Theory and Loop Space Index Theorems*, Commun. Math. Phys. **111** (1987) 1 [INSPIRE].

[27] F. Benini, R. Eager, K. Hori and Y. Tachikawa, *Elliptic genera of two-dimensional $N = 2$ gauge theories with rank-one gauge groups*, Lett. Math. Phys. **104** (2014) 465 [arXiv:1305.0533] [INSPIRE].

[28] A. Cabo-Bizet, D. Cassani, D. Martelli and S. Murthy, *Microscopic origin of the Bekenstein-Hawking entropy of supersymmetric AdS_5 black holes*, JHEP **10** (2019) 062 [arXiv:1810.11442] [INSPIRE].

[29] F. Larsen and S. Lee, *Microscopic entropy of AdS_3 black holes revisited*, JHEP **07** (2021) 038 [arXiv:2101.08497] [INSPIRE].

[30] J.D. Brown and M. Henneaux, *Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity*, Commun. Math. Phys. **104** (1986) 207 [INSPIRE].

[31] O. Coussaert and M. Henneaux, *Supersymmetry of the $(2 + 1)$ black holes*, Phys. Rev. Lett. **72** (1994) 183 [hep-th/9310194] [INSPIRE].

[32] B. Assel, D. Martelli, S. Murthy and D. Yokoyama, *Localization of supersymmetric field theories on non-compact hyperbolic three-manifolds*, JHEP **03** (2017) 095 [arXiv:1609.08071] [INSPIRE].

[33] K. Costello and N.M. Paquette, *Twisted Supergravity and Koszul Duality: A case study in AdS_3*, Commun. Math. Phys. **384** (2021) 279 [arXiv:2001.02177] [INSPIRE].

[34] T.D. Brennan, F. Carta and C. Vafa, *The String Landscape, the Swampland, and the Missing Corner*, PoS **TASI2017** (2017) 015 [arXiv:1711.00864] [INSPIRE].

[35] P. Benetti Genolini, J.M. Perez Ipiña and J. Sparks, *Localization of the action in AdS/CFT*, JHEP **10** (2019) 252 [arXiv:1906.11249] [INSPIRE].

JHEP07(2023)218

[36] D.Z. Freedman and A. Van Proeyen, *Supergravity*, Cambridge University Press, Cambridge, U.K. (2012).

[37] E. Bergshoeff, T. de Wit, R. Halbersma, S. Cucu, M. Derix and A. Van Proeyen, *Weyl multiplets of $N = 2$ conformal supergravity in five-dimensions*, JHEP **06** (2001) 051 [hep-th/0104113] [INSPIRE].

[38] E. Bergshoeff et al., *Superconformal $N = 2$, $D = 5$ matter with and without actions*, JHEP **10** (2002) 045 [hep-th/0205230] [INSPIRE].

[39] E. Bergshoeff, S. Cucu, T. de Wit, J. Gheerardyn, S. Vandoren and A. Van Proeyen, *$N = 2$ supergravity in five-dimensions revisited*, Class. Quant. Grav. **21** (2004) 3015 [hep-th/0403045] [INSPIRE].

[40] T. Kugo and K. Ohashi, *Supergravity tensor calculus in $5D$ from $6D$*, Prog. Theor. Phys. **104** (2000) 835 [hep-ph/0006231] [INSPIRE].

[41] T. Kugo and K. Ohashi, *Off-shell $D = 5$ supergravity coupled to matter Yang-Mills system*, Prog. Theor. Phys. **105** (2001) 323 [hep-ph/0010288] [INSPIRE].

[42] T. Fujita and K. Ohashi, *Superconformal tensor calculus in five-dimensions*, Prog. Theor. Phys. **106** (2001) 221 [hep-th/0104130] [INSPIRE].

[43] B. de Wit and S. Katmadas, *Near-Horizon Analysis of $D = 5$ BPS Black Holes and Rings*, JHEP **02** (2010) 056 [arXiv:0910.4907] [INSPIRE].

[44] R.K. Gupta and S. Murthy, *All solutions of the localization equations for $N = 2$ quantum black hole entropy*, JHEP **02** (2013) 141 [arXiv:1208.6221] [INSPIRE].

[45] R.K. Gupta, Y. Ito and I. Jeon, *Supersymmetric Localization for BPS Black Hole Entropy: 1-loop Partition Function from Vector Multiplets*, JHEP **11** (2015) 197 [arXiv:1504.01700] [INSPIRE].

[46] N. Hama and K. Hosomichi, *Seiberg-Witten Theories on Ellipsoids*, JHEP **09** (2012) 033 [*Addendum ibid.* **10** (2012) 051] [arXiv:1206.6359] [INSPIRE].

[47] S. Murthy and V. Reys, *Functional determinants, index theorems, and exact quantum black hole entropy*, JHEP **12** (2015) 028 [arXiv:1504.01400] [INSPIRE].

[48] H. Elvang, R. Emparan, D. Mateos and H.S. Reall, *A Supersymmetric black ring*, Phys. Rev. Lett. **93** (2004) 211302 [hep-th/0407065] [INSPIRE].

[49] B. Guo, M.R.R. Hughes, S.D. Mathur and M. Mehta, *Universal lifting in the D1-D5 CFT*, JHEP **10** (2022) 148 [arXiv:2208.07409] [INSPIRE].

[50] P. Kraus, *Lectures on black holes and the AdS_3/CFT_2 correspondence*, in *Lecture Notes in Physics* **755**, Springer (2008), pp. 193–247 [hep-th/0609074] [INSPIRE].

[51] S. Elitzur, G.W. Moore, A. Schwimmer and N. Seiberg, *Remarks on the Canonical Quantization of the Chern-Simons-Witten Theory*, Nucl. Phys. B **326** (1989) 108 [INSPIRE].

[52] C. Closset, L. Di Pietro and H. Kim, *'t Hooft anomalies and the holomorphy of supersymmetric partition functions*, JHEP **08** (2019) 035 [arXiv:1905.05722] [INSPIRE].

[53] M. Bianchi, D.Z. Freedman and K. Skenderis, *Holographic renormalization*, Nucl. Phys. B **631** (2002) 159 [hep-th/0112119] [INSPIRE].

[54] K.P. Tod, *All Metrics Admitting Supercovariantly Constant Spinors*, Phys. Lett. B **121** (1983) 241 [INSPIRE].

JHEP07(2023)218

[55] J.P. Gauntlett, J.B. Gutowski, C.M. Hull, S. Pakis and H.S. Reall, *All supersymmetric solutions of minimal supergravity in five-dimensions*, Class. Quant. Grav. **20** (2003) 4587 [hep-th/0209114] [INSPIRE].

[56] J. Scherk and J.H. Schwarz, *How to Get Masses from Extra Dimensions*, Nucl. Phys. B **153** (1979) 61 [INSPIRE].

[57] B. Sahoo and A. Sen, *Higher derivative corrections to non-supersymmetric extremal black holes in $N = 2$ supergravity*, JHEP **09** (2006) 029 [hep-th/0603149] [INSPIRE].

[58] H. Lu, C.N. Pope and J. Rahmfeld, *A Construction of Killing spinors on S^n*, J. Math. Phys. **40** (1999) 4518 [hep-th/9805151] [INSPIRE].